The Journal of Fourier Analysis and Applications

Volume 11, Issue 4, 2005

On Fejer and Bochner-Riesz Means

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Communicated by T. Körner

ABSTRACT. For the Fejer means on $L_p(R)$, $1 \le p \le \infty$ an equivalence between the rate of its convergence and an appropriate K-functional is established. For the Bochner-Riesz means on $L_p(R^d)$, $1 \le p \le \infty$, $d = 1, 2, \ldots$ an equivalence between the rate of convergence and the corresponding K-functional is obtained. The results are of the form of strong converse inequality of type A.

1. Introduction

Optimal (best up to a constant) quantitative estimates of the rate of approximation of an approximation process is a desired property in the study of the process. Such estimates are given here for the Fejer and the Bochner-Riesz means.

For an approximation process A_λ on a Banach space B of functions a quantitative estimate of the rate of approximation is usually given using a K-functional $K(f, Q, \lambda^{-\mu})_B$, that is,

$$\|f - A_\lambda f\|_B \le C \, \inf \left(\|f - g\|_B + \lambda^{-\mu} \|Qg\|_B \right) \equiv CK\left(f, Q, \lambda^{-\mu}\right)_B \tag{1.1}$$

where Q is an unbounded (usually differential) operator on B and $\mu > 0$. Estimates of this type [like (1.1)] are called direct results. It is the strong converse inequality in the terminology of [2] with the same K-functional that establishes it (the K-functional) with μ to be the appropriate measure of smoothness for investigating quantitatively the rate of convergence of $A_\lambda f$ to f.

The weakest form of a strong converse inequality is

$$K\left(f, Q, \lambda^{-\mu}\right) \le A \sup_{\nu \ge \lambda} \|A_\nu f - f\|_B \,, \tag{1.2}$$

Math Subject Classifications. 42A45, 42B15, 41A27.
Keywords and Phrases. Fejer and Bochner-Riesz means, direct estimates, strong converse inequalities.
Acknowledgements and Notes. Author was supported by NSERC grant of Canada A4816.

ISSN 1069-5869 DOI: 10.1007/s00041-005-5001-1

which is a strong converse inequality of type D in the terminology of [2], and which is still stronger than most converse inequalities appearing in the earlier literature.

In this article we will deal with strong converse inequality of type A in the terminology of [2] (the most refined) given by

$$\frac{1}{A}\, K\big(f, Q, \lambda^{-\mu}\big)_B \le \|A_\lambda f - f\|_B \le C K\big(f, Q, \lambda^{-\mu}\big)_B\,, \tag{1.3}$$

that is, without the supremum given in (1.2). In addition to being optimal and clearly superior to the combination of (1.1) and (1.2), this form has further benefits and establishes $\|A_\lambda f - f\|_B$ as another measure of smoothness.

For some approximation processes and appropriate K-functionals such inequalities were established in [2]. In several other articles using various techniques other strong converse inequalities were proved. Perhaps the most notable is Totik's result [5] yielding strong converse inequality of type A for the Bernstein polynomial approximation in $C[0, 1]$.

In the present article strong converse inequalities of type A are given for the well studied Fejer and Riesz-Bochner means. Reasonable estimates on the constants A and C of (1.3) are also achieved.

2. The Fejer Means

The Fejer means on R are given by

$$F_\lambda(f, x) \equiv G_\lambda * f(x) \equiv \frac{1}{2\pi\lambda} \int_{-\infty}^{\infty} \left(\frac{\sin\frac{\lambda y}{2}}{y/2}\right)^2 f(x-y)\,dy, \quad \lambda > 0\,. \tag{2.1}$$

The Hilbert transform of f is given by

$$Hf(x) = \text{ P.V. } \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x-u)}{u}\,du \equiv \lim_{\delta\to 0+} \frac{1}{\pi} \int_{|u|>\delta} \frac{f(x-u)}{u}\,du\,. \tag{2.2}$$

We can now state and prove the following theorem.

Theorem 1. *For $f \in L_p(R)$, $1 \le p < \infty$, or $C_0(R)$ for $p = \infty$, we have*

$$\frac{1}{2}\, K\Big(f, \frac{1}{\lambda}\Big)_{L_p(R)} \le \|F_\lambda(f,\cdot) - f(\cdot)\|_{L_p(R)} \le 3K\Big(f, \frac{1}{\lambda}\Big)_{L_p(R)} \tag{2.3}$$

where

$$K\Big(f, \frac{1}{\lambda}\Big)_{L_p(R)} = \inf\left(\|f-g\|_{L_p(R)} + \frac{1}{\lambda}\left\|\frac{d}{dx}\, Hg(x)\right\|_{L_p(R)}\right) \tag{2.4}$$

and the infimum is taken on all g such that $\frac{d}{dx}\, Hg(x)$, which is defined as an element of $\mathcal{S}'$ (the tempered distributions), is in $L_p(R)$.

Remarks. We note that (2.3) constitutes a strong converse result of type A (see [2]). In the definition of the K-functional the infimum could be taken on a much more restricted class of functions without changing the value of the K-functional.

Proof. We first establish the identity

$$G_\lambda * f - G_\lambda * G_\lambda * f = \frac{1}{\lambda}\, \frac{d}{dx}\, H(G_\lambda * f)\,. \tag{2.5}$$

The Fourier and inverse Fourier transforms $\widehat{f}$ and $\overset{\vee}{F}$ are given by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-ix\xi}\, dx \quad \text{and} \quad \overset{\vee}{F}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\xi) e^{ix\xi}\, d\xi \,. \tag{2.6}$$

Recalling that

$$(G_\lambda * f)^\wedge(\xi) = \Big(1 - \frac{|\xi|}{\lambda}\Big)_+ \widehat{f}(\xi), \quad [G_\lambda * (G_\lambda * f)]^\wedge(\xi) = \Big(1 - \frac{|\xi|}{\lambda}\Big)_+^2 \widehat{f}(\xi)$$

and

$$\Big[\frac{d}{dx} H(G_\lambda * f)\Big]^\wedge(\xi) = i\xi(-i \operatorname{sgn} \xi)\Big(1 - \frac{|\xi|}{\lambda}\Big)_+ \widehat{f}(\xi) = |\xi|\Big(1 - \frac{|\xi|}{\lambda}\Big)_+ \widehat{f}(\xi)\,,$$

we observe that the Fourier transforms of both sides of (2.5) are identical. We note that while for $f \in L_p(R)$, $1 \le p \le 2$, $\widehat{f}(\xi)$ is always a function, for $f \in L_p(R)$, $2 < p \le \infty$ we can only state that $\widehat{f}(\xi)$ is an element of $\mathcal{S}'$. For f in L_p, $G_\lambda * f$ and $G_\lambda * G_\lambda * f$ are in L_p and therefore so is $\frac{d}{dx} H(G_\lambda * f)$. We now use (2.5) and $\|G_\lambda\|_{L_1(R)} = 1$ to obtain

$$\begin{aligned} K\Big(f, \frac{1}{\lambda}\Big)_{L_p(R)} &\le \|f - G_\lambda * f\|_p + \frac{1}{\lambda} \Big\| \frac{d}{dx} H(G_\lambda * f)\Big\|_p \\ &= \|f - G_\lambda * f\|_p + \|G_\lambda * (f - G_\lambda * f)\|_p \\ &\le 2\|f - G_\lambda * f\|_p \,, \end{aligned}$$

which is the converse result and the left inequality of (2.3). To obtain the direct result we choose $g = g_{\varepsilon,\lambda,p}$ such that

$$(1+\varepsilon) K\Big(f, \frac{1}{\lambda}\Big)_p \ge \|f - g\|_p + \frac{1}{\lambda} \Big\| \frac{d}{dx} Hg \Big\|_p \,.$$

We now write

$$\begin{aligned} \|f - G_\lambda * f\|_p &\le \|(f-g) - G_\lambda(f-g)\|_p + \|g - G_\lambda * g\|_p \\ &\le 2\|f - g\|_p + \|g - G_\lambda * g\|_p \,. \end{aligned}$$

We will complete the proof when we show

$$\|g - G_\lambda * g\|_p \le 3\, \frac{1}{\lambda} \Big\| \frac{d}{dx} Hg \Big\|_p \,, \tag{2.7}$$

since then

$$\|f - G_\lambda * f\|_p \le 3(1+\varepsilon) K\Big(f, \frac{1}{\lambda}\Big)_p$$

and $\varepsilon > 0$ is arbitrary.

To establish (2.7) we write

$$\begin{aligned} \|g - G_\lambda * g\|_p &\le \|G_\lambda * g - G_\lambda * G_\lambda * g\|_p \\ &\quad + \|G_\lambda * G_\lambda * g - G_\Lambda * G_\Lambda * g\|_p + \|G_\Lambda * G_\Lambda * g - g\|_p \\ &\equiv I_1 + I_2(\Lambda) + I_3(\Lambda)\,. \end{aligned}$$

Since for any $f \in L_p$, $1 \le p < \infty$, or $f \in C_0(R)$, $\|G_\Lambda * G_\Lambda * f - f\|_p \to 0$ as $\Lambda \to \infty$, we may choose Λ so that $I_3 \le \varepsilon \frac{1}{\lambda} \left\| \frac{d}{dx} Hg \right\|_p$. (If $\frac{d}{dx} Hg = 0$, then $g =$ constant and (2.7) is redundant.) Following (2.5) and $\|G_\lambda\|_1 = 1$, we write

$$\begin{aligned} I_1 &\le \left\| \frac{1}{\lambda} \frac{d}{dx} H(G_\lambda * g \right\|_p \\ &= \left\| \frac{1}{\lambda} G_\lambda * \left(\frac{d}{dx} H(g) \right) \right\|_p \\ &\le \frac{1}{\lambda} \left\| \frac{d}{dx} H(g) \right\|_p . \end{aligned}$$

To estimate I_2, we first observe that

$$\frac{d}{d\mu} (G_\mu * G_\mu * g) = \frac{2}{\mu^2} G_\mu * \left(\frac{d}{dx} Hg \right)$$

which is obtained by comparing their Fourier transforms, the identity $\frac{d}{d\mu} \left(1 - \frac{|\xi|}{\mu}\right)_+^2 = 2 \frac{|\xi|}{\mu^2} \left(1 - \frac{|\xi|}{\mu}\right)_+$ and the fact that $\frac{d}{dx} Hg \in L_p$.

We now write

$$\begin{aligned} I_2 &= \left\| \int_\lambda^\Lambda \frac{d}{d\mu} (G_\mu * G_\mu * g)\, d\mu \right\|_p \\ &= 2 \left\| \int_\lambda^\Lambda G_\mu * \left(\frac{d}{dx} Hg \right) \frac{d\mu}{\mu^2} \right\|_p \\ &\le 2 \int_\lambda^\Lambda \left\| G_\mu * \left(\frac{d}{dx} Hg \right) \right\|_p \frac{d\mu}{\mu^2} \\ &\le 2 \int_\lambda^\Lambda \left\| \frac{d}{dx} Hg \right\|_p \frac{d\mu}{\mu^2} \\ &\le \frac{2}{\lambda} \left\| \frac{d}{dx} Hg \right\|_p . \end{aligned}$$

Combining the estimate for I_1, I_2 and I_3, we complete the proof of the direct result. □

3. Bochner-Riesz Means

For $f \in L_p(R^d)$, $1 \le p < \infty$, or $C_0(R^d)$ the Bochner-Riesz means are given by

$$R_{\lambda,b,d} f(x) = \left(\frac{1}{2\pi}\right)^d \int_{|\xi|<\lambda} \widehat{f}(\xi) \left(1 - \frac{|\xi|^2}{\lambda^2}\right)^b e^{ix\xi}\, d\xi \tag{3.1}$$

where

$$\widehat{f}(\xi) = \int_{R^d} f(x) e^{-ix\xi}\, dx .$$

While $\widehat{f}(\xi)$ is not necessarily a function, it is known (see [3, p. 380]) that, if $b > \frac{d-1}{2}$, $R_{\lambda,b,d} f$ is in L_p, if f is, since

$$R_{\lambda,b,d} f(x) = G_{\lambda,b,d} * f(x) \tag{3.2}$$

with

$$\|G_{\lambda,b,d}\|_{L_1} = M(b,d), \quad G_{\lambda,b,d}(x) = \lambda^d G_{1,b,d}(\lambda x) \equiv \lambda^d G_{b,d}(\lambda x)\,. \tag{3.3}$$

The analogue of the result of Section 2 is given by the following theorem.

Theorem 2. *Let $f \in L_p(R^d)$, $1 \le p < \infty$, or $C_0(R^d)$ for $p = \infty$, and let $b > \frac{d-1}{2}$ for $d = 1, 2, \ldots$. Then*

$$C_1(b,d) K_d\Big(f, \frac{1}{\lambda^2}\Big)_p \le \|R_{\lambda,b,d} f - f\|_p \le C_2(b,d) K_d\Big(f, \frac{1}{\lambda^2}\Big)_p \tag{3.4}$$

where $C_1(b,d) > 0$,

$$K_d\big(f, t^2\big)_p = \inf \big(\|f - g\|_p + t^2 \|\Delta g\|_p\big) \tag{3.5}$$

and Δ is the Laplacian (second derivative when $d = 1$).

Proof. The proof of the direct inequality [the second inequality of (3.4)] follows a similar pattern to the proof of that part of Theorem 1. We choose $g = g_{\varepsilon,\lambda,p,d}$ such that

$$(1+\varepsilon) K_d\Big(f, \frac{1}{\lambda^2}\Big)_p \ge \|f - g\|_p + \frac{1}{\lambda^2} \|\Delta g\|_p$$

and write

$$\begin{aligned} \|f - R_{\lambda,b,d} f\|_p &= \|f - G_{\lambda,b,d} * f\|_p \\ &\le \|(f-g) - G_{\lambda,b,d} * (f-g)\|_p + \|g - G_{\lambda,b,d} * g\|_p \\ &\equiv J_1 + J_2\,. \end{aligned}$$

We estimate J_1 by

$$J_1 \le \|f - g\|\big(1 + M(b,d)\big)\,.$$

To estimate J_2 we write

$$\begin{aligned} J_2 \le\ & \|G_{\lambda,b,d} * g - G_{\lambda,b+1,d} * g\|_p + \|G_{\lambda,b+1,d} * g - G_{\Lambda,b+1,d} * g\|_p \\ & + \|G_{\Lambda,b+1,d} * g - g\|_p \\ \equiv\ & I_1 + I_2 + I_3\,. \end{aligned}$$

We now write

$$I_1 = \Big\| \frac{1}{\lambda^2}\, \Delta(G_{\lambda,b,d} * g) \Big\|_p = \frac{1}{\lambda^2} \|G_{\lambda,b,d} * \Delta g\|_p \le \frac{1}{\lambda^2} M(b,d)\, \|\Delta g\|\,.$$

Using $\frac{d}{d\mu}\left(1-\frac{|\xi|^2}{\mu^2}\right)_+^{b+1} = 2(b+1)|\xi|^2\left(1-\frac{|\xi|^2}{\mu^2}\right)_+^{b}\frac{1}{\mu^3}$ for $b > \frac{d-1}{2}$, we have

$$\begin{aligned} I_2 &= \left\| \int_\lambda^\Lambda \frac{d}{d\mu}\,(G_{\lambda,b+1,d} * g)\,d\mu \right\|_p \\ &= 2(b+1)\left\| \int_\lambda^\Lambda \Delta(G_{\lambda,b,d} * g)\,\frac{d\mu}{\mu^3} \right\|_p \\ &= 2(b+1)\left\| \int_\lambda^\Lambda G_{\lambda,b,d} * (\Delta g)\,\frac{d\mu}{\mu^3} \right\|_p \\ &\le 2(b+1)\int_\lambda^\Lambda \|G_{\lambda,b,d} * \Delta g\|_p\,\frac{d\mu}{\mu^3} \\ &\le 2(b+1)M(b,d)\|\Delta g\|_p \int_\lambda^\infty \frac{d\mu}{\mu^3} \\ &\le (b+1)M(b,d)\,\frac{1}{\lambda^2}\,\|\Delta g\|_p\,. \end{aligned}$$

As $G_{\Lambda,b+1,d}(x) = \Lambda^d G_{1,b+1,d}(\Lambda x)$ [see (3.3)], we have $\|G_{\Lambda,b+1,d} * g - g\|_p \to 0$ as $\Lambda \to \infty$ for $g \in L_p(R^d)$, $1 \le p < \infty$, or $g \in C_0(R^d)$ when $p = \infty$, and using the estimate of I_2 which is independent of Λ, we derive the direct part of (3.4) with $C_2(b,d) = (b+2)M(b,d)$ as $M(b,d) \ge 2$.

For $b = 1$ (possible only for $d = 1$ and $d = 2$) we may follow the proof of the converse part of Theorem 1 and obtain

$$C_1(1,1) = \frac{1}{1+M(1,1)} \quad \text{and} \quad C_1(1,2) = \frac{1}{1+M(1,2)}$$

with $C_1(1,j)$ and $M(1,j)$ given in (3.4) and (3.3), respectively. For $b \ne 1$ we do not have such elegant identities and $C_1(b,d)$ will just be generic.

We define $\eta_\lambda(f)$ by

$$\eta_\lambda(f)^\wedge(\xi) = \eta\Big(\frac{|\xi|}{\lambda}\Big)\widehat{f}(\xi), \quad \lambda > 0$$

where $\eta(\xi) = 1$ for $0 \le \xi \le 1$, $\eta(\xi) = 0$ for $\xi > 2$ and $\eta(\xi) \in C^\infty(R_+)$. Following [1, Corollary 2.5], we have

$$K_d\big(f,t^2\big)_p \approx \|f - \eta_{1/t} f\|_p + t^2\|\Delta\eta_{1/t}\|_p\,. \tag{3.6}$$

Actually, in [1] $d \ge 2$ is assumed for many of the results on R^d, and Theorem 3.1 there is not valid for $d = 1$, but the equivalence (3.6) and its proof are valid for $d = 1$ and do not depend on the condition $d \ge 2$ assumed throughout in [1]. As

$$K_d\big(f,t^2\big)_p \approx K_d\big(f,(3t)^2\big)_p\,,$$

we have

$$K_d\big(f,t^2\big)_p \approx \|f - \eta_{1/(3t)} f\|_p + t^2\|\Delta\eta_{1/(3t)} f\|_p\,.$$

Hence, to prove the converse result it is sufficient to show that

$$\|f - \eta_{\lambda/3} f\|_p \le C\|f - R_{\lambda,b,d} f\|_p \tag{3.7}$$

and

$$\lambda^{-2}\|\Delta\eta_{\lambda/3}f\|_p \le C\|f - R_{\lambda,b,d}f\|_p\,. \tag{3.8}$$

We prove (3.7) first. We note that $\eta_{\lambda/3}$ and $R_{\lambda,b,d}$ (for $b > \frac{d-1}{2}$) are bounded multiplier operators on $L_p(R^d)$, $1 \le p \le \infty$, and hence,

$$\left\|(I-\eta_{\lambda/3})\big(I + R_{\lambda,b,d} + \cdots + R^{k-1}_{\lambda,b,d}\big)(f - R_{\lambda,b,d}f)\right\|_p \le C_1\|f - R_{\lambda,b,d}f\|_p\,.$$

Therefore, it is sufficient to prove now for some integer k that

$$\left\|(I-\eta_{\lambda/3})R^k_{\lambda,b,d}f\right\|_p \le C_2\|f - R_{\lambda,b,d}f\|_p\,.$$

In other words, it is sufficient to show that

$$\Phi_{\lambda,k}(\xi) = \left(1 - \eta\left(\frac{3|\xi|}{\lambda}\right)\right)\frac{\left(1-\left(\frac{|\xi|}{\lambda}\right)^2\right)_+^{bk}}{1-\left(1-\left(\frac{|\xi|}{\lambda}\right)^2\right)_+^{b}}$$

leads to a bounded multiplier operator on L_p. A change of variable implies that it is sufficient to deal with $\lambda = 1$ and show that $\overset{\vee}{\Phi}_{1,k}(x) \in L_1(R^d)$ where $\overset{\vee}{\Phi}_{1,k}(x)$ is the inverse Fourier transform of $\Phi_{1,k}(\xi)$. We observe that $\Phi_{1,k}(\xi) = 0$ for $|\xi| < \frac{1}{3}$ and for $|\xi| > 1$, and that it is bounded. Therefore, $\overset{\vee}{\Phi}_{1,k}(x)$ exists. To show that $\overset{\vee}{\Phi}_{1,k}$ is in L_1 we recall [4, p. 26] that

$$\left\|\overset{\vee}{\Phi}_{1,k}\right\|_{L_1(R^d)} \le C\sum_{|\alpha|<d+1}\left\|D^\alpha\Phi_{1,k}\right\|_{L_1(R^d)}\,,$$

and hence, we have to check only that $D^\alpha\Phi_{1,k} \in L_1$ for $|\alpha| < d+1$, and as $\Phi_{1,k}$ has compact support, boundedness of $D^\alpha\Phi_{1,k}$ for $|\alpha| < d+1$ is sufficient. For $|\xi| < \frac{1}{3}$, $1-\eta(3|\xi|) = 0$ and $D^\beta\big(1-\eta(3|\xi|)\big) = 0$. For $|\xi| > 1$ $\frac{(1-|\xi|)_+^{kb}}{1-(1-|\xi|)_+^b}$ and its derivatives are equal to zero. For $\frac{1}{3} \le |\xi| \le 1$ we note that $D^\beta\,\frac{(1-|\xi|)_+^{kb}}{1-(1-|\xi|)_+^b}$ behaves like $(1-|\xi|_+)^{kb-|\beta|}$ near $|\xi| = 2$. The possible singularity is avoided by choosing k so that $kb-d-1 > 0$. To prove (3.8) we have to show that

$$\Psi_\lambda(|\xi|) = \frac{\frac{|\xi|^2}{\lambda^2}\,\eta\left(\frac{3|\xi|}{\lambda}\right)}{1-\left(1-\left(\frac{|\xi|}{\lambda}\right)^2\right)_+^{b}}$$

is a bounded multiplier operator from L_p to L_p. It is sufficient to show that $\overset{\vee}{\Psi}_1(x) \in L_1(R^d)$.

The function $(1-z)^b$, $b > 0$ is analytic in z for $|z| < 1$ and equals 1 for $z = 0$, and hence, $(1-z)^b = 1 - \sum\limits_{n=1}^{\infty} c_n z^n$ converges for $|z| < 1$ and $1-(1-z)^b = z\sum\limits_{n=1}^{\infty} c_n z^{n-1}$ with $c_1 = b > 0$. Hence, we may write $\frac{1}{\sum_{n=1}^{\infty} c_n z^{n-1}} = \sum\limits_{n=0}^{\infty} d_n z^n$ which converges for $|z| < 1$ as

$|1-z|^b \neq 1$ for $|z|>0$ and $b>0$. Therefore, $\sum\limits_{n=0}^{\infty} d_n z^n$ converges uniformly for $|z| \leq \frac{2}{3}$. We now write

$$\Psi_1(|\xi|) = \eta(3|\xi|) \sum_{k=0}^{\infty} d_n |\xi|^{2n} ,$$

and clearly $D^\alpha \Psi_1(|\xi|)$ is a bounded function of compact support for $|\alpha| \leq d+2$. This implies (3.8) and the strong converse inequality [the first inequality of (3.4)]. □

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Received January 05, 2005

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