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# The Clifford-Fourier Transform

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*ABSTRACT. A pair of Clifford-Fourier transforms is defined in the framework of Clifford analysis, as operator exponentials with a Clifford algebra-valued kernel. It is a genuine Clifford analysis construct, which is shown to be a refinement of the classical multi-dimensional Fourier transform. An adequate operational calculus is developed.*

## 1. Introduction

The Fourier Transform is without any doubt one of the most powerful tools in pure and applied mathematics. Also in Clifford analysis — a direct and elegant generalization to higher dimension of the theory of holomorphic functions in the complex plane — extensive use is made of the classical multi-dimensional Fourier transform. The idea of generalizing the Fourier Transform to the Clifford analysis setting was already performed by Sommen in [8, 9] where a generalized Fourier transform was introduced in connection with similar generalizations of the Cauchy, Hilbert, and Laplace transforms; its definition is based on an exponential function which is a natural generalization of the classical Fourier kernel.

Clifford analysis has gained more and more interest over the years and has grown out to a proper branch of classical analysis. One of its most fundamental features is the factorization of the Laplace operator $(-\Delta_m)$. Whereas in general the square root of the Laplace operator is only a pseudo-differential operator, by embedding Euclidean space into a Clifford algebra, $\sqrt{-\Delta_m}$ can be realized as a first order, elliptic, rotation-invariant, Clifford-vector differential operator, the so-called Dirac operator $\partial_{\underline{x}}$, satisfying $-\Delta_m = \partial_{\underline{x}}^2$. It is precisely this Dirac operator $\partial_{\underline{x}}$ which underlies the notion of monogenicity of a function, a notion which is the multi-dimensional counterpart to that of holomorphicity in the complex plane. Monogenic functions have a special relationship with harmonic functions of several variables in that they are refining their properties. Note, for instance, that each harmonic function $h(\underline{x})$ can be split as $h(\underline{x}) = f(\underline{x}) + \underline{x}\ g(\underline{x})$ with $f, g$ monogenic, and that a real

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harmonic function is always the real part of a monogenic one, which does not need to be the case for a harmonic function of several complex variables.

It occurred to us that, in the same order of ideas, the classical multi-dimensional Fourier transform, should not be replaced nor improved by a Clifford analysis alternative. However, a refinement of the classical Fourier transform automatically appears within the language of Clifford analysis in much the same way as the notion of electron spin appears in the Pauli matrix formalism. It is what we call the ”Clifford-Fourier” transform.

In our approach the classical Fourier transform in $\mathbb{R}^m$:

$$\mathcal{F}[f](\underline{y}) = \left(\frac{1}{\sqrt{2\pi}}\right)^m \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{y} >\right) \, f(\underline{x}) \, dV(\underline{x})$$

with $< \underline{x}, \underline{y} >$ the standard inner product on $\mathbb{R}^m$, is seen as the operator exponential

$$\mathcal{F} = \exp\left(-i \frac{\pi}{2} \mathcal{H}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-i \frac{\pi}{2}\right)^n \mathcal{H}^n$$

where $\mathcal{H}$ is the scalar-valued operator

$$\mathcal{H} = \frac{1}{2}\left( - \Delta_m + r^2 - m\right) .$$

Splitting $\mathcal{H}$ into a sum of Clifford algebra-valued second order operators containing the angular Dirac operator $\Gamma$, leads in a natural way to a pair of transforms $\mathcal{F}_{\mathcal{H}^\pm}$, the harmonic average of which is precisely the standard Fourier transform $\mathcal{F}$. Introducing the square root of $\mathcal{F}_{\mathcal{H}^\pm}$ in the sense of the Fractional Fourier Transform (see e.g., [6] and [7]) the desired factorization of $\mathcal{F}$ is then obtained:

$$\sqrt{\mathcal{F}_{\mathcal{H}^+}} \, \sqrt{\mathcal{F}_{\mathcal{H}^-}} = \mathcal{F} .$$

Moreover, it is worth mentioning that in the two-dimensional case, which we treat in a separate article (see [4]), the newly introduced Clifford-Fourier transform may be qualified as a *co-axial Fourier transform,* since its integral kernel can be explicitly calculated to be, up to constants, $\exp(\underline{x} \wedge \underline{y}) = \cos(|\underline{x} \wedge \underline{y}|) + \dfrac{\underline{x} \wedge \underline{y}}{|\underline{x} \wedge \underline{y}|} \sin(|\underline{x} \wedge \underline{y}|)$ whereby $|\underline{x} \wedge \underline{y}|$ takes constant values on co-axial cylinders.

The outline of the article is as follows. For the reader who is not familiar with Clifford analysis, we recall some of its basic notions and results in Section 2. In Section 3 we discuss two alternative approaches to the classical Fourier transform. Next we discuss two commuting operators $O_1$ and $O_2$ (Section 4.1) which are used to split the scalar-valued kernel operator $\mathcal{H}$ and thus are crucial to the definition of the Clifford-Fourier transform (Section 4.2). The eigenfunctions of this new Clifford-Fourier transform are computed and its relation with the standard Fourier transform is established. In a final section we mention some operational formulae.

## 2. Some Basic Notions of Clifford Analysis

Clifford analysis (see e.g., [1] and [5]) offers a function theory which is a higher-dimensional analogue of the theory of the holomorphic functions of one complex variable.

The functions considered are defined in $\mathbb{R}^m$ $(m > 1)$ and take their values in the Clifford algebra $\mathbb{R}_m$ or its complexification $\mathbb{C}_m = \mathbb{R}_m \otimes \mathbb{C}$. If $(e_1, \ldots, e_m)$ is an orthonormal basis of $\mathbb{R}^m$, then a basis for the Clifford algebra $\mathbb{R}_m$ is given by $(e_A : A \subset \{1, \ldots, m\})$ where $e_\emptyset = 1$ is the identity element. The non-commutative multiplication in the Clifford algebra is governed by the rules:

$$\begin{aligned} e_j^2 &= -1 \ , \quad j = 1 \ , \ldots , m \ . \\ e_j e_k + e_k e_j &= 0 \ , \quad j \neq k \ , \quad j, k = 1, \ldots , m \ . \end{aligned}$$

Conjugation is defined as the anti-involution for which

$$\overline{e_j} = -e_j \ , \quad j = 1, \ldots , m$$

with the additional rule $\bar{i} = -i$ in the case of $\mathbb{C}_m$.

For $k = 0, 1, \ldots, m$ fixed, we call

$$\mathbb{R}_m^k = \left\{ \sum_{\#A=k} a_A e_A \ ; \ a_A \in \mathbb{R} \right\}$$

the subspace of $k$-vectors, i.e., the space spanned by the products of $k$ different basis vectors.

The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebras $\mathbb{R}_m$ and $\mathbb{C}_m$ by identifying $(x_1, \ldots, x_m)$ with the vector variable $\underline{x}$ given by

$$\underline{x} = \sum_{j=1}^{m} e_j x_j \ .$$

The product of two vectors splits up into a scalar part and a 2-vector, also called bivector, part:

$$\underline{x}\,\underline{y} = \underline{x} \cdot \underline{y} + \underline{x} \wedge \underline{y} \ ,$$

where

$$\underline{x} \cdot \underline{y} = - < \underline{x}, \underline{y} >= - \sum_{j=1}^{m} x_j y_j$$

and

$$\underline{x} \wedge \underline{y} = \sum_{i=1}^{m} \sum_{j=i+1}^{m} e_i e_j (x_i y_j - x_j y_i) \ .$$

Note that the square of a vector variable $\underline{x}$ is scalar-valued and is the norm squared up to a minus sign:

$$\underline{x}^2 = - < \underline{x}, \underline{x} >= -|\underline{x}|^2 \ .$$

The central notion in Clifford analysis is the notion of monogenicity, the higher-dimensional analogue of holomorphicity.

An $\mathbb{R}_m$- or $\mathbb{C}_m$-valued function $F(x_1, \ldots, x_m)$ is called left monogenic in an open region of $\mathbb{R}^m$, if in that region:

$$\partial_{\underline{x}} F = 0 \, .$$

Here $\partial_{\underline{x}}$ is the Dirac operator in $\mathbb{R}^m$:

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j} \ ,$$

an elliptic vector differential operator of the first order, splitting the Laplace operator:

$$\Delta_m = -\partial_{\underline{x}}^2 \, .$$

The notion of right monogenicity is defined in a similar way by letting act the Dirac operator from the right.

Introducing spherical co-ordinates in $\mathbb{R}^m$ by:

$$\underline{x} = r\underline{\omega} \ , \quad r = |\underline{x}| \in [0, +\infty[ \ , \quad \underline{\omega} \in S^{m-1}$$

with $S^{m-1}$ the unit sphere in $\mathbb{R}^m$, the Dirac operator takes the form:

$$\partial_{\underline{x}} = \underline{\omega} \left( \partial_r + \frac{1}{r} \, \Gamma \right) \ ,$$

where

$$\Gamma = \underline{x} \wedge \partial_{\underline{x}} = -\sum_{i=1}^{m} \sum_{j=i+1}^{m} e_i e_j (x_i \partial_{x_j} - x_j \partial_{x_i})$$

is the so-called *angular Dirac operator* which depends only on the angular co-ordinates.

Another fundamental operator is the so-called Euler operator

$$E = < \underline{x}, \partial_{\underline{x}} > = \sum_{i=1}^{m} x_i \partial_{x_i} \ ,$$

which measures the degree of homogeneity.

In the sequel the monogenic homogeneous polynomials will play an important rôle.

A left, respectively right, monogenic homogeneous polynomial $P_k$ of degree $k$ ($k \geq 0$) in $\mathbb{R}^m$ is called a left, respectively right, inner spherical monogenic of order $k$. The set of all left, respectively right, inner spherical monogenics of order $k$ will be denoted by $M_\ell^+(k)$, respectively $M_r^+(k)$. The dimension of $M_\ell^+(k)$ is given by

$$\dim\big(M_\ell^+(k)\big) = \frac{(m+k-2)!}{(m-2)! \, k!} \, .$$

The set

$$\left\{ \phi_{s,k,j}(\underline{x}) = \exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{s,m,k}\big(\sqrt{2}\underline{x}\big) \ P_k^{(j)}\big(\sqrt{2}\underline{x}\big) \right\} \ ,$$

$s, k \in \mathbb{N} \cup \{0\}$, $j = 1, \ldots, \dim\big(M_\ell^+(k)\big)$, constitutes an orthogonal basis for the space $L_2(\mathbb{R}^m)$ of square integrable functions (see [2]).

Here

$$\left\{ P_k^{(j)}(\underline{x});\ j = 1, 2, \ldots, \dim\big(M_\ell^+(k)\big) \right\}$$

denotes an orthonormal basis of $M_\ell^+(k)$. The polynomials $H_{s,m,k}(\underline{x})$ are the so-called generalized Clifford-Hermite polynomials introduced by Sommen in [10]; they are a multi-dimensional generalization to Clifford analysis of the classical Hermite polynomials on the real line. Note that $H_{s,m,k}(\underline{x})$ is a polynomial of degree $s$ in the variable $\underline{x}$ with real coefficients depending on $k$. Furthermore $H_{2s,m,k}(\underline{x})$ only contains even powers of $\underline{x}$, while $H_{2s+1,m,k}(\underline{x})$ only contains odd ones.

The basis functions $\phi_{s,k,j}$ satisfy the orthogonality relation

$$\begin{aligned} \big(\phi_{s,k_1,j_1}, \phi_{t,k_2,j_2}\big) &= \int_{\mathbb{R}^m} \overline{\phi_{s,k_1,j_1}}(\underline{x})\ \phi_{t,k_2,j_2}(\underline{x})\ dV(\underline{x}) \\ &= \frac{\gamma_{s,k_1}}{2^{m/2}}\ \delta_{s,t}\ \delta_{k_1,k_2}\ \delta_{j_1,j_2}\ , \end{aligned} \tag{2.1}$$

with $dV(\underline{x})$ the Lebesgue measure on $\mathbb{R}^m$ and $\gamma_{s,k_1}$ real constants depending on the parity of $s$.

Hence, a square integrable function $f \in L_2(\mathbb{R}^m)$ can be expanded in terms of the basis functions $\phi_{s,k,j}$:

$$f(\underline{x}) = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{\dim(M_\ell^+(k))} \phi_{s,k,j}(\underline{x})\ a_{s,k,j}\ . \tag{2.2}$$

The orthogonality relation (2.1) implies that the expansion coefficients $a_{s,k,j}$ are given by the integral representation

$$a_{s,k,j} = \frac{2^{m/2}}{\gamma_{s,k}} \int_{\mathbb{R}^m} \overline{\phi_{s,k,j}}(\underline{x})\ f(\underline{x})\ dV(\underline{x})\ . \tag{2.3}$$

Note that in general these coefficients are Clifford algebra-valued.

## 3. The Classical Fourier Transform

The classical Fourier transform in $\mathbb{R}^m$

$$\mathcal{F}[f](\underline{y}) = \left(\frac{1}{\sqrt{2\pi}}\right)^m \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{y} >\right)\ f(\underline{x})\ dV(\underline{x}) \tag{3.1}$$

has an interesting alternative representation as an operator exponential:

$$\mathcal{F}[f] = \exp\Big(-i\frac{\pi}{2}\mathcal{H}\Big)[f] = \sum_{n=0}^{\infty} \frac{1}{n!} \Big(-i\frac{\pi}{2}\Big)^n\ \mathcal{H}^n[f] \tag{3.2}$$

where the scalar-valued differential operator $\mathcal{H}$ is given by

$$\mathcal{H} = \frac{1}{2}\big( - \Delta_m + r^2 - m\big)\ .$$

Note that the operators $\mathcal{H}$ and $\exp\left(-i\frac{\pi}{2}\mathcal{H}\right)$ are Fourier invariant, i.e.,

$$\mathcal{F}\big[\mathcal{H}[f]\big] = \mathcal{H}\big[\mathcal{F}[f]\big] \quad \text{and} \quad \mathcal{F}\left[\exp\left(-i\frac{\pi}{2}\mathcal{H}\right)[f]\right] = \exp\left(-i\frac{\pi}{2}\mathcal{H}\right)\big[\mathcal{F}[f]\big] .$$

The equivalence of this operator exponential form (3.2) with the traditional integral form (3.1) can be found in classical textbooks such as [7]. However, it may be proved in a rather easy way in the framework of Clifford analysis. Indeed, in [2] we have shown that the $L_2(\mathbb{R}^m)$-basis functions $\phi_{s,k,j}$ are simultaneous eigenfunctions of the Fourier transform operator $\mathcal{F}$ in integral form and of the kernel operator $\mathcal{H}$. We thus have at the same time:

$$\begin{aligned}\mathcal{F}[\phi_{s,k,j}](\underline{y}) &= \left(\frac{1}{\sqrt{2\pi}}\right)^m \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{y} >\right) \phi_{s,k,j}(\underline{x})\, dV(\underline{x}) \\ &= \exp\left(-i(s+k)\frac{\pi}{2}\right) \phi_{s,k,j}(\underline{y})\end{aligned}$$

and

$$\mathcal{H}\big[\phi_{s,k,j}(\underline{x})\big] = (s+k)\, \phi_{s,k,j}(\underline{x}) .$$

It then follows that

$$\begin{aligned}\exp\left(-i\frac{\pi}{2}\mathcal{H}\right)[\phi_{s,k,j}] &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(-i\frac{\pi}{2}\right)^n \mathcal{H}^n[\phi_{s,k,j}] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(-i\frac{\pi}{2}\right)^n (s+k)^n \phi_{s,k,j} \\ &= \exp\left(-i\frac{\pi}{2}(s+k)\right)\phi_{s,k,j} \\ &= \mathcal{F}[\phi_{s,k,j}]\end{aligned}$$

which gives rise to the desired equivalence in $L_2(\mathbb{R}^m)$.

Moreover, if the function $f \in L_2(\mathbb{R}^m)$ is developed in terms of the basis functions $\phi_{s,k,j}$ according to (2.2), then its Fourier transform takes the series expansion form

$$\mathcal{F}[f](\underline{y}) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_\ell^+(k))} \exp\left(-i(s+k)\frac{\pi}{2}\right) \phi_{s,k,j}(\underline{y})\, a_{s,k,j} .$$

Note that from this series expansion the standard integral form is re-obtained by substituting expression (2.3) for the coefficients $a_{s,k,j}$ and applying the Mehler formula for the generalized Clifford-Hermite polynomials, established in [2].

## 4. The Clifford-Fourier Transform

As already explained in the introduction (Section 1) we aim at defining a Clifford-Fourier transform as an operator exponential involving a Clifford algebra-valued operator kernel. As $\partial_{\underline{x}}^2 = -\Delta_m$ and $\underline{x}^2 = -r^2$, the classical operator kernel $\mathcal{H}$ also reads

$$\mathcal{H} = \frac{1}{2}\left(\partial_{\underline{x}}^2 - \underline{x}^2 - m\right)$$

and in our quest for an appropriate operator, a "factorization" of $\mathcal{H}$ into linear operators seemed promising. This lead us to considering the operators

$$O_1 = \frac{1}{2}(\partial_{\underline{x}} - \underline{x})(\partial_{\underline{x}} + \underline{x}) \quad \text{and} \quad O_2 = \frac{1}{2}(\partial_{\underline{x}} + \underline{x})(\partial_{\underline{x}} - \underline{x}) \ ,$$

which turned out to be crucial in our approach.

## 4.1 The Operators $O_1$ and $O_2$

The operators $O_1$ and $O_2$ were already introduced in [3] while studying (anti-) monogenic operators in the generalized Clifford-Hermite polynomial setting. They satisfy the following properties.

***Proposition 1.*** *One has*

(i)

$$O_1 = \frac{1}{2}\left(\partial_{\underline{x}}^2 - \underline{x}^2\right) + \left(\Gamma - \frac{m}{2}\right) = \mathcal{H} + \Gamma$$

(ii)

$$O_2 = \frac{1}{2}\left(\partial_{\underline{x}}^2 - \underline{x}^2\right) - \left(\Gamma - \frac{m}{2}\right) = \mathcal{H} - \Gamma + m$$

(iii)

$$O_1 + O_2 = \partial_{\underline{x}}^2 - \underline{x}^2 = 2\left(\mathcal{H} + \frac{m}{2}\right)$$

(iv)

$$O_1 - O_2 = 2\left(\Gamma - \frac{m}{2}\right)$$

(v) *$O_1$ and $O_2$ are Fourier invariant operators*

(vi) *$O_1$ and $O_2$ are commuting operators*

(vii)

$$O_1\left[\phi_{s,k,j}(\underline{x})\right] = C_{s,m,k}\ \phi_{s,k,j}(\underline{x})$$

*with*

$$C_{s,m,k} = \begin{cases} s & \text{for} \quad s \ \text{even} \\ s - 1 + m + 2k & \text{for} \quad s \ \text{odd} \end{cases}$$

(viii)

$$O_2\left[\phi_{s,k,j}(\underline{x})\right] = C_{s+1,m,k}\ \phi_{s,k,j}(\underline{x}) \ .$$

**Proof.**

(i) (ii) Taking into account that the angular Dirac operator $\Gamma$ may be written as

$$\Gamma = -\frac{1}{2}(\underline{x}\ \partial_{\underline{x}} - \partial_{\underline{x}}\ \underline{x} - m) \ ,$$

the results follow from a straightforward computation.
(iii) (iv) Trivial.
(v) This property follows directly from the Fourier invariance of the operators $\mathcal{H}$ and $\Gamma$.
(vi) As $\Gamma$ commutes with the Laplace operator $\Delta_m$ and with the multiplication operator $r$, we have that

$$\left[\frac{1}{2}\big(\partial_{\underline{x}}^2 - \underline{x}^2\big), \Gamma\right] = \left[\frac{1}{2}\big(-\Delta_m + r^2\big), \Gamma\right] = 0$$

which, in view of (i) and (ii) yields $[O_1, O_2] = 0$.
(vii) (viii) It was proved in [3] that

$$(\partial_{\underline{x}} - \underline{x})\big[\phi_{s,k,j}(\underline{x})\big] = -\sqrt{2}\,\phi_{s+1,k,j}(\underline{x})$$

and

$$(\partial_{\underline{x}} + \underline{x})\big[\phi_{s,k,j}(\underline{x})\big] = -\sqrt{2}\,C_{s,m,k}\;\phi_{s-1,k,j}(\underline{x})\;.$$

By combining these results, the basis functions $\phi_{s,k,j}$ are found to be eigenfunctions of $O_1$ and $O_2$. □

**Remark.** Note that $(\partial_{\underline{x}} - \underline{x})$ increases the degree of the generalized Clifford-Hermite polynomial, so that it may be qualified as a *creation operator*. In the same order of ideas, $(\partial_{\underline{x}} + \underline{x})$ is an *annihilation operator*.

### 4.2 The Definition of the Clifford-Fourier Transform

In view of Proposition 1 (vii) and (viii), we define the Clifford-Fourier transform as the *pair* of transformations

$$\mathcal{F}_{\mathcal{H}^+} = \exp\left(-i\frac{\pi}{2}\mathcal{H}^+\right) \quad \text{and} \quad \mathcal{F}_{\mathcal{H}^-} = \exp\left(-i\frac{\pi}{2}\mathcal{H}^-\right)$$

with operators $\mathcal{H}^+$ and $\mathcal{H}^-$ closely linked to the operators $O_1$ and $O_2$. As we want the classical Fourier transform to be the harmonic average of the Clifford-Fourier transform pair $\{\mathcal{F}_{\mathcal{H}^+}, \mathcal{F}_{\mathcal{H}^-}\}$, i.e.,

$$\mathcal{F}^2 = \mathcal{F}_{\mathcal{H}^+}\,\mathcal{F}_{\mathcal{H}^-}$$

with $\mathcal{F}^2$ the parity operator:

$$\mathcal{F}^2[f](\underline{x}) = f(-\underline{x})\;,$$

the operators $\mathcal{H}^+$ and $\mathcal{H}^-$ must satisfy

$$\mathcal{H}^+ + \mathcal{H}^- = 2\,\mathcal{H}$$

or

$$\mathcal{H}^+ + \mathcal{H}^- = \partial_{\underline{x}}^2 - \underline{x}^2 - m = O_1 + O_2 - m\;.$$

This eventually inspires the following "symmetric" definition of $\mathcal{H}^+$ and $\mathcal{H}^-$.

**Definition 1.** One puts

$$\mathcal{H}^+ = O_1 - \frac{m}{2} \quad \text{and} \quad \mathcal{H}^- = O_2 - \frac{m}{2}\;.$$

Note that the operators $\mathcal{H}^+$ and $\mathcal{H}^-$ contain a scalar part and a bivector part. The following properties are easily proved.

***Proposition 2.*** *One has*

(i)

$$\mathcal{H}^+ = \frac{1}{2}\left(\partial_{\underline{x}}^2 - \underline{x}^2\right) + \Gamma - m = \mathcal{H} + \left(\Gamma - \frac{m}{2}\right)$$
$$\mathcal{H}^- = \frac{1}{2}\left(\partial_{\underline{x}}^2 - \underline{x}^2\right) - \Gamma = \mathcal{H} - \left(\Gamma - \frac{m}{2}\right)$$

(ii) $\mathcal{H}^+$ *and* $\mathcal{H}^-$ *are Fourier invariant*

(iii)

$$\mathcal{H}^{\pm}\left[\phi_{s,k,j}(\underline{x})\right] = C^{\pm}_{s,m,k}\ \phi_{s,k,j}(\underline{x})$$

*with*

$$C^+_{s,m,k} := C_{s,m,k} - \frac{m}{2} = \begin{cases} s - \frac{m}{2} & \text{for} \quad s \ \text{even} \\ s - 1 + \frac{m}{2} + 2k & \text{for} \quad s \ \text{odd} \end{cases}$$

*and*

$$C^-_{s,m,k} := C_{s+1,m,k} - \frac{m}{2} = \begin{cases} s + \frac{m}{2} + 2k & \text{for} \quad s \ \text{even} \\ s + 1 - \frac{m}{2} & \text{for} \quad s \ \text{odd}\ . \end{cases}$$

***Corollary 1.*** *The basis functions $\phi_{s,k,j}$ are eigenfunctions of the Clifford-Fourier transform:*

$$\mathcal{F}_{\mathcal{H}^{\pm}}\left[\phi_{s,k,j}\right](\underline{y}) = \exp\left(-i\frac{\pi}{2}C^{\pm}_{s,m,k}\right)\phi_{s,k,j}(\underline{y})\ .$$

Now if $f \in L_2(\mathbb{R}^m)$ is expanded w.r.t. the basis $\left\{\phi_{s,k,j}(\underline{x})\ ;\ s,k \in \mathbb{N}\cup\{0\},\ j = 1,\ldots,\dim\left(M_\ell^+(k)\right)\right\}$, the eigenvalue equation of Corollary 1 immediately yields the series representation of the Clifford-Fourier transform:

$$\mathcal{F}_{\mathcal{H}^{\pm}}[f](\underline{y}) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_\ell^+(k))} \exp\left(-i\frac{\pi}{2}C^{\pm}_{s,m,k}\right)\phi_{s,k,j}(\underline{y})\ a_{s,k,j}$$

the coefficients $a_{s,k,j}$ being given by (2.3).

Moreover, as the orthogonal $L_2$-basis $\left\{\phi_{s,k,j}(\underline{x})\ ;\ s,k \in \mathbb{N}\cup\{0\},\ j = 1,\ldots,\dim\left(M_\ell^+(k)\right)\right\}$ consists of eigenfunctions of both the operators $\mathcal{H}$ and $\Gamma$, one can easily verify the following properties.

***Proposition 3.***

(i) *The operators $\mathcal{H}$, $\Gamma$, $O_1$, $O_2$, $\mathcal{H}^+$, and $\mathcal{H}^-$ are self-adjoint, i.e., for all $f, g \in L_2(\mathbb{R}^m)$ and $T$ any of the mentioned operators one has*

$$(T[f], g) = (f, T[g])\ .$$

(ii) *The operators $\mathcal{H}$, $O_1$, and $O_2$ are non-negative, i.e., for each $f \in L_2(\mathbb{R}^m)$ and $T$ any of the mentioned operators one has*

$$\left[\left(T[f], f\right)\right]_0 \geq 0\ ,$$

*where* $[\lambda]_0$ *denotes the scalar part of the Clifford number* $\lambda$.

Next, by means of Proposition 2 (i), we obtain in terms of operator exponentials

$$\begin{aligned}\mathcal{F}_{\mathcal{H}^\pm} &= \exp\left(-i\frac{\pi}{2}\left(\mathcal{H} \pm \Gamma \mp \frac{m}{2}\right)\right) \\ &= \exp\left(\mp i\frac{\pi}{2}\left(\Gamma - \frac{m}{2}\right)\right) \exp\left(-i\frac{\pi}{2}\mathcal{H}\right) \\ &= \exp\left(\mp i\frac{\pi}{2}\left(\Gamma - \frac{m}{2}\right)\right) \mathcal{F}\,. \end{aligned} \tag{4.1}$$

This establishes the relationship between the classical Fourier transform and the newly introduced Clifford-Fourier transform. Note that use has been made of the commuting property of the operators $\mathcal{H}$ and $\Gamma$, so that indeed

$$\begin{aligned}\exp\left(-i\frac{\pi}{2}(\mathcal{H} \pm \Gamma)\right) &= \exp\left(-i\frac{\pi}{2}\mathcal{H}\right) \exp\left(\mp i\frac{\pi}{2}\Gamma\right) \\ &= \exp\left(\mp i\frac{\pi}{2}\Gamma\right) \exp\left(-i\frac{\pi}{2}\mathcal{H}\right). \end{aligned}$$

It thus turns out that the Clifford-Fourier transform is obtained as the composition of the classical Fourier transform with the operator exponential

$$\exp\left(\mp i\frac{\pi}{2}\left(\Gamma - \frac{m}{2}\right)\right).$$

As an immediate consequence, we obtain an integral representation for the Clifford-Fourier transform:

$$\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{y}) = \left(\frac{1}{\sqrt{2\pi}}\right)^m \int_{\mathbb{R}^m} \exp\left(\mp i\frac{\pi}{2}\left(\Gamma_{\underline{y}} - \frac{m}{2}\right)\right)\left[\exp\left(-i < \underline{x}, \underline{y} >\right)\right] f(\underline{x})\, dV(\underline{x})\,.$$

Introducing the square root of the Clifford-Fourier transforms, in the sense of the Fractional Fourier Transform (see [6] and [7]), by

$$\sqrt{\mathcal{F}_{\mathcal{H}^\pm}} = \exp\left(-i\frac{\pi}{4}\, \mathcal{H}^\pm\right)$$

we also obtain that

$$\sqrt{\mathcal{F}_{\mathcal{H}^+}}\, \sqrt{\mathcal{F}_{\mathcal{H}^-}} = \sqrt{\mathcal{F}_{\mathcal{H}^-}}\, \sqrt{\mathcal{F}_{\mathcal{H}^+}} = \exp\left(-i\frac{\pi}{4}\, \left(\mathcal{H}^+ + \mathcal{H}^-\right)\right) = \exp\left(-i\frac{\pi}{2}\, \mathcal{H}\right)$$

leading to the factorization of the standard Fourier transform:

$$\mathcal{F} = \sqrt{\mathcal{F}_{\mathcal{H}^+}}\, \sqrt{\mathcal{F}_{\mathcal{H}^-}} = \sqrt{\mathcal{F}_{\mathcal{H}^-}}\, \sqrt{\mathcal{F}_{\mathcal{H}^+}}\,.$$

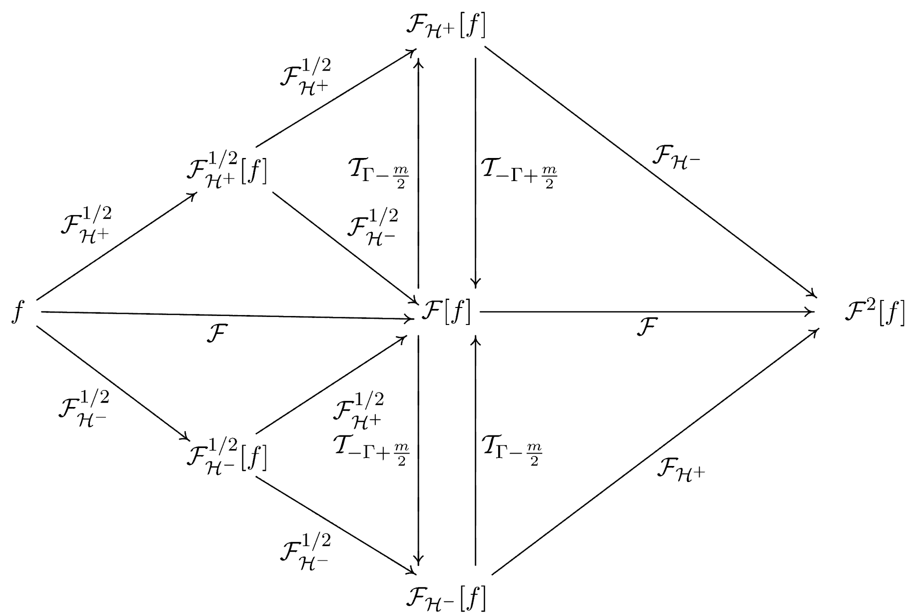
In [4] we study the two-dimensional Clifford-Fourier transform, since in this special two-dimensional case we succeed in finding a closed form for the kernel of the above mentioned integral representation. This closed form enables us to generalize the well-known properties of the classical multi-dimensional Fourier transform, as well in the $L_1$ as in the $L_2$ context.

Note that each operator which is (anti-) invariant under the classical Fourier transform and commutes with the angular Dirac operator $\Gamma$, is also (anti-) invariant under the Clifford-Fourier transform. For example, the operators $\Gamma - \frac{m}{2}$ and $E + \frac{m}{2}$ are, respectively, invariant and anti-invariant under the classical Fourier transform; as they both commute with the angular Dirac operator $\Gamma$, they show the (anti-)invariance property w.r.t the Clifford-Fourier transform.

For the inversion of the Clifford-Fourier transform, it suffices to observe that

$$(\mathcal{F}_{\mathcal{H}^{\pm}})^{-1} = \exp\left(i\frac{\pi}{2}\mathcal{H}^{\pm}\right) = \exp\left(\pm i\frac{\pi}{2}\left(\Gamma - \frac{m}{2}\right)\right)\mathcal{F}^{-1}\,.$$

Finally, using the notation $\mathcal{T}_T = \exp\big(-i\frac{\pi}{2}T\big)$, we can draw the following picture




## 5. Operational Calculus

As is the case for the classical Fourier transform, an operational calculus may be based upon the Clifford-Fourier transform. The operational formulae are derived from the relation (4.1) expressing the Clifford-Fourier transform in terms of the classical Fourier transform $\mathcal{F}$. One can easily prove the following results.

***Proposition 4.*** *The Clifford-Fourier transform satisfies:*

(i) *the linearity property*

$$\mathcal{F}_{\mathcal{H}^{\pm}}[f\lambda + g\mu] = \mathcal{F}_{\mathcal{H}^{\pm}}[f]\,\lambda + \mathcal{F}_{\mathcal{H}^{\pm}}[g]\,\mu \quad \textit{for} \quad \lambda, \mu \in \mathbb{C}_m$$

(ii) *the change of scale property*

$$\mathcal{F}_{\mathcal{H}^{\pm}}\big[f(a\underline{x})\big](\underline{y}) = \frac{1}{a^m}\,\mathcal{F}_{\mathcal{H}^{\pm}}\big[f(\underline{x})\big]\left(\frac{\underline{y}}{a}\right) \quad \textit{for} \quad a \in \mathbb{R}_+$$

(iii) *the multiplication rule*

$$\mathcal{F}_{\mathcal{H}^{\pm}}\left[\underline{x}\, f(\underline{x})\right](\underline{y}) = \mp\, \partial_{\underline{y}}\, \mathcal{F}_{\mathcal{H}^{\mp}}\left[f(\underline{x})\right](\underline{y})$$

*and more generally*

$$\mathcal{F}_{\mathcal{H}^{\pm}}\left[\underline{x}^{2n}\, f(\underline{x})\right](\underline{y}) = (-1)^n\, \partial_{\underline{y}}^{2n}\, \mathcal{F}_{\mathcal{H}^{\pm}}\left[f(\underline{x})\right](\underline{y})$$
$$\mathcal{F}_{\mathcal{H}^{\pm}}\left[\underline{x}^{2n+1}\, f(\underline{x})\right](\underline{y}) = \mp\, (-1)^n\, \partial_{\underline{y}}^{2n+1}\, \mathcal{F}_{\mathcal{H}^{\mp}}\left[f(\underline{x})\right](\underline{y})$$

(iv) *the differentiation rule*

$$\mathcal{F}_{\mathcal{H}^{\pm}}\left[\partial_{\underline{x}}\, f(\underline{x})\right](\underline{y}) = \mp\, \underline{y}\, \mathcal{F}_{\mathcal{H}^{\mp}}\left[f(\underline{x})\right](\underline{y})$$

*and more generally*

$$\mathcal{F}_{\mathcal{H}^{\pm}}\left[\partial_{\underline{x}}^{2n}\, f(\underline{x})\right](\underline{y}) = (-1)^n\, \underline{y}^{2n}\, \mathcal{F}_{\mathcal{H}^{\pm}}\left[f(\underline{x})\right](\underline{y})$$
$$\mathcal{F}_{\mathcal{H}^{\pm}}\left[\partial_{\underline{x}}^{2n+1}\, f(\underline{x})\right](\underline{y}) = \mp\, (-1)^n\, \underline{y}^{2n+1}\, \mathcal{F}_{\mathcal{H}^{\mp}}\left[f(\underline{x})\right](\underline{y})$$

(v) *the mixed product rule*

$$\mathcal{F}_{\mathcal{H}^{\pm}}\left[(\underline{x}\partial_{\underline{x}})^n\, f(\underline{x})\right](\underline{y}) = (-1)^n\, (\partial_{\underline{y}}\underline{y})^n\, \mathcal{F}_{\mathcal{H}^{\pm}}\left[f(\underline{x})\right](\underline{y})$$
$$\mathcal{F}_{\mathcal{H}^{\pm}}\left[(\partial_{\underline{x}}\underline{x})^n\, f(\underline{x})\right](\underline{y}) = (-1)^n\, (\underline{y}\partial_{\underline{y}})^n\, \mathcal{F}_{\mathcal{H}^{\pm}}\left[f(\underline{x})\right](\underline{y})\ .$$

As the Fourier transform of a radial function remains radial, and the angular Dirac operator $\Gamma$ does not affect radial functions, the next result follows readily.

***Proposition 5.*** *For a radial function $f$ one has*

$$\mathcal{F}_{\mathcal{H}^{\pm}}[f] = \exp\left(\pm i\frac{\pi}{4}m\right)\mathcal{F}[f]$$

*and in particular,*

$$\mathcal{F}_{\mathcal{H}^{\pm}}[\delta] = \exp\left(\pm i\frac{\pi}{4}m\right)\frac{1}{\left(\sqrt{2\pi}\right)^m}$$

*and*

$$\mathcal{F}_{\mathcal{H}^{\pm}}[1] = \exp\left(\pm i\frac{\pi}{4}m\right)\left(\sqrt{2\pi}\right)^m\delta\ .$$

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