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Bilinear Hilbert Transform on Measure Spaces

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ABSTRACT. In this article we obtain the boundedness of the periodic, discrete and ergodic bilinear Hilbert transform, from $L^{p_1} \times L^{p_2}$ into L^{p_3}, where $1/p_1 + 1/p_2 = 1/p_3$, $p_1, p_2 > 1$, and $p_3 \geq 1$. The main techniques are a bilinear version of the transference method of Coifman and Weiss and certain discretization of bilinear operators. In the periodic case, we also obtain the boundedness for $2/3 < p_3 < 1$.

1. Introduction

If $T : \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ is a continuous bilinear operator which commutes with simultaneous translations then, in the distributional sense, T can be represented as

$$T(f, g)(x) = \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\nu)m(\xi, \nu)e^{2\pi i x(\xi+\nu)}\, d\xi\, d\nu \ ,$$

for Schwarzt test functions f and g belonging to $\mathcal{S}(\mathbb{R})$. It has been of great interest in the last decades to find conditions on the symbol m such that T extends to a bounded operator from $L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R})$ whenever $1/r = 1/p + 1/q$ (see for example, the works of [7, 11, 12, 14], or [17]). In particular, if

$$T(f, g)(x) = \int_{\mathbb{R}^n} K(y) f(x-y)g(x+y)\, dy \ ,$$

where $K(y) = \frac{\Omega(y')}{|y|^n}$, $y' \in \Sigma_{n-1}$ and Ω is an odd-function then

$$T(f, g)(x) = \frac{1}{2}\int_{\Sigma_{n-1}} \Omega(\theta)\left(\int_{-\infty}^{\infty} f(x-t\theta)g(x+t\theta)\frac{dt}{t}\right) d\theta \ .$$

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The operator inside the brackets $H_\theta(f,g)(x) = \int_{-\infty}^{\infty} f(x-t\theta)g(x+t\theta)\frac{dt}{t}$ is the so-called uni-directional bilinear Hilbert transform whose boundedness can be proved directly from the boundedness of the bilinear Hilbert transform

$$H(f,g)(s) = \int_{-\infty}^{\infty} f(s-t)g(s+t)\frac{dt}{t},\, s \in \mathbb{R}\,.$$

This operator appeared for the first time in 1960, when A .P. Calderón was analyzing Cauchy integrals on Lipschitz curves and, in particular, the boundedness on $L^2(\mathbb{R})$ of the first commutator with a kernel $\frac{A(x)-A(y)}{(x-y)^2}$ where $A' \in L^\infty(\mathbb{R})$, and he needed to prove that the operator H maps boundedly $L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^1(\mathbb{R})$ (see [6, 16]).

After several articles concerning the problem, M. Lacey and C. Thiele (see [18, 20, 21]) proved Calderón's conjecture showing that $H : L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R})$ whenever $p, q > 1$ and $1/p + 1/q = 1/r < 3/2$, (see also [13]).

Since then, multilinear operators have become a matter of great interest in Harmonic Analysis.

In 2001, D. Fan and S. Sato [10] were able to show the boundedness of the bilinear Hilbert transform on the torus

$$H_{\mathbb{T}}(f,g)(x) = \int_{\mathbb{T}} f(x-y)g(x+y)\cot(\pi y)\,dy\;,$$

by transferring the result from $\mathbb{R}$. Their proof relies upon some DeLeeuw type transference methods for multilinear multipliers (see [9]). Similar techniques have been recently extended in [3] and [4].

Our aim in this article will be to study the boundedness of the bilinear Hilbert transform in different measure spaces. In particular, we shall obtain the boundedness (on the same range but $p_3 \geq 1$) of the discrete Hilbert transform

$$H_{\mathbb{Z}}(a,b)(m) = \sum_{n\neq 0} \frac{a_{m-n}b_{n+m}}{n}\;,$$

of the ergodic Hilbert transform

$$H_T(f,g)(x) = \sum_{n\neq 0} \frac{T^n f(x) T^{-n} g(x)}{n}\;,$$

where T is an ergodic transformation acting on $L^{p_i}(\Omega)$ for a certain σ-finite measure space Ω, and, hence also of

$$H_D(f,g)(x) = \sum_{n\neq 0} \frac{f(x-n)g(x+n)}{n}\;,$$

and, we shall also give a new proof of the result of Fan and Sato about the boundedness of the bilinear Hilbert transform in the torus.

The main technique is based in the so-called transference method of R. Coifman and G. Weiss (see [8]). This method was introduced in 1977 and since then, it has been developed and extended by many other people (see [2], or [1]) and has shown to be an extremely useful tool to prove the boundedness of many operators defined on certain measure space assuming that we know the boundedness of a related convolution operator on a certain group.

In 1996, L. Grafakos and G. Weiss (see [15]) proved a first result concerning a transference method for multilinear operators. They consider a multilinear operator T defined on an amenable group G by

$$T(g_1, \dots, g_k)(v) = \int_{G^k} K(u_1, \dots, u_k) g_1\big(u_1^{-1}v\big) \dots g_k\big(u_k^{-1}v\big)\, d\lambda(u_1) \dots d\lambda(u_k)\ ,$$

with g_j in some dense subset of $L^{p_j}(G)$ and where K is a kernel on G^k which may not be integrable, and they are able to transfer the boundedness of $T : L^{p_1}(G) \times \dots L^{p_k}(G) \to L^{p_0}(G)$ whenever $1/p_0 = 1/p_1 + \dots + 1/p_k$ to the boundedness of operator $\tilde{T} : L^{p_1}(\mu) \times \dots L^{p_k}(\mu) \to L^{p_0}(\mu)$ where (M, μ) is a measure space and

$$\tilde{T}(f_1, \dots, f_k)(x) = \int_{G^k} K(u_1, \dots, u_k)\big(R^1_{u_1} f_1\big)(x) \dots \big(R^k_{u_k} f_k\big)(x)\, d\lambda(u_1) \dots d\lambda(u_k)\ ,$$

where f_j is in some dense subset of $L^{p_j}(M)$, and $R^j : G \to \mathcal{B}(L^{p_j}(M))$ $(j = 0, 1, \dots, k)$ are representations which are connected through $R^0_v R^j_u = R^j_{uv}$ for all $u, v \in G$ and $1 \leq j \leq k$, and satisfy certain boundedness conditions.

In this article, we shall develop a transference method for bilinear operators in the same spirit as the one started by Coifman and Weiss for linear operators, which will allow us to transfer the boundedness of bilinear operators such as the bilinear Hilbert transform on $\mathbb{R}$ to other groups, recovering the Fan and Sato transference result from our general principle. We shall restrict ourselves to the two variable case, to locally compact abelian groups G and to integrable kernels (although our results will work in multilinear situation, amenable groups and more general kernels). A much more detailed study of this type of transference will be undertaken in [5]. Here, we shall be more concerned about the applications related to the bilinear Hilbert transform on measure spaces.

The second technique that we shall use concerns the discretization of bilinear operators.

2. Main Techniques: Transference Method and Discretization

2.1 Transference Method for Bilinear Operators

Let $K \in L^1(G)$ be a kernel with compact support and let $1 \leq p_1, p_2 < \infty$ and $0 < p_3 < \infty$ such that

$$\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}\cdot$$

From now on, p_1, p_2 and p_3 will satisfy the above relation.

Consider the mapping

$$B_K(\phi, \psi)(v) = \int_G \phi\big(u^{-1}v\big)\psi(uv)K(u)\, dm(u)\ ,$$

for $\phi \in L^{p_1}(G)$ and $\psi \in L^{p_2}(G)$, where m is the Haar measure on G, and let us define the transference operator $T_K : L^{p_1}(\mu) \times L^{p_2}(\mu) \to L^{p_3}(\mu)$ by

$$T_K(f, g)(x) = \int_G \big(R^1_{u^{-1}} f\big)(x)\big(R^2_u g\big)(x) K(u)\, dm(u)\ ,$$

where $R^j : G \to B(L^{p_j}(\mu))$ are strongly continuous mappings for $j = 1, 2$. Then:

Theorem 1. *Under the above conditions, if, for every $v \in G$, there exist $A_j > 0$ such that*

$$\left\| R_v^j f \right\|_{L^{p_j}} \le A_j \|f\|_{L^{p_j}} \tag{2.1}$$

and there exists a strongly continuous mapping $R^3 : G \to B(L^{p_3}(\mu))$ satisfying that, for every $u, v \in G$ and every $f \in L^{p_1}(M)$ and $g \in L^{p_2}(M)$,

$$R_v^3\big(R_{u^{-1}}^1 f R_u^2 g\big) = R_{vu^{-1}}^1 f R_{vu}^2 g \ , \tag{2.2}$$

and such that, for every $v \in G$, there exists $B > 0$ satisfying

$$\|f\|_{L^{p_3}(M)} \le B \left\| R_v^3 f \right\|_{L^{p_3}(M)} . \tag{2.3}$$

then, the bilinear operator $T_K : L^{p_1}(\mu) \times L^{p_2}(\mu) \to L^{p_3}(\mu)$ is bounded and it has norm bounded by $N_{p_1,p_2}(K) A_1 A_2 B$ where $N_{p_1,p_2}(K)$ stands for the norm of the bilinear map B_K in the corresponding spaces.

Proof. Using the continuity of R_v^3 and (2.2), we get that

$$\begin{aligned} R_v^3(T_K(f,g)) &= \int_G R_v^3\big(R_{u^{-1}}^1 f R_u^2 g\big) K(u)\, dm(u) \\ &= \int_G R_{vu^{-1}}^1 f R_{vu}^2 g K(u)\, dm(u) \ , \end{aligned}$$

and by (2.3), we obtain that, for every $f \in L^{p_1}(\Omega)$, $g \in L^{p_2}(\Omega)$, and every open set $V \subset G$,

$$\|T_K(f,g)\|_{L^{p_3}(\mu)}^{p_3} \le B^{p_3} \frac{1}{m(V)} \int_V \int_\Omega \left| R_v^3 T_K(f,g) \right|^{p_3} d\mu\, dm(v) \ .$$

Now, we can use similar arguments to those given in [8]. For any $\varepsilon > 0$, let $V \in \mathcal{V}$ such that

$$\max\left\{ \frac{m(VC)}{m(V)}, \frac{m\big(VC^{-1}\big)}{m(V)} \right\} \le 1 + \varepsilon \ ,$$

with $C = \operatorname{supp} K$. Then,

$$\begin{aligned} &\|T_K(f,g)\|_{L^{p_3}(\mu)}^{p_3} \\ &\le \frac{B^{p_3}}{m(V)} \int_\Omega \int_V \left| \int_G \big(R_{vu^{-1}}^1 f\big) \chi_{VC^{-1}}\big(vu^{-1}\big) \big(R_{vu}^2 g\big) \chi_{VC}(vu) K(u)\, dm(u) \right|^{p_3} d\mu\, dm(v) \\ &\le B^{p_3} \frac{1}{m(V)} \int_\Omega \left\| B_K\big(R_u^1 f \chi_{VC^{-1}}, R_u^2 g \chi_{CV}\big) \right\|_{p_3}^{p_3} d\mu \\ &\le B^{p_3} N_{p_1,p_2}(K)^{p_3} \\ &\quad \cdot \frac{1}{m(V)} \int_\Omega \left[\left(\int_{VC^{-1}} \left| R_v^1 f \right|^{p_1} dm(v) \right)^{p_3/p_1} \left(\int_{VC} \left| R_v^2 g \right|^{p_2} dm(v)) \right)^{p_3/p_2} \right] d\mu \\ &\le B^{p_3} N_{p_1,p_2}(K)^{p_3} \frac{1}{m(V)} \left(\int_{VC^{-1}} \left\| R_v^1 f \right\|_{L^{p_1}}^{p_1} \right)^{p_3/p_1} \left(\int_{VC} \left\| R_v^2 g \right\|_{L^{p_2}}^{p_2} \right)^{p_3/p_2} \\ &\le B^{p_3} A_1^{p_3} A_2^{p_3} N_{p_1,p_2}(K)^{p_3} (1+\varepsilon) ||f||_{p_1}^{p_3} ||g||_{p_2}^{p_3} \ , \end{aligned}$$

from which the result follows. □

2.2 Discretization Techniques

Let us denote $A_u = u + A$ where $u \in \mathbb{R}$ and A is an interval in $\mathbb{R}$ and let $I = (-1/4, 1/4)$ and $p \geq 1$. Denote by $Q : L^p(\mathbb{R}) \to \ell^p(\mathbb{Z})$ the bounded operator defined by

$$f \to \left(\int_{I_n} f(t)\, dt \right)_{n \in \mathbb{Z}}$$

and by $P : \ell^p(\mathbb{Z}) \to L^p(\mathbb{R})$ the map defined by

$$(a_n)_{n\in\mathbb{Z}} \to f = \sum_{n\in\mathbb{Z}} a_n \chi_{I_n} \; .$$

Observe that $\|Q\| = 2^{-1/p'}$ and $\|P\| = 2^{-1/p}$.

Proposition 1. *Let K be an integrable kernel in $L^1(\mathbb{R})$ and let us define*

$$K_n = \int_I \int_{(n+I_u)\cap(n-I_u)} K(t)\, dt\, du \; .$$

If

$$T_K(f,g)(x) = \int_{\mathbb{R}} f(x-t)g(x+t)K(t)\, dt$$

then

$$QT_K(Pa, Pb)(m) = T_{(K_n)}(a,b)(m) = \sum_{n\in\mathbb{Z}} a_{m-n} b_{m+n} K_n \; .$$

In particular, for $p_3 \geq 1$, one gets $\|T_{(K_n)}\|_{p_1,p_2} \leq \frac{1}{2}\|T_K\|_{p_1,p_2}$, where $\|T_{(K_n)}\|_{p_1,p_2}$ stands for the norm of the bilinear map $T_{(K_n)}$ from $\ell^{p_1}(\mathbb{Z}) \times \ell^{p_2}(\mathbb{Z})$ to $\ell^{p_3}(\mathbb{Z})$ and $\|T_K\|_{p_1,p_2}$ stands for the norm of the bilinear map T_K from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^{p_3}(\mathbb{R})$.

Proof. Given finite sequences a, b, we have that

$$\begin{aligned} T_K(Pa, Pb)(x) &= \sum_{n,m} a_n b_m T_K(\chi_{I_n}, \chi_{I_m})(x) \\ &= \sum_{n,m} a_n b_m \int_{(x-I_n)\cap(-x+I_m)} K(t)\, dt \\ &= \sum_{n,m} a_n b_m \int_{(x-n+I)\cap(-x+m+I)} K(t)\, dt \; . \end{aligned}$$

Now, it is clear that $(x-n+I)\cap(-x+m+I) \neq \emptyset$, if and only if $|2x-(n+m)| < 1/2$, and hence, given $k \in \mathbb{Z}$ and $x \in I_k$, $(x-n+I)\cap(-x+m+I) \neq \emptyset$ implies that $|2k-(n+m)| < 1$; that is $2k = n+m$. Thus,

$$\begin{aligned} \int_{I_k} T_K(Pa, Pb)(x)\, dx &= \sum_{n,m} a_n b_m \int_{k+I} \int_{(x-n+I)\cap(-x+m-I)} K(t)\, dt\, dx \\ &= \sum_{n,m} a_n b_m \int_I \int_{(k-n+I_u)\cap(-k+m-I_u)} K(t)\, dt\, du \\ &= \sum_{n+m=2k} a_n b_m \int_I \int_{(k-n+I_u)\cap(k-n-I_u)} K(t)\, dt\, du \\ &= \sum_{l\in\mathbb{Z}} a_{k-l} b_{k+l} \int_I \int_{(l+I_u)\cap(l-I_u)} K(t)\, dt\, du \; , \end{aligned}$$

and therefore, for every $m \in \mathbb{Z}$,

$$QT_K(Pa, Pb)(m) = \sum_{n\in\mathbb{Z}} a_{m-n} b_{m+n} K_n \ ,$$

as we wanted to see. □

3. Applications

3.1 Bilinear Hilbert Transform on $\mathbb{T}$

We shall apply our transference method to give a new proof of the result in [10] for the bilinear Hilbert transform on $\mathbb{T}$. For such a purpose take $G = \mathbb{R}$ with the Lebesgue measure, (Ω, Σ, μ) the measure space $(\mathbb{T}, \mathcal{B}(\mathbb{T}), m)$ the Lebesgue measure on $\mathbb{T}$ and $R^1 = R^2 = R^3 = R$, where

$$(R_u f)(e^{i\theta}) = f(e^{i(\theta-u)}) \ .$$

Trivially R^j, $j = 1, 2, 3$, satisfy conditions (2.1), (2.2), and (2.3).

Definition 1. A function $m \in L^\infty(\mathbb{R})$ is said to be normalized, if $m_n = \hat{\phi}_n * m$ is pointwise convergent to m where $\phi_n(x) = \frac{1}{2n}\chi_{[-n,n]} * \chi_{[-n,n]}$.

Theorem 2. *Let $K \in \mathcal{S}'(\mathbb{R})$ such that $\hat{K}(\xi) = m(\xi)$ for some normalized function m. Let*

$$T_K(f, g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\nu) m(\xi - \nu) e^{ix(\xi+\nu)} \, d\xi \, d\nu \ ,$$

for $f, g \in \mathcal{S}$ and let

$$\tilde{T}_K(P, Q)(x) = \sum_{k\in\mathbb{Z}} \sum_{k'\in\mathbb{Z}} \hat{P}(k)\hat{Q}(k) m(k - k') e^{ix(k+k')} \ ,$$

for P and Q trigonometric polynomials.

Then, if $T_K : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{p_3}(\mathbb{R})$ is bounded, we have that

$$\tilde{T}_K : L^{p_1}(\mathbb{T}) \times L^{p_2}(\mathbb{T}) \to L^{p_3}(\mathbb{T})$$

is also bounded, if $p_3 \geq 1$.

Proof. As in Lemma 3.5 of [8], let us take $\psi \in L^2(\mathbb{R})$ with compact support such that $\hat{\psi}(0) = 1$ and let us define $K_n(x) = (m_n \hat{h}_n)\check{}(x)$ where $h_n(x) = n\psi(nx)$. Then $K_n \in L^1(\mathbb{R})$, it has compact support and $\hat{K}_n(x) \to m(x)$ for all $x \in \mathbb{R}$.

Let

$$T_n(f, g)(x) = \int_{\mathbb{R}} K_n(t) f(x - t) g(x + t) \, dt$$

for $f, g \in \mathcal{S}(\mathbb{R})$.

We shall show first that $T_n : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{p_3}(\mathbb{R})$ and $\sup_{n\in\mathbb{N}} ||T_n|| < \infty$.

Now,

$$\begin{aligned} T_n(f, g)(x) &= \int_{\mathbb{R}} \hat{K}_n(\xi) [f(x - \cdot) g(x + \cdot)]\check{}(\xi) \, d\xi \\ &= \int_{\mathbb{R}} \hat{h}_n(\xi) m_n(\xi) [f(x - \cdot) g(x + \cdot)]\check{}(\xi) \, d\xi \\ &= \int_{\mathbb{R}} h_n(t) A_n(t, x) \, dt \end{aligned}$$

where $A_n(x,t) = (\check{m}_n * f(x+\cdot)g(x-\cdot))(t)$. We write

$$\begin{aligned} A_n(x,t) &= \int_{\mathbb{R}} \check{m}_n(y) f(x+t-y)g(x-t+y)\,dy \\ &= \int_{\mathbb{R}} m_n(y)[f(x+t-\cdot)g(x-t+\cdot)]\check{}\,dy \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \hat{\phi}_n(z)m(y-z)\,dz\right)[f(x+t-\cdot)g(x-t+\cdot)]\check{}\,dy \\ &= \int_{\mathbb{R}} \hat{\phi}_n(z)\left(\int_{\mathbb{R}} m(y-z)[f(x+t-\cdot)g(x-t+\cdot)]\check{}\,dy\right)dz\,. \end{aligned}$$

Now, observe that

$$\begin{aligned} B(z,x,t) &= \int_{\mathbb{R}} m(y-z)[f(x+t-\cdot)g(x-t+\cdot)]\check{}\,dy \\ &= \int_{\mathbb{R}} K(y)e^{2\pi izy} f(x+t-y)g(x-t+y)\,dy \\ &= \int_{\mathbb{R}} K(y)F_{z,t}(x-y)G_{z,t}(x+y)\,dy \end{aligned}$$

where $F_{z,t}(u) = e^{\pi izu}f(u+t)$ and $G_{z,t}(u) = e^{\pi izu}g(u-t)$.

Therefore,

$$T_n(f,g)(x) = \int_{\mathbb{R}} h_n(t)\left(\int_{\mathbb{R}} \hat{\phi}_n(z)B_K(F_{z,t},G_{z,t})(x)\,dz\right)dt$$

and, hence, since

$$||B_K(F_{z,t},G_{z,t})||_{p_3} \le C||F_{z,t}||_{p_1}||G_{z,t}||_{p_2} \le C||f||_{p_1}||g||_{p_2}\,,$$

and $p_3 \ge 1$, we obtain that

$$||T_n(f,g)||_{p_3} \le C||h_n||_1 \left\|\hat{\phi}_n\right\|_1 ||f||_{p_1}||g||_{p_2}\,.$$

Now, we can apply Theorem 1 with $R_u P(\theta) = P(\theta - u)$, to get that the transferred bilinear operator

$$\tilde{T}_n(P,Q)(\theta) = \int_{\mathbb{T}} \tilde{K}_n(u)P(\theta-u)Q(\theta+u)\,du\,,$$

where $\tilde{K}_n(u) = \sum_{m\in\mathbb{Z}} K_n(m+u)$, is bounded from $L^{p_1}(\mathbb{T}) \times L^{p_2}(\mathbb{T}) \to L^{p_3}(\mathbb{T})$ and the norms are uniformly bounded for $n \in \mathbb{N}$.

To finish the proof observe that, if $e_k(\theta) = e^{ik\theta}$ then

$$\tilde{T}_n\big(e_k, e_{k'}\big) = e_k e_{k'} \int_{\mathbb{R}} K_n(u)e^{iu(k'-k)}\,du = e_{k+k'}m_n\big(k-k'\big)\hat{h}_n\big(k-k'\big)\,,$$

and hence,

$$\lim_{n\to\infty} T_n(e_k,e_{k'}) = e_{k+k'}m\big(k-k'\big) = \tilde{T}_K(e_k,e_{k'})\,.$$

Therefore, by linearity, density and Fatou's lemma, we obtain the result. □

Now, in order to avoid the condition of $p_3 \geq 1$ in the case of the bilinear Hilbert transform, we need the following lemma that follows from the boundedness of the corresponding maximal bilinear Hilbert operator (see [19]).

Lemma 1. *Let $0 < A, A \leq \infty$ and let $p_1, p_2 > 1$. Define $K_{A,A'}(x) = \frac{1}{x}\chi_{A<|x|<A'}(x)$. Let*

$$B_{A,A'}(f,g)(x) = \int_{\mathbb{R}} f(x-t)g(x+t)K_{A,A'}(t)\,dt$$

and let $\|B_{A,A'}\|_{p_1,p_2}$ denote the norm as bounded bilinear map from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ into $L^{p_3}(\mathbb{R})$. Then

$$\sup_{A,A'} \|B_{A,A'}\|_{p_1,p_2} < \infty\,.$$

Let us give an easy proof of the above lemma in the case $p_3 \geq 1$. To this end, we first need the following lemma.

Lemma 2. *Let $p_3 \geq 1$. Let $h_1, h_2 \in L^1(\mathbb{R})$ and define $m(\xi) = \operatorname{sign}(\xi)\hat{h_1}(\xi) + \hat{h}_2(\xi)$. If*

$$B_{h_1,h_2}(f,g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)m(\xi-\eta)e^{i(\xi+\eta)x}\,d\xi\,d\eta\,.$$

Then B_h is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ into $L^{p_3}(\mathbb{R})$ and $\|B_{h_1,h_2}\|_{p_1,p_2} \leq \|H\|_{p_1,p_2}$ $\|h_1\|_1 + \|h_2\|_1$.

Proof.

$$\begin{aligned}
B_{h_1,h_2}(f,g)(x) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)m(\xi-\eta)e^{i(\xi+\eta)x}\,d\xi\,d\eta \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)\operatorname{sign}(\xi-\eta)\left(\int_{\mathbb{R}} h_1(y)e^{-i(\xi-\eta)y}\,dy\right)e^{i(\xi+\eta)x}\,d\xi\,d\eta \\
&\quad + \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)\left(\int_{\mathbb{R}} h_2(y)e^{-i(\xi-\eta)y}\,dy\right)e^{i(\xi+\eta)x}\,d\xi\,d\eta \\
&= \int_{\mathbb{R}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)e^{-i\xi y}\hat{g}(\eta)e^{i\eta y}\operatorname{sgn}(\xi-\eta)e^{i(\xi+\eta)x}\,d\xi\,d\eta\right)h_1(y)\,dy \\
&\quad + \int_{\mathbb{R}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)e^{-i\xi(y-x)}\hat{g}(\eta)e^{i\eta(y+x)}\,d\xi\,d\eta\right)h_2(y)\,dy \\
&= \int_{\mathbb{R}} H(f_y, g_{-y})(x)h_1(y)\,dy + \int_{\mathbb{R}} f(x-y)g(x+y)h_2(y)\,dy
\end{aligned}$$

where $f_x(y) = f(y-x)$. Now using the boundedness of the bilinear Hilbert transform, Hölder inequality and the integrability of h_1 and h_2 one gets the result. $\square$

Proof of Lemma 1 for $p_3 \geq 1$. It is known that

$$m_{A,A'}(\xi) = \hat{K}_{A,A'}(\xi) = \operatorname{sign}(\xi)\int_A^{A'} \frac{\sin(|\xi|u)}{u}\,du\,, \tag{3.1}$$

and hence,

$$m_{A,A'}(\xi) = m(A\xi) - m\big(A'\xi\big)$$

where $m(\xi) = \operatorname{sign}(\xi)\int_1^{\infty} \frac{\sin(|\xi|u)}{u}\,du$.

Denoting by $K(x) = K_{1,\infty}(x) = \frac{1}{x}\chi_{\{|x|>1\}}(x)$ and $Q(x) = \frac{x}{1+x^2}$, we have that $K - Q = h \in L^1(\mathbb{R})$. In particular,

$$m(\xi) = \hat{K}(\xi) = -i\ \operatorname{sign}(\xi)\hat{P}(\xi) + \hat{h}(\xi)\ ,$$

where $P(x) = \frac{1}{1+x^2}$ is the Poisson kernel.

Then,

$$m_{A,A'}(\xi) = -i\ \operatorname{sign}(\xi)\big(\hat{P}_A(\xi) - \hat{P}_{A'}(\xi)\big) + \hat{h}_A(\xi) - \hat{h}_{A'}(\xi)\ ,$$

where, as usual, $f_A(x) = \frac{1}{A} f(\frac{x}{A})$.

Finally, we can apply Lemma 2 to obtain that

$$\|B_{A,A'}\|_{p_1,p_2} \le \|H\|_{p_1,p_2}\|P_A - P_{A'}\|_1 + \|h_A - h_{A'}\|_1 \le 2\|H\|_{p_1,p_2} + 2\|h\|_1\ ,$$

as we wanted to see. □

As a consequence of the previous result, we obtain the following [10].

Corollary 1. *The bilinear Hilbert transform on the torus*

$$H_{\mathbb{T}}(f,g)(x) = \int_{\mathbb{T}} f(x-y)g(x+y)\cot(\pi y)\,dy\ ,$$

is bounded from $L^{p_1}(\mathbb{T}) \times L^{p_2}(\mathbb{T})$ into $L^{p_3}(\mathbb{T})$ whenever $p_1, p_2 > 1$ and $1/p_1 + 1/p_2 = 1/p_3 < 3/2$.

Proof. Let us take $A = 1/N$ and $A' = N$ in Lemma 1. Then, since the corresponding kernel $K_{A,A'}$ is in L^1 with compact support, we can apply our transference argument and this lemma to obtain that the operator

$$\tilde{T}_K^N(P,Q)(x) = \sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}} \hat{P}(k)\hat{Q}(k)m_{1/N,N}\big(k-k'\big)e^{ix(k+k')}\ ,$$

for P and Q trigonometric polynomials, satisfies that

$$\tilde{T}_K^N : L^{p_1}(\mathbb{T}) \times L^{p_2}(\mathbb{T}) \to L^{p_3}(\mathbb{T})$$

uniformly in N. Letting N goes to infinity, we obtain the result. □

3.2 Bilinear Hilbert Transform on $\mathbb{Z}$

Using now the discretization technique of Section 2.2, we obtain the following result, whenever $p_3 \ge 1$.

Proposition 2. *Let $N \in \mathbb{N}$, and let us define the truncated discrete bilinear Hilbert transform by*

$$H_{\mathbb{Z},N}(a,b)(m) = \sum_{k\neq 0,|n|\le N} \frac{a_{m-n}b_{m+n}}{n}\ .$$

Then, for $p_1, p_2 > 1$,

$$\sup_{N\in\mathbb{N}} \|H_{\mathbb{Z},N}\|_{p_1,p_2} < \infty\ .$$

Proof. Let us apply Lemma 1 with $K = K_{\frac{1}{2}, N-\frac{1}{2}}$ and Proposition 1, to obtain that

$$T_N(a, b)(m) = \sum_{n\in\mathbb{Z}} a_{m-n} b_{m+n} K_n \tag{3.2}$$

is bounded from $\ell^{p_1}(\mathbb{Z}) \times \ell^{p_2}(\mathbb{Z})$ into $\ell^{p_3}(\mathbb{Z})$. Let us now compute K_n in this particular case:

$$\begin{aligned} K_n &= \int_I \int_{(n+I_u)\cap(n-I_u)} K(t)\, dt\, du \\ &= \int_I \int_{(n+I_u)\cap(n-I_u)\cap(\frac{1}{2}, N-\frac{1}{2})} \frac{dt}{t}\, du + \int_I \int_{(n+I_u)\cap(n-I_u)\cap(-N+\frac{1}{2}, -\frac{1}{2})} \frac{dt}{t}\, du\,. \end{aligned}$$

Observe that for $u \in I$, we have that $(n + I_u) \cap (n - I_u) \subset (n - \frac{1}{2}, n + \frac{1}{2})$, and hence $K_n = 0$, if $|n| \geq N$ and $K_0 = 0$.

For $1 \leq n < N$, we can write

$$K_n = 2\int_0^{1/4} \int_{(n+u-\frac{1}{4}, n+u+\frac{1}{4})\cap(n-u-\frac{1}{4}, n-u+\frac{1}{4})\cap(\frac{1}{2}, N-\frac{1}{2})} \frac{dt}{t}\, du\,,$$

and, if $0 < u < \frac{1}{4}$, we obtain that

$$\left(n + u - \frac{1}{4}, n + u + \frac{1}{4}\right) \cap \left(n - u - \frac{1}{4}, n - u + \frac{1}{4}\right) = \left(n + u - \frac{1}{4}, n - u + \frac{1}{4}\right)\,.$$

Hence,

$$\begin{aligned} K_n &= 2\int_0^{1/4} \log\left(\frac{n - u + \frac{1}{4}}{n + u - \frac{1}{4}}\right) du = 2\int_0^{1/4} \log\Big(\frac{n + v}{n - v}\Big) dv \\ &= 2n\int_0^{1/4n} \log\Big(\frac{1 + x}{1 - x}\Big)\, dx\,. \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} \int_0^{1/4n} \log\left(\frac{1 + x}{1 - x}\right) dx &= \frac{1}{4n} \log\left(\frac{1 + \frac{1}{4n}}{1 - \frac{1}{4n}}\right) - \int_0^{1/4n} \frac{2x}{1 - x^2}\, dx \\ &= \frac{1}{4n} \log\left(\frac{1 + \frac{1}{4n}}{1 - \frac{1}{4n}}\right) + \log\left(1 - \frac{1}{16n^2}\right)\,, \end{aligned}$$

and hence,

$$K_n = \frac{1}{2}\log\left(\frac{n + \frac{1}{4}}{n - \frac{1}{4}}\right) - 2n\log\left(\frac{16n^2}{16n^2 - 1}\right)\,.$$

Since $\log(1 + x) = x + O(x^2)$, $(x \to 0)$, we finally get

$$\begin{aligned} K_n &= \frac{1}{4n - 1} + O\left(\frac{1}{\left(n - \frac{1}{4}\right)^2}\right) + \frac{2n}{16n^2 - 1} + 2n\,O\left(\frac{1}{\left(16n^2 - 1\right)^2}\right) \\ &= \frac{6n + 1}{16n^2 - 1} + O\Big(\frac{1}{n^2}\Big) = \frac{3}{8n} + O\Big(\frac{1}{n^2}\Big)\,. \end{aligned}$$

The case $-N \leq n \leq -1$ is obtained similarly, and the result follows from (3.2). $\square$

3.3 Ergodic Bilinear Hilbert Transform

The idea now is to transfer the boundedness of the discrete bilinear Hilbert transform to a measure space using our transference result.

Let $G = \mathbb{Z}$ and let (Ω, Σ, μ) be a σ-finite measure space. Let T be a bounded and invertible operator acting on $L^{p_i}(\Omega)$, such that

$$\max\left(\left\|T^{-1}\right\|_{L(L^{p_i}(\Omega))}, \|T\|_{L(L^{p_i}(\Omega))}\right) \leq 1 \ ,$$

for $i = 1, 2$. Let us assume that there exists a bounded and invertible operator S acting on $L^{p_3}(\Omega)$, such that $\max\left(\|S^{-1}\|_{L(L^{p_3}(\Omega))}, \|S\|_{L(L^{p_3}(\Omega))}\right) \leq 1$ and such that

$$S^m\left(T^n f T^{-n} g\right) = T^{m+n} f T^{m-n} g \ . \tag{3.3}$$

Then:

Theorem 3. *The bilinear ergodic Hilbert transform*

$$H_T(f, g)(x) = \sum_{n \neq 0} \frac{T^n f(x) T^{-n} g(x)}{n}$$

is bounded from $L^{p_1}(\Omega) \times L^{p_2}(\Omega)$ into $L^{p_3}(\Omega)$ whenever $p_1, p_2 > 1$ and $1/p_1 + 1/p_2 = 1/p_3 \leq 1$.

Proof. It is trivial to see that, if we take $R_n^1 = R_n^2 = T^n$ and $R_n^3 = S^n$ and use (3.3) then conditions (2.1), (2.2), and (2.3) hold and hence, we can transfer, using Theorem 1, the boundedness of the truncated discrete bilinear Hilbert transform proved in Proposition 2, to show that, in fact,

$$H_T^N(f, g)(x) = \sum_{n \neq 0, n=-N}^{N} \frac{T^n f(x) T^{-n} g(x)}{n}$$

is bounded uniformly in N. From this, the result follows. $\square$

In particular, using $Tf(x) = f(x-1)$ and $S = T$ one obtains the following:

Corollary 2. *The bilinear Hilbert transform*

$$H(f, g)(x) = \sum_{n \neq 0} \frac{f(x-n) g(x+n)}{n}$$

is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ into $L^{p_3}(\mathbb{R})$ whenever $p_1, p_2 > 1$ and $1/p_1 + 1/p_2 = 1/p_3 \leq 1$.

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