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Time-Frequency Mean and Variance Sequences of Orthonormal Bases

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ABSTRACT. We show that there exists an orthonormal basis $\{b_n\}_{n=1}^{\infty}$ for $L^2(\mathbb{R})$ such that $\{\Delta^2(b_n)\}_{n=1}^{\infty}$, $\{\mu(b_n)\}_{n=1}^{\infty}$ and $\{\mu(\widehat{b_n})\}_{n=1}^{\infty}$ are bounded sequences. We also show that there does not exist any orthonormal basis for $L^2(\mathbb{R})$ with $\{\Delta^2(b_n)\}_{n=1}^{\infty}$, $\{\Delta^2(\widehat{b_n})\}_{n=1}^{\infty}$ and $\{\mu(b_n)\}_{n=1}^{\infty}$ being bounded sequences. This is motivated by a question posed by H.S. Shapiro on the mean and variance sequences associated to orthonormal bases.

1. Introduction

In this article we shall examine a question posed by H.S. Shapiro on how general orthonormal bases for $L^2(\mathbb{R})$ cover the time-frequency plane. Let us begin by introducing some notation and necessary definitions.

For $f \in L^2(\mathbb{R})$ we formally define its *Fourier transform,* $\widehat{f} \in L^2(\widehat{\mathbb{R}})$, by

$$\forall \gamma \in \widehat{\mathbb{R}}, \quad \widehat{f}(\gamma) = \int f(t) e^{-2\pi i t \gamma}\, dt \ ,$$

where the integral is over $\mathbb{R}$, and $\widehat{\mathbb{R}} = \mathbb{R}$ is used to distinguish between the domains of f and $\widehat{f}$. With this notation, the *time-frequency plane* is $\mathbb{R} \times \widehat{\mathbb{R}}$.

Given $f \in L^2(\mathbb{R})$ with $||f||_{L^2(\mathbb{R})} = 1$, we define the following associated *mean*

$$\mu(f) = \mathrm{Mean}\big(|f|^2\big) = \int t|f(t)|^2\, dt \ , \tag{1.1}$$

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and the following associated *variance*

$$\Delta^2(f) = \mathrm{Var}\big(|f|^2\big) = \int |t - \mu(f)|^2 |f(t)|^2 \, dt \; . \tag{1.2}$$

It will frequently be convenient to also work with the *dispersion,* $\Delta(f) \equiv \sqrt{\Delta^2(f)}$.

If one thinks of f as a function of time and its Fourier transform as a function of frequency, then $\mu(f)$ and $\Delta^2(f)$ give insight into where f is concentrated in time; $\mu(\widehat{f})$ and $\Delta^2(\widehat{f})$ give insight into where it is concentrated in frequency. Namely, f is concentrated mostly around the point $(\mu(f), \mu(\widehat{f}))$ in the time-frequency plane, and the pair $(\Delta^2(f), \Delta^2(\widehat{f}))$ gives a measure of how spread out it is about $(\mu(f), \mu(\widehat{f}))$. The Heisenberg uncertainty principle,

$$\Delta^2(f)\Delta^2\big(\widehat{f}\big) \geq \frac{1}{16\pi^2} \; ,$$

states that a unit-norm function in $L^2(\mathbb{R})$ can not occupy an arbitrarily small region in the time-frequency plane. For more information on this and other uncertainty principles, see [9].

When we speak of an orthonormal basis, $\{\varphi_n\}_{n=1}^{\infty}$, covering the time-frequency plane, we refer specifically to the manner in which the sequences $\{(\mu(\varphi_n), \mu(\widehat{\varphi_n}))\}_{n=1}^{\infty}$ and $\{(\Delta^2(\varphi_n), \Delta^2(\widehat{\varphi_n}))\}_{n=1}^{\infty}$ determine regions in $\mathbb{R} \times \widehat{\mathbb{R}}$. For example, during the past two decades much work has been done on orthonormal bases given by wavelet and Gabor systems. One reason for the considerable interest in these systems is their respective affine and Heisenberg structures. These underlying group structures give rise to the statement that wavelets cover the time-frequency plane by an affine tiling and Gabor systems cover it by a rectangular tiling, [10].

The question of finding prescribed tilings of the time-frequency plane has been studied in the engineering literature by Bernardini and Kovačević, [4]. They use local bases to construct filters which realize a wide class of tilings of the time-frequency plane, and give applications to audio coding. We also refer the reader to [17]. The manner in which a basis covers the time-frequency plane is an important factor in determining which applications the basis is suited for. For example, the affine covering given by wavelet bases provides the "zooming" effect which has made wavelets so useful in image processing and compression, e.g., see [14].

2. Shapiro's Question

In 1991 Harold S. Shapiro posed the following question, [15].

Question 1 (Shapiro). *Given four sequences of real numbers,*

$$\{a_n\}_{n=1}^{\infty}, \; \{b_n\}_{n=1}^{\infty}, \; \{c_n\}_{n=1}^{\infty}, \; \{d_n\}_{n=1}^{\infty} \; ,$$

does there exist an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ *for* $L^2(\mathbb{R})$ *such that*

$$\mu(\varphi_n) = a_n, \;\; \mu\big(\widehat{\varphi_n}\big) = b_n, \;\; \Delta^2(\varphi_n) = c_n, \;\; \Delta^2\big(\widehat{\varphi_n}\big) = d_n \; ,$$

holds for all $n \in \mathbb{N}$*?*

This is essentially a question about how orthonormal bases can cover the time-frequency plane. The following theorem will serve as a starting point for our investigation.

Theorem 1. *There does not exist an infinite orthonormal sequence* $\{f_n\}_{n=1}^{\infty} \subseteq L^2(\mathbb{R})$ *such that all four of the mean and variance sequences are bounded.*

Shapiro, [15], derives this as a corollary of a compactness result of Kolmogorov. The result also follows from the theory of prolate spheroidal wavefunctions. Motivated by Theorem 1, we shall examine and answer the following question.

Question 2. *If* $\{\varphi_n\}_{n=1}^{\infty}$ *is an orthonormal basis for* $L^2(\mathbb{R})$*, how many of the sequences* $\{\mu(\varphi_n)\}_{n=1}^{\infty}, \{\mu(\widehat{\varphi_n})\}_{n=1}^{\infty}, \{\Delta^2(\varphi_n)\}_{n=1}^{\infty}, \{\Delta^2(\widehat{\varphi_n})\}_{n=1}^{\infty}$ *can be bounded? Which combinations of these sequences can be bounded?*

3. Main Results

We shall prove the following two theorems which answer Question 2. The first theorem makes use of a density calculation and the theory of prolate spheroidal wavefunctions, whereas the second theorem uses a constructive technique of Bourgain, [5].

Theorem 2. *There does not exist an orthonormal basis* $\{b_n\}_{n=1}^{\infty}$ *for* $L^2(\mathbb{R})$ *such that* $\{\Delta^2(b_n)\}_{n=1}^{\infty}, \{\Delta^2(\widehat{b_n})\}_{n=1}^{\infty}$ *and* $\{\mu(b_n)\}_{n=1}^{\infty}$ *are all bounded sequences.*

Theorem 3. *There exists a constant* $C > 0$ *such that for any* $\epsilon > 0$ *there exists an orthonormal basis,* $\{b_n\}_{n=1}^{\infty}$*, for* $L^2(\mathbb{R})$ *satisfying* $|\mu(b_n)| \leq \epsilon$*,* $|\mu(\widehat{b_n})| \leq \epsilon$ *and* $\Delta^2(b_n) \leq C$ *for all* $n \in \mathbb{N}$*.*

Theorem 2 says that it *is not* possible to have an orthonormal basis for $L^2(\mathbb{R})$ with both variance sequences and one mean sequence bounded. On the other hand, Theorem 3 says that it *is* possible to have orthonormal bases for $L^2(\mathbb{R})$ for which both mean sequences and one variance sequence are bounded.

4. Examples

Before proving the results in the previous section, we look at some important examples of orthonormal bases and their mean and variance sequences.

Example 1 (Wavelet basis). Take $\psi \in L^2(\mathbb{R})$ such that the *wavelet system* $\{\psi_{m,n} : m, n \in \mathbb{Z}\}$ defined by $\psi_{m,n}(t) = 2^{-m/2}\psi(2^{-m}t - n)$ is an orthonormal basis for $L^2(\mathbb{R})$. For further information on wavelets see [7, 11]. A direct calculation, e.g., [1], shows that for wavelet systems the three sequences

$$\{\mu(\psi_{m,n})\}_{m,n\in\mathbb{Z}}, \left\{\Delta^2(\psi_{m,n})\right\}_{m,n\in\mathbb{Z}}, \left\{\Delta^2\left(\widehat{\psi_{m,n}}\right)\right\}_{m,n\in\mathbb{Z}},$$

are unbounded.

Example 2 (Gabor basis). Let $g \in L^2(\mathbb{R})$ be any function such that the *Gabor system* $\{g_{m,n} : m, n \in \mathbb{Z}\}$ defined by $g_{m,n}(t) = e^{2\pi imt}g(t - n)$ is an orthonormal basis for $L^2(\mathbb{R})$. For more information on Gabor systems see [10, 3]. A direct computation shows that both $\{\mu(g_{m,n})\}_{m,n\in\mathbb{Z}}$ and $\{\mu(\widehat{g_{m,n}})\}_{m,n\in\mathbb{Z}}$ are unbounded sequences. Moreover, the Balian-Low theorem, e.g., [10, 6], shows that at least one of the two variance sequences,

$\{\Delta^2(g_{m,n})\}_{m,n\in\mathbb{Z}}$ and $\{\Delta^2(\widehat{g_{m,n}})\}_{m,n\in\mathbb{Z}}$, must be unbounded (in fact, constantly equal to ∞).

Example 3 (Hermite basis). Let $\{h_n\}_{n=0}^{\infty}$ be the Hermite functions defined by

$$h_k(t) = \frac{2^{1/4}}{\sqrt{k!}} \left(\frac{-1}{2\sqrt{\pi}}\right)^k e^{\pi t^2} \frac{d^k}{dt^k}\left(e^{-2\pi t^2}\right) .$$

We follow the notation of [9]. The Hermite functions are eigenfunctions of the Fourier transform, form an orthonormal basis for $L^2(\mathbb{R})$, and satisfy

$$2\sqrt{\pi}\; t h_k(t) = \sqrt{k+1} h_{k+1}(t) + \sqrt{k} h_{k-1}(t) . \tag{4.1}$$

By taking the inner product of (4.1) with h_k and using the orthonormality of the Hermite functions, it follows that $\mu(h_k) = 0$ for all k. Since each h_n is an eigenfunction of the Fourier transform, we also have $\mu(\widehat{h_k}) = 0$. In particular, both mean sequences are bounded. Using (4.1) again, one can show that $\Delta(h_k) = \Delta(\widehat{h_k}) = \frac{\sqrt{2k+1}}{2\sqrt{\pi}}$, so that both variance sequences are unbounded.

Example 4 (Bourgain basis). Let $\epsilon > 0$. In [5], Bourgain constructs an orthonormal basis, $\{f_n\}_{n=1}^{\infty}$, for $L^2(\mathbb{R})$ satisfying $\Delta^2(f_n) \le \left(\frac{1}{2\pi} + \epsilon\right)^2$ and $\Delta^2(\widehat{f_n}) \le \left(\frac{1}{2\pi} + \epsilon\right)^2$ for all $n \in \mathbb{N}$. However, the mean sequences are both unbounded. For recent work on Bourgain's theorem see [2].

Example 5 (Wilson basis). Let $g \in L^2(\mathbb{R})$ and define the associated *Wilson system,* $\{\psi_{l,k} : l, k \in \mathbb{Z}, \text{ and } 0 \le l\}$, by

$$\begin{aligned}
\psi_{0,k}(t) &= g(t-k), && k \in \mathbb{Z} \\
\psi_{l,k}(t) &= \sqrt{2} g(t-k/2)\cos(2\pi l t), && l > 0,\ k+l \text{ even} , \\
\psi_{l,k}(t) &= \sqrt{2} g(t-k/2)\sin(2\pi l t), && l > 0,\ k+l \text{ odd} .
\end{aligned}$$

See [10] for background on Wilson bases. For any nontrivial g one can verify that

$$\{\mu(\psi_{l,k})\}_{l,k\in\mathbb{Z},0\le l} \text{ and } \left\{\Delta^2\left(\widehat{\psi_{l,k}}\right)\right\}_{l,k\in\mathbb{Z},0\le l} \quad \text{are unbounded sequences} .$$

For an example of a Wilson basis whose generator g has exponential localization in time and frequency, see [8].

5. Two Variances and One Mean

In this section we shall prove Theorem 2. We begin with some background on the prolate spheroidal wavefunctions.

5.1 Prolate Spheroidal Wavefunctions

Fix $\Omega > 0$ and let $\{\psi_n\}_{n=0}^{\infty}$ be the associated prolate spheroidal wavefunctions, as defined in [16]. $\{\psi_n\}_{n=0}^{\infty}$ is an orthonormal basis for $PW_\Omega \equiv \{f \in L^2(\mathbb{R}) : \operatorname{supp}\widehat{f} \subseteq [-\Omega, \Omega]\}$. See [16, 12, 13] for background on prolate spheroidal wavefunctions. One of the main applications of prolate spheroidal wavefunctions is to study the set of almost timelimited and almost bandlimited functions, $S_{T,\Omega,\epsilon,\eta} \subseteq L^2(\mathbb{R})$.

Definition 1. Given constants $\epsilon, \eta, T, \Omega > 0$,

$$S_{T,\Omega,\epsilon,\eta} = \left\{ f \in L^2(\mathbb{R}) : \int_{|t|\geq T} |f(t)|^2\, dt \leq \epsilon^2 \text{ and } \int_{|\gamma|\geq\Omega} \left|\widehat{f}(\gamma)\right|^2 d\gamma \leq \eta^2 \right\} .$$

Our main interest in prolate spheroidal wavefunctions lies in the following theorem on the approximate dimension of $S_{T,\Omega,\epsilon,\eta}$, see Theorem 12 in [13].

Theorem 4 (Landau, Pollak). *If $f \in S_{T,\Omega,\epsilon,\eta}$ with $\|f\|_{L^2(\mathbb{R})} = 1$ then there exists $\{a_n\}_{n=0}^{\lfloor 2T\Omega\rfloor} \subseteq \mathbb{C}$, such that*

$$\left\| f - \sum_{n=0}^{\lfloor 2T\Omega\rfloor} a_n\psi_n \right\|_{L^2(\mathbb{R})}^2 \leq 12(\epsilon+\eta)^2 + \eta^2 .$$

In other words, the set of almost timelimited and almost bandlimited functions has approximate dimension $2T\Omega$. This yields the following corollary, whose proof we include for the sake of completeness.

Corollary 1. *Let $\epsilon, \eta > 0$ be sufficiently small, and let $T, \Omega > 0$. There exists $N \in \mathbb{N}$ such that $S_{T,\Omega,\epsilon,\eta}$ contains no orthonormal subset containing more than N elements.*

Proof. Suppose $\{f_k\}_{k=1}^N \subseteq S_{T,\Omega,\epsilon,\eta}$ is orthonormal. Let $\{\psi_n\}_{n=0}^{\infty}$ be the prolate spheroidal wavefunctions associated to Ω. For each $1 \leq k \leq N$, let $\{a_{n,k}\}_{n=0}^{\lfloor 2T\Omega\rfloor}$ be the coefficients given by Landau and Pollak's theorem which satisfy

$$\left\| f_k - \sum_{n=0}^{\lfloor 2T\Omega\rfloor} a_{n,k}\psi_n \right\|_{L^2(\mathbb{R})}^2 \leq 12(\epsilon+\eta)^2 + \eta^2 .$$

To simplify notation, we let

$$F_k = \sum_{n=0}^{\lfloor 2T\Omega\rfloor} a_{n,k}\psi_n .$$

Since the f_k are orthonormal, we have that for $j \neq k$

$$\begin{aligned}\left| \sum_{n=0}^{\lfloor 2T\Omega\rfloor} a_{n,j}\overline{a_{n,k}} \right| &= \left|\langle F_j, F_k\rangle\right| = \left|\langle F_j - f_j + f_j, F_k - f_k + f_k\rangle\right| \\ &\leq |\langle F_j - f_j, F_k - f_k\rangle| + |\langle F_j - f_j, f_k\rangle| + |\langle f_j, F_k - f_k\rangle| \\ &\leq 12(\epsilon+\eta)^2 + \eta^2 + 2\sqrt{12(\epsilon+\eta)^2+\eta^2} .\end{aligned}$$

Next, note that we also have

$$\begin{aligned}\left(\sum_{n=0}^{\lfloor 2T\Omega\rfloor} |a_{n,k}|^2 \right)^{\frac{1}{2}} &= \left\| \sum_{n=0}^{\lfloor 2T\Omega\rfloor} a_{n,k}\psi_n \right\|_{L^2(\mathbb{R})} \geq \|f_k\|_{L^2(\mathbb{R})} - \left\| f_k - \sum_{n=0}^{\lfloor 2T\Omega\rfloor} a_{n,k}\psi_n \right\|_{L^2(\mathbb{R})} \\ &\geq 1 - \sqrt{12(\epsilon+\eta)^2+\eta^2} .\end{aligned}$$

Thus, defining $v_k = (a_{0,k}, a_{1,k}, \ldots, a_{\lfloor 2T\Omega\rfloor,k}) \in \mathbb{C}^{\lfloor 2T\Omega\rfloor+1}$ for $k = 1, 2, \ldots, N$, we have

$$1 \geq ||v_k||_{l^2} \geq 1 - \sqrt{12(\epsilon+\eta)^2+\eta^2} , \tag{5.1}$$

and

$$12(\epsilon+\eta)^2+\eta^2+2\sqrt{12(\epsilon+\eta)^2+\eta^2} \geq |\langle v_j, v_k\rangle|, \quad \text{for} \quad j \neq k\,. \tag{5.2}$$

Take $\epsilon, \eta > 0$ small enough so that

$$\alpha \equiv 3\sqrt{12(\epsilon+\eta)^2+\eta^2} < \frac{1}{100}\,.$$

Now, for $j \neq k$, $||v_j||_{l^2} \geq 1-\alpha$ and $\alpha \geq |\langle v_j, v_k\rangle|$ imply that

$$||v_j - v_k||_{l^2}^2 \geq 2(1-\alpha)^2 - 2\alpha > 1\,.$$

A volume counting argument shows that

$$N \leq \frac{\text{Volume}\Big(\big\{v \in \mathbb{C}^{\lfloor 2T\Omega\rfloor+1} : ||v||_{l^2} \leq 3/2\big\}\Big)}{\text{Volume}\Big(\big\{v \in \mathbb{C}^{\lfloor 2T\Omega\rfloor+1} : ||v||_{l^2} \leq 1/2\big\}\Big)} \leq 3^{2\lfloor 2T\Omega\rfloor+2}\,.$$

While more refined estimates are possible, this suffices for our purposes. Note that the choice of sufficiently small $\epsilon, \eta > 0$ does not depend on T, Ω, but the size of N does depend on T, Ω. □

5.2 Preliminary Lemmas

Lemma 1. *Suppose $g \in L^2(\mathbb{R})$, $||g||_{L^2(\mathbb{R})} = 1$, satisfies*

$$|\mu(g)| < A,\ \left|\mu\big(\widehat{g}\big)\right| < B\ ,\ \Delta(g) < J\ ,\ \Delta\big(\widehat{g}\big) < K\,.$$

Fix $\epsilon > 0$. If $R > \max\{\frac{J}{\epsilon}, \frac{K}{\epsilon}\}$ then $g \in S_{A+R,B+R,\epsilon,\epsilon}$.

Proof. Since $R^2 > \frac{J^2}{\epsilon^2}$,

$$\int_{|t|\geq A+R} |g(t)|^2\,dt \leq \int_{|t|\geq|\mu(g)|+R} |g(t)|^2\,dt \leq \int_{|t-\mu(g)|\geq R} |g(t)|^2\,dt$$
$$\leq \frac{1}{R^2}\int_{\mathbb{R}} |t-\mu(g)|^2|g(t)|^2\,dt \leq \frac{J^2}{R^2} < \epsilon^2\,.$$

Likewise,

$$\int_{|\gamma|\geq B+R} \left|\widehat{g}(\gamma)\right|^2 d\gamma \leq \frac{K^2}{R^2} < \epsilon^2\,.$$

□

Lemma 2. *Suppose $f, g \in L^2(\mathbb{R})$, $||f||_{L^2(\mathbb{R})} = ||g||_{L^2(\mathbb{R})} = 1$, and that the means and variances*

$$\mu(f), \mu\big(\widehat{f}\big), \mu(g), \mu\big(\widehat{g}\big), \Delta^2(f), \Delta^2\big(\widehat{f}\big), \Delta^2(g), \Delta^2\big(\widehat{g}\big)$$

are all finite. Then,

$$|\langle f, g\rangle| \leq 2\,\frac{\Delta(f)+\Delta\big(\widehat{f}\big)+\Delta(g)+\Delta\big(\widehat{g}\big)}{|\mu(f)-\mu(g)| + \left|\mu\big(\widehat{f}\big)-\mu\big(\widehat{g}\big)\right|}\,.$$

Proof. Let

$$S_1 = \left\{ t : |t - \mu(f)| \geq \frac{1}{2}|\mu(f) - \mu(g)| \right\} \text{ and } S_2 = \left\{ t : |t - \mu(g)| \geq \frac{1}{2}|\mu(f) - \mu(g)| \right\}.$$

So,

$$\begin{aligned} |\langle f, g\rangle| &\leq \int |f(t)||g(t)|\,dt \leq \int_{S_1} |f(t)||g(t)|\,dt + \int_{S_2} |f(t)||g(t)|\,dt \\ &\leq \frac{2}{|\mu(f) - \mu(g)|} \int |t - \mu(f)||f(t)||g(t)|\,dt \\ &\quad + \frac{2}{|\mu(f) - \mu(g)|} \int |t - \mu(g)||f(t)||g(t)|\,dt \\ &\leq \frac{2\Delta(f)}{|\mu(f) - \mu(g)|} + \frac{2\Delta(g)}{|\mu(f) - \mu(g)|} = 2\frac{\Delta(f) + \Delta(g)}{|\mu(f) - \mu(g)|}\,. \end{aligned}$$

Likewise,

$$|\langle f, g\rangle| = \left|\langle \widehat{f}, \widehat{g}\rangle\right| \leq 2\frac{\Delta\left(\widehat{f}\right) + \Delta\left(\widehat{g}\right)}{\left|\mu\left(\widehat{f}\right) - \mu\left(\widehat{g}\right)\right|}\,.$$

Now, combining the previous two inequalities gives

$$|\langle f, g\rangle| \leq 2\frac{\Delta(f) + \Delta\left(\widehat{f}\right) + \Delta(g) + \Delta\left(\widehat{g}\right)}{|\mu(f) - \mu(g)| + \left|\mu\left(\widehat{f}\right) - \mu\left(\widehat{g}\right)\right|}\,,$$

as desired. □

5.3 Two Variances and One Mean: The Proof

We shall derive Theorem 2 as a corollary of the following theorem.

Theorem 5. *Suppose $\{b_n\}_{n=1}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R})$ with*

$$\forall n \in \mathbb{N}, \quad \Delta(b_n) \leq K \quad \text{and} \quad \Delta\left(\widehat{b_n}\right) \leq K\,.$$

Fix $a, b \in \mathbb{R}$. If

$$\left\{\left(\mu(b_n), \mu\left(\widehat{b_n}\right)\right)\right\}_{n=1}^{\infty} \subset W_{a,b} \equiv \left\{(x, y) \in \mathbb{R}^2 : x \leq a \text{ or } y \leq b\right\}, \tag{5.3}$$

then

$$\sum_{n=1}^{\infty} \frac{1}{\left(1 + |\mu(b_n)| + \left|\mu\left(\widehat{b_n}\right)\right|\right)^2} = \infty\,. \tag{5.4}$$

Proof. Without loss of generality assume $a = b = 0$. We proceed by contradiction and begin by assuming such a basis exists and satisfies

$$\sum_{n=1}^{\infty} \frac{1}{\left(1 + |\mu(b_n)| + \left|\mu\left(\widehat{b_n}\right)\right|\right)^2} < \infty\,. \tag{5.5}$$

Fix $f \in L^2(\mathbb{R})$ with $||f||_{L^2(\mathbb{R})} = 1$, $\mu(f) = \mu(\widehat{f}) = 0$, and $\Delta(f), \Delta(\widehat{f}) \leq K$. Let $f_N(t) = e^{2\pi i N t} f(t - N)$, so that $\mu(f_N) = \mu(\widehat{f_N}) = N$ and $\Delta(f_N), \Delta(\widehat{f_N}) \leq K$.

By Lemma 2,

$$1 = ||f_N||^2_{L^2(\mathbb{R})} = \sum_{n=1}^{\infty} |\langle f_N, b_n\rangle|^2 \le 64K^2 \sum_{n=1}^{\infty} \frac{1}{\left(|N-\mu(b_n)| + \left|N-\mu\left(\widehat{b_n}\right)\right|\right)^2} . \quad (5.6)$$

Now let

$$P_1 = \left\{ j \in \mathbb{N} : \mu\left(\widehat{b_j}\right) \le 0 \right\} \text{ and } P_2 = \{ j \in \mathbb{N} : \mu(b_j) \le 0 \} .$$

If $j \in P_1 \cup P_2 = \mathbb{N}$, and $N > 0$ then $|\mu(b_j)| + |\mu(\widehat{b_j})| + N \le 2|\mu(b_j) - N| + 2|\mu(\widehat{b_j}) - N|$.

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{\left(|N-\mu(b_n)| + \left|N-\mu\left(\widehat{b_n}\right)\right|\right)^2} \le 4 \sum_{n=1}^{\infty} \frac{1}{\left(|\mu(b_n)| + \left|\mu\left(\widehat{b_n}\right)\right| + N\right)^2} \equiv I(N) .$$

By assumption (5.5) we have $\lim_{N\to\infty} I(N) = 0$. Hence,

$$1 \le (16K)^2 I(N) \to 0, \quad \text{as } N \to \infty ,$$

which is a contradiction. □

We now show that Theorem 2 follows from Theorem 5.

Proof of Theorem 2. Suppose $\{b_n\}_{n=1}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R})$ with $\Delta(b_n) \le K$, $\Delta(\widehat{b_n}) \le K$ and $|\mu(b_n)| \le B$. For each $n \in \mathbb{Z}$, let

$$I_n = [-B, B] \times [n, n+1] \quad \text{and} \quad S_n = \left\{ j \in \mathbb{N} : \left(\mu(b_j), \mu\left(\widehat{b_j}\right)\right) \in I_n \right\} .$$

By translating to the origin and using Lemma 1 and Corollary 1

$$\exists M \text{ such that } \forall n \in \mathbb{Z}, \quad \text{card } S_n \le M .$$

Thus,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{\left(|\mu(b_n)| + \left|\mu\left(\widehat{b_n}\right)\right| + 1\right)^2} &= \sum_{k\in\mathbb{Z}} \sum_{j\in S_k} \frac{1}{\left(|\mu(b_j)| + \left|\mu\left(\widehat{b_j}\right)\right| + 1\right)^2} \\ &\le 2M \sum_{j=0}^{\infty} \frac{1}{(|j|+1)^2} < \infty . \end{aligned}$$

Since this contradicts Theorem 5, Theorem 2 holds. □

6. Two Means and One Variance

We prove Theorem 3 in this section. Our proof is based on Bourgain's method of construction in [5].

Proof of Theorem 3. We begin by defining a system of functions $\mathcal{G}(T, N)$, which we shall need for the proof. $C_c^{\infty}(\mathbb{R})$ denotes the set of infinitely differentiable, compactly supported functions on $\mathbb{R}$.

I. Let g be a function in the Schwartz class, $\mathcal{S}(\mathbb{R})$, satisfying

- $||g||_{L^2(\mathbb{R})} = 1$ and $\widehat{g} \in C_c^\infty(\mathbb{R})$,
- supp $\widehat{g} \subseteq [-1/2, 1/2]$,
- g is real and even,
- $\mu(g) = \mu(\widehat{g}) = 0$ and $\Delta(g) \equiv \delta < \infty$.

Regarding the third and fourth bullets, note that g is real and even, if and only if $\widehat{g}$ is real and even. Also, the mean of an even Schwartz class function is 0. Now define $g_n(t) = \sqrt{2}\cos(2\pi nt)g(t)$. The functions $\{g_n\}_{n=1}^\infty$ have the following properties which are easily verified:

- $\widehat{g_n}(\gamma) = \frac{\sqrt{2}}{2}(\widehat{g}(\gamma - n) + \widehat{g}(\gamma + n))$,
- $||g_n||_{L^2(\mathbb{R})} = 1$, and $\langle g_n, g_m\rangle = 0$, if $n \neq m$,
- $\mu(g_n) = 0 = \mu(\widehat{g_n})$,
- $\Delta(g_n) \leq (\sqrt{2})\Delta(g) = (\sqrt{2})\delta$,
- supp $\widehat{g_n} \subseteq [-n - \frac{1}{2}, -n + \frac{1}{2}] \bigcup [n - \frac{1}{2}, n + \frac{1}{2}]$.

Given $T, N \in \mathbb{N}$, we define the orthonormal system $\mathcal{G}(T, N) = \{g_n\}_{n=N}^{N+T-1}$.

II. Let $\{\varphi_n\}_{n=1}^\infty \subseteq \mathcal{S}(\mathbb{R})$ be dense in the unit sphere of $L^2(\mathbb{R})$ and satisfy

$$\forall n \in \mathbb{N}, \quad ||\varphi_n||_{L^2(\mathbb{R})} = 1 \text{ and } \widehat{\varphi_n} \in C_c^\infty(\mathbb{R}) .$$

The basis will be of the form $B = \bigcup_{j=1}^\infty B_j$, where each B_j is a finite set of Schwartz class functions whose Fourier transforms are compactly supported. We shall construct the B_j inductively.

Suppose we have already constructed $B_1, \ldots, B_{n-1}$. Let

$$\Phi_n = \varphi_n - P_{[B_1,\ldots,B_{n-1}]}\varphi_n ,$$

where $[S]$ is the notation in [5] which denotes the closed linear span of a set of functions S, and $P_{[S]}$ is the orthogonal projection of $L^2(\mathbb{R})$ onto the closed subspace $[S] \subseteq L^2(\mathbb{R})$. In particular, when $S = \{s_n\}_{n=1}^\infty \subseteq L^2(\mathbb{R})$ is any set of orthonormal functions then

$$P_{[S]}f = \sum_{n=1}^\infty \langle f, s_n\rangle s_n .$$

For the base case of our inductive construction let $\Phi_1 = \varphi_1$. Observe that $||\Phi_n||_{L^2(\mathbb{R})} \leq 1$, and Φ_n is orthogonal to the elements of B_j for each $j < n$. Note that $\widehat{\Phi_n} \in C_c^\infty(\mathbb{R})$ since φ_n and the elements of $\bigcup_{j=1}^{n-1} B_j$ also satisfy this property.

Take N_n large enough so that $[-N_n + 1, N_n - 1]$ contains the support of $\widehat{\Phi_n}$ and the supports of the Fourier transforms of the functions in $\bigcup_{j=1}^{n-1} B_j$. Take $T_n \in \mathbb{N}$ large enough so that:

$$\int |t|^2 |\Phi_n(t)|^2\, dt \leq T_n^2 , \tag{6.1}$$

and

$$\left|\int t|\Phi_n(t)|^2\, dt\right| \leq \epsilon T_n^2 \quad \text{and} \quad \left|\int \gamma \left|\widehat{\Phi_n}(\gamma)\right|^2 d\gamma\right| \leq \epsilon T_n^2 , \tag{6.2}$$

where $\epsilon > 0$ is as in the hypotheses of the theorem.

Enumerate the elements of $\mathcal{G}(T_n^2, N_n)$ as $\{g_{j,n}\}_{j=1}^{T_n^2}$. The support properties of $\mathcal{G}(T_n^2, N_n)$ ensure that the elements of $\mathcal{G}(T_n^2, N_n)$ are orthogonal to Φ_n and the elements of $\bigcup_{j=1}^{n-1} B_j$. We now define the elements of $B_n = \{b_{j,n}\}_{j=1}^{T_n^2}$ as

$$b_{1,n}(t) = \frac{\Theta}{T_n}\Phi_n(t) + \alpha_{1,n} g_{1,n}(t) \ ,$$

and for $1 < j \leq T_n^2$,

$$b_{j,n}(t) = \frac{\Theta}{T_n}\Phi_n(t) + \beta_{1,n} g_{1,n}(t) + \cdots + \beta_{j-1,n} g_{j-1,n}(t) + \alpha_{j,n} g_{j,n}(t) \ ,$$

where $0 < \Theta < \frac{1}{4}$ is a fixed constant, and $\{\alpha_{j,n}\}_{j=1}^{T_n^2}$ and $\{\beta_{j,n}\}_{j=1}^{T_n^2-1}$ are chosen to ensure that $B_n = \{b_{j,n}\}_{j=1}^{T_n^2} \subseteq L^2(\mathbb{R})$ is an orthonormal set. We may assume that

$$0 < \alpha_{j,n} \leq 1 \ . \tag{6.3}$$

Moreover, as in the proof of Bourgain's theorem in [5], one can show that

$$|\beta_{j,n}| \leq \frac{\Theta}{T_n^2} \ . \tag{6.4}$$

For further information, see Equation (3.8) in [5].

III. Let us now prove estimates for $\mu(b_{j,n})$. Recall that $\mu(g_{j,n}) = 0$ for all j, n. Using that $\widehat{\Phi_n}$ and the $\widehat{g_{j,n}}$ all have disjoint supports we have

$$\begin{aligned}
\mu(b_{j,n}) &= \int t|b_{j,n}(t)|^2\, dt = \frac{1}{2\pi i}\big\langle \big(\widehat{b_{j,n}}\big)', \widehat{b_{j,n}}\big\rangle \\
&= \frac{1}{2\pi i}\frac{\Theta^2}{T_n^2}\big\langle \big(\widehat{\Phi_n}\big)', \widehat{\Phi_n}\big\rangle + \frac{1}{2\pi i}\sum_{k=1}^{j-1}|\beta_{k,n}|^2\big\langle \big(\widehat{g_{k,n}}\big)', \widehat{g_{k,n}}\big\rangle + \frac{1}{2\pi i}|\alpha_{j,n}|^2\big\langle \big(\widehat{g_{j,n}}\big)', \widehat{g_{j,n}}\big\rangle \\
&= \frac{\Theta^2}{T_n^2}\int t|\Phi_n(t)|^2\, dt + \sum_{k=1}^{j-1}|\beta_{k,n}|^2\mu(g_{k,n}) + |\alpha_{j,n}|^2\mu(g_{j,n}) \\
&= \frac{\Theta^2}{T_n^2}\int t|\Phi_n(t)|^2\, dt \ .
\end{aligned}$$

Thus, by (6.2)

$$|\mu(b_{j,n})| \leq \frac{\Theta^2}{T_n^2}\left|\int t|\Phi_n(t)|^2\, dt\right| \leq \Theta^2 \epsilon < \epsilon \ .$$

IV. Next we estimate $|\mu(\widehat{b_{j,n}})|$. Recall that $\mu(\widehat{g_{j,n}}) = 0$ for all j, n. Using the support properties of the various functions, we have

$$\begin{aligned}
\mu\big(\widehat{b_{j,n}}\big) &= \int \gamma\big|\widehat{b_{j,n}}(\gamma)\big|^2\, d\gamma \\
&= \frac{\Theta^2}{T_n^2}\int \gamma\big|\widehat{\Phi_n}(\gamma)\big|^2\, d\gamma + \sum_{k=1}^{j-1}|\beta_{k,n}|^2\mu\big(\widehat{g_{k,n}}\big) + |\alpha_{j,n}|^2\mu\big(\widehat{g_{j,n}}\big) \\
&= \frac{\Theta^2}{T_n^2}\int \gamma\big|\widehat{\Phi_n}(\gamma)\big|^2\, d\gamma \ .
\end{aligned}$$

Thus, by (6.2)

$$\left|\mu\big(\widehat{b_{j,n}}\big)\right| \leq \frac{\Theta^2}{T_n^2}\left|\int \gamma \left|\widehat{\Phi_n}(\gamma)\right|^2 d\gamma\right| \leq \Theta^2\,\epsilon < \epsilon\,.$$

V. Now we estimate $\Delta^2(b_{j,n})$. Once again, using the disjointness of supports and proceeding as in part *III,* we have

$$\begin{aligned}
\Delta^2(b_{j,n}) &\leq \int |t|^2|b_{j,n}(t)|^2\,dt \\
&= \int |t|^2\left|\frac{\Theta}{T_n}\Phi_n(t)\right|^2 dt + \sum_{k=1}^{j-1}\int |t|^2|\beta_{k,n}g_{k,n}(t)|^2\,dt \\
&\quad + \int |t|^2|\alpha_{j,n}g_{j,n}(t)|^2\,dt \\
&= \frac{\Theta^2}{T_n^2}\int |t|^2|\Phi_n(t)|^2\,dt + \sum_{k=1}^{j-1}|\beta_{k,n}|^2\Delta^2(g_{k,n}) + |\alpha_{j,n}|^2\Delta^2(g_{j,n}) \\
&\leq \frac{\Theta^2}{T_n^2}\int |t|^2|\Phi_n(t)|^2\,dt + \sum_{k=1}^{j-1}\frac{\Theta^2}{T_n^4}2\delta^2 + |\alpha_{j,n}|^2 2\delta^2 \\
&\leq \frac{\Theta^2}{T_n^2}\int |t|^2|\Phi_n(t)|^2\,dt + T_n^2\left(\frac{2\delta^2\Theta^2}{T_n^4}\right) + (1)2\delta^2 \\
&\leq \frac{\Theta^2}{T_n^2}\int |t|^2|\Phi_n(t)|^2\,dt + 2\Theta^2\delta^2 + 2\delta^2 \\
&\leq \Theta + 2\Theta^2\delta^2 + 2\delta^2 < 1 + 4\delta^2\,.
\end{aligned}$$

The first step above holds since $\Delta^2(h) = \inf_{a\in\mathbb{R}}\int |t-a|^2|h(t)|^2\,dt$. The second step above follows by the same type of calculations as in part *III.* We used (6.1) in the final inequality.

Note that when $\Theta > 0$ is chosen sufficiently small, the upper bound on $\Delta^2(b_{j,n})$ can be made arbitrarily close to $2\delta^2 = 2\Delta^2(g)$, where $g \in \mathcal{S}(\mathbb{R})$ is the function used to define the system $\mathcal{G}(T, N)$.

VI. It only remains to show that $B = \bigcup_{j=1}^{\infty} B_j$ is complete. The verification is identical to that in [5], but we repeat the details here for the sake of completeness.

$$\begin{aligned}
||P_{[B_1,\cdots,B_k]}\varphi_k||^2_{L^2(\mathbb{R})} &= ||P_{[B_1,\cdots,B_{k-1}]}\varphi_k||^2_{L^2(\mathbb{R})} + ||P_{[B_k]}\varphi_k||^2_{L^2(\mathbb{R})} \\
&= ||P_{[B_1,\cdots,B_{k-1}]}\varphi_k||^2_{L^2(\mathbb{R})} + ||P_{[B_k]}(\Phi_k + P_{[B_1,\cdots,B_{k-1}]}\varphi_k)||^2_{L^2(\mathbb{R})} \\
&= 1 - ||\Phi_k||^2_{L^2(\mathbb{R})} + ||P_{[B_k]}\Phi_k||^2_{L^2(\mathbb{R})} \\
&= 1 - ||\Phi_k||^2_{L^2(\mathbb{R})} + \sum_{j=1}^{(T_k)^2}|\langle\Phi_k, b_{j,k}\rangle|^2 \\
&= 1 - ||\Phi_k||^2_{L^2(\mathbb{R})} + (T_k)^2\left(\frac{\Theta}{T_k}||\Phi_k||^2_{L^2(\mathbb{R})}\right)^2 \\
&= 1 - ||\Phi_k||^2_{L^2(\mathbb{R})} + \Theta^2||\Phi_k||^4_{L^2(\mathbb{R})} \\
&\geq \Theta^2\,.
\end{aligned}$$

To see the final inequality, let $h(t) = 1 - t^2 + a^2 t^4$ be defined on [0, 1], where $0 < a < \frac{1}{4}$ is fixed. It is easy to see that $h(t) \geq a^2$. Since $||\Phi_k||_{L^2(\mathbb{R})} \leq 1$ and $\Theta < \frac{1}{4}$, the last step follows.

Now, suppose $y \in L^2(\mathbb{R})$ satisfies $\langle y, b \rangle = 0$ for all $b \in B$. If y is not identically zero, then $\tilde{y} = y/||y||_{L^2(\mathbb{R})}$ is in the unit sphere of $L^2(\mathbb{R})$ and there exists φ_{n_k} such that $\varphi_{n_k} \to \tilde{y}$ in $L^2(\mathbb{R})$ as $k \to \infty$. Thus,

$$0 < \Theta \leq ||P_{[B_1, \cdots, B_{n_k}]}\varphi_{n_k}||_{L^2(\mathbb{R})} \leq ||P_{[B]}\varphi_{n_k}||_{L^2(\mathbb{R})} \to \left\| P_{[B]}\tilde{y} \right\|_{L^2(\mathbb{R})} = 0 ,$$

where the limit is as $k \to \infty$. This contradiction shows that the orthonormal set $B \subseteq L^2(\mathbb{R})$ is complete and hence is an orthonormal basis for $L^2(\mathbb{R})$. □

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References

[1] Benedetto, J. J. (1994). Frame decompositions, sampling, and uncertainty principle inequalities, in *Wavelets: Mathematics and Applications, Stud. Adv. Math.* CRC, Boca Raton, FL.

[2] Benedetto, J. J. and Powell, A. M. (2004). A (p, q) version of Bourgain's theorem, to appear in *Trans. AMS.*

[3] Benedetto, J. J. and Walnut, D. F. (1994). Gabor frames for L^2 and related spaces, in *Wavelets: Mathematics and Applications, Stud. Adv. Math.* CRC, Boca Raton, FL.

[4] Bernardini, R. and Kovačević, J. (1999). Arbitrary tilings of the time-frequency plane using local bases, *IEEE Trans. Signal Proc.* **47**(8), 2293–2304.

[5] Bourgain, J. (1988). A remark on the uncertainty principle for Hilbertian basis, *J. Funct. Anal.* **79**(1), 136–143.

[6] Czaja, W. and Powell, A. M. (2005). Recent developments in the Balian-Low theorem, to appear in *Harmonic Analysis and Applications,* Heil, C., Ed., Birkhäuser, Boston, MA.

[7] Daubechies, I. (1992). *Ten Lectures on Wavelets,* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA.

[8] Daubechies, I, Jaffard, S, and Journé, J.-L. (1991). A simple Wilson orthonormal basis with exponential decay, *SIAM J. Math. Anal.* **22**(2), 554–573.

[9] Folland, G. B. and Sitaram, A. (1997). The uncertainty principle: A mathematical survey, *J. Fourier Anal. Appl.* **3**(3), 207–238.

[10] Gröchenig, K. (2001). *Foundations of Time-Frequency Analysis,* Applied and Numerical Harmonic Analysis, Birkäuser, Boston, MA.

[11] Hernández, E. and Weiss, G. (1996). *A First Course on Wavelets,* CRC Press, Boca Raton, FL.

[12] Landau, H. J. and Pollak, H. O. (1961). Prolate spheroidal wave functions, Fourier analysis and uncertainty II, *Bell System Tech. J.* **40**, 65–84.

[13] Landau, H. J. and Pollak, H. O. (1962). Prolate spheroidal wave functions, Fourier analysis and uncertainty III. The dimension of the space of essentially time- and band-limited signals, *Bell System Tech. J.* **41**, 1295–1336.

[14] Mallat, S. (1998). *A Wavelet Tour of Signal Processing,* Academic Press, San Diego, CA.

[15] Shapiro, H. S. (1991). Uncertainty principles for bases in $L^2(\mathbb{R})$, unpublished manuscript.

[16] Slepian, D. and Pollak, H. O. (1961). Prolate spheroidal wave functions, Fourier analysis and uncertainty I, *Bell System Tech. J.* **40**, 43–63.

[17] Thiele, C. and Villemoes, L. (1996). A fast algorithm for adapted time-frequency tilings, *Appl. Comput. Harmon. Anal.* **3**(2), 91–99.

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