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L^p-Integrability, Supports of Fourier Transforms and Uniqueness for Convolution Equations

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ABSTRACT. *It is proved that a nonzero function is not in $L^p(R^n)$ with $p \leq 2n/d$ if its Fourier transform is supported by a $d-$ dimensional submanifold. It is shown that the assertion fails for $p > 2n/d$ and $d \geq n/2$. The result is applied for obtaining uniqueness theorems for convolution equations in L^p-spaces.*

1. Introduction

We will start with a motivation of the problems we are going to study. Let us consider the spherical means $f * \mu_r$ of a continuous function f on $\mathbb{R}^n$ which are defined as

$$f * \mu_r(x) = \int_{|x|=r} f(x-y)\, d\mu_r(y), \qquad r > 0\,,$$

where μ_r is the normalized surface measure on the sphere $\{x : |x| = r\}$. Note that the above expression is the average of f over a sphere of radius r centred at x.

One may ask whether f is uniquely determined by the above averages. In other words we are asking if the operator taking f into $f * \mu_r$ is injective. In general this is not true. A counterexample is provided by the Bessel functions. If

$$\phi_\lambda(x) = c_n \frac{J_{\frac{n}{2}-1}(\lambda|x|)}{(\lambda|x|)^{\frac{n}{2}-1}}\,,$$

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then it is well known that

$$\phi_\lambda * \mu_r(x) = \phi_\lambda(r)\phi_\lambda(x) . \tag{1.1}$$

So it suffices to choose $r > 0$ such that $\phi_\lambda(r) = 0$. Here c_n is a constant such that $\phi_\lambda(0) = 1$. Note that, from the asymptotics of the Bessel functions, it is easy to see that $\phi_\lambda \in L^p$ iff $p > 2n/(n-1)$. Hence injectivity fails for $L^p,\ p > 2n/(n-1)$.

Nevertheless, a *two radii* theorem holds. Namely, Zalcman proved [11] that two spherical averages $f * \mu_{r_1},\ f * \mu_{r_2}$ uniquely determine a locally integrable function f if and only if r_1/r_2 is not a ratio of the zeros of a Bessel function.

However, if f decays at infinity faster than the Bessel function, more precisely, if $f \in L^p(\mathbb{R}^n)$ for $1 \leq p \leq 2n/(n-1)$, then a *one radius* theorem is true, i.e., for such functions f the condition $f * \mu_r = 0$ for some $r > 0$ implies $f = 0$. This result was obtained by Thangavelu [10].

In this article we look at the equations of the general form $f * T = 0$, where f belongs to some L^p class and T is a compactly supported distribution.

We obtain sharp estimates for the index of summability p, providing uniqueness of the solution in the space $L^p(\mathbb{R}^n)$. This critical index appears as $p \leq 2n/d$, where d is the dimension of the zero set of the Fourier transform $\hat{T}$.

Thus, we show that the phenomenon of uniqueness is related only to the dimension d of zero set of the Fourier transform of the kernel T. In the particular case of spherical averages we have $d = n - 1$ and the result of [10] follows.

This article is organized as follows. In Section 2 we relate the membership of a nonzero function in the space $L^p(\mathbb{R}^n)$ with the dimension d of the support of Fourier transform of f and obtain corresponding estimates for p. We prove sharpness of the estimates under the restriction $2d \geq n$. In Section 3 we apply the obtained results to convolution equations and prove uniqueness theorems. In Section 4 we give an application of our results to characterization of stationary sets of evolution equations, namely we generalize the result of [2] on closed stationary hypersurfaces, from the wave equation to a wider class of equations.

2. L^p-Integrability and Dimension of Supports of Fourier Transforms

In this section we relate the dimension of the support of the Fourier transform of a function with its membership in L^p.

Theorem 1.

If $f \in L^p(\mathbb{R}^n)$ and supp $\hat{f}$ is carried by a C^1 manifold M of dimension $d < n$ then $f = 0$ provided $1 \leq p \leq 2n/d$ and $d > 0$. If $d = 0$ then $f = 0$ for $1 \leq p < \infty$.

Proof. Denote $k = n - d$ the codimension of the manifold M. For $k = n$ the Fourier transform of f is supported on a discrete set, which is not possible for $f \in L^p,\ p < \infty$.

For $k < n$ we will show that f vanishes identically by refining the proof of Theorem 7.1.27 of [6].

First, note that by convolving f with a compactly supported smooth function we may assume that f belongs to L^{p_0} where $p_0 = 2n/(n-k)$.

Now choose an even function $\chi \in C_c^\infty(\mathbb{R}^n)$ with support in the unit ball and $\int_{\mathbb{R}^n} \chi(x)\, dx = 1$. Let $\chi_\varepsilon(x) = \varepsilon^{-n}\chi(x/\varepsilon)$ and $u_\varepsilon = u * \chi_\varepsilon$ where $u = \hat{f}$. Then by the Plancherel theorem

$$
\begin{aligned}
\|u_\varepsilon\|_2^2 &= \int_{\mathbb{R}^n} |f(x)|^2 \left|\hat{\chi}(\varepsilon x)\right|^2 \, dx \\
&\leq C\,\varepsilon^{-k} \sum_{j=-\infty}^{\infty} 2^{jk} \sup_{2^j \leq |x| \leq 2^{j+1}} \left|\hat{\chi}(y)\right|^2 \left(2^{-j}\varepsilon\right)^k \int_{2^j \leq |\varepsilon x| \leq 2^{j+1}} |f(x)|^2 \, dx \\
&= C\,\varepsilon^{-k} \sum_{j=-\infty}^{\infty} a_j \, b_j^\varepsilon
\end{aligned}
$$

where

$$
a_j = 2^{jk} \sup_{2^j \leq |x| \leq 2^{j+1}} \left|\hat{\chi}(y)\right|^2
$$

and

$$
b_j^\varepsilon = \left(2^{-j}\varepsilon\right)^k \int_{2^j\varepsilon^{-1} \leq |x| \leq 2^{j+1}\varepsilon^{-1}} |f(x)|^2 \, dx \, .
$$

Applying Holder's inequality,

$$
\left|b_j^\varepsilon\right| \leq C \left(2^{-j}\varepsilon\right)^k \left(2^{-j}\varepsilon\right)^{-n(p_0-2)/p_0} \left(\int_{2^j\varepsilon^{-1} \leq |x| \leq 2^{j+1}\varepsilon^{-1}} |f(x)|^{p_0} \, dx\right)^{2/p_0}
$$

which goes to zero as $\varepsilon \to 0$, for any fixed j, as $k - n(p_0 - 2)/p_0 = 0$. Also note that we have $|b_j^\varepsilon| \leq C \, \|f\|_{p_0}^2 < \infty$ for some constant C independent of ε and j. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be arbitrary and denote by M_ε the set of points at a distance $< \varepsilon$ from the intersection of supp u_ε with supp ψ. Then, as supp u is carried by a C^1 manifold of codimension k it can be easily proved by a change of variables that

$$
\varepsilon^{-k} \int_{M_\varepsilon} |\psi|^2 \to C_k \int_{\text{supp } u} |\psi|^2, \; \varepsilon \to 0 \, ,
$$

where C_k is the volume of the unit ball in $\mathbb{R}^k$. Hence, by the Cauchy–Schwarz inequality

$$
| < u, \psi > |^2 = \lim_{\varepsilon \to 0} | < u_\varepsilon, \psi > |^2 \leq C \int_{\text{supp } u} |\psi|^2 dS \, \lim_{\varepsilon \to 0} K(\varepsilon)
$$

where $K(\varepsilon) = \sum_{j=-\infty}^{\infty} a_j b_j^\varepsilon$. Since $\sum_{j=-\infty}^{\infty} |a_j|$ is finite, by the dominated convergence theorem, we have $\lim_{\epsilon \to 0} K(\varepsilon) = 0$. Hence $< u, \psi >= 0$ for any $\psi \in C_c^\infty(\mathbb{R}^n)$ which implies that $f = 0$. □

Remark. Related type results can be found in [1]. If u is a temperate solution of $P(D)u = 0$, where $P(D)$ is a differential operator with constant coefficients, then the Fourier transform $\hat{u}$ is a density on the manifold of zeros of P. In [1] the authors prove that the L^2 norm of this density can be exactly determined.

Now we will show that the estimate for the summability index p given by Theorem 1 is sharp when the dimension d of the support of the Fourier transform satisfies $n/2 \leq d \leq n$:

Theorem 2.

For any n and d such that $d \geq n/2$, there exists a smooth manifold $M \subset \mathbb{R}^n$, dim $M = d$, and a measure μ supported on M such that the Fourier transform $f = \hat{\mu} \in L^p(\mathbb{R}^n)$ for all $p > 2n/d$.

We will need the following elementary fact:

Lemma 1.
Let $l > m$. Then one can choose l vectors $\alpha_1, \alpha_2, \cdots\cdots \alpha_l$ in $\mathbb{R}^m$ such that any m of them form a linearly independent set in $\mathbb{R}^m$.

Proof. If $l = m + 1$, we choose $\{\alpha_1, \alpha_2, \cdots\cdots \alpha_m\}$ to be a basis for $\mathbb{R}^m$ and define $\alpha_{m+1} = \sum_{i=1}^{m} \alpha_i$.

To prove the general case we use induction. Assume that we have a collection $B = \{\alpha_1, \alpha_2, \cdots\cdots \alpha_l\}$ such that any m of them form a linearly independent set in $\mathbb{R}^m$. If F is a subcollection of precisely $m-1$ members of B, let V_F denote the $m-1$ dimensional subspace spanned by the members of F. Choose a vector α_{l+1} which is not in the union of the subspaces V_F where F runs over all the subsets of B with cardinality $m-1$. It is easy to see that the set $\{\alpha_1, \alpha_2, \cdots\cdots \alpha_l, \alpha_{l+1}\}$ has the required property. This finishes the proof of lemma. □

Proof of Theorem 2. When $n = d$ the assertion is trivial. Hence assume that $n/2 \leq d < n$.

Consider the measure μ defined by,

$$\mu(\varphi) = \int_{[-1,1]^d} \varphi\left(t_1, t_2, \cdots, t_d, \sum_{j=1}^{d} \lambda_j^1 t_j^2, \sum_{j=1}^{d} \lambda_j^2 t_j^2, \cdots \sum_{j=1}^{d} \lambda_j^{n-d} t_j^2\right) dt_1 \cdots dt_d$$

for $\varphi \in C_c^\infty(\mathbb{R}^n)$, where λ_j^s are certain constants to be chosen later. Clearly μ is supported on a smooth d-dimensional manifold, which is the graph of the quadratic mapping

$$(t_1, t_2, \cdots t_d) \to \left(\sum_{j=1}^{d} \lambda_j^1 t_j^2, \sum_{j=1}^{d} \lambda_j^2 t_j^2, \cdots \sum_{j=1}^{d} \lambda_j^{n-d} t_j^2\right).$$

We shall show that the Fourier transform of μ belongs to L^p for all $p > 2n/d$. Denoting $\hat{\mu}$ by f we have

$$f(x) = \Pi_{j=1}^{d} \int_{[-1,1]} e^{i(x_j t_j + L_j(x') t_j^2)}\, dt_j\,, \tag{2.1}$$

where $x' = (x_{d+1} \cdots\cdots x_n)$ and

$$L_k\left(x'\right) = \sum_{i=1}^{n-d} \lambda_k^i\, x_{d+i}, \quad k = 1, 2, \cdots d\,.$$

Note that the L_j are linear forms in the last $n-d$ variables and are determined by $\alpha_j = (\lambda_j^1, \cdots\cdots \lambda_j^{n-d})$. We choose α_j according to the conclusion of Lemma 1. Now it is well known that (see [7] and [9])

$$\left|\int_{-1}^{1} e^{i(y_1 t + y_2 t^2)}\, dt\right| \leq C\left(1 + y_1^2 + y_2^2\right)^{-\frac{1}{4}} \tag{2.2}$$

with a C independent of y_1 and y_2. From (2.1) and (2.2) we have

$$|f(x)| \leq C\, \Pi_{j=1}^{d} \left(1 + x_j^2 + \left|L_j\left(x'\right)\right|^2\right)^{-\frac{1}{4}}. \tag{2.3}$$

We shall show that the above product belongs to L^p for any $p > 2n/d$.

Let us start with a simple lemma.

Lemma 2.

Let $g(a) = \int_0^\infty (1 + u^2 + a^2)^{-\frac{p}{4}}\, du \quad a \geq 0$. Then

$$|g(a)| \leq C(1 + |a|)^{-(\frac{p}{2}-1)}$$

with C independent of a, provided $p > 2$.

Proof. The proof follows from the inequalities

$$|g(a)| \leq \int_0^\infty \left(1 + u^2\right)^{-\frac{p}{4}}\, du \leq C < \infty$$

and

$$\begin{aligned} |g(a)| &\leq \int_0^\infty \left(u^2 + a^2\right)^{-\frac{p}{4}}\, du \\ &\leq C\, a^{-(\frac{p}{2}-1)} \int_0^\infty \left(1 + s^2\right)^{-\frac{p}{4}}\, ds \\ &\leq C\, a^{-(\frac{p}{2}-1)}\,. \end{aligned}$$

□

Applying Lemma 2 to the right hand side of (2.3) and taking into account that $p > 2n/d\; > 2$ we have:

$$\int_{\mathbb{R}^d} |f(x)|^p\, dx \leq C \int_{\mathbb{R}^{n-d}} \Pi_{j=1}^d \left(1 + \left|L_j\left(x'\right)\right|\right)^{-(\frac{p}{2}-1)}\, dx'\,. \tag{2.4}$$

Define

$$G(y) = G(y_1, y_2, \cdots y_{n-d}) = \Pi_{j=1}^d (1 + |L_j(y)|)^{-(\frac{p}{2}-1)}\,.$$

We need to show that G is integrable on $\mathbb{R}^{n-d}$. Since G is a bounded function it is enough to consider the integral over $\{|y| \geq 1\}$. Now

$$\begin{aligned} \int_{|y|\geq 1} |G(y)|\, dy &\leq \int_{|y|\geq 1} \Pi_{j=1}^d |L_j(y)|^{-(\frac{p}{2}-1)}\, dy \\ &\leq C \int_1^\infty r^{-d(\frac{p}{2}-1)+n-d-1}\, dr \left(\int_{S^{n-d-1}} \Pi_{j=1}^d |L_j(\theta)|^{-(\frac{p}{2}-1)}\, d\sigma(\theta)\right). \end{aligned}$$

The integral with respect to r is finite provided $p > 2n/d$. We shall show that the integral over S^{n-d-1} too is finite which will complete the proof.

To this end note first that, not more than $n - d - 1$ linear forms can vanish simultaneously on the unit sphere. This is because, if $L_j(x) = 0$ for some x and $n - d$ many forms then $x = 0$ due to the linear independence of α_j's.

Let θ_0 be a point on the unit sphere where the function $\Pi_{j=1}^d |L_j(\theta)|^{-(\frac{p}{2}-1)}$ vanishes. For the sake of simplicity let us assume that $L_1(\theta_0) = L_2(\theta_0) = \cdots\cdots = L_r(\theta_0) = 0$ for some $r < n - d$.

Let $N(\theta_0)$ be a small enough neighborhood of θ_0 on the unit sphere where the remaining linear forms are bounded away from zero. We first show that the above product is integrable over this neighborhood. We have,

$$\int_{N(\theta_0)} \Pi_{j=1}^{d} |L_j(\theta)|^{-(\frac{p}{2}-1)}\, d\sigma(\theta) \leq C \int_{N(\theta_0)} \Pi_{j=1}^{r} |L_j(\theta)|^{-(\frac{p}{2}-1)}\, d\sigma(\theta) .$$

Since the L_j are linearly independent, shrinking $N(\theta_0)$ if necessary, we may transfer the above integral to a small neighborhood of the origin in R^{n-d-1} via a diffeomorphism in such a way that $L_j(\theta)$ becomes x_j for $j = 1, 2, \cdots r$.

Since the Jacobian of the diffeomorphism is bounded above and below it suffices to consider the integral $\int_{(-\varepsilon,\varepsilon)^r} \Pi_{j=1}^{r} |x_j|^{-(\frac{p}{2}-1)}\, dx_j$ which is finite provided $p < 4$.

Recall that $p > 2n/d$ and $n/2 < d < n$. Hence for $p \in (2n/d, 4)$, we have proved that the product $\Pi_{j=1}^{d} |L_j(\theta)|^{-(\frac{p}{2}-1)}$ is integrable over the neighborhood $N(\theta_0)$. Arguing similarly at the other points on the sphere where the function $\Pi_{j=1}^{d} |L_j(\theta)|^{-(\frac{p}{2}-1)}$ vanishes and using the compactness of the sphere we conclude that this product is integrable over the whole sphere.

Hence the function f defined by (2.1) belongs to $L^p(\mathbb{R}^n)$ for $p \in (\frac{2n}{d}, 4)$. Since f is clearly bounded we have that $f \in L^p$ for all $p > 2n/d$ which finishes the proof of Theorem 2. □

Remark. For large codimension $k = n - d$ the estimate for p in Theorem 1 is not sharp. For example, let $\gamma(t)$ be a polynomial curve (codimension $n - 1$). Then it is known that the Fourier transform of a measure supported on γ can belong to L^p only for p very large, in fact $p > O(n^2)$ (see [7] and [9]), whereas Theorem 1 suggests the range $p > 2n$.

3. Convolution Equations in $L^p(\mathbb{R}^n)$

Theorem 1 implies the following uniqueness theorem for convolution equations:

Theorem 3.

Let T be a compactly supported distribution on $\mathbb{R}^n$ and assume that the zero set of $\hat{T}$ in $\mathbb{R}^n$ is carried by a C^1 manifold of codimension k. If $f \in L^p$ satisfies, $f * T = 0$ then f vanishes identically provided $1 \leq p \leq 2n/(n-k)$ and $k < n$. If $k = n$ then $f = 0$ for $1 \leq p < \infty$.

Proof. Since T is a compactly supported distribution, $\hat{T}$ is a smooth function of tempered growth on $\mathbb{R}^n$. Hence by taking Fourier transform on $f * T = 0$ we can easily conclude that supp $\hat{f}$ is contained in the zero set of $\hat{T}$. Now the proof follows from Theorem 1. □

Theorem 4.

Let T be a compactly supported distribution on $\mathbb{R}^n$ and $f \in L^p$. If $f * T = 0$ then f vanishes identically provided $1 \leq p \leq 2n/(n-1)$.

Proof. As above, the equation $f * T = 0$ implies that the support of the Fourier transform of f is contained in the zero set of the function $\hat{T}$. The function $\hat{T}$ extends to the space $\mathbb{C}^n$ as an entire function whose zero set $Z(\hat{T})$ is an analytic set in $\mathbb{C}^n$.

This set admits a stratification (see [3], Chapter 1, p. 60]

$$Z = \cup M_j$$

into complex manifolds M_j. The stratification is locally finite, the strata M_j are pairwise disjoint and $M_i \subset \overline{M}_j$, $\dim M_i < \dim M_j$ if $i < j$ and $M_i \cap \overline{M}_j \neq \emptyset$.

For the zero set $N = Z \cap \mathbb{R}^n$ of the function $\hat{T}$ we have the corresponding stratification $N = \cup N_j$, $N_j = M_j \cap \mathbb{R}^n$ into real submanifolds of $\mathbb{R}^n$.

Assume that $f \neq 0$ so that its support is nonempty. Let N_p be the stratum of the maximal dimension. The set $V = N_p \setminus \cup_{j<p} \overline{N}_j$ is open in N_p and therefore $V = U \cap N_p$ where U is an open subset of $\mathbb{R}^n$ which can be chosen disjoint with all the sets $\overline{N}_j$, $j < p$.

Choose a function η in the Schwartz class $S(\mathbb{R}^n)$ such that supp $\hat{\eta} \subset U$ and define

$$g = f * \eta .$$

Then $\hat{g} = \hat{\eta} \cdot \hat{f}$ and therefore

$$\text{supp } \hat{g} \subset U \cap \text{supp } \hat{f} \subset U \cap N \subset N_p .$$

By condition, $f \in L^p(\mathbb{R}^n)$ for some $p \leq 2n/(n-1)$ and so g is of the same class. Since $2n/(n-1) \leq 2n/\dim N_p$ and $\hat{g}$ is supported on the manifold N_p, Theorem 1 implies $g = 0$. Due to the arbitrariness of choice of η we conclude that supp $\hat{f} \cap N_p = \emptyset$.

Now we can proceed with the next strata of less dimension and consequently sweep out the support of the function $\hat{f}$ from all the strata $N_j's$. Therefore $f = 0$.

Remark 1. The injectivity result of Thangavelu mentioned in the introduction easily follows from Theorem 4.

Remark 2. 1) Since $\hat{T}$ is an entire function on $\mathbb{C}^n$, its zero set $Z(\hat{T})$ is of dimension $n-1$ as an analytic set in $\mathbb{C}^n$. But $Z(\hat{T}) \cap \mathbb{R}^n$ can be of any dimension. For example one may consider the distribution T given by

$$T(\varphi) = \sum_{j=1}^{n-k} \frac{\partial^2 \varphi}{\partial x_j^2}(0) .$$

Then $\hat{T}$ is given by the polynomial $\sum_{j=1}^{n-k} x_j^2$ whose zero set has codimension k.
2) Theorem 3 may also be rephrased as a uniqueness result for a system of convolution equations. More precisely if T_j, $j = 1, 2, \cdots l$ are compactly supported distributions on $\mathbb{R}^n$ and the intersection of the zero sets of $\hat{T}_j$ is carried by a C^1 manifold of codimension k, then any solution in $L^p(\mathbb{R}^n)$ of the system $f * T_j = 0$, $j = 1, 2, \cdots l$ vanishes identically, provided $p \leq 2n/(n-k)$.
3) Choosing the above distributions to be supported at origin one may interpret the above as a uniqueness result for a system of differential equations. We leave the details to the reader.

Remark 3. In the particular case when the codimension k of the support of the Fourier transform divides n, the sharpness of the estimate for the index of summability p can be shown much easier than in Theorem 2, just by considering the surface measure on a torus and using the asymptotics of the Bessel functions.

Namely, for $k = 1$ we may use the example in (1.1). Now let $n = kl$ for some positive integer l. Write $\mathbb{R}^n = \mathbb{R}^l \times \cdots\cdots \times \mathbb{R}^l$ (k times). Let ν be the normalized surface measure on $M = S_r^{l-1} \times \cdots\cdots \times S_r^{l-1}$, where S_r^{l-1} is the sphere of radius r in $\mathbb{R}^l$ centred at the origin. Note that M has codimension k. For $x \in \mathbb{R}^n$ write $x = (x_1, x_2, \ldots\ldots, x_k)$ where each x_j is in the j'th l dimensional space. If

$$f(x) = \frac{J_{\frac{l}{2}-1}(r|x_1|)}{(r|x_1|)^{\frac{l}{2}-1}} \cdots\cdots \frac{J_{\frac{l}{2}-1}(r|x_k|)}{(r|x_k|)^{\frac{l}{2}-1}}$$

then $f \in L^p$ for every $p > 2l/(l-1) = 2n/(n-k)$. Here C is a constant such that $f(0) = 1$. Note that the Fourier transform of f is supported on M.

Now let $r > 0$ be a zero of $J_{\frac{l}{2}-1}(t)$. Then we have

$$f * \nu(x) = C \left(\frac{J_{\frac{l}{2}-1}(r)}{r^{\frac{l}{2}-1}} \right)^k f(x) = 0 .$$

Hence injectivity too fails for $p > 2n/(n-k)$ which shows that Theorem 3 can not be improved. □

As another corollary to Theorem 1 we have the following Wiener–Tauberian type theorem.

Corollary 1.

Let h be a compactly supported continuous function on $\mathbb{R}^n$. Then the linear span of translates of h forms a dense subset of $L^p(\mathbb{R}^n)$ as long as $2n/(n+k) \leq p < \infty$ where k is the minimal codimension of the zero set of $\hat{h}$ in $\mathbb{R}^n$. In particular linear span of translates of h span a dense susbet of L^p for $2n/(n+1) \leq p < \infty$.

Proof. Suppose that a function f in the dual space annihilates all the translates of h. We need to show that $f = 0$. Since f is orthogonal to all the translates of h we have $f * h = 0$. Taking Fourier transform we have supp $\hat{f} \subset \{x : \hat{h}(x) = 0\}$. Now proceeding as in Theorem 4 we finish the proof. □

Remark. We remark that the above corollary answers a question posed by C.S. Herz in [5] (see p. 727).

4. An Application: Stationary Sets of Evolution Equations

In [2] the authors studied injectivity sets for the spherical means on $\mathbb{R}^n$. It was proved that the boundary Γ of any bounded domain $\Omega \subset \mathbb{R}^n$ is a set of injectivity for the spherical means operator in $L^p(\mathbb{R}^n)$ as long as $p \leq 2n/(n-1)$. In other words if $f * \mu_r(x) = 0$ for all $x \in \Gamma$ and for all $r > 0$ then $f = 0$ provided $p \leq 2n/(n-1)$.

This result is equivalent to the nonexistence of closed stationary sets for the wave equation when the initial velocity vanishes at infinity too fast, more precisely, belongs to L^p with p as above (Theorem 3 in [2]). The estimate for the index of summability p came from the asymptotic of Bessel functions which are eigenfunctions of the Laplace operator.

Theorem 1 enables us to obtain similar result for evolution equation for more general differential operators than Laplacian.

Let $P(D)$ be a second order elliptic partial differential operator with constant coefficients which has a non negative self adjoint extension to $L^2(\mathbb{R}^n)$. Assume that the level sets $P(x) = \lambda$, $\lambda \in R$ are smooth manifolds and the minimal codimension of the level sets is k.

Consider the associated wave equation

$$u_{tt} + P(D)u = 0,\ u(x,0) = 0,\ u_t(x,0) = f(x)\ , \tag{4.1}$$

with the initial velocity $f \in L^p(\mathbb{R}^n)$ for some p. We may extend the solution uniquely to the whole time axis by assuming that $u(x, -t) = -u(x, t)$ for all $t \in \mathbb{R}$.

Corollary 2.
Let Γ be the boundary of any bounded domain in $\mathbb{R}^n$, $n \geq 2$. Suppose that the solution $u(x, t)$ of the Cauchy problem (4.1) with the initial data $f \in L^p(\mathbb{R}^n)$ satisfies the condition

$$u(x, t) = 0 \text{ for all } x \in \Gamma \text{ at any time } t > 0 .$$

Then $u = 0$ as long as $p \leq 2n/(n-k)$ for $k < n$ and $1 \leq p < \infty$ for $k = n$ where k is the minimal codimension of the the level sets of P.

The result of [2] corresponds to the case when the differential operator $P(D)$ is the Laplace operator, the level sets $P(D) = \lambda$ are spheres in $\mathbb{R}^n$ and $k = 1$.

Proof. Denote by Ω the domain bounded by Γ. Since the operator $P(D)$ with Dirichlet boundary condition is self-adjoint, there exists an orthonormal basis $\{\psi_l\}_{l=0}^{\infty}$ in $L^2(\Omega)$ consisting of Dirichlet eigenfunctions of $P(D)$, $P(D)\psi_l = \lambda_l \psi_l$, $\lambda_l > 0$.

Now we can *verbatim* follow the arguments in [2]. Namely, using convolution in $t-$variable one reduces the problem to the case of separable solution

$$u(x, t) = c_l \sin t\sqrt{\lambda_l}\ \psi_l(x) \tag{4.2}$$

with $u(., t) \in L^p(\mathbb{R}^n)$. The eigenfunctions ψ_l are built from a global solution $u(x, t)$ to (4.1) and this allows to extend ψ_l to $\mathbb{R}^n$ as a global eigenfunction of $P(D)$ with the eigenvalue λ_l.

Then the Fourier transform of ψ_l is supported on the level set $P = \lambda_l$ which has codimension greater than k by assumption. Since $\psi_l \in L^p(\mathbb{R}^n)$ and $p \leq 2n/(n-k)$ it now follows from Theorem 1 that the constant c_l appearing in (4.2) is zero, which finishes the proof. □

Remark. The above corollary can be formulated for a higher order elliptic partial differential operator with an appropriate Cauchy problem instead of (4.1).

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