The Journal of Fourier Analysis and Applications

Volume 10, Issue 4, 2004

The Multichannel Deconvolution Problem: A Discrete Analysis

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Communicated by D.F. Walnut

ABSTRACT. We study the multichannel deconvolution problem (MDP) in a discrete setting by developing the theory for converting the method used in the continuous setting in [36]. We give a method for solving the MDP when the convolvers are characteristic functions, derive the explicit form of the linear system, and obtain an upper bound on the condition number of the system in a particular case. We compare the Schiske reconstruction [28] to our solution in the discrete setting, and give an explicit formula for the corresponding error. We then give the algorithm for solving the general MDP and discuss in detail the local reconstruction aspects of the problem. Finally, we describe a method for improving the reconstruction by regularization and give some explicit estimates on error bounds in the presence of noise.

1. Introduction

In this article we study several aspects of the **multichannel deconvolution problem (MDP)** in a discrete setting. This problem, also known in engineering as **multichannel restoration**, **multichannel equalization**, or **superresolution**, belongs to a class of problems known as **single-input multi-output (SIMO) deconvolution** and **multi-input multi-output (MIMO) deconvolution**. The problem of finding solutions with finite support is known as **restoration by FIR (finite impulse response) filters**.

Deconvolution, that is, recovering f from $s = f * h$ when h is a given compactly supported function, is a well known ill-posed inverse problem. The convolution operator $\mathcal{C}_h(f) = f * h$, viewed as a functional over $C(\mathbb{R})$, fails to be injective. When the operator is viewed as a functional over $L^2(\mathbb{R})$, the inverse operator is unbounded [13]. A successful strategy to avoid these obstacles is to receive m ($m \geq 2$) signals/images when the convolvers $h_1, \ldots, h_m$ are chosen so that the information lost in one of the channels can be retrieved

Math Subject Classifications. 42A85, 65F22.
Keywords and Phrases. Bezout equation, deconvolution.

ISSN 1069-5869 DOI: 10.1007/s00041-004-0953-0

from the other channels. To the best of our knowledge, the idea of doing multichannel deconvolution in the field of micro-electroscopy was first conceived by P. Schiske in [28] and [29]. Recently, variations of Schiske's ideas have been rediscovered [4, 34, 39] and a new interest in multichannel deconvolution has re-emerged. For an overview of some recent articles on the MDP, see [30].

One application of the MDP is the simultaneous exposure of a moving object by a number of independent cameras [22]. Other applications include multiple images of astronomical objects with varying focal settings, remote sensing through a time varying atmosphere [35], and wide-band radar calibration [27].

Attempts to demonstrate the theory of multichannel deconvolution envisioned by Berenstein and coauthors have not been very successful. In this work we study the MDP in a discrete setting by converting the work done in the continuous setting in [13] and [36]. Finding and applying compactly supported or time-limited deconvolvers allows us to obtain the true underlying signal or image without having to make any additional assumptions. We derive computationally efficient formulas that can be calculated with the Fast Fourier Transform which give a perfect reconstruction in the simulations carried out in [37]. Moreover, we gain an extended boundary of the original image that other methods cannot provide. We devise methods which allow us to demonstrate this theory in a digital setting for any conceivable compactly supported convolver. Our technique to handle noise is based on combining a regularization method with a de-noising routine. The results of this approach seem very promising and further improvements are likely.

In Section 2, after giving the introductory definitions and notation, we formulate the MDP in the discrete setting.

In Section 3, we discuss the solvability of the discrete MDP for characteristic convolvers (impulses) and give a discrete nonperiodic sampling theorem (Theorem 4) that allows us to obtain formulas for the deconvolvers in the characteristic case (Theorem 5) with certain restrictions. In Theorem 6 we extend Theorem 5 to the general case.

In Section 4 (Proposition 1 and in Theorem 7), we give a computational solution to the general MDP and a particular solution to the characteristic MDP in matrix form. In Theorem 8, we obtain an upper bound on the condition number of the system for the characteristic MDP. In Theorem 10, we give the analogue of Theorem 6 for the m-channel MDP. We compare Schiske's solution [28] to ours in the two-channel case and give an explicit formula for the corresponding error.

In Section 5, we give the algorithm for solving the general MDP, discuss issues about the support of the deconvolvers, and analyze the local reconstruction aspects. We outline a method for improving the reconstruction by regularization and give some explicit estimates on error bounds in the presence of noise.

In Section 6, we provide the illustrations of Schiske's reconstruction and ours and show the extended boundary recovered using the local reconstruction.

Berenstein and coauthors [7, 10, 8], and for more references see [4] considered the following MDP: *Given a collection $\{h_i\}_{i=1}^m$ of compactly supported distributions on $\mathbb{R}^d$ (or* ***convolvers****), find a collection $\{\tilde{h}_i\}_{i=1}^m$ of compactly supported distributions (or* ***deconvolvers****) such that*

$$\sum_{i=1}^{m} h_i * \tilde{h}_i = \delta \,, \tag{1.1}$$

where δ is a Dirac delta distribution.

We are interested in finding compactly supported deconvolvers because they allow for local reconstruction and for the recovery of an extended boundary of the original image, as we shall see in Section 5.2.

In [37] progress was made in finding a computationally efficient solution to the MDP for particular cases of impulse response functions, where the notation 1_X is used for the characteristic function of the set X.

Theorem 1 ([37]).

Let $0 < r_1 < r_2$ be such that r_1/r_2 is poorly approximated by rationals, i.e., there exist $C, N > 0$ such that $|r_1/r_2 - (m/n)| \geq C|n|^{-N}$, for all integers $m, n, n \neq 0$. Given $\psi \in C_c^\infty(\mathbb{R})$ satisfying supp $\psi \subseteq [-r_1 - r_2, r_1 + r_2]$, $h_1 = 1_{[-r_1,r_1]}$ *and* $h_2 = 1_{[-r_2,r_2]}$, *define*

$$\begin{aligned}\widehat{\tilde{h}_{1,\psi}}\left(\frac{n}{2r_2}\right) &= \frac{\widehat{\psi}\left(\frac{n}{2r_2}\right)}{\widehat{h_1}\left(\frac{n}{2r_2}\right)} \text{ if } n \neq 0,\ \widehat{\tilde{h}_{1,\psi}}(0) = \eta\,\frac{\widehat{\psi}(0)}{\widehat{h_1}(0)}\ ,\\ \widehat{\tilde{h}_{2,\psi}}\left(\frac{n}{2r_1}\right) &= \frac{\widehat{\psi}\left(\frac{n}{2r_1}\right)}{\widehat{h_2}\left(\frac{n}{2r_1}\right)} \text{ if } n \neq 0,\ \widehat{\tilde{h}_{2,\psi}}(0) = (1-\eta)\,\frac{\widehat{\psi}(0)}{\widehat{h_2}(0)}\ ,\end{aligned}$$

for some $\eta \in \mathbb{R}$. The functions defined by

$$\begin{aligned}\tilde{h}_{1,\psi}(t) &= \frac{1}{2r_2}\sum_{n=-\infty}^{\infty}\widehat{\tilde{h}_{1,\psi}}\left(\frac{n}{2r_2}\right)e^{\pi\frac{n}{r_2}t}\cdot 1_{[-r_2,r_2]}\\ \tilde{h}_{2,\psi}(t) &= \frac{1}{2r_1}\sum_{n=-\infty}^{\infty}\widehat{\tilde{h}_{2,\psi}}\left(\frac{n}{2r_1}\right)e^{\pi\frac{n}{r_1}t}\cdot 1_{[-r_1,r_1]}\end{aligned}$$

are solutions to $h_1 * \tilde{h}_{1,\psi} + h_2 * \tilde{h}_{2,\psi} = \psi$ satisfying the conditions

(a) supp $\tilde{h}_{1,\psi} \subseteq [-r_2, r_2]$, supp $\tilde{h}_{2,\psi} \subseteq [-r_1, r_1]$,

(b) $\tilde{h}_{i,\psi} \in L^\infty(\mathbb{R}),\ i = 1, 2.$

If $r_1/r_2 = \sqrt{p}$ where p is a positive integer which is not a perfect square, then r_1/r_2 is poorly approximated by rationals. The auxiliary function ψ is used in place of the δ distribution in order for the series representation of the solutions to be convergent.

In Section 3 we shall derive an analogous solution to this problem in the digital setting.

2. Discrete Setting

A digital signal consists of a finite sequence of values, $\{f[n]\}_{n=0}^{N-1}$, where $N \in \mathbb{N}$. Identifying the domain with the cyclic group $\mathbb{Z}_N$, the signal f is then understood as a periodic sequence with period N. The inner product of $f, g : \mathbb{Z}_N \to \mathbb{C}$ is defined as $\langle f, g\rangle = \sum_{n=0}^{N-1} f[n]\overline{g[n]}$. The **discrete Fourier transform** (DFT) of f is

$$\widehat{f}[k] = \frac{1}{N}\sum_{n=0}^{N-1} f[n]e^{-2\pi i nk/N},\ \text{ for } k = 0, \cdots, N-1\ ,$$

and its inverse is

$$f[n] = \sum_{k=0}^{N-1} \widehat{f}[k] e^{2\pi i n k/N}, \text{ for } n = 0, \cdots, N-1 .$$

Let $\omega_N = e^{2\pi i/N}$. If f and $\widehat{f}$ are the vector of input data $(f[0], \ldots, f[N-1])^T$ and the vector of output values $(\widehat{f}[0], \ldots, \widehat{f}[N-1])^T$, respectively, then the DFT can be written as $\widehat{f} = Wf$, where W is the $N \times N$ nonsingular matrix whose (j, k)-entry is $\frac{1}{N}\omega_N^{-(j-1)(k-1)}$.

For $M \in \mathbb{N}$, define the space of **discrete band-limited signals of bandwidth** M by $\mathcal{B}_M = \{f \in \ell^2(\mathbb{Z}_N) | \widehat{f}[k] = 0 \text{ for } k \geq M\}$, and the space of **time-limited signals of duration** M by $\mathcal{T}_M = \{f \in \ell^2(\mathbb{Z}_N) | f[k] = 0 \text{ for } k \geq M\}$.

The **discrete M-sinc function** is defined by

$$\mathrm{sinc}_M[t] = \begin{cases} \frac{\sin(\pi M t/N)}{M \sin(\pi t/N)} e^{-\pi i(1-M)t/N}, & \text{if } t = 1, 2, \ldots, N-1 , \\ 1, & \text{if } t = 0 . \end{cases}$$

Theorem 2 (Discrete Classical Sampling Formula. [15]).

Given $f \in \mathcal{B}_M$, if d is a positive integer which divides N such that $d \leq N/M$ and $r = (N/d) - 1$, then

$$f[n] = \sum_{j=0}^{r} f[dj] \mathrm{sinc}_M[n - dj] .$$

The **discrete linear convolution** of $f, h \in \ell^2(\mathbb{Z}_N)$ is defined as

$$f * h[n] = \sum_{p=0}^{N-1} f[p] h[n-p] \text{ for } n = 0, \ldots, N-1 .$$

To avoid the ill-posed nature of deconvolution we propose to solve the discrete MDP below.

Discrete Multichannel Deconvolution Problem. *Given time-limited functions $\{h_i\}_{i=1}^m$, find time-limited functions $\{\tilde{h}_i\}_{i=1}^m$ such that*

$$\sum_{i=1}^{m} h_i * \tilde{h}_i[k] = \delta[k] ,$$

for all $k \in \mathbb{Z}$, where $\delta[k] = \delta_{0,k}$.

By choosing a positive integer N large enough we may think of h_i and $\tilde{h}_i$ as functions defined in $\mathbb{Z}_N$. Thus, the MDP can now be expressed as follows: *Find time-limited functions $\tilde{h}_1, \ldots, \tilde{h}_m$ such that $\sum_{i=1}^m h_i * \tilde{h}_i[k] = \delta_N[k]$, $(k \in \mathbb{Z}_N)$, where δ_N is the Kronecker delta in $\mathbb{Z}_N$.*

In this context, the discrete Schiske deconvolvers,

$$\widehat{\tilde{h}}_i[k] = \frac{1}{N^2} \frac{\overline{\widehat{h}_i[k]}}{\sum_{j=1}^m \left|\widehat{h}_j[k]\right|^2}, \; k = 0, \ldots, N-1 , \tag{2.1}$$

are candidates for solutions according to the Discrete Convolution Theorem [12], but they are not time-limited, just as their continuous analogues are not compactly supported. We shall show the boundary effects of Schiske's reconstruction in Section 4.

3. Solvability of the Discrete MDP

3.1 Discrete MDP for Characteristic Impulses

When a characteristic function is sampled, we get a sequence of 1's with padded 0's. We denote by 1_n the characteristic function of $\{0, \ldots, n-1\}$.

Given positive integers n and m, we wish to find solutions to the MDP problem where the convolvers are given by $h_1 = 1_n$ and $h_2 = 1_m$.

The z-**transform** of a doubly infinite sequence $\{a_n\}_{n=-\infty}^{\infty}$ is the complex function defined by the Laurent expansion $A(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. We only consider **causal** sequences, that is, infinite sequences $\{a_n\}_{n=-\infty}^{\infty}$ such that $a_n = 0$ for $n < 0$, in order for the discrete convolution to satisfy the associative property (see p. 156 of [23]), which is needed to do deconvolution. The z-transform of $y_n = \sum_{m=0}^{\infty} x_m b_{n-m}$ is the product of the z-transforms of $\{x_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$. Thus, using the z-transform, the problem of finding time-limited solutions $\tilde{h}_1$ and $\tilde{h}_2$ such that $h_1 * \tilde{h}_1 + h_2 * \tilde{h}_2 = \delta$ becomes the problem of finding polynomials $\tilde{H}_1(z)$ and $\tilde{H}_2(z)$ such that

$$H_1(z)\tilde{H}_1(z) + H_2(z)\tilde{H}_2(z) = 1 \,. \tag{3.1}$$

In the characteristic case, $H_1(z) = 1 + z + \cdots + z^{n-1}$, $H_2(z) = 1 + z + \cdots + z^{m-1}$. Thus,

$$H_1(z) = \begin{cases} \frac{1-z^n}{1-z} & \text{if } z \neq 1\,, \\ n & \text{if } z = 1, \end{cases} \quad \text{and } H_2(z) = \begin{cases} \frac{1-z^m}{1-z} & \text{if } z \neq 1\,, \\ m & \text{if } z = 1\,. \end{cases}$$

For n, m nonzero integers, let (n, m) be the greatest common divisor of n and m.

Theorem 3.

For $h_1 = 1_n$ and $h_2 = 1_m$, with $(n, m) \neq 1$, the equation

$$h_1 * \tilde{h}_1 + h_2 * \tilde{h}_2 = \delta$$

is unsolvable for time-limited functions $\tilde{h}_1$ and $\tilde{h}_2$.

Proof. If $(n, m) = d \neq 1$, then there exist $n_1, m_1 \in \mathbb{N}$ such that $n = n_1 d$, $m = m_1 d$, $(n_1, m_1) = 1$, so that for $z \neq 1$, $1 - z^n = (1 - z^d) \sum_{j=0}^{n_1-1} z^{dj}$ and $1 - z^m = (1 - z^d) \sum_{j=0}^{m_1-1} z^{dj}$. Thus, $\frac{1-z^d}{1-z}$ is a nonconstant polynomial factor of both $H_1(z)\tilde{H}_1(z)$ and $H_2(z)\tilde{H}_2(z)$, and hence, of 1, by (3.1). □

In Theorem 5 we shall see that if $(n, m) = 1$, time-limited solutions $\tilde{h}_1$ and $\tilde{h}_2$ can be constructed. Thus, $(n, m) = 1$ is a necessary and sufficient condition for the solvability of the discrete MDP.

Using the z-transform, we obtain the following discrete version of the Paley–Weiner–Schwartz Theorem: *The z-transform of a finite causal sequence is a polynomial (entire function) and, conversely, the inverse z-transform of a polynomial is a finite (causal) sequence.*

Notice that given $a(z)$ and $b(z)$ polynomials with no common zeros, solutions to the analytic Bezout equation $a(z)p(z) + b(z)q(z) = 1$ are the Schiske deconvolvers (in the sense of the z-transform)

$$p(z) = \frac{\overline{a(z)}}{|a(z)|^2 + |b(z)|^2}, \; q(z) = \frac{\overline{b(z)}}{|a(z)|^2 + |b(z)|^2} \,.$$

Since $p(z)$ and $q(z)$ are not polynomials, the inverse z-transforms of $p(z)$ and $q(z)$ are not finite (causal) sequences. Thus, the z-transform analogues of the Schiske deconvolvers are not of finite support. But, if we restrict the domain of the z-transform to be $z = e^{2\pi ik/N}$ for $k \in \mathbb{Z}_N$ for some N large enough, these solutions become the Schiske deconvolvers in (2.1).

3.2 Solving the MDP from the Discrete Fourier Axis

Using the geometric sum, the discrete Fourier transforms of $h_1 = 1_n$ and $h_2 = 1_m$ become

$$\begin{aligned}\widehat{h_1}[k] &= \frac{1}{N}\sum_{p=0}^{n-1} e^{-2\pi ipk/N} = \begin{cases} \frac{1}{N}\frac{1-e^{-2\pi ink/N}}{1-e^{-2\pi ik/N}} & \text{if } k \neq 0\,, \\ \frac{n}{N} & \text{if } k = 0\,, \end{cases} \\ \widehat{h_2}[k] &= \begin{cases} \frac{1}{N}\frac{1-e^{-2\pi imk/N}}{1-e^{-2\pi ik/N}} & \text{if } k \neq 0\,, \\ \frac{m}{N} & \text{if } k = 0\,, \end{cases}\end{aligned}$$

for $k = 0, \ldots, N-1$. By Discrete Convolution Theorem, the discrete MDP on the discrete Fourier axis can be formulated as

$$N\widehat{h_1}[k]\widehat{\tilde{h}_1}[k] + N\widehat{h_2}[k]\widehat{\tilde{h}_2}[k] = \frac{1}{N},\ k = 0, 1, \ldots, N-1\,, \tag{3.2}$$

which can be written as the underdetermined system $M\widehat{\tilde{h}} = \frac{1}{N}1_N$, where $\widehat{\tilde{h}}$ is the column vector obtained by stacking the vectors $\widehat{\tilde{h}_1}$ and $\widehat{\tilde{h}_2}$ and M is the $N \times 2N$ matrix whose $N \times N$ blocks are diagonal matrices with diagonal vectors $N\widehat{h_1}$ and $N\widehat{h_2}$.

When $h_1 = 1_n$ and $h_2 = 1_m$, a possible method for finding the minimally supported deconvolvers is based on looking at the points k such that $\widehat{h_1}[k] = 0$ or $\widehat{h_2}[k] = 0$, since the values of the deconvolvers are known there. The zero sets of $\widehat{h_1}$ and $\widehat{h_2}$ are $\mathcal{Z}_1 = \{k \in \{1, \ldots, N-1\} : \frac{nk}{N} \in \mathbb{Z}\}$, and $\mathcal{Z}_2 = \{k \in \{1, \ldots, N-1\} : \frac{mk}{N} \in \mathbb{Z}\}$. If $N = nm$ and $(n, m) = 1$, then $|\mathcal{Z}_1| = n-1$, $|\mathcal{Z}_2| = m-1$, and $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are disjoint.

The following result is the discrete version of the classical Nonperiodic Sampling Theorem [36].

Theorem 4 (The Discrete Nonperiodic Sampling Theorem).

Suppose that $(n, m) = 1$, $N \geq mn$, and $\varphi \in \mathcal{B}_{m+n-1}$. If $\varphi[kn] = 0$ for $k = 0, \ldots, m-1$ and $\varphi[km] = 0$ for $k = 0, \ldots, n-1$, then $\varphi[k] = 0$ for all $k \in \{0, \ldots, N-1\}$.

Proof. Since $\varphi[j] = \sum_{k=0}^{m+n-2} \widehat{\varphi}[k]e^{2\pi ijk/N}$, for each $j \in \{0, \ldots, N-1\}$, it suffices to show that $\widehat{\varphi}[k] = 0$ for all $k \in \{0, \ldots, m+n-2\}$. Since $(n, m) = 1$, there are at least $n+m-1$ distinct values of j for which $\varphi[j] = 0$. The system $\sum_{k=0}^{m+n-2} \widehat{\varphi}[k]e^{2\pi ijk/N} = \varphi[j]$ for $j = dn$, $(0 \leq d \leq m-1)$ or $j = dm$, $(0 \leq d \leq n-1)$ is homogeneous and the coefficient matrix is a square matrix of full rank, since it is a Vandermonde matrix. Thus, the system has only the trivial solution $\widehat{\varphi}[k] = 0$ for all $k \in \{0, \ldots, m+n-2\}$. □

Theorem 5.

Given $h_1 = 1_n$ and $h_2 = 1_m$ with $(n, m) = 1$ and $N = nm$, the solutions of the *equation* $h_1 * \tilde{h}_1[k] + h_2 * \tilde{h}_2[k] = \delta_N[k]$ *are given by*

$$\begin{aligned}
\tilde{h}_1[s] &= \left[\frac{a}{mn} + \frac{1}{m}\sum_{k=1}^{m-1}\left(\frac{\sin(\pi k/m)}{\sin(\pi nk/m)}e^{-\pi ik(1-n)/m}\right)e^{2\pi isk/m}\right]1_m[s] \\
\tilde{h}_2[s] &= \left[\frac{1-a}{nm} + \frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{\sin(\pi k/n)}{\sin(\pi mk/n)}e^{-\pi ik(1-m)/n}\right)e^{2\pi isk/n}\right]1_n[s]
\end{aligned}$$

for any $a \in \mathbb{R}$.

Proof. Since $N\widehat{h_1}[k]\widehat{\tilde{h}_1}[k] + N\widehat{h_2}[k]\widehat{\tilde{h}_2}[k] = \frac{1}{N}$ for $k = 0, \ldots, N-1$, and $N = nm$ with $(n, m) = 1$, $nj \in \mathcal{Z}_2$ ($j = 1, \ldots, m-1$) and $mj \in \mathcal{Z}_1$ ($j = 1, \ldots, n-1$). Thus,

$$\widehat{\tilde{h}_1}[nj] = \frac{1}{N^2\widehat{h_1}[nj]},\ j = 1, \ldots, m-1,\ \widehat{\tilde{h}_2}[mj] = \frac{1}{N^2\widehat{h_2}[mj]},\ j = 1, \ldots, n-1\ .$$

Given any $a \in \mathbb{R}$, define $\widehat{\tilde{h}_1}[0] = \frac{a}{N^2n}$ and $\widehat{\tilde{h}_2}[0] = \frac{1-a}{N^2m}$. By Theorem 2, we have

$$\widehat{\tilde{h}_1}[k] = \sum_{j=0}^{m-1}\widehat{\tilde{h}_1}[nj]\,\mathrm{sinc}_m[k-nj],\ \widehat{\tilde{h}_2}[k] = \sum_{j=0}^{n-1}\widehat{\tilde{h}_2}[mj]\,\mathrm{sinc}_n[k-mj]\ .$$

If we let $\varphi[k] = N\widehat{h_1}[k]\widehat{\tilde{h}_1}[k] + N\widehat{h_2}[k]\widehat{\tilde{h}_2}[k] - 1/N$, we get $\varphi[nj] = 0$ for $j = 0, \ldots, m-1$ and $\varphi[mj] = 0$ for $j = 0, \ldots, n-1$. By Theorem 4, $\varphi[k] = 0$ for all $k \in \{0, \ldots, N-1\}$ and thus, the interpolated functions $\widehat{\tilde{h}_1}$ and $\widehat{\tilde{h}_2}$ are solutions to the MDP when $N = nm$. Now

$$\begin{aligned}
\tilde{h}_1[k] &= \sum_{s=0}^{N-1}\widehat{\tilde{h}_1}[s]e^{2\pi iks/N} = \sum_{s=0}^{N-1}\left(\sum_{j=0}^{m-1}\widehat{\tilde{h}_1}[nj]\,\mathrm{sinc}_m[s-nj]\right)e^{2\pi iks/N} \\
&= n\sum_{j=0}^{m-1}\widehat{\tilde{h}_1}[nj]e^{2\pi ik(nj)/N}1_m[k] = n\sum_{j=0}^{m-1}\widehat{\tilde{h}_1}[nj]e^{2\pi ikj/m}1_m[k]\ .
\end{aligned}$$

Similarly $\tilde{h}_2[k] = m\sum_{j=0}^{n-1}\widehat{\tilde{h}_2}[mj]e^{2\pi ikj/n}1_n[k]$. Furthermore, for $k \neq 0$

$$\widehat{h_1}[k] = \frac{1}{N}\frac{\sin(\pi kn/N)}{\sin(\pi k/N)}e^{\pi ik(1-n)/N},\ \widehat{h_2}[k] = \frac{1}{N}\frac{\sin(\pi km/N)}{\sin(\pi k/N)}e^{\pi ik(1-m)/N}\ ,$$

and the final form of the solutions follows by substitution. □

Remark 1. Theorem 5 can be adapted to include the case $N = nmd$, $d \in \mathbb{N}$. In Theorem 6, we shall see a method for solving the convolution equation of Theorem 5 for a general $N \in \mathbb{N}$.

3.3 Discrete Fourier Series Solution to the MDP

Discrete convolution can be interpreted as a product of two polynomials (see [12]). Thus, an alternative description of the MDP is the following: *Given two convolvers represented by the polynomials $a(x)$ and $b(x)$ of degrees n and m, respectively, find polynomials $p(x)$ and $q(x)$ that satisfy the algebraic Bezout equation*

$$a(x)p(x) + b(x)q(x) = 1 . \tag{3.3}$$

Berenstein and coauthors studied the problem of finding solutions $q_1, \ldots, q_m$ to the algebraic Bezout equation $p_1q_1 + \cdots + p_mq_m = 1$ (where $p_1, \ldots, p_m$ are polynomials with no common zeros in $\mathbb{C}^n$) and its connection to the MDP in [3, 5], and [6].

One approach to solving (3.3) is the following. Let $\zeta_1, \ldots, \zeta_n$ and $\mu_1, \ldots, \mu_m$ be the roots of $a(x)$ and $b(x)$, respectively. Then $q(\zeta_j) = 1/b(\zeta_j)$, $j = 1, \ldots, n$, $p(\mu_j) = 1/a(\mu_j)$, $j = 1, \ldots, m$. The polynomials $p(x)$ and $q(x)$ can be determined by solving a Vandermonde matrix equation. However, finding the roots of high degree polynomials can be numerically ill-conditioned (see [38]). In addition, the speed of the algorithm depends on the speed of finding the roots explicitly.

Although this procedure is not recommended for solving the MDP for general convolvers, it can be used effectively in the special case when the convolvers are discrete characteristic impulses. The polynomials $a(x)$ and $b(x)$ which correspond to the discrete characteristic impulses have roots $\zeta_k = e^{-2\pi ki/n}$ for $k = 1, \ldots, n-1$ and $\mu_k = e^{-2\pi ki/m}$ for $k = 1, \ldots, m-1$, respectively. Therefore

$$\begin{aligned} p\big(e^{-2\pi ki/m}\big) &= \frac{1}{a\big(e^{-2\pi ki/m}\big)} = \frac{e^{-2\pi ik/m}-1}{e^{-2\pi ink/m}-1} = \frac{\sin(\pi k/m)}{\sin(\pi nk/m)} e^{-\pi i(1-n)k/m} , \\ q\big(e^{-2\pi ki/n}\big) &= \frac{1}{b\big(e^{-2\pi ki/n}\big)} = \frac{e^{-2\pi ik/n}-1}{e^{-2\pi imk/n}-1} = \frac{\sin(\pi k/n)}{\sin(\pi mk/n)} e^{-\pi i(1-m)k/n} . \end{aligned}$$

For the root 1 we have $np(1)+mq(1) = 1$. Thus, we may choose $p(1) = \frac{a}{n}$ and $q(1) = \frac{1-a}{m}$ for any $a \in \mathbb{R}$.

The Vandermonde matrix equation for finding the coefficients of $p(x)$ is

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_m^{-1} & \cdots & \omega_m^{-(m-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_m^{-(m-1)} & \cdots & \omega_m^{-(m-1)^2} \end{pmatrix} \mathbf{p} = \begin{pmatrix} a/n \\ \frac{\sin(\pi 1/m)}{\sin(\pi n/m)} e^{-\pi i(1-n)/m} \\ \vdots \\ \frac{\sin(\pi(m-1)/m)}{\sin(\pi n(m-1)/m)} e^{-\pi i(1-n)(m-1)/m} \end{pmatrix} ,$$

where $\mathbf{p} = (p_0, \ldots, p_{m-1})^T$. The $m \times m$ matrix on the left-hand side is the DFT matrix after rescaling the system. Thus, the solution can be expressed as the product of the inverse DFT matrix and the right-hand side.

Theorem 6.

Given $h_1 = 1_n$ and $h_2 = 1_m$ with $(n, m) = 1$ and $p \in \mathbb{Z}_{m+n}$, the solutions of

$h_1 * \tilde{h}_1[k] + h_2 * \tilde{h}_2[k] = \delta_N[k-p],\ k = 0, \ldots, N-1$, *are given by*

$$\begin{aligned}
\tilde{h}_1[s] &= \left[\frac{a}{nm} + \frac{1}{m}\sum_{k=1}^{m-1}\left(\frac{\sin(\pi k/m)}{\sin(\pi nk/m)} e^{-\pi ik((1-n)+2p)/m}\right) e^{2\pi isk/m}\right] 1_m[s]\ , \\
\tilde{h}_2[s] &= \left[\frac{1-a}{nm} + \frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{\sin(\pi k/n)}{\sin(\pi mk/n)} e^{-\pi ik((1-m)+2p)/n}\right) e^{2\pi isk/n}\right] 1_n[s]\ ,
\end{aligned}$$

for any $a \in \mathbb{R}$.

Proof. We derived the solution above for the case $a(x)p(x) + b(x)q(x) = 1$. As a polynomial, the right-hand side of the Bezout equation has coefficient vector $[1,\ 0,\ \ldots,\ 0]$, where 1 stands for the constant term. To solve the Bezout equation when the right-hand side is $\delta_N[k-p]$ for $p \in [0, n+m-1]$, the corresponding polynomial equation is $a(x)p(x) + b(x)q(x) = x^p$. The conclusion follows at once. □

4. Computational Solution to the General MDP

4.1 Closed-Form Solution

The discrete MDP can be written as a matrix based problem.

Proposition 1.
Given $\{h_1[i]\}_{i=0}^{n-1}$ *and* $\{h_2[j]\}_{j=0}^{m-1}$, *let*

$$\tilde{h} = \left[\tilde{h}_1[0]\,\tilde{h}_1[1]\,\cdots\,\tilde{h}_1[m-1]\,\tilde{h}_2[0]\,\tilde{h}_2[1]\,\cdots\,\tilde{h}_2[n-1]\right]^T .$$

Then the i*th column of the* $(n+m-1)\times(n+m)$ *matrix* M *such that* $M\tilde{h} = h_1 * \tilde{h}_1 + h_2 * \tilde{h}_2$ *is given by*

$$M_i = \begin{cases} \left[\underbrace{0\ \cdots\ 0}_{i-1}\ \underbrace{h_2[0]\ , h_2[1]\ \cdots\ h_2[m-1]}_{m}\ \underbrace{0\ \cdots\ 0}_{n-i}\right]^T & \text{if } i = 0, \cdots, n-1\ , \\ \left[\underbrace{0\ \cdots\ 0}_{i-n-1}\ \underbrace{h_1[0]\, h_1[1]\ \cdots\ h_1[n-1]}_{n}\ \underbrace{0\ \cdots\ 0}_{m+n-i}\right]^T & \text{if } i = n, \cdots, n+m-1\ . \end{cases}$$

Proof. By definition of discrete convolution and using the fact that $h_1[i] = \tilde{h}_2[i] = 0$ for $i \notin \{0, \ldots, n-1\}$ and $h_2[j] = \tilde{h}_1[j] = 0$ for $j \notin \{0, \ldots, m-1\}$, we have

$$M\tilde{h}[k] = \sum_{i=0}^{n+m-1}\left(h_1[k+n-i]\tilde{h}_1[n+i] + h_2[k-i]\tilde{h}_2[i]\right),$$

for all $k = 1, \cdots, n+m-1$. Hence, the $(i+1)^{st}$ column of M is

$$\begin{aligned}
&\left[h_2[-i]\, h_2[1-i]\ \ldots\ h_2[n+m-1-i]\right]^T, && \text{if } 0 \le i \le n-1\ , \\
&\left[h_1[n-i]\, h_1[n+1-i]\ \ldots\ h_1[n+m-1-i]\right]^T, && \text{if } n \le i \le n+m-1\ ,
\end{aligned}$$

proving the result. $\square$

Fixing $t = 1, \cdots, n+m-1$, the linear system $M\tilde{h} = \delta_t$ is underdetermined. The addition of a row to M and an entry to the column vector δ_t fixes the degree of freedom: we consider the system with augmented matrix

$$\begin{pmatrix} M_1 & \dots & M_{n+m-1} & M_{n+m} & \delta_t \\ 0 & \dots & 0 & 1 & a \end{pmatrix}$$

where a is a real parameter and M_i is the ith column of M, so that $\tilde{h}_2[n] = a$.

Variations of this technique can be used to specify other coordinates of one of the deconvolvers. This constraint guarantees the uniqueness of the solution.

Assume $h_1 = 1_n$ and $h_2 = 1_m$, and let $\{e_k\}_{k=1,\dots,n+m-1}$ be the standard basis of $\mathbb{R}^{n+m-1}$ where $m > n$ so that the first n equations of the system $M\tilde{h} = e_k$ form the matrix equation $\begin{pmatrix} A_n & A_n & O \end{pmatrix} \tilde{h} = e_k$, where A_n is a lower triangular matrix of order n whose nonzero elements are all 1. If we subtract the i^{th} row from the $(i+1)^{st}$ row for $i = 1, \dots, n-1$, we get $\begin{pmatrix} I_n & I_n & O \end{pmatrix} \tilde{h} = D_{n,k}$, where $D_{n,k}$ is the kth column of the $n \times n$ matrix D_n given by $D_n(i,i) = 1$ for $i = 1, \dots, n$, $D_n(i+1, i) = -1$ for $i = 1, \dots, n-1$. For $i = 1, \dots, n$, we obtain

$$\tilde{h}_1[i] + \tilde{h}_2[i] = \begin{cases} e_k[i] - e_{k+1}[i] & \text{if } k = 1, \dots, n-1 \;, \\ e_n[i] & \text{if } k = n \;, \\ 0 & \text{if } k > n \;. \end{cases}$$

In particular,

$$\tilde{h}_1 = -\tilde{h}_2 1_n, \text{ for } k > n \;. \tag{4.1}$$

Notice that (4.1) can be visually observed in the plots of the deconvolvers shown in [37].

In the case $h_1 = 1_n$, $h_2 = 1_m$ and $a = 0$, the augmented MDP matrix of order $(n+m)$ in Proposition 1 can be expressed as

$$M_{n,m} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$$

where $A_{1,1}$, $A_{2,2}$ are square matrices of order n and m, respectively. Observe that

$$\begin{pmatrix} I & O \\ -A_{2,1}A_{1,1}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ O & A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2} \end{pmatrix} .$$

The matrix $A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2}$ is known as the **Schur complement** of $A_{1,1}$ in $M_{n,m}$. If $m > n$, then $A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2} = M_{n,m-n}S_{m-n}^m$, where S_{m-n}^m is the diagonal matrix of order m whose first $m-n$ diagonal entries are -1 and the remaining entries are 1. In addition

$$-A_{2,1}A_{1,1}^{-1}(i,j) = \begin{cases} -1 & \text{if } 1 \le i \le m-1, \; j = n \;, \\ 1 & \text{if } i = j+n-m, \; m-n+1 \le j \le n-1 \;, \\ 0 & \text{otherwise} \;. \end{cases}$$

Theorem 7.

The i^{th} column of the inverse of the MDP matrix $M_{n,n+1}$ is given by

$$\left(M_{n,n+1}^{-1}\right)_i = \begin{cases} \Big[\underbrace{0\cdots 0}_{i-1}\ 1\ \underbrace{0\cdots 0}_{n}\ -1\ \underbrace{0\cdots 0}_{n-i+1}\Big]^T & \textit{for } i = 1\ldots n-1\,, \\ \Big[\underbrace{1\cdots 1}_{n}\ \underbrace{-1\cdots -1}_{n}\ 0\Big]^T & \textit{for } i = n\,, \\ \Big[\underbrace{0\cdots 0}_{i-n-1}\ 1\ \underbrace{0\cdots 0}_{n}\ -1\ \underbrace{0\cdots 0}_{i+1}\Big]^T & \textit{for } i = n+1,\ldots,2n\,, \\ \Big[\underbrace{-1\cdots -1}_{n}\ \underbrace{1\cdots 1}_{n+1}\Big]^T & \textit{for } i = 2n+1\,. \end{cases}$$

Proof. We need to solve $M_{n,n+1}\tilde{h} = e_k$ for $k = 1,\ldots,2n+1$. Left multiplication by $\begin{pmatrix} I & O \\ -A_{2,1}A_{1,1}^{-1} & I \end{pmatrix}$ yields

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ O & M_{n,1}S_1^{n+1} \end{pmatrix}\begin{pmatrix} \tilde{h}_1 \\ \tilde{h}_2 \end{pmatrix} = \begin{pmatrix} I & O \\ -A_{2,1}A_{1,1}^{-1} & I \end{pmatrix} e_k\,.$$

Because of (4.1), it suffices to solve for $\tilde{h}_2$. Thus, for $n < k \le 2n+1$, we need to solve $M_{n,1}S_1^{n+1}\tilde{h}_2 = e_{k-n}^{n+1}$, where e_i^j denotes the $j \times 1$ matrix whose only nonzero entry 1 is in the i^{th} row. Now $(M_{n,1}S_1^{n+1})^{-1} = S_1^{n+1}M_{n,1}^{-1}$, so

$$\tilde{h}_2 = \begin{cases} -S_1^{n+1}M_{n,1}^{-1}A_{2,1}A_{1,1}^{-1}e_k^{n+1} & \text{for } 1 \le k \le n\,, \\ S_1^{n+1}M_{n,1}^{-1}e_{k-n}^{n+1} & \text{for } n < k \le m+n\,. \end{cases}$$

Observe that $M_{n,1}^{-1}$ is the upper triangular matrix with entries

$$\alpha_{i,j} = \begin{cases} 1 & \text{if } 1 \le i = j \le n+1 \\ -1 & \text{if } 1 \le i \le n,\ j = n+1 \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.2}$$

The result follows immediately by inspection for the various values of k. □

4.2 Conditioning of the MDP for the Characteristic Case

Lemma 1.

Let $h_1 = 1_n$, $h_2 = 1_m$ where $n < m$ and $(m,n) = 1$. Then the entries of the vector $\tilde{h} = (\tilde{h}_1\ \tilde{h}_2)^T$ are in the set $\{-1, 0, 1\}$, where $\tilde{h}_1$ and $\tilde{h}_2$ are the solutions to the MDP $h_1 * \tilde{h}_1 + h_2 * \tilde{h}_2 = \delta_{n+m-1}$ found by the MDP matrix with $\tilde{h}_2[n] = 0$.

The proof is based on a recursive argument in the spirit of the Euclidean algorithm and on the use of the Schur complement to turn the MDP into solving a matrix equation involving only $\tilde{h}_2$.

Proof. Let q and r be the unique positive integers such that $m = nq + r$ with $r < n$. We wish to solve

$$M_{n,m}\tilde{h} = e^{n+m}_{n+m-1} , \tag{4.3}$$

or, using the block matrix representation

$$M_{n,m} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$$

with $A_{1,1}$ and $A_{2,2}$ square matrices of order n and m, respectively,

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ O & M_{n,m-n}S^m_{m-n} \end{pmatrix}\begin{pmatrix} \tilde{h}_1 \\ \tilde{h}_2 \end{pmatrix} = \begin{pmatrix} I & O \\ -A_{2,1}A^{-1}_{1,1} & I \end{pmatrix} e^{n+m}_{n+m-1} ,$$

where, we recall, S^m_{m-n} is the $m \times m$ diagonal matrix where the first $m - n$ entries are -1 and the remaining entries are 1.

In order to show that the entries of $\tilde{h}$ are in $\{-1, 0, 1\}$, it suffices to show that the entries of $\tilde{h}_2$ are 0 or ±1, since $\tilde{h}_1$ and $\tilde{h}_2$ are related by (4.1). Thus, the problem reduces to solving the system $M_{n,m-n}S^m_{m-n}\tilde{h}_2 = e^m_{m-1}$. Since multiplication by S^m_{m-n} only produces a possible sign change, we may view the above as a matrix equation analogous to (4.3). Letting $\tilde{h}^{(2)} = S^m_{m-n}\tilde{h}_2 = (\tilde{h}^{(2)}_1\ \tilde{h}^{(2)}_2)^T$ where $\tilde{h}^{(2)}_2$ has dimension $m - n$, we may repeat the above procedure q times. Hence, the problem reduces to solving $M_{n,r}S^{n+r}_r\tilde{h}^{(q)}_2 = e^{n+r}_{n+r-1}$, where, we recall, $r < n$. Swap the first n columns with the next r columns of $M_{n,r}$ and modify accordingly the order of the affected coordinates of the solution vector. The problem can now be formulated as $M_{r,n}S^{n+r}_r\tilde{h}' = e^{n+r}_{n+r-1}$, where $\tilde{h}'$ is the permuted solution vector and $(r, n) = 1$. Reapply the Euclidean algorithm and the argument outlined above until the MDP matrix has the form $M_{t,1}$ for some $t > 1$. Since the entries of $M^{-1}_{t,1}$ are 0, 1, or -1, so are the coordinates of $\tilde{h}'$, completing the proof. □

Theorem 8.

Let $h_1 = 1_n$, $h_2 = 1_m$, where m and n are positive integers such that $m > n$ and $(m, n) = 1$. The problem of computing δ_{n+m-1} from $h_1 * \tilde{h}_1 + h_2 * \tilde{h}_2$, with $\tilde{h}_j$ $(j = 1, 2)$ computed using the MDP matrix $M_{n,m}$ with the constraint $\tilde{h}_2[n] = 0$, has condition number bounded by $m(m + n)$ with respect to the norm defined by

$$\|A\|_1 = \max_{1\le j\le|\mathrm{Row}(A)|} \|a_j\|_1 ,$$

where a_j is the j^{th} column of A.

Thus, the problem is well-conditioned for small values of n and m.

Proof. Viewing the above as the problem of computing $e_{n+m-1} = M_{n,m}\tilde{h}$ from $\tilde{h} = (\tilde{h}_1\ \tilde{h}_2)^T$, the condition number is $\kappa = \|M_{n,m}\|_1 \frac{\|\tilde{h}\|_1}{\|e_{n+m-1}\|_1} = \|M_{n,m}\|_1\|\tilde{h}\|_1$. From the form of $M_{n,m}$ we obtain

$$\|M_{n,m}\|_1 = \max_{1\le j\le n+m}\left(\sum_{i=1}^{n+m-1}|m_{i,j}|\right) = m ,$$

where $m_{i,j}$ is the (i, j)th entry of $M_{n,m}$. Since, by Lemma 1, $\tilde{h}$ is a column vector whose entries are in $\{-1, 0, 1\}$, $\|\tilde{h}\|_1 \le n + m$. Thus, $\kappa \le m(n + m)$. □

Remark 2. In the special case $m = n + 1$, Theorem 7 yields $\kappa = (n + 1)(2n + 1)$.

4.3 The m-Channel MDP

We now solve the MDP when $h_k = 1_{n_k}$ for $n_k \in \mathbb{N}$, following the procedure used in the two-channel case. Suppose $a_k(x) = \sum_{j=0}^{n_k-1} x^j$, $k = 1, \ldots, m$, for some positive integers $n_1, \ldots, n_k$. We wish to find polynomials $b_k(x)$, $k = 1, \ldots, m$, such that

$$\sum_{k=1}^{m} a_k(x) b_k(x) = 1 \ .$$

Let us assume that each $b_k(x)$ is a polynomial of degree $\sum_{j \neq k} n_j - m + 1$, and that

$$b_k(x) = c_k(x) \prod_{j \neq k, \sigma(k)} a_j(x) \ ,$$

where σ is a permutation of $\{1, \ldots, m\}$ such that $\sigma(k) \neq k$ for each $k \in \{1, \ldots, m\}$ and each $c_k(x)$ is a polynomial of degree $\sigma(k) - 1$, $k = 1, \ldots, m$. We need to solve for $c_1(x), \ldots, c_m(x)$ the algebraic Bezout equation

$$\sum_{k=1}^{m} c_k(x) \prod_{j \neq \sigma(k)} a_j(x) = 1 \ , \tag{4.4}$$

which is solvable if no two polynomials among the $a_k(x)$ have a common factor.

Theorem 9.
Given $h_k = 1_{n_k}$ for $k = 1, \ldots, m$ and a permutation σ of $\{1, \ldots, m\}$ such that $\sigma(k) \neq k$ for all $k \in \{1, \ldots, m\}$, if $(n_k, n_j) \neq 1$ for some $k \neq j$, then the equation

$$\sum_{k=1}^{m} h_k * c_k * h_{\sigma(k)} = \delta \tag{4.5}$$

is unsolvable for time-limited functions $c_1, \ldots, c_m$.

Proof. If (4.5) is solvable, then no two of the polynomials $a_k(x) = (1 - x^{n_k})/(1 - x)$ $(k = 1, \ldots, m)$ can have a common factor. Thus, the numbers $n_1, \ldots, n_m$ must be pairwise relatively prime. □

In Theorem 10 we shall see that the condition $(n_k, n_j) = 1$ for all $k \neq j$ is necessary and sufficient for the existence of time-limited solutions to (4.4).

Let $\mu_{k,\nu}$ be the roots of $a_k(x)$, $(\nu \in \{1, \ldots, n_k - 1\})$, for each $k = 1, \ldots, m$. Then, evaluating (4.4) at these roots yields

$$c_k(\mu_{\sigma(k),\nu}) = \frac{1}{\prod_{j \neq \sigma(k)} a_j\big(\mu_{\sigma(k),\nu}\big)}, \text{ for } \nu = 1, \ldots, n_{\sigma(k)} - 1 \ .$$

Furthermore,

$$\sum_{k=1}^{m} \left(\prod_{j \neq \sigma(k)} n_j \right) c_k(1) = 1 \ .$$

Thus, we may define

$$c_k(1) = \frac{t_k}{\prod_{j\neq\sigma(k)} n_j}, \text{ for } k = 1, \dots, m \ ,$$

for any constants $t_1, \dots, t_m$ such that $\sum_{k=1}^m t_k = 1$. Calculations similar to those done in the two-channel case yield

$$a_k(\mu_{j,\nu}) = \frac{\sin(\pi k\nu/\sigma(k))}{\sin(\pi\nu/\sigma(k))} e^{\pi i\nu(k-1)/\sigma(k)} \ ,$$

for $\nu = 1, \dots, \sigma(k) - 1$, $k = 1, \dots, m$. Setting up the Vandermonde matrix equations and rescaling the equations yields the following DFT solutions.

Theorem 10.

Let $n_1, \dots, n_m$ be pairwise relatively prime positive integers and set $M = \sum_{k=1}^m n_k$, $N = \prod_{k=1}^m n_k$, and $N_k = N/n_k$ for each $k = 1, \dots, m$. Given $\eta \in \mathbb{Z}_{M-1}$, and $h_k = 1_{n_k}$, $k = 1, \dots, m$, the solutions $\tilde{h}_1, \dots, \tilde{h}_m$ of the equation

$$\sum_{k=1}^{m} h_k * \tilde{h}_k[s] = \delta_N[s - \eta], \ \textit{for} \ s = 0, \dots, N - 1 \ ,$$

are given by

$$\tilde{h}_k[s] = \frac{1}{N_k} \sum_{j=1}^{N_k-1} \prod_{\nu\neq k} \tilde{H}_k\left(\frac{j}{n_\nu}\right) e^{2\pi isj/N_k} \cdot 1_{N_k}[s] \ ,$$

where

$$\tilde{H}_k(\omega) = \frac{\sin(\pi\omega)}{\sin(\pi n_k\omega)} e^{-\pi i\omega(1-n_k+2\eta)} \ ,$$

and $\tilde{H}_k(0) = \frac{t_k}{N_k}$, with $t_1, \dots, t_m$ arbitrarily chosen reals such that $\sum_{k=1}^m t_k = 1$.

4.4 Boundary Effect of the Discrete Schiske Reconstruction

We compare the performance of the Schiske deconvolvers to the deconvolvers found using the associated MDP matrix.

Assume the original signal f we wish to simulate using the multi-channel deconvolution procedure is an array of length L, and the impulse responses h_1 and h_2 are arrays of lengths n and m, respectively, with $m > n$. If we take as our data $s_1 = h_1 * f$ and $s_2 = h_2 * f$ considered as arrays of lengths $L + n - 1$ and $L + m - 1$, respectively, the Schiske deconvolution procedure will yield a perfect reconstruction for f within machine precision, because s_1 and s_2 can be considered as periodic signals of some length greater than $L + m - 1$ by padding s_1 and s_2 appropriately with zeros.

Let h be a vector of length n and consider $h * f$ as a matrix multiplication Hf, that is

$$h * f = \begin{pmatrix} h[1] & & & \\ h[2] & h[1] & & \\ \vdots & h[2] & \ddots & \\ h[n] & \vdots & & h[1] \\ & h[n] & & h[2] \\ & & \ddots & \vdots \\ & & & h[n] \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_L \end{pmatrix},$$

where H has dimension $(n + L - 1) \times L$ and all unmarked entries are zero. If we pad the vector f with $n - 1$ trailing 0's, we may replace H by a $(L + n - 1) \times (L + n - 1)$ cyclic extension which is diagonalizable by the DFT matrix.

In order to diagonalize the matrices associated with $h_1 * f$ and $h_2 * f$, we pad f with at least $m - 1$ zeros and then view the corresponding matrices as cyclic convolution matrices, which are thus, diagonalizable by the DFT matrices. In this case, the Schiske deconvolvers are clearly solutions, and, of course, they would still be solutions if $s_1 = h_1 * f$ and $s_2 = h_2 * f$ were computed by cyclic convolution, with h_1, h_2 and f viewed as periodic functions of length L. Thus, to demonstrate the effectiveness of the time-limited deconvolvers compared to the Schiske deconvolvers, we truncate s_1 and s_2 to be of sizes smaller than $L + n - 1$.

To gain a little insight about the boundary effects that arise, let us analyze the case when the inverse filtering method is used. Let us solve for f the equation $Hf = s$, where H is a truncated convolution matrix assumed to be invertible. Embed H into a larger cyclic convolution matrix H_c, pad s with trailing zeros, and call the extended vector s_c. Our estimated solution $\tilde{f}$ can now be expressed as the truncation of the vector $\begin{pmatrix} \tilde{f} \\ \tilde{f}_e \end{pmatrix} = H_c^{-1} s_c$, where H_c^{-1} is found by means of a DFT matrix. We can then write the system as

$$\begin{pmatrix} H & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \tilde{f} \\ \tilde{f}_e \end{pmatrix} = \begin{pmatrix} s \\ 0 \end{pmatrix}.$$

The inverse of H_c is given by the **Schur–Banachiewicz inverse formula**

$$H_c^{-1} = \begin{pmatrix} H^{-1} + H^{-1} H_{12} S^{-1} H_{21} H^{-1} & -H^{-1} H_{12} S^{-1} \\ -S^{-1} H_{21} H^{-1} & S^{-1} \end{pmatrix},$$

where $S = H_{22} - H_{21} H^{-1} H_{12}$. This implies that $\tilde{f} = f + H^{-1} H_{12} S^{-1} H_{21} f$ and that the boundary effects are caused by the term $H^{-1} H_{12} S^{-1} H_{21} f$.

We now carry out the analysis of the type of procedure used to implement the Schiske deconvolvers in the two-channel case. As for the case of the inverse filtering method, in the formula that yields the estimated solution there is a nontrivial term added to the original signal which is responsible for the boundary effects.

Given $H^1 f = s_1$, $H^2 f = s_2$, and proceeding as in the one-channel case, we embed H^1 and H^2 into cyclic matrices, thus obtaining the two matrix equations

$$\begin{pmatrix} H^j & H^j_{12} \\ H^j_{21} & H^j_{22} \end{pmatrix} \begin{pmatrix} \tilde{f}^j \\ \tilde{f}^j_e \end{pmatrix} = \begin{pmatrix} s_j \\ 0 \end{pmatrix}, \; j = 1, 2, \tag{4.6}$$

where the vectors $\tilde{f}^j$ and $\tilde{f}_e^j$ are the approximate modifications of f and corresponding errors due to the padding necessary in order to get cyclic matrices. From the one channel-case, letting $S_j = H_{22}^j - H_{21}^j (H^j)^{-1} H_{12}^j$ $(j = 1, 2)$, we obtain

$$\tilde{f}^j = f + \left(H^j\right)^{-1} H_{12}^j \left(S^j\right)^{-1} H_{21}^j f,\ \tilde{f}_e^j = -\left(S^j\right)^{-1} H_{21}^j f\ . \tag{4.7}$$

We may write the left-hand side of (4.6) as

$$\begin{pmatrix} H^j & H_{12}^j \\ H_{21}^j & H_{22}^j \end{pmatrix} W_N^T W_N \begin{pmatrix} \tilde{f}^j \\ \tilde{f}_e^j \end{pmatrix}, \tag{4.8}$$

where N is the order of the cyclic matrix and W_N is the $N \times N$ DFT matrix. Let h^j be the first column of the cyclic matrix containing H^j, $j = 1, 2$. Taking the DFT of (4.8) by multiplying by W_N, and using (4.7) we get

$$\mathrm{Diag}\left(\widehat{h^j}\right) W_N \left[\begin{pmatrix} f \\ 0 \end{pmatrix} + \begin{pmatrix} f_e^j \\ \tilde{f}_e^j \end{pmatrix} \right], \tag{4.9}$$

where $\mathrm{Diag}(\widehat{h^j})$ is the diagonal matrix with diagonal entries $\widehat{h^j}[1], \ldots, \widehat{h^j}[N]$.

Let V^j $(j = 1, 2)$ be the $N \times N$ Schiske diagonal matrices with diagonal entries $\overline{\widehat{h^j}[i]} / \sum_{k=1}^2 |\widehat{h^k}[i]|^2$. Left multiplying (4.9) by V^j, $j = 1, 2$, and adding the two expressions yields

$$\begin{aligned} &\left[V^1 \mathrm{Diag}\left(\widehat{h^1}\right) + V^2 \mathrm{Diag}\left(\widehat{h^2}\right) \right] W_N \begin{pmatrix} f \\ 0 \end{pmatrix} + \sum_{j=1}^2 V^j \mathrm{Diag}\left(\widehat{h^j}\right) W_N \begin{pmatrix} f_e^j \\ \tilde{f}_e^j \end{pmatrix} \\ &= W_N \begin{pmatrix} f \\ 0 \end{pmatrix} + \sum_{j=1}^2 V^j \mathrm{Diag}\left(\widehat{h^j}\right) W_N \begin{pmatrix} f_e^j \\ \tilde{f}_e^j \end{pmatrix}. \end{aligned}$$

After left multiplying by W_N^T, we get

$$\begin{pmatrix} f \\ 0 \end{pmatrix} + \sum_{j=1}^2 W_N^T V^j \mathrm{Diag}\left(\widehat{h^j}\right) W_N \begin{pmatrix} f_e^j \\ \tilde{f}_e^j \end{pmatrix}. \tag{4.10}$$

Let L be the length of f, and denote by $[A]_L$ the first L rows of a matrix A. Then the first L rows of (4.10) yield the expression for the errors involved when carrying out the Schiske reconstruction method in the discrete setting when the convolution matrices are not cyclic:

$$f + \sum_{j=1}^2 \left[W_N^T V^j \mathrm{Diag}\left(\widehat{h^j}\right) W_N \begin{pmatrix} \left(H^j\right)^{-1} H_{12}^j \left(S^j\right)^{-1} H_2 1^j f \\ -\left(S^j\right)^{-} 1 H_{21}^j f \end{pmatrix} \right]_L .$$

5. Algorithms for Solving the MDP and Local Deconvolution

5.1 Algorithms for Solving the MDP in Dimensions 1 and 2

For $p \in \mathbb{N}$, let δ_p be the infinite vector whose only non-zero entry is 1 at the pth entry. The discrete MDP in the one-dimensional setting can be formulated as $H_1[\tilde{h}_1]^T + H_2[\tilde{h}_2]^T + \cdots + H_m[\tilde{h}_m]^T = \delta_p$, where $H_i[\tilde{h}_i]^T = h_i * \tilde{h}_i$ for $i = 1, \ldots, m$.

In the two-dimensional case, the matrix $\tilde{h}_i$ is turned into a lexicographically ordered column vector by stacking the columns. Specifically, if h_i is an $n_i \times m_i$ matrix with columns $h_i^1\ h_i^2\ \ldots\ h_i^{m_i}$ and $\tilde{h}_i$ is an $\tilde{n}_i \times \tilde{m}_i$ matrix, then $h_i * \tilde{h}_i$ can be written as $H_i[\tilde{h}_i]$, where H_i is the Toeplitz-block-Toeplitz matrix of size $(n_i + \tilde{n}_i - 1)(m_i + \tilde{m}_i - 1) \times \tilde{n}_i \tilde{m}_i$ defined by

$$H_i = \begin{pmatrix} H_i^1 & & & \\ H_i^2 & H_i^1 & & \\ \vdots & H_i^2 & \ddots & \\ H_i^{m_i} & \vdots & & H_i^1 \\ & H_i^{m_i} & & H_i^2 \\ & & \ddots & \vdots \\ & & & H_i^{m_i} \end{pmatrix} .$$

Here H_i^j is the one-dimensional convolution matrix of the column vector h_i^j and $[\tilde{h}_i]$ is the vector formed by concatenating the columns of $\tilde{h}_i$. We then consider the linear system

$$\begin{pmatrix} H_1\ H_2\ \ldots H_m \end{pmatrix} \begin{pmatrix} [\tilde{h}_1]^T \\ [\tilde{h}_2]^T \\ \vdots \\ [\tilde{h}_m]^T \end{pmatrix} = \delta_p .$$

Remove all entries not included in the intended support of the deconvolvers, the corresponding elements of $(H_1\ H_2\ \ldots H_m)$ and rows of δ_p. Denote this new system by $H\tilde{h} = \delta_p'$, where H is an $l \times k$ matrix of rank l. Of course, we are assuming that the convolvers h_i are chosen so that this matrix system is consistent with a coefficient matrix of rank l. For instance, when choosing characteristic functions as convolvers (i.e., square matrices of 1's where the trailing zeros are ignored), a sufficient condition is that the orders of these matrices be pairwise relatively prime by Theorem 10 for the three-channel case. This is due to the fact that one can derive the two-dimensional solution from the three-channel case (see [37]).

Note that this finite realization puts a constraint on the location of the non-zero entry of δ_p because we cannot allow δ_p' to be the zero vector. We now solve for $\tilde{h}$ by means of the pseudoinverse of H, i.e., $\tilde{h} = H^T(HH^T)^{-1}\delta_p'$, and extract the deconvolvers incorporated in $\tilde{h}$.

To be more specific about the matrix formulation in the two-dimensional case, let us consider the case of three channels. Let h_i, $1 \leq i \leq 3$, be described by an $n_i \times n_i$ matrix and let $s \in \mathbb{Z}$, be such that $s > 1 - n_1 - n_2 - n_3$. Then $\tilde{h}_1$, $\tilde{h}_2$, and $\tilde{h}_3$ are square matrices of order $(n_2 + n_3 + s)$, $(n_1 + n_3 + s)$, and $(n_1 + n_2 + s)$, respectively. H_1 is the convolution matrix of h_1 to be convolved with a square matrix of order $(n_2 + n_3 + s)$. The matrices H_2 and H_3 are formed similarly. So $H = (\, H_1\, H_2\, H_3)$ and $\delta_p^{'}$ is a column vector of length $(n_1 + n_2 + n_3 + s - 1)^2$ containing only one nonzero entry. $\tilde{h}_1$ is formed by taking rows 1 through $(n_2 + n_3 + s)^2$ and the resulting vector is reshaped into a square matrix of order $(n_2 + n_3 + s)$. $\tilde{h}_2$ is formed by taking rows $(n_2 + n_3 + s)^2 + 1$ through $(n_2 + n_3 + s)^2 + (n_1 + n_3 + s)^2$ and the resulting vector is reshaped into a square matrix of order $(n_1 + n_3 + s)$. Finally, $\tilde{h}_3$ is formed by taking rows $(n_2 + n_3 + s)^2 + (n_1 + n_3 + s)^2 + 1$ through $(n_2 + n_3 + s)^2 + (n_1 + n_3 + s)^2 + (n_1 + n_2 + s)^2$ and the resulting vector is reshaped into a square matrix of order $(n_1 + n_2 + s)$. The parameter s indicates the amount of extra length that can be added to the smallest possible lengths of the deconvolvers.

The solution $\tilde{h}$ found by the pseudoinverse of H is the solution of smallest ℓ^2-norm [11].

To understand the reason we are interested in finding the solution with smallest ℓ^2-norm, let us recall the reconstruction method. For the two-channel case, we are given $s_1 = H_1 f + \epsilon_1$ and $s_2 = H_2 f + \epsilon_2$, where ϵ_1 and ϵ_2 are white Gaussian noise vectors. Thus,

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} f + \epsilon \,,$$

where $\epsilon = (\epsilon_1\, \epsilon_2)^T$. Form the matrix $(\tilde{H}_1\, \tilde{H}_2)$, where the convolution matrices $\tilde{H}_1$ and $\tilde{H}_2$, are formed from $\tilde{h}_1$ and $\tilde{h}_2$, respectively. So

$$\big(\tilde{H}_1\, \tilde{H}_2\big) \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = f + \big(\tilde{H}_1\, \tilde{H}_2\big)\epsilon \,.$$

The expectation of the error in the reconstruction is $E[\|(\tilde{H}_1\, \tilde{H}_2)\epsilon\|_2^2] = \operatorname{trace}((\tilde{H}_1\, \tilde{H}_2)\ (\tilde{H}_1\, \tilde{H}_2)^T)$. If the lengths of f, h_1, and h_2 are L, n, and m, respectively, then

$$E\Big[\big\|\big(\tilde{H}_1\, \tilde{H}_2\big)\epsilon\big\|_2^2\Big] = \big(n + L - 1\big)^2 \big\|\tilde{h}_1\big\|_2^2 + \big(m + L - 1\big)^2 \big\|\tilde{h}_2\big\|_2^2 \,.$$

This means that the deconvolvers that are optimal in terms of noise performance are the solutions $\tilde{h}_1$ and $\tilde{h}_2$ with smallest ℓ^2-norm. But this objective is achieved by finding the solution $\tilde{h} = (\tilde{h}_1 \tilde{h}_2)^T$ of smallest ℓ^2-norm since $\|\tilde{h}\|_2^2 = \|\tilde{h}_1\|_2^2 + \|\tilde{h}_2\|_2^2$.

If we express H in terms of its singular value decomposition, $H = U \Sigma V^T$, then the pseudoinverse of H is $H^+ = V \Sigma^{-1} U^T$. This formulation has numerical advantages when the conditioning of the matrix involved is large because it avoids having to find the inverse of $H H^T$, whose conditioning is $\kappa(H H^T) = \kappa(H)^2$.

Remark 3. In the continuous case for the characteristic convolvers, the supports of the deconvolvers were determined by the only sampling rates that were possible on the Fourier transform axis [36]. The same principle was used in the case of the discrete characteristic convolvers. Since the support of a convolution of two finitely supported functions is the sum of the supports, one possible choice for the support of the deconvolvers is the choice that yields equal support lengths for all convolution terms. As in the characteristic case, in the general convolver case we chose the supports of each expected deconvolver $\tilde{h}_i$ to be the sum of the supports of all the convolvers h_j for $j \neq i$.

There are cases when the support length is smaller than the length we chose. For instance, if the convolvers $\{h_i\}_{i=1}^p$, have the same support length L, then there exist $\{\tilde{h}_i\}_{i=1}^p$ of support length greater than or equal to the ceiling of $\frac{L-1}{p-1}$ (cf. [21]).

There is an upper bound to the lengths of the supports by the following result.

Theorem 11 ([3] and [9]).

Let $\{f_i\}_{i=1}^p \in \mathbb{C}[x_1, \ldots, x_n]$ *with no common zeros. Then there exist polynomials* $g_1, \ldots, g_p$ *such that* $\sum_{i=1}^p f_i g_i = 1$ *and* $\deg(g_i) \leq 2(2d)^{2^{n-1}}$*, where* $d = \max\limits_{1\leq i\leq p} \deg(f_i)$.

5.2 Local Deconvolution

We wish to clarify an aspect in the discrete setting that allows us to recover lost segments, a phenomenon that is not possible by traditional reconstruction methods.

Consider the case when the discrete convolutions are truncated. Assume the length of f is N and that only N values of $\{s_i\}_i$ are known. (We can also consider the number of known values of s_i to be smaller than N and recover the values of f by our method). Thus, $s_i = H_{i,N} f$, where $H_{i,N}$ is the convolution matrix of h_i to be multiplied by a vector of length N. The superscript (a_i, b_i) means that rows a_i through b_i are taken from $H_{i,N}$, where the number of rows from a_i to b_i is N.

Consider the two-channel problem where h_1 has length n, h_2 has length m, and $n < m$. Set $a_1 = 1 + \lceil (n-1)/2 \rceil$, $b_1 = N + n - \lfloor (n-1)/2 \rfloor$, $a_2 = 1 + \lceil (m-1)/2 \rceil$, and $b_2 = N + m - \lfloor (m-1)/2 \rfloor$, where $\lceil x \rceil$ and $\lfloor x \rfloor$ are the ceiling and floor of x, respectively. Form truncated convolver matrices $\tilde{H}_{i,N}^{(c_i,d_i)}$ from $\tilde{h}_i$ with $c_1 = 1 + \lceil (m-1)/2 \rceil$, $d_1 = N+m-\lfloor (m-1)/2 \rfloor$, $c_2 = 1+\lceil (n-1)/2 \rceil$, and $d_2 = N+n-\lfloor (n-1)/2 \rfloor$. The reconstruction is carried out by taking $\tilde{H}_{1,N}^{(c_1,d_1)} s_1 + \tilde{H}_{2,N}^{(c_2,d_2)} s_2$. To determine what coordinates of f are recovered, we analyze $\tilde{H}_{1,N}^{(c_1,d_1)} H_{1,N}^{(a_1,b_1)} + \tilde{H}_{2,N}^{(c_2,d_2)} H_{2,N}^{(a_2,b_2)}$. Recalling that the deconvolvers depend on the value of k corresponding to the position of the 1 in δ_k, the resulting matrix Δ_k is a banded matrix of the form

$$\begin{pmatrix}
d_{1,1} & \ldots & d_{1,p} & \delta_k[1] & & & & \\
\vdots & & \delta_k[2] & & \delta_k[1] & & & \\
d_{p,1} & & & \delta_k[2] & & \ddots & & \\
\delta_k[L] & & & & \ddots & & \delta_k[1] & \\
& \delta_k[L] & & & & \delta_k[2] & d_{N-p,N} & \\
& & \ddots & & & & \vdots & \\
& & & \delta_k[L] & d_{N,N-p} & \ldots & d_{N,N} &
\end{pmatrix},$$

where $L = n+m-1$, $p = \lceil (m-1)/2 \rceil + 1$, and $d_{i,j} \in \mathbb{R}$. The result of the reconstruction is $\Delta_k f$. For example, if $f(i) = i$ for $i = 1, \ldots, 6$, $h_1 = 1_5$, $h_2 = 1_3$, then

$$\Delta_1 = \begin{pmatrix} -1 & -\frac{3}{8} & 0 & 1 & 0 & 0 \\ \frac{3}{8} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{5}{8} \\ 0 & 0 & 0 & 0 & -\frac{3}{8} & 0 \end{pmatrix} .$$

The row containing the ith entry 1 and all other entries 0 allows us to recover $f(i)$. Thus, $f(6) = 6$ is recovered in the reconstruction from Δ_1. Changing the values of k, we can recover the other coordinates of f. Eventually, we find

$$\Delta_7 = \begin{pmatrix} 0 & -\frac{3}{8} & 0 & 0 & 0 & 0 \\ -\frac{5}{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{3}{8} \\ 0 & 0 & 1 & 0 & -\frac{3}{8} & -1 \end{pmatrix}$$

which yields $f(1) = 1$.

It is still possible to recover all the coordinates of f if the convolution matrices are truncated further. If we remove the first row from all the convolver and deconvolver matrices in the above example, the resulting matrix is the old Δ_k with the first row removed. In this case, s_1 and s_2 are vectors of length 5 and, by changing values of k, we can still recover f, a vector of length 6. Thus, this reconstruction method allows us to recover segments that could not be recovered by other traditional methods. For instance, if the system $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} H_{1,6}^{(a_1,b_1)} \\ H_{2,6}^{(a_2,b_2)} \end{pmatrix} f$ is solved by the pseudoinverse solution, only 5 approximate coordinates of f can be recovered.

5.3 Regularized Local Multichannel Deconvolution

We expect to use a de-noising routine directly after the reconstruction by the local deconvolvers in order to improve performance of the reconstruction method. If the noise level of the blurred images is very high, however, the reconstruction tends to also have a high level of noise. A better approach is to perturb the deconvolvers at a cost of slightly approximating the original image to improve noise performance and then apply a good de-noising routine. Noting the shortcomings in the deconvolution techniques suggested in [14] and [20], Neelamini, Choi, and Baraniuk in [25] suggested a type of hybridization consisting of applying first a Wiener filter with a moderate bias and then a wavelet-based de-noising routine. Because of practical constraints such as analog and digital approximation and computation time, the MDP was slightly modified by Berenstein and coauthors in [1]. Instead of trying to solve for deconvolvers that would yield an ideal reconstruction, their aim was to solve

$$h_1 * \tilde{h}_1 + \cdots + h_m * \tilde{h}_m = \psi \ , \tag{5.1}$$

where $\psi \in C_c^\infty(\mathbb{R})$. The function ψ was then modified, for example by dilation, so that $\psi \to \delta$, as supp $\psi \to \{0\}$. In this case, the reconstructed signal $f * \psi$ is an arbitrarily close approximation of f.

One of the advantages of modifying the problem in this way is that the deconvolvers are now functions (as opposed to distributions) and hence, numerically likely to be smoother. For example, if the h_i are integrable functions, then the $\tilde{h}_i$ cannot be integrable functions, otherwise $\sum_{i=1}^m h_i * \tilde{h}_i$ would be an integrable function, and thus, not a δ distribution.

But when the sampling rate is such that the deconvolvers have relatively small support lengths, a sampled version of ψ is inadequate. Instead we use a computational trick for deriving an appropriate δ approximation. We found that if H is the MDP matrix, then for some $\lambda \geq 0$, $\tilde{h} = H^T(HH^T + \lambda I)^{-1}\delta_k$ are in a sense solutions to (5.1).

The matrix regularized solution can be viewed as a solution to the least-squares formulation for an underdetermined system, i.e., find $\tilde{h}$ that minimizes $\lambda^{-1}\|H\tilde{h} - \delta_k\|_2^2 + \|L\tilde{h}\|_2^2$, where L is a regularized operator that represents constraints on the solution $\tilde{h}$. Let us consider the case when $L = I$. Differentiating with respect to $\tilde{h}$ and setting the derivative equal to zero yields the standard least squares solution $\tilde{h} = (H^T H + \lambda I)^{-1} H^T \delta_k$. Using $I - H^T(HH^T + \lambda I)^{-1}H = \lambda(H^T H + \lambda T)^{-1}$, we obtain

$$\begin{aligned}
\tilde{h} &= \lambda^{-1}\Big(I - H^T\big(HH^T + \lambda I\big)^{-1}H\Big)H^T\delta_k \\
&= \lambda^{-1}H^T\Big(I - \big(HH^T + \lambda I\big)^{-1}HH^T\Big)\delta_k \\
&= \lambda^{-1}H^T\Big(I - \big(HH^T + \lambda I\big)^{-1}\big(HH^T + \lambda I\big) + \big(HH^T + \lambda I\big)^{-1}\lambda I\Big)\delta_k \\
&= H^T\big(HH^T + \lambda I\big)^{-1}\delta_k \, .
\end{aligned}$$

This method, called **Tikhonov regularization**, is known to be the stochastically optimal regularization method if the solution is reasonably bounded [26] and is equivalent to filtering the singular values $1/\sigma_i$ in the pseudoinverse solution by multiplying them by $\sigma_i^2/(\sigma_i^2 + \lambda)$.

If the noise level is unknown, there is a simple method for estimating the best value for λ. Typical methods for estimating the regularization parameter λ are either generalized cross validation or by means of the L-curve (see [26] for further details and references). Since the method of reconstruction we consider is local in nature, we can find out how well the regularization works by considering how it performs on a more global scale. To be specific, let f_λ be the regularized reconstruction of the underlying image. Since $s_1 = h_1 * f$, the regularized solution $\tilde{h}$ is best when $\|f_\lambda * h_1 - s_1\|$ is minimized. Additional information from the other channels could be used when trying to find the best value of λ, yet we found no numerical difference.

6. Numerical Results and Illustrations

We use as our discrete point spread function the $n \times n$ matrix $\vec{1}_n$ whose entries are all $1/n^2$. This is commonly used to model out-of-focus blur [17].

The **power** of an $m \times n$ image matrix I is given by

$$P(I) = \sum_{i=1}^{m}\sum_{j=1}^{n}\big(I_{i,j} - \mu_I\big)^2 \, ,$$

where μ_I is the mean of all pixel values.

Let N be the additive Gaussian noise with zero mean and standard deviation σ_{noise}. Denote by $S_{\text{noise}} = S + N$ the received blurred image, where S is the blurred image without noise. The **blurred signal-to-noise ratio** (BSNR) of S_{noise} in decibels (dB) is defined by $BSNR = 10 \log_{10} \frac{P(S)}{P(N)}$.

Let O be the original image and let O_{res} be the restored image from S_{noise}. To analyze the performance of the method we shall use the **peak signal-to-noise ratio** (PSNR) in decibels (dB):

$$PSNR = 20 \log_{10} \left(255\sqrt{nm} / \| O - O_{\text{res}} \|_F \right) ,$$

where $\| \cdot \|_F$ is the Frobenius norm.

In Figure 1, we demonstrate the two-dimensional local deconvolution algorithm and how we can gain an extended boundary. All three blurred images were truncated to be the same size after convolution with the respective point spread functions. We reconstructed the upper left-hand portion of the blurred image by applying the deconvolvers that were found by carrying out the above algorithm for the appropriately chosen value p of δ'_p. Recall that δ'_p represents, a lexicographically ordered delta matrix ordered by stacking either the rows or columns so that the chosen value of p corresponds to the nonzero point of the delta matrix in the lower right-hand corner of the matrix. By changing the value of p we can reconstruct other parts of the original image, as shown. By assembling the parts together we obtain an image that is larger in size than the blurred images. If the sizes of the convolvers are $n_1 \times n_1, n_2 \times n_2, n_3 \times n_3$, with $n_1 < n_2 < n_3$ and the blurred images are all of size $n \times n$, then the reconstructed image will be of size $(n + n_1 - 1) \times (n + n_1 - 1)$, independent of the support parameter. For this particular demonstration, we took the convolvers to be $\vec{1}_9$, $\vec{1}_{11}$, and $\vec{1}_{13}$ (so that $n_1 - 1 = 8$). If we reconstruct this extended boundary by applying Schiske's method, we get the result presented at the bottom right of Figure 1.

In Figure 2, we demonstrate the performance of the reconstruction method when noise is incorporated. We can see a large improvement in noise performance when the regularization method proposed earlier is used with the estimated value for λ found by our estimation procedure. We then apply a translation invariant, or equivalently, an undecimated wavelet de-noising routine to the regularized reconstructed image shown at the bottom right of Figure 2 (see [24] for details and `http://www-dsp.rice.edu/software/rwt.shtml` for available software that does this type of denoising).

Table 2.1 gives the numerical data on the experimental results of the above reconstruction for the image of "Lena" and two other well-known images of "Cameraman" and "Barbara." In our experiments we obtained the following BSNR levels in dB. We used as convolvers $\vec{1}_5$, $\vec{1}_7$, and $\vec{1}_9$.

TABLE 2.1

	BSNR	BSNR	BSNR
Lena	25.89	25.59	25.30
Cameraman	28.28	28.09	27.92
Barbara	25.65	25.45	25.27

The following are the numerical results of the reconstructions in terms of their peak signal to noise ratios.

FIGURE 1 On top are the blurred images used for the reconstruction. The center images are the partial reconstructions. At the bottom left is the total reconstructed image. At the bottom right is the reconstructed image from Schiske's method.

	$\lambda = 0$	$\lambda = 0.0045$	UWT denoised	Schiske
Lena	3.41	26.45	29.42	10.53
Cameraman	2.19	24.96	28.47	8.90
Barbara	2.25	23.55	24.55	8.92

Finally, we wish to point out some similarities between our algorithm and the algorithm presented in [18]. Ours was inspired by the solutions found for the case of the characteristic convolvers. The algorithm presented in [18] assumes that all convolvers can be embedded into a larger vector or matrix. The authors assume the convolvers of equal support length, since the main interest in the algorithm is for the blind MDP. We found, however, that when we assume that the characteristic convolvers are represented as matrices of equal support size, the matrix equation involved becomes singular. Ordinarily, one would expect an improvement in noise performance when the support length of the deconvolvers (i.e., the value of s) is increased. Yet, the level of improvement is not at all comparable to how well the regularization technique works. In fact, we found that if we choose different values of s and find the best value for λ in all cases when $s \neq 0$ using the noisy blurred images of Lena mentioned above, we get the following results.

FIGURE 2 On top are the blurred noisy images. At the center left is the image derived from the local reconstruction without regularization. At the center right is the image from the regularized reconstruction. At the bottom left is the denoised regularized reconstructed image. At the bottom right is the reconstructed image from Schiske's method.

	$s=-2$	$s=0$	$s=2$	s=5	s=8
PSNR	23.66	29.44	16.41	19.62	17.89

Thus, the best performance occurs when $s = 0$.

Acknowledgments

Part of this work was done by the second author in his doctoral dissertation [15]. We wish to thank David Walnut for his very helpful suggestions.

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Received June 22, 2002

Revision received February 15, 2004

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