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Cyclic Functions in $L^p(\mathbb{R})$, $1 \leq p < \infty$

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ABSTRACT. Functions whose translates span $L^p(\mathbb{R})$ are called L^p-cyclic functions. For a fixed $p \in [1, \infty)$, we construct Schwartz-class functions which are L^r-cyclic for $r > p$ and not L^r-cyclic for $r \leq p$. We then construct Schwartz-class functions which are L^r-cyclic for $r \geq p$ and not L^r-cyclic for $r < p$. The constructions differ for $p \in (1, 2)$ and $p > 2$.

1. Introduction

Functions whose translates span $L^p(\mathbb{R})$ are called L^p-cyclic functions. Equivalently, $f \in L^p(\mathbb{R})$ is cyclic if and only if the only $g \in L^q(\mathbb{R})$ such that $f * g = 0$ is $g = 0$ (p and q are conjugate exponents). In the 1920s and 1930s, Wiener proved two classic results about cyclic functions in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ [19]. If $f \in L^1(\mathbb{R})$, then f is cyclic in $L^1(\mathbb{R})$ if and only if the Fourier transform $\hat{f}$ is never zero, and if $f \in L^2(\mathbb{R})$, then f is cyclic in $L^2(\mathbb{R})$ if and only if $\hat{f}$ is zero only on a set of Lebesgue measure zero. Wiener's theorems make it possible to characterize cyclic functions completely for $p = 1$ and $p = 2$; however, no complete characterization exists for L^p-cyclic functions when $1 < p < 2$ and $p > 2$. Attempts at classifying L^p-cyclic functions have been made throughout the twentieth century; the work of Bary, Segal, Beurling, Salem, Kahane, Herz, Edwards, and many others has touched on this problem. The partial classifications which exist involve zero sets of Fourier transforms. In fact, a complete characterization of cyclic functions in terms of Fourier transforms may not be possible, since even defining the zero set of a Fourier transform is a delicate task when a function is not in $L^1(\mathbb{R})$. However, when the zero set is well-defined, the critical issue is whether a zero set can support a distribution whose Fourier transform belongs to $L^q(\mathbb{R})$, where q is the conjugate index to p. Different

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names are given to zero sets which cannot support distributions with this property: sets of type U^q, q-thin sets, q-negligible sets, and sets of L^q-uniqueness.

If $f \in L^p(\mathbb{R}) \cap L^r(\mathbb{R})$, where $1 \leq p \leq r < \infty$ and f is L^p-cyclic, then f is L^r-cyclic. See [7] and [5] for two different approaches. A natural first question to ask in response to this feature of cyclic functions is if it is possible to construct a function in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ which is not cyclic in $L^r(\mathbb{R})$ for $r < p$ but cyclic in $L^r(\mathbb{R})$ for $r > p$. We answer the first question (Theorem 3) and then show that one of the strict inequalities can always be replaced with $\leq$ or $\geq$ (Theorems 4, 6, 10, and 11).

2. Useful Constructions and Definitions

2.1 Smooth Functions with Specified Zero Sets

We construct functions in the Schwartz class which are zero *only* on specified compact sets.

Lemma 1.

Let $K \subset \mathbb{R}$ be a compact set. There exists $\phi \in \mathcal{S}(\mathbb{R})$ such that $\phi(x) = 0$ for all $x \in K$ and $\phi(x) > 0$ for all $x \in \mathbb{R}\backslash K$.

Proof. Let $k_1 = \min\{x : x \in K\}$ and let $k_2 = \max\{x : x \in K\}$ and let U denote the complement of K in $[\,k_1, k_2\,]$. Since U is open, U is a countable disjoint union of open intervals. For each $n \in \mathbb{Z}^+$, there are only finitely many intervals in U with length greater than $1/n$. Order these intervals by non-increasing length: $|I_n| \geq |I_{n+1}|$.

If $I = (a, b)$, define $\tilde{f}(x)$ by

$$\tilde{f}(x) = \begin{cases} e^{-1/(1-g(x)^2)} & x \in (a,b) \\ 0 & x \notin (a,b) \end{cases} \tag{2.1}$$

where $g(x) = \frac{2(x-a-(b-a)/2)}{b-a}$. Then $\tilde{f} \in \mathcal{S}(\mathbb{R})$ and the support of f and all its derivatives is the closed interval $[\,a, b\,]$. Folland constructs this function on $[-1, 1]$ [8, p. 228–230]; the dilation and translation shift the support to $[\,a, b\,]$. Set

$$s_{d,k} = \sup_{x \in I_k} \ \tilde{f}_k^{(d)}(x) \qquad \text{and} \qquad c_n = \max_{d,k \leq n} \ s_{d,k}\ ,$$

and define functions

$$f_k = k^{-1} c_k^{-1} \tilde{f}_k, \qquad \phi_n = \sum_{k=1}^{n} f_k, \qquad \text{and} \qquad \phi = \lim_{n \to \infty} \phi_n\ . \tag{2.2}$$

For a fixed n,

$$\sup_{x \in C} \ |\phi(x) - \phi_n(x)| = \sup_{x \in C} \ \left| \ \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{n} f_k(x) \ \right| \leq \frac{1}{n+1}\ , \tag{2.3}$$

which implies that $\phi_n \to \phi$ uniformly. For $d \leq n$, we have

$$\sup_{x \in C} \ \left| \sum_{k=1}^{\infty} f_k^{(d)}(x) - \phi_n^{(d)}(x) \right| = \sup_{x \in C} \ \left| \sum_{k=n+1}^{\infty} f_k^{(d)} \right| \leq \frac{1}{n} \tag{2.4}$$

because we have bounded the first j derivatives of f_j by $\frac{1}{j}$. The dth derivative $\phi^{(d)}$ therefore exists and equals the sum of the dth derivatives by [1, Theorem 9.13].

Define g by

$$g(x) = \begin{cases} e^{-1/(-x-k_1)} e^{-(x-k_1)^2} & x < k_1 \\ \phi(x) & x \in [k_1, k_2] \\ e^{-1/(x-k_2)} e^{-(x-k_2)^2} & x > k_2 \, . \end{cases} \tag{2.5}$$

Then $g = 0$ precisely on K, and $g \in \mathcal{S}(\mathbb{R})$ because the functions $e^{-1/(-x-k_1)}$ and $e^{-1/(x-k_2)}$ are bounded by 1. □

2.2 Distributions and Their Supports

We work with distributions whenever we take the Fourier transform of a function $f \in L^p$, $p > 2$. As a result, we state some basic facts about distributions on $\mathbb{R}$ which we will use later.

The support of a measure ν is the complement of the union of all ν-null sets. This definition agrees with the following definition of the support of a distribution.

Definition 1. The closed set K is the support of the distribution D provided that for every open set U such that $U \cap K = \emptyset$ and for every compactly supported smooth function ϕ such that $\operatorname{supp}(\phi) \subset U$, $D(\phi) = 0$.

Borel measures provide examples of distributions; another class of distributions we will use is the class of pseudo-measures. Pseudo-measures are obtained by taking Fourier transforms of bounded functions. An important property of pseudo-measures and measures is the following.

Lemma 2 ([11]).
Suppose ν is a pseudo-measure and $f \in L^1$. Then

$$\operatorname{supp}\left(\hat{f}\nu\right) \subseteq \operatorname{supp}\left(\hat{f}\right) \cap \operatorname{supp}(\nu) \, .$$

3. Cyclic Functions in $L^p(\mathbb{R})$, $p \in (1, 2)$

3.1 Functions Spanning $L^r(\mathbb{R})$, $r > p$, but not $L^r(\mathbb{R})$, $r < p$

Because $L^1(\mathbb{R})$-cyclic functions must have Fourier transforms which never vanish and $L^2(\mathbb{R})$-cyclic functions can have Fourier transforms which vanish on sets of measure zero, we need a way to compare sizes of sets, all of which have Lebesgue measure zero. Suppose $f \in \mathcal{S}(\mathbb{R})$. Since f has a continuous Fourier transform, the closed set $\mathcal{Z}(f) = \{\xi : \hat{f}(\xi) = 0\}$ is well-defined. If, for example, $f \in L^2$, the zero set of $\hat{f}$ would be defined up to a set of measure zero, which is too "big" for what we are about to do.

Theorem 1 (Beurling, p. 154, [5]).
If for some $p \in (1, 2)$ the linear span of translates of f is not dense in $L^p(\mathbb{R})$, then the Hausdorff dimension of $\mathcal{Z}(f)$, $\dim_H(\mathcal{Z}(f))$, is at least $2 - \frac{2}{p}$.

Stated another way, if $p \in (1,2)$ and $\dim_H(\mathcal{Z}(f)) < 2 - \frac{2}{p}$, then f is cyclic in $L^p(\mathbb{R})$. Beurling's proof shows that if K is compact and $\dim_H(K) = \alpha$, then K cannot support a distribution D with $\hat{D} \in L^q$ for any $q < 2/\alpha$. See [6] for more details. We also use the following theorem of Salem.

Theorem 2 (Theorems 1 and 4, [17]).

Suppose $\alpha \in (0,1)$ and $\varepsilon > 0$.

1. *There exists a measure μ whose support E_μ is a perfect set with $\alpha = \dim_H(E_\mu)$ and $\hat{\mu} \in L^q(\mathbb{R})$ for all $q \geq \frac{2}{\alpha} + \varepsilon$.*
2. *There exists a measure ν whose support E_ν is a perfect set with $\alpha = \dim_H(E_\nu)$ and $\hat{\nu} \in L^q(\mathbb{R})$ for all $q > \frac{2}{\alpha}$.*

Lemma 3.

Suppose $p \in (1,2)$, μ is a measure whose support E_μ is a perfect set with $\dim_H(E_\mu) = 2 - \frac{2}{p}$, and $\check{\mu} \in L^q(\mathbb{R})$ for all $q > p'$. Suppose $f \in \mathcal{S}(\mathbb{R})$ and $\mathcal{Z}(f) = E_\mu$. Then f is not cyclic in $L^r(\mathbb{R})$ for $r < p$ and f is cyclic in $L^r(\mathbb{R})$ for $r > p$.

Proof. We first show that f is not cyclic in $L^r(\mathbb{R})$ for $r < p$. Consider the convolution $f * \check{\mu}$. At the outset, the only thing we can say about this convolution is that $f * \check{\mu} \in L^q(\mathbb{R})$, where $q > p'$. A Fubini–Tonelli argument shows that

$$f * \check{\mu}(x) = \int \hat{f}(t) e^{2\pi i x t} d\mu(t) \,, \tag{3.1}$$

so that the convolution $f * \check{\mu}$ is the inverse Fourier transform of the bounded Borel measure $\hat{f}\mu$. The total variation of $\hat{f}\mu$ is zero, so the convolution $f * \check{\mu}$ is zero. Therefore, $\check{\mu}$ is orthogonal to all translates of f, and the function f cannot be cyclic in $L^r(\mathbb{R})$ for any $r < p$. On the other hand, by Theorem 1, f must be cyclic in $L^r(\mathbb{R})$ for $r > p$ since the Hausdorff dimension of $\mathcal{Z}(f)$, $2 - \frac{2}{p}$, is strictly less than $2 - \frac{2}{r}$. □

Theorem 3.

Let $p \in (1,2)$. There exists a function $f \in \mathcal{S}(\mathbb{R})$ which is cyclic in $L^r(\mathbb{R})$ for $r > p$ and which is not cyclic in $L^r(\mathbb{R})$ for $r < p$.

Proof. Let $\alpha = 2 - \frac{2}{p}$ and let $K = E_\mu$ be a compact set with Hausdorff dimension α which supports the measure μ. Apply Lemma 1 to construct $f \in \mathcal{S}(\mathbb{R})$ which is zero precisely on K. By Lemma 3, the inverse Fourier transform $\check{f}$ is an example of a function which is cyclic in $L^r(\mathbb{R})$ for $r > p$ and which is not cyclic in $L^r(\mathbb{R})$ for $r < p$. □

3.2 Functions Spanning $L^r(\mathbb{R})$, $r \geq p$, but not $L^r(\mathbb{R})$, $r < p$

Salem's results do not immediately address what happens at the critical index $\frac{2}{\alpha}$. The next construction will allow us to build a set which cannot support a pseudo-measure whose Fourier transform is in $L^{\frac{2}{\alpha}}(\mathbb{R})$.

Choose $\alpha \in (0,1)$. Choose $\{\alpha_m\}$, a sequence in $(0,1)$, with each $\alpha_m < \alpha_{m+1} < \alpha$ and $\lim_{m\to\infty} \alpha_m = \alpha$. For each α_m, use Salem's method to construct a compact set $E_m \subset (\frac{1}{m+1}, \frac{1}{m})$ so that $\dim_H(E_m) = \alpha_m$, and such that E_m supports a non-zero positive regular Borel measure μ_m such that $\widehat{\mu_m} \in L^s$ for all $s > \frac{2}{\alpha_m}$. Let $E = \{0\} \cup \bigcup_{m=1}^{\infty} E_m$. The set

E is compact by construction. Clearly, $\dim_H(E) \geq \alpha_m$ for all $m \geq 1$ and $\dim_H(E) = \alpha$ because $\text{meas}_{H,\alpha}(E_m)$, the Hausdorff measure of order α, is 0 for all $m \geq 1$.

For any $s > \frac{2}{\alpha}$, we have $s > \frac{2}{\alpha_m}$ for sufficiently large m. Hence, E_m and thus E contains the support of a non-zero pseudo-measure whose Fourier transform g is in $L^s(\mathbb{R})$. On the other hand, E cannot support a pseudo-measure whose Fourier transform is in $L^{\frac{2}{\alpha}}$. Indeed, suppose $g \in L^{\frac{2}{\alpha}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\hat{g}$ is supported in E. Take a function $\psi_m \in \mathcal{S}$, such that $\widehat{\psi_m} = 1$ on E_m and $\widehat{\psi_m} = 0$ on $E \backslash E_m$. Then $g * \psi_m$ is in $L^{\frac{2}{\alpha}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ too and by Lemma 2, its Fourier transform is supported on E_m. But $g * \psi_m \in L^{\frac{2}{\alpha}}$. By the proof of Beurling's theorem, E_m cannot support a distribution with Fourier transform in L^q for any $q < \frac{2}{\alpha_m}$. Since $\frac{2}{\alpha} < \frac{2}{\alpha_m}$, $g * \psi_m = 0$. It follows that the support of $\hat{g}$ is contained in $E \backslash E_m$. Since this is true for all m, the support of $\hat{g}$ is contained in $\{0\}$. But $g \in L^{\frac{2}{\alpha}}(\mathbb{R})$, so $g = 0$.

Theorem 4.

Let $p \in (1, 2)$ and let $\alpha = 2 - \frac{2}{p}$, so that $\frac{2}{\alpha}$ is the conjugate index to p. There exists $f \in \mathcal{S}(\mathbb{R})$ such that f is cyclic in $L^r(\mathbb{R})$, $r \geq p$ and such that f is not cyclic in $L^r(\mathbb{R})$, $r < p$.

Proof. Let E be the compact set Hausdorff dimension α constructed above. Let $f \in \mathcal{S}$ be a function such that the zero set of $\hat{f}$ is E. Because of the support properties of E above, f is cyclic in $L^r(\mathbb{R})$ for all $r \geq p$. Indeed, fix $r \geq p$ and let s be conjugate to r. Then $s \leq \frac{2}{\alpha}$. Let $g \in L^s(\mathbb{R})$ with $f * g = 0$. Let $\phi \in \mathcal{S}$ be arbitrary. We have $f * g * \phi = 0$. Since $g * \phi \in L^r(\mathbb{R}) \cap L^\infty(\mathbb{R})$ too, $g * \phi$ has a Fourier transform which is a non-zero pseudo-measure supported in E by Lemma 2. But then the result about E above shows that $g * \phi = 0$. Since ϕ is arbitrary, this means $g = 0$. Hence, f is cyclic in $L^r(\mathbb{R})$ for any $r \geq p$. On the other hand, f is not cyclic in $L^r(\mathbb{R})$ for any $r < p$. Let s be the conjugate index to r; then $s > \frac{2}{\alpha}$. Hence, E contains the support of a non-zero pseudo-measure whose Fourier transform is in $L^s(\mathbb{R})$. Therefore, f cannot be cyclic in $L^r(\mathbb{R})$ since $\hat{f}$ vanishes on E. □

3.3 Functions Spanning $L^r(\mathbb{R})$, $r > p$, but not $L^r(\mathbb{R})$, $r \leq p$

In the previous section, we constructed sets with Hausdorff dimension α which cannot support a pseudo-measure whose Fourier transform is in $L^{\frac{2}{\alpha}}(\mathbb{R})$. In this section, we describe a class of sets with Hausdorff dimension α which support measures whose Fourier transforms are in $L^{\frac{2}{\alpha}}(\mathbb{R})$. The constructions here depend on results of Kahane. We thank Robert Kaufman for showing us how to use these results.

We note that Salem himself seemed to be attempting to prove Theorem 2 in the case $q = \frac{2}{\alpha}$; however, with his probabilistic techniques he was only able to show that if μ is a measure supported on E where $\dim_H(E) = \alpha$, then $\hat{\mu}$ decays like $\Omega(x)x^{-\frac{\alpha}{2}}$ where Ω grows slower than any polynomial.

Let $\mathbb{W}$ denote a Wiener process on $[0, 1]$ and let $\mathbb{W}(E)$ denote the image of $E \subseteq [0, 1]$ under the process $\mathbb{W}$. In other words, $\mathbb{W}(E)$ is the set of all images of E under each W in the sample space of $\mathbb{W}$. We let $W(E)$ denote one of these images.

Suppose $\dim_H(E) = \alpha < \frac{1}{2}$. By [10, Theorem 2, p. 236], the image of E under the Wiener process, $\mathbb{W}(E)$, will almost surely have Hausdorff dimension 2α. If E supports a measure θ with certain growth properties depending on a concave function h, then $W(E)$

almost surely supports a measure $W(\theta)$ such that the growth of $\widehat{W(\theta)}$ is controlled by h as well [10, Lemmas 3 and 4, p. 255]. This measure $\mu = W(\theta)$, called a random measure, is formally defined by

$$\int_{\mathbb{R}} f(x)\, d\mu(x) = \int_0^1 f(\mathbb{W}(t))\, d\theta(t) \ . \tag{3.2}$$

With this definition, the Fourier transform of a measure μ associated with one sample path $W(t)$ is

$$\hat{\mu}(\xi) = \int e^{-2\pi i \xi x} d\mu(x) = \int_0^1 e^{-2\pi i \xi W(t)}\, d\theta(t) \ .$$

Kahane shows

$$\hat{\mu}(\xi) = O\left(\sqrt{\log|\xi| h\left(|\xi|^{-2}\right)}\right)$$

where h is a positive concave function h on E such that E has non-zero h-measure and E supports a measure θ such that $\theta(|I|) \le Ch(|I|)$ for each interval $I \subseteq [0, 1]$.

As a result, we want to find a function h which is of the same order as the function g, where

$$g(x) = \sup_{0 \le u < \infty} \theta(u, u + x) \ , \tag{3.3}$$

because it is not hard to show that the g-measure of E is positive. We also look for a set $E \subseteq [0, 1]$ and a measure θ supported on E which satisfies the inequality $\theta(|I|) \le Ch(|I|)$ for each interval $I \subseteq [0, 1]$. Also, $\sqrt{\log|\xi| h(|\xi|^{-2})}$ must be in $L^{\frac{2}{\alpha}}(\mathbb{R})$. It is possible to satisfy all these conditions using a symmetric Cantor set E. Some of the ideas for this proof are in [13].

Theorem 5.

There exists a set $F \subset \mathbb{R}$ such that $\dim_H(F) = \alpha$ and such that F supports a measure μ with $\hat{\mu} \in L^{\frac{2}{\alpha}}(\mathbb{R})$.

Proof. Let $E \subset [0, 1]$ be a symmetric set of measure zero and let θ be the standard Lebesgue–Stieltjes measure on E. Suppose we have a function concave function h such that the h-measure of E is non-zero. Let $\mu = \mathbb{W}(\theta)$. Then $\hat{\mu} \in L^{\frac{2}{\alpha}}(\mathbb{R})$ provided that

$$h(x) = O\left(x^{\alpha/2} \log^{-2\alpha-1}\left(x^{-1}\right)\right) \ \text{as} \ \ x \to 0 \ . \tag{3.4}$$

Set

$$h(x) = x^{\alpha/2} \log^{-2\alpha-1}\left(x^{-1}\right) \ . \tag{3.5}$$

Then h is concave for small enough positive values of x. The construction of E is controlled by the ratios of dissection $\xi_1, \xi_2, \dots$ where each $\xi_n < 1/2$. We want to choose ξ_n such that $h(\xi_1 \cdot \dots \cdot \xi_n) = O(2^{-n})$ as $n \to \infty$. We look for a formula for ξ_n so that

$$(\xi_1 \cdot \dots \cdot \xi_n)^{\alpha/2} \log^{-2\alpha-1}\left(\left(\xi_1 \cdot \dots \cdot \xi_n\right)^{-1}\right) = O\left(2^{-n}\right) \ \text{as} \ \ n \to \infty \ . \tag{3.6}$$

Since it will be hard to work with $\xi_1 \cdot \dots \cdot \xi_n$ in both factors, we make the estimate $\log(\xi_1 \cdot \dots \cdot \xi_n)^{-1} = O(n)$. Equation (3.6) is valid when

$$\xi_n = 2^{-\frac{2}{\alpha}} \left(1 + \frac{1}{n}\right)^{\frac{2}{\alpha}(2\alpha+1)} \ . \tag{3.7}$$

With this choice of ξ_n, $\dim_H(E) = \frac{\alpha}{2}$ and h and g are of the same order. Let F be the image of E under the Wiener process $\mathbb{W}$ and let $\mu = \mathbb{W}(\theta)$. Then $\dim_H F = \alpha$ and $\hat{\mu} \in L^{\frac{2}{\alpha}}$. $\square$

As a result, we can state the following theorem.

Theorem 6.

Let $p \in (1, 2)$ and let $\alpha = 2 - \frac{2}{p}$, so that the index conjugate to p is $\frac{2}{\alpha}$. There exists $f \in \mathcal{S}(\mathbb{R})$ such that f is cyclic in $L^r(\mathbb{R})$ for $r > p$ and such that f is not cyclic in $L^r(\mathbb{R})$ for $r \leq p$.

Proof. Choose F with Hausdorff dimension α such that F supports a random measure with Fourier transform in $L^{\frac{2}{\alpha}}$ and let $F = \mathcal{Z}(f)$. $\square$

4. Cyclic functions in $L^p(\mathbb{R})$, $p \in (2, \infty)$

In this section, we modify a theorem of Katznelson and Hirschman about sets of uniqueness and multiplicity for the circle $\mathbb{T}$. We prove that these same sets are sets of uniqueness or multiplicity for $L^q(\mathbb{R})$, $q \in (1, 2)$. We will then obtain theorems analogous to Theorems 4 and 6 for $p > 2$.

4.1 Functions Spanning $L^r(\mathbb{R})$, $r > p$, but not $L^r(\mathbb{R})$, $r \leq p$

We say that a set $K \subset \mathbb{T}$ is a set of l^q-uniqueness provided that if D is a distribution supported on K and $\{\hat{D}(n)\} \in l^q(\mathbb{Z})$, then $D = 0$. Otherwise, the set K is called a set of l^q-multiplicity. Similarly, we say that the set $E \subset \mathbb{R}$ is a set of L^q-uniqueness provided that if D is a distribution supported on E and $\hat{D} \in L^q(\mathbb{R})$, then $D = 0$. Otherwise, the set E is called a set of L^q-multiplicity. In 1964, Katznelson proved that there exist sets of positive Lebesgue measure which are sets of l^q-uniqueness for all $q \in (1, 2)$ [12]. In 1965, Katznelson and Hirschman proved that for $s, q \in (1, 2)$ with $s < q$, there exist sets of positive measure which are sets of l^s-uniqueness and l^q-multiplicity [9, Theorems 3 and 5]).

Theorem 7.

Let $q \in (1, 2)$. There exists a set E of positive measure in $\mathbb{T}$ which is a set of l^s-uniqueness for all $s < q$ and a set of l^q-multiplicity.

Proof. Choose sequences $\{s_n\}$ and $\{\varepsilon_n\}$ such that $s_n < s_{n+1}$, $s_n \to q$, and $\sum \varepsilon_n < 1/2$. For each n, choose a closed set $\tilde{E}_n \subset \mathbb{T}$ and functions $\tilde{F}_n, \tilde{G}_n \in C^\infty(\mathbb{T})$ such that $\tilde{E}_n$, $\tilde{F}_n$, and $\tilde{G}_n$ satisfy Theorems 4 and 5 in [9] with $\varepsilon = \varepsilon_n$ and $q = q_n$. We must dilate these sets and functions further. We show that there exists a sequence $\{\Lambda_k\}_{k=1}^\infty$ of positive integers such that for each $N \in \mathbb{Z}^+$,

$$\left\| \left(\prod_{k=1}^N G_k \right)^{\wedge} \right\|_q \leq \exp\left(\sum_{k=1}^{N-1} \varepsilon_k \right) \prod_{k=1}^N \|\widehat{G_k}\|_q$$

where $G_k(x) = \tilde{G}_k(\Lambda_k x)$.

Suppose that $\Lambda_1, \Lambda_2, \ldots, \Lambda_{M-1}$ have been chosen so that

$$\left\|(G_1 G_2)\hat{}\right\|_q \le \exp(\varepsilon_1)\left\|\widehat{G_1}\right\|_q \left\|\widehat{G_2}\right\|_q$$

$$\vdots$$

$$\left\|\left(\prod_{k=1}^{M-1} G_k\right)\hat{}\right\|_q \le \exp\left(\sum_{k=1}^{M-2}\varepsilon_k\right)\prod_{k=1}^{M-1}\|\widehat{G_k}\|_q \ .$$

Let $\phi = \prod_{k=1}^{M-1} G_k$ and $\psi = \tilde{G}_M$. Since each $\tilde{G}_k \in C^\infty(\mathbb{T})$, each $G_k \in C^\infty(\mathbb{T})$ as well, so $\phi = \prod_{k=1}^{M-1} G_k \in C^\infty(\mathbb{T})$. Furthermore, $\{\hat{\phi}(n)\} \in l^1 \cap l^q$ because $\phi \in C^\infty(\mathbb{T})$ implies that ϕ and all of its derivatives are in $L^1(\mathbb{T})$; we have $\hat{\phi}(n) = o(n^{-k})$ as $n \to \infty$ for each $k \in \mathbb{Z}^+$. We are now able to apply Lemma 2 in [9] to the functions ϕ and ψ: choose Λ_M such that

$$\left\|\widehat{\phi\psi_{\Lambda_M}}\right\|_q \le e^{\varepsilon_{M-1}}\left\|\hat{\phi}\right\|_q \left\|\hat{\psi}_{\Lambda_M}\right\|_q \ .$$

Then

$$\left\|\left(\prod_{k=1}^{M} G_k\right)\hat{}\right\|_q \le \exp\left(\sum_{k=1}^{M-1}\varepsilon_k\right)\prod_{k=1}^{M}\|\widehat{G_k}\|_q \ .$$

Let $E_n = (\tilde{E}_n)_{\Lambda_n}$ and let $F_n(x) = \tilde{F}_n(\Lambda_n x)$. Then E_n supports G_n and F_n is identically 1 on E_n. Also, $\|F_n\|_{s_n'} = \|\tilde{F}_n\|_{s_n'} < \varepsilon_n$ and $\log\|G_n\|_q = \log\|\tilde{G}_n\|_q < \varepsilon_n$. Set $E = \cap_{k=1}^\infty E_k$. The set E is a set of l^s-uniqueness for all $s < q$. Suppose that μ is a distribution supported on E such that $\{\hat{\mu}(m)\} \in l^{s_n}$ for some $s_n < q$. Since $s_n < s_{n+1} < \ldots < q$, we have $\|\hat{\mu}\|_q \le \ldots \le \|\hat{\mu}\|_{s_{n+1}} \le \|\hat{\mu}\|_{s_n}$. The distribution μ is supported on E_n for all n and F_n is 1 on E_n, so $\mu = \mu F_n$ and

$$\left|\hat{\mu}(m)\right| \le \left\|\hat{\mu}\right\|_1 = \left\|\widehat{\mu F_n}\right\|_1 = \left\|\hat{\mu} * \widehat{F_n}\right\|_1 \le \left\|\hat{\mu}\right\|_{s_n}\left\|\widehat{F_n}\right\|_{s_n'} \le \left\|\hat{\mu}\right\|_{s_n}\varepsilon_n \ . \tag{4.1}$$

Since $\{\|\hat{\mu}\|_{s_n}\}$ is a non-increasing sequence and $\varepsilon_n \to 0$ as $n \to \infty$, $|\hat{\mu}(m)| = 0$ for each m.

The set E is also a set of l^q-multiplicity. Let $g_N(x) = \prod_{k=1}^N G_k(x)$. We have chosen $\{\varepsilon_n\}$ and G_n such that for each $N \in \mathbb{Z}^+$,

$$\|\widehat{g_N}\|_q \le \exp\left(\sum_{k=1}^{N-1}\varepsilon_k\right)\prod_{k=1}^{N}\|G_k\|_q \le e \ . \tag{4.2}$$

The sequences $\{\widehat{g_N}(n)\}$ live inside the closed ball, radius e, in $l^q(\mathbb{Z})$. Because l^q is separable, $\{\widehat{g_N}(n)\}$ has a weak limit point $\{g(n)\}$. We can choose a subsequence $\{\widehat{g_{N_k}}(n)\}$ such that $\lim_{k\to\infty}\langle\widehat{g_{N_k}}, \psi\rangle = \langle g, \psi\rangle$ for each $\psi \in l^{q'}$ by [16, Theorem 11.29]. Let ϕ be the distribution on $\mathbb{T}$ defined by $\hat{\phi}(n) = g(n)$. First, the distribution ϕ is nontrivial since

$$\hat{\phi}(0) = g(0) \ge 1 - 2\sum_{n=1}^{\infty}\varepsilon_n > 0 \ . \tag{4.3}$$

Second, the support of ϕ is contained in E. Let $\psi \in C_c^\infty(E^c)$ with support K, and let U be an open set containing E which is disjoint from K. We have $E \subset \ldots \subset E_{N_{k+1}} \subset E_{N_k}$.

Choose k_0 such that for all $k > k_0$, $E_{N_k} \subset U$. Then for $k > k_0$, $\langle g_{N_k}, \psi \rangle = 0$ because $K \cap E_{N_k} = \emptyset$. But

$$0 = \lim_{k\to\infty} \langle g_{N_k}, \psi \rangle = \lim_{k\to\infty} \left\langle \widehat{g_{N_k}}, \hat{\psi} \right\rangle = \left\langle g, \hat{\psi} \right\rangle = \left\langle \check{g}, \psi \right\rangle = \langle \phi, \psi \rangle \ , \tag{4.4}$$

which implies that the support of ϕ is contained in E. □

We now show that the sets of uniqueness and multiplicity in Theorem 7 are also sets of uniqueness and multiplicity for $L^q(\mathbb{R})$ and vice versa.

Theorem 8.

If $K \subset [0, 1)$ is a set of uniqueness for $l^q(\mathbb{Z})$ for some $q \in (1, 2)$, then K is a set of uniqueness for $L^q(\mathbb{R})$.

Proof. Fix $q \in (1, 2)$. Let D be a distribution on $\mathbb{R}$ whose support is K and let $F(t) = \langle D, e^{2\pi ixt} \rangle$. The function $F(t)$ is the Fourier transform of D. We want to show that if $F \in L^q(\mathbb{R})$, then $F = 0$. Set $d_n = \int_n^{n+1} F(t)\,dt$. We have $\sum_{n=-\infty}^{\infty} d_n = \int_{\mathbb{R}} F(t)\,dt$. We will show that the sequence $\{d_n\}$ is the set of Fourier coefficients for a distribution on $\mathbb{T}$ whose support is K. We can switch the order of integration and the application of the distribution D because D can be interchanged with Riemann sums, and the Riemann sums for $\int_n^{n+1} e^{2\pi ixt} dt$ converge uniformly to $\int_n^{n+1} e^{2\pi ixt} dt$. We have

$$d_n = \int_n^{n+1} \left\langle D, e^{2\pi ixt} \right\rangle dt = \left\langle D, \int_n^{n+1} e^{2\pi ixt} dt \right\rangle = \frac{e^{2\pi ix} - 1}{2\pi ix} \left\langle D, e^{2\pi inx} \right\rangle \ ; \tag{4.5}$$

this calculation tells us that d_n is the nth Fourier coefficient of the distribution on $\mathbb{T}$ defined by

$$\widetilde{D} = \frac{e^{2\pi ix} - 1}{2\pi ix} D \ . \tag{4.6}$$

The support of $\widetilde{D}$ is the same as the support of D because $\frac{e^{2\pi ix}-1}{2\pi ix}$ has a removable singularity at $x = 0$. There are no other zeros or singularities in $\frac{e^{2\pi ix}-1}{2\pi ix}$. At $x = 0$, $\frac{e^{2\pi ix}-1}{2\pi ix} = 1$. By Katznelson's theorem [12], if the Fourier coefficients $\{d_n\}$ are in l^q, then $\widetilde{D} = 0$. Hölder's inequality allows us to conclude that

$$|\, d_n \,|^q \leq \int_n^{n+1} |F(t)|^q dt \ , \tag{4.7}$$

and summing over $\mathbb{Z}$ yields

$$\sum_n |\, d_n \,|^q = \|F\|_q^q < \infty \ . \tag{4.8}$$

The sequence $\{d_n\} \in l^q$, so $\widetilde{D} = D = 0$. □

Theorem 9.

Suppose $K \subset (0, 1)$ is a compact set which is a set of L^q-uniqueness. Then K is also a set of l^q-uniqueness.

Part of the idea for this proof came from Mantero, who extends Katznelson's and Hirschman's construction to locally compact groups [14].

Proof. Let $P \in C^\infty(\mathbb{R})$, $\operatorname{supp}(P) \subset [0,1]$, $P(x) > 0$, $x \in (0,1)$, and let P be real-valued. Suppose D is a distribution supported on K such that $\{\hat{D}(n)\} \in l^q(\mathbb{Z})$. Because the function P is smooth and compactly supported, P is slowly increasing on $\mathbb{R}$ (P and all of its derivatives are bounded on $\mathbb{R}$). As a result, the product DP is a distribution on $\mathbb{R}$ as well as a distribution on $\mathbb{T}$. The Fourier transform of DP is given by $\langle DP, e^{-2\pi i\xi x}\rangle$. We have

$$\left\langle DP, e^{-2\pi i\xi x}\right\rangle = \sum_{n\in\mathbb{Z}} \hat{D}(n)\overline{\left(Pe^{2\pi i\xi x}\right)\hat{}\,(n)} \tag{4.9}$$

and

$$\overline{\left(Pe^{2\pi i\xi x}\right)\hat{}\,(n)} = \int P(x)e^{-2\pi i(\xi-n)}dx = \hat{P}(\xi-n)\ , \tag{4.10}$$

so

$$(DP)\hat{}\,(\xi) = \sum_{n\in\mathbb{Z}} \hat{D}(n)\hat{P}(\xi-n)\ . \tag{4.11}$$

We want to show that $(DP)\hat{} \in L^q(\mathbb{R})$; the iterated integral

$$\left\|(DP)\hat{}\,\right\|_{L^q(\mathbb{R})} \le \int \sum_n \left|\hat{D}(n)\hat{P}(\xi-n)\right|^q d\xi = \sum_n \left|\hat{D}(n)\right|^q \int \left|\hat{P}(\xi-n)\right|^q d\xi, \tag{4.12}$$

is finite since $P \in \mathcal{S}(\mathbb{R})$ and $\sum_n |\hat{D}(n)|^q$ are both finite. The set K contains the support of the distribution DP. Let U be an open set on $\mathbb{R}$ which does not intersect K, and let $\phi \in C_c^\infty(U)$. Then $\operatorname{supp}(P\phi) = [0,1] \cap \operatorname{supp}(\phi)$. The function $P\phi$ can be considered an element of $C^\infty(\mathbb{T})$ because $\operatorname{supp}(P\phi) \subset [0,1]$ and $P(0) = P(1) = 0$. Since $\operatorname{supp}(P\phi)$ is compact and disjoint from K in $\mathbb{T}$, there is an open set V containing $\operatorname{supp}(P\phi)$ which is also disjoint from K. Then $P\phi \in C_c^\infty(V)$, and $\langle DP, \phi\rangle = \langle D, P\phi\rangle = 0$. □

Lemma 4.

Suppose $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, $2 \le p < \infty$ and $g \in L^q(\mathbb{R})$, $1 < q \le 2$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then $\widehat{f * g} = \hat{f}\hat{g}$ in $L^p(\mathbb{R})$.

Proof. Since $f \in L^1$ and $g \in L^q$, $f * g \in L^q$ by Young's inequality. By Riesz–Thorin interpolation, $\widehat{f * g} \in L^p$. The product $\hat{f}\hat{g} \in L^p$ as well. Choose a sequence $g_n \in L^1 \cap L^q$ such that $\|g - g_n\|_q \to 0$ as $n \to \infty$. Since the Fourier transform is a continuous mapping from L^q to L^p, $\|\hat{g} - \hat{g}_n\|_p \to 0$ as $n \to \infty$.

Consider the sequence $f * g_n$. We have

$$\|f * g_n - f * g\|_q = \|f * (g_n - g)\|_q \le \|f\|_1\|g_n - g\|_q \to 0 \tag{4.13}$$

as $n \to \infty$. As a result, $f * g_n \to f * g$ in L^q, which in turn implies that $\widehat{f * g_n} \to \widehat{f * g}$ in L^p by the continuity of the Fourier transform. Since $f \in L^1$ and $g_n \in L^1$ for all n, $\widehat{f * g_n} = \hat{f}\hat{g}_n$. Finally, $g_n \to g$ in L^q implies $\hat{g}_n \to \hat{g}$ in L^p. We have

$$\widehat{f * g_n} \to \widehat{f * g} \qquad \text{and} \qquad \widehat{f * g_n} = \hat{f}\hat{g}_n \to \hat{f}\hat{g} \tag{4.14}$$

so $\widehat{f * g} = \hat{f}\hat{g}$. □

We note that the sets in Theorem 7 are contained within the open interval $(0, 1)$ when $\mathbb{T}$ is viewed as a subset of $\mathbb{R}$.

Theorem 10.

Let $p > 2$. There exists $f \in \mathcal{S}(\mathbb{R})$ such that f is cyclic in $L^r(\mathbb{R})$ for all $r > p$ and such that f is not cyclic in $L^r(\mathbb{R})$, $r \leq p$.

Proof. Fix $p > 2$; then the conjugate exponent $q \in (1, 2)$. By Theorem 7, there exists $E \subset \mathbb{T}$ which is a set of l^s-uniqueness for all $s < q$ and which is a set of l^q-multiplicity. By Theorems 8 and 9, E is a set of uniqueness for $L^s(\mathbb{R})$ for all $s < q$ and E is a set of multiplicity for $L^q(\mathbb{R})$. By Lemma 1 choose $f \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{Z}(f) = E$. Then f is cyclic in $L^r(\mathbb{R})$ for all $r > p$, and f is not cyclic in $L^p(\mathbb{R})$. Suppose there exists $\check{g} \in L^s$ such that $f * \check{g} = 0$. Apply Lemma 4 to the functions f and $\check{g}$: $(f * \check{g})\hat{} = \hat{f} g = 0$ in L^r. This means that $g \in L^r$ must be supported in the zero set of $\hat{f}$, which is E. But $\check{g} \in L^s$ and E is a set of L^s-uniqueness, so $g = 0$. To see that f is not cyclic in $L^p(\mathbb{R})$, consider the distribution $DP \in L^q(\mathbb{R})$ supported in E in Theorem 9. Then $(f * (DP)\check{})\hat{} = \hat{f} DP = 0$ even though $DP \neq 0$. □

4.2 Functions Spanning $L^r(\mathbb{R})$, $r \geq p$, but not $L^r(\mathbb{R})$, $r < p$

This case is handled in the way as in the case $p \in (1, 2)$, but we use the machinery of the previous section.

Theorem 11.

Let $p > 2$. There exists a function $f \in \mathcal{S}$ such that f is cyclic in $L^r(\mathbb{R})$, $r \geq p$ and such that f is not cyclic in $L^r(\mathbb{R})$, $r < p$.

Proof. Choose $\{p_m\}$, an increasing sequence with limit p. Let q_m be the index conjugate to p_m. Use the results of Section 4.1 to select compact sets $E_m \subset (\frac{1}{m+1}, \frac{1}{m})$ such that each E_m is a set of $L^s(\mathbb{R})$-uniqueness for any $s < q_m$ and is also a set of $L^{q_m}(\mathbb{R})$-multiplicity. Again, let $E = \{0\} \cup \bigcup_{m=1}^{\infty} E_m$. The arguments in Section 3.2 show that E is a set of $L^s(\mathbb{R})$-uniqueness for any $s \leq q$ and is a set of $L^s(\mathbb{R})$-multiplicity for any $s > q$. Hence, if we select a function $f \in \mathcal{S}$ whose Fourier transform has E as the zero set, then f will be cyclic in $L^r(\mathbb{R})$ for all $r \geq p$ and is not cyclic in $L^r(\mathbb{R})$ for any $r < p$. □

5. Concluding Remarks

The question of whether a function spans $L^p(\mathbb{R})$ is closely connected to the theory of translation-invariant subspaces. We have examined translation-invariant subspaces generated by a single Schwartz-class function. In general, translation-invariant subspaces are not always singly generated; in 1973 and 1975, Atzmon showed that there are translation-invariant subspaces of $L^p(\mathbb{R})$ which are not generated by a single function for $1 \leq p < 2$ [2] and [3].

The construction of a cyclic or non-cyclic function f boils down to constructing a distribution D whose support can "hide" in $\mathcal{Z}(f)$ and proving that some version of the formula $(f * \check{D})\hat{} = \hat{f} D$ is correct. In general, even if $\mathcal{Z}(f)$ supports a distribution D, it is not always the case that $\text{supp}(D) \subseteq \mathcal{Z}(f)$ implies that $f * \check{D} = 0$ (see [6, p. 231–232]). The study of spectral synthesis properties of sets helps settle this question; the implication does hold if we know that D is a pseudomeasure and $\text{supp}(D)$ is a set of spectral synthesis. For example, Wiener's original theorem about cyclic functions in $L^1(\mathbb{R})$—that $\hat{f}$ never vanishes if and only if f spans $L^1(\mathbb{R})$—can be rephrased in the language of spectral synthesis by stating that the empty set is a set of spectral synthesis. See [4] and [15] for more details.

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