The Journal of Fourier Analysis and Applications
Volume 8, Issue 6, 2002

On a BMO-Property for Subharmonic Functions

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Communicated by Carlos A. Berenstein

ABSTRACT. We study traces of subharmonic functions to Ahlfors regular subsets of $\mathbb{R}^2$. In particular, we establish for the traces a generalized BMO-property and the reverse Hölder inequality.

1. Introduction

We recall the following.

Definition 1. Let X be a complete metric space equipped with a regular Borel measure μ. A locally integrable on X function f belongs to $BMO(X, \mu)$ if

$$|f|_* := \sup \left\{ \frac{1}{\mu(B)} \int_B |f - f_B|\, d\mu \right\} < \infty ;$$

here supremum is taken over all metric balls $B \subset X$ and $f_B = \frac{1}{\mu(B)} \int_B f\, d\mu$.

Let ν be another regular Borel measure on X. Clearly the condition $f \in BMO(X, \mu)$ does not necessarily imply that $f \in BMO(X, \nu)$. In general such f is not even locally integrable with respect to ν (consider e. g., $\log|x| \in BMO(\mathbb{R}^2,\ dx\, dy)$, and any ν supported on the line $x = 0$). In this article we show that if f is a subharmonic function defined in an open set $U \subset \mathbb{R}^2$ then $f \in BMO(\mathbb{R}^2, \mu)$ for a wide class of measures μ with $\operatorname{supp}(\mu) \subset U$.

To formulate the result we, first, introduce some notations.
Set $\mathbb{D}_s := \{z \in \mathbb{C} : |z| < s\}$ and $\mathbb{D}(x, t) := \{z \in \mathbb{C} : |z - x| < t\}$. For a $K \subset \mathbb{R}^2$ denote $K_{x,t} := \mathbb{D}(x, t) \cap K$.

Definition 2. A compact subset $K \subset \mathbb{R}^2$ is said to be (Ahlfors) d-*regular* if there is a positive number a such that for any $x \in K$, $0 < t \leq \operatorname{diam}(K)$

$$\mathcal{H}^d(K_{x,t}) \leq at^d\ . \tag{1.1}$$

Math Subject Classifications. primary 31B05; secondary 46E30.
Keywords and Phrases. subharmonic function, d-regular set, *BMO*-function.

ISSN 1069-5869

Here $\mathcal{H}^d(\omega)$ denotes the d-Hausdorff measure of ω.

This class will be denoted by $\mathcal{A}(d, a)$.

A compact subset $K \in \mathcal{A}(d, a)$ is said to be a *d-set* if there is a positive number b such that for any $x \in K$, $0 < t \le \mathrm{diam}(K)$

$$bt^d \le \mathcal{H}^d(K_{x,t}) \, . \tag{1.2}$$

We denote this class by $\mathcal{A}(d, a, b)$.

The class of d-sets, in particular, contains Lipschitz d-manifolds (with d integer), Cantor type sets and self-similar sets (with arbitrary d), see, e. g., [5], p. 29 and [8], Sect. 4.13.

Theorem 1.

Let $K \subset \mathbb{R}^2$ be a compact d-set. Assume that f is a subharmonic function defined in an open neighborhood of K. Then restriction $f|_K$ belongs to BMO$(K, \mathcal{H}^d)$.

We deduce this result from the following distributional inequality.

Theorem 2.

Assume that f is a subharmonic in $\mathbb{D}_1$ function satisfying

$$\sup_{\mathbb{D}_1} f \le M_1 \quad \text{and} \quad \sup_{\mathbb{D}_r} f \ge M_2 \qquad (r < 1) \, . \tag{1.3}$$

Let K be a compact from $\mathcal{A}(d, a, b)$. We set $f_{x,t} := \sup_{K_{x,t}} f$, $x \in K$. Assume that $\mathbb{D}(x, t/r) \subset \mathbb{D}_r$.

There is a constant $c = c(r) > 0$ such that

$$\mathcal{H}^d \left\{ y \in K_{x,t} \ : \ f_{x,t} - f(y) \ge \lambda \right\} \le \frac{(4e)^d a}{r^d \, db} e^{-\lambda d/(c(M_1 - M_2))} \cdot \mathcal{H}^d(K_{x,t}) \, .$$

As a consequence of the inequality of Theorem 2 we also prove the corresponding reverse Hölder inequality.

Theorem 3.

Let $K \in \mathcal{A}(d, a, b)$. Then for any $K_{x,t}$, $x \in K$, $t > 0$, and any $1 \le p \le \infty$ the inequality

$$\left(\frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} e^{pf} \, d\mathcal{H}^d \right)^{1/p} \le C(K, f, d) \frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} e^{f} \, d\mathcal{H}^d \tag{1.4}$$

holds.

2. Abstract Version of Cartan's Lemma

Our proofs are based on estimates for subharmonic functions which generalize well-known Cartan's Lemma for polynomials (see [2]). We use a version of the generalized Cartan's Lemma proved by Gorin (see [3]).

Let X be a complete metric space and let μ be a finite Borel measure on X. We consider a continuous, strictly increasing, nonnegative function ϕ on $[0, +\infty[$, $\phi(0) = 0$, $\lim_{x \to \infty} \phi(x) > \mu(X)$. The function ϕ will be called a *majorant*.

For each point $x \in X$ we set $\tau(x) = \sup\{t : \mu(B(x,t)) \geq \phi(t)\}$, where $B(x,t)$ is the closed ball in X with center x and radius t. It is easy to see that $\mu(B(x,\tau(x)) = \phi(\tau(x))$ and $\sup_x \tau(x) \leq \phi^{-1}(\mu(X)) < \infty$.

A point $x \in X$ is said to be *regular* (with respect to μ and ϕ) if $\tau(x) = 0$, i. e., $\mu(B(x,t)) < \phi(t)$ for all $t > 0$. The next result shows that the set of regular points is sufficiently large for an arbitrary majorant ϕ.

Lemma 1 (Gorin).

Let $0 < \gamma < 1/2$. There exists a sequence of balls $B_k = B(x_k, t_k)$, $k = 1, 2, \ldots$, which collectively cover all the irregular points and which are such that $\sum_{k\geq 1} \phi(\gamma t_k) \leq \mu(X)$ (i. e., $t_k \to 0$).

For the sake of completeness we present Gorin's proof of the lemma.

Proof. Let $0 < \alpha < 1$, $\beta > 2$ but $\gamma < \alpha/\beta$. We set $B_0 = \emptyset$ and assume that the balls $B_0, \ldots, B_{k-1}$ have been constructed. If $\tau_k = \sup\{\tau(x) : x \notin B_0 \cup \cdots \cup B_{k-1}\}$, then there exists a point $x_k \notin B_0 \cup \cdots \cup B_{k-1}$, such that $\tau(x_k) \geq \alpha\tau_k$. We set $t_k = \beta\tau_k$ and $B_k = B(x_k, t_k)$. Clearly, the sequence τ_k (and thus also t_k) does not increase. The balls $B(x_k, \tau_k)$ are pairwise disjoint. Indeed, if $l > k$, then $x_l \notin B_k$, i. e., the distance between x_l and x_k is greater than $\beta\tau_k > 2\tau_k \geq \tau_k + \tau_l$. Then,

$$\sum_{k=1}^{\infty} \phi(\gamma t_k) \leq \sum_{k=1}^{\infty} \phi\alpha\tau_k) \leq \sum_{k=1}^{\infty} \phi(\tau(x_k)) = \sum_{k=1}^{\infty} \mu(B(x_k,\tau_k)) \leq \mu(X)\ ;$$

consequently, $\tau_k \to 0$, i. e., for each point x, not belonging to the union of the balls B_k, $\tau(x) = 0$, x is a regular point. In addition, $t_k = \beta\tau_k \to 0$. □

Remark 1. If X is a locally compact metric space then one can take $\gamma = 1/2$ (for similar arguments see, e. g., [7], Th. 11.2.3).

We now apply Lemma 1 to obtain estimates for logarithmic potentials of measures.

Assume that X is a locally compact metric space with metric $d(.,.)$.

Theorem 4.

Let

$$u(z) = \int_X \log d(x,\xi)\, d\mu(\xi)$$

where μ is a Borel measure, $\mu(X) = k < \infty$.

Given $H > 0, d > 0$ there exists a system of balls such that

$$\sum r_j^d \leq \frac{(2H)^d}{d} \tag{2.1}$$

where r_j are radii of these balls, and

$$u(z) \geq k \log \frac{H}{e}$$

everywhere outside these balls.

Proof. Let $\phi(t) = (pt)^d$ be a majorant with $p = \frac{(kd)^{1/d}}{H}$. We cover all irregular points of X by balls according to Gorin's Lemma 1 and Remark 1. It remains to estimate the potential

u outside of these balls, i. e., at any regular point z. Let $n(t;z) = \mu(\{\xi : d(z,\xi) \le t\})$. Clearly, for any $N \ge \max\{1, H\}$

$$u(z) \ge \int_{d(z,\xi)\le N} \log d(z,\xi)\, d\mu(\xi) = \int_0^N \log t\, dn(t;z) = n(t;z)\log t|_0^N - \int_0^N \frac{n(t;z)}{t}\, dt\ .$$

Since $n(t;z) < (pt)^d$, we then have

$$u(z) \ge n(N;z)\log N - \int_0^N \frac{n(t;z)}{t}\, dt\ .$$

In addition, $n(t;z) \le n(N;z)$ for $t \le N$. Therefore,

$$\begin{aligned} u(z) &\ge n(N;z)\log N - \int_0^H \frac{(pt)^d}{t}\, dt - \int_H^N \frac{n(N;z)}{t}\, dt \\ &= n(N;z)\log N - \frac{(pH)^d}{d} - n(N;z)\log N + n(N;z)\log H = -k + n(N;z)\log H\ . \end{aligned}$$

Letting here $N \to \infty$ and taking into account that $\lim_{N\to\infty} n(N;z) = k$ we obtain the required result. □

3. Proof of Theorem 2

We deduce Theorem 2 from the following result.

Theorem 5.

Let $\omega \subset \mathbb{D}(x,t)$ be a compact set of $\mathcal{A}(d,a)$ satisfying $\mathcal{H}^d(\omega) \ge \epsilon > 0$. Assume that $\mathbb{D}(x,t/r) \subset \mathbb{D}_r$. Then there is a constant $c = c(r) > 0$ such that inequality

$$\sup_{\mathbb{D}(x,t)} f \le \sup_{\omega} f + (M_1 - M_2)c\log\frac{4\eta^{1/d}}{r(d\epsilon)^{1/d}}$$

holds for any subharmonic f satisfying (1.3).

Proof. We begin with

Proposition 1.

Let u be a nonpositive subharmonic function on $\mathbb{D}_1$ satisfying

$$\sup_{\mathbb{D}_r} u \ge -1 \qquad \textit{for some } r < 1\ .$$

Then for any $H > 0$, $d > 0$ there is a set of disks such that

$$\sum r_j^d \le \frac{(2H)^d}{d}\ , \tag{3.1}$$

where r_j are radii of these disks, and

$$u(z) \ge c\log\frac{H}{e}$$

outside these disks in $\mathbb{D}_r$. Here $c = c(r) > 0$ depends on r only.

Proof. Clear we can consider $H \leq e$ for otherwise the statement is trivial.

Let κ be a nonnegative radial C^∞-function on $\mathbb{C}$ satisfying

$$\int\int_{\mathbb{C}} \kappa(x, y)\,dx\,dy = 1 \quad \text{and} \quad \operatorname{supp}(\kappa) \subset \mathbb{D}_1 \quad (z = x + iy)\ . \tag{3.2}$$

Let u_k denote the function defined on $\mathbb{D}_{1-1/k}$ by

$$u_k(w) := \int\int_{\mathbb{C}} \kappa(x, y)u(w - z/k)\,dx\,dy\ . \tag{3.3}$$

It is well known, see, e. g., [6], Theorem 2.9.2, that u_k is subharmonic on $\mathbb{D}_{1-1/k}$ of the class C^∞ and that $u_k(w)$ monotonically decreases and tends to $u(w)$ for each $w \in \mathbb{D}_1$ as $k \to \infty$. Let $K := \{z \in \mathbb{D}_1 \ :\ \frac{1+r}{2} \leq |z| \leq \frac{3+r}{4}\}$ be an annulus in $\mathbb{D}_1$ and $k \geq k_0 = [\frac{8}{1-r}] + 1$. We are based on the following result (see, e. g., [1], Lemma 2.3).

There are a constant $A = A(r) > 0$ and numbers t_k, $k \geq k_0$, satisfying $\frac{1+r}{2} \leq t_k \leq \frac{3+r}{4}$ such that $u_k(z) \geq -A$ for any $z \in \mathbb{C},\ |z| = t_k$.

Then we can construct functions f_k subharmonic on $\mathbb{C}$ by

$$f_k(z) := \begin{cases} u_k(z) & \left(z \in \mathbb{D}_{t_k}\right)\ ; \\ \max\left\{u_k(z), \dfrac{-2A\log|z|}{\log t_k}\right\} & \left(z \in \mathbb{D}_1 \setminus \mathbb{D}_{t_k}\right)\ ; \\ \dfrac{-2A\log|z|}{\log t_k} & (z \in \mathbb{C} \setminus \mathbb{D}_1)\ . \end{cases}$$

Without loss of generality we may assume that $t_k \to t \in [\frac{1+r}{2}, \frac{3+r}{4}]$ as $k \to \infty$. Finally, define

$$f(z) = \left(\overline{\lim_{k\to\infty}} f_k(z)\right)^*\ ,$$

where g^* denotes upper semicontinuous regularization of g. Then f is subharmonic in $\mathbb{C}$ satisfying

$$f(z) = u(z) \quad (z \in \mathbb{D}_r) \quad \text{and} \quad f(z) = \frac{-2A\log|z|}{\log t} \quad \left(z \in \mathbb{C} \setminus \overline{\mathbb{D}_1}\right)\ .$$

Consider now $\mu = \Delta f$. Then μ is a finite Borel measure on $\mathbb{C}$ supported in $\overline{\mathbb{D}_1}$. According to F. Riesz's theorem (see, e. g., [4], Th. 3.9)

$$\tilde{f}(z) := \frac{1}{2\pi}\int\int_{\mathbb{C}} \log|z - \xi|\,d\mu(\xi)$$

is subharmonic in $\mathbb{C}$ and satisfies $\Delta\tilde{f} = \Delta f = \mu$. Thus $h = \tilde{f} - f$ is a real-valued harmonic in $\mathbb{C}$ function. Moreover, h goes to infinity as $\left(\frac{\mu(\mathbb{C})}{2\pi} - \frac{-2A}{\log t}\right)\log|z|$. This immediately implies (by arguments involving Liouville's theorem) that $h = 0$ and $\frac{\mu(\mathbb{C})}{2\pi} = \frac{-2A}{\log t}$. Now according to Theorem 4 applied to $f (= \tilde{f})$, for any $0 < H \leq e$, $d > 0$ there is a system of disks with radii r_j satisfying $\sum r_j^d \leq \frac{(2H)^d}{d}$ such that

$$f \geq \frac{-2A}{\log t}\log\frac{H}{e} \geq \frac{-2A}{\log[(3+r)/4]}\log\frac{H}{e}$$

outside these disks. It remains to set $c = \frac{-2A}{\log[(3+r)/4]}$.

The proof of the proposition is complete. □

Assume now that f is subharmonic and satisfies (1.3). Then by the main theorem in [1] there is a constant $C = C(r) > 0$ such that the inequality

$$\sup_{\mathbb{D}(x,t/r)} f \le C(M_1 - M_2) + \sup_{\mathbb{D}(x,t)} f$$

holds for any pair of disks $\mathbb{D}(x,t) \subset \mathbb{D}(x,t/r)(\subset \mathbb{D}_r)$. Applying inequality of Proposition 1 to the function

$$u(z) = \frac{f(tz/r) - \sup_{\mathbb{D}(x,t/r)} f}{C(M_1 - M_2)} \qquad (z \in \mathbb{D}_1)$$

and then going back to f we obtain the following.

Proposition 2.

There is a constant $c = c(r) > 0$ such that for any disk $\mathbb{D}(x,t)$ satisfying $\mathbb{D}(x,t) \subset \mathbb{D}(x,t/r) \subset \mathbb{D}_r$ and any $H > 0, d > 0$ there is a system of disks such that

$$\sum r_j^d \le \frac{(2tH/r)^d}{d} ,$$

where r_j are radii of these disks, and

$$f(z) \ge \sup_{\mathbb{D}(x,t)} f + c(M_1 - M_2) \log \frac{H}{e}$$

outside these disks in $\mathbb{D}(x,t)$.

Remark 2. A particular case of Proposition 2 for functions $u = \log|f|$ with holomorphic f and for $d = 1$ was proved in [7].

We proceed to the proof of Theorem 5. First we show that ω can not be covered by a system of disks such that

$$\sum r_j^d \le \frac{(1-1/n)\epsilon}{2^d a} \qquad (n \ge 1) \tag{3.4}$$

where r_j are radii of these disks. Assume to the contrary that there exists a system of disks $\{\mathbb{D}(x_j, r_j)\}$ whose radii satisfy (3.4) which covers ω. For any x_j choose $y_j \in \omega$ so that $|x_j - y_j| \le r_j$. Then the system of disks $\{\mathbb{D}(y_j, 2r_j)\}$ also covers ω. Since $\omega \in \mathcal{A}(d,a)$, we obtain inequality

$$\mathcal{H}^d(\omega) \le \sum \mathcal{H}^d(\omega \cap \mathbb{D}(y_j, 2r_j)) \le 2^d a \sum r_j^d < \epsilon$$

which contradicts to $\mathcal{H}^d(\omega) \ge \epsilon$.

We now apply Proposition 2 with $H_n = \frac{(d(1-1/n)\epsilon)^{1/d} r}{4ta^{1/d}}$. Since any system of disks with $\sum r_j^d \le \frac{(2tH_n/r)^d}{d}$ can not cover ω, Proposition 2 implies that there is a point $x_n \in \omega$ such that

$$\sup_{\omega} f \ge f(x_n) \ge \sup_{\mathbb{D}(x,t)} f + c(M_1 - M_2) \log \frac{H_n}{e} .$$

Letting $n \to \infty$ we get the required inequality.

Theorem 5 is proved. □

Let us also show that the condition of d-regularity is necessary for the set to satisfy the inequality of Theorem 5.

Proposition 3.
Let $K \subset \mathbb{D}_{1/2}$ be a compact set with $\mathcal{H}^d(K) < \infty$. Assume that the inequality

$$\sup_{\mathbb{D}(x,t)} f \leq \sup_{\omega} f + L + C \log \frac{t}{\epsilon^{1/d}}$$

holds for any $\omega \subset K \cap \mathbb{D}(x,t) \subset \mathbb{D}(x, 3t/2) \subset \mathbb{D}_{2/3}$, $x \in K$, with $\mathcal{H}^d(\omega) = \epsilon$ and any f subharmonic in $\mathbb{D}_1$ satisfying (1.3) *with $r = 2/3$ and some M_1, M_2. Here L and $C > 0$ depend on K, d, M_1, M_2. Then $K \in \mathcal{A}(d, c)$ for some $c > 0$.*

Proof. For any $f, \omega, t \leq 1/9$ satisfying assumptions of the proposition the inequality

$$-C \log \frac{t}{\epsilon^{1/d}} \leq \sup_{\mathbb{D}(x,t)} f - \sup_{\omega} f - C \log \frac{t}{\epsilon^{1/d}} \leq L < \infty$$

holds. For a point $x \in K$ we set $f_x(z) = \log |z - x|$ and $\epsilon_t := \mathcal{H}^d(\mathbb{D}(x,t) \cap K)$. Clearly the family $\{f_x\}$ satisfies inequality (1.3) with $r = 2/3$, $M_1 = 3/2$ and $M_2 = 1/6$. Then from the above inequality applied to f_x we obtain

$$L \geq -C \log \frac{t}{\epsilon_t^{1/d}} \ ,$$

that is equivalent to $\epsilon_t \leq \widetilde{L} t^d$ for $\widetilde{L} = e^{\frac{dL}{C}}$. Thus the definition of d-regularity is checked for $t \leq 1/9$. For $t > 1/9$ the inequality is obvious. □

Proof of Theorem 2. The proof is an easy consequence of the inequality of Theorem 5 where we choose $\omega := \mathcal{H}^d\{y \in K_{x,t} \ : \ f_{x,t} - f(y) \geq \lambda\}$ and the definition of d-sets. We leave the details to the reader. □

4. Proofs

Proof of Theorem 1. First, we prove a local version of the theorem. Assume that $K \subset \mathbb{D}_r$ is a compact from $\mathcal{A}(d, a, b)$ and f, $\mathbb{D}(x,t)$ satisfy conditions of Theorem 2. Set $f' = f_{x,t} - f$ and $D_{f'}(\lambda) := \mathcal{H}^d\{y \in K_{x,t} \ : \ f_{x,t} - f(y) \geq \lambda\}$. Then from the inequality of Theorem 2 it follows that

$$\frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} f' \, d\mathcal{H}^d = \frac{1}{\mathcal{H}^d(K_{x,t})} \int_0^\infty D_{f'}(x) \, dx \leq \frac{ca(4e)^d (M_1 - M_2)}{br^d d^2} \ . \qquad (4.1)$$

Now we have

$$\frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} \left| f - f_{K_{x,t}} \right| d\mathcal{H}^d \leq \frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} \left| (f - f_{x,t}) - (f - f_{x,t})_{K_{x,t}} \right| d\mathcal{H}^d$$
$$\leq \frac{2}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} f' \, d\mathcal{H}^d \leq \frac{2ca(4e)^d (M_1 - M_2)}{br^d d^2} \ .$$

This gives the estimate of the BMO-norm in each ball $K_{x,t} = \mathbb{D}(x,t) \cap K$ with $\mathbb{D}(x,t) \subset \mathbb{D}(x, t/r) (\subset \mathbb{D}_r)$. In the general case, we cover K by a finite number of open disks $\mathbb{D}(x_i, R)$, $i = 1, \dots, N$ such that f is defined in the union of these disks, the set $\cup_{i=1}^{N} \mathbb{D}(x_i, R/2)$ also covers K and any disk of radius $\leq R/4$ centered at a point of K belongs to one of $\mathbb{D}(x_i, R/2)$. Then the estimate of the BMO-norm in any $\mathbb{D}(x,t) \cap K$, $x \in K$, $t \leq R/4$, follows from Theorem 2 and inequality (4.1). To estimate BMO-norms for $\mathbb{D}(x,t) \cap K$ with $t \geq R/4$ we write

$$\frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} |f - f_{K_{x,t}}| \, d\mathcal{H}^d \leq \frac{4^d}{bR^d} \int_{K_{x,t}} 2|f| \, d\mathcal{H}^d < C \int_K |f| \, d\mathcal{H}^d < \infty \, .$$

Here we used that $\int_K |f| \, d\mathcal{H}^d < \infty$ by Theorem 2. □

We now formulate another corollary of Theorem 2.

Corollary 1.

Assume that a subharmonic function f defined on $\mathbb{C}$ satisfies

$$f(z) \leq c' + \log(1 + |z|) \qquad (z \in \mathbb{C})$$

for some $c' \in \mathbb{R}$. Assume also that $S \in \mathcal{A}(d, a, b)$. Then $f|_S \in BMO(S, \mathcal{H}^d)$ and the BMO norm $|f|_S|_* \leq \frac{\tilde{c}a}{bd^2}$ with an absolute constant $\tilde{c}$.

Proof. For functions f satisfying conditions of the corollary the Bernstein–Walsh inequality

$$\sup_{\mathbb{D}(x,qt)} f \leq \log q + \sup_{\mathbb{D}(x,t)} f \tag{4.2}$$

holds for any $x \in \mathbb{C}$, $t \geq 0$, $q \geq 1$. (The proof is based on the classical Bernstein inequality for polynomials and the polynomial representation of the $\mathcal{L}$-extremal function of the disk (see, e. g., [6]).) Then the estimate of the BMO-norm in $f|_{\mathbb{D}(x,t) \cap S}$ follows from inequality (4.1) with $r = 1/2$ and $M_1 - M_2 = \log 2$. □

Proof of Theorem 3. As in the proof of Theorem 1 we, first, consider a local version of the theorem. Assume that $K \subset \mathbb{D}_r$ is a compact from $\mathcal{A}(d, a, b)$ and f, $\mathbb{D}(x,t)$ satisfy conditions of Theorem 2. Denote $g_t = e^{-f'} = e^f / e^{f_{x,t}}$. Consider the distribution function $d_g(\lambda) := \mathcal{H}^d\{y \in K_{x,t} \ : g_t(y) \leq \lambda\}$. Then from the inequality of Theorem 2 we deduce

$$d_g(\lambda) \leq \frac{(4e)^d a}{r^d \, db} (\lambda)^{d/(c(M_1 - M_2))} \cdot \mathcal{H}^d(K_{x,t}) \, .$$

Let $g_*(s) = \inf\{\lambda \ : d_g(\lambda) \geq s\}$. From the previous inequality we obtain

$$g_*(s) \geq \left(\frac{sr^d \, db}{(4e)^d \mathcal{H}^d(K_{x,t}) a} \right)^{c(M_1 - M_2)/d} \, .$$

In particular,

$$\frac{1}{\mathcal{H}^d(K_{x,t})} \int_{K_{x,t}} g_t \, d\mathcal{H}^d = \frac{1}{\mathcal{H}^d(K_{x,t})} \int_0^{\mathcal{H}^d(K_{x,t})} g_*(s) \, ds$$
$$\geq \int_0^1 \left(\frac{sr^d \, db}{(4e)^d a} \right)^{c(M_1 - M_2)/d} ds = \frac{1}{1 + c(M_1 - M_2)/d} \left(\frac{r^d \, db}{(4e)^d a} \right)^{c(M_1 - M_2)/d} \, .$$

Thus we obtain

$$\sup_{K_{x,t}} e^f \le (1 + c(M_1 - M_2)/d)\left(\frac{(4e)^d a}{r^d\, db}\right)^{c(M_1-M_2)/d} \frac{1}{\mathcal{H}^d(K_{x,t})}\int_{K_{x,t}} e^f \, d\mathcal{H}^d \qquad (4.3)$$

which implies the required local reverse Hölder inequality. In the general case, we cover again K by a finite number of open disks $\mathbb{D}(x_i, R)$, $i = 1, \dots, N$ such that f is defined in the union of these disks, the set $\cup_{i=1}^N \mathbb{D}(x_i, R/2)$ also covers K and any disk of radius $\le R/4$ centered at a point of K belongs to one of $\mathbb{D}(x_i, R/2)$. Then the reverse Hölder inequality of the form (4.3) holds for any $K_{x,t} = \mathbb{D}(x,t) \cap K$, $x \in K$, $t \le R/4$. Assume now that $t > R/4$ and set

$$m := \inf_{x\in K, t > R/4}\left\{\frac{1}{\mathcal{H}^d(K_{x,t})}\int_{K_{x,t}} e^f \, d\mathcal{H}^d\right\}.$$

Then $m > 0$. Indeed, let $x_i, t_i > R/4$, be a sequence for which the expression on the right above converges to m. Without loss of generality we may assume also that x_i tends to $x \in K$ and t_i tends to $t \ge R/4$. Then there is i_0 such that for any $i \ge i_0$, the ball K_{x_i,t_i} contains $K_{x,R/8}$. Note that $\sup_{K_{x,R/8}} e^f > 0$ because $K_{x,R/8}$ is not a polar set. Then inequality (4.3) applied to $K_{x,R/8}$ and the d-regularity of K show that

$$m \ge \frac{C}{\mathcal{H}^d(K_{x,R/8})}\int_{K_{x,R/8}} e^f \, d\mathcal{H}^d > 0$$

for a constant $C := C(K)$. Finally, since $\sup_{K_{x,t}} e^f \le M := \sup_K e^f < \infty$, inequality (4.3) for $t > R/4$ is valid with the constant M/m.

The proof of the theorem is complete. □

Corollary 2.

Assume that a subharmonic function f defined on $\mathbb{C}$ satisfies

$$f(z) \le c' + \log(1 + |z|) \qquad (z \in \mathbb{C})$$

for some $c' \in \mathbb{R}$. Assume also that $S \in \mathcal{A}(d, a, b)$. Then for $e^f|_S$ the reverse Hölder inequality (4.3) *holds with the constant $\frac{c_1}{d}\left(\frac{a}{db}\right)^{c_2/d}$, where c_1, c_2 are absolute positive constants.*

Proof. The proof follows directly from the Bernstein–Walsh inequality (4.2) and Theorem 3. □

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Received July 10, 2001

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