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The Generalized Multifractional Field: A Nice Tool for the Study of the Generalized Multifractional Brownian Motion

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ABSTRACT. *The Generalized Multifractional Brownian Motion (GMBM) is a continuous Gaussian process $\{X(t)\}_{t\in[0,1]}$ that extends the classical Fractional Brownian Motion (FBM) and the Multifractional Brownian Motion (MBM) [15, 4, 1, 2]. Its main interest is that, its Hölder regularity can change widely from point to point. In this article we introduce the Generalized Multifractional Field (GMF), a continuous Gaussian field $\{Y(x,y)\}_{(x,y)\in[0,1]^2}$ that satisfies for every t, $X(t) = Y(t,t)$. Then, we give a wavelet decomposition of Y and using this nice decomposition, we show that Y is β-Hölder in y, uniformly in x. Generally speaking this result seems to be quite important for the study of the GMBM. In this article, it will allow us to determine, without any restriction, its pointwise, almost sure, Hölder exponent and to prove that two GMBM's with the same Hölder regularity differ by a "smoother" process.*

1. Introduction

The pointwise Hölder exponent of a function allows to measure the local variations of its irregularity. This notion will be fundamental in this article, so let us recall it precisely. A complex-valued function defined on $\mathbb{R}^d$, is said to satisfy a Hölder condition of exponent α, $m < \alpha \leq m+1$, $m \in \mathbb{N}$ at a point t_0, if there are a polynomial P_m of degree at most m and a constant $c > 0$ (that generally depend on t_0), such that the inequality

$$|f(t) - P_m(t-t_0)| \leq c|t-t_0|^{\alpha}\ , \tag{1.1}$$

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holds when $|t - t_0|$ is small enough. The pointwise Hölder exponent of f at t_0 is defined to be the supremum of the α that satisfy (1.1) and denoted $\alpha(t_0)$. The function $s \mapsto \alpha(s)$ is called the pointwise Hölder exponent of f.

K. Daoudi, J. Lévy Véhel and Y. Meyer have completely described the class of pointwise Hölder exponents of continuous functions over a compact interval. More precisely, they have shown that this class is that of lower-limits of continuous functions [5]. In [6] S. Jaffard gives a wavelet proof of this fundamental result. However, the deterministic continuous functions with the most general Hölder exponent, given in [5] and [6], seems to be extremely peculiar. Therefore, the problem of constructing continuous stochastic processes, with the most general Hölder exponent that extend the FBM has been raised in [7]. Such processes would allow to model random signals with very erratic Hölder exponents and non-stationary increments, as for instance some signals that occur in finance or in turbulence. A partial answer to this problem is supplied by the GMBM [1, 2]. This continuous Gaussian process has been introduced by the author and J. Lévy Véhel. Roughly speaking it is obtained "by substituting" to the Hurst parameter of the FBM a sequence of Hölder functions. So, at least for this reason, the GMBM is an extension of the FBM and thus seems to be a good candidate for modeling.

Let us now introduce some notations that will allow us to define precisely the GMBM. These notations will be used extensively in the sequel (throughout this article, we will be using non-random as well as random constants. To ease distinction, we use small letters (e. g., c) to denote non-random constants and capital letters (e. g., $C = C(\omega)$) to denote random constants).

(A) The GMBM will mainly depend of one parameter a sequence $(H_n(.))_{n\in\mathbb{N}}$ of $(\beta, \tilde{c}_n)$-Hölder functions with values in an interval $[a, b] \subset]0, 1[$, fixed once and for all. Recall that f is said to be a (β, c)-Hölder function if

$$|f(t_1) - f(t_2)| \le c|t_1 - t_2|^{\beta} \ ,$$

whenever $|t_1 - t_2|$ is small enough. The admissible sequences $(H_n(.))_{n\in\mathbb{N}}$ will satisfy the condition:

$$\sup_{t\in[0,1]} H(t) < \beta \text{ and } \tilde{c}_n = O\left(2^{n(a-\eta)}\right) \ , \qquad (\mathcal{C})$$

where the fixed real $\eta > 0$ is arbitrarily small and for every t, $H(t) = \liminf_{n\to\infty} H_n(t)$.

(B) We will denote $(\hat{f}_n(.))$ a sequence of functions of the Schwartz class $S(\mathbb{R})$, satisfying for every ξ

$$\hat{f}_{-1}(\xi) = \hat{f}_{-1}(-\xi) = \begin{cases} 1 \text{ if } |\xi| \le \frac{2\pi}{3} \\ 0 \text{ if } |\xi| \ge \pi \end{cases} \qquad (1.2)$$

and for every $n \in \mathbb{N}$,

$$\hat{f}_n(\xi) = \hat{f}_{-1}\left(2^{-n-1}\xi\right) - \hat{f}_{-1}\left(2^{-n}\xi\right) = \hat{f}_0\left(2^{-n}\xi\right) \ . \qquad (1.3)$$

Recall that $S(\mathbb{R})$ is the space of all infinitely differentiable functions $u(t)$ which satisfy for all integers m and n, $\lim_{|t|\to\infty} t^m \left(\frac{d}{dt}\right)^n u(t) = 0$. Let us mention that to characterize the classical functional spaces one constructs Littlewood–Paley Analyses via such functions $\hat{f}_n$.

(C) The Generalized Multifractional Field (GMF) of parameter $(H_n(.))_{n\in\mathbb{N}}$, or more simply the GMF, will be the mean-zero Gaussian field $Y = \{Y(x, y)\}_{(x,y)\in[0,1]^2}$, with continuous realizations and the covariance kernel

$$E\left(Y(x, y)Y\left(x', y'\right)\right) = \int_{\mathbb{R}} K(x, y, \xi)\overline{K\left(x', y', \xi\right)}\, d\xi\ , \tag{1.4}$$

where for every reals x, y and ξ we have set

$$K(x, y, \xi) = \sum_{n=0}^{\infty} \frac{\left(e^{ix\xi} - 1\right)}{|\xi|^{H_n(y)+1/2}} \hat{f}_{n-1}(\xi)\ . \tag{1.5}$$

This definition makes sense. Indeed, for every, $(s_1, t_1), \ldots, (s_p, t_p) \in [0, 1]^2$ the matrix $\left(\int_{\mathbb{R}} K(s_i, t_i, \xi)K(s_j, t_j, \xi)\, d\xi\right)_{1\le i,j\le p}$ is positively defined: for all complex numbers $h_1, \ldots, h_p$, we have

$$\sum_{1\le i,j\le p} h_i\overline{h_j} \int_{\mathbb{R}} K(s_i, t_i, \xi)\overline{K(s_j, t_j, \xi)}\, d\xi = \int_{\mathbb{R}} \left|\sum_{k=1}^{p} h_k K(s_k, t_k, \xi)\right|^2 d\xi \ge 0\ .$$

The existence of a continuous version of Y follows from the Kolmogorov criterion, since for every (x, y) and (x', y')

$$E\left(\left|Y(x, y) - Y\left(x', y'\right)\right|^2\right) \le c\left(\left|x - x'\right|^2 + \left|y - y'\right|^2\right)^a\ . \tag{1.6}$$

The inequality (1.6) even implies that for any $\epsilon > 0$, with probability 1, the trajectories of Y are $(a - \epsilon)$-Hölder functions (see Lemma 8). At last, it is useful to note that Y can be represented as the Wiener integral

$$Y(x, y) = \int_{\mathbb{R}} K(x, y, \xi)\, dW(\xi)\ . \tag{1.7}$$

The Brownian measure dW, will be chosen so that Y be real-valued (see for example [2]). The GMBM $\{X(t)\}_{t\in[0,1]}$ can be defined as the restriction of the field $\{Y(x, y)\}_{(x,y)\in[0,1]^2}$ to the diagonal: for every $t \in [0, 1]$

$$X(t) = Y(t, t)\ . \tag{1.8}$$

One the main interest of this Gaussian process is that its Hölder exponent $\alpha_X(.)$ can be prescribed via the sequence $(H_n(.))_{n\in\mathbb{N}}$ and may differ widely from point to point. Indeed, it is shown in [1] and [2] that under the conditions (i) and (ii) below, one has for every t_0, almost surely (a.s. in short),

$$\alpha_X(t_0) = \liminf_{n\to\infty} H_n(t_0) = H(t_0)\ . \tag{1.9}$$

(i) For each $t \in [0, 1]$ and each $\epsilon > 0$, there are $n_0 \in \mathbb{N}$ and $\eta > 0$ (depending only on t and ϵ) such that the inequality $H_n(t + h) \ge H(t) - \epsilon$ holds for every $|h| \le \eta$ and $n \ge n_0$.

(ii) For all t, $H(t) < \beta$ and $\tilde{c}_n = O(n)$ when n goes to infinity.

Remarks.

1) The definition of the GMBM is not canonical. It depends on the choice of the function $\hat{f}_{-1}(.)$ in the Littlewood–Paley construction.
2) The GMBM depends on the whole sequence $(H_n(.))$ and not only on its lower-limit $H(.)$. However, we will show that when two admissible sequences $(H_n(.))$ and $(\tilde{H}_n(.))$ satisfy for all t and n, $|H_n(t) - \tilde{H}_n(t)| \leq c(t)2^{-n\delta(t)}$, then the corresponding GMBM's differ by "a smoother" process (see Theorem 2).
3) The GMBM is an extension of the FBM. Mainly for the following reasons.
   - It is defined "by substituting" to the Hurst parameter, in the harmonizable representation of the FBM, a sequence of admissible Hölder functions.
   - According to Proposition 3 in [1], under some technical condition on $(H_n(.))$, the GMBM is Locally Asymptotically Self-Similar in the sense of Benassi, Jaffard and Roux [4] i. e.,

$$\lim_{\rho \to 0^+} \text{law} \left\{ \frac{X(t_0 + \rho u) - X(t_0)}{\rho^{H(t_0)}} \right\}_u = \text{law}\{B_{H(t_0)}(u)\}_u \ ,$$

     where the process $\{B_{H(t_0)}(u)\}_u$ is an FBM of parameter $H(t_0)$. Roughly speaking this property means that at t_0, there is an FBM of parameter $H(t_0)$ tangent to the GMBM.
4) The GMBM is an extension of the MBM in the sense that when all the $H_n(.)$ are equal to the same Hölder function $H(.)$, then the GMBM is an MBM of parameter $H(.)$. Recall that the MBM has been introduced independently in [15] and in [4]. It is defined by substituting to the Hurst parameter of the FBM a Hölder function.
5) One can construct sequences $(H_n(.))$ satisfying (i) and (ii) whose lower-limits $H(.)$ have sets of discontinuities with accumulation points (see [1] Proposition 2). For instance $H(.)$ can be equal to a on the Cantor set and to b on its complementary.
6) The condition (i) implies that the function $H(.)$ is lower-semi-continuous, which may be restrictive in some situations. In contrast with (i), the conditions (ii) and $(\mathcal{C})$ are not restrictive at all: any lower-limit of continuous functions on $[0, 1]$ can be written as a lower-limit of a sequence of Lipschitz functions (or even a sequence of polynomials) satisfying (ii) or satisfying $(\mathcal{C})$ [5, 6]. Observe that $(\mathcal{C})$ is weaker than (ii). One of the main result of our article will be that the Relation (1.9) remains true, only under the condition $(\mathcal{C})$.

The goal of our article is to study some local properties of the GMBM (see Section 3). However, we think that the main point of it, is to introduce $\{Y(x, y)\}_{(x,y)\in[0,1]^2}$ the GMF and to show that it is β-Hölder in y, uniformly in x i. e., it satisfies almost surely for every $x, y, y' \in [0, 1]$,

$$\sup_{x\in[0,1]} \left|Y(x, y) - Y\left(x, y'\right)\right| \leq C \left|y - y'\right|^{\beta} \ .$$

Indeed, this nice property of Y, turns out to be very useful in some problems on the GMBM, as for instance the determination or the indentification of its Hölder exponent.

Our article is organized as follows. In Section 2, we introduce a "wavelet decomposition" of the GMF and using this nice decomposition we show that the GMF is β-Hölder in y uniformly in x. Then, this important result, allows us, in Section 3 to determine the pointwise (a.s.) Hölder exponent of the GMBM and to prove that two GMBM's with the same Hölder regularity differ by "a smoother" process.

2. The GMF is β-Hölder in y Uniformly in x

In this section our objective will be to show that

Proposition 1.
Let Y be the GMF, then we have almost surely,

$$\sup_{x\in[0,1]} \left|Y(x,y) - Y\left(x,y'\right)\right| \le C_1 \left|y-y'\right|^{\beta} \ , \tag{2.1}$$

where C_1 is a positive random variable of finite moment of arbitrary order.

To prove Proposition 1 we need to introduce "a wavelet decomposition" of the GMF.

2.1 "A Wavelet Decomposition" of the GMF

Let $\{\psi_{j,k}(x)\}_{j,k\in\mathbb{Z}}$ be a Lemarié–Meyer orthonormal wavelet basis of $L^2(\mathbb{R})$ [11, 13]. Recall that for every $j,k\in\mathbb{Z}$,

$$\psi_{j,k}(x) = 2^{j/2}\psi\left(2^j x - k\right) \ ,$$

where the mother wavelet ψ belongs to $S(\mathbb{R})$ and has a Fourier transform $\hat{\psi}$ with a support in the domain $\{\xi,\ 2\pi/3 \le |\xi| \le 8\pi/3\}$. In this subsection our goal will be to show that.

Proposition 2.
The GMF can be expressed as the random series,

$$Y(x,y) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} s_{j,k}(x,y)\epsilon_{j,k} \ , \tag{2.2}$$

where for every $(x,y)\in[0,1]^2$ and $(j,k)\in\mathbb{Z}^2$

$$s_{j,k}(x,y) = \int_{\mathbb{R}} \left(\sum_{n=0}^{\infty} \frac{\left(e^{ix\xi}-1\right)}{|\xi|^{H_n(y)+1/2}} \hat{f}_{n-1}(\xi)\right) \hat{\psi}_{j,k}(\xi)\, d\xi \ , \tag{2.3}$$

and $\{\epsilon_{j,k}\}_{j,k\in\mathbb{Z}}$ is a sequence of $\mathcal{N}(0,1)$ independent Gaussian variables. This series is almost surely convergent, for every $(x,y)\in[0,1]^2$. The deterministic coefficients $s_{j,k}(.,.)$ will be called "the wavelet coefficients."

To prove Proposition 2 we need the following results.

Remark 1. Let an arbitrary $(x,y)\in[0,1]^2$, the wavelet coefficients $s_{j,k}(x,y)$ can be expressed as:

- for every $j\in\mathbb{N}$ and $k\in\mathbb{Z}$,

$$s_{j,k}(x,y) = \sum_{l=0}^{2} 2^{-jH_{j+l}(y)} \left(g_{l-1}\left(2^j x + k, H_{j+l}(y)\right) - g_{l-1}(k, H_{j+l}(y))\right) \ , \tag{2.4}$$

- for every $k\in\mathbb{Z}$

$$s_{-1,k}(x,y) = \sum_{l=0}^{1} 2^{H_l(y)} \left(g_{l-1}\left(2^j x + k, H_l(y)\right) - g_{l-1}(k, H_l(y))\right) \ , \tag{2.5}$$

- for every $j \geq 2$ and $k \in \mathbb{Z}$

$$s_{-j,k}(x,y) = 2^{jH_0(y)}\left(g_{-1}\left(2^j x + k, H_0(y)\right) - g_{-1}(k, H_0(y))\right) , \qquad (2.6)$$

where for all $l \in \{-1, 0, 1\}$, $X \in \mathbb{R}$ and $\alpha \in [a,b] \subset]0,1[$, we have set

$$g_l(X,\alpha) = \int_{\mathbb{R}} e^{iX\xi}\left(\frac{\hat{f}_l(\xi)\hat{\psi}(\xi)}{|\xi|^{\alpha+1/2}}\right) d\xi \ . \qquad (2.7)$$

Proof of Remark 1. We will only give the proof of (2.4) that of (2.5) and (2.6) are quite similar. The intersection of the supports of $\hat{f}_n$ and $\hat{\psi}_{j,k}$ being a negligible set when $j \notin \{n-1, n, n+1\}$, we get that

$$s_{j,k}(x,y) = \sum_{l=0}^{2} 2^{-j/2}\int_{\mathbb{R}} \frac{(e^{ix\xi}-1)}{|\xi|^{H_{j+l}(y)+1/2}} e^{k2^{-j}\xi}\hat{\psi}\left(2^{-j}\xi\right)\hat{f}_{j+l-1}(\xi)\, d\xi \ .$$

Then setting $u = 2^{-j}\xi$ in this last integral and using (1.3) we obtain (2.4). □

Lemma 1.

Let a, b as in paragraph (A) of Section 1 and $\tilde{h}$ the function defined for all $(X,\alpha) \in \mathbb{R} \times [a,b]$ as

$$\tilde{h}(X,\alpha) = \int_{\mathbb{R}} e^{iX\xi}\frac{\hat{h}(\xi)}{|\xi|^{\alpha+1/2}}\, d\xi \ ,$$

where $\hat{h}$ is a function of $S(\mathbb{R})$, with support in the domain $\Delta = \{\xi \in \mathbb{R},\ 0 < d \leq |\xi| \leq D\}$. Then for any integer $L \geq 0$, there is a constant $c_2 > 0$, depending only on L, a and b, such that for all $(X,\alpha) \in \mathbb{R} \times [a,b]$,

$$\left|\tilde{h}(X,\alpha)\right| \leq c_2(2+|X|)^{-L} \ . \qquad (2.8)$$

Proof of Lemma 1. We will suppose that $X \geq 0$, the proof is quite similar when $X < 0$. Let us set $\hat{U}(\xi) = e^{-i2\xi}\hat{h}(\xi)$, $\hat{H}(\xi,\alpha) = \frac{\hat{U}(\xi)}{|\xi|^{\alpha+1/2}}$ if $(\xi,\alpha) \in \Delta \times [a,b]$ and $\hat{H}(\xi,\alpha) = 0$ else. It is clear that for any $\alpha \in [a,b]$ the function $\xi \mapsto \hat{H}(\xi,\alpha)$ belongs to $S(\mathbb{R})$. Let us first prove that

$$c = \sup\left\{\left|\left(\frac{\partial}{\partial\xi}\right)^L \hat{H}(\xi,\alpha)\right|,\ (\xi,\alpha) \in \Delta \times [a,b]\right\} < \infty \, . \qquad (2.9)$$

Using the Leibniz Formula we obtain that for any $(\xi,\alpha) \in \Delta$

$$\left(\frac{\partial}{\partial\xi}\right)^L \hat{H}(\xi,\alpha) = \sum_{k=0}^{L}(-1)^{k\sigma(\xi)} e_{L,k}(\alpha)\frac{\hat{U}^{(L-k)}(\xi)}{|\xi|^{\alpha+1/2+k}} \ , \qquad (2.10)$$

where $\sigma(\xi) = 1$ if $\xi < 0$ and $\sigma(\xi) = 0$ else, $e_{L,0}(\alpha) = 1$ and $e_{L,k}(\alpha) = \begin{pmatrix} L \\ k \end{pmatrix}\prod_{n=1}^{k}(\alpha + n - 1/2)$ else. Then, as for every integers $0 \leq k \leq L$ and $(\xi,\alpha) \in \Delta \times [a,b]$

$$|e_{L,k}(\alpha)| \leq \begin{pmatrix} L \\ k \end{pmatrix}\prod_{n=1}^{k}(b+n-1/2) + 1 \ ,$$

and

$$\frac{\left|\hat{U}^{(L-k)}(\xi)\right|}{|\xi|^{\alpha+1/2+k}} \leq \left\|\hat{U}^{(L-k)}\right\|_{\infty}\left(\frac{1}{|d|^{a+1/2+k}}+\frac{1}{|d|^{b+1/2+k}}+\frac{1}{|D|^{a+1/2+k}}+\frac{1}{|D|^{b+1/2+k}}\right)$$

we get (2.9). Now let us prove (2.8), for any $(X,\alpha) \in \mathbb{R}_+ \times [a,b]$, we have

$$\tilde{h}(X,\alpha) = \int_{\mathbb{R}} e^{i(2+|X|)\xi}\,\hat{H}(\xi,\alpha)\,d\xi\ .$$

After L integrations by parts we obtain that

$$\left|\tilde{h}(X,\alpha)\right| = (2+|X|)^{-L}\left|\int_{\mathbb{R}} e^{i(2+|X|)\xi}\left(\frac{\partial}{\partial\xi}\right)^{L}\hat{H}(\xi,\alpha)\,d\xi\right|\ . \tag{2.11}$$

At last the support of the function $\xi \mapsto \left(\frac{\partial}{\partial\xi}\right)^{L}\hat{H}(\xi,\alpha)$ being contained in the domain Δ for any $\alpha \in [a,b]$, it follows from (2.9) and (2.11) that for each $(X,\alpha) \in \mathbb{R}_+ \times [a,b]$,

$$\begin{aligned}\left|\tilde{h}(X,\alpha)\right| &= (2+|X|)^{-L}\int_{\Delta}\left|\left(\frac{\partial}{\partial\xi}\right)^{L}\hat{H}(\xi,\alpha)\right|d\xi \\ &\leq 2(D-d)c(2+|X|)^{-L}\ .\end{aligned}$$

□

Lemma 2.

Let a and b be as in paragraph (A) of Section 1. For every integer $L>0$ there is a constant $c_3>0$ (depending only on a, b and L) such that the inequalities

$$|s_{j,k}(x,y)| \leq c_3 2^{-ja}\left\{\left(1+\left|2^j x+k\right|\right)^{-L}+(1+|k|)^{-L}\right\} \tag{2.12}$$

and

$$|s_{-j,k}(x,y)| \leq c_3 2^{-j(1-b)}(1+|k|)^{-L}\ , \tag{2.13}$$

hold for all $j\in\mathbb{N}$, $k\in\mathbb{Z}$ and $x, y, y' \in [0,1]$.

Proof of Lemma 2. First, let us prove (2.12). It follows from Remark 1 that it is sufficient to show that there is a constant $c>0$, such that for any $l \in \{-1,0,1\}$ and any $(X,\alpha) \in \mathbb{R}\times[a,b]$,

$$|g_l(X,\alpha)| \leq c(2+|X|)^{-L}\ .$$

This inequality will result from Lemma 1 by taking for each $l \in \{-1,0,1\}$, $\hat{h}(\xi) = f_l(\xi)\hat{\psi}(\xi)$. Let us now prove (2.13). We will restrict to the case, $j \geq 2$, the case $j=1$ can be treated similarly. Applying the Mean Value Theorem to the right member of (2.6), we obtain that for some real $c \in]k,k+1[$ (depending on x, y, j, k), we have

$$\begin{aligned}|s_{-j,k}(x,y)| &= 2^{-j(1-H_0(y))}\left|\left(\frac{\partial}{\partial X}\right)g_{-1}(c,H_0(y))\right| \\ &\leq 2^{-j(1-b)}\left|\left(\frac{\partial}{\partial X}\right)g_{-1}(c,H_0(y))\right|\ .\end{aligned}$$

Thus, it is sufficient to prove that there is a constant $c' > 0$ such that for any $(X, \alpha) \in \mathbb{R} \times [a, b]$,

$$\left| \left(\frac{\partial}{\partial X} \right) g_{-1}(X, \alpha) \right| \le c'(2 + |X|)^{-L} .$$

As $\left(\frac{\partial}{\partial X}\right) g_{-1}(X, \alpha) = i \int_R e^{iX\xi} \left(\frac{\xi \hat{f}_{-1}(\xi)\hat{\psi}(\xi)}{|\xi|^{\alpha+1/2}} \right) d\xi$, this last inequality will result from Lemma 1 by taking $\hat{h}(\xi) = \xi \hat{f}_{-1}(\xi)\hat{\psi}(\xi)$. □

Now we will state two lemmas that allow to bound the sequence $\{\epsilon_{j,k}\}_{j,k\in\mathbb{Z}}$. These lemmas are similar to that given [14, 3].

Lemma 3.

Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a Gaussian process such that for every n, ϵ_n is an $\mathcal{N}(0, 1)$ Gaussian variable. Then, there is a positive random variable C of finite moment of arbitrary order such that, almost surely, for every n

$$|\epsilon_n| \le C\sqrt{\log(2 + n)} .$$

Proof of Lemma 3. We set for any $n \in \mathbb{N}$, $\eta_n = \log^{-1/2}(2+n)\epsilon_n$ and $C = \sup |\eta_n|$. If we show that the random variable C is finite (a.s.), then its moment of arbitrary order will be finite as well (see [8, 10]) and the lemma will be proved. To prove that $C < \infty$ (a.s.) we will use the Borel–Cantelli Lemma. Let a fixed real $\alpha > \sqrt{2}$, for any integer $n \ge 1$ we have,

$$P(|\eta_n| \ge \alpha) = \sqrt{\frac{2}{\pi}} \int_{\alpha\sqrt{\log(2+n)}}^{+\infty} e^{-x^2/2}\, dx .$$

Integration by parts yields

$$\begin{aligned}
&\int_{\alpha\sqrt{\log(2+n)}}^{+\infty} x^{-1} \left(xe^{-x^2/2} \right) dx \\
&= \left[-x^{-1}e^{-x^2/2} \right]_{\alpha\sqrt{\log(2+n)}}^{+\infty} - \int_{\alpha\sqrt{\log(2+n)}}^{+\infty} x^2 e^{-x^2/2}\, dx \\
&\le \left(\alpha\sqrt{\log(2 + n)} \right)^{-1} e^{-\alpha^2 \log(2+n)/2} \\
&\le (2 + n)^{-\alpha^2/2} ,
\end{aligned}$$

so that $\sum_{n=0}^{\infty} P(|\eta_n| > 1) < \infty$. By the Borel–Cantelli Lemma, $|\eta_n| > \alpha$ occurs almost surely only for finitely many n. This implies that $C < \infty$ (a.s.). □

Lemma 4.

Let $\{\epsilon_M\}_{M\in\mathbb{Z}^d}$ be a Gaussian process such that for every M, ϵ_M is an $\mathcal{N}(0, 1)$ Gaussian variable. Then, there is a positive random variable C of finite moment of arbitrary order, such that, almost surely, for each $M = (m_1, \ldots , m_d) \in \mathbb{Z}^d$,

$$|\epsilon_M| \le C \sqrt{\log\left(2 + \sum_{i=1}^{d} |m_i|\right)} . \tag{2.14}$$

Proof of Lemma 4. This lemma is essentially a consequence of Lemma 3. We will suppose that $d \geq 2$ and proceed by induction on d. Recall that the map ϕ:$\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ defined for all (p, q) as

$$\phi(p, q) = \frac{(p+q)(p+q+1)}{2} + p + 1 \,, \tag{2.15}$$

is a bijection. It was introduced by Cantor and is often used to prove that the set $\mathbb{N} \times \mathbb{N}$ is countable. Thus, the map φ: $\mathbb{Z}^{d-2} \times \mathbb{N} \times \mathbb{N} \to \mathbb{Z}^{d-2} \times \mathbb{N}$ defined for all $M = (m_1, \ldots, m_{d-2}, m_{d-1}, m_d)$ as

$$\varphi(M) = (m_1, \ldots, m_{d-2}, \phi(m_{d-1}, m_d)) \,, \tag{2.16}$$

is a bijection. Hence by replacing M by $\varphi(M)$, one can view the sequence $\{\epsilon_M,\ M \in \mathbb{Z}^{d-2} \times \mathbb{N}^2\}$ as a sequence indexed by $\mathbb{Z}^{d-2} \times \mathbb{N}$ and thus by using the induction hypothesis, we obtain that, almost surely, for each $M = (m_1, \ldots, m_{d-2}, m_{d-1}, m_d) \in \mathbb{Z}^{d-2} \times \mathbb{N}^2$

$$|\epsilon_M| \leq \tilde{C} \sqrt{\log\left(2 + \sum_{i=2}^{d-2} |m_i| + \phi(m_{d-1}, m_d)\right)} \,, \tag{2.17}$$

where $\tilde{C}$ is a positive random variable of finite moment of arbitrary order. Then, as $\phi(m_{d-1}, m_d) \leq c(|m_{d-1}| + |m_d|)^2$, using (2.17), we obtain (2.14). Similarly, one can view the sequences $\{\epsilon_M,\ M \in \mathbb{Z}^{d-2} \times (\mathbb{Z}_-) \times \mathbb{N}\}$, $\{\epsilon_M,\ M \in \mathbb{Z}^{d-2} \times \mathbb{N} \times (\mathbb{Z}_-)\}$ and $\{\epsilon_M,\ M \in \mathbb{Z}^{d-2} \times (\mathbb{Z}_-) \times (\mathbb{Z}_-)\}$ as sequences indexed by $\mathbb{Z}^{d-2} \times \mathbb{N}$ and thus we can prove that (2.14) is satisfied for any $M \in \mathbb{Z}^d$. □

Proof of Proposition 2. It follows from Lemmas 2 and 4 that there is a random variable $C > 0$ of finite moment of arbitrary order such that, almost surely for all $(x, y) \in [0, 1]^2$

$$\begin{aligned}
&|Y(x, y)| \\
&\quad \leq C \textstyle\sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-ja} \left[\left(1 + \left|2^j x - k\right|\right)^{-2} + (1 + |k|)^{-2}\right] \log^{1/2}(2 + j + |k|) \\
&\qquad + C \textstyle\sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-j(1-b)} (1 + |k|)^{-2} \log^{1/2}(2 + j + |k|) \,.
\end{aligned}$$

Let us show that the series $\sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} 2^{-ja} \frac{\log^{1/2}(2+j+|k|)}{(1+|2^j x - k|)^2}$, is convergent for every $x \in [0, 1]$. One can similarly prove that the series $\sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} 2^{-ja} \frac{\log^{1/2}(2+j+|k|)}{(1+|k|)^2}$ and $\sum_{j=1}^{+\infty} \sum_{k=-\infty}^{+\infty} 2^{-j(1-b)} \frac{\log^{1/2}(2+j+|k|)}{(1+|k|)^2}$ are convergent. Using the sub-additivity of the function $x \mapsto \log^{1/2}(2 + x)$ we obtain that for any $j \in \mathbb{N}$,

$$\begin{aligned}
&\textstyle\sum_{k=-\infty}^{\infty} \frac{\log^{1/2}(2+j+|k|)}{\left(1+\left|2^j x - k\right|\right)^2} = \sum_{k=-\infty}^{\infty} \frac{\log^{1/2}\left(2+j+\left|k+\left[2^j x\right]\right|\right)}{\left(1+\left|2^j x - \left[2^j x\right] - k\right|\right)^2} \\
&\quad \leq \textstyle\sum_{k=-\infty}^{\infty} \frac{\log^{1/2}(2+|k|)}{\left(1+\left|2^j x - \left[2^j x\right] - k\right|\right)^2} + \sum_{k=-\infty}^{\infty} \frac{\log^{1/2}\left(2+j+2^j\right)}{\left(1+\left|2^j x - \left[2^j x\right] - k\right|\right)^2} \,, \\
&\quad \leq 2e \log^{1/2}\left(2 + j + 2^j\right) \,,
\end{aligned}$$

where $e = 2 \sup_{X \in [0,1]} \sum_{k=-\infty}^{\infty} \frac{\log^{1/2}(2+|k|)}{(1+|X-k|)^2} < \infty$ and $[2^j x]$ denotes the integer part of $2^j x$. Thus,

$$\sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-ja} \frac{\log^{1/2}(2 + j + |k|)}{\left(1 + |2^j x - k|\right)^2} \leq e \sum_{j=0}^{\infty} 2^{-ja} \log^{1/2}\left(2 + j + 2^j\right) < \infty \,. \qquad \square$$

To conclude this subsection, let us mention that following the same ideas as in [3], one can show that the series (2.2) is almost surely uniformly convergent in $(x, y) \in [0, 1]^2$. This result is clearly stronger than Proposition 2. We will not give its proof, since we do not need it in the present article.

2.2 Proof of Proposition 1

The following estimations of the "wavelet coefficients" $s_{j,k}(x, y)$ will be very useful.

Lemma 5.

Let a, b, β and η be as in paragraph (A) of Section 1. For any integer $L > 0$, there is a constant $c_4 > 0$ (depending only on a, b and L) such that the inequalities

$$\left|s_{j,k}(x, y)-s_{j,k}\left(x, y'\right)\right| \leq c_4 2^{-j\eta/2}\left\{\frac{1}{\left(1+\left|2^j x-k\right|\right)^L}+\frac{1}{(1+|k|)^L}\right\}\left|y-y'\right|^{\beta} \quad (2.18)$$

and

$$\left|s_{-j,k}(x, y)-s_{-j,k}\left(x, y'\right)\right| \leq c_4 \frac{(1+j)2^{-j(1-b)}}{(1+|k|)^L}\left|y-y'\right|^{\beta} \quad (2.19)$$

hold for all $x, y, y' \in [0, 1]$, $j \in \mathbb{N}$ and $k \in \mathbb{Z}$.

Note that Lemma 5 is to a certain extent, inspired from Proposition 4.1, p. 79 in [4]. To prove this lemma we need the following result.

Lemma 6.

Let a, b be as in paragraph (A) of Section 1 and for any $(X, \alpha) \in \mathbb{R} \times [a, b]$, let

$$g(X, \alpha)=\int_{\mathbb{R}} e^{iX\xi} \frac{\hat{\varphi}(\xi)}{|\xi|^{\alpha+1/2}}\, d\xi\ ,$$

where $\hat{\varphi}$ is a function of $S(\mathbb{R})$ with support in the domain $\{\xi \in \mathbb{R},\ 0 < d \leq |\xi| \leq D\}$. Then, for any integer $L > 0$, there is a constant $c_5 > 0$ such that the inequalities

$$\left|2^{-j\alpha} g\left(2^j x+k, \alpha\right)-2^{-j\alpha'} g\left(2^j x+k, \alpha'\right)\right| \leq c_5 \frac{(1+j)2^{-ja}}{\left(1+\left|2^j x+k\right|\right)^L}\left|\alpha-\alpha'\right| \quad (2.20)$$

and

$$\begin{aligned}&\Big|2^{j\alpha}\left(g\left(2^{-j}x+k, \alpha\right)-g(k, \alpha)\right)\\ &\quad -2^{j\alpha'}\left(g\left(2^{-j}x+k, \alpha'\right)-g\left(k, \alpha'\right)\right)\Big| \leq c_5 \tfrac{(1+j)2^{-j(1-b)}}{(1+|k|)^L}\left|\alpha-\alpha'\right|\end{aligned} \quad (2.21)$$

hold for all $j \in \mathbb{N}$, $k \in \mathbb{Z}$, $\alpha, \alpha' \in [a, b]$ and $x \in [0, 1]$.

Proof of Lemma 6. First we will prove the inequality (2.20). We have for all $j \in \mathbb{N}$, $k \in \mathbb{Z}$, $\alpha, \alpha' \in [a, b]$ and $x \in [0, 1]$.

$$\left|2^{-j\alpha} g\left(2^j x+k, \alpha\right)-2^{-j\alpha'} g\left(2^j x+k, \alpha'\right)\right| \leq S_{j,k}\left(x, \alpha, \alpha'\right)+T_{j,k}\left(x, \alpha, \alpha'\right) \quad (2.22)$$

where

$$S_{j,k}\left(x, \alpha, \alpha'\right)=2^{-j\alpha'}\left|g\left(2^j x+k, \alpha\right)-g\left(2^j x+k, \alpha'\right)\right| \quad (2.23)$$

and

$$T_{j,k}\left(x,\alpha,\alpha'\right) = \left|2^{-j\alpha} - 2^{-j\alpha'}\right| \left|g\left(2^j x + k, \alpha\right)\right| . \tag{2.24}$$

Let us give an ad hoc upper bound of $S_{j,k}(x,\alpha,\alpha')$. Using the Mean Value Theorem we obtain that

$$S_{j,k}\left(x,\alpha,\alpha'\right) = 2^{-j\alpha'} \left|\left(\frac{\partial}{\partial\alpha}\right) g\left(2^j x + k, \gamma\right)\right| \left|\alpha - \alpha'\right| , \tag{2.25}$$

for some $\gamma \in]a,b[$ depending on j, k, x, α, α'. Moreover, since

$$\left(\frac{\partial}{\partial\alpha}\right) g(X,\alpha) = -\int_{\mathbb{R}} e^{iX\xi} \left(\frac{\hat{\varphi}(\xi)\log|\xi|}{|\xi|^{\alpha+1/2}}\right) d\xi .$$

Taking in Lemma 1 $\hat{h}(\xi) = \hat{\varphi}(\xi)\log|\xi|$, we obtain that for some constant $c > 0$ and all $(X,\alpha) \in \mathbb{R}\times[a,b]$,

$$\left|\left(\frac{\partial}{\partial\alpha}\right) g(X,\alpha)\right| \le c(2+|X|)^{-L} .$$

Therefore, it follows from (2.25) that for any $x \in [0,1]$, $j \in \mathbb{N}$, $k \in \mathbb{Z}$ and $\alpha, \alpha' \in [a,b]$,

$$S_{j,k}\left(x,\alpha,\alpha'\right) \le c\frac{2^{-ja}}{\left(1+\left|2^j x + k\right|\right)^L} \left|\alpha - \alpha'\right| . \tag{2.26}$$

Now let us give an ad hoc upper bound of $T_{j,k}(x,\alpha,\alpha')$. Taking in Lemma 1, $\hat{h}(\xi) = \hat{\varphi}(\xi)$, we obtain that for some constant $c' > 0$ and all $(X,\alpha) \in \mathbb{R}\times[a,b]$,

$$|g(X,\alpha)| \le c'(1+|X|)^{-L} . \tag{2.27}$$

Moreover, using the Mean Value Theorem we get that

$$\left|2^{-j\alpha} - 2^{-j\alpha'}\right| \le (\log 2) j 2^{-ja} \left|\alpha - \alpha'\right| . \tag{2.28}$$

Thus, it follows from (2.24), (2.27), and (2.28) that for any $x \in [0,1]$, $j \in \mathbb{N}$, $k \in \mathbb{Z}$ and $\alpha, \alpha' \in [a,b]$

$$T_{j,k}\left(x,\alpha,\alpha'\right) \le c''\frac{j2^{-ja}}{\left(1+\left|2^j x + k\right|\right)^L} \left|\alpha - \alpha'\right| .$$

Now, let us prove the inequality (2.21). We have for all $j \in \mathbb{N}$, $k \in \mathbb{Z}$, $\alpha, \alpha' \in [a,b]$ and $x \in [0,1]$

$$\begin{aligned}&\left|2^{j\alpha}\left(g\left(2^{-j}x + k, \alpha\right) - g(k,\alpha)\right) - 2^{j\alpha'}\left(g\left(2^{-j}x + k, \alpha'\right) - g\left(k,\alpha'\right)\right)\right| \\ &\quad \le K_{j,k}\left(x,\alpha,\alpha'\right) + L_{j,k}\left(x,\alpha,\alpha'\right) ,\end{aligned}$$

where

$$\begin{aligned}K_{j,k}\left(x,\alpha,\alpha'\right) = 2^{j\alpha'} &\left|\left(g\left(2^{-j}x + k, \alpha\right) - g(k,\alpha)\right)\right. \\ &\left. - \left(g\left(2^{-j}x + k, \alpha'\right) - g\left(k,\alpha'\right)\right)\right| ,\end{aligned} \tag{2.29}$$

and

$$L_{j,k}\left(x,\alpha,\alpha'\right) = \left|2^{j\alpha} - 2^{j\alpha'}\right| \left|g\left(2^{-j}x+k,\alpha\right) - g(k,\alpha)\right| . \tag{2.30}$$

Let us give an ad hoc upper bound of $K_{j,k}(x,\alpha,\alpha')$. Using the Mean Value Theorem, we obtain that

$$K_{j,k}\left(x,\alpha,\alpha'\right) = 2^{j\alpha'} \left|\left(\frac{\partial}{\partial\alpha}\right) g\left(2^{-j}x+k,\delta\right) - \left(\frac{\partial}{\partial\alpha}\right) g(k,\delta)\right| \left|\alpha-\alpha'\right| , \tag{2.31}$$

for some $\delta \in]a,b[$ depending on j,k,x,α,α'. Then using again the Mean Value Theorem, we get that

$$K_{j,k}\left(x,\alpha,\alpha'\right) = 2^{-j(1-\alpha')} \left|\left(\frac{\partial^2}{\partial X\partial\alpha}\right) g(e,\delta)\right| \left|\alpha-\alpha'\right| |x| , \tag{2.32}$$

for some $e \in]k,k+1[$ depending on j,k,x,α,α'. Moreover, since

$$\left(\frac{\partial^2}{\partial X\partial\alpha}\right) g(X,\alpha) = -i\int_{\mathbb{R}} e^{iX\xi}\left(\frac{\xi\hat{\varphi}(\xi)\log|\xi|}{|\xi|^{\alpha+1/2}}\right) d\xi ,$$

taking in Lemma 1, $\hat{h}(\xi) = \xi\hat{\varphi}(\xi)\log|\xi|$, it follows that for some constant $\lambda > 0$ and all $(X,\alpha) \in \mathbb{R}\times[a,b]$

$$\left|\left(\frac{\partial^2}{\partial X\partial\alpha}\right) g(X,\alpha)\right| \le \lambda(2+|X|)^{-L} .$$

Therefore, as $x \in [0,1]$ and $e \in]k,k+1[$, (2.32) implies that for all $j \in \mathbb{N}$, $k \in \mathbb{Z}$, $x \in [0,1]$ and $\alpha,\alpha' \in [a,b]$,

$$K_{j,k}\left(x,\alpha,\alpha'\right) \le \lambda\frac{2^{-j(1-b)}}{(1+|k|)^L}\left|\alpha-\alpha'\right| .$$

Now, let us give an ad hoc upper bound of $L_{j,k}(x,\alpha,\alpha')$. It follows from the Mean Value Theorem that

$$\left|2^{j\alpha} - 2^{j\alpha'}\right| \le (\log 2)j2^{jb}\left|\alpha-\alpha'\right| \tag{2.33}$$

and

$$\left|g\left(2^{-j}x+k,\alpha\right) - g(k,\alpha)\right| = \left|\left(\frac{\partial}{\partial X}\right) g\left(e',\alpha\right)\right| \left|2^{-j}x\right| , \tag{2.34}$$

for some $e' \in]k,k+1[$ depending on j,k,x,α,α'. Moreover, since

$$\left(\frac{\partial}{\partial X}\right) g(X,\alpha) = i\int_{\mathbb{R}} e^{iX\xi}\left(\frac{\xi\hat{\varphi}(\xi)}{|\xi|^{\alpha+1/2}}\right) d\xi ,$$

taking in Lemma 1 $\hat{h}(\xi) = \xi\hat{\varphi}(\xi)$, we obtain that for some constant $\mu > 0$ and all $(X,\alpha) \in \mathbb{R}\times[a,b]$,

$$\left|\left(\frac{\partial}{\partial X}\right) g(X,\alpha)\right| \le \mu(2+|X|)^{-L} .$$

Therefore, as $x \in [0,1]$ and $e' \in]k, k+1[$, (2.34) implies that

$$\left| g\left(2^{-j}x + k, \alpha\right) - g(k, \alpha) \right| \leq \mu \frac{2^{-j}}{(1+|k|)^L} \left|\alpha - \alpha'\right| \; . \tag{2.35}$$

At last it follows from (2.30), (2.33), and (2.35), that for all $j \in \mathbb{N}$, $k \in \mathbb{Z}$, $x \in [0,1]$ and $\alpha, \alpha' \in [a,b]$,

$$L_{j,k}\left(x, \alpha, \alpha'\right) \leq \mu \frac{2^{-j(1-b)}}{(1+|k|)^L} \left|\alpha - \alpha'\right| \; . \qquad \square$$

Proof of Lemma 5. Let us first prove the inequality (2.18). It follows from (2.4) that for all $j \in \mathbb{N}$, $k \in \mathbb{Z}$ and $x, y, y' \in [0,1]$,

$$\left| s_{j,k}(x,y) - s_{j,k}\left(x, y'\right) \right| \leq T_{j,k}\left(x, y, y'\right) + T_{j,k}\left(0, y, y'\right) \; ,$$

where

$$\begin{aligned} T_{j,k}\left(x, y, y'\right) = \textstyle\sum_{l=0}^{2} \Big| & 2^{-jH_{j+l}(y)} g_{l-1}\left(2^j x + k, H_{j+l}(y)\right) \\ & - 2^{-jH_{j+l}(y')} g_{l-1}\left(2^j x + k, H_{j+l}\left(y'\right)\right) \Big| \; . \end{aligned}$$

Let us give an ad hoc upper bound of $T_{j,k}(x, y, y')$. Taking in (2.20), for each $l \in \{0,1,2\}$, $g = g_{l-1}$, $\alpha = H_{j+l}(y)$ and $\alpha' = H_{j+l}(y')$, we obtain that

$$T_{j,k}\left(x, y, y'\right) \leq 2c_5 \sum_{l=0}^{2} \frac{(1+j)2^{-ja}}{\left(1 + \left|2^j x + k\right|\right)^L} \left| H_{j+l}(y) - H_{j+l}\left(y'\right) \right| \; . \tag{2.36}$$

Then, since the sequence $(H_n(.))$ satisfies the condition $(\mathcal{C})$ (see paragraph (A) of Section 1), it follows from (2.36) that

$$\begin{aligned} T_{j,k}\left(x, y, y'\right) &\leq 2c_5 \sum_{l=0}^{2} \frac{(1+j)2^{-ja}\tilde{c}_{j+l}}{\left(1 + \left|2^j x + k\right|\right)^L} \left| y - y' \right|^\beta \\ &\leq c \sum_{l=0}^{2} \frac{(1+j)2^{-ja}2^{(j+l)(a-\eta)}}{\left(1 + \left|2^j x + k\right|\right)^L} |y - y'|^\beta \\ &\leq c_4 \frac{2^{-j\eta/2}}{\left(1 + \left|2^j x + k\right|\right)^L} \left| y - y' \right|^\beta \; . \end{aligned}$$

Now, let us prove the inequality (2.19). We will only deal with the case $j \geq 2$, the case $j = 1$, can be studied similarly. It follows from (2.6) that for all $j \geq 2$, $k \in \mathbb{Z}$ and $x, y, y' \in [0,1]$

$$\begin{aligned} \left| s_{-j,k}(x,y) - s_{-j,k}\left(x, y'\right) \right| \leq \Big| & 2^{jH_0(y)} \left(g_{-1}\left(2^{-j}x + k, H_0(y)\right) - g_{-1}(k, H_0(y)) \right) \\ & - 2^{jH_0(y')} \left(g_{-1}\left(2^{-j}x + k, H_0\left(y'\right)\right) - g_{-1}\left(k, H_0\left(y'\right)\right) \right) \Big| \; . \end{aligned}$$

Taking in (2.21) $g = g_{-1}$, $\alpha = H_0(y)$ and $\alpha' = H_0(y')$, it follows that,

$$\left| s_{-j,k}(x,y) - s_{-j,k}\left(x, y'\right) \right| \leq c_5 \frac{(1+j)2^{-j(1-b)}}{(1+|k|)^L} \left| H_0(y) - H_0\left(y'\right) \right| \; . \tag{2.37}$$

Then since $H_0(.)$ is a $(\beta, \tilde{c}_0)$-Hölder function, (2.38) implies that

$$\left|s_{-j,k}(x,y) - s_{-j,k}\left(x,y'\right)\right| \le c_4 \frac{(1+j)2^{-j(1-b)}}{(1+|k|)^L} \left|y-y'\right|^{\beta} . \qquad \square \tag{2.38}$$

As a conclusion, let us give the proof Proposition 1, the main result of this section.

Proof of Proposition 1. Using (2.2), Lemma 4, and Lemma 5 with $L = 2$, there is a positive random variable of finite moment of arbitrary order, such that, almost surely, for any $x, y, y' \in [0,1]$

$$\left|Y(x,y) - Y\left(x,y'\right)\right| \le C(R(x) + R(0) + B)\left|y-y'\right|^{\beta} , \tag{2.39}$$

where

$$R(x) = \sum_{j=0}^{+\infty}\sum_{k=-\infty}^{+\infty} 2^{-j\eta/2}\left(1+\left|2^j x - k\right|\right)^{-2}\log^{1/2}(2+j+|k|)$$

and

$$B = 4\sum_{j=0}^{\infty}\sum_{k=-\infty}^{\infty}(1+j)2^{-j(1-b)}(1+|k|)^{-2}\log^{1/2}\left(2+2^j\right)\log^{1/2}(2+|k|) .$$

Clearly, this last series is convergent. Moreover, following the same lines as in the proof of Proposition 2, one can show that there is a constant $e > 0$, such that for every $x \in [0,1]$,

$$R(x) \le e\sum_{j=0}^{+\infty} 2^{-j\eta/2}\log^{1/2}\left(2+j+2^j\right) < \infty ,$$

which completes the Proof of Proposition 1. $\square$

3. Some Local Properties of the GMBM

3.1 Pointwise Hölder Regularity of the GMBM

The aim of this subsection is to show the following result.

Theorem 1.
Let $X = \{X(t)\}_{t\in[0,1]}$ be a GMBM of parameter an arbitrary admissible sequence $(H_n(.))$. Then at any t_0, the pointwise Hölder exponent of X satisfies almost surely,

$$\alpha_X(t_0) = \liminf_{n\to\infty} H_n(t_0) . \tag{3.1}$$

To prove Theorem 1, we need the following results. First, we will give a lemma that allows to bound above, the pointwise, almost sure, Hölder exponent of an arbitrary second order stochastic process. For the shake of generality, we have not restricted to Gaussian processes.

Lemma 7.

Let $\{S(t)\}_{t\in[0,1]}$ be a real-valued, second order and mean-zero stochastic process satisfying

$$\lim_{n\to\infty} E\left(\frac{|S(s+h_n)-S(s)|^2}{|h_n|^{2\mu}}\right) = +\infty \ , \tag{3.2}$$

where s is some point of $[0,1]$, μ a fixed positive exponent and (h_n) a sequence of non-vanishing reals that converges to 0. For every integer n, we set $Z_n = 1 + \frac{|S(s+h_n)-S(s)|}{|h_n|^{\mu}}$, $\sigma_n^2 = E(Z_n^2)$ and $Y_n = \frac{Z_n}{\sigma_n}$. Then a subsequence of (Y_n) converges in distribution to a non-negative, square integrable, random variable Y. Moreover, when $P(Y=0)=0$, the Hölder exponent of S at s, satisfies almost surely,

$$\alpha_S(s) \le \mu \ . \tag{3.3}$$

Proof of Lemma 7. The unit ball of $L^2(\Omega)$ being weakly compact, one can extract from the sequence (Y_n) a subsequence (Y_{n_k}) converging in distribution to a non-negative, square integrable random variable Y. Let a real $\eta > 0$ and an integer m be arbitrary and fixed. Since $\lim_{k\to\infty} \sigma_{n_k} = +\infty$, we have for k big enough $\frac{1}{\sigma_{n_k}\eta} \le \frac{1}{m}$, which entails that,

$$\begin{aligned} 1 \ge P\left(\frac{1}{Z_{n_k}} < \eta\right) &= P\left(Z_{n_k} > \frac{1}{\eta}\right) \\ &= P\left(Y_{n_k} > \frac{1}{\sigma_{n_k}\eta}\right) \ge P\left(Y_{n_k} > \frac{1}{m}\right) \end{aligned}$$

and consequently that

$$1 \ge \limsup_{k\longrightarrow\infty} P\left(\frac{1}{Z_{n_k}} < \eta\right) \ge \liminf_{k\longrightarrow\infty} P\left(\frac{1}{Z_{n_k}} < \eta\right) \ge P\left(Y > \frac{1}{m}\right) \ .$$

By letting m goes to infinity, since $P(Y>0)=1$ we obtain $\lim_{k\to\infty} P\left(\frac{1}{Z_{n_k}} < \eta\right) = 1$. This means that $\left\{\frac{1}{Z_{n_k}}\right\}_k$ converges in probability to 0 and therefore a subsequence of it converges almost surely to 0. This clearly implies that $\alpha_S(s) \le \mu$ (a.s.). □

Remark 2. When a real-valued and mean-zero Gaussian process S satisfies (3.2), the inequality $\alpha_S(s) \le \mu$ holds almost surely.

Proof of Remark 2. We take the same notations as in Lemma 7. To prove that $P(Y=0)=0$, let us set $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and for every integer n, $\tau_n^2 = E\left(\frac{|S(s+h_n)-S(s)|^2}{|h_n|^{2\mu}}\right)$. Observe that $\lim_{n\to+\infty} \sigma_n/\tau_n = 1$. We have for any real $\eta > 1$ and integer n big enough

$$\begin{aligned} P(Y_n > \eta) &= P\left(\frac{|S(s+h_n)-S(s)|}{|h_n|^{\mu}} > \sigma_n\eta - 1\right) \\ &= \frac{2}{\tau_n}\int_{\sigma_n\eta-1}^{\infty} f\left(\frac{x}{\tau_n}\right) dx \\ &= 2\int_{\frac{\sigma_n\eta-1}{\tau_n}}^{\infty} f(x)\, dx \ , \end{aligned}$$

and this last equality implies that

$$P(Y > \eta) = \lim_{k\to\infty} P(Y_{n_k} > \eta) = 2\int_{\eta}^{+\infty} f(x)\,dx\ .$$

Therefore, we get that $P(Y = 0) = 0$. □

To bound below the pointwise Hölder exponent of a continuous Gaussian field, we will use Remark 3. This remark is a straightforward consequence of the following lemma.

Lemma 8.

Let $G = \{G(u)\}_{u\in[0,T]^d}$ be a mean-zero real-valued Gaussian field such that for all $u,\ u' \in [0,T]^d$

$$E\left(\left|G(u) - G\left(u'\right)\right|^2\right) \le c\left|u - u'\right|^{2\nu}\ , \tag{3.4}$$

where $c > 0$ and $\nu \in]0,1[$ are two constants. Then, the trajectories of some version of G, are, with probability 1, $(\nu - \epsilon)$-Hölder functions, with $\epsilon > 0$ arbitrarily small.

Remark 3. Let $G = \{G(u)\}_{u\in[0,T]^d}$ be a mean-zero real-valued continuous Gaussian field that satisfies (3.4) and α_G its Hölder exponent. Then, at any point s, one has, almost surely,

$$\alpha_G(s) \ge \nu\ . \tag{3.5}$$

Proof of Lemma 8. We will only give the main lines of this proof (see [2] for the details). Let $u, u' \in [0,T]^d$ be arbitrary, as the random variable $G(u) - G(u')$ is Gaussian, it follows that

$$E\left(\left|G(u) - G\left(u'\right)\right|^d\right) = c'\left(E\left|G(u) - G\left(u'\right)\right|^2\right)^{d/2}$$

where the constant $c' > 0$, only depends on d. Then using (3.4) and a strong version of the Kolmogorov criterion (see for example Chapter 2 of [9], or [2]) we obtain Lemma 8. □

Lemma 9.

Let us fix an arbitrary $t_0 \in [0,1]$ and let $\{Y(x,y)\}_{(x,y)\in[0,1]^2}$ be a GMF of parameter an arbitrary admissible sequence $(H_n(.))$. Then the Hölder exponent of the Gaussian process $\{Y(s,t_0)\}_{s\in[0,1]}$, at t_0, is almost surely equal to $H(t_0) = \liminf_{n\to\infty} H_n(t_0)$.

Proof of Lemma 9. Step 1: Let us show that the Hölder exponent at t_0 of the process $\{Y(s,t_0)\}_{s\in[0,1]}$ is (a.s.) bounded above by $H(t_0) + \epsilon$, with $\epsilon > 0$ arbitrarily small. It follows from Remark 2 that it sufficient to prove that

$$\lim_{n\to\infty} E\left(\frac{\left|Y\left(t_0 + 2^{-n}, t_0\right) - Y(t_0,t_0)\right|^2}{2^{-2n(H(t_0)+\epsilon)}}\right) = +\infty\ . \tag{3.6}$$

Since $H(t_0) = \liminf_{n\to\infty} H_n(t_0)$, one can extract from the sequence $(H_n(t_0))_{n\in\mathbb{N}}$ a subsequence $(H_{n_l}(t_0))_{l\in\mathbb{N}}$ satisfying for every l,

$$H_{n_l}(t_0) \le H(t_0) + \epsilon/2\ . \tag{3.7}$$

For simplicity, we set $n = n_l$ in the sequel. The function $\hat{f}_n$ being equal to 1, on $[2^n\pi, \frac{2^{n+2}\pi}{3}]$, it follows from (1.7) and (3.7) that

$$\begin{aligned} E\left(\left|Y\left(t_0+2^{-n}, t_0\right)-Y(t_0,t_0)\right|^2\right) &\geq \int_{2^n\pi}^{2^{n+2}\pi/3} \frac{\left|e^{i2^{-n}\xi}-1\right|^2}{|\xi|^{2H_n(t_0)+1}}\,d\xi \\ &\geq \int_{2^n\pi}^{2^{n+2}\pi/3} \frac{\left|e^{i2^{-n}\xi}-1\right|^2}{|\xi|^{2H(t_0)+1+\epsilon}}\,d\xi \\ &= 2^{-2n(H(t_0)+\epsilon/2)} \int_{\pi}^{4\pi/3} \frac{\left|e^{iu}-1\right|^2}{|u|^{2H(t_0)+1+\epsilon}}\,du \end{aligned}$$

and we obtain (3.6).

Step 2: Let us show that the Hölder exponent at t_0 of the process $\{Y(s,t_0)\}_{s\in[0,1]}$ is (a.s.) bounded below by $H(t_0)-\epsilon$, with $\epsilon > 0$ arbitrarily small. It follows from Lemma 8 that it is sufficient to prove that for every $s, s+h \in [0,1]$, one has

$$E\left(|Y(s+h,t_0)-Y(s,t_0)|^2\right) \leq c|h|^{2H(t_0)-\epsilon}\ . \tag{3.8}$$

Since $H(t_0) = \liminf_{n\to\infty} H_n(t_0)$, there is an integer N, such that for every $n \geq N+1$,

$$H_n(t_0) \geq H(t_0) - \epsilon/2\ . \tag{3.9}$$

Then it follows from (1.7), (3.9), (1.2), and (1.3) that

$$\begin{aligned} E\left(|Y(s+h,t_0)-Y(s,t_0)|^2\right) &= \int_{\mathbb{R}} \left|e^{ih\xi}-1\right|^2 \left(\sum_{n=0}^{+\infty} \frac{\hat{f}_{n-1}(\xi)}{|\xi|^{H_n(t_0)+1/2}}\right)^2 d\xi\ . \\ &\leq 2\int_{\mathbb{R}} \left|e^{ih\xi}-1\right|^2 \left(\sum_{n=0}^{N} \frac{\hat{f}_{n-1}(\xi)}{|\xi|^{H_n(t_0)+1/2}}\right)^2 d\xi \\ &\quad + 2\int_{\mathbb{R}} \frac{\left|e^{ih\xi}-1\right|^2}{|\xi|^{2H(t_0)+1-\epsilon}} \left(\sum_{n=N+1}^{+\infty} \hat{f}_{n-1}(\xi)\right)^2 d\xi \\ &\leq 2|h|^2 \int_{\mathbb{R}} \left(\sum_{n=0}^{N} |\xi|^2 \frac{\hat{f}_{n-1}(\xi)}{|\xi|^{H_n(t_0)+1/2}}\right)^2 d\xi \\ &\quad + 2|h|^{2(H(t_0)-\epsilon/2)} \int_{\mathbb{R}} \frac{\left|e^{iu}-1\right|^2}{|u|^{2H(t_0)+1-\epsilon}}\,du\ , \end{aligned}$$

and we obtain (3.9). □

Proof of Theorem 1. Let $\{Y(x,y)\}_{(x,y)\in[0,1]^2}$ be a GMF of parameter an arbitrary admissible sequence $(H_n(.))$ and let $\{X(t)\}_{t\in[0,1]}$ be the corresponding GMBM. We will denote t_0 an arbitrary fixed point of $[0,1]$. As for every $t \in [0,1]$, $X(t) = Y(t,t)$, it follows from Proposition 1, that with probability 1, for all $t_0 + h \in [0,1]$

$$\begin{aligned} &|Y(t_0+h,t_0)-Y(t_0,t_0)| - C_1|h|^\beta \\ &\quad \leq |X(t_0+h)-X(t_0)| \leq |Y(t_0+h,t_0)-Y(t_0,t_0)| + C_1|h|^\beta\ . \end{aligned} \tag{3.10}$$

Then as $\beta > H(t_0)$ (see condition $\mathcal{C}$ in paragraph (A) of Section 1), (3.10), and Lemma 9 imply that, $\alpha_X(t_0) = H(t_0)$ almost surely. □

3.2 On the Difference of Two GMBM's with the Same Hölder Regularity

The aim of this subsection is to show the following result.

Theorem 2.

Let $\{X(t)\}_{t\in[0,1]}$ *and* $\{\tilde{X}(t)\}_{t\in[0,1]}$ *be two GMBM's of parameters, respectively* $(H_n(.))_{n\in\mathbb{N}}$ *and* $(\tilde{H}_n(.))_{n\in\mathbb{N}}$. *Let* $\{R(t)\}_{t\in[0,1]}$ *be the difference of* $\{X(t)\}_{t\in[0,1]}$ *and* $\{\tilde{X}(t)\}_{t\in[0,1]}$ *i. e., the Gaussian process defined for every* t *as* $R(t) = X(t) - \tilde{X}(t)$. *Suppose that for some* $t_0 \in [0,1]$ *and all integer* $n \geq 0$,

$$\left| H_n(t_0) - \tilde{H}_n(t_0) \right| \leq c2^{-n\delta} , \tag{3.11}$$

where $c > 0$ *and* $\delta > 0$ *are two constants (that generally depend on* t_0*). Then, the Hölder exponent of* R *at* t_0 *satisfies almost surely,*

$$\alpha_R(t_0) > H(t_0) = \alpha_X(t_0) = \alpha_{\tilde{X}}(t_0) . \tag{3.12}$$

Proof of Theorem 2. Let $\{Y(x,y)\}_{(x,y)\in[0,1]^2}$ and $\{\tilde{Y}(x,y)\}_{(x,y)\in[0,1]^2}$ be the GMF's that satisfy $X(t) = Y(t,t)$ and $\tilde{X}(t) = \tilde{Y}(t,t)$. We will denote $\{T(t)\}_{t\in[0,1]}$ the Gaussian process defined for every t as,

$$T(t) = Y(t,t_0) - \tilde{Y}(t,t_0) . \tag{3.13}$$

Let us first show that the Hölder exponent of T at t_0, satisfies, almost surely,

$$\alpha_T(t_0) > H(t_0) . \tag{3.14}$$

Thanks to Remark 3 and Lemma 8, it is sufficient to show that for some $\eta > 0$ and $c > 0$ the inequality

$$E\left(|T(t+h) - T(t)|^2\right) \leq c_1 |h|^{2H(t_0)+\eta} , \tag{3.15}$$

holds for every $t, t+h \in [0,1]$. It follows from (1.7), (1.5) the Mean Value Theorem and (3.11) that

$$\begin{aligned} E\left(|T(t+h) - T(t)|^2\right) &= 4\int_{\mathbb{R}} \frac{\sin^2(h\xi/2)}{|\xi|} \left(\sum_{n=0}^{\infty} \left(\frac{1}{|\xi|^{H_n(t_0)}} - \frac{1}{|\xi|^{\tilde{H}_n(t_0)}} \right) \hat{f}_{n-1}(\xi) \right)^2 d\xi , \\ &\leq 4\int_{\mathbb{R}} \frac{(\log|\xi|)^2 \sin^2(h\xi/2)}{|\xi|} \left(\sum_{n=0}^{\infty} \frac{\left| H_n(t_0) - \tilde{H}_n(t_0) \right|}{|\xi|^{\theta_n}} \hat{f}_{n-1}(\xi) \right)^2 d\xi \\ &\leq c_2 \int_{\mathbb{R}} \frac{(\log|\xi|)^2 \sin^2(h\xi/2)}{|\xi|} \left(\sum_{n=0}^{\infty} \frac{2^{-n\delta}}{|\xi|^{\theta_n}} \hat{f}_{n-1}(\xi) \right)^2 d\xi , \end{aligned} \tag{3.16}$$

where for every $n \in \mathbb{N}$, $\theta_n \in [\min(H_n(t_0), \tilde{H}_n(t_0)), \max(H_n(t_0), \tilde{H}_n(t_0))]$.

Now, as

$$H(t_0) = \liminf_{n\to\infty} H_n(t_0) = \liminf_{n\to\infty} \tilde{H}_n(t_0) ,$$

for all $\epsilon > 0$ arbitrarily small, there is an integer n_0 such that for any $n \geq n_0 + 1$,

$$\theta_n \geq H(t_0) - \epsilon/2 \, . \tag{3.17}$$

Thus, it follows from (3.16), (1.2), and (1.3) that

$$E\left(|T(t+h) - T(t)|^2\right) \leq c_3(I_1(h) + I_2(h)) \, , \tag{3.18}$$

where

$$I_1(h) = \int_{\mathbb{R}} \frac{(\log|\xi|)^2 \sin^2(h\xi/2)}{|\xi|} \left(\sum_{n=0}^{n_0} \frac{2^{-n\delta}}{|\xi|^{\theta_n}} \hat{f}_{n-1}(\xi) \right)^2 d\xi \, , \tag{3.19}$$

and

$$I_2(h) = \int_{\mathbb{R}} \frac{\sin^2(h\xi/2)}{|\xi|^{2H(t_0)+1-2\epsilon}} \left(\sum_{n=0}^{\infty} 2^{-n\delta} \hat{f}_{n-1}(\xi) \right)^2 d\xi \, . \tag{3.20}$$

It is clear that

$$I_1(h) \leq |h|^2 \int_{\mathbb{R}} |\xi| (\log|\xi|)^2 \left(\sum_{n=0}^{n_0} \frac{2^{-n\delta}}{|\xi|^{\theta_n}} \hat{f}_{n-1}(\xi) \right)^2 d\xi \, . \tag{3.21}$$

Let us give an ad hoc upper bound of $I_2(h)$. We will suppose that $h \neq 0$ and n_1 will be the integer such that

$$2^{-n_1-1} < |h| \leq 2^{-n_1} \, . \tag{3.22}$$

We have

$$I_2(h) \leq 2(J_2(h) + L_2(h)) \, , \tag{3.23}$$

where

$$J_2(h) = |h|^2 \int_{\mathbb{R}} |\xi|^{-2H(t_0)+1+2\epsilon} \left(\sum_{n=0}^{n_1} 2^{-n\delta} \hat{f}_{n-1}(\xi) \right)^2 d\xi \tag{3.24}$$

and

$$L_2(h) = 2^{-2(n_1+1)\delta} \int_{\mathbb{R}} \frac{\sin^2(h\xi/2)}{|\xi|^{2H(t_0)+1-2\epsilon}} \left(\sum_{n=n_1+1}^{\infty} \hat{f}_{n-1}(\xi) \right)^2 d\xi \, . \tag{3.25}$$

It follows from (1.2) and (1.3) that supp $\hat{f}_{-1} \subset [-\frac{4\pi}{3}, \frac{4\pi}{3}]$, that for every $n \in \mathbb{N}$, supp $\hat{f}_n \subset \{\xi, \frac{2^{n+1}\pi}{3} \leq |\xi| \leq \frac{2^{n+3}\pi}{3}\}$ and that the functions $\hat{f}_n$ are with values in $[0, 1]$.

Therefore, using (3.24) and (3.22), we obtain that

$$\begin{aligned}J_2(h) &\le (n_1+1)2^{-2n_1}\sum_{n=0}^{n_1} 2^{-2n\delta}\int_{\mathbb{R}} \hat{f}_{n-1}^2(\xi)|\xi|^{-2H(t_0)+1+2\epsilon}\,d\xi \\ &\le 2(n_1+1)2^{-2n_1}\Bigg(\int_0^{4\pi/3}\frac{d\xi}{\xi^{2H(t_0)-1-2\epsilon}} \\ &\quad + \sum_{n=1}^{n_1} 2^{-2n\delta}\int_{2^n\pi/3}^{2^{n+2}\pi/3}\frac{d\xi}{\xi^{2H(t_0)-1-2\epsilon}}\Bigg) \\ &\le c_4(n_1+1)2^{-2n_1}\left(\sum_{n=0}^{n_1} 2^{-2n\delta}\left(\frac{2^{n+2}\pi}{3}\right)^{2(1-H(t_0)+\epsilon)}\right) \\ &\le c_5(n_1+1)2^{-2n_1}2^{2n_1(1-H(t_0)-\delta+\epsilon)} \\ &\le c_6|h|^{2(H(t_0)+\delta-2\epsilon)}\,.\end{aligned} \tag{3.26}$$

As for every real ξ, $\sum_{n=0}^{\infty}\hat{f}_{n-1}(\xi)=1$, it follows from (3.25) and (3.22) that

$$L_2(h) \le |h|^{2\delta}\int_{\mathbb{R}}\frac{\sin^2(h\xi/2)}{|\xi|^{2H(t_0)+1-2\epsilon}}\,d\xi$$

and setting $u=h\xi$ in this last integral, we obtain that

$$L_2(h) \le c_7|h|^{2(H(t_0)+\delta-\epsilon)}\,. \tag{3.27}$$

Thus if we take $\eta = 2\delta - 4\epsilon$, then (3.15) follows from (3.18), (3.21), (3.23), (3.26), and (3.27). At last, let us show that (3.14) and Proposition 1, imply that, almost surely

$$\alpha_R(t_0) > H(t_0)\,. \tag{3.28}$$

Using (3.13), we obtain that

$$\begin{aligned}R(t_0+h)-R(t_0) &= \Big(Y(t_0+h,t_0+h)-\tilde{Y}(t_0+h,t_0+h)\Big)-\Big(Y(t_0,t_0)-\tilde{Y}(t_0,t_0)\Big) \\ &= (Y(t_0+h,t_0+h)-Y(t_0+h,t_0)) - \Big(\tilde{Y}(t_0+h,t_0+h)-\tilde{Y}(t_0+h,t_0)\Big) \\ &\quad + (Y(t_0+h,t_0)-Y(t_0,t_0)) - \Big(\tilde{Y}(t_0+h,t_0)-\tilde{Y}(t_0,t_0)\Big) \\ &= (Y(t_0+h,t_0+h)-Y(t_0+h,t_0)) - \Big(\tilde{Y}(t_0+h,t_0+h)-\tilde{Y}(t_0+h,t_0)\Big) \\ &\quad + (T(t_0+h)-T(t_0))\,,\end{aligned}$$

which entails that

$$\begin{aligned}|R(t_0+h)-R(t_0)| &\le |T(t_0+h)-T(t_0)| \\ &\quad + \sup_{x\in[0,1]}|Y(x,t_0+h)-Y(x,t_0)| + \sup_{x\in[0,1]}\Big|\tilde{Y}(x,t_0+h)-\tilde{Y}(x,t_0)\Big|\,.\end{aligned}$$

Then it follows from these inequalities and Proposition 1, that almost surely, for every $|h|$ small enough

$$|R(t_0+h)-R(t_0)| \le C_8|h|^{\beta} + |T(t_0+h)-T(t_0)|\,, \tag{3.29}$$

where C_8 is a positive random variable. This last, inequality and (3.14) imply that almost surely

$$\alpha_R(t_0) \ge \min(\alpha_T(t_0),\beta) > H(t_0)\,. \qquad \square$$

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