

# Continuous Wavelet Transforms from Semidirect Products: Cyclic Representations and Plancherel Measure

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**ABSTRACT.** Continuous wavelet transforms arising from the quasiregular representation of a semidirect product group  $G = \mathbb{R}^k \rtimes H$  have been studied by various authors. Recently the attention has shifted from the irreducible case to include more general dilation groups  $H$ , for instance cyclic (more generally: discrete) or one-parameter groups. These groups do not give rise to irreducible square-integrable representations, yet it is possible (and quite simple) to give admissibility conditions for a large class of them. We put these results in a theoretical context by establishing a connection to the Plancherel theory of the semidirect products, and show how the admissibility conditions relate to abstract admissibility conditions which use Plancherel theory.

## Introduction

In one of the initial articles of wavelet analysis [18], Grossmann, Morlet and Paul proved that the continuous wavelet transform on  $L^2(\mathbb{R})$  and its inversion formula rests on a certain representation of the semidirect product  $\mathbb{R} \rtimes \mathbb{R}^+$  acting on  $L^2(\mathbb{R})$ . More precisely, they showed that the square-integrability (in the sense of [9]) of that representation guarantees the existence of an inverse wavelet transform, and they showed how the admissibility conditions relate to the so-called *Duflo–Moore operators*, which are naturally associated to square-integrable representations. This realization opened the way to analogous constructions in a variety of settings. One class of groups and representations attracting particular attention are the semidirect products of the type  $\mathbb{R}^k \rtimes H$ . Here  $H$  is a closed matrix group (the so-called *dilation group*).  $G$  has a natural unitary representation on  $L^2(\mathbb{R}^k)$ , which

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is the chief object of study for the construction of wavelet transforms. Concrete higher-dimensional examples of such transforms were produced by Murenzi [29] and Bohnke [6]. Later the problem was studied in greater generality by Bernier and Taylor [5], who gave sufficient conditions for the existence of inversion formulas, and also calculated the Duflo–Moore operators for these cases. The general approach was further pursued in [12, 1].

All the references cited so far restrict attention to the case of *irreducible* square-integrable representations (also called *discrete series representations*). But several authors produced evidence that inversion formulas could also be obtained for non-irreducible representations [19, 21, 28]. All results were given for the semidirect product setting, and they turned out to be rather simple generalizations of what had been obtained in the irreducible case. A more general family of dilation groups was studied by Gröchenig, Kaniuth and Taylor [17], who focused on certain one-parameter groups. Their so-called “projection generating functions” turn out to be particular admissible vectors. Quite recently, G. Weiss and collaborators announced in [32] and proved in [26] an almost characterization of those dilation groups which admit an inversion formula; in particular all of the aforementioned examples fall under the class described in [26].

One intriguing aspect of the concrete results in [19, 21, 28] (and the initial motivation for this paper) was that, until recently, no general representation-theoretic framework, comparable to the characterization for irreducible representations, existed in which the concrete admissibility conditions would fit. The theoretical framework has now been provided in [14], where general admissibility conditions are formulated by use of the Plancherel decomposition of the regular representation of the underlying group. It is the main purpose of this paper to exhibit the relationship between the concrete results on semidirect products and the abstract admissibility conditions, and thus to provide a bridge between [26] and [14].

Our paper has three sections. In the first section, we consider semidirect products  $\mathbb{R}^k \rtimes H$  and their quasi-regular representations on  $L^2(\mathbb{R}^k)$ . We derive admissibility criteria for subrepresentations of quasi-regular representations, and give sufficient conditions for the existence of admissible vectors. The results in that section have to a large extent been proved in [26]. We include proofs for several reasons: Our results are slightly more general in that we consider arbitrary subrepresentations of the quasiregular representations. Secondly, the bulk of our results was obtained independently from [32, 26]. The third and most important reason is that most of the constructions and objects used in the proof of the concrete results turn up again in the abstract setting; most notably the dual orbit space  $\widehat{\mathbb{R}^k}/H$  and certain measures on the orbits and the orbit space. Hence including the proofs should facilitate understanding the relationship between abstract and concrete results.

Section two is devoted to a short review of Plancherel theory and its relation to admissibility conditions. Section three then connects the results of the previous two sections, by identifying the Duflo–Moore operators and the Plancherel measure with certain operators and measures obtained in the concrete setting. We wish to point out, though, that in the course of our argument the concrete admissibility conditions do not appear as mere corollaries of the abstract ones. We use the concrete results to show that they are special cases of the abstract results. Hence the results of the third section might be considered redundant, but we maintain that it is useful to work out this class of examples and to explicitly compute the various objects of Plancherel theory, i. e., the Duflo–Moore operators and the corresponding Plancherel measure.

Apart from serving as an illustration for the results in [14], the discussion in Section 3 also ties in nicely with the results of Kleppner and Lipsman [22] on the Plancherel theory of semidirect products in general (see the discussion in Section 3 for a more detailed account of their results). We wish to stress that while the results of Kleppner and Lipsman provide

an orientation for our arguments in Section 3, our proof does not use their calculation of Plancherel measure.

In the remainder of the introduction we wish to collect some generalities concerning wavelet transforms and admissible vectors. For the representation-theoretic notions used in this paper, confer [11]. All throughout the paper,  $G$  denotes a (second countable) locally compact group and  $(\pi, \mathcal{H}_\pi)$  a (strongly continuous, unitary) representation of  $G$  on a separable Hilbert space  $\mathcal{H}_\pi$ .  $L^2(G)$  denotes the usual  $L^2$ -space on  $G$  with respect to left Haar measure  $\mu_G$ . The left regular representation  $\lambda_G$  acts on  $L^2(G)$  by left translations.  $\widehat{G}$  denotes the dual space of  $G$ , i. e., the equivalence classes of irreducible representations of  $G$ .  $\widehat{G}$  carries a natural Borel structure, the so-called *Mackey Borel structure*. We will not distinguish explicitly between representations and their equivalence classes.

For a given representation  $(\pi, \mathcal{H}_\pi)$ , two vectors  $\eta, \phi \in \mathcal{H}_\pi$  and  $x \in G$  define

$$V_\eta \phi(x) := \langle \phi, \pi(x)\eta \rangle .$$

Then  $V_\eta : \phi \mapsto V_\eta \phi$  is a bounded operator from  $\mathcal{H}_\pi$  into  $C_b(G)$ , the space of continuous bounded functions on  $G$ . We call  $\eta$  **admissible** if  $V_\eta$  is an isometry from  $\mathcal{H}_\pi$  into  $L^2(G)$ . Note that usually even the well-definedness of  $V_\eta : \mathcal{H}_\pi \rightarrow L^2(G)$  is a non-trivial issue. The interest in isometries comes from the fact that in this case there exists—at least formally—an inversion formula, in the form of the weak operator integral

$$\phi = \int_G V_\eta \phi(x) \pi(x)\eta \, d\mu_G(x) .$$

A further pleasant feature of admissible vectors is that the orthogonal projection onto the image  $V_\eta(\mathcal{H}_\pi) \subset L^2(G)$  is given by convolution with  $V_\eta \eta$ , which entails that  $V_\eta(\mathcal{H}_\pi)$  is a reproducing kernel Hilbert space.

A vector  $\eta$  is called **weakly admissible** if  $V_\eta$  is a bounded one-to-one mapping into  $L^2(G)$ . Clearly, the condition that  $V_\eta$  be one-to-one is equivalent to cyclicity of  $\eta$ . We call a representation **weakly square-integrable** if a weakly admissible vector exists, and **strongly square-integrable** if an admissible vector exists. It is well-known that, for irreducible representations, the notions of weakly admissible and admissible vectors coincide, and so do the weakly and strongly square-integrable irreducible representations. But for the general case, the two notions may well differ.

Both notions of square-integrability lead to subrepresentations of the left regular representations: The wavelet transform clearly intertwines  $\pi$  with the left action of the group (whichever function space on the group is under consideration). If  $\eta$  is weakly admissible, then the unitary part of the polar decomposition of  $V_\eta$  is a unitary equivalence between  $\pi$  and a subrepresentation of  $\lambda_G$ . Conversely, Losert and Rindler proved [24], that there exists a cyclic vector  $\eta$  for  $\lambda_G$  iff  $G$  is first countable. Moreover they showed that the cyclic vector may be chosen continuous with compact support (in particular, in  $L^1(G)$ ), which implies that it is in fact weakly admissible for  $\lambda_G$ , by Young's inequality. Hence  $\lambda_G$  is weakly square-integrable iff  $G$  is first countable. In order to obtain a weakly admissible vector for some invariant subspace, we only have to orthogonally project the weakly admissible vector for  $\lambda_G$  into the subspace. Thus we see that for first countable groups (and only for these) weakly square-integrable representations are precisely those which are equivalent to some subrepresentation of  $\lambda_G$ .

By contrast, strong square-integrability is more complicated to check, as it will be seen to depend on the modular function of the group: If  $G$  is non-unimodular with type-I regular representation, the notions strongly and weakly square-integrable coincide, whereas

the only unimodular groups for which this is true are the discrete ones. In the unimodular case, the existence of admissible vectors can be characterized by a finite Plancherel measure condition, at least when  $\lambda_G$  is type-I (for all these results, see [14]).

## 1. Admissibility Conditions for Quasiregular Representations

In this section we consider a well-studied setting, where  $G = \mathbb{R}^k \rtimes H$  is a semidirect product of a vector group with a matrix group, and  $\pi$  is the quasiregular representation of that group acting on  $L^2(\mathbb{R}^k)$ . Various articles have been concerned with constructing irreducible square integrable representations for these groups [18, 29, 6, 5, 12] or with wavelet transforms obtained from reducible representations [19, 21, 32].

Throughout this section  $H < \text{GL}(k, \mathbb{R})$  is a closed subgroup. Elements of  $G$  are denoted by  $(x, h)$  with  $x \in \mathbb{R}^k$  and  $h \in H$ ; the group law is then given by  $(x_0, h_0)(x_1, h_1) = (x_0 + h_0x_1, h_0h_1)$ . A left Haar measure of  $G$  is given by  $d\mu_G(x, h) = |\det(h)|^{-1} dx d\mu_H(h)$ , and the modular function is computed as  $\Delta_G(x, h) = \Delta_H(h)|\det(h)|^{-1}$ . For simplicity we will sometimes write  $\Delta_G(h)$  instead of  $\Delta_G(0, h)$ . The quasiregular representation  $\pi$  of  $G$  acts on  $L^2(\mathbb{R}^k)$  by

$$(\pi(x, h)f)(y) = |\det(h)|^{-1/2} f(h^{-1}(y - x)) .$$

The closedness of  $H$  in  $\text{GL}(k, \mathbb{R})$  may seem a somewhat arbitrary condition (Lie subgroups might also work), but it is in fact not a real restriction, because of the following:

### **Proposition 1.**

*Let  $H$  be a subgroup of  $\text{GL}(n, \mathbb{R})$ , endowed with some locally compact group topology. Assume that the semidirect product  $\mathbb{R}^k \rtimes H$  is a topological semidirect product, and that the quasiregular representation has a nontrivial subrepresentation with a weakly admissible vector. Then  $H$  is a closed subgroup of  $\text{GL}(n, \mathbb{R})$ , and the topology on  $H$  is the relative topology.*

**Proof.** By the observations made in the Introduction, the subrepresentation having a weakly admissible vector is also (equivalent to) a subrepresentation of the regular representation. Since all matrix coefficients of the latter vanish at infinity, we thus obtain a nontrivial  $C_0$  matrix coefficient for  $\pi$ . Now we can proceed as in the proof of [13, Proposition 5] to conclude that  $H$  must be closed.  $\square$

The dual group  $\widehat{\mathbb{R}^k}$  is the character group of  $\mathbb{R}^k$ , suitably identified with the space of row vectors. We define the Fourier transform as a mapping  $\mathcal{F} : L^2(\mathbb{R}^k) \rightarrow L^2(\widehat{\mathbb{R}^k})$  by letting

$$\mathcal{F}(f) = \widehat{f}(\omega) = \int_{\mathbb{R}^k} f(x) e^{-2\pi i \omega x} dx ,$$

on  $L^1(\mathbb{R}^k) \cap L^2(\mathbb{R}^k)$ . Consequently the Plancherel formula is given by  $\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle$ . Subrepresentations of the quasiregular representation are best described in terms of the dual action of  $H$  on  $\widehat{\mathbb{R}^k}$ , which arises by duality from the action on  $\mathbb{R}^k$ . Under our identification of  $\widehat{\mathbb{R}^k}$  with row vectors, the dual action is matrix multiplication on the right. For any  $H$ -invariant subset  $U \subset \widehat{\mathbb{R}^k}$ , we denote by  $U/H$  the quotient space, endowed with the quotient

Borel structure. In particular  $\widehat{\mathbb{R}^k}/H$  denotes the *dual orbit space*. For  $\gamma \in \widehat{\mathbb{R}^k}$ , we let  $H_\gamma$  denote the stabilizer of  $\gamma$  in  $H$ , also called the *little fixed group* of  $\gamma$ ; it is a closed subgroup of  $H$ . For the discussion of subrepresentations of  $\pi$ , it is useful to introduce the representation  $\widehat{\pi}$  obtained by conjugating  $\pi$  with the Fourier transform on  $\mathbb{R}^k$ . It is readily seen to operate on  $L^2(\widehat{\mathbb{R}^k})$  via

$$(\widehat{\pi}(x, h)f)(\gamma) = |\det(h)|^{1/2} e^{2\pi i \gamma \cdot x} f(\gamma h). \tag{1.1}$$

The action of  $\widehat{\pi}$  allows to identify subrepresentations in a simple way: Every invariant closed subspace  $\mathcal{H} \subset L^2(\widehat{\mathbb{R}^k})$  is of the form

$$\mathcal{H} = \mathcal{H}_U = \left\{ g \in L^2(\widehat{\mathbb{R}^k}) : \widehat{g} \text{ vanishes outside of } U \right\},$$

where  $U \subset \widehat{\mathbb{R}^k}$  is a measurable,  $H$ -invariant subset (see [12] for a detailed argument). We let  $\pi_U$  denote the subrepresentation acting on  $\mathcal{H}_U$ . In view of the correspondence between subrepresentations and invariant subsets of the dual, the dual orbit space becomes a natural object of study.

**Remark 1 (Dual Orbit Space).** The structure of the dual orbit space is not only important for the decomposition of the quasi-regular representation and for the construction of admissible vectors, but also for the decomposition of the regular representation of  $G$ , i. e., the Plancherel decomposition. For our discussion, the following two sets will be central

$$\Omega_c = \left\{ \omega \in \widehat{\mathbb{R}^k} : H_\omega \text{ is compact} \right\}, \quad \Omega_{rc} = \{ \omega \in \Omega_c : \omega H \text{ is locally closed} \}.$$

The set  $\Omega_{rc} \subset \Omega_c$  consists of the “regular” orbits in  $\Omega_c$ ; i. e., it is the “well-behaved” part of  $\Omega_c$ . Loosely speaking,  $\Omega_c$  is the set we have to deal with, and  $\Omega_{rc}$  is the set we can deal with. Put more precisely: While Theorem 1 below shows that subrepresentations with admissible vectors necessarily correspond to invariant subsets  $U$  of  $\Omega_c$ , the existence result in Theorem 2 only considers subsets of the smaller set  $\Omega_{rc}$ . However, this distinction is not due to a shortcoming of our approach: As witnessed by Example 3 below, there are subsets of  $\Omega_c$  that do not allow admissible vectors for the corresponding subrepresentations.

Let us now collect some measure-theoretic properties of the two sets.  $\Omega_c$  can be shown to be measurable; we have included a proof of that statement in the Appendix. But usually  $\Omega_c$  is not open, even when it is conull, as is illustrated by the example of  $SL(2, \mathbb{Z})$ : It is easy to see that  $\Omega_c$  consists of all the vectors  $(\omega_1, \omega_2)$  such that  $\omega_1/\omega_2$  is irrational. This is a conull set with dense complement in  $\widehat{\mathbb{R}^2}$ .

By contrast,  $\Omega_{rc}$  is always open (cf. the Appendix for a proof). A pleasant consequence of this is that Glimm’s Theorem [16] applies (since  $\Omega_{rc}$  is locally compact), which entails a number of useful properties of the orbit space  $\Omega_{rc}/H$ : It is a standard Borel space having a measurable cross section  $\Omega_{rc}/H \rightarrow \Omega_{rc}$ , and there exists a measurable transversal, i. e., a Borel subset  $A \subset \Omega_{rc}$  meeting each orbit in precisely one point.

Unfortunately, the example of  $SL(2, \mathbb{Z})$  shows that  $\Omega_{rc}$  can be empty even when  $\Omega_c$  is conull: Since the complement of  $\Omega_c$  is dense,  $\Omega_c$  contains no nonempty open set.

Let us now derive the admissibility condition for the quasiregular representation, or more generally for subrepresentations. As a matter of fact, the proof of the admissibility condition does not involve any measure-theoretic subtleties. It is only when we address the existence of functions fulfilling the condition that we are forced to use more involved arguments. The theorem was derived for certain concrete groups  $H$  in [19, 21, 28]; the general

version given here appears also in [26]. Note that the admissibility condition also figures as a part of the definition of the notion of “projection generating function” in [17, Definition 2.1]. Thus the following theorem also answers a question raised in [17, Remark 2.6(b)]: There the authors observe that taking a projection generating function as wavelet gives rise to orthogonality relations among the wavelet coefficients which closely resemble those for irreducible square-integrable representations, even though the representation at hand is not irreducible. This is readily explained by the isometry property guaranteed by the admissibility condition. That these orthogonality relations also arise in the non-irreducible setting is due to the fact that the representation can be identified with a subrepresentation of the Plancherel decomposition of the group. This is precisely the topic of our paper.

**Theorem 1.**

Let  $(\pi_U, \mathcal{H}_U)$  be a subrepresentation of  $\pi$  corresponding to some invariant measurable subset  $U$ . Then

$$g \in \mathcal{H}_U \text{ is weakly admissible} \Leftrightarrow \text{the mapping } \gamma \mapsto \int_H |\widehat{g}(\gamma h)|^2 d\mu_H(h) \text{ is positive}$$

$$\text{and essentially bounded on } U,$$

$$g \in \mathcal{H}_U \text{ is admissible} \Leftrightarrow \int_H |\widehat{g}(\gamma h)|^2 d\mu_H(h) = 1 \text{ (for almost every } \gamma \in U).$$

In particular, if  $\pi_U$  has a weakly admissible vector, then  $U \subset \Omega_c$  (up to a null set).

**Proof.** We start by explicitly calculating the  $L^2$ -norm of  $V_g f$ , for  $f, g \in \mathcal{H}_U$ . The following computations are standard, see also [5, 12, 32]; we include them for convenience.

$$\begin{aligned} \|V_g f\|_{L^2(G)}^2 &= \int_G |\langle f, \pi(x, h)g \rangle|^2 d\mu_G(x, h) \\ &= \int_G |\langle \widehat{f}, (\pi(x, h)g)^\wedge \rangle|^2 d\mu_G(x, h) \\ &= \int_G \left| \int_{\widehat{\mathbb{R}^k}} \widehat{f}(\gamma) |\det(h)|^{1/2} e^{-2\pi i \gamma x} \widehat{g}(\gamma h) d\lambda(\gamma) \right|^2 d\mu_G(x, h) \\ &= \int_H \int_{\mathbb{R}^k} \left| \int_{\widehat{\mathbb{R}^k}} \widehat{f}(\gamma) e^{-2\pi i \gamma x} \widehat{g}(\gamma h) d\lambda(\gamma) \right|^2 d\lambda(x) d\mu_H(h) \\ &= \int_H \int_{\mathbb{R}^k} |\mathcal{F}(\phi_h)(x)|^2 d\lambda(x) d\mu_H(h). \end{aligned}$$

Here  $\phi_h(\gamma) = \widehat{f}(\gamma) \widehat{g}(\gamma h)$ , and  $\mathcal{F}$  denotes the Fourier transform on  $L^1(\widehat{\mathbb{R}^k})$ . An application of Plancherel’s formula to the last expression yields

$$\begin{aligned} \int_H \int_{\widehat{\mathbb{R}^k}} |\phi_h(\gamma)|^2 d\lambda(\gamma) d\mu_H(h) &= \int_H \int_{\widehat{\mathbb{R}^k}} |\widehat{f}(\gamma)|^2 |\widehat{g}(\gamma h)|^2 d\lambda(\gamma) d\mu_H(h) \\ &= \int_{\widehat{\mathbb{R}^k}} |\widehat{f}(\gamma)|^2 \left( \int_H |\widehat{g}(\gamma h)|^2 d\mu_H(h) \right) d\lambda(\gamma). \end{aligned}$$

Now the admissibility criteria are obvious. Moreover, it is easily seen that whenever the stabilizer  $H_\gamma$  is noncompact, we have

$$\int_H |\widehat{g}(\gamma h)|^2 d\mu_H(h) \in \{0, \infty\},$$

(cf. also the proof of [12, Theorem 10]), hence  $V_g f \in L^2(G)$  entails that the pointwise product  $\widehat{f\hat{g}}$  vanishes a.e. outside of  $\Omega_c$ . In particular, a weakly admissible vector vanishes almost everywhere outside of  $\Omega_c$ , hence we obtain in such a case that  $U \subset \Omega_c$  (up to a null set).  $\square$

For the construction of admissible vectors we need to decompose Lebesgue-measure  $\lambda$  on  $\Omega_{rc}$  into certain measures on the orbits and a measure on  $\Omega_{rc}/H$ . Then we address the relationship of the measures on the orbits to the Haar measure of  $H$ .

**Lemma 1.**

- (a) *There exists a measure  $\bar{\lambda}$  on  $\Omega_{rc}/H$  and on each orbit  $\gamma H$  a measure  $\beta_{\gamma H}$  such that for every measurable  $A \subset \Omega_{rc}$  the mapping*

$$\gamma H \mapsto \int_{\gamma H} \chi_A(\omega) d\beta_{\gamma H}(\omega)$$

*is  $\bar{\lambda}$ -measurable, and in addition*

$$\lambda(A) = \int_{\widehat{\mathbb{R}^k}/H} \int_{\gamma H} \chi_A(\omega) d\beta_{\gamma H}(\omega) d\bar{\lambda}(\gamma H).$$

- (b) *Let  $(\bar{\lambda}, (\beta_{\gamma H})_{\gamma H \in \Omega_{rc}/H})$  be as in (a). For  $\gamma \in \Omega_{rc}$  define  $\mu_{\gamma H}$  as the image measure of  $\mu_H$  under the projection map  $p_\gamma : h \mapsto \gamma h$ .  $\mu_{\gamma H}$  is a  $\sigma$ -finite measure, and its definition is independent of the choice of representative  $\gamma$ . Then, for almost all  $\gamma \in \Omega_{rc}$ , the  $\mu_{\gamma H}$  and  $\beta_{\gamma H}$  are equivalent, with globally Lebesgue-measurable Radon–Nikodym-derivatives: There exists an (essentially unique) Lebesgue-measurable function  $\kappa : \Omega_{rc} \rightarrow \mathbb{R}^+$  such that for  $\omega \in \gamma H$ ,*

$$\frac{d\beta_{\gamma H}}{d\mu_{\gamma H}}(\omega) = \kappa(\omega).$$

- (c) *The function  $\kappa$  fulfills the semi-invariance relation*

$$\kappa(\omega h) = \kappa(\omega) \Delta_G(h)^{-1}. \tag{1.2}$$

*In particular,  $\kappa$  is  $H$ -invariant iff  $G$  is unimodular. In that case, we can in fact assume that  $\kappa = 1$  almost everywhere. This choice determines the measure  $\bar{\lambda}$  uniquely.*

**Proof.** Statement (a) is a classical result from measure theory; see for instance [22, Theorem 2.1]. The standardness of  $\Omega_{rc}/H$  is decisive.

In order to prove part (b), well-definedness and  $\sigma$ -finiteness of  $\mu_{\gamma H}$  follow from compactness of  $H_\gamma$ . The independence of the representative  $\gamma$  of the orbit follows from the fact that  $\mu_H$  is left-invariant, whereas the dual action is on the right. To compute the Radon–Nikodym derivative  $\kappa$ , we first introduce an auxiliary function  $\ell : \Omega_{rc} \rightarrow \mathbb{R}_0^+$ : Fix a Borel-measurable transversal  $A \subset \Omega_{rc}$  of the  $H$ -orbits. Then the mapping  $\tau : A \times H \rightarrow \Omega_{rc}$ ,  $\tau(\omega, h) = \omega h$  is bijective and continuous, hence, since  $A \times H$  is a standard Borel space,  $\tau^{-1} : \Omega_{rc} \rightarrow A \times H$  is Borel as well, by [3, Theorem 3.3.2]. If we let  $\tau^{-1}(\gamma)_H$  denote the  $H$ -valued coordinate of  $\tau^{-1}(\gamma)$ , then  $\ell(\gamma) := \Delta_G(\tau^{-1}(\gamma)_H)$  is a Borel-measurable mapping. Since  $\Delta_G$  is constant on every compact subgroup (in particular on all the little fixed groups of elements in  $\Omega_{rc}$ ), a straightforward calculation shows that  $\ell$  satisfies the semi-invariance relation  $\ell(\omega h) = \ell(\omega) \Delta_G(h)^{-1}$ .

Next fix an orbit  $\gamma H$  and let us compare the measures  $\beta_{\gamma H}$  and  $\ell\mu_{\gamma H}$ : Since

$$d\mu_{\gamma H}(\omega h) = \Delta_H(h) d\mu_{\gamma H}(\omega) \quad \text{and} \quad d\beta_{\gamma H}(\omega h) = |\det(h)| d\beta_{\gamma H}(\omega),$$

the definition of  $\ell$  ensures that  $\ell\mu_{\gamma H}$  and  $\beta_{\gamma H}$  behave identically under the action of  $H$ . Moreover, they are  $\sigma$ -finite and quasi-invariant, hence equivalent. Since they have the same behavior under the operation of  $H$ , the Radon–Nikodym derivative turns out to be a positive constant on the orbit. Summarizing, we find for  $\omega \in \gamma H$  that

$$\frac{d\beta_{\gamma H}}{d\mu_{\gamma H}}(\omega) = \ell(\omega)c_{\gamma H},$$

with  $\ell, c_{\gamma H} > 0$ , and it remains to show that  $c_{\gamma H}$  depends measurably on the orbit.

For this purpose pick a relatively compact open neighborhood  $B \subset H$  of the identity. Then  $AB = \tau(A \times B) \subset \Omega_{rc}$  is Borel-measurable, as a continuous image of a standard space, hence  $\chi_{AB}$ , the indicator function of  $AB$ , is a Borel-measurable function. Both

$$\phi_1 : \gamma H \mapsto \int_{\gamma H} \chi_{AB}(\omega) d\beta_{\gamma H}(\omega)$$

and

$$\phi_2 : \gamma H \mapsto \int_{\gamma H} \chi_{AB}(\omega)\ell(\omega) d\mu_{\gamma H}(\omega)$$

are measurable functions: The first one is by choice of the  $\beta_{\gamma H}$ , see part (a). The second one is measurable by Fubini's theorem, applied to the mapping  $(\omega, h) \mapsto \chi_{AB}(\omega h)\ell(\omega h)$  on  $\widehat{\mathbb{R}}^k \times H$  (recall the definition of  $\mu_{\gamma H}$ ).

In addition, both functions are finite and positive on  $\Omega_{rc}$ . We have

$$\phi_2(\gamma H) = \int_{\gamma H} \chi_{AB}(\omega)\ell(\omega) d\mu_{\gamma H}(\omega) = \int_{p_\gamma^{-1}(AB)} \Delta_G(h) d\mu_H(h),$$

and  $p_\gamma^{-1}(AB)$  is relatively compact and open, hence it has finite and positive Haar measure. Since in addition  $\Delta_G$  is positive and bounded on  $p_\gamma^{-1}(AB)$ , we find  $0 < \phi_2(\gamma H) < \infty$ . Hence

$$\phi_1(\gamma H) = c_{\gamma H}\phi_2(\gamma H)$$

can be solved for  $c_{\gamma H}$ , which thus turns out to depend measurably upon  $\gamma H$ . Hence

$$\kappa(\omega) = \frac{d\beta_{\gamma H}}{d\mu_{\gamma H}} = \ell(\omega)c_{\gamma H}$$

is a Lebesgue-measurable function.

The remaining part (c) is simple to prove: The semi-invariance relation of  $\ell$  entails the relation for  $\kappa$ . The normalization is easily obtained: If  $\kappa$  is constant on the orbits, it defines a measurable mapping  $\bar{\kappa}$  on  $\Omega_{rc}/H$ . If we replace each  $\beta_{\gamma H}$  by  $\mu_{\gamma H}$ , we can make up for it by taking  $\bar{\kappa}(\gamma H) d\bar{\lambda}(\gamma H)$  as the new measure on the orbit space. The new choice has the desired properties. The uniqueness of  $\bar{\lambda}$  follows from the usual Radon–Nikodym arguments.  $\square$



**Remark 2.** Let us for the rest of the paper fix a choice of  $\bar{\lambda}$ . Note that this also uniquely determines the function  $\kappa$ . In the unimodular case we take  $\kappa$  to be 1, which in turn determines  $\bar{\lambda}$  uniquely.

As we shall later see, the choice of a pair  $(\bar{\lambda}, \kappa)$  corresponds exactly to a choice of Plancherel measure and the associated family of Duflo–Moore operators (at least on a subset of  $\widehat{G}$ ).

Before we turn to the construction of admissible vectors, we introduce some notation to help clarify the construction: To a function  $\widehat{g}$  on  $U$  we associate two auxiliary  $H$ -invariant functions  $T_H(\widehat{g})$  and  $S_H(\widehat{g})$  such that admissibility of  $g$  translates to a condition on  $T_H(\widehat{g})$  and square-integrability of  $g$  to a condition on  $S_H(\widehat{g})$ .

**Definition 1.** For a measurable function  $\widehat{g}$  on  $\Omega_{rc}$ , let  $T_H(\widehat{g})$  denote the function

$$T_H(\widehat{g})(\omega) := \left( \int_{\omega H} |\widehat{g}(\gamma)|^2 d\mu_{\omega H}(\gamma) \right)^{1/2} = \left( \int_{\omega H} |\kappa(\gamma)^{-1/2} \widehat{g}(\gamma)|^2 d\beta_{\omega H}(\gamma) \right)^{1/2}.$$

$T_H(\widehat{g})$  is a measurable,  $H$ -invariant mapping  $\Omega_{rc} \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$ . The admissibility condition can then be reformulated:

$$g \in L^2(U) \text{ is admissible} \Leftrightarrow T_H(\widehat{g}) \equiv 1 \text{ ( a.e. on } U \text{ )}. \tag{1.3}$$

Similarly, weak admissibility is equivalent to the requirement that  $T_H(\widehat{g}) \in L^\infty(U)$  and  $T_H(\widehat{g}) > 0$  almost everywhere. We can also define

$$S_H(\widehat{g})(\omega) := \left( \int_{\omega H} |\widehat{g}(\gamma)|^2 d\beta_{\omega H}(\gamma) \right)^{1/2}.$$

By our choice of measures,  $S_H$  and  $T_H$  coincide iff  $G$  is unimodular. Both  $T_H(\widehat{g})$  and  $S_H(\widehat{g})$  may (and will) be regarded as functions on the quotient space  $U/H$ . By the choice of the  $\beta_{\omega H}$ ,

$$\int_U |\widehat{g}(\omega)|^2 d\omega = \int_{U/H} |S_H(\widehat{g})(\omega H)|^2 d\bar{\lambda}(\omega H), \tag{1.4}$$

so that  $\widehat{g}$  is square-integrable iff  $S_H(\widehat{g})$  is a square-integrable function on  $U/H$ .

Now we can address the existence of admissible vectors. The following theorem is essentially the same as [26, Theorem 1.8].

**Theorem 2.**

Let  $U \subset \Omega_{rc}$  be measurable and  $H$ -invariant. Then  $\pi_U$  has a weakly admissible vector. It has an admissible vector iff either

- (i)  $G$  is unimodular and  $\bar{\lambda}(U/H) < \infty$ .
- (ii)  $G$  is non-unimodular.

Note that the strategy for the construction of admissible vectors in the following proof is similar to the arguments in [26], but also to the construction in [14]. It amounts to treating the admissibility condition—involving  $T_H$ —first, and then adjusting the construction to fulfill the square-integrability condition (involving  $S_H$ ) as well.

**Proof.** Recall that by the last remark we have for each admissible vector  $g$  that  $T_H(\widehat{g})$  is constant almost everywhere. At the same time, in the unimodular case it is square-integrable as a function on  $U/H$ , because of  $S_H = T_H$ . This shows the necessity of (i) in the unimodular case.

To prove the existence of admissible vectors, we first construct a function  $\widehat{g}$  on  $U$  fulfilling the admissibility condition (1.3), and then modify the construction to provide for square-integrability.

For this purpose we recycle the sets  $A \subset \Omega_{rc}$  and  $B \subset H$  from the proof of Lemma 1. We already observed there that  $\widehat{f} = \chi_{AB}$  is Lebesgue-measurable, and that  $T_H(\widehat{f})$  is positive and finite almost everywhere on  $U$ . Hence we may define  $\widehat{g} = \widehat{f}/T_H(\widehat{f})$ , which fulfills the admissibility criterion. In the unimodular case, Equation (1.4) together with  $S_H = T_H$  shows that  $\widehat{g} \in L^2(U)$ .

In the non-unimodular case, we modify  $g$  as follows: For every  $\gamma \in U$ , the compactness of  $p_\gamma^{-1}(AB)$  entails that  $\Delta_G$  is bounded on that set. Thus  $S_H(\widehat{g})$  is positive and finite almost everywhere. Since  $\bar{\lambda}$  is  $\sigma$ -finite, we can write  $U/H = \bigcup_{n \in \mathbb{N}} V_n$ , with disjoint  $V_n$  of finite measure, such that in addition  $S_H(\widehat{g})$  is bounded on each  $V_n$  (here we regard  $S_H(\widehat{g})$  as a function on the quotient). In particular,  $S_H(\widehat{g}) \cdot \chi_{V_n}$  is square-integrable on  $U/H$ . Now let  $U_n \subset U$  be the inverse image of  $V_n$  under the quotient map, and for  $h_0 \in H$  and  $n, k_n \in \mathbb{N}$ , denote by

$$\widehat{g}_n(\omega) := \Delta_H(h_0)^{k_n/2} \widehat{f}_2(\omega h_0^{k_n}) \cdot \chi_{U_n}(\omega).$$

Then the normalization ensures that  $\widehat{g}_n$  has the following properties:

$$T_H(\widehat{g}_n) = \chi_{U_n} \tag{1.5}$$

and

$$S_H(\widehat{g}_n) = \Delta_H(h_0)^{k_n/2} |\det(h_0)|^{-k_n/2} S_H(\widehat{g}) \cdot \chi_{U_n} = \Delta_G(h_0)^{k_n/2} S_H(\widehat{g}) \cdot \chi_{U_n}. \tag{1.6}$$

Hence the following construction gives an admissible vector: Choose  $h_0 \in H$  such that  $\Delta_G(h_0) < 1/2$ , pick  $k_n \in \mathbb{N}$  satisfying

$$2^{-k_n} \|S_H(\widehat{g}) \cdot \chi_{U_n}\|_2^2 < 2^{-n} \tag{1.7}$$

and let  $\widehat{g}(\omega) := \Delta_H(h_0)^{k_n/2} \widehat{f}_2(\omega h_0^{k_n})$ , for  $\omega \in U_n$ . Then (1.5) implies that  $T_H(\widehat{g}) = 1$  a.e., whereas (1.6) and (1.7) ensure that  $S_H(\widehat{g}) \in L^2(U/H, \bar{\lambda})$ .

A weakly admissible vector for  $\pi_U$  (which is missing in the unimodular case) can be obtained by similar (somewhat simpler) methods.  $\square$

**Remark 3.** In Theorem 1 we cannot replace  $\Omega_{rc}$  by the bigger set  $\Omega_c$ . To give a non-unimodular example, let  $H = \{2^k h : k \in \mathbb{Z}, h \in \text{SL}(2, \mathbb{Z})\}$ , which is a discrete subgroup of  $\text{GL}(2, \mathbb{R})$ . Whenever  $(\gamma_1, \gamma_2) \in \widehat{\mathbb{R}}^2$  is such that  $\gamma_1/\gamma_2$  is irrational, the stabilizer of  $(\gamma_1, \gamma_2)$  in  $H$  is finite. Hence the set  $\Omega_c$  is a conull subset in  $\widehat{\mathbb{R}}^2$ , whereas (as we already noted)  $\Omega_{rc}$  is empty.  $H$  operates ergodically on  $\widehat{\mathbb{R}}^2$  (already  $\text{SL}(2, \mathbb{Z})$  does, [33, 2.2.9]), and hence  $\pi$  is an irreducible representation. But it has been shown that for discrete dilation groups  $\pi$  is never irreducible and square-integrable, see [12, Remark 12].

Let us now give a short summary of the steps which have to be carried out for the construction of wavelet transforms from semidirect products:

1. Compute the  $H$ -orbits in  $\widehat{\mathbb{R}}^k$ , possibly by giving a parametrization of  $\widehat{\mathbb{R}}^k/H$ .
2. Determine the set  $\Omega_{rc}$ . If  $\lambda(\Omega_{rc}) = 0$ , stop.
3. Parametrize each orbit in  $\Omega_{rc}$  and determine the image  $\mu_{\gamma H}$  of Haar measure under the projection map  $h \mapsto \gamma h$ .

4. Compute the measure decomposition  $d\lambda(\gamma) = d\beta_{\omega H}(\gamma)d\bar{\lambda}(\omega H)$ .
5. Compute the Radon–Nikodym derivative  $\kappa$ .
6. The admissibility condition can then be formulated for subsets of  $\Omega_{rc}$  just as in Theorem 1. Theorem 2 ensures the existence of admissible vectors.

Since the final step—the actual construction of admissible vectors—is missing, the description is somewhat incomplete. Clearly the construction given in the proof of Theorem 2 is not very practical, but it seems doubtful to us that a more explicit method is available which works in full generality. However, in many concrete cases where parametrizations of orbits and orbit spaces are possible, they can be given differentiably. Then computing the various measures and Radon–Nikodym derivatives reduces to computing the Jacobians of those parametrizations. We expect that in such a setting the construction of admissible vectors should also be facilitated.

For the remainder of this section we want to focus on the case that  $G$  is unimodular. The main motivation for the following proposition is to show that certain subrepresentations of  $\pi$  do not have admissible vectors. In the light of Theorem 2, this amounts to proving that  $\bar{\lambda}(U/H)$  is infinite, for the  $H$ -invariant set  $U \subset \Omega_{rc}$  under consideration.

The argument proving the following proposition employs the action of the scalars on the orbit space  $\Omega_{rc}/H$ . The group of scalars could be replaced by any group  $A \subset GL(k, \mathbb{R})$  which normalizes  $H$ . Symmetry arguments of this type could also simplify the steps 1. through 6. sketched above.

$\mathbb{R}^+$  operates on  $\widehat{\mathbb{R}^k}/H$  by multiplication, i. e., if  $a \in \mathbb{R}^+$  then  $a \cdot (\gamma H) = (a\gamma)H$  is an operation. Obviously  $\Omega_{rc}$  is invariant, so that we obtain an operation on  $\Omega_{rc}/H$ . The next proposition gives the behavior of  $\bar{\lambda}$  under this action.

**Proposition 2.**

Assume that  $G$  is unimodular. Let the measures  $\bar{\lambda}$  and  $\mu_{\gamma H}$  be as in Lemma 1. For  $a \in \mathbb{R}^+$  and  $\gamma \in \widehat{\mathbb{R}^k}$  let  $a^*(\mu_{\gamma H})$  denote the image measure of  $\mu_{\gamma H}$  on  $\gamma Ha$ , i. e., for measurable  $B \subset \gamma Ha$  let  $a^*(\mu_{\gamma H})(B) := \mu_{\gamma H}(Ba^{-1})$ . Moreover let the measure  $\bar{\lambda}_a$  be given by  $\bar{\lambda}_a(B) := \bar{\lambda}(Ba)$  ( $B \subset \widehat{\mathbb{R}^k}/H$  measurable). Then on  $\Omega_{rc}/H$  the following relations hold:

$$\begin{aligned} \mu_{a\gamma H} &= a^*(\mu_{\gamma H}), \\ \bar{\lambda}_a &= a^k \bar{\lambda}. \end{aligned}$$

**Proof.** The first equality is immediate from the definitions of  $\mu_{\gamma H}$  and  $\mu_{a\gamma H}$ . For the second equation let us introduce the following notation: If  $f : \Omega_{rc} \rightarrow \mathbb{R}$  is a positive, measurable function, let  $q(f)$  denote the function on  $\Omega_{rc}/H$  defined by

$$q(f)(\gamma H) := \int_{\gamma H} f(\omega) d\mu_{\gamma H}(\omega).$$

Moreover let  $f_a(\omega) := f(\omega a^{-1})$ , for all  $\omega \in \Omega_{rc}$  and  $a \in \mathbb{R}^+$ . From the first equation we obtain

$$\begin{aligned} q(f_a)(\gamma H) &= \int_{\gamma H} f(\omega a^{-1}) d\mu_{\gamma H}(\omega) \\ &= \int_{\gamma Ha^{-1}} f(\omega) d\mu_{\gamma Ha^{-1}}(\omega) \\ &= q(f)(\gamma Ha^{-1}). \end{aligned}$$

Using this equation, we compute

$$\begin{aligned}
 \int_{\Omega_{rc}/H} q(f)(\gamma H) d\bar{\lambda}(\gamma H) &= a^{-n} \int_{\Omega_{rc}} f_a(\omega) d\lambda(\omega) \\
 &= a^{-n} \int_{\Omega_{rc}/H} q(f_a)(\gamma H) d\bar{\lambda}(\gamma H) \\
 &= a^{-n} \int_{\Omega_{rc}/H} q(f)(\gamma H a^{-1}) d\bar{\lambda}(\gamma H) \\
 &= a^{-n} \int_{\Omega_{rc}/H} q(f)(\gamma H) d\bar{\lambda}_a(\gamma H) .
 \end{aligned}$$

Using arguments similar to the one used in the proof of Theorem 2, it is readily seen that for each measurable  $A \subset \Omega_{rc}/H$  there exists a positive measurable  $f$  on  $\Omega_{rc}$  with  $q(f) = \chi_A$ . Hence we have shown the second equation.  $\square$

As a first consequence we obtain that admissible vectors exist only for proper subsets of  $\Omega_{rc}$ . This was already noted (in the special case where  $\Omega_{rc}$  is conull in  $\widehat{\mathbb{R}^k}$ ) in [26], Theorem 1.8.

**Corollary 1.**

*Assume that  $G$  is unimodular, and that  $U := \Omega_{rc}$  is not a nullset. Then the subrepresentation  $\pi_U$  is not strongly square integrable.*

**Proof.** By assumption we have  $\bar{\lambda}(\Omega_{rc}/H) > 0$ , and we need to show that  $\bar{\lambda}(\Omega_{rc}/H) = \infty$ . But for all  $a \in \mathbb{R}^+$ ,  $a\Omega_{rc} = \Omega_{rc}$ , and Proposition 2 yields  $\bar{\lambda}(\Omega_{rc}/H) = \bar{\lambda}(a \cdot \Omega_{rc}/H) = |a|^{-k} \bar{\lambda}(\Omega_{rc}/H)$ .  $\square$

It is well known that, given a square integrable representation  $\sigma$  of a locally compact group  $G$ , every vector in  $\mathcal{H}_\sigma$  is admissible iff  $\sigma$  is irreducible and  $G$  is unimodular. Hence irreducible representations would be particularly useful, having no restrictions at all on admissible vectors. But the following corollary excludes irreducible representations from our setting. The statement was proved first in [15] by a technique employing the Fell topology on  $\hat{G}$ .

**Corollary 2.**

*Let  $G$  be unimodular. Then the quasiregular representation  $\pi$  does not contain any irreducible square-integrable subrepresentations.*

**Proof.** Assume the contrary and let  $\pi_U$  be an irreducible square integrable subrepresentation. Here  $U$  denotes the corresponding  $H$ -invariant subset of  $\widehat{\mathbb{R}^k}$ . Then, by [2, Theorem 1.1],  $U$  is (up to a null set) an orbit of positive measure, hence open (by Sard's Theorem). In particular  $U \subset \Omega_{rc}$ , and  $\bar{\lambda}(\{U\}) > 0$ .

From the fact that  $\mathcal{H}_U$  has admissible vectors we conclude that  $\bar{\lambda}(\{U\}) < \infty$ . On the other hand, an easy connectedness argument shows that for each  $\gamma \in U$ , the ray  $\mathbb{R}^+\gamma$  is contained in the open orbit  $U$ . Hence the same argument which proved the previous corollary shows that  $\bar{\lambda}(\{U\}) = \infty$ , which yields the desired contradiction.  $\square$

## 2. Plancherel Measure and Admissibility

This section is devoted to a short review of Plancherel theory and its relations to generalized wavelet transforms. For a more detailed (yet still short) account of Plancherel theory, consult [11, Section 7.5]. Unfortunately, there does not seem to exist a widely accessible exposition of Plancherel theory that covers non-unimodular groups in sufficient detail; a good source for the unimodular case is [8]. For the non-unimodular theory, the original articles [22, 30, 9] are probably still the best sources (though rather technical at times).

All throughout this section,  $G$  is a second countable locally compact group. The starting point for the definition of the Plancherel transform is the operator valued Fourier transform on  $L^1(G)$ . Given  $f \in L^1(G)$  and  $\sigma \in \widehat{G}$ , we define

$$\mathcal{F}(f)(\sigma) := \sigma(f) := \int_G f(x)\sigma(x) d\mu_G(x),$$

where the integral is taken in the weak operator sense. As direct consequences of the definition we have  $\|\sigma(f)\|_\infty \leq \|f\|_1$  and  $\sigma(f * g) = \sigma(f) \circ \sigma(g)$ .

The Plancherel transform is obtained by extending the Fourier transform from  $L^1(G) \cap L^2(G)$  to  $L^2(G)$ . The non-unimodular part of the following Plancherel theorem is due to Duflo and Moore [9, Theorem 5], whereas the unimodular version may be found in [8].

### Theorem 3.

Let  $G$  be a second countable locally compact group having a type-I regular representation. Then there exists a measure  $\nu_G$  on  $\widehat{G}$  and a measurable field  $(K_\sigma)_{\sigma \in \widehat{G}}$  of selfadjoint positive operators with densely defined inverses, with the following properties:

- (i) For  $f \in L^1(G) \cap L^2(G)$  and  $\nu_G$ -almost all  $\sigma \in \widehat{G}$ , the closure of the operator  $\sigma(f)K_\sigma^{1/2}$  is a Hilbert–Schmidt operator on  $\mathcal{H}_\pi$ .
- (ii) The map  $L^1(G) \cap L^2(G) \ni f \mapsto ([\sigma(f)K_\sigma^{1/2}]_{\sigma \in \widehat{G}})$  extends to a unitary equivalence

$$\mathcal{P} : L^2(G) \rightarrow \mathcal{B}_2^\oplus := \int_{\widehat{G}}^\oplus \mathcal{B}_2(\mathcal{H}_\sigma) d\nu_G(\sigma).$$

This unitary operator is called the **Plancherel transform** of  $G$ . It intertwines the two-sided representation  $\lambda_G \times \rho_G$  with  $\int_{\widehat{G}}^\oplus \sigma \otimes \bar{\sigma} d\nu_G(\sigma)$ .

- (iii)  $G$  is unimodular iff almost all  $K_\sigma$  are scalar multiples of the identity operator. In this case we fix  $K_\sigma = \text{Id}_{\mathcal{H}_\sigma}$ .
- (iv) Once a measurable choice  $(K_\sigma)_{\sigma \in \widehat{G}}$  of Duflo–Moore operators is made, the Plancherel measure  $\nu_G$  is uniquely determined. This applies in particular to the unimodular case, where  $K_\sigma = \text{Id}_{\mathcal{H}_\sigma}$  leads to a unique definition of  $\nu_G$ .

In the following  $\widehat{f}$  denotes the Plancherel transform of the  $L^2$ -function  $f$ ; in particular in the non-unimodular case it should not be confused with the Fourier transform.

A further important feature of the Plancherel transform is the decomposition of intertwining operators: If  $T : L^2(G) \rightarrow L^2(G)$  is a bounded operator which commutes with left translations, then there exists a measurable field of bounded operators  $(T_\sigma)_{\sigma \in \widehat{G}}$  with  $\|T_\sigma\|_\infty$  uniformly bounded, such that

$$T = \int_{\widehat{G}}^\oplus \text{Id}_{\mathcal{H}_\sigma} \otimes T_\sigma d\nu_G(\sigma).$$

This applies in particular to projections onto invariant subspaces. The obvious analogue for the right action of  $G$  holds as well. This decomposition property is the key feature from the point of view of generalized wavelet transforms. First of all it provides systematic access to the subrepresentations of the regular representations, and by the discussion at the end of the Introduction, those subrepresentations exhaust all situations of interest. Moreover, for a given admissible vector  $\eta$ , there are two Plancherel transforms to consider: The Plancherel transform of  $\eta$  as  $L^2$ -function, and the decomposition of the intertwining operator  $V_\eta$ . Relating those two objects enables us to formulate admissibility conditions and to construct vectors which fulfill them. The following theorem summarizes the results of [14]. Note that [14] uses the operators  $C_\sigma = K_\sigma^{-1/2}$ . Note also that the admissibility conditions in part (b) are only sufficient, though we expect them to be necessary as well. The admissibility conditions are not explicitly stated in [14], but they are simple consequences of [14, Lemma 1.4].

**Theorem 4.**

Assume that  $\lambda_G$  is type-I. Let  $\mathcal{H} \subset \mathcal{B}_2^\oplus$  be an invariant subspace and denote by  $P$  the projection onto  $\mathcal{H}$ . Then there exists a measurable family of projections  $(P_\sigma)_{\sigma \in \widehat{G}}$  such that

$$P = \int_{\widehat{G}}^\oplus \text{Id}_{\mathcal{H}_\sigma} \otimes P_\sigma \, d\nu_G(\sigma).$$

(a) Assume that  $G$  is unimodular. Then  $\eta \in \mathcal{H}$  is

weakly admissible iff  $\widehat{\eta}(\sigma)$  is injective on  $\nu_G$ -a.e.  $P_\sigma(\mathcal{H}_\sigma)$ ,  
and  $\sigma \mapsto \|\widehat{\eta}(\sigma)\|_\infty$  is in  $L^\infty(\widehat{G})$ .

admissible iff  $\widehat{\eta}(\sigma)$  is an isometry on  $P_\sigma(\mathcal{H}_\sigma)$   
for  $\nu_G$ -a.e.  $\sigma$ .

$\mathcal{H}_U$  has admissible vectors iff  $\int_{\widehat{G}} \dim(P_\sigma(\mathcal{H}_\sigma)) \, d\nu_G(\sigma) < \infty$

$\lambda_G$  has an admissible vector iff  $G$  is discrete.

(b) Assume that  $G$  is non-unimodular. Assume that  $\eta \in \mathcal{H}$  is such that  $\widehat{\eta}(\sigma)^* K_\sigma^{-1/2}$  extends to a bounded operator on  $\mathcal{H}_\sigma$ , for  $\nu_G$ -almost every  $\sigma \in \widehat{G}$ . Then  $\eta \in \mathcal{H}$  is

weakly admissible if  $\widehat{\eta}(\sigma)$  is injective on  $\nu_G$ -a.e.  $P_\sigma(\mathcal{H}_\sigma)$ ,  
and  $\sigma \mapsto \left\| \left[ \widehat{\eta}(\sigma)^* K_\sigma^{-1/2} \right] \right\|_\infty$  is in  $L^\infty(\widehat{G})$ .

admissible if  $\left[ \widehat{\eta}(\sigma)^* K_\sigma^{-1/2} \right]^*$  is an isometry on  $P_\sigma(\mathcal{H}_\sigma)$ ,  
for  $\nu_G$ -a.e.  $\sigma$ .

There exists a vector  $\eta \in \mathcal{B}_2^\oplus$  fulfilling the admissibility condition for  $\mathcal{H} = \mathcal{B}_2^\oplus$ . Hence  $\lambda_G$ —and thus every subrepresentation thereof—has an admissible vector.

The statements somewhat simplify when we consider multiplicity-free representations. Since the quasi-regular representation will turn out to be multiplicity-free, we find it useful to work out this particular case in some detail:

**Theorem 5.**

Assume that  $\lambda_G$  is type-I. Let  $\pi$  be a multiplicity-free weakly square-integrable representation of  $G$ . Then there exists a  $\nu_G$ -measurable set  $\Sigma \subset \widehat{G}$  such that

$$\pi \simeq \int_{\Sigma} \sigma \, d\nu_G(\sigma).$$

For the following, we assume that  $\pi$  is in fact realized as the direct integral.

- (a) Assume that  $G$  is unimodular.  $\eta = (\eta_{\sigma})_{\sigma \in \Sigma} \in \mathcal{H}_{\pi}$  is admissible iff  $\|\eta_{\sigma}\| = 1$  for  $\nu_G$ -every  $\sigma \in \Sigma$ .  $\pi$  is strongly square-integrable iff  $\nu_G(\Sigma) < \infty$ .
- (b) Assume that  $G$  is non-unimodular. Let  $\eta = (\eta_{\sigma})_{\sigma \in \Sigma}$  be such that for  $\nu_G$ -a.e.  $\sigma \in \Sigma$ ,  $\eta_{\sigma} \in \text{dom}(K_{\sigma}^{-1/2})$ , with  $\|K_{\sigma}^{-1/2}\eta_{\sigma}\| = 1$ . Then  $\eta$  is admissible. There exist vectors  $\eta \in \mathcal{H}_{\pi}$  fulfilling this admissibility condition.

A consequence of the theorem is the following proposition, which shows that Plancherel measure is in fact characterized by the admissibility condition. This observation will allow us to identify the quotient measure  $\bar{\lambda}$  obtained in Section 1 with Plancherel measure.

**Proposition 3.**

Assume that  $\lambda_G$  is type-I. Let  $(K_{\sigma})_{\sigma \in \widehat{G}}$  be a measurable choice of Duflo–Moore operators—for  $G$  unimodular,  $K_{\sigma} = \text{Id}_{\mathcal{H}_{\sigma}}$ —and let  $\nu_G$  be the corresponding Plancherel measure. Let  $\pi, \Sigma$  be as in Theorem 5, and assume in addition that there exists an admissible vector for  $\pi$ ; i. e., assume in the unimodular case that  $\nu_G(\Sigma) < \infty$ . Let  $\tilde{\nu}$  be a Borel measure on  $\Sigma$  which is equivalent to  $\nu_G$ , and consider the representation

$$\tilde{\pi} = \int_{\Sigma}^{\oplus} \sigma \, d\tilde{\nu}(\sigma).$$

Assume that for all measurable vector fields  $\eta = (\eta_{\sigma})_{\sigma \in \Sigma} \in \mathcal{H}_{\tilde{\pi}}$  fulfilling  $\eta_{\sigma} \in \text{dom}(K_{\sigma}^{-1/2})$   $\tilde{\nu}$ -a. e., the admissibility criterion

$$\eta \text{ admissible for } \tilde{\pi} \iff \|K_{\sigma}^{-1/2}\eta_{\sigma}\| = 1 \text{ } \tilde{\nu}\text{-almost everywhere}$$

is valid. Then  $\tilde{\nu} = \nu_G$  on  $\Sigma$ .

**Proof.** Denote by  $T : \int_{\Sigma}^{\oplus} \sigma \, d\nu_G(\sigma) \rightarrow \int_{\Sigma}^{\oplus} \sigma \, d\tilde{\nu}(\sigma)$  the unitary intertwining operator obtained by

$$T((\phi_{\sigma})_{\sigma \in \Sigma}) = \left( \sqrt{\frac{d\tilde{\nu}}{d\nu_G}(\sigma)} \phi_{\sigma} \right)_{\sigma \in \Sigma}.$$

Clearly, a unitary equivalence maps admissible vectors onto admissible vectors. Moreover, if  $\eta = (\eta_{\sigma})_{\sigma \in \Sigma}$  fulfills  $\eta_{\sigma} \in \text{dom}(K_{\sigma}^{-1/2})$   $\nu_G$ -almost everywhere, then  $T\eta$  does as well. Hence, if we let  $\eta \in \mathcal{H}_{\pi}$  be any vector fulfilling the admissibility criterion from Theorem 5, the necessary admissibility criterion for  $\tilde{\pi}$  gives

$$\|K_{\sigma}^{-1/2}\eta_{\sigma}\| = 1 = \left\| \sqrt{\frac{d\tilde{\nu}}{d\nu_G}(\sigma)} K_{\sigma}^{-1/2}\eta_{\sigma} \right\| \text{ almost everywhere .}$$

Hence the Radon–Nikodym-derivative is 1 almost everywhere. □

### 3. Concrete and Abstract Admissibility Conditions

Now we are in a position to establish the connection between the results in Sections 1 and 2. In order to apply the theorems from Section 2, let us assume that the semidirect product has a type-I regular representation. In addition, we restrict attention to the case that  $\widehat{\mathbb{R}^k}/H$  is standard.

The first and crucial step consists in writing  $\pi$  as the direct integral of monomial representations, as described in the following result (which is [23, Theorem 2.1]). The direct integral decomposition is obtained by looking at the dual representation (1.1) in the light of the measure decomposition from Lemma 1 (a). Note that if  $\widehat{\mathbb{R}^k}/H$  is standard, the measure disintegration described in Lemma 1 (a) can be given for all of  $\widehat{\mathbb{R}^k}$  instead of  $\Omega_{rc}$ , i. e.,  $\bar{\lambda}$  is given on  $\widehat{\mathbb{R}^k}/H$ , and the  $\beta_{\gamma H}$  exist also for orbits  $\gamma H$  without compact stabilizer.

**Lemma 2.**

*The quasiregular representation  $\pi$  is the direct integral*

$$\pi \simeq \int_{\widehat{\mathbb{R}^k}/H}^{\oplus} \left( \text{Ind}_{G_\gamma}^G \gamma \times 1 \right) d\bar{\lambda}(\gamma H),$$

*of irreducible representations, where  $G_\gamma = \mathbb{R}^k \rtimes H_\gamma$ . In particular  $\pi$  is multiplicity-free.*

**Proof.** The disintegration  $d\lambda(\omega) = d\beta_{\gamma H}(\omega)d\bar{\lambda}(\gamma H)$  allows us to interpret the conjugated representation  $\widehat{\pi}$  as a direct integral over  $\widehat{\mathbb{R}^k}/H$  of representations acting on the spaces  $L^2(\gamma H, \mu_{\gamma H})$ , by identifying each  $f$  with the family of restrictions  $(\widehat{f}|_{\gamma H})_{\gamma H \in \widehat{\mathbb{R}^k}/H}$ . The representations over the orbits are easily seen to be induced in the indicated manner, using a concrete realization of  $\text{Ind}_{G_\gamma}^G \gamma \times 1$  on  $L^2(\gamma H, \mu_{\gamma H})$  via cross sections. The representations are irreducible and pairwise inequivalent by Mackey's theory.  $\square$

An algorithm for the construction of Plancherel measures for semidirect products was provided by Kleppner and Lipsman in the fundamental articles [22] I, II. Let us give an outline: By Mackey's theory, the dual of  $G$  may be described by

$$\widehat{G} = \bigcup_{\gamma H \in \widehat{\mathbb{R}^k}/H} \left\{ \text{Ind}_{G_\gamma}^G \gamma \times \sigma : \sigma \in \widehat{H}_\gamma \right\},$$

whenever  $\widehat{\mathbb{R}^k}/H$  is standard. An intuitive interpretation is that  $\widehat{G}$  may be considered as a “fibred space,” with base space  $\widehat{\mathbb{R}^k}/H$  and the fibre over  $\gamma H$  is (identified with)  $\widehat{H}_\gamma$ , and it is the central result of [22] (see part II, Theorem 2.3) that the Plancherel measure is obtained as a “fibred measure” as well: Take any pseudo-image  $\bar{\lambda}_\nu$  of standard Lebesgue measure, choose Plancherel measures  $\nu_{\gamma H}$  of the little groups  $H_\gamma$  which exist if we assume that almost every  $H_\gamma$  has a type-I regular representation. Then the measure  $\nu$  defined by

$$\nu(A) = \int_{\widehat{\mathbb{R}^k}/H} \int_{\widehat{H}_\gamma} \chi_A \left( \text{Ind}_{G_\gamma}^G \gamma \times \sigma \right) d\nu_{\gamma H}(\sigma) d\bar{\lambda}_\nu(\gamma H) \quad (3.1)$$

is equivalent to the Plancherel measure. This determines the measure class. If  $G$  is unimodular, the proof of [22, II, Theorem 2.3] in fact gives a recipe how to obtain the correct normalization; whereas in the non-unimodular case the results of [22] only give the measure class, and no access to the Duflo–Moore operators.



The “fibred measure”-view of Plancherel measure provides a neat interpretation of the set  $\Omega_c$ . As calculated in the previous lemma, the quasi-regular representation is a direct integral, and the measure defining it is supported on a subset of  $\widehat{G}$  which meets each fibre in precisely one point, that is in  $\text{Ind}_{G_\gamma}^G \gamma \times 1$ , which corresponds to the trivial representation in  $\widehat{H}_\gamma$ . The Plancherel measure of the trivial representation in  $\widehat{H}_\gamma$  is positive if and only if the  $H_\gamma$  is compact. Hence the inner integral in (3.1) vanishes except for  $\gamma \in \Omega_c$ . It follows that the part of the support corresponding to the complement of  $\Omega_c$  is a Plancherel-null set. On the other hand, the construction of Plancherel measure shows that on the part of the support corresponding to  $\Omega_c$ , Plancherel measure and  $\bar{\lambda}$  are equivalent. Indeed, we are free to take  $\bar{\lambda}_v = \bar{\lambda}$ , and the inner integral in (3.1) is positive if  $H_\gamma$  is compact. In short,  $\widehat{\mathbb{R}^k}/H = \Omega_c/H \cup (\widehat{\mathbb{R}^k} \setminus \Omega_c)/H$  gives the decomposition of  $\bar{\lambda}$  into a part which is absolutely continuous with respect to Plancherel measure, and a part which is singular. This yields an abstract explanation of the role of  $\Omega_c$ , in particular of the necessary condition in Theorem 1.

Before we describe the relationship between concrete and abstract admissibility conditions, it is useful to relate the Borel space  $\Omega_{rc}/H$  to a suitable subset of  $\widehat{G}$ .

**Proposition 4.**

*There exists a conull, Borel subset  $U_0 \subset \Omega_{rc}$ , such that the mapping*

$$\Phi : U_0/H \ni \gamma H \mapsto \text{Ind}_{G_\gamma}^G \gamma \times 1$$

*is a Borel isomorphism onto a standard measurable subset  $\Sigma \subset \widehat{G}$ .*

**Proof.** We just noted that  $\pi_U$  is multiplicity-free with

$$\pi_U \simeq \int_{U/H}^\oplus \left( \text{Ind}_{G_\gamma}^G \gamma \times 1 \right) d\bar{\lambda}(\gamma H).$$

On the other hand, the existence of a weakly admissible vector for  $\pi_U$  implies that  $\pi_U$  is equivalent to a subrepresentation of the regular representation. This gives the alternative decomposition

$$\pi_U \simeq \int_{\Sigma_1}^\oplus \sigma d\nu_G(\sigma).$$

Hence we may invoke [27, Theorem, p. 117] to see that  $T$  arises from an isomorphism of the underlying Borel spaces: There exist conull subsets  $\Sigma \subset \Sigma_1$ ,  $U_0 \subset \Omega_{rc}/H$ , a Borel isomorphism  $\Phi : U_0 \rightarrow \Sigma$ , a measurable field of unitary operators  $T_{\gamma H} : L^2(\gamma H, d\beta_{\gamma H}) \rightarrow \mathcal{H}_{\Phi(\gamma H)}$  and a Radon–Nikodym derivative  $\Psi : U_0/H \rightarrow \mathbb{R}_0^+$  such that  $T$  decomposes into  $(\Psi(\gamma H)T_{\gamma H})_{\gamma H}$ . Since  $T$  is an intertwining operator, so is almost every  $T_{\gamma H}$ , and thus  $\Phi(\gamma H) = \text{Ind}_{G_\gamma}^G (\gamma \times 1)$ .  $\square$

Let us now summarize the transfer between concrete and abstract admissibility conditions.

**Theorem 6.**

*Let  $\Phi$ ,  $U_0$  and  $\Sigma_0$  be as in Proposition 4. For  $\gamma H \subset U_0$  let  $\mathcal{K}_{\gamma H}$  denote the operator on  $L^2(\gamma H, d\beta_{\gamma H})$  given by pointwise multiplication with  $\kappa|_{\gamma H}$ .  $\Phi$  gives rise to*

the following correspondences between the objects in Section 1 and those in Section 2:

$$\begin{aligned}
 U_0/H &\longleftrightarrow \Sigma_0, \\
 \gamma H &\longleftrightarrow \sigma, \\
 L^2(\gamma H, d\beta_{\gamma H}) &\longleftrightarrow \mathcal{H}_\sigma, \\
 \widehat{f}|_{\gamma H} &\longleftrightarrow \eta_\sigma, \\
 S_H(\widehat{f})(\gamma H) &\longleftrightarrow \|\eta_\sigma\|, \\
 \bar{\lambda} &\longleftrightarrow \nu_G, \\
 \mathcal{K}_{\gamma H} &\longleftrightarrow K_\sigma, \\
 T_H(\widehat{f})(\gamma H) &\longleftrightarrow \left\| K_\sigma^{-1/2} \eta_\sigma \right\|.
 \end{aligned}$$

These correspondences exhibit Theorems 1 and 2 as special instances of Theorem 5.

**Proof.** It remains to check that the Duflo–Moore  $K_\sigma$  corresponds to  $\mathcal{K}_{\gamma H}$ , and that the Plancherel measure  $\nu_G$  belonging to this particular choice of Duflo–Moore operators corresponds to  $\bar{\lambda}$ . Straightforward calculation, using relation (1.2) from Lemma 1, shows that  $\mathcal{K}_{\gamma H}$  satisfies the quasi-invariance relation

$$\left( \text{Ind}_{G_\gamma}^G(\gamma \times 1)(x, h) \right) \mathcal{K}_{\gamma H} \left( \text{Ind}_{G_\gamma}^G(\gamma \times 1)(x, h) \right)^* = \Delta_G(x, h)^{-1} \mathcal{K}_{\gamma H}.$$

By [9, Corollary 1 to Theorem 5], the same relation has to be fulfilled by the Duflo–Moore operators; in fact by [9, Lemma 1], the relation characterizes the Duflo–Moore operators up to a scalar multiple. Since in addition, the measurability of  $\kappa$  ensures that  $(\mathcal{K}_{\gamma H})_{\gamma H \in U_0/H}$  is measurable, we may take this operator field as a realization of the Duflo–Moore operators.

Then it remains to prove that, given this particular choice of Duflo–Moore operators, the measure  $\bar{\lambda}$  is the corresponding Plancherel measure. But this is provided by the concrete admissibility condition in Theorem 1 together with Proposition 3.  $\square$

## Concluding Remarks

The use of discrete dilation groups can be seen as a first discretization step; for a fully discrete wavelet transform we would have to discretize the translations as well. Recent publications [4, 31, 25] document the increasing interest in the use of direct integral decompositions for the study and construction of fully discrete wavelet (or wavelet-like) systems. We expect that a connection between our results and the results contained in these articles would provide a better understanding of the discretization problem.

Our main motivation was to study admissibility conditions both by more or less elementary methods and in connection with Plancherel theory. Our paper can be seen as a natural continuation of the article by Bernier and Taylor. Our construction of Duflo–Moore operators is entirely analogous to what is done in [5] for the case of open free orbits. Irreducible square-integrable subrepresentations of  $\pi$  correspond to open orbits, which appear as atoms in the quotient space  $\Omega_{rc}/H$ . Passing from irreducible representations (or a finite direct sum of irreducibles) to the general case thus corresponds to the transition from counting measure on a finite set of open orbits to a certain measure on the orbit space. The additional technical cost consists mainly in keeping track of the various measures and their respective normalizations.

As we mentioned in the Introduction, Plancherel theory did not help establish admissibility conditions or construct admissible vectors for the quasi-regular representation. It was rather the other way round: The close connection between the admissibility conditions and Plancherel measure, as observed in Proposition 3, turned out to be a useful tool for the explicit calculation of Plancherel measure and the Duflo–Moore operators. Nevertheless, given an arbitrary reducible representation, the connection to Plancherel theory can provide an orientation and a possible strategy for the construction of admissible vectors. We believe that the semidirect products studied in this paper are very well suited to explicitly see these aspects of Plancherel theory at work.

The results given here and in [14] could probably be generalized to hold for the type-I part of  $\lambda_G$ . Duflo and Moore devised their Plancherel theory for this more general setting, and we believe that the proofs in [14] should go through as well. Similarly, the requirement that  $\widehat{\mathbb{R}^k}/H$  be standard could probably be replaced by something weaker. Generally speaking, the type-I requirement is not needed for the existence of admissible vectors. For instance, admissible vectors exist for the regular representation of an arbitrary discrete group (simply take the Kronecker- $\delta$  at the origin), but only very few of these groups have a type-I regular representation; confer [20]. Also, as examples in the literature illustrate, Plancherel measure can exist in the non-type-I setting, but it is no longer unique. This will probably have consequences for the necessary condition in Theorem 4(a) for the existence of admissible vectors. However, for the construction of admissible vectors for the regular representation of a non-unimodular group, it is conceivable that any Plancherel decomposition might work. (Note however that the construction in [14] rests on the concrete description of the Duflo–Moore operators in [9], which is valid only for the type-I setting.)

Despite the discussion for the general case, the existence of admissible vectors for the quasiregular representation (or some subrepresentation) seems to be tied more closely to a regularity condition on the orbit space, which in turn is related to the type of the regular representation. In this context there is essentially one open question left, and that regards the role of the set  $\Omega_c$ , or rather,  $\Omega_c \setminus \Omega_{rc}$ . As we have seen in Remark 3, the existence of admissible vectors corresponding to subsets of the latter set is not guaranteed. The following conjecture, which is a sharpening of a statement from Theorem 1, would neatly resolve the question; unfortunately discrete dilation groups acting ergodically (such as in Remark 3) are so far our only evidence for its truth. However, in the observations following [26, Proposition 2.8] the authors make a conjecture similar to the one we give here.

**Conjecture:** If  $\pi_U$  has an admissible vector, then  $U \subset \Omega_{rc}$ .

## Appendix

### A. The Sets $\Omega_c$ and $\Omega_{rc}$

In this Appendix we prove the measurability of  $\Omega_c$  and the openness of  $\Omega_{rc}$ . The proof for the first result uses the subgroup space of  $H$ , as introduced by Fell [10].

**Definition and Remark A.1.** Let  $G$  be a locally compact group. The **subgroup space of  $G$**  is the set  $K(G) := \{L < G : L \text{ is closed}\}$ , endowed with the topology generat-

ed by the sets

$$U(V_1, \dots, V_n; C) := \{L \in K(G) : L \cap V_i \neq \emptyset, \forall 1 \leq i \leq n, L \cap C = \emptyset\},$$

where  $V_1, \dots, V_n$  denotes any finite family of open subsets of  $G$  and  $C \subset G$  is compact. With this topology  $K(G)$  is a compact Hausdorff space.

The next few results are probably well known, but we were not able to find references for them:

**Proposition A.2.**

*If  $G$  is second countable, then so is  $K(G)$ .*

**Proof.** Clearly it suffices to consider only  $U(V_1, \dots, V_n; C)$  with  $V_i$  belonging to a countable base of the topology on  $G$ , hence we are done when we show that for each  $C \subset G$  compact we have  $U(G; C) = \bigcup_{n \in \mathbb{N}} U(G; C_n)$ , with the  $C_n$  belonging to a fixed countable collection of compact sets. For this purpose, let  $A \subset G$  be countable and dense and let  $\mathcal{U}$  be a countable neighborhood base of unity consisting of compact sets. Then, for a given compact  $C$  and a closed subgroup  $H$  with  $H \in U(G; C)$  there exist  $a_1, \dots, a_m \in A$  and  $U_1, \dots, U_m \in \mathcal{U}$  with  $C \subset \bigcup_{i=1}^m a_i U_i \subset G \setminus H$ . Hence  $H \in U(G; \bigcup_{i=1}^m a_i U_i) \subset U(G; C)$ , which shows the claim.  $\square$

**Proposition A.3.**

*Let  $G$  be a Lie group and  $H < G$  be closed. Then the intersection mapping*

$$\mathcal{I} : K(G) \rightarrow K(H), L \mapsto L \cap H$$

*is Borel.*

**Proof.** Since  $H$  is second countable, the sets of the type  $U(V; \emptyset)$  ( $V \subset H$  open) and  $U(H; C)$  ( $C \subset H$  compact) generate the Borel structure of  $K(H)$ , hence we need only take care of these. Clearly  $\mathcal{I}^{-1}(U(H; C)) = \{L \in K(G) : L \cap C = \emptyset\}$  is open. Now let  $V \subset H$  be open,  $V = \tilde{V} \cap H$  for some open set  $\tilde{V} \subset G$ . Since  $G$  is a Lie group,  $\tilde{V} = \bigcup_{i \in \mathbb{N}} F_i$  with closed subset  $F_i$ . Furthermore  $H = \bigcup_{j \in \mathbb{N}} C_j$  with compact sets  $C_j$ . Hence,

$$\begin{aligned} \mathcal{I}^{-1}(U(V; \emptyset)) &= \{L \in K(G) : L \cap H \cap \tilde{V} \neq \emptyset\} \\ &= \bigcup_{i, j \in \mathbb{N}} \{L \in K(G) : L \cap C_j \cap F_i \neq \emptyset\} \end{aligned}$$

is a countable union of closed sets, since the sets  $C_j \cap F_i$  are compact.  $\square$

Now we can consider the stabilizer mapping  $\gamma \mapsto H_\gamma$ :

**Proposition A.4.**

*Let  $H < \text{GL}(k, \mathbb{R})$  be closed. Then the stabilizer mapping  $\widehat{\mathbb{R}^k} \ni \gamma \mapsto H_\gamma \in K(H)$  is a Borel mapping. The set  $\Omega_c$  is a Borel subset of  $\widehat{\mathbb{R}^k}$ .*

**Proof.** By Proposition A.3 it is sufficient to consider  $H = \text{GL}(k, \mathbb{R})$ . For this case fix  $\gamma_1 \in \widehat{\mathbb{R}^k} \setminus \{0\}$  and let  $\rho : \widehat{\mathbb{R}^k} \setminus \{0\} \rightarrow H$  be any measurable cross section, i. e.,  $\rho$  fulfills  $\gamma_1 \rho(\omega) = \omega$  for all  $\omega \in \widehat{\mathbb{R}^k} \setminus \{0\}$ . Then we have  $H_\omega = \rho(\omega) H_{\gamma_1} \rho(\omega)^{-1}$ , hence the stabilizer mapping equals  $c \circ \rho$ , with  $c(x) := x H_{\gamma_1} x^{-1}$ . The mapping  $c : H \rightarrow K(H)$  is easily seen to be continuous, hence the stabilizer mapping is measurable.

For the last statement it suffices to show that  $K_c(H) := \{L \in K(H) : L \text{ is compact}\}$  is a Borel subset of  $K(H)$ . For this purpose let  $(C_n)_{n \in \mathbb{N}}$  be a countable family of compact subsets of  $H$  with the property that for each  $K \subset H$  compact there exists an  $n \in \mathbb{N}$  with  $K \subset C_n$ . Such a family was constructed in the proof of Proposition A.2. We then have

$$K_c(H) = \bigcup_{n \in \mathbb{N}} \{L \in K(H) : L \cap (H \setminus C_n) = \emptyset\},$$

whence we see that  $K_c(H)$  is the countable union of closed sets. □

The proof of the following proposition uses ideas from [26].

**Proposition A.5.**

$\Omega_{rc}$  is open.

**Proof.** Define the  $\epsilon$ -stabilizer

$$H_\omega^\epsilon = \{h \in H : |\omega h - \omega| \leq \epsilon\},$$

where  $|\cdot|$  denotes the Euclidean norm on  $\widehat{\mathbb{R}^k}$ . If  $H_\omega^\epsilon$  is compact for some  $\epsilon > 0$ , then  $B_\epsilon(\omega) \cap \omega H = \omega H_\omega^\epsilon$  is compact. (Here  $B_\epsilon(x)$  denotes the closed  $\epsilon$ -ball around  $x$ .) Hence the orbit  $\omega H$  is locally closed. Conversely, assume that  $B_\epsilon(\omega) \cap \omega H$  is compact for some  $\epsilon > 0$  and that  $H_\omega$  is compact. There exists a measurable cross-section  $\tau : \omega H \rightarrow H$  which maps compact sets in  $\omega H$  to relatively compact sets in  $H$ . Hence  $H_\omega^\epsilon \subset H_\omega \tau(B_\epsilon(\omega))$  is relatively compact and closed, hence compact. In short, we have shown

$$\omega \in \Omega_{rc} \iff \exists \epsilon > 0 : H_\omega^\epsilon \text{ is compact},$$

and we are going to use this characterization to prove the openness of  $\Omega_{rc}$ .

If the origin is in  $\Omega_{rc}$ , then  $H$  is compact, and  $\Omega_{rc} = \widehat{\mathbb{R}^k}$ . In the other case, pick  $\omega$  in  $\Omega_{rc}$  and  $\epsilon > 0$  with  $H_\omega^\epsilon$  compact. Since  $\text{GL}(k, \mathbb{R})$  acts transitively on  $\widehat{\mathbb{R}^k} \setminus \{0\}$ , we may (possibly after passing to a smaller  $\epsilon$ ) assume that there exists a continuous cross-section  $\sigma : B_\epsilon(\omega) \rightarrow \text{GL}(k, \mathbb{R})$  with relatively compact image, i. e.,  $\omega \sigma(\gamma) = \gamma$ , for all  $\gamma \in B_\epsilon(\omega)$ , and  $\sigma(B_\epsilon(\gamma)) \subset U$ , where  $U$  is a compact neighborhood of the identity in  $\text{GL}(k, \mathbb{R})$ . We are going to show that  $B_\epsilon(\omega) \subset \Omega_{rc}$ . For this purpose let  $\gamma \in B_\epsilon(\omega)$ . Clearly it is enough to prove that

$$C := \{h \in H : \gamma h \in B_\epsilon(\omega)\} = \{h \in \text{GL}(k, \mathbb{R}) : \gamma h \in B_\epsilon(\omega)\} \cap H$$

is relatively compact. By assumption,

$$H_\omega^\epsilon = \{h \in \text{GL}(k, \mathbb{R}) : \omega h \in B_\epsilon(\omega)\} \cap H$$

is compact. Hence

$$\begin{aligned} C &= \{h \in \text{GL}(k, \mathbb{R}) : \omega \sigma(\gamma) h \in B_\epsilon(\omega)\} \cap H \\ &= \sigma(\gamma)^{-1} \{h \in \text{GL}(k, \mathbb{R}) : \omega h \in B_\epsilon(\omega)\} \cap H \\ &\subset U^{-1} (\{h \in \text{GL}(k, \mathbb{R}) : \omega h \in B_\epsilon(\omega)\} \cap H) \end{aligned}$$

i. e.,  $C$  is contained in the product of two compact sets, and thus relatively compact. (Note that we used here that  $H$  is a closed subgroup of  $\text{GL}(k, \mathbb{R})$ , hence compactness in  $H$  is the same as compactness in  $\text{GL}(k, \mathbb{R})$ .) □

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