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Uniformly Elliptic PDEs with Bounded, Measurable Coefficients

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ABSTRACT. Let $L[\cdot]$ be a nondivergent linear second-order uniformly elliptic partial differential operator defined on functions with domain Ω. Consider the question, "When is a function u a solution of $L[u] = 0$ on Ω?" The naive answer, "u is a solution of $L[u] = 0$ on Ω if $u \in C^2(\Omega)$ and $L[u](x) = 0$ for all $x \in \Omega$," is clearly too limited. Indeed, if the coefficients of L are in $W^{1,2} \cap L^\infty$, then L can be rewritten in divergence form for which the notion of a "weak" solution can be applied. In this case there could be infinitely many functions that are "weak" but not classical solutions. More importantly, even if the coefficients of L are just bounded and measurable, the recent results of Krylov permit us to construct "solutions" of $L[u] = 0$ on Ω, and these "solutions" are generally no better than continuous; the "weak" solutions previously mentioned can be obtained by this construction, too.

The preceding discussion provides us with an adequate extrinsic definition of solution (i.e., given a function u we either prove that it is *or* is not *the result of such a construction) that has been used by several authors, but one that is not particularly satisfying or illuminating. Our major contribution in this paper is to show the following.*

I. *There is an intrinsic definition of solution that is equivalent to the extrinsic one.*

II. *Furthermore, the intrinsic definition is just the (now) well-known Crandall–Lions viscosity solution, modified in a natural way to accommodate measurable coefficients.*

1. Introduction

In this paper we examine the properties of and interrelationships between several definitions for a solution of a nondivergence structure, linear, second-order, uniformly elliptic pde *with bounded, measurable coefficients*. To be precise, we consider definitions for a solution of

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} = g(x) \quad \text{in } \Omega \tag{1.1}$$

under the assumption—implicit throughout this paper—that there exist positive constants K_1 and e_1 such that

$$\begin{cases} |a_{ij}(x)| \le K_1 & \text{for all } 1 \le i, j \le n \text{ and } x \in \Omega, \\ |g(x)| \le K_1 & \text{for all } x \in \Omega, \\ e_1 \le \displaystyle\sum_{i,j=1}^{n} a_{ij}(x) q_i q_j & \text{for all unit vectors } \mathbf{q} \in \mathbf{S}^{n-1} \text{ and } x \in \Omega, \end{cases} \tag{1.2}$$

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where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega \in \mathrm{C}^{2,\alpha}$, $a_{ij}\colon \Omega \to \mathbf{R}$ for $1 \le i, j \le n$ are measurable functions, $g\colon \Omega \to \mathbf{R}$ is a measurable function, and $\mathbf{S}^{n-1} \subset \mathbf{R}^n$ is the $(n-1)$-dimensional unit sphere.

Remark. Our results remain valid with the inclusion of bounded, measurable lower order terms in (1.1). We have excluded such lower order terms from our exposition to minimize nonessential, technical complications in our proofs. $\square$

Elliptic pdes of the form (1.1) have been studied extensively for almost a century, and the results pertaining to them are voluminous. However, almost all of these results require significantly stronger structure than (1.2)—typically, uniform continuity of the coefficients a_{ij} for $1 \le i, j \le n$. (This, of course, is peculiar to the nondivergence structure of (1.1). The dichotomy between divergence structure and nondivergence structure elliptic pdes is well known; in contrast with the nondivergence case, research in divergence structure, elliptic pdes with measurable coefficients has been a fertile field of study.) Still, important progress has been made within the last decade or so, expanding our understanding of (1.1) under (1.2). One of the most important results in this area is the Hölder continuity estimate of Krylov, strongly suggesting the existence of some *generalized* notion of solution of (1.1). In the pages to follow we shall study four such competing definitions of solution. It is the fundamental result of this paper (Theorem 1.14), that all four of these definitions are, in fact, equivalent.

Before turning to these generalized notions of solution, we review the inherent limitations of the more traditional $\mathrm{W}^{2,p}$ solution of (1.1). These limitations supply much of the motivation for studying the less conventional definitions which we next display. We close this section with Theorem 1.14, delineating the relationships among all five definitions. The remaining sections are devoted to the proof of Theorem 1.14.

Definition 1.3. For $1 \le p \le \infty$ a $\mathrm{W}^{2,p}$ solution of (1.1) is a function w in the Sobolev space $\mathrm{W}^{2,p}(\Omega)$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j}(x) = g(x) \quad \text{for almost every } x \in \Omega,$$

where $\frac{\partial^2 w}{\partial x_i \partial x_j}$ is the distribution derivative of w. $\square$

While there are advantages to $\mathrm{W}^{2,p}$ solutions, the current context also breeds several serious drawbacks. To demonstrate, consider the function

$$w(x_1, x_2) = 1 - \sqrt[4]{x_1{}^2 + x_2{}^2} \quad \text{for} \quad x_1{}^2 + x_2{}^2 \le 1.$$

We see that $w \in \mathrm{W}^{2,p}(\mathrm{B}(0,1))$ for any $1 \le p < 4/3$, and for every point x in the punctured disk $\mathrm{B}(0,1) \setminus \{(0,0)\}$ w satisfies

$$\left(1 + \frac{x_1{}^2}{x_1{}^2 + x_2{}^2}\right)\frac{\partial^2 w}{\partial x_1{}^2} + \frac{2x_1x_2}{x_1{}^2 + x_2{}^2}\frac{\partial^2 w}{\partial x_1 \partial x_2} + \left(1 + \frac{x_2{}^2}{x_1{}^2 + x_2{}^2}\right)\frac{\partial^2 w}{\partial x_2{}^2} = 0.$$

Clearly, w is a $\mathrm{W}^{2,p}$ solution of the preceding pde; just as obviously, w fails to satisfy the Hopf maximum principle. Furthermore, examples like this can be constructed on the unit ball in $\mathbf{R}^n$ for any $p < n$. To retain the Hopf maximum principle, we are forced to require $p \ge n$. Unfortunately, this leads to another problem. As first exhibited by Safonov, there are linear, elliptic pdes of the form

(1.1), satisfying (1.2), that do not have solutions in $W^{1,p}(\Omega)$ for any $p > n$. By the Sobolev Embedding Theorem such pdes do not have solutions in $W^{2,n-\delta}(\Omega)$ (for some $\delta > 0$). Consequently, if we insist on using $W^{2,p}$ solutions, then either we give up the Hopf maximum principle—unappealing, due to the deep connections with diffusion processes and stochastic differential equations—or we lose the existence of solutions—equally unpalatable, due to the result of Krylov to which we previously alluded. Finally, we note that the existence of $W^{2,p}$ solutions of (1.1) is not known even for $p = 1$.

Having "bonged" the $W^{2,p}$ solution, we examine our less conventional alternatives. We begin with a constructive definition for a solution of (1.1). This is a notion favored by several authors, including Escauriaza, Fabes, Krylov, and Safonov.

Definition 1.4. $u \in C(\overline{\Omega})$ is an α-solution of (1.1) if there is a sequence of $W^{2,p}$ solutions $\{u^k\}_{k=1}^{\infty}$ for $p > n$ of the linear pdes

$$\sum_{i,j=1}^{n} a_{ij}^k(x) \frac{\partial^2 u^k}{\partial x_i \partial x_j} = g^k(x) \quad \text{ in } \Omega, \tag{1.5}$$

satisfying (1.2) uniformly with respect to k, and such that

$$a_{ij}^k \in C(\overline{\Omega}) \text{ for all } 1 \le i, j \le n \text{ and all } k \in \mathbf{Z}^+, \tag{1.6}$$

$$a_{ij}^k \to a_{ij} \text{ in } L^1(\Omega) \text{ as } k \to \infty \text{ for all } 1 \le i, j \le n, \tag{1.7}$$

$$g^k \to g \text{ in } L^1(\Omega) \text{ as } k \to \infty, \tag{1.8}$$

and

$$u^k \to u \text{ in the sup-norm on } C(\overline{\Omega}) \text{ as } k \to \infty. \quad \square \tag{1.9}$$

The major advantage of this definition is the fact that an α-solution of (1.1) can be approximated by classical solutions of arbitrarily small perturbations of (1.1). Indeed, using the version of Pucci's L^∞ estimates found in [4], any $W^{2,p}$ solution of (1.1) with $p > n$ can be approximated by a classical solution as claimed. By Definition 1.4, the same must be true for α-solutions. As an immediate consequence:

α-solutions satisfy the Hopf maximum principle.

Additionally, it is not too difficult to prove that given continuous boundary data:

There exists an α-solution of the Dirichlet boundary value problem associated to (1.1).

On the other hand, the extrinsic nature of this definition makes it difficult to verify whether a given function $u \in C(\overline{\Omega})$ is an α-solution. It is also frustrating that the superposition principle remains unresolved for α-solutions—a fact that leaves the uniqueness of α-solutions as an open question too.

The remaining three definitions for a solution of (1.1) are motivated by [9] and the principles on which the viscosity solutions of M. G. Crandall and P.-L. Lions are based; see [2] for an introduction to the theory of viscosity solutions. The first of these definitions can be described as a strong version of such a solution, while the last two are apparently weak versions. We use $B(x, \varepsilon)$ to denote the open ball in $\mathbf{R}^n$ with center x and radius ε, $[t]^+$ to denote $\max\{0, t\}$, and $[t]^-$ to denote $\max\{0, -t\}$.

Definition 1.10. Let $u \in \mathrm{C}(\overline{\Omega})$; and for positive parameters e, K, and R set

$$\gamma_{e,K,R}(\eta) = \left(\frac{ne\Gamma_n}{2}\right) \cdot \left(\frac{ne\eta}{2K[n\eta + n^2R + 1]}\right)^{n-1} \cdot \eta,$$

where Γ_n is the volume of the unit ball in $\mathbf{R}^n$.

i. u is a β-subsolution of (1.1) if for e_1 and K_1 from (1.2)

$$\int_{\mathrm{B}(x,\varepsilon)} \left[\sum_{i,j=1}^{n} a_{ij}(y)\left(\frac{\partial^2\phi}{\partial x_i \partial x_j}(y) + \eta\delta_{ij}\right) - g(y)\right]^+ dy \geq \gamma_{e_1,K_1,R}(\eta)\varepsilon^n$$

for all $\eta > 0$, $\varepsilon <$ distance$(x, \partial\Omega)$, and $R > ||\phi||_{W^{2,\infty}(\Omega)}$

whenever $x \in \Omega$ and $\phi \in \mathrm{C}^2(\Omega)$ are such that

$$0 = (u - \phi)(x) \geq (u - \phi)(y) \quad \text{for all } y \in \Omega.$$

ii. u is a β-supersolution of (1.1) if for e_1 and K_1 from (1.2)

$$\int_{\mathrm{B}(x,\varepsilon)} \left[\sum_{i,j=1}^{n} a_{ij}(y)\left(\frac{\partial^2\phi}{\partial x_i \partial x_j}(y) - \eta\delta_{ij}\right) - g(y)\right]^- dy \geq \gamma_{e_1,K_1,R}(\eta)\varepsilon^n$$

for all $\eta > 0$, $\varepsilon <$ distance$(x, \partial\Omega)$, and $R > ||\phi||_{W^{2,\infty}(\Omega)}$

whenever $x \in \Omega$ and $\phi \in \mathrm{C}^2(\Omega)$ are such that

$$0 = (u - \phi)(x) \leq (u - \phi)(y) \quad \text{for all } y \in \Omega.$$

iii. u is a β-solution of (1.1) if it is both a β-subsolution and a β-supersolution of (1.1). □

Remark. It follows from a careful examination of the arguments appearing later that the function $\gamma_{e,K,R}(\cdot)$ must be modified to extend the scope of the preceding definition to include pdes with lower order terms. However, the qualitative properties of $\gamma_{e,K,R}(\cdot)$ are preserved; once modified, the remaining body of the definition remains unchanged. □

An important property of β-solutions is their stability. That is, under very general conditions, if (1.1) is the "limit" of (1.5) and u is the limit of a sequence of β-solutions $\{u^k\}_{k=1}^{\infty}$ corresponding to (1.5), then u is a β-solution of (1.1). Unfortunately, as in the α-solution case, the superposition principle for β-solutions remains unresolved.

Completing our list, we present the final two definitions for a solution of (1.1).

Definition 1.11. Let $u \in \mathrm{C}(\overline{\Omega})$.

i. u is a γ-subsolution of (1.1) if

$$\limsup_{\varepsilon \to 0^+} \frac{1}{\varepsilon^n} \int_{\mathrm{B}(x,\varepsilon)} \left[\sum_{i,j=1}^{n} a_{ij}(y)\left(\frac{\partial^2\phi}{\partial x_i \partial x_j}(x) + \eta\delta_{ij}\right) - g(y)\right]^+ dy > 0 \quad \text{for all } \eta > 0$$

whenever $x \in \Omega$ and $\phi \in \mathrm{C}^2(\Omega)$ are such that

$$0 = (u - \phi)(x) \geq (u - \phi)(y) \quad \text{for all } y \in \Omega.$$

ii. u is a γ-supersolution of (1.1) if

$$\limsup_{\varepsilon \to 0^+} \frac{1}{\varepsilon^n} \int_{\mathrm{B}(x,\varepsilon)} \left[\sum_{i,j=1}^{n} a_{ij}(y) \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) - \eta \delta_{ij} \right) - g(y) \right]^- dy > 0 \quad \text{for all } \eta > 0$$

whenever $x \in \Omega$ and $\phi \in \mathrm{C}^2(\Omega)$ are such that

$$0 = (u - \phi)(x) \leq (u - \phi)(y) \quad \text{for all } y \in \Omega.$$

iii. u is a γ-solution of (1.1) if it is both a γ-subsolution and a γ-supersolution of (1.1). □

Before proceeding to our last definition, we recall the notions of the second-order "superjet" and "subjet" found in [2]—equivalent with [6, Definition 1.4]. We use $\mathcal{S}(n)$ to denote the $n \times n$, real-valued, symmetric matrices.

Definition 1.12. Let $u \in \mathrm{C}(\Omega)$ be bounded.

i. The second-order "superjet" of u at $\hat{x} \in \Omega$ is the set

$$\begin{aligned} J_\Omega^{2,+} u(\hat{x}) = \{ (\mathbf{p}, \mathbf{M}) \in \mathbf{R}^n \times \mathcal{S}(n) : u(x) \leq u(\hat{x}) + \langle \mathbf{p}, x - \hat{x} \rangle \\ + \tfrac{1}{2} \langle \mathbf{M}(x - \hat{x}), x - \hat{x} \rangle + o(|x - \hat{x}|^2) \text{ as } x \to \hat{x} \}. \end{aligned}$$

ii. The second-order "subjet" of u at $\hat{x} \in \Omega$ is the set

$$\begin{aligned} J_\Omega^{2,-} u(\hat{x}) = \{ (\mathbf{p}, \mathbf{M}) \in \mathbf{R}^n \times \mathcal{S}(n) : u(x) \geq u(\hat{x}) + \langle \mathbf{p}, x - \hat{x} \rangle \\ + \tfrac{1}{2} \langle \mathbf{M}(x - \hat{x}), x - \hat{x} \rangle - o(|x - \hat{x}|^2) \text{ as } x \to \hat{x} \}. \end{aligned}$$ □

Definition 1.13. Let $u \in \mathrm{C}(\overline{\Omega})$.

i. u is a δ-subsolution of (1.1) if

$$\limsup_{\varepsilon \to 0^+} \frac{1}{\varepsilon^n} \int_{\mathrm{B}(x,\varepsilon)} \left[\sum_{i,j=1}^{n} a_{ij}(y) \left(m_{ij} + \eta \delta_{ij} \right) - g(y) \right]^+ dy > 0 \quad \text{for all } \eta > 0$$

whenever $x \in \Omega$ and there exists a vector $\mathbf{p} \in \mathbf{R}^n$ such that $(\mathbf{p}, \mathbf{M}) \in J_\Omega^{2,+} u(x)$ for $\mathbf{M} = (m_{ij})$.

ii. u is a δ-supersolution of (1.1) if

$$\limsup_{\varepsilon \to 0^+} \frac{1}{\varepsilon^n} \int_{\mathrm{B}(x,\varepsilon)} \left[\sum_{i,j=1}^{n} a_{ij}(y) \left(m_{ij} - \eta \delta_{ij} \right) - g(y) \right]^- dy > 0 \quad \text{for all } \eta > 0$$

whenever $x \in \Omega$ and there exists a vector $\mathbf{p} \in \mathbf{R}^n$ such that $(\mathbf{p}, \mathbf{M}) \in J_\Omega^{2,-} u(x)$ for $\mathbf{M} = (m_{ij})$.

iii. u is a δ-solution of (1.1) if it is both a δ-subsolution and delta-supersolution of (1.1). □

Which definition is the proper one? They all are! According to the following theorem, the four preceding definitions are all equivalent.

1.14. *Theorem*

The following implications hold.

- **i.** *If $p > n$, then Definition* 1.3 $\Longrightarrow$ *Definition* 1.10—$W^{2,p}$ *solutions are β-solutions for $p > n$.*
- **ii.** *Definition* 1.10 $\Longrightarrow$ *Definition* 1.11—*β-solutions are γ-solutions.*
- **iii.** *Definition* 1.11 $\Longrightarrow$ *Definition* 1.13—*γ-solutions are δ-solutions.*
- **iv.** *Definition* 1.13 $\Longrightarrow$ *Definition* 1.4—*δ-solutions are α-solutions.*
- **v.** *Definition* 1.4 $\Longrightarrow$ *Definition* 1.10—*α-solutions are β-solutions.*

In particular, the definitions for α-, β-, γ-, and δ-solutions are equivalent.

This theorem allows us to refer to the α-, β-, γ-, and δ-solutions collectively by a single name. Furthermore, if $a_{ij} \in C(\Omega)$ for $1 \leq i, j \leq n$ and $g \in C(\Omega)$, then it is not too hard to show that u is a δ-solution of (1.1) if and only if it is a viscosity solution of (1.1)—in the sense of Crandall and Lions [2]. Therefore, it makes sense to extend the scope of "viscosity solution" to include the α-, β-, γ-, and δ-solutions. However, "viscosity solution" is regarded by many (including the author) as a misleading and undesirable name. In deference to the fundamental contributions made by M. G. Crandall and P.-L. Lions, we henceforth refer to α-, β-, γ-, and δ-solutions *and* viscosity solutions as Crandall–Lions solutions (C-L solutions, for short).

2. α, β-, and γ-Solutions: Existence, Stability, and Other Properties

The stability of β-solutions was claimed in the introduction, and this result will be useful later in establishing the fifth implication of Theorem 1.14. We therefore find it expedient to begin the current section with a proof of this claim.

2.1. *Theorem*

(Stability) *Let* $\{u^k\}_{k=1}^{\infty}$ *be a sequence of β-solutions corresponding to* (1.5)*, satisfying* (1.2) *uniformly in k. If* (1.7)–(1.9) *hold, then u is a β-solution of* (1.1)*, where u is the function determined by* (1.9)*, and* (1.1) *is the pde with coefficients and inhomogeneous term determined by* (1.7) *and* (1.8).

Proof. We need to show that u is a β-subsolution and β-supersolution, but since the arguments are symmetric, it is sufficient to prove the former. To this end let $x_0 \in \Omega$ and $\phi_0 \in C^2(\Omega)$ satisfy

$$0 = (u - \phi_0)(x_0) \geq (u - \phi_0)(y) \quad \text{for all } y \in \Omega.$$

To ensure that x_0 is the only point with the preceding property, we replace ϕ_0 by $\phi_0^\nu(y) = \phi_0(y) + \nu|y - x_0|^2$ for $\nu > 0$. Next, we define $x_k \in \overline{\Omega}$ and $\phi_k^\nu \in C^2(\Omega)$ by the requirements

$$\left[\begin{array}{ll} \phi_k^\nu(y) = \phi_0^\nu(y) + d_k & \text{for some } d_k \in \mathbf{R}, \\ 0 = (u^k - \phi_k^\nu)(x_k) \geq (u^k - \phi_k^\nu)(y) & \text{for all } y \in \Omega. \end{array}\right.$$

Based on our construction we note that

$$\begin{cases} x_k \to x_0 & \text{as } k \to \infty, \\ \phi_k^\nu \to \phi_0^\nu & \text{in } W^{2,\infty}(\Omega) \text{ uniformly in } \nu \text{ as } k \to \infty, \\ \phi_0^\nu \to \phi_0 & \text{in } W^{2,\infty}(\Omega) \text{ as } \nu \to 0^+. \end{cases} \tag{2.2}$$

Proceeding to the next stage of our proof, fix values of ε and R such that

$$\varepsilon > \text{distance}(x_0, \partial\Omega) \quad \text{and} \quad R > ||\phi_0||_{W^{2,\infty}(\Omega)}.$$

By (2.2) there are constants $\hat{k} > 0$ and $\hat{\nu} > 0$ such that

$$\varepsilon > \text{distance}(x_k, \partial\Omega) \quad \text{and} \quad R > ||\phi_k^\nu||_{W^{2,\infty}(\Omega)}$$

for all $k > \hat{k}$ and $0 < \nu < \hat{\nu}$. It follows from Definition 1.10 that

$$\int_{\mathrm{B}(x_k,\varepsilon)} \left[\sum_{i,j=1}^n a_{ij}^k(y) \left(\frac{\partial^2 \phi_k^\nu}{\partial x_i \partial x_j}(y) + \eta\delta_{ij} \right) - g^k(y) \right]^+ dy \geq \gamma_{e_1,K_1,R}(\eta)\varepsilon^n$$

for all $\eta > 0$, $k > \hat{k}$, and $0 < \nu < \hat{\nu}$. Taking the limit first as $k \to \infty$ and then as $\nu \to 0^+$, we complete the proof of this theorem. $\square$

Our next task is to prove the first implication of Theorem 1.14, which will also be used in the proof of the fifth implication of Theorem 1.14. This will be accomplished with the assistance of a version of the L^∞ estimates of C. Pucci [9]. (These estimates can also be extracted from either [1] or [4].) The particular version we use is as follows.

2.3. *Lemma*

If u is a $\mathrm{W}^{2,p}$ solution of (1.1) for some $p > n$, then

i. $\sup\left\{u^+(x) : x \in \Omega\right\} \leq \sup\left\{u^+(x) : x \in \partial\Omega\right\} + \frac{\rho}{ne_1\sqrt[n]{\Gamma_n}}||g^-||_{L^n(\Omega)}$,

ii. $\sup\left\{u^-(x) : x \in \Omega\right\} \leq \sup\left\{u^-(x) : x \in \partial\Omega\right\} + \frac{\rho}{ne_1\sqrt[n]{\Gamma_n}}||g^+||_{L^n(\Omega)}$,

where, as before, $u^+(x) = \max\{u(x), 0\}$ is the positive part of u, $u^-(x) = \max\{-u(x), 0\}$ is the negative part of u, $g^+(x) = \max\{g(x), 0\}$ is the positive part of g, $g^-(x) = \max\{-g(x), 0\}$ is the negative part of g, and ρ is the radius of the smallest ball containing Ω.

With the aid of this lemma we easily prove the following result.

2.4. *Theorem*

(1.14i) *If $p > n$, then Definition* 1.3 $\Longrightarrow$ *Definition* 1.10.

Proof. Given a $\mathrm{W}^{2,p}$ solution u of (1.1) with $p > n$, assume $x \in \Omega$ and $\phi \in \mathrm{C}^2(\Omega)$ are such that

$$0 = (u - \phi)(x) \geq (u - \phi)(y) \quad \text{for all } y \in \Omega.$$

Keeping careful track of all constants, apply Lemma 2.3 with $\hat{u}$ in place of u, $\hat{g}$ in place of g, and $\mathrm{B}(x,\varepsilon)$ in place of Ω where

$$\varepsilon < \text{distance}(x, \partial\Omega),$$
$$\hat{u}(y) = u(y) - \phi(y) - \frac{\eta}{2}|y - x|^2 + \frac{\eta\varepsilon^2}{2},$$

and

$$\hat{g}(y) = g(y) - \sum_{i,j=1}^n a_{ij}(y) \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}(y) + \eta\delta_{ij} \right).$$

Noting that $\sup\left\{\hat{u}^+(y) : y \in \mathrm{B}(x,\varepsilon)\right\} \geq \frac{\eta\varepsilon^2}{2}$ and $\sup\left\{\hat{u}^+(y) : y \in \partial\mathrm{B}(x,\varepsilon)\right\} = 0$, this leads to

$$\int_{\mathrm{B}(x,\varepsilon)} \left(\hat{g}^-(y)\right)^n \, dy \geq \Gamma_n \left(\frac{ne_1\eta\varepsilon}{2}\right)^n .$$

Using the definition of $\hat{g}$ and the estimate

$$||\hat{g}(y)||_{L^\infty(\Omega)} \leq K_1(n^2R + n\eta + 1) \quad \text{for} \quad R > ||\phi||_{W^{2,\infty}(\Omega)},$$

this establishes u as a β-subsolution of (1.1). A similar argument proves that u is a β-supersolution. □

Having acquired the necessary tools, it is now easy to prove the fifth implication of Theorem 1.14.

2.5. *Theorem*
(1.14v) *Definition* 1.4 $\Longrightarrow$ *Definition* 1.10.

Proof. Assume u is an α-solution of (1.1); then u is the limit of a sequence $\left\{u^k\right\}_{k=1}^\infty$ of $\mathrm{W}^{2,p}$ solutions corresponding to (1.5), satisfying (1.8), (1.7), (1.6), and (1.2) uniformly in k. By Theorem 2.4 $\left\{u^k\right\}_{k=1}^\infty$ is a sequence of β-solutions. An application of Theorem 2.1 now completes our proof. □

In the introduction we claimed the existence of an α-solution to the Dirichlet boundary value problem for continuous boundary data. We shall now prove this claim. To do so, we appeal to the Harnack inequality of Krylov and Safonov; a proof of this may be found at the beginning of [1].

2.6. *Theorem*
(Existence) *Assume $h \in \mathrm{C}(\partial\Omega)$; then there exists an α-solution u of* (1.1) *such that*

$$u(y) = h(y) \quad \textit{for all } y \in \partial\Omega. \tag{2.7}$$

Proof. Given (1.1) and $h \in \mathrm{C}(\partial\Omega)$, we construct sequences $\left\{(a_{ij}^k)\right\}_{k=1}^\infty \subset \mathrm{C}^\infty(\overline{\Omega}; \mathcal{S}(n))$, $\left\{g^k\right\}_{k=1}^\infty \subset \mathrm{C}^\infty(\overline{\Omega})$, and $\left\{h^k\right\}_{k=1}^\infty \subset \mathrm{C}^{2,\alpha}(\partial\Omega)$ such that

$\left\{(a_{ij}^k)\right\}_{k=1}^\infty$ and $\left\{g^k\right\}_{k=1}^\infty$ satisfy (1.2) uniformly in k,
$(a_{ij}^k) \to (a_{ij})$ in $\mathrm{L}^1(\Omega; \mathcal{S}(n))$ as $k \to \infty$,
$g^k \to g$ in $\mathrm{L}^1(\Omega)$ as $k \to \infty$,
$\left\{h^k\right\}_{k=1}^\infty$ has a uniform modulus of continuity for all k,
$h^k \to h$ in the sup-norm on $\mathrm{C}(\partial\Omega)$ as $k \to \infty$.

It follows from standard theory (see the development found in [4]) that there exists a sequence of solutions $\left\{u^k\right\}_{k=1}^\infty \subset \mathrm{C}^{2,\alpha}(\overline{\Omega})$ for the system of boundary value problems

$$\begin{cases} \displaystyle\sum_{i,j=1}^n a_{ij}^k(x) \frac{\partial^2 u^k}{\partial x_i \partial x_j} = g^k(x) & \text{in } \Omega \\ u^k(x) = h^k(x) & \text{for } x \in \partial\Omega \end{cases}$$

generated by the sequences above. Lemma 2.3 shows us that $\{u^k\}_{k=1}^{\infty}$ is uniformly bounded, and the Harnack inequality of Krylov and Safonov in conjunction with standard barrier arguments at the boundary implies it is also equicontinuous. As a consequence, $\{u^k\}_{k=1}^{\infty}$ is precompact and we may assume without loss of generality that the sequence converges to some function $u \in \mathrm{C}(\overline{\Omega})$ in the sup-norm on $\mathrm{C}(\overline{\Omega})$. The function u is obviously an α-solution of (1.1), satisfying (2.7). □

Remark. It is very important to note that we are *not* able to prove $\{u^k\}_{k=1}^{\infty}$ has a unique limit. Indeed, this would be equivalent to proving the uniqueness of α-solutions, a significant conjecture that is still undecided. The ramifications of this fact become clear in the process of proving that δ-solutions are α-solutions. Indeed, it becomes apparent that the δ-solution u itself must be be involved in constructing $\{u^k\}_{k=1}^{\infty}$, the defining sequence of an α-solution, for there is no guarantee that (1.6), (1.7), and (1.8) will force the solutions of (1.5) to converge to u. We deal with this difficulty in the last section. □

We close this section with several easy results. The first two of these are the second and third implications of Theorem 1.14.

2.8. *Theorem*

(1.14ii) *Definition* 1.10 $\Longrightarrow$ *Definition* 1.11.

Proof. By symmetry it is sufficient to prove that β-subsolutions are γ-subsolutions. To this end let u be a β-subsolution of (1.1) and let $x \in \Omega$ and $\phi \in \mathrm{C}^2(\Omega)$ be such that

$$0 = (u - \phi)(x) \geq (u - \phi)(y) \quad \text{for all } y \in \Omega.$$

This implies that for $\eta > 0$, $\varepsilon <$ distance$(x, \partial\Omega)$, and $R > ||\phi||_{W^{2,\infty}(\Omega)}$

$$\frac{1}{\varepsilon^n}\int_{\mathrm{B}(x,\varepsilon)}\left[\sum_{i,j=1}^{n} a_{ij}(y)\left(\frac{\partial^2\phi}{\partial x_i \partial x_j}(x) + \eta\delta_{ij}\right) - g(y)\right]^+ dy \geq \gamma_{e_1,K_1,R}(\eta) - o_\varepsilon(1)$$

where e_1 and K_1 are from (1.2). It is now obvious that u is a γ-subsolution of (1.1). □

2.9. *Theorem*

(1.14iii) *Definition* 1.11 $\Longrightarrow$ *Definition* 1.13.

Proof. This follows immediately from the fact, as noted in [2], that for $\hat{x} \in \Omega$, $(\mathbf{p}, \mathbf{M}) \in J_{\Omega}^{2,+}u(\hat{x})$ if and only if there exists $\phi \in \mathrm{C}^2(\Omega)$ such that

$$\mathbf{D}\phi(\hat{x}) = \mathbf{p}, \qquad \mathbf{D}^2\phi(\hat{x}) = \mathbf{M}$$
$$0 = (u - \phi)(\hat{x}) \geq (u - \phi)(y) \quad \text{for all } y \in \Omega.$$

Our last result indicates the relationship between γ-solutions and the generally accepted definition of C–L solutions, AKA viscosity solutions. □

2.10. *Theorem*

Assume $u \in \mathrm{C}(\overline{\Omega})$.

i. *If u is a γ-subsolution of* (1.1), *then*

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2\phi}{\partial x_i \partial x_j}(x) \geq g(x)$$

whenever $x \in \Omega$ is a Lebesgue point of (a_{ij}) and g and $\phi \in \mathrm{C}^2(\Omega)$ are such that

$$0 = (u - \phi)(x) \geq (u - \phi)(y) \quad \text{for all } y \in \Omega.$$

ii. *If u is a γ-supersolution of* (1.1)*, then*

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) \leq g(x)$$

whenever $x \in \Omega$ is a Lebesgue point of (a_{ij}) and g and $\phi \in \mathrm{C}^2(\Omega)$ are such that

$$0 = (u - \phi)(x) \leq (u - \phi)(y) \quad \textit{for all } y \in \Omega.$$

Proof. Due to the symmetry in proving conditions i and ii, it will be sufficient to prove i. To this end let u be a γ-subsolution of (1.1), $x \in \Omega$ a Lebesgue point of (a_{ij}) and g, and $\phi \in \mathrm{C}^2(\Omega)$. The definition for a γ-subsolution implies that given $\eta > 0$ there exist a constant $d > 0$, a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$, and sets $E_k \subset \mathrm{B}(x, \varepsilon_k)$ such that

$$\begin{aligned}
&\varepsilon_k \to 0^+ \text{ as } k \to \infty, \\
&\sum_{i,j=1}^{n} a_{ij}(y) \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) + \eta \delta_{ij} \right) - g(y) > 0 \text{ for all } y \in E_k, \\
&|E_k| \geq d |\mathrm{B}(x, \varepsilon_k)| \text{ for all } k.
\end{aligned}$$

In conjunction with the assumption that x is a Lebesgue point of both (a_{ij}) and g, the preceding observation demonstrates

$$\sum_{i,j=1}^{n} a_{ij}(x) \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) + \eta \delta_{ij} \right) - g(x) > 0.$$

Since η is arbitrary, this proves our theorem. □

Remark. It should be noted that the preceding properties of γ-solutions are too weak to be used as the basis for another definition of a solution of (1.1). Indeed, these properties hold for the example we constructed in the introduction when we examined the limitations of $\mathrm{W}^{2,p}$ solutions. □

3. δ-Solutions: Constructing Approximations

To complete our proof of Theorem 1.14 we borrow some tools from the theory of C–L solutions and convex analysis. The first of these are known as inf- and sup-convolutions.

Definition 3.1. Given a bounded, continuous function $w \in \mathrm{C}(\Omega)$, define the operators $A_\varepsilon^+[w]$ and $A_\varepsilon^-[w]$ by the formulas

$$\begin{aligned}
A_\varepsilon^+[w](x) &= \sup \left\{ w(\xi) - \frac{1}{2\varepsilon}|x - \xi|^2 : \xi \in \Omega \right\}, \\
A_\varepsilon^-[w](x) &= \inf \left\{ w(\xi) + \frac{1}{2\varepsilon}|x - \xi|^2 : \xi \in \Omega \right\}. \quad \square
\end{aligned}$$

With the definition conveniently nearby for reference, we state the fundamental properties of $A_\varepsilon^+[\,\cdot\,]$ and $A_\varepsilon^-[\,\cdot\,]$.

3.2. *Lemma*

If $u \in C(\overline{\Omega})$ has modulus of continuity $\omega(\cdot)$ and $\varepsilon, \delta > 0$, then for $v^+ = A^+_\varepsilon[u]$ and $v^- = A^-_\varepsilon[u]$

i.

$$A^+_{\varepsilon+\delta}[u] = A^+_\delta[A^+_\varepsilon[u]], \textit{ i.e., } A^+_\varepsilon[\,\cdot\,] \textit{ is a semigroup in } \varepsilon;$$

$$u \le v^+ \in W^{1,\infty}(\Omega);$$

$$|\mathbf{D}v^+(y)| \le \frac{\sqrt{2}\omega(\sqrt{8}\varepsilon^{1/2}||v||_{L^\infty(\Omega)})^{1/2}}{\varepsilon^{1/2}} \textit{ for all } y \in \Omega;$$

$$\frac{-1}{\varepsilon}\mathbf{I} \le \mathbf{D}^2 v^+ \textit{ in the sense of distributions on } \Omega;$$

$$(v^+ - u)(y) \le \omega\big(\sqrt{2}\varepsilon^{1/2}\omega(\sqrt{8}\varepsilon^{1/2}||v||_{L^\infty(\Omega)})^{1/2}\big) \textit{ for all } y \in \Omega.$$

ii.

$$A^-_{\varepsilon+\delta}[u] = A^-_\delta[A^-_\varepsilon[u]], \textit{ i.e., } A^-_\varepsilon[\,\cdot\,] \textit{ is a semigroup in } \varepsilon;$$

$$u \ge v^- \in W^{1,\infty}(\Omega);$$

$$|\mathbf{D}v^-(y)| \le \frac{\sqrt{2}\omega(\sqrt{8}\varepsilon^{1/2}||v||_{L^\infty(\Omega)})^{1/2}}{\varepsilon^{1/2}} \textit{ for all } y \in \Omega;$$

$$\frac{1}{\varepsilon}\mathbf{I} \ge \mathbf{D}^2 v^- \textit{ in the sense of distributions on } \Omega;$$

$$(u - v^-)(y) \le \omega\big(\sqrt{2}\varepsilon^{1/2}\omega(\sqrt{8}\varepsilon^{1/2}||v||_{L^\infty(\Omega)})^{1/2}\big) \textit{ for all } y \in \Omega.$$

Proof. These are all standard results that may be verified directly or extracted from sources such as [6] and [8]. □

We shall also use another pair of regularizing operators, generalizations of the convex hull and concave hull. First, we introduce special sets of polynomial functions.

Definition 3.3. Given $\delta > 0$ define Q^+_δ and Q^-_δ by

$$Q^+_\delta = \left\{ p \in C(\mathbf{R}^n) : p \text{ is a quadratic polynomial, and } \mathbf{D}^2 p = \frac{1}{\delta}\mathbf{I} \right\},$$

$$Q^-_\delta = \left\{ p \in C(\mathbf{R}^n) : p \text{ is a quadratic polynomial, and } \mathbf{D}^2 p = \frac{-1}{\delta}\mathbf{I} \right\}. \quad \square$$

We now define what we call the semiconvex and semiconcave hull operators.

Definition 3.4. Given $w \in C(\overline{\Omega})$, define the operators $SC^+_\delta[w]$ and $SC^-_\delta[w]$ by

$$SC^+_\delta[w](x) = \inf\left\{p(x) : p \in Q^+_\delta, \text{ and } p \ge w\right\},$$
$$SC^-_\delta[w](x) = \sup\left\{p(x) : p \in Q^-_\delta, \text{ and } p \le w\right\}. \quad \square$$

Again it is convenient to follow the definitions of $SC^+_\delta[\,\cdot\,]$ and $SC^-_\delta[\,\cdot\,]$ with their fundamental properties.

3.5. *Lemma*

If $v \in \mathrm{C}(\overline{\Omega})$ and $\delta > 0$, then for $w^+ = SC^+_\delta[v]$ and $w^- = SC^-_\delta[v]$

i.

$$v \le w^+ \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega);$$

$$w^+ \le A^+_\delta[v];$$

$$\frac{1}{\delta}\mathbf{I} \ge \mathbf{D}^2 w^+ \textit{ in the sense of distributions on } \Omega;$$

$$\mathbf{D}^2 v \ge -c_0\mathbf{I} \textit{ and } c_0 > 0 \implies \mathbf{D}^2 w^+ \ \ge\ -c_0\mathbf{I} \textit{ in the sense of distributions on } \Omega.$$

ii.

$$v \ge w^- \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega);$$

$$w^- \ge A^-_\delta[v];$$

$$\frac{-1}{\delta}\mathbf{I} \le \mathbf{D}^2 w^- \textit{ in the sense of distributions on } \Omega;$$

$$\mathbf{D}^2 v \le c_0\mathbf{I} \textit{ and } c_0 > 0 \implies \mathbf{D}^2 w^- \ \le\ c_0\mathbf{I} \textit{ in the sense of distributions on } \Omega.$$

Proof. The statements are clearly symmetrical, so it is sufficient to prove i. Setting $w^+ = SC^+_\delta[v]$, it is clear that $v \le w^+$. Nor is it too difficult to see that $w^+ \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega)$ and $\frac{1}{\delta}\mathbf{I} \ge \mathbf{D}^2 w^+$ in the sense of distributions. To prove the remaining two properties we claim

$$w^+ = SC^+_\infty[v - p_0] + p_0, \tag{3.6}$$

where $p_0(y) = \frac{1}{2\delta}|y|^2$ and (by a mild abuse of notation) $SC^+_\infty[\,\cdot\,]$ denotes the usual concave hull operator. Indeed, if $p \in \mathrm{Q}^+_\delta$, then $l = p - p_0$ is a linear function; $p \ge v$ if and only if $l \ge (v - p_0)$. Our claim follows immediately from this.

Continuing from (3.6), for any $x_0 \in \Omega$ there are positive numbers $\{\alpha_1, \ldots, \alpha_k\}$ and points $\{x_1, \ldots, x_k\} \subset \overline{\Omega}$ for some $k \le (n+1)$ and a linear function $l_0 \ge (v - p_0)$ such that

$$\begin{cases} \displaystyle\sum_{j=1}^{k} \alpha_j = 1, \\ \displaystyle\sum_{j=1}^{k} \alpha_j x_j = x_0, \\ (w^+ - p_0)(x_0) = l_0(x_0), \\ l_0(x_j) = (v - p_0)(x_j) \quad \text{for all } 1 \le j \le k. \end{cases} \tag{3.7}$$

We claim $w^+ \le v^+ = A^+_\delta[v]$. For the sake of contradiction, suppose $x_0 \in \Omega$ is a point for which

$$w^+(x_0) > v^+(x_0) = A^+_\delta[v](x_0). \tag{3.8}$$

As demonstrated by (3.6), there are positive numbers $\{\alpha_1, \ldots, \alpha_k\}$ and points $\{x_1, \ldots, x_k\} \subset \overline{\Omega}$ for some $k \le (n+1)$ and a linear function l_0 satisfying (3.7). In particular, for $\hat{p} = l_0 + p_0$

$$v \le \hat{p}, \quad w^+(x_0) = \hat{p}(x_0) \quad \text{, and} \quad \hat{p}(x_j) = v(x_j) \text{ for all } 1 \le j \le k.$$

Based on the definitions of v^+ and $\hat{p}$, for all $1 \le j \le k$

$$\begin{aligned} w^+(x_0) > v^+(x_0) \ge v(x_j) - \frac{1}{2\delta}|x_0 - x_j|^2 &= \hat{p}(x_j) - \frac{1}{2\delta}|x_0 - x_j|^2 \\ &= l_0(x_j) + \frac{1}{\delta}\langle x_j, x_0\rangle - \frac{1}{2\delta}|x_0|^2. \end{aligned}$$

Multiplying by the corresponding α_j and summing yield

$$\begin{aligned} w^+(x_0) &> \sum_{j=1}^{k} \alpha_j l_0(x_j) + \sum_{j=1}^{k} \frac{\alpha_j}{\delta}\langle x_j, x_0\rangle - \frac{1}{2\delta}|x_0|^2 \\ &= l_0(x_0) + \frac{1}{\delta}|x_0|^2 - \frac{1}{2\delta}|x_0|^2 = \hat{p}(x_0) = w^+(x_0). \end{aligned}$$

This contradiction denies (3.8) and establishes our claim.

The last statement to prove is

$\mathbf{D}^2 v \ge -c_0\mathbf{I}$ and $c_0 > 0 \implies \mathbf{D}^2 w^+ \ge -c_0\mathbf{I}$
in the sense of distributions.

Keeping (3.6) in mind, we see that it is sufficient to prove

$\mathbf{D}^2 v \ge -c_0\mathbf{I}$ and $c_0 > 0 \implies \mathbf{D}^2(SC^+_\infty[v - p_0] + p_0) \ge -c_0\mathbf{I}$
in the sense of distributions.

This in turn is equivalent to proving that if $\mathbf{D}^2 v \ge -c_0\mathbf{I}$ in the sense of distributions with $c_0 > 0$, then for each $x_0 \in \Omega$ there is a quadratic function $\hat{p} \in \mathrm{Q}^-_{\delta/(1+c_0\delta)}$ such that

$$\hat{p} \le SC^+_\infty[v - p_0] \quad \text{and} \quad \hat{p}(x_0) = SC^+_\infty[v - p_0](x_0). \tag{3.9}$$

Given x_0 we note that this is trivial if $v(x_0) = w^+(x_0)$. On the other hand, if $v(x_0) < w^+(x_0)$, then (3.6) implies the existence of positive numbers $\{\alpha_1, \dots, \alpha_k\}$ and points $\{x_1, \dots, x_k\} \subset \overline{\Omega}$ for some $k \le (n+1)$ and of a linear function $l_0 \ge (v - p_0)$ satisfying (3.7). From the assumption $\mathbf{D}^2 v \ge -c_0\mathbf{I}$ in the sense of distributions and $c_0 > 0$, there is a set of functions $\{p_1, \dots, p_k\} \subset \mathrm{Q}^-_{\delta/(1+c_1\delta)}$ such that

$$p_j(x_j) = (v - p_0)(x_j) \quad \text{and} \quad p_j \le (v - p_0) \qquad \text{in } \Omega \text{ for } 1 \le j \le k.$$

Due to the concavity of $SC^+_\infty[v - p_0]$, we may construct $\hat{p}$ satisfying (3.9) by the formula

$$\hat{p}(x) = \sum_{j=1}^{k} \alpha_j p_j(x - x_0 + x_j),$$

completing the proof of the lemma. □

The previously defined operators are regularizing approximations to the identity. They will be used to construct "smooth" approximates of δ-solutions of (1.1). However, due to the fact that these operators do not commute with either differential operators or products of functions, they invariably perturb the approximates from true solutions of (1.1). Given a function $v \in \mathrm{W}^{2,\infty}(\Omega)$ we define mappings $T^+_\rho[v]$ and $T^-_\rho[v]$ from Ω to $\mathbf{R}^n$. These maps will be used to correct the errors introduced by the previous approximations.

Definition 3.10. Given a continuous function $v \in \mathrm{W}^{2,\infty}(\Omega)$, define the mappings $T_\rho^+[v]$ and $T_\rho^-[v]$ by

$$T_\rho^+[v](x) = x + \rho \mathbf{D}v(x), \qquad T_\rho^-[v](x) = x - \rho \mathbf{D}v(x). \qquad \square$$

An elementary corollary exposes a basic property of the preceding maps.

3.11. *Corollary*

Let $u \in \mathrm{C}(\overline{\Omega})$, and set $T_\varepsilon^+ = T_\varepsilon^+[A_\varepsilon^+[u]]$ and $T_\varepsilon^- = T_\varepsilon^-[A_\varepsilon^-[u]]$. There exists a constant $C = C(u)$ such that for any subdomain $\Omega^* \subset \Omega$ with distance$(\Omega^*, \partial\Omega) \geq C\varepsilon^{1/2}$, $T_\varepsilon^\pm\left(\overline{\Omega^*}\right) \subset \Omega$.

Proof. This is obvious from the estimates in Lemma 3.2 on $|\mathbf{D}(A_\varepsilon^+[u])|$ and $|\mathbf{D}(A_\varepsilon^-[u])|$. $\square$

Finally, for the last of our definitions we present the Pucci extremal operators. Given an $n \times n$, real-valued, symmetric matrix $\mathbf{M} \in \mathcal{S}(n)$ we let $\Lambda^+[\mathbf{M}]$ and $\Lambda^-[\mathbf{M}]$ be the sets of positive and negative eigenvalues of $\mathbf{M}$, respectively (including repetitions).

Definition 3.12. Given $\Theta > \theta > 0$ we define functions $P_{\Theta,\theta}^+$ and $P_{\Theta,\theta}^-$ from the $n \times n$, real-valued, symmetric matrices to the reals by

$$\begin{aligned} P^+(M) = P_{\Theta,\theta}^+(M) &= \Theta \cdot \sum_{\lambda \in \Lambda^+[\mathbf{M}]} \lambda + \theta \cdot \sum_{\lambda \in \Lambda^-[\mathbf{M}]} \lambda, \\ P^-(M) = P_{\Theta,\theta}^-(M) &= \Theta \cdot \sum_{\lambda \in \Lambda^-[\mathbf{M}]} \lambda + \theta \cdot \sum_{\lambda \in \Lambda^+[\mathbf{M}]} \lambda. \qquad \square \end{aligned}$$

Using P^+ and P^-, the next lemma establishes two differential inequalities satisfied by solutions of (1.1).

3.13. *Lemma*

If u is a δ-solution of (1.1)*, then there exist constants $C_0 > 0$ and $\Theta_0 > \theta_0 > 0$ such that with $P^+ = P_{\Theta_0,\theta_0}^+$ and $P^- = P_{\Theta_0,\theta_0}^-$*

i. *$P^+(\mathbf{M}) + C_0 \geq 0$ for all $(\mathbf{p}, \mathbf{M}) \in J_\Omega^{2,+}u(x)$ for all $x \in \Omega$,*
ii. *$P^-(\mathbf{M}) - C_0 \leq 0$ for all $(\mathbf{p}, \mathbf{M}) \in J_\Omega^{2,-}u(x)$ for all $x \in \Omega$.*

Proof. Define the constants appearing in the lemma as

$$\begin{cases} C_0 = ||g||_{\mathrm{L}^\infty(\Omega)}, \\ \Theta_0 = \sup\left\{ ||\sum_{i,j=1}^n a_{ij} q_i q_j ||_{\mathrm{L}^\infty(\Omega)} : \mathbf{q} \in \mathbf{S}^{n-1} \right\}, \\ \theta_0 = \inf\left\{ ||\sum_{i,j=1}^n a_{ij} q_i q_j ||_{\mathrm{L}^\infty(\Omega)} : \mathbf{q} \in \mathbf{S}^{n-1} \right\}, \end{cases}$$

which are positive and finite by (1.2). Given $(\mathbf{p}, \mathbf{M}) \in J_\Omega^{2,+}u(x)$ for some $x \in \Omega$, Definition 1.13 implies the existence of a set $E \subset \Omega$ of positive measure such that

$$\sum_{i,j=1}^n a_{ij}(y) \left(m_{ij} + \eta \delta_{ij} \right) - g(y) > 0 \quad \text{for all } y \in E.$$

Based on our choices of C_0, Θ_0, and θ_0 it follows that the first statement of the lemma is true. In as much as the second statement is a mirror image of the first (and can be proved by an argument symmetric with the preceding one), this completes the proof of our lemma. $\square$

As an immediate corollary, we obtain the following result.

3.14. ***Corollary***

If u is a δ-solution of (1.1)*, then there exist constants $C_0 > 0$ and $\Theta_0 > \theta_0 > 0$ such that with $P^+ = P^+_{\Theta_0,\theta_0}$ and $P^- = P^-_{\Theta_0,\theta_0}$*

i. *u is a C–L subsolution of*

$$P^+(\mathbf{D}^2 u) + C_0 = 0 \quad \text{in } \Omega; \tag{3.15}$$

ii. *u is a C–L supersolution of*

$$P^-(\mathbf{D}^2 u) - C_0 = 0 \quad \text{in } \Omega. \tag{3.16}$$

Proof. Apply Lemma 3.13. $\square$

At this point we begin the technical work necessary to complete the proof of Theorem 1.14.

3.17. ***Lemma***

Let u be a δ-solution of (1.1)*, and let $\Omega^* \subset\subset \Omega$ denote a compactly contained subdomain. Set $v^+ = A^+_\varepsilon[u]$ and $v^- = A^-_\varepsilon[u]$ for $0 < \varepsilon < \varepsilon_0$ where ε_0 is determined from Corollary 3.11 by $\varepsilon_0 = (\text{distance}(\Omega^*, \partial\Omega)/C)^2$. Then $\mathbf{D}^2 v^\pm(x)$ exists for almost every point $x \in \Omega^*$ (in the sense of a quadratic approximation* [5, Lemma 3.15]*), and there exists a constant $\beta \in (0, 1)$ independent of ε such that*

i. *at every point x where $\mathbf{D}^2 v^+(x)$ exists*

$$\frac{-\beta}{\varepsilon}\mathbf{I} \le \mathbf{D}^2 v^+(x) \quad \text{and} \quad \det\left|\mathbf{I} + \varepsilon \mathbf{D}^2 v^+(x)\right| \ge (1-\beta)^n,$$

ii. *at every point x where $\mathbf{D}^2 v^-(x)$ exists*

$$\mathbf{D}^2 v^-(x) \le \frac{\beta}{\varepsilon}\mathbf{I} \quad \text{and} \quad \det\left|\mathbf{I} - \varepsilon \mathbf{D}^2 v^-(x)\right| \ge (1-\beta)^n.$$

Proof. As noted previously, by symmetry the proof of i is sufficient. The almost everywhere existence of $\mathbf{D}^2 v^+(x)$ as a quadratic approximation to v^+ follows from [5, Lemmas 3.3 and 3.15]. To prove the existence of β we use Corollary 3.14 and [6, Lemma 2.14]. Specifically, let $x \in \Omega^*$ be a point at which $\mathbf{M}_0 = \mathbf{D}^2 v^+(x)$ exists. Using [6, Lemma 2.14], we conclude that for any $\delta > 0$

$$\left(\mathbf{p}_0, \mathbf{M}_\delta - \varepsilon \mathbf{M}_\delta(\mathbf{I} + \varepsilon \mathbf{M}_\delta)^{-1}\mathbf{M}_\delta\right) \in J^{2,+}_\Omega u\left(T^+_\varepsilon(x)\right),$$

where $\mathbf{p}_0 = \mathbf{D}v^+(x)$, $\mathbf{M}_\delta = \mathbf{M}_0 + \delta\mathbf{I}$, and T^+_ε is the map defined in Corollary 3.11. According to Corollary 3.14, u is a C–L subsolution of (3.15); so the definition of $J^{2,+}_\Omega u\left(T^+_\varepsilon(x)\right)$ implies

$$P^+(\mathbf{M}_\delta - \varepsilon \mathbf{M}_\delta(\mathbf{I} + \varepsilon \mathbf{M}_\delta)^{-1}\mathbf{M}_\delta) + C \ge 0.$$

Calculating the left-hand side of the previous inequality by the definition of P^+ we obtain

$$\Theta_0 \cdot \sum_{\lambda \in \Lambda^+[\mathbf{M}_\delta]} \frac{\lambda}{1+\varepsilon\lambda} + \theta_0 \cdot \sum_{\lambda \in \Lambda^-[\mathbf{M}_\delta]} \frac{\lambda}{1+\varepsilon\lambda} \geq -C.$$

After some modest manipulation, the previous inequality gives rise to

$$\theta_0 \cdot \sum_{\lambda \in \Lambda^-[\mathbf{M}_\delta]} \frac{\varepsilon\lambda}{1+\varepsilon\lambda} \geq -C \cdot \varepsilon - n\Theta_0,$$

which reveals the existence of $\beta \in (0, 1)$ independent of δ such that

$$\lambda \geq \frac{-\beta}{\varepsilon} \quad \text{for all } \lambda \in \Lambda^-[\mathbf{M}_\delta].$$

Passing to the limit as $\delta \to 0^+$ we conclude that the preceding inequality remains valid for $\delta = 0$, which proves i. $\square$

A couple of important corollaries are immediately evident from the preceding lemma.

3.18. *Corollary*

Let u be a δ-solution of (1.1), *and let $\Omega^* \subset\subset \Omega$ (as in the Lemma 3.17). Set $w^+ = SC_\delta^+[A_\varepsilon^+[u]]$ and $w^- = SC_\delta^-[A_\varepsilon^-[u]]$ for $0 < \varepsilon < \varepsilon_0$ and $0 < \delta < \varepsilon$ where ε_0 is the same constant as determined in Lemma 3.17. Then $\mathbf{D}^2 w^\pm(x)$ exist for almost every $x \in \Omega^*$ (simultaneously, in both the classical sense and as a distribution in $\mathrm{L}^p(\Omega^*; \mathcal{S}(n))$), and for the same constant $\beta \in (0, 1)$ as appearing in Lemma 3.17*

i. *at every point x where $\mathbf{D}^2 w^+(x)$ exists*

$$\frac{-\beta}{\varepsilon}\mathbf{I} \leq \mathbf{D}^2 w^+(x) \leq \frac{1}{\delta}\mathbf{I} \quad \textit{and} \quad \det\left|\mathbf{I} + \varepsilon\mathbf{D}^2 w^+(x)\right| \geq (1-\beta)^n,$$

ii. *at every point x where $\mathbf{D}^2 w^-(x)$ exists*

$$\frac{-1}{\delta}\mathbf{I} \leq \mathbf{D}^2 w^-(x) \leq \frac{\beta}{\varepsilon}\mathbf{I} \quad \textit{and} \quad \det\left|\mathbf{I} - \varepsilon\mathbf{D}^2 w^-(x)\right| \geq (1-\beta)^n.$$

Proof. Apply Lemmas 3.17 and 3.5 to obtain the distribution estimates. Now simply appeal to standard results on Sobolev spaces. $\square$

3.19. *Corollary*

Let u be a δ-solution of (1.1), *and let $\Omega^* \subset\subset \Omega$. Set $w^+ = SC_\delta^+[A_\varepsilon^+[u]]$ and $w^- = SC_\delta^-[A_\varepsilon^-[u]]$ for $0 < \varepsilon < \varepsilon_0$ and $0 < \delta < \varepsilon$ where ε_0 is the same constant as determined in Lemma 3.17. Set $T_{\varepsilon,\delta}^+ = T_\varepsilon^+[w^+]$ and $T_{\varepsilon,\delta}^- = T_\varepsilon^-[w^-]$. Then for the same constant $\beta \in (0, 1)$ as appearing in Lemma 3.17*

$$|T_{\varepsilon,\delta}^+(x) - T_{\varepsilon,\delta}^+(y)| \geq (1-\beta)|x-y| \quad \textit{for all } x, y \in \Omega*,$$

$$\det\left|\mathbf{D}T_{\varepsilon,\delta}^+\right|(x) \geq (1-\beta)^n \quad \textit{for almost every } x \in \Omega*,$$

$$|T_{\varepsilon,\delta}^-(x) - T_{\varepsilon,\delta}^-(y)| \geq (1-\beta)|x-y| \quad \textit{for all } x, y \in \Omega*,$$

$$\det\left|\mathbf{D}T_{\varepsilon,\delta}^-\right|(x) \geq (1-\beta)^n \quad \textit{for almost every } x \in \Omega*.$$

Proof. Use Corollary 3.18 and standard results in analysis from, for example, [3]. □

Our next lemma is a technical observation to be applied later.

3.20. *Lemma*

Let u be a δ-solution of (1.1). *Set* $v^+ = A_\varepsilon^+[u]$, $v^- = A_\varepsilon^-[u]$, $w^+ = SC_\delta^+[A_\varepsilon^+[u]]$, *and* $w^- = SC_\delta^-[A_\varepsilon^-[u]]$ *for* $0 < \delta < \varepsilon$. *Then there exists a constant* $\mu \in (0, 1)$ *independent of* ε *and* δ *such that when* $0 < \delta \leq \mu\varepsilon$

i.

$$\sum_{i,j=1}^{n} a_{ij}(y)\frac{\partial^2 w^+}{\partial x_i \partial x_j}(x) - g(y) \geq 0 \quad \textit{for all } y \in \Omega$$

for any point $x \in \Omega$ *at which both* $\mathbf{D}^2 w^+(x)$ *exists and* $w^+(x) > v^+(x)$,

ii.

$$\sum_{i,j=1}^{n} a_{ij}(y)\frac{\partial^2 w^-}{\partial x_i \partial x_j}(x) - g(y) \leq 0 \quad \textit{for all } y \in \Omega$$

for any point $x \in \Omega$ *at which both* $\mathbf{D}^2 w^-(x)$ *exists and* $w^-(x) < v^-(x)$.

Proof. Again, it will suffice to prove i. Assuming $\mathbf{D}^2 w^+(x)$ exists and $w^+(x) > v^+(x)$ we claim that

$$\text{there exists } \lambda_0 \in \Lambda^+[\mathbf{D}^2 w^+(x)] \text{ such that } \lambda = \frac{1}{\delta}. \tag{3.21}$$

Indeed, by (3.6) $w^+ = SC_\infty^+[v^+ - p_0] + p_0$, and the preceding is equivalent to claiming that $SC_\infty^+[v^+ - p_0]$ has an eigenvalue of zero. This, of course, is well known; our claim is established.

Using (3.21), the constant C_0 and function $P^-(\,\cdot\,)$ defined in Lemma 3.13 are such that

$$\sum_{i,j=1}^{n} a_{ij}(y)\frac{\partial^2 w^+}{\partial x_i \partial x_j}(x) - g(y) \geq P^-(\mathbf{D}^2 w^+(x)) - C \geq \frac{\theta_0}{\delta} - (n-1)\frac{\Theta_0}{\varepsilon} - C.$$

The existence of μ now follows easily from this last inequality. □

With our preparation the following important lemma is easily established.

3.22. *Lemma*

Let u be a δ-solution of (1.1), *and let* $\Omega^* \subset\subset \Omega$. *Set* $w^+ = SC_\delta^+[A_\varepsilon^+[u]]$ *and* $w^- = SC_\delta^-[A_\varepsilon^-[u]]$ *for* $0 < \varepsilon < \varepsilon_0$ *and* $\delta = \mu\varepsilon$ *where* ε_0 *is the same constant as determined in Lemma 3.17 and* μ *is the constant from Lemma 3.20. Set* $T_\varepsilon^+ = T_{\varepsilon,\delta}^+$ *and* $T_\varepsilon^- = T_{\varepsilon,\delta}^-$ *where* $T_{\varepsilon,\delta}^\pm$ *are from Corollary 3.19. Then*

i.

$$\sum_{i,j=1}^{n} a_{ij}(T_\varepsilon^+(x))\frac{\partial^2 w^+}{\partial x_i \partial x_j}(x) - g(T_\varepsilon^+(x)) \geq 0$$

for any point $x \in \Omega^*$ *such that* $T_\varepsilon^+(x)$ *is a Lebesgue point of both* (a_{ij}) *and* g *and at which* $\mathbf{D}^2 w^+(x)$ *exists,*

ii.

$$\sum_{i,j=1}^{n} a_{ij}(T_\varepsilon^-(x))\frac{\partial^2 w^-}{\partial x_i \partial x_j}(x) - g(T_\varepsilon^-(x)) \le 0$$

for any point $x \in \Omega^*$ such that $T_\varepsilon^-(x)$ is a Lebesgue point of both (a_{ij}) and g and at which $\mathbf{D}^2 w^-(x)$ exists.

Proof. To prove i, let $x \in \Omega^*$ be a point such that $T_\varepsilon^+(x)$ is a Lebesgue point of both (a_{ij}) and g and $\mathbf{D}^2 w^+(x)$ exists. If $w^+(x) > v^+(x)$, then Lemma 3.20 proves the validity of i. On the other hand, if $w^+(x) = v^+(x)$, then $(\mathbf{D}w^+(x), \mathbf{D}^2 w^+(x)) \in J_\Omega^{2,+} v^+(x)$. By [6, Lemma 2.14],this in turn implies

$$(\mathbf{D}w^+(x), \mathbf{D}^2 w^+(x)) \in J_\Omega^{2,+} u(T_\varepsilon^+(x)).$$

The proof of Theorem 2.10 applies to δ-solutions with virtually no modifications and implies

$$\sum_{i,j=1}^{n} a_{ij}(T_\varepsilon^+(x))\frac{\partial^2 w^+}{\partial x_i \partial x_j}(x) \ge g(T_\varepsilon^+(x)).$$

A similar argument establishes ii, completing our proof. □

Our next lemma shows that the preceding one is actually valid almost everywhere.

3.23. *Lemma*

Let u be a δ-solution of (1.1), and let $\Omega^* \subset\subset \Omega$. Set $w^+ = SC_\delta^+[A_\varepsilon^+[u]]$ and $w^- = SC_\delta^-[A_\varepsilon^-[u]]$ for $0 < \varepsilon < \varepsilon_0$ and $\delta = \mu\varepsilon$ where ε_0 is the same constant as determined in Lemma 3.17 and μ is the constant from Lemma 3.20. Set $T_\varepsilon^+ = T_{\varepsilon,\delta}^+$ and $T_\varepsilon^- = T_{\varepsilon,\delta}^-$ where $T_{\varepsilon,\delta}^\pm$ are from Corollary 3.19. Then for almost every $x \in \Omega^*$

- **i.** *$T_\varepsilon^+(x)$ is a Lebesgue point of both (a_{ij}) and g,*
- **ii.** *$T_\varepsilon^-(x)$ is a Lebesgue point of both (a_{ij}) and g,*
- **iii.** *$\mathbf{D}^2 w^+(x)$ and $\mathbf{D}^2 w^-(x)$ both exist.*

Proof. Since iii follows from standard results on Sobolev spaces, by symmetry it is once again sufficient to prove i. To this end let E be the measurable set given by

$$E = \left\{ y \in \Omega^* : y \text{ is not a Lebesgue point of either } (a_{ij}) \text{ or } g \right\}.$$

Due to Corollary 3.19, we see that

$$\begin{aligned} 0 = |E| &= \int_E dy = \int_{(T_\varepsilon^+)^{-1}(E)} \det \left|\mathbf{D}T_\varepsilon^+\right|(x)\, dx \\ &\ge \int_{(T_\varepsilon^+)^{-1}(E)} (1-\beta)^n\, dx = (1-\beta)^n \left|(T_\varepsilon^+)^{-1}(E)\right|, \end{aligned}$$

which clearly implies the desired result. □

Combining the two preceding lemmas, we have the following result.

3.24. *Corollary*

Let u be a δ-solution of (1.1), *and let $\Omega^* \subset\subset \Omega$. Set $w^+ = SC_\delta^+[A_\varepsilon^+[u]]$ and $w^- = SC_\delta^-[A_\varepsilon^-[u]]$ for $0 < \varepsilon < \varepsilon_0$ and $\delta = \mu\varepsilon$ where ε_0 is the same constant as determined in Lemma 3.17 and μ is the constant from Lemma 3.20. Set $T_\varepsilon^+ = T_{\varepsilon,\delta}^+$ and $T_\varepsilon^- = T_{\varepsilon,\delta}^-$ where $T_{\varepsilon,\delta}^\pm$ are from Corollary 3.19. Then for almost every point $x \in \Omega^*$*

$$\sum_{i,j=1}^n a_{ij}(T_\varepsilon^+(x))\frac{\partial^2 w^+}{\partial x_i \partial x_j}(x) - g(T_\varepsilon^+(x)) \geq 0$$

and

$$\sum_{i,j=1}^n a_{ij}(T_\varepsilon^-(x))\frac{\partial^2 w^-}{\partial x_i \partial x_j}(x) - g(T_\varepsilon^-(x)) \leq 0.$$

Our next step is to bracket a δ-solution by a $W^{2,\infty}$ subsolution and supersolution with appropriate properties.

3.25. *Theorem*

(Approximation) *Let u be a δ-solution of* (1.1), *and let $\Omega^* \subset\subset \Omega$. Given $\nu > 0$ there exist functions $w^{\nu+} > w^{\nu-}$ in $W^{2,\infty}(\Omega^*)$, $a_{ij}^{\nu+}$ and $a_{ij}^{\nu-}$ in $L^\infty(\Omega^*)$ for $1 \leq i, j \leq n$, and $g^{\nu+}$ and $g^{\nu-}$ in $L^\infty(\Omega^*)$ such that*

i. *$w^{\nu+}$ is a $W^{2,\infty}$ supersolution of*

$$\sum_{i,j=1}^n a_{ij}^{\nu+}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} = g^{\nu+}(x) \quad \text{in } \Omega^*, \tag{3.26}$$

which satisfies (1.2) *with the same constants as* (1.1)*;*

ii. *$w^{\nu-}$ is a $W^{2,\infty}$ subsolution of*

$$\sum_{i,j=1}^n a_{ij}^{\nu-}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} = g^{\nu-}(x) \quad \text{in } \Omega^*, \tag{3.27}$$

which satisfies (1.2) *with the same constants as* (1.1)*;*

iii.

$$\begin{cases} ||a_{ij} - a_{ij}^{\nu\pm}||_{L^n(\Omega^*)} \leq \nu & \text{for } 1 \leq i, j \leq n \\ ||g - g^{\nu\pm}||_{L^n(\Omega^*)} \leq \nu; \end{cases} \tag{3.28}$$

iv.

$$u + \frac{\nu}{2} \geq w^{\nu+} - \frac{\nu}{2} \geq u \geq w^{\nu-} + \frac{\nu}{2} \geq u - \frac{\nu}{2}. \tag{3.29}$$

Proof. We prove the preceding statements in reverse order. To start, take ε_0 as the constant as determined in Lemma 3.17 and μ as the constant from Lemma 3.20. Set $\delta = \mu\varepsilon$, $w^+ = SC_\delta^+[A_\varepsilon^+[u]]$, and $w^- = SC^-_\delta[A^-_\varepsilon[u]]$. We claim there exists $0 < \varepsilon_1 < \varepsilon_0$ such that for any $0 < \varepsilon < \varepsilon_1$ iv is satisfied with

$$w^{\nu+} = w^- + \frac{3\nu}{4} \quad \text{and} \quad w^{\nu-} = w^+ - \frac{3\nu}{4}.$$

Indeed, by Lemmas 3.2 and 3.5 we can choose ε_1 so that

$$|w^\pm - u| \le \frac{\nu}{4} \quad \text{for all } 0 < \varepsilon < \varepsilon_1,$$

which establishes our claim.

Fix ε_0, ε_1, μ, $w^{\nu+}$, and $w^{\nu-}$ as just defined. Set $T^+_\varepsilon = T^+_{\varepsilon,\delta}$ and $T^-_\varepsilon = T^-_{\varepsilon,\delta}$ where $T^\pm_{\varepsilon,\delta}$ are from Corollary 3.19. We claim there exists $0 < \varepsilon_2 < \varepsilon_1$ such that for any $0 < \varepsilon < \varepsilon_2$ iii is satisfied with

$$\begin{aligned} &a^{\nu+}_{ij}(x) = a_{ij}(T^+_\varepsilon(x)) \quad \text{for } 1 \le i, j \le n, \\ &a^{\nu-}_{ij}(x) = a_{ij}(T^-_\varepsilon(x)) \quad \text{for } 1 \le i, j \le n, \\ &g^{\nu+}(x) = g(T^+_\varepsilon(x)), \quad \text{and} \quad g^{\nu-}(x) = g(T^-_\varepsilon(x)). \end{aligned}$$

Indeed (once again) by Lemmas 3.2 and 3.5 $T^\pm_\varepsilon \to \mathbf{I}$ as $\varepsilon \to 0^+$. Approximating (a_{ij}) and g by continuous functions, the claim is clearly true.

Closing out, i and ii now follow easily by Corollary 3.24. $\square$

Now let us approximate a δ-solution with a $\mathrm{W}^{2,p}$ solution with $p > n$.

3.30. *Theorem*

Let u be a δ-solution of (1.1)*, and let $\Omega^* \subset\subset \Omega$ with smooth boundary $\partial\Omega^* \in \mathrm{C}^{2,\alpha}$. Given $\nu > 0$ and $p > n$ there exist functions w^ν in $\mathrm{W}^{2,p}(\Omega^*)$, a^ν_{ij} in $\mathrm{C}(\overline{\Omega^*})$ for $1 \le i, j \le n$, and g^ν in $\mathrm{C}(\overline{\Omega^*})$ such that*

i. *w^ν is a $\mathrm{W}^{2,p}$ solution of*

$$\sum_{i,j=1}^n a^\nu_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = g^\nu(x) \quad \text{in } \Omega^*, \tag{3.31}$$

which satisfies (1.2) *with the same constants as* (1.1)*;*

ii.

$$\begin{cases} \|a_{ij} - a^\nu_{ij}\|_{L^n(\Omega^*)} \le 4\nu \quad \text{for } 1 \le i, j \le n \\ \|g - g^\nu\|_{L^n(\Omega^*)} \le 4\nu; \end{cases} \tag{3.32}$$

iii.

$$u + 2\nu \ge w^\nu \ge u - 2\nu. \tag{3.33}$$

Proof. The trick is to construct a special quasi-linear pde. To this end let $w^{\nu\pm}$, $(a^{\nu\pm}_{ij})$, and $g^{\nu\pm}$ be the functions produced by Theorem 3.25. Given $\delta > 0$ there exists an open set $G_\delta \subset \Omega^*$ such that

$$\begin{cases} |G_\delta| < \delta, \\ (a^{\nu\pm}_{ij}) \text{ and } g^{\nu\pm} \text{ are continuous on } \overline{\Omega^*} \setminus G_\delta. \end{cases} \tag{3.34}$$

Construct continuous extensions of $(\tilde{a}_{ij}^{\nu\pm})$ of $(a_{ij}^{\nu\pm})$ and $\tilde{g}^{\nu\pm}$ of $g^{\nu\pm}$ satisfying (1.2) with the same constants as for (1.1). Using this, we see that

$$\begin{cases} \|\tilde{a}_{ij}^{\nu\pm} - a_{ij}^{\nu\pm}\|_{L^n(\Omega^*)} \le 2K_1\sqrt[n]{\delta} & \text{for } 1 \le i, j \le n \\ \|\tilde{g}^{\nu\pm} - g^{\nu\pm}\|_{L^n(\Omega^*)} \le 2K_1\sqrt[n]{\delta}. \end{cases}$$

Clearly, there exists $\delta_0 > 0$ such that $2K_1\sqrt[n]{\delta} < \nu$ if $0 < \delta < \delta_0$, which by (3.28) implies

$$\begin{cases} \|\tilde{a}_{ij}^{\nu\pm} - a_{ij}\|_{L^n(\Omega^*)} \le 2\nu & \text{for } 1 \le i, j \le n \text{ if } 0 < \delta < \delta_0 \\ \|\tilde{g}^{\nu\pm} - g\|_{L^n(\Omega^*)} \le 2\nu & \text{if } 0 < \delta < \delta_0. \end{cases} \tag{3.35}$$

By (3.29) it is possible to find a function $\psi \in C^\infty(\overline{\Omega^*} \times \mathbf{R})$ such that

$$\begin{aligned} &0 \le \psi(\cdot,\cdot) \le 1, \\ &\psi(x,t) = 1 \quad \text{if } t \ge w^{\nu+}, \\ &\psi(x,t) = 0 \quad \text{if } t \le w^{\nu-}. \end{aligned}$$

From this we construct the coefficients and nonhomogeneous term of a quasi-linear pde. Namely,

$$\begin{aligned} \hat{a}_{ij}(x,t) &= \Big(\psi(x,t)\tilde{a}_{ij}^{\nu+}(x) + (1-\psi(x,t))\tilde{a}_{ij}^{\nu-}(x)\Big) \quad \text{for } 1 \le i, j \le n, \\ \hat{g}(x,t) &= \big(\psi(x,t)\tilde{g}^{\nu+} + (1-\psi(x,t))\tilde{g}^{\nu-}\big). \end{aligned}$$

This leads to the pde

$$\sum_{i,j=1}^{n} \hat{a}_{ij}(x,u)\frac{\partial^2 u}{\partial x_i \partial x_j} - \hat{g}(x,u) = 0 \quad \text{in } \Omega^*. \tag{3.36}$$

Finally let $h \in C^{2,\alpha}(\partial\Omega^*)$ be a function such that

$$u\big|_{\partial\Omega^*} + \nu \ge h \ge u\big|_{\partial\Omega^*} - \nu. \tag{3.37}$$

Define a mapping $\Phi : C(\overline{\Omega^*}) \to C(\overline{\Omega^*})$ by setting $\Phi[v] = u$ where u is the solution of

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} \hat{a}_{ij}(x,v)\frac{\partial^2 u}{\partial x_i \partial x_j} - \hat{g}(x,v) = 0 & \text{in } \Omega^*, \\ u\big|_{\partial\Omega^*} = h. \end{cases}$$

The Compact Mapping Theorem guarantees the existence of a fixed point of Φ. Let w^ν denote one such fixed point. Then $w^\nu \in W^{2,p}$ is a $W^{2,p}$ solution of (3.36) satisfying the boundary condition $w^\nu\big|_{\partial\Omega^*} = h$. Using w^ν, we define (a_{ij}^ν) and g^ν by

$$\begin{aligned} a_{ij}^\nu(x) &= \hat{a}_{ij}(x, w^\nu(x)) \text{ for } 1 \le i, j \le n \\ g^\nu(x) &= \hat{g}(x, w^\nu(x)). \end{aligned}$$

A careful examination of our construction allows us to conclude the validity of i, and establishing the inequalities of ii follows from (3.35) and the construction of $(\hat{a}_{ij})$ and $\hat{g}$.

To complete our proof, we need to establish iii. We begin by setting

$$\Omega^+ = \left\{x \in \Omega^* : w^\nu(x) > w^{\nu+}(x)\right\}.$$

From our construction it follows that $w^\nu - w^{\nu+}$ is a $\mathrm{W}^{2,p}$ subsolution of

$$\sum_{i,j=1}^n a_{ij}^\nu(x)\frac{\partial^2 u}{\partial x_i \partial x_j} = -2K_1(1+n^2||w^{\nu+}||_{W^{2,\infty}})\mathcal{X}_{G_\delta} \quad \text{in } \Omega^+.$$

As such the L^∞ estimates of Lemma 2.3 apply. If ρ_0 is the radius of the smallest ball containing Ω, then

$$\sup\left\{(w^\nu - w^{\nu+})^+(x) : x \in \Omega^+\right\} \le \frac{\rho_0}{ne_1\sqrt[n]{\Gamma_n}} 2K_1(1+n^2||w^{\nu+}||_{W^{2,\infty}})\sqrt[n]{|G_\delta|}.$$

From this it is obvious that there exists a $0 < \delta_1 < \delta_0$ so that if $0 < \delta < \delta_1$, then $w^{\nu+} + \nu \ge w^\nu$; and by a similar argument $w^\nu \ge w^{\nu-} - \nu$. In conjunction with (3.29) this proves iii. □

We may now prove the following result.

3.38. ***Theorem***
(1.14iv) *Definition* 1.13 $\Longrightarrow$ *Definition* 1.4.

Proof. Let u be a δ-solution of (1.1). It will be sufficient to prove that given $\nu > 0$ there exists a $\mathrm{W}^{2,p}$ solution $\hat{w}$ of

$$\begin{cases} \displaystyle\sum_{i,j=1}^n \hat{a}_{ij}(x)\frac{\partial^2 \hat{w}}{\partial x_i \partial x_j} = \hat{g}(x) \quad \text{in } \Omega, \\ \hat{w}\big|_{\partial\Omega} = u\big|_{\partial\Omega} \end{cases} \tag{3.39}$$

such that $\hat{a}_{ij}$ and $\hat{g}$ satisfy (1.2), and

$$\hat{w} \in \mathrm{W}^{2,p}(\Omega) \quad \text{for some } p > n,$$

$$\hat{a}_{ij} \in C(\overline{\Omega}) \quad \text{for all } 1 \le i, j \le n,$$

$$\hat{g} \in C(\overline{\Omega}),$$

$$\sup_{x\in\Omega} |(u - \hat{w})(x)| \le 3\nu,$$

$$\|a_{ij} - \hat{a}_{ij}\|_{L^n(\Omega)} \le 5\nu \quad \text{for all } 1 \le i, j \le n,$$

$$\|g - \hat{g}\|_{L^n(\Omega)} \le 5\nu.$$

We begin by defining a pseudodistance function $d(\Omega, \Omega^*)$ where $\Omega^* \subset \Omega$ is an arbitrary subdomain of Ω:

$$d(\Omega, \Omega^*) = \sup\left\{\inf\{|x-y| : x \in \mathbf{R}^n \setminus \Omega\} : y \in \Omega \setminus \Omega^*\right\}.$$

Assuming $\hat{a}_{ij}$ and $\hat{g}$ satisfy (1.2), standard barrier arguments and elementary measure theory establish the existence of a number $\mu > 0$ such that for any $\mathrm{W}^{2,p}$ solution $\hat{w}$ of (3.39)

$$|\, u(x) - \hat{w}(x)| \;\le\; \nu \qquad \text{for all } x \in \Omega \setminus \Omega^*; \tag{3.40}$$

$$|\, \Omega \setminus \Omega^*| \;\le\; (\nu/(2K_1))^n \,. \tag{3.41}$$

Fixing a subdomain $\Omega^* \subset \Omega$ with $d(\Omega, \Omega^*) < \mu$ and $\partial\Omega^* \in C^{2,\alpha}$, let w^ν, $\left(a^\nu_{ij}\right)$, and g^ν be the functions produced by Theorem 3.30 for this subdomain (with ν as given). We construct $\left(\hat{a}_{ij}\right)$ and $\hat{g}$ by extending $\left(a^\nu_{ij}\right)$ and g^ν continuously to $\overline{\Omega}$ so that (1.2) remains satisfied. Standard modern pde theory now produces a $\mathrm{W}^{2,p}$ solution $\hat{w}$ of (3.39) with $p > n$. By the maximum principle for $\mathrm{W}^{2,p}$ solutions with $p > n$ in conjunction with (3.40) it follows that

$$\|w^\nu(x) - \hat{w}(x)\| \le \nu \qquad \text{for all } x \in \Omega^*.$$

Consequently, by (3.40) and Theorem 3.30

$$\sup_{x\in\Omega} |\,(u - \hat{w})(x)| \le 3\nu.$$

Due to (3.41) and Theorem 3.30 we also have

$$\|a_{ij} - \hat{a}_{ij}\|_{L^n(\Omega)} \;\le\; 5\nu \qquad \text{for all } 1 \le i,\, j \le n,$$

$$\|g - \hat{g}\|_{L^n(\Omega)} \;\le\; 5\nu.$$

This completes the proof of the theorem. $\square$

References

[1] Caffarelli, L. (1989). Interior a priori estimates for solutions of fully nonlinear equations. *Ann. of Math.* (2), **130**, 180–213.

[2] Crandall, M. G., Ishii, H., and Lions, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.* **27**, 1–67.

[3] Evans, C. and Gariepy, R. (1992). *Measure Theory and Fine Properties of Functions*, first ed. CRC Press, Ann Arbor.

[4] Gilbarg, D. and Trudinger, N. S. (1983). *Elliptic Partial Differential Equations of Second Order*, second ed. Springer-Verlag, New York.

[5] Jensen, R. R. (1988). The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations. *Arch. Rational Mech. Anal.* **101**, 1–27.

[6] ———. (1989). Uniqueness criteria for viscosity solutions of fully nonlinear elliptic partial differential equations. *Indiana Univ. Math. J.* **38**, 629–667.

[7] Jensen, R. R., Lions, P.-L., and Souganidis, P.E. (1988). A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations. *Proc. Amer. Math. Soc.* **102**, 975–978.

[8] Lasry, J. M. and Lions, J.-L. (1986). A remark on regularization in Hilbert spaces. *Israel J. Math.* **55**, 257–266.

[9] Pucci, C. (1966). Su una limitazone per soluzioni di equazioni ellitche. *Boll. Un. Mat. Ital.* **21**, 228–233.

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