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An Almost Orthogonal Radial Wavelet Expansion for Radial Distributions

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ABSTRACT. This paper presents an expansion for radial tempered distributions on $\mathbf{R}^n$ *in terms of smooth, radial analyzing and synthesizing functions with space-frequency localization properties similar to standard wavelets. Scales of quasi-norms are defined for the coefficients of the expansion that characterize, via Littlewood–Paley–Stein theory, when a radial distribution belongs to a Triebel–Lizorkin or Besov space. These spaces include, for example, the* L^p *spaces,* $1 < p < \infty$*, Hardy spaces* H^p*,* $0 < p \leq 1$*, Sobolev spaces* L_k^p*, and Lipschitz spaces* Λ_α*,* $\alpha > 0$*. We also present a smooth radial atomic decomposition and norm estimates for sums of smooth radial molecules. The radial wavelets, atoms, and molecules that we consider are localized near certain annuli, as opposed to cubes in the usual, nonradial setting. The radial wavelet expansion is multiscale, where the functions in the different scales are related by dilation. However, there is no translation structure within a given scale, unlike the situation with standard wavelet systems.*

0. Introduction

We develop an expansion of wavelet type adapted to the study of radial functions on $\mathbf{R}^n$, or more generally, radial tempered distributions on $\mathbf{R}^n$. By "radial" we mean spherically symmetric. If $f : \mathbf{R}^n \to \mathbf{C}$ and $R \in O(n)$, let $Rf : \mathbf{R}^n \to \mathbf{C}$ be given by $Rf(x) = f(R^{-1}x)$. We consider a tempered distribution $f \in \mathcal{S}'(\mathbf{R}^n)$ to be radial if $f(g) = f(Rg)$ for every test function $g \in \mathcal{S}(\mathbf{R}^n)$ and $R \in O(n)$. More specifically, a radial *function* is of the form $f(x) = f_0(|x|)$, $f_0 : [0, \infty) \to \mathbf{C}$. For $f : \mathbf{R}^n \to \mathbf{C}$, radial or not, we could consider its expansion in terms of an orthonormal wavelet basis $\{\psi_{\mu k}^{(j)}\}$, $\mu \in \mathbf{Z}$, $k \in \mathbf{Z}^n$, $j = 1, \ldots, 2^n - 1$, where $\psi_{\mu k}^{(j)}(x) = 2^{\mu n/2}\psi^{(j)}(2^\mu x - k)$ for appropriate $\psi^{(j)}$ (see [3, 15, 17]), or we could consider its (nonorthogonal) φ-transform decomposition

$$f = \sum_{\mu \in \mathbf{Z}} \sum_{k \in \mathbf{Z}^n} \langle f, \varphi_{\mu k} \rangle \psi_{\mu k}$$

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for $\varphi_{\mu k}$ and $\psi_{\mu k}$ determined by certain φ and ψ as above (see [9, 11]). One advantage of these expansions is that most traditional function space norms are precisely characterized via Littlewood–Paley–Stein theory by certain expressions involving the magnitudes of the coefficients in these expressions [11, 12, 17]. Another advantage is that the $\psi_{\mu k}$ and $\varphi_{\mu k}$ can be taken to be both spatially and frequency localized so that the coefficients give a simultaneous space-frequency analysis of f. In applications, such as signal analysis or numerical partial differential equations, this allows one to combine the advantages of a Fourier decomposition—such as diagonalization of translation-invariant operators, which becomes near-diagonalization for these and many other operators in the φ-transform and wavelet cases—with the ability to focus on interesting local behavior, such as shock formation. In many real-world applications the essential characteristics of a signal are carried by relatively few terms of the wavelet expansion as compared to the Fourier expansion, leading to quick computation and rapid convergence of various numerical schemes. (See, e.g., [1, 2, 19].)

For a radial function, it should be possible to exploit the radial symmetry for mathematical or computational advantage. However, the expansions above are not suitable for this. Even if ψ is radial, its translates and dilates $\psi_{\mu k}$ are not. Roughly speaking, the sum indexed by all translates over $\mathbf{Z}^n$ introduces many unnecessary degrees of freedom in the expansion of a radial function f. We will demonstrate the existence of *radial* functions $\{\varphi_{\mu k}\}_{\mu\in\mathbf{Z},k\in\mathbf{Z}^+}$ and $\{\psi_{\mu k}\}_{\mu\in\mathbf{Z},k\in\mathbf{Z}^+}$ that are smooth, frequency localized (supp $\hat{\varphi}_{\mu k}, \hat{\psi}_{\mu k} \subseteq \{\xi : 2^{\mu-2} \le |\xi| \le 2^{\mu}\}$), and spatially localized near certain annuli $A_{\mu k}$, such that a radial $f : \mathbf{R}^n \to \mathbf{C}$ can be expanded as

$$f = \sum_{\mu\in\mathbf{Z}} \sum_{k=1}^{\infty} \langle f, \varphi_{\mu k}\rangle \psi_{\mu k}. \tag{1}$$

See §2 for the exact statements. Note that the range of k is independent of the dimension n, reflecting the radiality.

We cannot obtain a satisfactory version of (1) simply by applying the standard one-dimensional wavelet results to the restriction of f to $\mathbf{R}$ and extending radially to $\mathbf{R}^n$ since this process destroys Fourier transform localization, vanishing moment conditions, and norm estimates. Also, we cannot apply the methods of the "T of b" theorem [4] to a restriction of f since the relevant measure $r^{n-1}dr$ is not para-accretive. Moreover, the methods used by Meyer, Mallat, and Daubechies to construct orthonormal wavelets do not apply here (as far as we can see) since these methods depend on an underlying translation (or at least lattice or group) structure. So although we are interested in exploiting symmetry (radiality), the main difficulty is to remove translation symmetry from the construction. Further work in this direction is done in [6], where we develop a wavelet decompostion adapted to polar coordinates in $\mathbf{R}^2$. In that case we have neither translation nor dilation symmetry.

The identity (1) is a nonorthogonal expansion similar to the φ-transform decomposition noted above; in fact, its derivation is similar. We start with a sampling formula (Theorem 1.1) for radial band-limited functions—see the related results in [14]. This formula is obtained by a method similar to the proof of the Shannon sampling formula (which lies behind the φ-transform—see, e.g., [13]), except that we use a Fourier–Bessel expansion instead of a Fourier series expansion. The sampling formula and a standard partition of unity on the frequency side lead to (1).

As for the φ-transform and wavelet expansions, Littlewood–Paley techniques can be applied to (1) to obtain norm characterizations of the function spaces covered by Littlewood–Paley theory. These include the L^p spaces, $1 < p < \infty$, the Hardy spaces H^p, $0 < p \le 1$, the Riesz potential spaces, and the homogeneous Lipschitz spaces. To treat all these cases systematically, we state our results for the homogeneous Besov and homogeneous Triebel–Lizorkin spaces, denoted $\dot{B}_p^{\alpha q}$ and

$\dot{F}_p^{\alpha q}$, respectively. See §2 for precise definitions and results for $\dot{F}_p^{\alpha q}$; the Besov spaces are considered in §5. In §6 we state results for the inhomogeneous Besov and Triebel–Lizorkin spaces, which include as special cases the Bessel potential spaces L^p_α, $\alpha > 0$, $1 < p < \infty$, and in particular the Sobolev spaces L^p_k. See [12] for a discussion of all these spaces in the context of Littlewood–Paley theory, the φ-transform, and wavelets. For a sequence $s = \{s_{\mu k}\}_{\mu\in\mathbf{Z},k\in\mathbf{Z}^+}$, let

$$\|s\|_{\dot{r}_p^{\alpha q}} = \left\| \left(\sum_{\mu\in\mathbf{Z}} \sum_{k=1}^{\infty} (2^{\mu\alpha}|s_{\mu k}|\tilde{\chi}_{A_{\mu k}})^q \right)^{1/q} \right\|_{L^p(\mathbf{R}^n)},$$

where $\tilde{\chi}_{A_{\mu k}} = |A_{\mu k}|^{-1/2}\chi_{A_{\mu k}}$ is the L^2-normalized characteristic function of an annulus $A_{\mu k}$ defined in §2. Our basic result in the case of homogeneous Triebel–Lizorkin spaces is that for radial f, $\alpha \in \mathbf{R}$, $0 < q \le +\infty$, and $0 < p < +\infty$,

$$\|f\|_{\dot{F}_p^{\alpha q}(\mathbf{R}^n)} \approx \|\{\langle f, \varphi_{\mu k}\rangle\}\|_{\dot{r}_p^{\alpha q}}, \tag{2}$$

where $\approx$ means the two quasi-norms are equivalent. (See Corollary 2.1 and similar results for the Besov spaces in §5.) This is much like the case with the usual φ-transform or wavelet characterizations, with cubes being replaced by annuli; in particular, the right side of (2) depends only on the magnitudes of the coefficients in (1). These results are proved in §§1 and 2.

In §3 we obtain sufficient conditions for a family of radial functions $m_{\mu k}$, $\mu \in \mathbf{Z}$, $k \in \mathbf{Z}^+$, to be a "family of radial molecules" for $\dot{F}_p^{\alpha q}$; by definition this means that for every sequence $s \in \dot{r}_p^{\alpha q}$,

$$\left\| \sum_{\mu\in\mathbf{Z}} \sum_{k=1}^{\infty} s_{\mu k} m_{\mu k} \right\|_{\dot{F}_p^{\alpha q}} \le c\|s\|_{\dot{r}_p^{\alpha q}}, \tag{3}$$

where $c < \infty$ is a constant independent of the sequence s. By Theorem 2.2, the collection $\{\psi_{\mu k}\}$ in (1) is a family of radial molecules for $\dot{F}_p^{\alpha q}$. Such families are important, for example, because any linear operator that is radial (i.e., takes radial functions to radial functions) and has the property that $\{T\psi_{\mu k}\}$ is a family of radial molecules for $\dot{F}_p^{\alpha q}$ is by (1), (2), and (3), bounded on the space $\dot{R}_p^{\alpha q}$ of radial elements of $\dot{F}_p^{\alpha q}$. A radial function $m_{\mu k}$ is defined to be a smooth radial molecule over $A_{\mu k}$ for $\dot{R}_p^{\alpha q}$ if it satisfies certain size, smoothness, decay, and cancellation conditions depending on α, p, and q—see §3. The main result in §3 is that if each $m_{\mu k}$ is a smooth radial molecule over $A_{\mu k}$ for $\dot{R}_p^{\alpha q}$, then $\{m_{\mu k}\}$ is a family of radial molecules for $\dot{F}_p^{\alpha q}$. This is the content of Theorem 3.1.

The conditions assumed for smooth molecules $m_{\mu k}$ in Theorem 3.1 are similar to those in the rectangular case, with cubes replaced by annuli, except for the cancellation condition:

$$\int_{\mathbf{R}^n} m_{\mu k}(x)|x|^j\,dx = 0, \tag{4}$$

required for $j = 0, 1, \ldots, l$, where $l > n/\min\{1, p, q\} - n - 1 - \alpha$ (the condition is void if $l < 0$). To understand (4) consider the case $\alpha = 0$, $p, q \ge 1$ (e.g., $\dot{F}_p^{02} \approx L^p$, $1 < p < \infty$), so that the minimum acceptable l is zero. Then condition (4) is just that the molecules $m_{\mu k}$ have mean zero. The proof of the φ-transform analogue of (3) in [11] involves using the cancellation assumptions to subtract off constants in certain convolution estimates, leading to geometric decay terms arising from the diameter $c_n \cdot 2^{-\mu}$ of the dyadic cubes $\{Q_{\mu k}\}_{k\in\mathbf{Z}^n}$ in $\mathbf{R}^n$ of side length $2^{-\mu}$. Now it happens that the diameters of the annuli $A_{\mu k}$ go to infinity with k, for fixed μ. Nevertheless, we obtain a positive result because of the fact that a radial function with mean zero is orthogonal to any function that is homogeneous of degree zero, not just to constants. Thus, in the proof of (3) we can subtract a function of the angle, leading to estimates involving the width of the annulus $A_{\mu k}$ (i.e., its outer radius minus its inner radius), which is bounded by $c \cdot 2^{-\mu}$ independent of k. We emphasize the case $l = 0$ because then (4) is just the usual mean zero condition, but this principle of compensating for lack of localization by using stronger cancellation is behind the use of (4) to prove (3) in the general case.

As we take $l > 0$ (in particular $\alpha \to -\infty$ or $p, q \to 0$), condition (4) becomes unfamiliar and quite restrictive. Unlike the cancellation condition in [11], (4) does not follow from the assumption $0 \notin \operatorname{supp} \hat{m}_{\mu k}$, for the odd values of j. In particular, the functions $\psi_{\mu k}$ in (1) do not necessarily satisfy (4) for odd values of j. Consequently, from Theorems 2.2 and 3.1, we have two families of molecules, neither of which contains the other.

In §4 we consider the radial analogue of the smooth atomic decomposition results in [9] and [11], which held there for all α, q, p indices. For the radial case, if $l \le 0$ in (4), we can imitate these methods using annuli instead of cubes to get analogous results (see Theorem 4.1). However, if $l \ge 1$, the restrictiveness of (4) does not allow this method to succeed and we do not know if there is an appropriate radial result in this case. Although the cases where $l \le 0$ are of main interest (e.g., L^p spaces and Sobolev spaces), this phenomenon is curious and perhaps merits further study.

In §5 we consider the Besov spaces, in §6 the inhomogeneous spaces, and conclude with some further questions in §7.

1. Preliminaries

For the Fourier transform, we use the conventions

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-ix\cdot\xi}\, dx \quad \text{and} \quad \check{f}(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} f(\xi) e^{ix\cdot\xi}\, d\xi.$$

Let $d\sigma_t$ denote (unnormalized) Lebesgue surface measure on the sphere $\{x \in \mathbf{R}^n : |x| = t\}$, and let ω_{n-1} denote the measure of the unit sphere $\{x \in \mathbf{R}^n : |x| = 1\}$. Then we have $\int d\sigma_t = \omega_{n-1} t^{n-1}$. The usual convention is in effect, i.e., that the symbol c denotes a constant that can change from line to line in a sequence of inequalities. We take as our definition of Bessel functions

$$J_\nu(x) = \frac{(x/2)^\nu}{\pi^{1/2}\Gamma(\nu + 1/2)} \int_{-1}^{1} (1 - t^2)^{\nu - 1/2} e^{ixt}\, dt \quad \text{for } \nu > -\frac{1}{2},$$

and $J_{-1/2}(x) = (2/\pi x)^{1/2} \cos x$. From now on we let $\nu = (n-2)/2$. It follows from the definition of Bessel functions that

$$\widehat{(d\sigma_t)}(\xi) = (2\pi)^{n/2} t^{n-1} \frac{J_\nu(t|\xi|)}{(t|\xi|)^\nu}. \tag{5}$$

This may be used to show that if f is radial on $\mathbf{R}^n$, i.e., $f(x) = f_0(|x|)$, then

$$\hat{f}(\xi) = \frac{(2\pi)^{n/2}}{|\xi|^\nu} \int_0^\infty f_0(s) J_\nu(s|\xi|) s^{n/2}\, ds. \tag{6}$$

For proofs of (5) and (6), see [5, Lemma 5.1] and [20, Theorem 3.3], taking into account our convention for the Fourier transform.

Let $j_{\nu,1} < j_{\nu,2} < j_{\nu,3} < \cdots$ denote the positive zeroes of J_ν. We will use the result of McMahon's asymptotic expansion, that

$$j_{\nu,k} = (k + \nu/2 - 1/4)\pi + O(1/k).$$

(See [21, pp. 505–506].) For $\lambda > 0$, let

$$h_{\nu,k}(\lambda) = \frac{\sqrt{2}}{J_{\nu+1}(j_{\nu,k})} \frac{J_\nu(j_{\nu,k}\lambda)}{\lambda^\nu}.$$

Then $\{h_{\nu,k}\}_{k=1}^\infty$ is known to be a complete orthonormal basis for $L^2([0,1], \lambda^{n-1} d\lambda)$. (See [8, p. 147].) The following pointwise convergence lemma is obtained by a change of weight in [21, Theorem 18.24].

1.1. Lemma

Suppose $\nu = (n-2)/2$. Let $f(\lambda)$ be a measurable function on the interval $[0, 1]$ such that

$$\int_0^1 \lambda^{\nu+1/2} |f(\lambda)|\, d\lambda < \infty.$$

If $\lambda^\nu f(\lambda)$ is continuous on (a, b) and has bounded variation on (a, b), where $0 < a < b < 1$, then the series

$$\sum_{k=1}^\infty \langle f, h_{\nu,k} \rangle_{L^2([0,1],\lambda^{n-1}d\lambda)} h_{\nu,k}(x)$$

converges to $f(x)$ for every $x \in (a, b)$.

The φ-transform expansion [9, 10, 11] was derived from an n-dimensional generalization of the Shannon sampling formula. The next theorem provides an n-dimensional *radial* version of the sampling formula. (Compare with [14].)

1.1. Theorem

Suppose f, g are radial, $f \in \mathcal{S}'(\mathbf{R}^n)$, $g \in \mathcal{S}(\mathbf{R}^n)$, $\operatorname{supp} \hat{f} \subset \{\xi \in \mathbf{R}^n : |\xi| < B\}$ for some $B > 0$, and $\operatorname{supp} \hat{g} \subset \{\xi \in \mathbf{R}^n : |\xi| \le B\}$. Let e be a fixed unit vector in $\mathbf{R}^n$. Then for every $x \in \mathbf{R}^n$

$$(f * g)(x) = 2B^{-1} \sum_{k=1}^{\infty} (j_{\nu,k} J_{\nu+1}^2(j_{\nu,k}))^{-1} f(B^{-1} j_{\nu,k} e)\, g * d\sigma_{B^{-1} j_{\nu,k}}(x). \tag{7}$$

Proof. Suppose first that $f \in \mathcal{S}$. We will show that the right side of (7) is a continuous, integrable function, whose Fourier transform equals $\hat{f}\hat{g}$.

The principal asymptotic formula for $J_\nu(x)$, as $x \to +\infty$, is

$$J_\nu(x) = (2/\pi x)^{1/2} \cos(x - \nu\pi/2 - \pi/4) + O(x^{-3/2}).$$

(See, [21, 7.21].) The asymptotics for $j_{\nu,k}$ therefore imply that

$$\lim_{k\to\infty} j_{\nu,k} J_{\nu+1}^2(j_{\nu,k}) = 2/\pi.$$

Since the zeros of J_ν and $J_{\nu+1}$ are interlaced, we conclude that there exist constants $c_1, c_2 > 0$ such that

$$c_1 \le j_{\nu,k} J_{\nu+1}^2(j_{\nu,k}) \le c_2 \tag{8}$$

for all k. Also, since $f \in \mathcal{S}$, the sequence of numbers $\{f(B^{-1} j_{\nu,k} e)\}_{k=1}^{\infty}$ has rapid decay as $k \to \infty$. Finally, the family of continuous functions $\{g * d\sigma_{B^{-1} j_{\nu,k}}\}_{k=1}^{\infty}$ has the property that

$$\|g * d\sigma_{B^{-1} j_{\nu,k}}\|_{L^\infty(\mathbf{R}^n)} \le \omega_{n-1} (B^{-1} j_{\nu,k})^{n-1} \|g\|_{L^\infty(\mathbf{R}^n)} \le c k^{n-1}.$$

We conclude that the right side of (7) converges uniformly to a continuous function. As for integrability, note that

$$\|g * d\sigma_{B^{-1} j_{\nu,k}}\|_{L^1(\mathbf{R}^n)} \le \omega_{n-1} (B^{-1} j_{\nu,k})^{n-1} \|g\|_{L^1(\mathbf{R}^n)} \le c k^{n-1}.$$

Using the rapid decay of the numbers $f(B^{-1} j_{\nu,k})$ and the Lebesgue dominated convergence theorem, we see that the Fourier transform of the right side of (7) is

$$2B^{-1} \sum_{k=1}^{\infty} (j_{\nu,k} J_{\nu+1}^2(j_{\nu,k}))^{-1} f(B^{-1} j_{\nu,k} e) \hat{g}(\xi) \widehat{d\sigma}_{B^{-1} j_{\nu,k}}(\xi).$$

Now we show that for $0 < \lambda < B$,

$$\hat{f}(\lambda e) = 2B^{-1} \sum_{k=1}^{\infty} (j_{\nu,k} J_{\nu+1}^2(j_{\nu,k}))^{-1} f(B^{-1} j_{\nu,k} e) \widehat{d\sigma}_{B^{-1} j_{\nu,k}}(\lambda e).$$

By scaling, the functions $\{B^{-n/2} h_{\nu,k}(B^{-1}\lambda)\}_{k=1}^{\infty}$ form a complete orthonormal basis for $L^2([0, B], \lambda^{n-1} d\lambda)$. Since $f \in \mathcal{S}$, the function $F(\lambda) = \hat{f}(\lambda e)$ satisfies the hypotheses of Lemma 1.1, suitably scaled to $[0, B]$. Hence, the Fourier–Bessel series

$$\sum_{k=1}^{\infty} B^{-n/2} \langle F, h_{\nu,k}(B^{-1}\cdot)\rangle_{L^2([0,B],\lambda^{n-1}d\lambda)} B^{-n/2} h_{\nu,k}(B^{-1}\lambda)$$

converges to $F(\lambda)$ for every $0 < \lambda < B$. Now,

$$\begin{aligned}
&\langle F, h_{\nu,k}(B^{-1}\cdot)\rangle_{L^2([0,B],\lambda^{n-1}d\lambda)} \\
&\quad = \frac{\sqrt{2}}{J_{\nu+1}(j_{\nu,k})} \int_0^B F(\lambda) \frac{J_\nu(j_{\nu,k}B^{-1}\lambda)}{(B^{-1}\lambda)^\nu} \lambda^{n-1}\, d\lambda \\
&\quad = \frac{\sqrt{2}(2\pi)^{n/2} j_{\nu,k}^{\nu}}{J_{\nu+1}(j_{\nu,k})} \cdot \frac{(2\pi)^{-n/2}}{(B^{-1}j_{\nu,k})^\nu} \int_0^\infty F(\lambda) J_\nu(j_{\nu,k}B^{-1}\lambda)\lambda^{n/2}\, d\lambda \\
&\quad = \frac{\sqrt{2}(2\pi)^{n/2} j_{\nu,k}^{\nu}}{J_{\nu+1}(j_{\nu,k})} f(B^{-1}j_{\nu,k}e),
\end{aligned}$$

where the last equality comes from the Fourier inverse version of (6). Also,

$$h_{\nu,k}(B^{-1}\lambda) = \frac{\sqrt{2}(2\pi)^{-n/2} j_{\nu,k}^{-n/2} B^{n-1}}{J_{\nu+1}(j_{\nu,k})} (\widehat{d\sigma}_{B^{-1}j_{\nu,k}})(\lambda e).$$

This completes the proof if $f \in \mathcal{S}$.

In the general case $f \in \mathcal{S}'$, we apply a regularization argument. According to the Paley–Wiener theorem, f is a smooth function of exponential type. For $\varepsilon > 0$ let $f(x;\varepsilon) = \gamma(\varepsilon x) f(x)$, where $\gamma \in \mathcal{S}$ satisfies $\gamma(0) = 1$ and $\operatorname{supp}\hat{\gamma} \subset \{\xi : |\xi| \le 1\}$. Then $f(\cdot\,;\varepsilon) \in \mathcal{S}$, so that if $\varepsilon > 0$ is sufficiently small we may apply the above result to get

$$(f(\cdot\,;\varepsilon) * g)(x) = 2B^{-1} \sum_{k=1}^{\infty} (j_{\nu,k} J_{\nu+1}^2(j_{\nu,k}))^{-1} f(B^{-1}j_{\nu,k}e;\varepsilon) g * d\sigma_{B^{-1}j_{\nu,k}}(x).$$

It is elementary now to obtain (7) by taking the limit $\varepsilon \to 0$. $\square$

2. The Radial Wavelet Transform

Throughout this section e denotes a fixed unit vector in $\mathbf{R}^n$. Let $\varphi, \psi \in \mathcal{S}(\mathbf{R}^n)$ be radial functions such that $\operatorname{supp}\hat{\varphi}, \hat{\psi} \subset \{\xi : 1/4 < |\xi| < 1\}$, $|\hat{\varphi}(\xi)|, |\hat{\psi}(\xi)| \ge c > 0$ if $3/10 \le |\xi| \le 5/6$, and

$$\sum_{\mu \in \mathbf{Z}} \overline{\hat{\varphi}(2^{-\mu}\xi)} \hat{\psi}(2^{-\mu}\xi) = 1 \quad \text{for } \xi \ne 0.$$

(Such functions exist—see, e.g., [12, p. 54].) Let $\varphi_\mu(x) = 2^{\mu n}\varphi(2^\mu x)$ and $\psi_\mu(x) = 2^{\mu n}\psi(2^\mu x)$. Recall that if $f \in \mathcal{S}'(\mathbf{R}^n)$ and $\hat{f}$ is supported at the origin, then f must be a polynomial on $\mathbf{R}^n$. Hence, if $f \in \mathcal{S}'$, then the identity

$$f = \sum_{\mu\in\mathbf{Z}} \tilde{\varphi}_\mu * \psi_\mu * f$$

holds in $\mathcal{S}'/\mathcal{P}$, where we use the notation $\tilde{\varphi}_\mu(x) \equiv \overline{\varphi_\mu(-x)}$. Now applying Theorem 1.1 to the functions $\tilde{\varphi}_\mu * f$ and ψ_μ, using $B = 2^\mu$, we obtain

$$f = \sum_{\mu\in\mathbf{Z}} 2^{-\mu+1} \sum_{k=1}^{\infty} (j_{\nu,k} J_{\nu+1}^2(j_{\nu,k}))^{-1} \tilde{\varphi}_\mu * f(2^{-\mu} j_{\nu,k} e)\psi_\mu * d\sigma_{2^{-\mu} j_{\nu,k}}.$$

Since $\tilde{\varphi}_\mu$ and f are radial, so is their convolution $\tilde{\varphi}_\mu * f$. We claim that

$$\tilde{\varphi}_\mu * f(2^{-\mu} j_{\nu,k} e) = \frac{2^{\mu(n-1)}}{j_{\nu,k}^{n-1}\omega_{n-1}} \langle f, \varphi_\mu * d\sigma_{2^{-\mu} j_{\nu,k}}\rangle. \tag{9}$$

By a regularization argument, it suffices to prove this claim for bounded, measurable functions f. In this case

$$\begin{aligned}
\tilde{\varphi}_\mu * f(2^{-\mu} j_{\nu,k} e) &= \omega_{n-1}^{-1} \int \tilde{\varphi}_\mu * f(2^{-\mu} j_{\nu,k} y)\, d\sigma_1(y) \\
&= \omega_{n-1}^{-1} \int\int \tilde{\varphi}_\mu(2^{-\mu} j_{\nu,k} y - x) f(x)\, dx\, d\sigma_1(y) \\
&= \frac{2^{\mu(n-1)}}{j_{\nu,k}^{n-1}\omega_{n-1}} \int f(x) \int \overline{\varphi_\mu(x-y)}\, d\sigma_{2^{-\mu} j_{\nu,k}}(y)\, dx \\
&= \frac{2^{\mu(n-1)}}{j_{\nu,k}^{n-1}\omega_{n-1}} \langle f, \varphi_\mu * d\sigma_{2^{-\mu} j_{\nu,k}}\rangle,
\end{aligned}$$

by an application of Fubini's theorem and a change of variables. Define

$$\varphi_{\mu k} = \left(\frac{2^{(\mu(n-2)+1)}}{j_{\nu,k}^n J_{\nu+1}^2(j_{\nu,k})\omega_{n-1}}\right)^{1/2} \varphi_\mu * d\sigma_{2^{-\mu} j_{\nu,k}}.$$

Similarly define $\psi_{\mu k}$, using ψ_μ in place of φ_μ. Then we obtain the radial wavelet expansion

$$f = \sum_{\mu\in\mathbf{Z}} \sum_{k=1}^{\infty} \langle f, \varphi_{\mu k}\rangle \psi_{\mu k}. \tag{10}$$

Note that $\varphi_{\mu k}(x) = 2^{\mu n/2}\varphi_{0k}(2^\mu x)$ and $\psi_{\mu k}(x) = 2^{\mu n/2}\psi_{0k}(2^\mu x)$.

The remainder of this section is devoted to the characterization of function spaces in terms of the coefficients in the expansion (10). Recall that the homogeneous Triebel–Lizorkin space $\dot{F}_p^{\alpha q}$ is defined to be the set of all $f \in \mathcal{S}'/\mathcal{P}$ for which

$$\|f\|_{\dot{F}_p^{\alpha q}} = \left\| \left(\sum_{\mu \in \mathbf{Z}} (2^{\mu\alpha} |\varphi_\mu * f|)^q \right)^{1/q} \right\|_{L^p(\mathbf{R}^n)} < \infty.$$

(Equivalent norms are obtained from different choices of the function φ satisfying the conditions given above.) The range of allowed indices is $\alpha \in \mathbf{R}$, $0 < q \leq \infty$, and $0 < p < \infty$. Let $\dot{R}_p^{\alpha q}$ denote the subspace of radial elements $f \in \dot{F}_p^{\alpha q}$, with $\|f\|_{\dot{R}_p^{\alpha q}} := \|f\|_{\dot{F}_p^{\alpha q}}$. Since the elements of $\dot{F}_p^{\alpha q}$ are actually equivalence classes modulo polynomials, what we mean by a "radial element" is that there exists a radial distribution in its class. We introduce the nonstandard convention that $j_{\nu,0} = 0$, and we define, for $\mu \in \mathbf{Z}$, $k \in \mathbf{Z}^+$,

$$A_{\mu k} = \{x \in \mathbf{R}^n : 2^{-\mu} j_{\nu,k-1} \leq |x| \leq 2^{-\mu} j_{\nu,k}\}.$$

Thus, if $k = 1$, this set is a ball; otherwise the set is an annulus. Let $\dot{r}_p^{\alpha q}$ denote the space of sequences $s = \{s_{\mu k}\}_{\mu \in \mathbf{Z}, k \in \mathbf{Z}^+}$ for which

$$\|s\|_{\dot{r}_p^{\alpha q}} = \left\| \left(\sum_{\mu \in \mathbf{Z}} \sum_{k=1}^{\infty} (2^{\mu\alpha} |s_{\mu k}| \tilde{\chi}_{A_{\mu k}})^q \right)^{1/q} \right\|_{L^p(\mathbf{R}^n)} < \infty,$$

where $\tilde{\chi}_{A_{\mu k}} = |A_{\mu k}|^{-1/2} \chi_{A_{\mu k}}$ denotes the L^2-normalized characteristic function of $A_{\mu k}$.

2.1. Theorem

Let $\alpha \in \mathbf{R}$, $0 < q \leq \infty$, *and* $0 < p < \infty$. *The operator*

$$S : \dot{R}_p^{\alpha q} \to \dot{r}_p^{\alpha q}$$

defined by

$$S(f) = \{\langle f, \varphi_{\mu k} \rangle\}$$

is bounded.

2.2. Theorem

Let $\alpha \in \mathbf{R}$, $0 < q \leq \infty$, *and* $0 < p < \infty$. *The operator*

$$T : \dot{r}_p^{\alpha q} \to \dot{R}_p^{\alpha q}$$

defined by

$$T(\{s_{\mu k}\}) = \sum_{\mu\in\mathbf{Z}} \sum_{k=1}^{\infty} s_{\mu k}\psi_{\mu k}$$

is bounded.

Note that (10) implies that $T \circ S$ is the identity map. Thus we have the following.

2.1. Corollary

Let $\alpha \in \mathbf{R}$, $0 < q \le \infty$, and $0 < p < \infty$. Then for a radial $f \in \mathcal{S}'(\mathbf{R}^n)$,

$$\|f\|_{\dot{F}_p^{\alpha q}} \approx \|\{\langle f, \varphi_{\mu k}\rangle\}\|_{\dot{r}_p^{\alpha q}}.$$

Using these results we can address the convergence properties of (10).

2.2. Corollary

If $f \in \dot{R}_p^{\alpha q}$, $q < \infty$, and

$$f_n = \sum_{\mu=-n}^{+n} \sum_{k=1}^{\infty} \langle f, \varphi_{\mu k}\rangle \psi_{\mu k},$$

then $f_n \to f$ in $\dot{R}_p^{\alpha q}$ as $n \to \infty$.

To prove this corollary, apply Theorem 2.2 and the dominated convergence theorem to $f - f_n$.

Proof of Theorem 2.1. First, note that because of the asymptotics of $j_{\nu,k}$,

$$|A_{\mu k}| = \frac{\omega_{n-1}}{n}((2^{-\mu} j_{\nu,k})^n - (2^{-\mu} j_{\nu,k-1})^n) \approx 2^{-\mu n} k^{n-1}. \tag{11}$$

Suppose $x \in A_{\mu,k}$. Then using (11), (9), and (8)

$$\begin{aligned}
&2^{\mu\alpha}|\langle f, \varphi_{\mu k}\rangle|\tilde{\chi}_{A_{\mu k}}(x) \\
&\quad \le c 2^{\mu\alpha} 2^{\mu n/2} k^{-(n-1)/2} |\langle f, \varphi_{\mu k}\rangle| \\
&\quad = c 2^{\mu\alpha} 2^{\mu n/2} k^{-(n-1)/2} \left(\frac{2^{(\mu(n-2)+1)}}{j_{\nu,k}^n J_{\nu+1}^2(j_{\nu,k})\omega_{n-1}}\right)^{1/2} |\langle f, \varphi_\mu * d\sigma_{2^{-\mu} j_{\nu,k}}\rangle| \\
&\quad = c 2^{\mu\alpha} 2^{\mu n/2} k^{-(n-1)/2} \left(\frac{2^{(\mu(n-2)+1)}}{j_{\nu,k}^n J_{\nu+1}^2(j_{\nu,k})\omega_{n-1}}\right)^{1/2} \left(\frac{2^{\mu(n-1)}}{j_{\nu,k}^{n-1}\omega_{n-1}}\right)^{-1} |\tilde{\varphi}_\mu * f(2^{-\mu} j_{\nu,k} e)| \\
&\quad \le c 2^{\mu\alpha} \left|\tilde{\varphi}_\mu * f\left(2^{-\mu} j_{\nu,k} \frac{x}{|x|}\right)\right|.
\end{aligned}$$

(The constants c may change from line to line but depend only on the dimension n.) Since $x \in A_{\mu k}$,

$$\left| x - 2^{-\mu} j_{\nu,k} \frac{x}{|x|} \right| \leq 2^{-\mu}(j_{\nu,k} - j_{\nu,k-1}) \leq c2^{-\mu}.$$

Therefore, for every $\gamma > 0$, there exists a constant $c_\gamma < \infty$ such that

$$2^{\mu\alpha} \left| \tilde{\varphi}_\mu * f\left(2^{-\mu} j_{\nu,k} \frac{x}{|x|}\right) \right| \leq c_\gamma 2^{\mu\alpha} \sup_{y \in \mathbf{R}^n} \frac{|\tilde{\varphi}_\mu * f(x-y)|}{(1+2^\mu |y|)^\gamma} := c_\gamma 2^{\mu\alpha} \varphi_\mu^{**} f(x).$$

The last object is Peetre's maximal function [18]. Now with μ fixed,

$$\left(\sum_{k=1}^{\infty} (2^{\mu\alpha} |\langle f, \varphi_{\mu,k} \rangle| \tilde{\chi}_{A_{\mu k}}(x))^q \right)^{1/q} \leq c |2^{\mu\alpha} \varphi_\mu^{**} f(x)| \quad \text{a.e.,}$$

by the essential disjointness of the sets $\{A_{\mu k}\}_{k=1}^{\infty}$. Hence, by Peetre's inequality [18] (for γ chosen sufficiently large)

$$\|S(f)\|_{\dot{r}_p^{\alpha q}} \leq c \left\| \left(\sum_{\mu \in \mathbf{Z}} |2^{\mu\alpha} \varphi_\mu^{**} f|^q \right)^{1/q} \right\|_{L^p} \leq c \|f\|_{\dot{R}_p^{\alpha q}}. \qquad \square$$

For the proof of Theorem 2.2 we require three lemmas. Let M denote the Hardy–Littlewood maximal operator, whose action on $f \in L^1_{\text{loc}}(\mathbf{R}^n)$ is given by

$$Mf(x) = \sup_B |B|^{-1} \int_B |f|,$$

where the supremum is taken over all balls in $\mathbf{R}^n$ centered at x. Note that this version of the maximal operator maps radial functions to radial functions.

2.1. Lemma

Let $e \in S^{n-1}$. There exists $c > 0$, independent of $k \in \mathbf{N}$, such that

$$M\chi_{A_{0k}}(re) \geq c(1 + |r - j_{\nu,k}|)^{-n}.$$

Proof. This is easy if $k = 1$, so assume $k \geq 2$. Suppose $0 < a < b$, and let $A(a, b)$ denote the set $\{x \in \mathbf{R}^n : a \leq |x| \leq b\}$. Also let $B(x, r)$ denote the ball of radius r, centered at $x \in \mathbf{R}^n$. If $r > 0$, then we have

$$B\left(\frac{a+b}{2}e, \frac{b-a}{2}\right) \subset A(a, b) \cap B(re, b - a + |a - r|).$$

Hence

$$M\chi_{A(a,b)}(re) \geq |B(re, b-a+|a-r|)|^{-1} \int_{B(re,b-a+|a-r|)} \chi_{A(a,b)}(y)dy$$
$$\geq \frac{|B((a+b)e/2, (b-a)/2)|}{|B(re, b-a+|a-r|)|} = \frac{((b-a)/2)^n}{(b-a+|a-r|)^n}.$$

Applying this with $a = j_{\nu,k-1}$ and $b = j_{\nu,k}$, and using the asymptotics of $j_{\nu,k}$, the result is

$$M\chi_{A_{0k}}(re) \geq \frac{2^{-n}|j_{\nu,k} - j_{\nu,k-1}|^n}{(j_{\nu,k} - j_{\nu,k-1} + |r - j_{\nu,k-1}|)^n} \geq c(1+|r-j_{\nu,k}|)^{-n}. \qquad \square$$

In §3 we will require sharper estimates on $M\chi_{A_{0k}}$. However, the above elementary result suffices for this section.

2.2. Lemma

Let $e \in S^{n-1}$, and let $0 < \eta \leq 1$. For every $m > n/\eta + 1$, there exists a constant $c < \infty$, depending only on n, η, and m, such that for every $\mu \in \mathbf{Z}$, $\{s_k\}_{k \in \mathbf{Z}^+}$, and $r > 0$

$$\sum_{k=1}^{\infty} |s_k|(1 + 2^{\mu}|r - 2^{-\mu} j_{\nu,k}|)^{-m} \leq c \left(M \left(\sum_{k=1}^{\infty} |s_k|^{\eta} \chi_{A_{\mu k}} \right) \right)^{1/\eta} (re).$$

Proof. It suffices to prove the lemma for $\mu = 0$, because that case implies

$$\sum_{k=1}^{\infty} |s_k|(1 + 2^{\mu}|r - 2^{-\mu} j_{\nu,k}|)^{-m} = \sum_{k=1}^{\infty} |s_k|(1 + |2^{\mu} r - j_{\nu,k}|)^{-m}$$
$$\leq c \left(M \left(\sum_{k=1}^{\infty} |s_k|^{\eta} \chi_{A_{0k}} \right) \right)^{1/\eta} (2^{\mu} re)$$
$$= c \left(M \left(\sum_{k=1}^{\infty} |s_k|^{\eta} \chi_{A_{\mu k}} \right) \right)^{1/\eta} (re).$$

For the $\mu = 0$ case we use Lemma 2.1,

$$|s_k|(1 + |r - j_{\nu,k}|)^{-m} = ((1 + |r - j_{\nu,k}|)^{-m\eta+n} |s_k|^{\eta} (1 + |r - j_{\nu,k}|)^{-n})^{1/\eta}$$
$$\leq c(1 + |r - j_{\nu,k}|)^{-m+n/\eta} (M(|s_k|^{\eta} \chi_{A_{0k}}))^{1/\eta}(re)$$
$$\leq c(1 + |r - j_{\nu,k}|)^{-m+n/\eta} \left(M \left(\sum_{l=1}^{\infty} |s_l|^{\eta} \chi_{A_{0l}} \right) \right)^{1/\eta} (re).$$

Now we sum over k. Since $m > n/\eta + 1$, there exists a constant $c < \infty$ independent of r such that

$$\sum_{k=1}^{\infty}(1 + |r - j_{\nu,k}|)^{-m+n/\eta} < c. \qquad \square$$

2.3. Lemma

Suppose $f : \mathbf{R}^n \to \mathbf{C}$ is radial. If $m > 0$ and $|f(x)| \le (1+|x|)^{-(m+n-1)}$ for all $x \in \mathbf{R}^n$, then for every $\epsilon > 0$ there exists a constant $c < \infty$ such that

$$|f * d\sigma_t(x)| \le c(1 + |\,|x| - t|)^{-m+\epsilon}$$

for all $x \in \mathbf{R}^n$.

Proof. If $n = 1$, then we trivially get the bound $|f * d\sigma_t(x)| \le 2(1 + |\,|x| - t|)^{-m}$, so assume $n \ge 2$. Since $f * d\sigma_t$ is radial, it suffices to check the estimate at $x \in \mathbf{R}^n$ of the form $re_1 = (r, 0, \ldots, 0)$. Let $d\sigma_{r,t}$ denote the Lebesgue surface measure on the sphere $S(r,t) = \{x \in \mathbf{R}^n : |x - re_1| = t\}$. We have

$$f * d\sigma_t(re_1) = \int f(y)\, d\sigma_{r,t}(y).$$

Using $|f(y)| \le c(1+|y|^2)^{-(m+n-1)/2}$ and the law of cosines $|y|^2 = r^2 + t^2 - 2rt\cos\theta$, where θ is the angle between $\overline{0\,re_1}$ and $\overline{re_1\,y}$, we get

$$\begin{aligned}
|f * d\sigma_t(re_1)| &\le \int_0^{\pi} c(1 + r^2 + t^2 - 2rt\cos\theta)^{-(m+n-1)/2}\omega_{n-2}(t\sin\theta)^{n-2}t\, d\theta \\
&= \int_0^{\pi/2}(\cdot)d\theta + \int_{\pi/2}^{\pi}(\cdot)\, d\theta \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Clearly $\mathrm{II} \le \mathrm{I}$. If $t < 2/\pi$, then using

$$r^2 + t^2 - 2rt\cos\theta \ge (r-t)^2 \quad \text{for } 0 \le \theta \le \pi/2, \tag{12}$$

we get

$$\mathrm{I} \le c(1 + (r-t)^2)^{-(m+n-1)/2} \le c(1 + |r-t|)^{-(m+n-1)},$$

which is more than satisfactory. If $t \ge 2/\pi$, then we write

$$\mathrm{I} = \int_0^{1/t}(\cdot)\, d\theta + \int_{1/t}^{\pi/2}(\cdot)\, d\theta = \mathrm{III} + \mathrm{IV}.$$

Using $\sin\theta \le \theta$ and (12),

$$\mathrm{III} \le c(1+(r-t)^2)^{-(m+n-1)/2} t^{n-1} \int_0^{1/t} \theta^{n-2}\,d\theta \le c(1+|r-t|)^{-(m+n-1)}.$$

For IV, we use (12), $\sin\theta \le \theta$, and

$$r^2+t^2-2rt\cos\theta \ge (r^2+t^2)(1-\cos\theta) \ge ct^2\theta^2, \qquad 0 \le \theta \le \pi/2,$$

to obtain for $\epsilon > 0$

$$\begin{aligned} \mathrm{IV} &\le \int_{1/t}^{\pi/2} c(1+(r-t)^2)^{-(m-\epsilon)/2}(t^2\theta^2)^{-(n-1+\epsilon)/2} t^{n-1}\theta^{n-2}\,d\theta \\ &\le c(1+|r-t|)^{-m+\epsilon} t^{-\epsilon} \int_{1/t}^{\pi/2} \theta^{-(1+\epsilon)}\,d\theta \\ &\le c(1+|r-t|)^{-m+\epsilon}. \qquad \square \end{aligned}$$

From Lemma 2.3 we see that for every $m > 0$ and every multi-index γ,

$$|D^\gamma \varphi_{\mu k}(x)|, |D^\gamma \psi_{\mu k}(x)| \le c_{\gamma,m} 2^{\mu n/2} j_{\nu,k}^{-(n-1)/2} 2^{\mu|\gamma|}(1+2^\mu||x|-2^{-\mu} j_{\nu,k}|)^{-m},$$

where $c_{\gamma,m}$ is a constant independent of μ and k.

Proof of Theorem 2.2. Let

$$f = \sum_{\mu\in\mathbf{Z}} \sum_{k=1}^{\infty} s_{\mu k}\psi_{\mu k}.$$

Since

$$\operatorname{supp} \hat{\psi}_{\mu k}, \hat{\varphi}_\mu \subset \{\xi : 2^{\mu-2} \le |\xi| \le 2^\mu\},$$

we have $\varphi_\mu * \psi_{\lambda k} = 0$ unless $\lambda = \mu - 1$, μ, or $\mu + 1$. Thus

$$\varphi_\mu * f(re) = \sum_{\lambda=\mu-1}^{\mu+1} \sum_{k=1}^{\infty} s_{\lambda k}\, \varphi_\mu * \psi_{\lambda k}(re).$$

Now, by writing out the convolutions and changing variables, we see that

$$\begin{aligned}\varphi_\mu * \psi_{\lambda k}(re) &= \left(\frac{2^{(\lambda(n-2)+1)}}{j_{\nu,k}^n J_{\nu+1}^2(j_{\nu,k})\omega_{n-1}}\right)^{1/2} \varphi_\mu * \psi_\lambda * d\sigma_{2^{-\lambda} j_{\nu,k}}(re) \\ &= \left(\frac{2^{(\lambda n+1)}}{j_{\nu,k}^n J_{\nu,k}^2(j_{\nu,k})\omega_{n-1}}\right)^{1/2} \varphi_\zeta * \psi_0 * d\sigma_{j_{\nu,k}}(2^\lambda re),\end{aligned}$$

where $\zeta = \mu - \lambda$. Restricting to $\zeta = -1, 0, 1$, we have

$$|\varphi_\zeta * \psi_0(y)| \le c_m(1+|y|)^{-(m+n)}$$

for all $m > 0$, since $\varphi_{-1}, \varphi_0, \varphi_1, \psi_0 \in \mathcal{S}$. According to Lemma 2.3 this implies

$$|\varphi_\zeta * \psi_0 * d\sigma_{j_{\nu,k}}(re)| \le c_m(1+|r-j_{\nu,k}|)^{-m}.$$

Hence, by (8),

$$|\varphi_\mu * \psi_{\lambda k}(re)| \le c2^{\lambda n/2}k^{-(n-1)/2}(1+2^\lambda|r-2^{-\lambda}j_{\nu,k}|)^{-m}.$$

Choose $\eta \in (0, 1]$ such that $p/\eta, q/\eta > 1$. By Lemma 2.2

$$\begin{aligned}|\varphi_\mu * f(re)| &\le c\sum_{\lambda=\mu-1}^{\mu+1}\sum_{k=1}^{\infty} 2^{\lambda n/2}k^{-(n-1)/2}|s_{\lambda k}|(1+2^\lambda|r-2^{-\lambda}j_{\nu,k}|)^{-m} \\ &\le c\sum_{\lambda=\mu-1}^{\mu+1}(M(\sum_{k=1}^{\infty}2^{\lambda n\eta/2}k^{-(n-1)\eta/2}|s_{\lambda k}|^\eta\chi_{A_{\lambda k}}))^{1/\eta}(re).\end{aligned}$$

So, we have

$$\begin{aligned}\|f\|_{\dot{R}_p^{\alpha q}} &= \left\|\left(\sum_{\mu\in\mathbf{Z}}(2^{\mu\alpha}|\varphi_\mu * f|)^q\right)^{1/q}\right\|_{L^p} \\ &\le c\left\|\left(\sum_{\mu\in\mathbf{Z}}\left(2^{\mu\alpha}\sum_{\lambda=\mu-1}^{\mu+1}\left(M\left(\sum_{k=1}^{\infty}2^{\lambda n\eta/2}k^{-\eta(n-1)/2}|s_{\lambda k}|^\eta\chi_{A_{\lambda k}}\right)\right)^{1/\eta}\right)^q\right)^{1/q}\right\|_{L^p} \\ &\le c\left\|\left(\sum_{\mu\in\mathbf{Z}}\left(M\left(\sum_{k=1}^{\infty}(2^{\mu\alpha}|s_{\mu k}|\tilde{\chi}_{A_{\mu k}})^\eta\right)\right)^{q/\eta}\right)^{\eta/q}\right\|_{L^{p/\eta}}^{1/\eta}.\end{aligned}$$

Applying the Fefferman–Stein vector-valued maximal inequality [7], since $p/\eta, q/\eta > 1$, we remove the operator M and use the essential disjointness of the sets $\{A_{\mu k}\}_{k=1}^{\infty}$ for μ fixed, with the result

$$\begin{aligned}\|f\|_{\dot{R}_p^{\alpha q}} &\leq \left\| \left(\sum_{\mu\in\mathbf{Z}} \left(\sum_{k=1}^{\infty} (2^{\mu\alpha}|s_{\mu k}|\tilde{\chi}_{A_{\mu k}})^{\eta} \right)^{q/\eta} \right)^{\eta/q} \right\|_{L^{p/\eta}}^{1/\eta} \\ &= c \left\| \left(\sum_{\mu\in\mathbf{Z}} \sum_{k=1}^{\infty} (2^{\mu\alpha}|s_{\mu k}|\tilde{\chi}_{A_{\mu k}})^{q} \right)^{1/q} \right\|_{L^p} \\ &= c\|\{s_{\mu k}\}\|_{\dot{r}_p^{\alpha q}}. \qquad \square\end{aligned}$$

3. Smooth Radial Molecules

The purpose of this section is to describe a family of smooth radial molecules for $\dot{R}_p^{\alpha q}$. We assume that $\alpha \in \mathbf{R}$, $0 < q \leq \infty$, and $0 < p < \infty$ are *fixed*. Let $0 < \eta \leq 1$ be such that $p/\eta, q/\eta > 1$, let $J > \max\{n/\eta, n/\eta - \alpha\}$, let $S > n/\eta$, and let l be the smallest integer in the set $\{-1, 0, 1, 2, \ldots\}$ such that $n + l + 1 + \alpha - n/\eta > 0$. Also let $N = [\alpha]$, and let $\delta \in (0, 1]$ be such that $N + \delta > \alpha$. A radial function $m_{\mu k} : \mathbf{R}^n \to \mathbf{C}$ is said to be a *smooth radial molecule over* $A_{\mu k}$ *for* $\dot{R}_p^{\alpha q}$ if

$$\int_{\mathbf{R}^n} m_{\mu k}(x)|x|^a \, dx = 0, \qquad a = 0, 1, \ldots, l; \tag{13}$$

$$|m_{\mu k}(x)| \leq 2^{\mu n/2} j_{\nu,k}^{-(n-1)/2} (1 + 2^{\mu} |\,|x| - 2^{-\mu} j_{\nu,k}|)^{-J}; \tag{14}$$

$$|D^{\gamma} m_{\mu k}(x)| \leq 2^{\mu n/2} j_{\nu,k}^{-(n-1)/2} 2^{\mu|\gamma|} (1 + 2^{\mu} |\,|x| - 2^{-\mu} j_{\nu,k}|)^{-S}, \qquad |\gamma| \leq N; \tag{15}$$

$$\begin{aligned}|D^{\gamma} m_{\mu k}(x) - D^{\gamma} m_{\mu k}(x')| \leq\ & 2^{\mu n/2} j_{\nu,k}^{-(n-1)/2} 2^{\mu|\gamma|} 2^{\mu\delta} |x - x'|^{\delta} \\ & \cdot \sup_{0<\theta<1} (1 + 2^{\mu} |\,|x + \theta(x' - x)| - 2^{-\mu} j_{\nu,k}|)^{-S}, \qquad |\gamma| = N.\end{aligned} \tag{16}$$

The superscript γ denotes an ordinary multi-index. Our convention is that (13) is void if $l = -1$ and that (15) and (16) are void if $N < 0$. Note that the functions $\varphi_{\mu k}$, $\psi_{\mu k}$ from §2 satisfy, to within a constant, (14) through (16), but not necessarily (13).

3.1. Theorem

Let $\alpha \in \mathbf{R}$, $0 < q \leq \infty$, and $0 < p < \infty$. Suppose that for each $\mu \in \mathbf{Z}$ and $k \in \mathbf{Z}^+$, $m_{\mu k}$ is a smooth radial molecule over $A_{\mu k}$ for $\dot{R}_p^{\alpha q}$. Then there exists a constant $c < \infty$ independent of $\{s_{\mu k}\}$ and $\{m_{\mu k}\}$ such that

$$\left\| \sum_{\mu \in \mathbf{Z}} \sum_{k \in \mathbf{Z}^+} s_{\mu k} m_{\mu k} \right\|_{\dot{R}_p^{\alpha q}} \leq c \|\{s_{\mu k}\}\|_{\dot{r}_p^{\alpha q}}.$$

The proof follows the method of proof for the φ-transform in [10]. We require estimates on $|\varphi_\mu * m_{\lambda k}(x)|$, followed by some maximal function estimates. First, though, we make some observations about using the vanishing moments hypothesis (13). Since the molecules are radial, (13) implies that for every $y' \in S^{n-1}$ and $a = 0, 1, \ldots, l$,

$$\int_0^\infty m_{\lambda k}(ry') r^{a+n-1} dr = 0. \tag{17}$$

For $x \in \mathbf{R}^n$, $y' \in S^{n-1}$, define

$$\begin{aligned} h_{\mu,x,y'}(r) &= \varphi_\mu(x - ry'), \\ h_{\mu,x,y'}^{(a)} &= \frac{d^a}{dr^a}(h_{\mu,x,y'}). \end{aligned}$$

Now using the vanishing moments condition (17),

$$\begin{aligned} \varphi_\mu * m_{\lambda k}(x) &= \int_{S^{n-1}} \int_0^\infty h_{\mu,x,y'}(r) m_{\lambda k}(ry') r^{n-1} dr d\sigma_1(y') \qquad (18) \\ &= \int_{S^{n-1}} \int_0^\infty \left(h_{\mu,x,y'}(r) - \sum_{a=0}^{l} h_{\mu,x,y'}^{(a)}(2^{-\lambda} j_{\nu,k}) \frac{(r - 2^{-\lambda} j_{\nu,k})^a}{a!} \right) m_{\lambda k}(ry') r^{n-1}\, dr\, d\sigma_1(y'). \end{aligned}$$

There are two ways of estimating $|\varphi_\mu * m_{\lambda k}|$ using (18). First, we claim that if $0 < \beta \leq 1$, then

$$\begin{aligned} &\left| h_{\mu,x,y'}(r) - \sum_{a=0}^{l} h_{\mu,x,y'}^{(a)}(2^{-\lambda} j_{\nu,k}) \frac{(r - 2^{-\lambda} j_{\nu,k})^a}{a!} \right| \qquad (19) \\ &\quad \leq c|r - 2^{-\lambda} j_{\nu,k}|^{l+\beta} 2^{\mu(n+l+\beta)} \sup_{0<\theta<1} (1 + 2^\mu |x - 2^{-\lambda} j_{\nu,k} y' - \theta(r - 2^{-\lambda} j_{\nu,k}) y'|)^{-L}, \end{aligned}$$

where L can be taken arbitrarily large. To prove (19), note that

$$h_{\mu,x,y'}^{(a)}(r) = \sum_{\gamma:|\gamma|=a} (-1)^a (D^\gamma \varphi_\mu)(x - ry')(y')^\gamma.$$

Therefore, using Taylor's theorem, we have

$$\begin{aligned}
&\left| h_{\mu,x,y'}(r) - \sum_{a=0}^{l-1} h^{(a)}_{\mu,x,y'}(2^{-\lambda} j_{\nu,k}) \frac{(r-2^{-\lambda} j_{\nu,k})^a}{a!} - h^{(l)}_{\mu,x,y'}(2^{-\lambda} j_{\nu,k}) \frac{(r-2^{-\lambda} j_{\nu,k})^l}{l!} \right| \\
&\quad = \left| h^{(l)}_{\mu,x,y'}(\xi) \frac{(r-2^{-\lambda} j_{\nu,k})^l}{l!} - h^{(l)}_{\mu,x,y'}(2^{-\lambda} j_{\nu,k}) \frac{(r-2^{-\lambda} j_{\nu,k})^l}{l!} \right| \\
&\quad \le c|r-2^{-\lambda} j_{\nu,k}|^l \sum_{|\gamma|=l} |D^\gamma \varphi_\mu(x-\xi y') - D^\gamma \varphi_\mu(x-2^{-\lambda} j_{\nu,k} y')| \\
&\quad = c|r-2^{-\lambda} j_{\nu,k}|^l \sum_{|\gamma|=l} 2^{\mu(n+l)} |D^\gamma \varphi(2^\mu(x-\xi y')) - D^\gamma \varphi(2^\mu(x-2^{-\lambda} j_{\nu,k} y'))|,
\end{aligned}$$

where ξ is some number contained in the open interval I with endpoints r and $2^{-\lambda} j_{\nu,k}$. If $2^\mu|r-2^{-\lambda} j_{\nu,k}| \le 1$, then by the mean value theorem and the fact that $\varphi \in \mathcal{S}$, we have

$$\begin{aligned}
&|D^\gamma \varphi(2^\mu(x-\xi y')) - D^\gamma \varphi(2^\mu(x-2^{-\lambda} j_{\nu,k} y'))| \\
&\quad \le 2^\mu |\xi - 2^{-\lambda} j_{\nu,k}| \sup_{0<\theta<1} |D(D^\gamma \varphi)((1-\theta)2^\mu(x-\xi y') + \theta 2^\mu(x-2^{-\lambda} j_{\nu,k} y'))| \\
&\quad \le c(2^\mu|2^{-\lambda} j_{\nu,k} - r|)^\beta \sup_{\theta \in I} (1+2^\mu|x-\theta y'|)^{-L}.
\end{aligned}$$

On the other hand, if $2^\mu|2^{-\lambda} j_{\nu,k} - r| > 1$, then we have

$$|D^\gamma \varphi(2^\mu(x-\xi y')) - D^\gamma \varphi(2^\mu(x-2^{-\lambda} j_{\nu,k} y'))| \le c \sup_{\theta \in I} (1+2^\mu|x-\theta y'|)^{-L},$$

which suffices for (19). The other way of estimating (18) is simply to use the triangle inequality:

$$\begin{aligned}
&\left| h_{\mu,x,y'}(r) - \sum_{a=0}^{l} h^{(a)}_{\mu,x,y'}(2^{-\lambda} j_{\nu,k}) \frac{(r-2^{-\lambda} j_{\nu,k})^a}{a!} \right| \qquad (20) \\
&\quad \le c2^{\mu n}(1+2^\mu|x-ry'|)^{-L} + c\sum_{a=0}^{l} |r-2^{-\lambda} j_{\nu,k}|^a 2^{\mu(n+a)} (1+2^\mu|x-2^{-\lambda} j_{\nu,k} y'|)^{-L},
\end{aligned}$$

where L may be taken arbitrarily large. In general, (19) gives the better estimate if $|r-2^{-\lambda} j_{\nu,k}| \le 2^{-\mu}$ while (20) gives the better estimate if $|r-2^{-\lambda} j_{\nu,k}| > 2^{-\mu}$. Note that (19) and (20) are still correct when $l = -1$, if we take $\beta = 1$.

We will repeatedly use the next elementary lemma.

3.1. Lemma

Suppose $s, t \ge 0$ and $L > s+t+1$. Then there exists a constant $c < \infty$ independent of λ, k such that

$$\int_0^\infty \frac{|r-2^{-\lambda} j_{\nu,k}|^s r^t \, dr}{(1+2^\lambda|r-2^{-\lambda} j_{\nu,k}|)^L} \le c2^{-\lambda(s+t+1)} j^t_{\nu,k}.$$

Proof. By a change of variables, the integral in the lemma equals

$$2^{-\lambda(s+t+1)} \int_0^\infty \frac{|r - j_{\nu,k}|^s r^t \, dr}{(1 + |r - j_{\nu,k}|)^L}.$$

We split the last integral into three parts. First,

$$\int_0^{j_{\nu,k}} \frac{|r - j_{\nu,k}|^s r^t \, dr}{(1 + |r - j_{\nu,k}|)^L} = \int_0^{j_{\nu,k}} \frac{r^s (j_{\nu,k} - r)^t \, dr}{(1 + r)^L} \le j_{\nu,k}^t \int_0^\infty \frac{r^s \, dr}{(1 + r)^L} \le c j_{\nu,k}^t.$$

Similarly,

$$\int_{j_{\nu,k}}^{2 j_{\nu,k}} \frac{|r - j_{\nu,k}|^s r^t \, dr}{(1 + |r - j_{\nu,k}|)^L} = \int_0^{j_{\nu,k}} \frac{r^s (r + j_{\nu,k})^t \, dr}{(1 + r)^L} \le c j_{\nu,k}^t \int_0^\infty \frac{r^s dr}{(1 + r)^L} \le c j_{\nu,k}^t.$$

Finally,

$$\int_{2 j_{\nu,k}}^\infty \frac{|r - j_{\nu,k}|^s r^t \, dr}{(1 + |r - j_{\nu,k}|)^L} \le c \int_{2 j_{\nu,k}}^\infty r^{s+t-L} \, dr \le c j_{\nu,k}^{s+t-L+1} \le c \le c j_{\nu,k}^t. \qquad \square$$

To state the next two lemmas we require more notation. Let $x \in \mathbf{R}^n$ be fixed, and define

$$\begin{aligned} A_\lambda &= \{k \in \mathbf{N} : 2 \cdot 2^{-\lambda} j_{\nu,k} \le |x|\}, \\ B_\lambda &= \{k \in \mathbf{N} : 2^{-\lambda} j_{\nu,k-1}/2 < |x| < 2 \cdot 2^{-\lambda} j_{\nu,k}\}, \\ C_\lambda &= \{k \in \mathbf{N} : |x| \le 2^{-\lambda} j_{\nu,k-1}/2\}. \end{aligned}$$

Let $\beta = \max\{k : k \in B_\lambda\}$, and let T be the smallest integer such that $2^T > 2 j_{\nu,\beta+1}$. Let

$$S_{\lambda 0} = \{k \in B_\lambda : |\,|x| - 2^{-\lambda} j_{\nu,k}| \le 2^{-\lambda}\}$$

and, for $m = 1, 2, \ldots, T$ let

$$S_{\lambda m} = \{k \in B_\lambda : 2^{m-1-\lambda} < |\,|x| - 2^{-\lambda} j_{\nu,k}| \le 2^{m-\lambda}\}.$$

Note that

$$B_\lambda = \bigcup_{m=0}^T S_{\lambda m}.$$

3.2. Lemma

Suppose $\mu \le \lambda$. Then there exist constants $R > n/\eta - \alpha$, $s > n/\eta$, and $c < \infty$, such that

i. *if $k \in A_\lambda$, then*

$$|\varphi_\mu * m_{\lambda k}(x)| \le c2^{(\mu-\lambda)R}(1+2^\mu|x|)^{-s}2^{\lambda n/2}j_{\nu,k}^{(n-1)/2};$$

ii. *if $m \in \{0,\dots,T\}$ and $k \in S_{\lambda m}$, then*

$$|\varphi_\mu * m_{\lambda k}(x)| \le c2^{(\mu-\lambda)(R+1-n)}(1+2^{m+\mu-\lambda})^{-s}2^{\lambda n/2}j_{\nu,k}^{-(n-1)/2};$$

iii. *if $k \in C_\lambda$, then*

$$|\varphi_\mu * m_{\lambda k}(x)| \le c2^{(\mu-\lambda)R}(1+2^{\mu-\lambda}j_{\nu,k})^{-s}2^{\lambda n/2}j_{\nu,k}^{(n-1)/2}.$$

Here c is independent of μ, λ, k, x, and in (ii) m.

Proof. Let $\rho^- = \max\{0, 2^{-\lambda}j_{\nu,k} - 2^{-\mu}\}$ and $\rho^+ = 2^{-\lambda}j_{\nu,k} + 2^{-\mu}$. From (18) and (14)

$$\begin{aligned}|\varphi_\mu * m_{\lambda k}(x)| \\ &\le \int_0^{\rho^-}\int_{S^{n-1}}(\cdot)\,d\sigma_1(y')\,dr + \int_{\rho^-}^{\rho^+}\int_{S^{n-1}}(\cdot)\,d\sigma_1(y')\,dr + \int_{\rho^+}^{\infty}\int_{S^{n-1}}(\cdot)\,d\sigma_1(y')\,dr \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III},\end{aligned}$$

where

$$\begin{aligned}(\cdot) = &\left| h_{\mu,x,y'}(r) - \sum_{a=0}^{l} h^{(a)}_{\mu,x,y'}(2^{-\lambda}j_{\nu,k})\frac{(r-2^{-\lambda}j_{\nu,k})^a}{a!}\right| \\ &\cdot 2^{\lambda n/2}j_{\nu,k}^{-(n-1)/2}(1+2^\lambda|r-2^{-\lambda}j_{\nu,k}|)^{-J}r^{n-1}.\end{aligned}$$

We consider II first. Then $|r - 2^{-\lambda}j_{\nu,k}| \le 2^{-\mu}$. Hence for $y' \in S^{n-1}$,

$$\sup_{0<\theta<1}(1+2^\mu|x-2^{-\lambda}j_{\nu,k}y' - \theta(r-2^{-\lambda}j_{\nu,k})y'|)^{-L} \le c(1+2^\mu|x-2^{-\lambda}j_{\nu,k}y'|)^{-L}. \tag{21}$$

Let

$$\begin{aligned}(A) &= \int_{S^{n-1}}(1+2^\mu|x-2^{-\lambda}j_{\nu,k}y'|)^{-L}\,d\sigma_1(y') \\ (B) &= \int_0^\infty |r-2^{-\lambda}j_{\nu,k}|^{l+\beta}(1+2^\lambda|r-2^{-\lambda}j_{\nu,k}|)^{-J}r^{n-1}\,dr.\end{aligned}$$

We will take $\beta \in (0, 1]$ such that $n/\eta - \alpha < n + l + \beta < J$. This is possible, since $n + l \leq n/\eta - \alpha < n + l + 1$ and $n/\eta - \alpha < J$. Note that if $l = -1$, then we can (and will) take $\beta = 1$, since in this case we have $n - 1 \leq n/\eta - \alpha < n < J$. Now, by (19), (21), and Lemma 3.1,

$$\begin{aligned} \mathrm{II} &\leq c2^{\mu(n+l+\beta)} 2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2}(A)(B) \\ &\leq c2^{(\mu-\lambda)(n+l+\beta)} 2^{\lambda n/2} j_{\nu,k}^{(n-1)/2}(A). \end{aligned} \tag{22}$$

If $k \in A_\lambda$, then for all $y' \in S^{n-1}$, $|x - 2^{-\lambda} j_{\nu,k} y'| \geq |x|/2$, so

$$(A) \leq c(1 + 2^\mu |x|)^{-L}.$$

For the case $k \in S_{\lambda m}$, we use Lemma 2.3 and a change of variables to obtain

$$\int_{S^{n-1}} (1 + 2^\mu |x - ry'|)^{-L} \, d\sigma_1(y') \leq c(2^\mu r)^{-(n-1)} (1 + 2^\mu | \, |x| - r|)^{-L+n}. \tag{23}$$

Thus, for $k \in S_{\lambda m}$, we have

$$\begin{aligned} (A) &\leq c(2^{\mu-\lambda} j_{\nu,k})^{-(n-1)} (1 + 2^\mu | \, |x| - 2^{-\lambda} j_{\nu,k}|)^{-L+n} \\ &\leq c(2^{\mu-\lambda} j_{\nu,k})^{-(n-1)} (1 + 2^{m+\mu-\lambda})^{-L+n}. \end{aligned}$$

(This holds even for $m = 0$ since $\lambda \geq \mu$.) If $k \in C_\lambda$, then for $y' \in S^{n-1}$, $|x - 2^{-\lambda} j_{\nu,k} y'| \geq 2^{-\lambda} j_{\nu,k}/2$, so

$$(A) \leq c(1 + 2^{\mu-\lambda} j_{\nu,k})^{-L}.$$

Substituting these estimates above, taking $L > n + n/\eta$, and recalling that $n + l + \beta > n/\eta - \alpha$, we obtain satisfactory estimates for II in all cases.

We now consider III. In this case $2^\mu |r - 2^{-\lambda} j_{\nu,k}| \geq 1$, so the term with $a = l$ in the sum over a on the right side of (20) dominates the terms with $a = 0, 1, \ldots, l - 1$. Thus $\mathrm{III} = \mathrm{IV} + \mathrm{V}$, where

$$\mathrm{IV} \leq c2^{\mu n} \int_{\rho^+}^{\infty} \int_{S^{n-1}} (1 + 2^\mu |x - ry'|)^{-L} 2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2} (1 + 2^\lambda |r - 2^{-\lambda} j_{\nu,k}|)^{-J} \, d\sigma_1(y') r^{n-1} \, dr,$$

and, with (A) as above,

$$\mathrm{V} \leq c2^{\mu(n+l)} 2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2} (A)(D),$$

where

$$(D) = \int_{\rho^+}^{\infty} |r - 2^{-\lambda} j_{\nu,k}|^l (1 + 2^\lambda |r - 2^{-\lambda} j_{\nu,k}|)^{-J} r^{n-1} \, dr.$$

By the change of variables $2^\lambda r - j_{\nu,k} \to r$,

$$\begin{aligned}(D) &= \int_{2^{\lambda-\mu}}^{\infty} 2^{-\lambda(n+l)} r^l (1+r)^{-J} (r + j_{\nu,k})^{n-1}\, dr \\ &\le c2^{-\lambda(n+l)} \int_{2^{\lambda-\mu}}^{\infty} (r^{l+n-1-J} + j_{\nu,k}^{n-1} r^{l-J})\, dr \\ &\le c2^{-\lambda(n+l)} (2^{(\mu-\lambda)(J-n-l)} + j_{\nu,k}^{n-1} 2^{(\mu-\lambda)(J-l)}) \\ &\le c2^{-\lambda(n+l)} j_{\nu,k}^{n-1} 2^{(\mu-\lambda)(J-n-l)}. \end{aligned} \tag{24}$$

Hence $\mathrm{V} \le c2^{(\mu-\lambda)J} 2^{\lambda n/2} j_{\nu,k}^{(n-1)/2}(A)$. This is the same as (22), with J in place of $n + l + \beta$. Since $J > n/\eta - \alpha$, the estimates above for (A) yield the desired estimates for V.

To estimate IV, let

$$(E) = \int_{S^{n-1}} (1 + 2^\mu |x - ry'|)^{-L}\, d\sigma_1(y').$$

If $k \in C_\lambda$, then for $r > \rho^+ > 2^{-\lambda} j_{\nu,k} > 2|x|$, we have $|x - ry'| \ge cr$ for $y' \in S^{n-1}$, so

$$(E) \le c(1 + 2^\mu r)^{-L} \le c(1 + 2^{\mu-\lambda} j_{\nu,k})^{-L},$$

as for (A) above. Using the $l = 0$ case of (24), we obtain

$$\mathrm{IV} \le c2^{(\mu-\lambda)J} (1 + 2^{\mu-\lambda} j_{\nu,k})^{-L+n} 2^{\lambda n/2} j_{\nu,k}^{(n-1)/2}$$

for $k \in C_\lambda$, as desired, since $J > n/\eta - \alpha$ and we can take $L > n/\eta$.

Now suppose $k \in A_\lambda$. Then by (23),

$$(E) \le c(2^\mu r)^{-(n-1)} (1 + 2^\mu \,||x| - r|)^{-L+n}. \tag{25}$$

Substituting this above, we obtain

$$\mathrm{IV} \le \int_{\rho^+}^{\max\{\rho^+, 3|x|/4\}} (\cdot)\, dr + \int_{\max\{\rho^+, 3|x|/4\}}^{\infty} (\cdot)\, dr = \mathrm{VI} + \mathrm{VII},$$

where

$$(\cdot) = c2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2} 2^\mu (1 + 2^\mu \,||x| - r|)^{-L+n} (1 + 2^\lambda |r - 2^{-\lambda} j_{\nu,k}|)^{-J}.$$

For VI, since $r < 3|x|/4$, we have $||x| - r| \ge c|x|$. So

$$\mathrm{VI} \le c2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2} (1 + 2^\mu |x|)^{-L+n} 2^\mu \int_{\rho^+}^{\infty} (1 + 2^\lambda |r - 2^{-\lambda} j_{\nu,k}|)^{-J}\, dr.$$

Using the $l = 0$, $n = 1$ case of (24) and taking $L > n + n/\eta$, we obtain satisfactory estimates for VI. For VII, note that $r > 3|x|/4 > 3 \cdot 2^{-\lambda} j_{\nu,k}/2$, so that $1 + 2^{\lambda}|r - 2^{-\lambda} j_{\nu,k}| \geq c(1 + 2^{\lambda} r)$. If $|x| < 2^{-\mu-1}$, then since $\rho^{+} > 2^{-\mu}$, we obtain

$$\begin{aligned} \mathrm{VII} &\leq c 2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2} 2^{\mu} \int_{2^{-\mu}}^{\infty} (2^{\lambda} r)^{-J}\, dr \\ &\leq c 2^{(\mu-\lambda)J} 2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2}, \end{aligned}$$

as needed. If $|x| \geq 2^{-\mu-1}$, then since $r > 3|x|/4$, it follows that $1 + 2^{\lambda}|r - 2^{-\lambda} j_{\nu,k}| \geq c 2^{\lambda} r \geq c 2^{\lambda}|x|$, so taking $L > n + 1$,

$$\begin{aligned} \mathrm{VII} &\leq c 2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2} (2^{\lambda}|x|)^{-J} 2^{\mu} \int_{-\infty}^{\infty} (1 + 2^{\mu}|\,|x| - r|)^{-L+n} dr \\ &\leq c 2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2} 2^{(\mu-\lambda)J} (2^{\mu}|x|)^{-J}, \end{aligned}$$

as desired in this case.

Now suppose $k \in S_{\lambda m}$, $m \in \{0, \ldots, T\}$. Substituting (25) and making the change of variables $2^{\mu}(r - 2^{-\lambda} j_{\nu,k}) \to r$, we get

$$\mathrm{IV} \leq c 2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2} \int_{1}^{\infty} (1 + |r - 2^{\mu}(|x| - 2^{-\lambda} j_{\nu,k})|)^{-L+n} (1 + 2^{\lambda-\mu} r)^{-J}\, dr.$$

If $m \leq \lambda - \mu + 2$, we simply use $(1 + 2^{\lambda-\mu} r)^{-J} \leq 2^{(\mu-\lambda)J}$ for $r \geq 1$ to get (taking $L > n + 1$) $\mathrm{IV} \leq c 2^{(\mu-\lambda)J} 2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2}$, which is more than satisfactory since $J + n - 1 \geq J > n/\eta - \alpha$. If $m > \lambda - \mu + 2$ and $1 \leq r \leq 2^{m+\mu-\lambda-2}$, then $r \leq 2^{\mu}|\,|x| - 2^{-\lambda} j_{\nu,k}|/2$, so that $1 + |r - 2^{\mu}(|x| - 2^{-\lambda} j_{\nu,k})| \geq 1 + 2^{\mu}|\,|x| - 2^{-\lambda} j_{\nu,k}|/2 \geq c(1 + 2^{m+\mu-\lambda})$. Hence

$$\begin{aligned} &\int_{1}^{2^{m+\mu-\lambda+2}} (1 + |r - 2^{\mu}(|x| - 2^{-\lambda} j_{\nu,k})|)^{-L+n} (1 + 2^{\lambda-\mu} r)^{-J}\, dr \\ &\qquad \leq c(1 + 2^{m+\mu-\lambda})^{-L+n} 2^{(\mu-\lambda)J}, \end{aligned}$$

as desired, by taking $L > n + n/\eta$. If $r \geq 2^{m+\mu-\lambda-2} \geq 1$, then $(1 + 2^{\lambda-\mu} r)^{-J} \leq c 2^{(\mu-\lambda)J}(1 + 2^{m+\mu-\lambda})^{-J}$, and

$$\int_{2^{m+\mu-\lambda-2}}^{\infty} (1 + |r - 2^{\mu}(|x| - 2^{-\lambda} j_{\nu,k})|)^{-L+n}\, dr \leq c,$$

by taking $L > n + 1$. Since $J > n/\eta$, this is sufficient, giving the needed estimate for IV in the final case. This completes the estimate for III.

The estimates for I are similar to those for III. We assume $\rho^{-} > 0$, i.e., $2^{\mu-\lambda} j_{\nu,k} > 1$. As for III, we have $2^{\mu}|r - 2^{-\lambda} j_{\nu,k}| \geq 1$, which leads to $\mathrm{I} \leq \mathrm{IV}' + \mathrm{V}'$, where

$$\mathrm{IV}' = c2^{\mu n} \int_0^{\rho^-} (E) 2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2} (1 + 2^\lambda |r - 2^{-\lambda} j_{\nu,k}|)^{-J} r^{n-1}\, dr$$

and

$$\mathrm{V}' = c2^{\mu(n+l)} 2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2} (A)(D').$$

Here

$$\begin{aligned}(D') &= \int_0^{\rho^-} |r - 2^{-\lambda} j_{\nu,k}|^l (1 + 2^\lambda |r - 2^{-\lambda} j_{\nu,k}|)^{-J} r^{n-1}\, dr \\ &= \int_{2^{\lambda-\mu}}^{j_{\nu,k}} 2^{-\lambda(l+n)} r^l (1+r)^{-J} (j_{\nu,k} - r)^{n-1}\, dr,\end{aligned}$$

by the change of variables $j_{\nu,k} - 2^\lambda r \to r$. Clearly $(D') \le (D)$, so we obtain the required estimate for V' as above.

To estimate IV', suppose first $k \in A_\lambda$. Then $r < 2^{-\lambda} j_{\nu,k} \le |x|/2$, so for $y' \in S^{n-1}$, $|x - ry'| \ge c|x|$, which implies $(E) \le c(1 + 2^\mu |x|)^{-L}$. Thus, by the $l = 0$ cases of $(D') \le (D)$ and (24),

$$\begin{aligned}\mathrm{IV}' &\le c2^{\mu n} (1 + 2^\mu |x|)^{-L} 2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2} \int_0^{\rho^-} (1 + 2^\lambda |r - 2^{-\lambda} j_{\nu,k}|)^{-J} r^{n-1}\, dr \\ &\le c2^{(\mu-\lambda)J} 2^{\lambda n/2} j_{\nu,k}^{(n-1)/2} (1 + 2^\mu |x|)^{-L}.\end{aligned}$$

This is acceptable if we take $L > n/\eta$.

Now suppose $k \in C_\lambda$. Then by (25),

$$\mathrm{IV}' \le c2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2} 2^\mu \int_0^{\rho^-} (1 + 2^\mu |\,|x| - r|)^{-L+n} (1 + 2^\lambda |r - 2^{-\lambda} j_{\nu,k}|)^{-J}\, dr.$$

If $r \le 3 \cdot 2^{-\lambda} j_{\nu,k}/4$, then $|r - 2^{-\lambda} j_{\nu,k}| \ge 2^{-\lambda} j_{\nu,k}/4$. Hence, taking $L > n + 1$,

$$\begin{aligned}&\int_0^{3\cdot 2^{-\lambda} j_{\nu,k}/4} (1 + 2^\mu |\,|x| - r|)^{-L+n} (1 + 2^\lambda |r - 2^{-\lambda} j_{\nu,k}|)^{-J}\, dr \\ &\qquad \le j_{\nu,k}^{-J} \int_{-\infty}^{\infty} (1 + 2^\mu |\,|x| - r|)^{-L+n}\, dr \\ &\qquad \le c2^{-\mu} j_{\nu,k}^{-J} \le c2^{-\mu} (1 + 2^{\mu-\lambda} j_{\nu,k})^{-J} 2^{(\mu-\lambda)J}\end{aligned}$$

since $2^{\mu-\lambda} j_{\nu,k} > 1$. Now suppose $3 \cdot 2^{-\lambda} j_{\nu,k}/4 \le r \le \rho^-$. Then since $k \in C_\lambda$, it follows that $r > 3|x|/2$, so $|\,|x| - r| \ge cr \ge c2^{-\lambda} j_{\nu,k}$. Hence

$$\int_{3\cdot 2^{-\lambda} j_{\nu,k}/4}^{\rho^-} (1+2^\mu||x|-r|)^{-L+n}(1+2^\lambda|r-2^{-\lambda} j_{\nu,k}|)^{-J}\,dr$$
$$\le c(1+2^{\mu-\lambda} j_{\nu,k})^{-L+n}\int_{3\cdot 2^{-\lambda} j_{\nu,k}/4}^{\rho^-}(1+2^\lambda|r-2^{-\lambda} j_{\nu,k}|)^{-J}\,dr$$
$$\le c(1+2^{\mu-\lambda} j_{\nu,k})^{-L+n}2^{-\lambda}\int_{2^{\lambda-\mu}}^{j_{\nu,k}/4} r^{-J}\,dr$$
$$= c(1+2^{\mu-\lambda} j_{\nu,k})^{-L+n}2^{-\mu}2^{(\mu-\lambda)J}.$$

Taking $L > n + n/\eta$, we have a satisfactory estimate in this case.

Finally, suppose $k \in S_{\lambda m}$, $m \in \{0, \dots, T\}$. Using (25),

$$\mathrm{IV}' \le c2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2} 2^\mu \int_0^{\rho^-} (1+2^\mu||x|-r|)^{-L+n}(1+2^\lambda|r-2^{-\lambda} j_{\nu,k}|)^{-J}\,dr$$
$$= c2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2}\int_1^{2^{\mu-\lambda} j_{\nu,k}} (1+|r-2^\mu(2^{-\lambda} j_{\nu,k}-|x|)|)^{-L+n}(1+2^{\lambda-\mu}r)^{-J}\,dr,$$

by the change of variables $2^\mu(2^{-\lambda} j_{\nu,k} - r) \to r$. Now it is clear that the estimates above for IV hold for IV$'$ as well. $\square$

In the next lemma we apply Taylor's theorem to the molecules $m_{\lambda k}$, instead of φ_μ. Since φ, and hence φ_μ, have vanishing moments of all orders, we have

$$\varphi_\mu * m_{\lambda k}(x) = \int_{\mathbf{R}^n} \varphi_\mu(x-y)\Big(m_{\lambda k}(y) - \sum_{|\gamma|\le N} D^\gamma m_{\lambda k}(x)\frac{(y-x)^\gamma}{\gamma!}\Big)\,dy. \tag{26}$$

Since $\varphi \in \mathcal{S}$,

$$|\varphi_\mu(x-y)| \le c_L 2^{\mu n}(1+2^\mu|x-y|)^{-L}, \tag{27}$$

where L can be taken arbitrarily large. When $N \ge 0$, we can use (16) to estimate the other term in (26),

$$\left|m_{\lambda k}(y) - \sum_{|\gamma|\le N} D^\gamma m_{\lambda k}(x)\frac{(y-x)^\gamma}{\gamma!}\right| \tag{28}$$
$$= \left|m_{\lambda k}(y) - \sum_{|\gamma|\le N-1} D^\gamma m_{\lambda k}(x)\frac{(y-x)^\gamma}{\gamma!} - \sum_{|\gamma|=N} D^\gamma m_{\lambda k}(x)\frac{(y-x)^\gamma}{\gamma!}\right|$$
$$= \left|\sum_{|\gamma|=N}(D^\gamma m_{\lambda k}(x+\rho(y-x)) - D^\gamma m_{\lambda k}(x))\frac{(y-x)^\gamma}{\gamma!}\right| \quad \text{for some } \rho \in (0,1)$$
$$\le c2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2} 2^{\lambda N}2^{\lambda\delta}|y-x|^{N+\delta}\sup_{0<\theta<1}(1+2^\lambda||x+\theta(y-x)|-2^{-\lambda} j_{\nu,k}|)^{-S}.$$

This estimate is true for all x, y, but when $2^\lambda|y-x| \le 1$ we have

$$\sup_{0<\theta<1}(1+2^\lambda|\,|x+\theta(y-x)|-2^{-\lambda}j_{\nu,k}|)^{-S} \le c(1+2^\lambda|\,|x|-2^{-\lambda}j_{\nu,k}|)^{-S},$$

in which case (28) is bounded by

$$c2^{\lambda n/2}j_{\nu,k}^{-(n-1)/2}2^{\lambda N}2^{\lambda\delta}|y-x|^{N+\delta}(1+2^\lambda|\,|x|-2^{-\lambda}j_{\nu,k}|)^{-S}. \tag{29}$$

When $N<0$, then $N+\delta \le 0$, so that when $2^\lambda|y-x| \le 1$, the bound (29) for (28) still holds, with S replaced by $J > n/\eta$. On the other hand, just using the triangle inequality and (14), we have for $2^\lambda|y-x| \ge 1$,

$$\begin{aligned}&\left|m_{\lambda k}(y)-\sum_{|\gamma|\le N}D^\gamma m_{\lambda k}(x)\frac{(y-x)^\gamma}{\gamma!}\right| \qquad (30)\\ &\quad\le c2^{\lambda n/2}j_{\nu,k}^{-(n-1)/2}(1+2^\lambda|\,|y|-2^{-\lambda}j_{\nu,k}|)^{-J}\\ &\qquad+\ c2^{\lambda n/2}j_{\nu,k}^{-(n-1)/2}2^{\lambda N}|y-x|^N(1+2^\lambda|\,|x|-2^{-\lambda}j_{\nu,k}|)^{-S}.\end{aligned}$$

3.3. Lemma

Suppose $\lambda \le \mu$. Then there exist constants $R > \alpha$ and $c < \infty$ such that

i. *if $k \in A_\lambda$, then*

$$|\varphi_\mu * m_{\lambda k}(x)| \le c2^{(\lambda-\mu)R}(2^\lambda|x|)^{-s}2^{\lambda n/2}j_{\nu,k}^{-(n-1)/2};$$

ii. *if $m \in \{0,\dots,T\}$ and $k \in S_{\lambda m}$, then*

$$|\varphi_\mu * m_{\lambda k}(x)| \le c2^{(\lambda-\mu)R}2^{-ms}2^{\lambda n/2}j_{\nu,k}^{-(n-1)/2};$$

iii. *if $k \in C_\lambda$, then*

$$|\varphi_\mu * m_{\lambda k}(x)| \le c2^{(\lambda-\mu)R}j_{\nu,k}^{-s}2^{\lambda n/2}j_{\nu,k}^{-(n-1)/2}.$$

Here c is independent of μ, λ, k, x, and m, and $s = \min\{S, J\} > n/\eta$, with J, S as in (15),(16).

Proof. By (26)–(30),

$$|\varphi_\mu * m_{\lambda k}(x)| \le c(\mathrm{I}+\mathrm{II}+\mathrm{III}),$$

where

$$\begin{aligned}
\mathrm{I} &= 2^{\lambda(n/2+N+\delta)} j_{\nu,k}^{-(n-1)/2} (1+2^{\lambda}|\,|x|-2^{-\lambda} j_{\nu,k}|)^{-s} \\
&\quad \cdot \int_{B(x,2^{-\lambda})} 2^{\mu n}|x-y|^{N+\delta}(1+2^{\mu}|x-y|)^{-L}\,dy, \\
\mathrm{II} &= 2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2} \int_{\mathbf{R}^n\setminus B(x,2^{-\lambda})} 2^{\mu n}(1+2^{\mu}|x-y|)^{-L}(1+2^{\lambda}|\,|y|-2^{-\lambda} j_{\nu,k}|)^{-s}\,dy, \\
\mathrm{III} &= 2^{\lambda(n/2+N)} j_{\nu,k}^{-(n-1)/2} (1+2^{\lambda}|\,|x|-2^{-\lambda} j_{\nu,k}|)^{-S} \cdot \\
&\quad \cdot \int_{\mathbf{R}^n\setminus B(x,2^{-\lambda})} 2^{\mu n}|x-y|^{N}(1+2^{\mu}|x-y|)^{-L}\,dy.
\end{aligned}$$

We consider I and III first. By the change of variables $2^{\mu}(x-y) \to y$,

$$\begin{aligned}
&\int_{B(x,2^{-\lambda})} 2^{\mu n}|x-y|^{N+\delta}(1+2^{\mu}|x-y|)^{-L}\,dy \\
&\qquad = \int_{B(0,2^{\mu-\lambda})} 2^{-\mu(N+\delta)}|y|^{N+\delta}(1+|y|)^{-L}\,dy \le c2^{-\mu(N+\delta)},
\end{aligned}$$

by taking L sufficiently large. Similarly, assuming $L > N+n$,

$$\begin{aligned}
&\int_{\mathbf{R}^n\setminus B(x,2^{-\lambda})} 2^{\mu n}|x-y|^{N}(1+2^{\mu}|x-y|)^{-L}\,dy \\
&\qquad \le c2^{-\mu N}\int_{\mathbf{R}^n\setminus B(0,2^{\mu-\lambda})} |y|^{N-L}\,dy = c2^{-\mu N}2^{(\lambda-\mu)(L-N-n)}.
\end{aligned} \tag{31}$$

If $k \in A_{\lambda}$, then

$$(1+2^{\lambda}|\,|x|-2^{-\lambda} j_{\nu,k}|)^{-s} \le c(1+2^{\lambda}|x|)^{-s};$$

if $k \in S_{\lambda m}$, then

$$(1+2^{\lambda}|\,|x|-2^{-\lambda} j_{\nu,k}|)^{-s} \le c(1+2^{m})^{-s};$$

and if $k \in C_{\lambda}$, then

$$(1+2^{\lambda}|\,|x|-2^{-\lambda} j_{\nu,k}|)^{-s} \le c(1+j_{\nu,k})^{-s}.$$

Taking $L > n+\alpha$ and recalling that $N+\delta > \alpha$, we obtain satisfactory estimates for I and III in all cases.

To estimate II, we define a set A in each case as follows:

i. If $k \in A_\lambda$, let $A = \{y : |x-y| > 2^{-\lambda}$ and $|\,|y| - 2^{-\lambda} j_{\nu,k}| \geq |x|/4\}$.

ii. If $k \in S_{\lambda m}$, let $A = \{y : |x-y| > 2^{-\lambda}$ and $|\,|y| - 2^{-\lambda} j_{\nu,k}| \geq 2^{m-\lambda}/4\}$.

iii. If $k \in C_\lambda$, let $A = \{y : |x-y| > 2^{-\lambda}$ and $|\,|y| - 2^{-\lambda} j_{\nu,k}| \geq 2^{-\lambda} j_{\nu,k}/4$.

In each case, define $B = \{y : |x-y| > 2^{-\lambda}\} \setminus A$. Then

$$\int_{\mathbf{R}^n \setminus B(x,2^{-\lambda})} (\cdot)\, dy = \int_A (\cdot)\, dy + \int_B (\cdot)\, dy = \mathrm{IV} + \mathrm{V},$$

where

$$(\cdot) = 2^{\mu n}(1 + 2^\mu |x-y|)^{-L}(1 + 2^\lambda |\,|y| - 2^{-\lambda} j_{\nu,k}|)^{-s}.$$

For IV, we can estimate $(1 + 2^\lambda |\,|y| - 2^{-\lambda} j_{\nu,k}|)^{-s}$, for $y \in A$, by $c(2^\lambda |x|)^{-s}$ if $k \in A_\lambda$, by $c2^{-ms}$ if $k \in S_{\lambda m}$, and by $cj_{\nu,k}^{-s}$ if $k \in C_\lambda$. Estimating

$$\int_{\mathbf{R}^n \setminus B(x,2^{-\lambda})} 2^{\mu n}(1 + 2^\mu |x-y|)^{-L}\, dy$$

by the $N = 0$ case of (31), we obtain the desired estimates for IV.

For V, we make the trivial estimate $(1 + 2^\lambda |\,|y| - 2^{-\lambda} j_{\nu,k}|)^{-s} \leq 1$. We write

$$|x-y| \geq |\,|x| - |y|\,| \geq |\,|x| - 2^{-\lambda} j_{\nu,k}| - |\,|y| - 2^{-\lambda} j_{\nu,k}|.$$

Examining the definitions in each case, we obtain, for $y \in B$,

i. $|x-y| \geq |x|/4$ if $k \in A_\lambda$,

ii. $|x-y| \geq 2^{m-\lambda}/4$ if $k \in S_{\lambda m}$,

iii. $|x-y| \geq 2^{-\lambda} j_{\nu,k}/4$ if $k \in C_\lambda$.

We write

$$(1 + 2^\mu |x-y|)^{-L} = (1 + 2^\mu |x-y|)^{-s}(1 + 2^\mu |x-y|)^{-L+s},$$

and estimate $(1 + 2^\mu |x-y|)^{-s}$ by $2^{(\lambda-\mu)s}(2^\lambda |x|)^{-s}$ if $k \in A_\lambda$, by $2^{(\lambda-\mu)s}2^{-ms}$ if $k \in S_{\lambda m}$, and by $2^{(\lambda-\mu)s} j_{\nu,k}^{-s}$ if $k \in C_\lambda$. By (31), assuming $L > s + n$,

$$\int_{\mathbf{R}^n \setminus B(x,2^{-\lambda})} 2^{\mu n}(1 + 2^\mu |x-y|)^{-L+s}\, dy \leq c2^{(\lambda-\mu)(L-s-n)}.$$

Taking $L > n + \alpha$, we have the required estimate for V. $\square$

3.4. Lemma

There exist constants $\varepsilon > 0$ *and* $c > 0$ *such that for any sequence* $\{s_{\lambda k}\}_{\lambda\in\mathbf{Z},k\in\mathbf{Z}^+}$,

$$2^{\mu\alpha}\left|\varphi_\mu * \sum_{\lambda\in\mathbf{Z}}\sum_{k=1}^{\infty} s_{\lambda k}m_{\lambda k}(x)\right| \le c\sum_{\lambda\in\mathbf{Z}} 2^{-|\mu-\lambda|\varepsilon}\left(M\left(\sum_{k=1}^{\infty} 2^{\lambda\alpha\eta}|s_{\lambda k}|^\eta \tilde{\chi}^\eta_{A_{\lambda k}}\right)(x)\right)^{1/\eta}. \tag{32}$$

Here $\eta \in (0, 1]$ *is such that* $p/\eta, q/\eta > 1$, *as above,* M *is the Hardy–Littlewood maximal operator, and* c *and* ε *depend on* α, p, q *and* η, *but not on* x, $\{s_{\lambda k}\}$, *or* $\{m_{\lambda k}\}$ *as in* (13)–(16).

Proof. Let A_λ, B_λ, $\{S_{\lambda m}\}_{m=0}^T$, and C_λ be as in Lemmas 3.2 and 3.3. First suppose $k \in A_\lambda$. If $\mu \le \lambda$, we use Lemma 3.2(i) and $(1 + 2^\mu|x|)^{-s} \le (2^\mu|x|)^{-n/\eta}$ (since $s > n/\eta$) to obtain

$$2^{\mu\alpha}|\varphi_\mu * m_{\lambda k}(x)| \le c2^{(\mu-\lambda)(R+\alpha-n/\eta)}2^{\lambda\alpha}2^{-\lambda n/\eta}|x|^{-n/\eta}2^{\lambda n/2}j_{\nu,k}^{(n-1)/2},$$

with $R > n/\eta - \alpha$. If $\lambda \le \mu$, we use Lemma 3.3(i), $(2^\lambda|x|)^{-s} \le c(2^\lambda|x|)^{-n/\eta}$ (since $s > n/\eta$ and $|x| \ge c2^{-\lambda}$ for $x \in A_\lambda$), and $j_{\nu,k}^{-(n-1)/2} \le cj_{\nu,k}^{(n-1)/2}$ to obtain

$$2^{\mu\alpha}|\varphi_\mu * m_{\lambda k}(x)| \le c2^{(\lambda-\mu)(R-\alpha)}2^{\lambda\alpha}2^{-\lambda n/\eta}|x|^{-n/\eta}2^{\lambda n/2}j_{\nu,k}^{(n-1)/2},$$

with $R > \alpha$ in this case. Thus we have $\varepsilon > 0$ such that in both cases,

$$2^{\mu\alpha}|\varphi_\mu * m_{\lambda k}(x)| \le c2^{-|\mu-\lambda|\varepsilon}2^{\lambda\alpha}2^{-\lambda n/\eta}|x|^{-n/\eta}2^{\lambda n/2}j_{\nu,k}^{(n-1)/2}.$$

By the continuous imbedding $l^\eta \to l^1$, and since $j_{\nu,k}^{(n-1)\eta/2} \le cj_{\nu,k}^{n-1-(n-1)\eta/2}$, we obtain

$$\begin{aligned}
&2^{\mu\alpha}|\varphi_\mu * \sum_{\lambda\in\mathbf{Z}}\sum_{k\in A_\lambda} s_{\lambda k}m_{\lambda k}(x)| \\
&\quad\le c\sum_{\lambda\in\mathbf{Z}}\sum_{k\in A_\lambda} 2^{-|\mu-\lambda|\varepsilon}|s_{\lambda k}|2^{\lambda\alpha}2^{-\lambda n/\eta}|x|^{-n/\eta}2^{\lambda n/2}j_{\nu,k}^{(n-1)/2} \\
&\quad\le c\sum_{\lambda\in\mathbf{Z}} 2^{-|\mu-\lambda|\varepsilon}\left(\sum_{k\in A_\lambda}|s_{\lambda k}|^\eta 2^{\lambda\alpha\eta}2^{-\lambda n}|x|^{-n}2^{\lambda n\eta/2}j_{\nu,k}^{n-1-(n-1)\eta/2}\right)^{1/\eta}.
\end{aligned}$$

By considering $B(x, 3|x|/2)$, we see that the term inside the parentheses above is bounded by $cM(\sum_{k\in A_\lambda}|s_{\lambda k}|^\eta 2^{\lambda\alpha\eta}\tilde{\chi}^\eta_{A_{\lambda k}})(x)$. This yields the part of (32) for $k \in A_\lambda$.

Now suppose $k \in S_{\lambda m}$, for some $m \in \{0, 1, \ldots, T\}$. If $\mu \le \lambda$, note that in Lemma 3.2(ii) we have $R + 1 - n > n/\eta - \alpha + 1 - n = 1/\eta - \alpha + (n-1)(1/\eta - 1) \ge 1/\eta - \alpha$. Hence, by reducing s if necessary, we can assume $R + 1 - n > s - \alpha$ and $s > 1/\eta$. We obtain

$$2^{\mu\alpha}|\varphi_\mu * m_{\lambda k}(x)| \le c2^{(\mu-\lambda)(R+1-n+\alpha-s)}2^{\lambda\alpha}2^{-ms}2^{\lambda n/2}j_{\nu,k}^{-(n-1)/2}.$$

If $\lambda \le \mu$, Lemma 3.3(ii) gives

$$2^{\mu\alpha}|\varphi_\mu * m_{\lambda k}(x)| \le c2^{(\lambda-\mu)(R-\alpha)}2^{\lambda\alpha}2^{-ms}2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2},$$

where $s > n/\eta \ge 1/\eta$ and $R > \alpha$. Thus, in either case we have

$$2^{\mu\alpha}|\varphi_\mu * m_{\lambda k}(x)| \le c2^{-|\mu-\lambda|\varepsilon}2^{\lambda\alpha}2^{-ms}2^{\lambda n/2} j_{\nu,k}^{-(n-1)/2},$$

with $s > 1/\eta$ and $\varepsilon > 0$ independent of m. By the imbedding $l^\eta \to l^1$, then

$$\begin{aligned}
&2^{\mu\alpha}\Big|\varphi_\mu * \sum_{\lambda\in\mathbf{Z}}\sum_{k\in B_\lambda} s_{\lambda k} m_{\lambda k}(x)\Big| \\
&\quad = 2^{\mu\alpha}\Big|\varphi_\mu * \sum_{\lambda\in\mathbf{Z}}\sum_{m=0}^{T}\sum_{k\in S_{\lambda m}} s_{\lambda k} m_{\lambda k}(x)\Big| \\
&\quad \le c\sum_{\lambda\in\mathbf{Z}} 2^{-|\mu-\lambda|\varepsilon}\sum_{m=0}^{T} 2^{-ms}\left(\sum_{k\in S_{\lambda m}} |s_{\lambda k}|^\eta 2^{\lambda\alpha\eta}2^{\lambda n\eta/2} j_{\nu,k}^{-(n-1)\eta/2}\right)^{1/\eta}.
\end{aligned}$$

We claim that

$$\sum_{k\in S_{\lambda m}} |s_{\lambda k}|^\eta 2^{\lambda\alpha\eta}2^{\lambda n\eta/2} j_{\nu,k}^{-(n-1)\eta/2} \le c2^m M\left(\sum_{k\in S_{\lambda m}} 2^{\lambda\alpha\eta}|s_{\lambda k}|^\eta \tilde{\chi}_{A_{\lambda k}}^\eta\right)(x). \tag{33}$$

This follows easily from Lemma 3.5, which we state and prove at the end of this section. Using (33) in the estimate above and replacing the sum over $k \in S_{\lambda m}$ by $k \in B_\lambda$, we can sum over m to obtain the part of (32) for $k \in B_\lambda$ since $s > 1/\eta$.

Now suppose $k \in C_\lambda$. Let

$$T_{\lambda 0} = \{k \in C_\lambda : 2^{-\lambda} j_{\nu,k} \le 2^{-\mu}\}$$

and, for $m = 1, 2, 3, \ldots,$

$$T_{\lambda m} = \{k \in C_\lambda : 2^{m-\mu-1} < 2^{-\lambda} j_{\nu,k} \le 2^{m-\mu}\}.$$

Then $C_\lambda = \bigcup_{m=0}^\infty T_{\lambda m}$. Suppose $k \in T_{\lambda m}$. If $\mu \le \lambda$, then $1 + 2^{\mu-\lambda} j_{\nu,k} \approx 2^m$, so by Lemma 3.2(iii),

$$\begin{aligned}
2^{\mu\alpha}|\varphi_\mu * m_{\lambda k}(x)| &\le c2^{(\mu-\lambda)(R+\alpha)}2^{\lambda\alpha}2^{-ms}2^{\lambda n/2} j_{\nu,k}^{(n-1)/2} \\
&= c2^{(\mu-\lambda)(R+\alpha-n/\eta)}2^{(\mu-\lambda)n/\eta}2^{\lambda\alpha}2^{-ms}2^{\lambda n/2} j_{\nu,k}^{(n-1)/2},
\end{aligned}$$

with $R > n/\eta - \alpha$ and $s > n/\eta$. Now suppose $\lambda \le \mu$. In Lemma 3.3(iii) we have $R > \alpha$ and $s > n/\eta$, so by reducing s if necessary, we can assume $R > \alpha + s - n/\eta$ and $s > n/\eta$. Since $k \in T_{\lambda m}$, we have $j_{\nu,k} \ge c2^{m+\lambda-\mu}$ (when $m = 0$ this follows since $j_{\nu,k} \ge c$ and $\lambda \le \mu$). Hence

$$\begin{aligned}2^{\mu\alpha}|\varphi_\mu * m_{\lambda k}(x)| &\le c2^{(\lambda-\mu)(R-\alpha-s)}2^{\lambda\alpha}2^{-ms}2^{\lambda n/2}j_{\nu,k}^{(n-1)/2}\\ &= c2^{(\lambda-\mu)(R-\alpha-s+n/\eta)}2^{(\mu-\lambda)n/\eta}2^{\lambda\alpha}2^{-ms}2^{\lambda n/2}j_{\nu,k}^{(n-1)/2}.\end{aligned}$$

Hence in either case there exists $\varepsilon > 0$ (independent of m) and $s > n/\eta$ such that

$$2^{\mu\alpha}|\varphi_\mu * m_{\lambda k}(x)| \le c2^{-|\mu-\lambda|\varepsilon}2^{(\mu-\lambda)n/\eta}2^{\lambda\alpha}2^{-ms}2^{\lambda n/2}j_{\nu,k}^{(n-1)/2}.$$

Using the imbedding $l^\eta \to l^1$ again,

$$\begin{aligned}&2^{\mu\alpha}|\varphi_\mu * \sum_{\lambda\in\mathbf{Z}}\sum_{k\in C_\lambda} s_{\lambda k}m_{\lambda k}(x)|\\ &\quad= 2^{\mu\alpha}|\varphi_\mu * \sum_{\lambda\in\mathbf{Z}}\sum_{m=0}^{\infty}\sum_{k\in T_{\lambda m}} s_{\lambda k}m_{\lambda k}(x)|\\ &\quad\le c\sum_{\lambda\in\mathbf{Z}}2^{-|\mu-\lambda|\varepsilon}\sum_{m=0}^{\infty}2^{-ms}\left(\sum_{k\in T_{\lambda m}}2^{(\mu-\lambda)n}|s_{\lambda k}|^\eta 2^{\lambda\alpha\eta}2^{\lambda n\eta/2}j_{\nu,k}^{n-1-(n-1)\eta/2}\right)^{1/\eta}.\end{aligned}$$

Since $k \in T_{\lambda m} \subseteq C_\lambda$, we have $|x| \le 2^{-\lambda}j_{\nu,k}/2 \le 2^{m-\mu-1}$. Therefore $|B(x, |x| + 2^{m-\mu})| \approx 2^{(m-\mu)n}$, which implies that the term inside the last parentheses above is bounded by $c2^{mn}M(\sum_{k\in T_{\lambda m}} 2^{\lambda\alpha\eta}|s_{\lambda k}|^\eta\tilde{\chi}_{A_{\lambda k}}^\eta)(x)$. Using this estimate above and then replacing the sum over $k \in T_{\lambda m}$ by the sum over $k \in C_\lambda$, we can sum over m to obtain the portion of (32) for $k \in C_\lambda$, since $s > n/\eta$. $\square$

Proof of Theorem 3.1. We take the l^q norm over $\mu \in \mathbf{Z}$ of both sides of (32). If $q \ge 1$, we use Young's inequality $\|a * b\|_{l^q} \le \|a\|_{l^1}|b\|_{l^q}$, while if $0 < q < 1$ we first use the q-triangle inequality $|\sum_\lambda a_\lambda|^q \le \sum_\lambda |a_\lambda|^q$ followed by the $q = 1$ version of Young's inequality. Since $\varepsilon > 0$, the result is

$$\left(\sum_{\mu\in\mathbf{Z}}\left(2^{\mu\alpha}\left|\varphi_\mu * \sum_{\lambda\in\mathbf{Z}}\sum_{k=1}^{\infty}s_{\lambda k}m_{\lambda k}(x)\right|\right)^q\right)^{1/q} \le c\left(\sum_{\mu\in\mathbf{Z}}\left(M\left(\sum_{k=1}^{\infty}2^{\mu\alpha\eta}|s_{\mu k}|^\eta\tilde{\chi}_{A_{\mu k}}^\eta\right)(x)\right)^{q/\eta}\right)^{1/q}.$$

Taking the L^p norm of both sides and applying the Fefferman–Stein vector-valued maximal inequality as in the proof of Theorem 2.2, we obtain the desired result

$$\left\|\sum_{\mu\in\mathbf{Z}}\sum_{k=1}^{\infty}s_{\mu k}m_{\mu k}\right\|_{\dot{R}_p^{\alpha q}} \le c\|\{s_{\mu k}\}\|_{\dot{r}_p^{\alpha q}}. \qquad \square$$

It remains now to prove the inequality (33).

3.5. Lemma

Let $x \in \mathbf{R}^n$, $x \ne 0$, $\lambda \in \mathbf{Z}$, $m \in \{0, 1, \dots, T\}$, and $k \in S_{\lambda m}$. Then there exist constants $a, c > 0$ independent of x, m, λ, and k, such that

$$|A_{\lambda k} \cap B(x, 2^{m-\lambda}a)| \ge c2^{m(n-1)}2^{-\lambda n}. \tag{34}$$

Proof. By a dilation argument we may assume that $\lambda = 0$. The $n = 1$ case is easy (just take $a > 1$), so we assume that $n \geq 2$. By symmetry we may assume that x is a positive multiple of the unit vector $e_1 = (1, 0, \ldots, 0)$. Note that the quantity $|A_{0k} \cap B(x, 2^m a)|$ is increasing in a. Let $\gamma = \sup_{k\in\mathbf{N}}(j_{\nu,k} - j_{\nu,k-1}) < \infty$ and take $a \geq 1 + \gamma$. Then $B((j_{\nu,k-1} + j_{\nu,k})e_1/2, (j_{\nu,k} - j_{\nu,k-1})/2) \subset A_{0k} \cap B(x, 2^m a)$, since $||x| - j_{\nu,k}| \leq 2^m$, and $2^m a \geq 2^m + \gamma \geq 2^m + (j_{\nu,k} - j_{\nu,k-1})$. Hence $|A_{0k} \cap B(x, 2^m a)| \geq c(\inf_{k\in\mathbf{N}}(j_{\nu,k} - j_{\nu,k-1}))^n > 0$, which proves (34) for small values of m.

If $2^m \geq j_{\nu,k-1}/4$, take $a \geq 9 + 2\gamma$. Then

$$2^m a \geq 2 \cdot 4 \cdot 2^m + 2\gamma + 2^m \geq 2 \cdot j_{\nu,k-1} + 2\gamma + 2^m \geq 2 j_{\nu,k} + 2^m.$$

Since $||x| - j_{\nu,k}| \leq 2^m$, we see that $A_{0k} \subset A_{0k} \cap B(x, 2^m a)$. Thus, $|A_{0k} \cap B(x, 2^m a)| \approx j_{\nu,k}^{n-1}$. On the other hand, since $k \in S_{0m} \subset B_0$, we have (*without* using $2^m \geq j_{\nu,k-1}/4$) that when $\beta > 1$

$$j_{\nu,k} \geq |x|/2 \geq j_{\nu,\beta-1}/4 \geq c j_{\nu,\beta+1} > c2^T > c2^m.$$

If $\beta = 1$ and $m > 0$ we use

$$3 j_{\nu,1} > |x| + j_{\nu,1} \geq ||x| - j_{\nu,1}| > 2^m/2.$$

Finally, if $\beta = 1$ and $m = 0$, we use the trivial fact that $|A_{01}| \geq c$. These observations prove (34) when $2^m \geq j_{\nu,k-1}/4$.

Now suppose that $\max\{1, 4\gamma\} < 2^m < j_{\nu,k-1}/4$ (which rules out $k = 1$) and, as a first subcase, that $j_{\nu,k} \leq |x|$. We will consider the effect of taking $a > \gamma + 2$. Let $j_{\nu,k-1} \leq r \leq j_{\nu,k}$. The ball $B(x, 2^m a)$ intersects the $n - 1$ sphere $\{y \in R^n : |y| = r\}$ in a nonempty spherical cap $\mathcal{C}_r$ whose boundary $\partial\mathcal{C}_r$ is either an $n - 2$ sphere (meaning a pair of points in the $n = 2$ case) or else empty. If $\partial\mathcal{C}_r \neq \emptyset$, then the points in $\partial\mathcal{C}_r$ have a common first coordinate, say $t(r)$. In this case the hyperplane $\{y \in \mathbf{R}^n : y \cdot e_1 = t(r)\}$ intersects $B(x, 2^m a)$ in an $n - 1$ disk D whose boundary coincides with $\partial\mathcal{C}_r$. Let $\rho(r)$ denote the radius of D. Then the $(n-1)$-dimensional measure $|\mathcal{C}_r|_{n-1}$ of the spherical cap $\mathcal{C}_r$ is bounded below by a quantity of the form $c_n(\rho(r))^{n-1}$ if $t(r) > 0$ (this is an isoperimetric inequality) and by a quantity of the form $c j_{\nu,k-1}^{n-1} \geq c2^{m(n-1)}$ if $t(r) \leq 0$ or if $\partial\mathcal{C}_r = \emptyset$. Using spherical coordinates,

$$|A_{0k} \cap B(x, 2^m a)| = c_n \int_{j_{\nu,k-1}}^{j_{\nu,k}} |\mathcal{C}_r|_{n-1}\, dr.$$

Thus, it suffices in this case to show that

$$\inf\{\rho(r) : r \in [j_{\nu,k-1}, j_{\nu,k}] \text{ and } t(r) > 0\} \geq c2^m$$

for some $c > 0$ independent of k, x, m. First, suppose that $j_{\nu,k-1} - 2^m \leq t(r) \leq j_{\nu,k}$. Then $|x - t(r)e_1| \leq 2 \cdot 2^m + \gamma$. Since $(\rho(r))^2 + |x - t(r)e_1|^2 = (2^m a)^2$, we obtain

$$(\rho(r))^2 \geq (2^m a)^2 - (2 \cdot 2^m + \gamma)^2 = (a^2 - (4 + \frac{4\gamma}{2^m} + \frac{\gamma^2}{2^{2m}}))2^{2m} \geq (a^2 - (\gamma + 2)^2)2^{2m}.$$

Next, if $0 < t(r) \leq j_{\nu,k-1} - 2^m$, then we use $(\rho(r))^2 + (t(r))^2 = r^2 \geq j_{\nu,k-1}^2$ to get

$$(\rho(r))^2 \geq j_{\nu,k-1}^2 - (j_{\nu,k-1} - 2^m)^2 = 2 \cdot j_{\nu,k-1} 2^m - 2^{2m} \geq 7 \cdot 2^{2m}.$$

Finally, we consider the subcase where $j_{\nu,k} > |x|$. Since $2^m > \max\{1, 4\gamma\}$, it must be that $m > 0$. Therefore, $||x| - j_{\nu,k}| > 2^{m-1} > 2\gamma$, which implies that $|x| < j_{\nu,k-1}$. This time we consider the effect of taking $a > 1$. If $|x| \leq t(r)$, then

$$0 \leq t(r) - |x| \leq j_{\nu,k} - |x| \leq 2^m.$$

Hence, from $(t(r) - |x|)^2 + (\rho(r))^2 = (2^m a)^2$, we obtain $(\rho(r))^2 \geq 2^{2m}(a^2 - 1)$, as desired. If $t(r) < |x|$, then we get a lower bound for $|\mathcal{C}_r|_{n-1}$ by considering a ball of smaller radius than $2^m a$, centered at x, such that $t(r) = |x|$. Then the value of ρ corresponding to this smaller ball satisfies $|x| + \rho \geq r$. But $r - |x| \geq j_{\nu,k-1} - |x| \geq j_{\nu,k} - |x| - \gamma \geq 2^{m-1} - 2^{m-2} = 2^{m-2}$, so $\rho \geq 2^{m-2}$. $\square$

By considering $B(x, 2^{m-\lambda} a)$, the inequality (33) now follows.

4. The Smooth Radial Atomic Decomposition

In some applications (e.g., trace theorems—see [9] or [11]) it is useful to have representing functions of compact support in the standard rectangular Littlewood–Paley decompositions. Nonorthogonal decompositions of this type can be obtained easily from the Calderón reproducing formula. Such results have a long history, part of which is described in [12]. More recently, Daubechies [3] constructed smooth orthonormal wavelets of compact support. Here we present a radial version of the older, nonorthogonal decomposition. However, because of the requirement (13) in the proof of Theorem 3.1, we obtain norm equivalences for $\dot{R}_p^{\alpha q}$ only for α large enough; e.g., for $p, q \geq 1$, we require $\alpha > -1$.

For $\mu \in \mathbf{Z}$ and $k \in \mathbf{Z}^+$, we let $\tilde{A}_{\mu k}$ be the *fattening* of $A_{\mu k}$: $\tilde{A}_{\mu k} = \{x \in \mathbf{R}^n : 2^{-\mu}(j_{\nu,k-1} - 1) \leq |x| \leq 2^{-\mu}(j_{\nu,k} + 1)\}$. Since $j_{\nu,k} - j_{\nu,k-1} = \pi + O(1/k)$, this fattening is of the same order as the width of the annulus $A_{\mu k}$. For $\alpha > (n/\min\{1, p, q\}) - n - 1$, we define a *smooth radial atom for* $A_{\mu k}$ to be any radial function $a_{\mu k}$ satisfying

i. $\operatorname{supp} a_{\mu k} \subseteq \tilde{A}_{\mu k}$,

ii. $\displaystyle\int a_{\mu k} = 0$,

iii. $|D^\gamma a_{\mu k}(x)| \leq c_\gamma 2^{\mu n/2} 2^{\mu|\gamma|} j_{\nu,k}^{-(n-1)/2}$,

for all multi-indices γ, where $c_\gamma = 1$ for $|\gamma| \leq \alpha + 1$ and the c_γ are independent of μ and k for $|\gamma| > \alpha + 1$ (with fixed values to be determined by the proof below).

4.1. Theorem

Suppose $0 < q \leq \infty$, $0 < p < \infty$, *and* $\alpha > (n/\min\{1, p, q\}) - n - 1$. *For* $f \in \dot{R}_p^{\alpha q}$, *there exists a sequence* $s = \{s_{\mu k}\} \in \dot{r}_p^{\alpha q}$ *and functions* $\{a_{\mu k}\}_{\mu \in \mathbf{Z}, k \in \mathbf{Z}^+}$ *(both depending on* f*) such that each* $a_{\mu k}$ *is a smooth radial atom for* $A_{\mu k}$,

$$f = \sum_{\mu \in \mathbf{Z}} \sum_{k \in \mathbf{Z}^+} s_{\mu k} a_{\mu k}, \tag{35}$$

and

$$\|f\|_{\dot{R}_p^{\alpha q}} \approx \|s\|_{\dot{r}_p^{\alpha q}}, \tag{36}$$

with equivalence constants independent of f. The identity (35) has the interpretation that the partial sums

$$f_N = \sum_{\mu=-N}^{\mu=+N} \sum_{k \in \mathbf{Z}^+} s_{\mu k} a_{\mu k}$$

converge to f in the $\dot{R}_p^{\alpha q}$ quasi-norm as $N \to \infty$ if $q < +\infty$ and in $\mathcal{S}'/\mathcal{P}$ if $q = +\infty$.

Proof. Select a radial function $\theta \in \mathcal{D}(\mathbf{R}^n)$ satisfying supp $\theta \subseteq B(0,1)$, $\int \theta = 0$, and $|\hat{\theta}(\xi)| \geq c > 0$ if $1/4 \leq |\xi| \leq 1$. (See, e.g., [9, p. 783].) Then there exists a radial function $\varphi \in \mathcal{S}(\mathbf{R}^n)$ such that supp $\hat{\varphi} \subset \{\xi : 1/4 \leq |\xi| \leq 1\}$, $|\hat{\varphi}(\xi)| \geq c > 0$ if $3/10 \leq |\xi| \leq 5/6$, and

$$\sum_{\mu \in \mathbf{Z}} \hat{\varphi}(2^{-\mu}\xi)\hat{\theta}(2^{-\mu}\xi) = 1 \quad \text{if } \xi \in \mathbf{R}^n \setminus \{0\}. \tag{37}$$

For $\mu \in \mathbf{Z}$, let $\varphi_\mu(x) = 2^{\mu n}\varphi(2^\mu x)$ and $\theta_\mu(x) = 2^{\mu n}\theta(2^\mu x)$. By (37), we have

$$f = \sum_{\mu \in \mathbf{Z}} \theta_\mu * \varphi_\mu * f, \tag{38}$$

with convergence in $\mathcal{S}'/\mathcal{P}$. Since $\{A_{\mu k}\}_{k=1}^{\infty}$ is an essentially disjoint cover of $\mathbf{R}^n$,

$$\theta_\mu * \varphi_\mu * f(x) = \sum_{k=1}^{\infty} \int_{A_{\mu k}} \theta_\mu(x-y)\varphi_\mu * f(y)\, dy. \tag{39}$$

For a sufficiently large constant c' to be determined later, define

$$s_{\mu k} = c'|A_{\mu k}|^{1/2} \sup_{y \in A_{\mu k}} |\varphi_\mu * f(y)|$$

and, if $s_{\mu k} \neq 0$, define

$$a_{\mu k}(x) = \frac{1}{s_{\mu k}} \int_{A_{\mu k}} \theta_\mu(x-y)\varphi_\mu * f(y)\, dy. \tag{40}$$

Thus (38) and (39) yield (35). Since $\theta_\mu, \varphi_\mu, f$, and $A_{\mu k}$ are spherically symmetric, $a_{\mu k}$ is a radial function. Since supp $\theta_\mu \subseteq B(0, 2^{-\mu})$, we have supp $a_{\mu k} \subseteq A_{\mu k} + B(0, 2^{-\mu}) = \tilde{A}_{\mu k}$, i.e., (i) holds. Since $\int \theta = 0$, (ii) also holds. Next,

$$
\begin{aligned}
|D^\gamma a_{\mu k}(x)| &\leq \frac{1}{s_{\mu k}} \int_{A_{\mu k}} 2^{\mu n} 2^{\mu|\gamma|} |(D^\gamma \theta)(2^\mu x - 2^\mu y)|\, |\varphi_\mu * f(y)|\, dy \\
&\leq \frac{2^{\mu|\gamma|}}{c'} |A_{\mu k}|^{-1/2} \int_{A_{0k}} |D^\gamma \theta(2^\mu x - y)|\, dy \\
&= \frac{2^{\mu|\gamma|}}{c'} |A_{\mu k}|^{-1/2} \int_{j_{\nu,k-1}}^{j_{\nu,k}} |D^\gamma \theta| * d\sigma_t(2^\mu x)\, dt.
\end{aligned}
$$

By Lemma 2.3,

$$
|D^\gamma \theta| * d\sigma_t(2^\mu x) \leq c'_{\gamma,m} (1 + |2^\mu |x| - t|)^{-m}.
$$

Since $|A_{\mu k}|^{-1/2} \approx 2^{\mu n/2} j_{\nu,k}^{-(n-1)/2}$, we have $|D^\gamma a_{\mu k}(x)| \leq \frac{c'_{\gamma m}}{c'} 2^{\mu n/2} 2^{\mu|\gamma|} j_{\nu,k}^{-(n-1)/2}$. Taking c' large enough gives (iii). Hence each $a_{\mu k}$ is a smooth radial atom for $A_{\mu k}$.

If $x \in A_{\mu k}$, then by the radiality of $|\varphi_\mu * f|$ we have

$$
|A_{\mu k}|^{-1/2} |s_{\mu k}| = c' \sup\{|\varphi_\mu * f(rx/|x|)| : 2^{-\mu} j_{\nu,k-1} \leq r \leq 2^{-\mu} j_{\nu,k}\} \leq c\varphi_\mu^{**} f(x).
$$

As in the proof of Theorem 2.1, we obtain

$$
\|s\|_{\dot{r}_p^{\alpha q}} \leq c \|f\|_{\dot{R}_p^{\alpha q}}. \tag{41}
$$

The converse estimates are obtained from Theorem 3.1. We check that for the assumed indices, radial smooth atoms $a_{\mu k}$ are, up to a constant factor, radial smooth molecules as defined before Theorem 3.1. By the assumption on α in Theorem 4.1, we can take $0 < \eta \leq 1$ sufficiently close to $\min\{1, p, q\}$ so that $p/\eta, q/\eta > 1$ and $n + 1 + \alpha - n/\eta > 0$. This allows us to take $l = 0$ in (13), so (13) reduces to (ii). Then (14),(15), and (16) follow, up to a constant multiple, from (i) and (iii). Thus, by (35) and Theorem 3.1, we have

$$
\|f\|_{\dot{R}_p^{\alpha q}} \leq c \|s\|_{\dot{r}_p^{\alpha q}}. \tag{42}
$$

The remark about convergence in the quasi-norm when $q < \infty$ follows by applying Theorem 3.1 to

$$
f - f_N = \sum_{|\mu| > N} \sum_{k=1}^{\infty} s_{\mu k} a_{\mu k}
$$

and using the dominated convergence theorem. $\square$

The obstruction to such a result if $\alpha \le (n/\min\{1, p, q\}) - n - 1$ is as follows. To apply Theorem 3.1 to get (42), we need to assume (13) with $l \ge 1$ for $a_{\mu k}$ in place of $m_{\mu k}$. Even if we find a θ satisfying such a condition, (40) does not guarantee that this condition will be "inherited" by $a_{\mu k}$, unlike the simple mean-zero condition. Finding a version of Theorem 4.1 for α below the stated value remains open.

5. Results for the Besov Spaces

In this section we briefly describe some of the results for Besov spaces. Recall that the homogeneous Besov space $\dot{B}_p^{\alpha q}$, $\alpha \in \mathbf{R}$, $0 < p, q \le \infty$, is defined to be the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{B}_p^{\alpha q}} = \left(\sum_{\mu \in \mathbf{Z}} (2^{\mu\alpha} \|\varphi_\mu * f\|_{L^p(\mathbf{R}^n)})^q\right)^{1/q} < \infty.$$

Let $\dot{H}_p^{\alpha q}$ denote the space of all radial elements of $\dot{B}_p^{\alpha q}$, together with the same quasi norm as $\dot{B}_p^{\alpha q}$. Again, by a radial element, we mean that that there exists a radial element in its class. For $s = \{s_{\mu k}\}_{\mu \in \mathbf{Z}, k \in \mathbf{Z}^+}$, define

$$\|s\|_{\dot{h}_p^{\alpha q}} = \left(\sum_{\mu \in \mathbf{Z}} \left\| \sum_{k=1}^{\infty} 2^{\mu\alpha} |s_{\mu k}| \tilde{\chi}_{A_{\mu k}} \right\|_{L^p}^q\right)^{1/q}.$$

Let $\{\varphi_{\mu k}\}_{\mu \in \mathbf{Z}, k \in \mathbf{Z}^+}$ and $\{\psi_{\mu k}\}_{\mu \in \mathbf{Z}, k \in \mathbf{Z}^+}$ be as in §2.

5.1. Theorem

Let $\alpha \in \mathbf{R}$ and $0 < p, q \le \infty$. The operator

$$S : \dot{H}_p^{\alpha q} \to \dot{h}_p^{\alpha q}$$

defined by

$$S(f) = \{\langle f, \varphi_{\mu k} \rangle\}$$

is bounded.

5.2. Theorem

Let $\alpha \in \mathbf{R}$ and $0 < p, q \le \infty$. The operator

$$T : \dot{h}_p^{\alpha q} \to \dot{H}_p^{\alpha q}$$

defined by

$$T(\{s_{\mu k}\}) = \sum_{\mu \in \mathbf{Z}} \sum_{k=1}^{\infty} s_{\mu k} \psi_{\mu k}$$

is bounded.

5.1. Corollary

Let $\alpha \in \mathbf{R}$ and $0 < p, q \le \infty$. Then for a radial $f \in \mathcal{S}'(\mathbf{R}^n)$,

$$\|f\|_{\dot{B}_p^{\alpha q}} \approx \|\{\langle f, \varphi_{\mu k}\rangle\}\|_{\dot{h}_p^{\alpha q}}.$$

Proof of Theorem 5.1. As in the proof of Theorem 2.1, we have

$$\sum_{k=1}^{\infty} 2^{\mu\alpha} |\langle f, \varphi_{\mu k}\rangle| \tilde{\chi}_{A_{\mu k}}(x) \le c 2^{\mu\alpha} \varphi_\mu^{**} f(x) \quad \text{a.e.}$$

We will take $\gamma > n/p$ in the definition of φ_μ^{**}. According to Peetre's estimate [18],

$$\varphi_\mu^{**} f(x) \le c(M(|\tilde{\varphi}_\mu * f|^{n/\gamma}))^{\gamma/n}(x).$$

Applying the ordinary maximal inequality (since $\gamma p/n > 1$) gives

$$\|\varphi_\mu^{**} f\|_{L^p} \le c\|\tilde{\varphi}_\mu * f\|_{L^p}$$

with c independent of μ and f. The desired inequality $\|S(f)\|_{\dot{h}_p^{\alpha q}} \le c\|f\|_{\dot{H}_p^{\alpha q}}$ now follows easily, since we obtain an equivalent norm to $\|\cdot\|_{\dot{B}_p^{\alpha q}}$ using $\tilde{\varphi}_\mu$ in place of φ_μ. $\square$

Proof of Theorem 5.2. By examining the proof of Theorem 2.2 we see that

$$\|\varphi_\mu * f\|_{L^p} \le c \sum_{\lambda=\mu-1}^{\mu+1} \left\| \left(M\left(\sum_{k=1}^{\infty} |s_{\lambda k}|^\eta \tilde{\chi}_{A_{\mu k}}^\eta \right)\right)^{1/\eta} \right\|_{L^p},$$

where, as before,

$$f = \sum_{\mu \in \mathbf{Z}} \sum_{k=1}^{\infty} s_{\mu k} \psi_{\mu k}.$$

Taking $0 < \eta \le 1$ such that $p/\eta > 1$, applying the ordinary maximal inequality, and using the essential disjointness of the sets $\{A_{\lambda k}\}_{k=1}^{\infty}$ (for fixed λ), we see that the right side of the last inequality is bounded by

$$c \sum_{\lambda=\mu-1}^{\mu+1} \left\| \sum_{k=1}^{\infty} |s_{\lambda k}| \tilde{\chi}_{A_{\lambda k}} \right\|_{L^p}.$$

This immediately gives the desired inequality $\|f\|_{\dot{H}_p^{\alpha q}} \le c\|s\|_{\dot{h}_p^{\alpha q}}$. □

For the molecular estimates, the only change in the definition of a molecule is that we need only assume that $0 < \eta \le 1$ satisfies $p/\eta > 1$. From Lemma 3.4 we get, with $p^* = \min\{1, p\}$, by either Minkowski's inequality if $p \ge 1$ or the p-triangle inequality if $p < 1$,

$$\begin{aligned} 2^{\mu\alpha} \left\| \varphi_\mu * \sum_{\lambda\in\mathbf{Z}} \sum_{k=1}^{\infty} s_{\lambda k} m_{\lambda k} \right\|_{L^p} &\le c \left(\sum_{\lambda\in\mathbf{Z}} 2^{-|\lambda-\mu|\varepsilon p^*} \left\| M\left(\sum_{k=1}^{\infty} 2^{\lambda\alpha\eta} |s_{\lambda k}|^{\eta} \tilde{\chi}^{\eta}_{A_{\lambda k}} \right) \right\|_{L^{p/\eta}}^{p^*/\eta} \right)^{1/p^*} \\ &\le c \left(\sum_{\lambda\in\mathbf{Z}} 2^{-|\lambda-\mu|\varepsilon p^*} \left\| \sum_{k=1}^{\infty} 2^{\lambda\alpha} |s_{\lambda k}| \tilde{\chi}_{A_{\lambda k}} \right\|_{L^p}^{p^*} \right)^{1/p^*} \\ &\le c \sum_{\lambda\in\mathbf{Z}} 2^{-|\lambda-\mu|\varepsilon/2} \left\| \sum_{k=1}^{\infty} 2^{\lambda\alpha} |s_{\lambda k}| \tilde{\chi}_{A_{\lambda k}} \right\|_{L^p}. \end{aligned}$$

The second inequality follows from the ordinary maximal inequality (since $p/\eta > 1$) and the essential disjointness of the sets $\{A_{\lambda k}\}_{k=1}^{\infty}$ for fixed λ. The last inequality is trivial if $p^* = 1$ and follows from Hölder's inequality if $p^* < 1$. We now have

$$\begin{aligned} \left\| \sum_{\lambda\in\mathbf{Z}} \sum_{k=1}^{\infty} s_{\lambda k} m_{\lambda k} \right\|_{\dot{H}_p^{\alpha q}} &\le c \left(\sum_{\mu\in\mathbf{Z}} \left(\sum_{\lambda\in\mathbf{Z}} 2^{-|\lambda-\mu|\delta/2} \left\| \sum_{k=1}^{\infty} 2^{\lambda\alpha} |s_{\lambda k}| \tilde{\chi}_{A_{\lambda k}} \right\|_{L^p} \right)^q \right)^{1/q} \\ &\le c \left(\sum_{\mu\in\mathbf{Z}} \left\| \sum_{k=1}^{\infty} 2^{\mu\alpha} |s_{\mu k}| \tilde{\chi}_{A_{\mu k}} \right\|_{L^p}^q \right)^{1/q} = c\|s\|_{\dot{h}_p^{\alpha q}}, \end{aligned}$$

as desired. Here we have used $\|a * b\|_{l^q} \le \|a\|_{l^1} \|b\|_{l^q}$ if $q \ge 1$ and $\|a * b\|_{l^q} \le \||a|^q * |b|^q\|_{l^1}^{1/q} \le \||a|^q\|_{l^1}^{1/q} \||b|\|_{l^1}^{1/q} = \|a\|_{l^q} \|b\|_{l^q}$ if $q < 1$.

Finally, the atomic decomposition described in Theorem 4.1 is valid for the pair $\dot{H}_p^{\alpha q}, \dot{h}_p^{\alpha q}$, with the same proof, except in this case we only require $\alpha > (n/\min\{1, p\}) - n - 1$ and p is allowed to take the value $+\infty$. Let $s_{\mu k}$ be defined as in the proof of Theorem 4.1. Then, using Peetre's estimate again,

$$
\begin{aligned}
\|s\|_{\dot{h}_p^{\alpha q}} &\le c\left(\sum_{\mu\in\mathbf{Z}}\left\|\sum_{k=1}^{\infty}2^{\mu\alpha}(\sup_{y\in A_{\mu k}}|\varphi_\mu * f(y)|)\chi_{A_{\mu k}}\right\|_{L^p}^q\right)^{1/q}\\
&\le c\left(\sum_{\mu\in\mathbf{Z}}\|2^{\mu\alpha}\tilde{\varphi}_\mu^{**}f\|_{L^p}^q\right)^{1/q}\\
&\le c\left(\sum_{\mu\in\mathbf{Z}}(2^{\mu\alpha}\|\varphi_\mu * f\|_{L^p})^q\right)^{1/q} = c\|f\|_{\dot{H}_p^{\alpha q}}.
\end{aligned}
$$

The converse estimate follows from the $\dot{H}_p^{\alpha q}$ version of Theorem 3.1, which, as discussed above, only requires that there exists some $0 < \eta \le 1$ such that $p/\eta > 1$ and $n + 1 + \alpha - n/\eta > 0$. This is guaranteed by our modified hypotheses.

6. Results for the Inhomogeneous Spaces

In addition to the homogeneous Besov and Triebel–Lizorkin spaces $\dot{B}_p^{\alpha q}$ and $\dot{F}_p^{\alpha q}$, there are also the inhomogeneous spaces $B_p^{\alpha q}$ and $F_p^{\alpha q}$. The Bessel potential spaces $L_\alpha^p \approx F_p^{\alpha 2}$, $1 < p < +\infty$, $\alpha > 0$, arise in this scale; in particular, when $\alpha = k \in \mathbf{Z}^+$ we have the usual Sobolev space L_k^p. One advantage is that for the inhomogeneous spaces we have quasi-norms on all of $\mathcal{S}'$ and not just the quotient $\mathcal{S}'/\mathcal{P}$. As usual in this subject (see, e.g., [11, §12]), the results for $\dot{B}_p^{\alpha q}$ and $\dot{F}_p^{\alpha q}$ have complete analogues for the inhomogeneous spaces. Since the proofs are virtually the same, we will be brief.

To set notation, pick radial functions $\Phi, \Psi \in \mathcal{S}(\mathbf{R}^n)$ satisfying supp $\hat{\Phi}, \hat{\Psi} \subseteq \{\xi : |\xi| \le 1\}$, $|\hat{\Phi}(\xi)|, |\hat{\Phi}(\xi)| \ge c > 0$ if $|\xi| \le 5/6$, and

$$
\overline{\hat{\Phi}}(\xi)\ \hat{\Psi}(\xi) + \sum_{\mu=1}^{\infty}\overline{\hat{\varphi}}_\mu(\xi)\hat{\psi}_\mu(\xi) = 1 \quad \text{for all } \xi \in \mathbf{R}^n, \tag{43}
$$

where φ_μ, ψ_μ are as in §2. For $0 < q \le \infty$, $0 < p < \infty$, $\alpha \in \mathbf{R}$, and $f \in \mathcal{S}'(\mathbf{R}^n)$, let

$$
\|f\|_{F_p^{\alpha q}} = \|\Phi * f\|_{L^p} + \left\|\left(\sum_{\mu=1}^{\infty}(2^{\mu\alpha}|\varphi_\mu * f|)^q\right)^{1/q}\right\|_{L^p}
$$

(see, e.g., [11, Lemma 12.1]). For the same q, α, f, and $0 < p \le \infty$, let

$$
\|f\|_{B_p^{\alpha q}} = \|\Phi * f\|_{L^p} + \left(\sum_{\mu=1}^{\infty}(2^{\mu\alpha}\|\varphi_\mu * f\|_{L^p})^q\right)^{1/q}.
$$

Let $R_p^{\alpha q}$ denote the space of radial elements in $F_p^{\alpha q}$ and $H_p^{\alpha q}$ the space of radial elements in $B_p^{\alpha q}$, with the same quasi-norms, respectively.

From (43) we obtain

$$f = \tilde{\Phi} * \Psi * f + \sum_{\mu=1}^{\infty} \tilde{\varphi}_\mu * \psi_\mu * f, \tag{44}$$

with convergence in $\mathcal{S}'$. For $k \in \mathbf{Z}^+$, let

$$\Phi_k = \left(\frac{2}{j_{\nu,k}^n J_{\nu+1}^2(j_{\nu,k})\omega_{n-1}} \right)^{1/2} \Phi * d\sigma_{j_{\nu,k}},$$

and similarly define $\{\Psi_k\}_{k=1}^{\infty}$. Then each Φ_k and Ψ_k is radial. If f is radial, then by applying Theorem 1.1 as in §2 we obtain the identity

$$f = \sum_{k=1}^{\infty} \langle f, \Phi_k \rangle \Psi_k + \sum_{\mu=1}^{\infty} \sum_{k=1}^{\infty} \langle f, \varphi_{\mu k} \rangle \psi_{\mu k}.$$

For a sequence $s = \{s_{\mu k}\}_{k \in \mathbf{Z}^+, \mu = 0,1,2,\ldots}$ and indices as above, define

$$\|s\|_{r_p^{\alpha q}} = \left\| \left(\sum_{\mu=0}^{\infty} \sum_{k=1}^{\infty} (2^{\mu\alpha} |s_{\mu k}| \tilde{\chi}_{A_{\mu k}})^q \right)^{1/q} \right\|_{L^p}$$

and

$$\|s\|_{h_p^{\alpha q}} = \left(\sum_{\mu=0}^{\infty} \left\| \sum_{k=1}^{\infty} 2^{\mu\alpha} |s_{\mu k}| \tilde{\chi}_{A_{\mu k}} \right\|_{L^p}^q \right)^{1/q}.$$

For $f \in \mathcal{S}'$, let $S(f) = \{s_{\mu k}\}$, where $s_{0k} = \langle f, \Phi_k \rangle$, and for $\mu \geq 1$, $s_{\mu k} = \langle f, \varphi_{\mu k} \rangle$. For $s = \{s_{\mu k}\}_{\mu,k}$, let

$$T(s) = \sum_{k=1}^{\infty} s_{0k} \Psi_k + \sum_{\mu=1}^{\infty} \sum_{k=1}^{\infty} s_{\mu k} \psi_{\mu k}.$$

6.1. Theorem

The maps $S : R_p^{\alpha q} \to r_p^{\alpha q}$ ($S : H_p^{\alpha q} \to h_p^{\alpha q}$) and $T : r_p^{\alpha q} \to R_p^{\alpha q}$ ($T : h_p^{\alpha q} \to H_p^{\alpha q}$) are bounded, and the composition $T \circ S$ is the identity on $R_p^{\alpha q}$ ($H_p^{\alpha q}$). In particular, $\|f\|_{R_p^{\alpha q}} \approx \|S(f)\|_{r_p^{\alpha q}}$ ($\|f\|_{H_p^{\alpha q}} \approx \|S(f)\|_{h_p^{\alpha q}}$).

Proof. Modify the proof in §2 by dropping negative indices and replacing φ_0 with Φ. $\square$

A family of smooth radial molecules has the form $\{m_{\mu k}\}_{\mu,k}$, where k ranges over $\mathbf{Z}^+$ and $\mu = 0, 1, 2, \ldots$. The only change in the definition is that when $\mu = 0$, (13) is not required.

6.2. Theorem

If $\{m_{\mu k}\}_{\mu,k}$ is a family of smooth molecules for $R_p^{\alpha q}$ $(H_p^{\alpha q})$, then there exists a constant $c > 0$ such that for all sequences $s = \{s_{\mu k}\}_{k\in\mathbf{Z}^+,\mu=0,1,2,\ldots}$,

$$\left\| \sum_{\mu=0}^{\infty} \sum_{k=1}^{\infty} s_{\mu k} m_{\mu k} \right\|_{R_p^{\alpha q}} \leq c \|s\|_{r_p^{\alpha q}}$$

(and similarly with $H_p^{\alpha q}$, $h_p^{\alpha q}$).

Proof. Note that in the proof of Theorem 3.1, we only need (13) to estimate $\varphi_\mu * m_{\lambda k}$ when $\mu < \lambda$. (The $\mu = \lambda$ case is covered by Lemma 3.3.) Here $\mu \geq 0$, so for $\lambda = 0$, (13) is never required. $\square$

For the smooth radial atomic decomposition, the decomposition takes the form

$$f = \sum_{\mu=0}^{\infty} \sum_{k=1}^{\infty} s_{\mu k} a_{\mu k},$$

where the $a_{\mu k}$ are as above, except that (ii) is not required for $\mu = 0$. The norm equivalence is, of course, $\|f\|_{R_p^{\alpha q}} \approx \|s\|_{r_p^{\alpha q}}$ $(\|f\|_{H_p^{\alpha q}} \approx \|s\|_{h_p^{\alpha q}})$.

7. Conclusion

We conclude with some problems and directions for further study. The first and most obvious question is whether there exists an orthonormal radial wavelet decomposition having good space-frequency localization and allowing norm characterizations for the spaces occuring in Littlewood–Paley theory. The "rectangular" wavelet theory, as developed by Lemarié and Meyer [15], Mallat [16], and others is based on dyadic translations and dilations. To work by analogy in the radial setting seems impossible.

Theorems 2.1 and 2.2 can be used to study radial linear operators (i.e., linear operators mapping radial functions to radial functions) on $\dot{R}_p^{\alpha q}$, in the same way that the φ-transform and wavelet transform can be used to study linear operators defined on all of $\dot{F}_p^{\alpha q}$. There is a notion of *radial almost diagonality* parallel to the standard notion of almost diagonality. In another direction, the radial atomic decomposition given by Theorem 4.1 together with the method of proving Theorem 3.1 can be used to study the restriction of a (not necessarily radial) linear operator to radial functions. We plan to discuss some of these issues elsewhere.

In reference [6] we describe a wavelet-type transform adapted to polar coordinates on $\mathbf{R}^2$, which generalizes the transform in this paper. The polar wavelets are not related to each other by translation or dilation. Some of them are rotations of each other. Continuing in the direction of removing symmetries, we could consider the general problem of developing a wavelet-type theory, including Littlewood–Paley norm equivalences, on an arbitrary compact Riemannian manifold.

References

[1] Benedetto, J. and Frazier, M., Eds., (1993). *Wavelets: Mathematics and Applications.* CRC Press, Boca Raton, FL.

[2] Chui, C. K., Ed., (1992). *Wavelets: A Tutorial in Theory and Applications*. Academic Press, Boston, MA.

[3] Daubechies, I., (1988). Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.* **41**, 909–996.

[4] David, G., Journé, J.-L., Semmes, S., (1985). Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation. *Rev. Mat. Iberoamericana* **1**, 1–56.

[5] Davis, K. and Chang, Y. C., (1987). Lectures on Bochner-Riesz means. *London Math. Soc. Lecture Note Ser.*, vol. 114, Cambridge Univ. Press, Cambridge.

[6] Epperson, J. and Frazier, M., (to appear). Polar wavelets and associated Littlewood–Paley theory, Dissertationes Mathematicae.

[7] Fefferman C. and Stein, E. M., (1971). Some maximal inequalities. *Amer. J. Math.* **93**, 107–115.

[8] Folland, G. B., (1992). *Fourier Analysis and Its Applications.* Wadsworth Inc., Belmont, CA.

[9] Frazier, M. and Jawerth, B., (1985). Decomposition of Besov spaces. *Indiana Univ. Math. J.* **34**, 777–799.

[10] ———, (1988). The φ-transform and applications to distribution spaces. Function Spaces and Applications (M. Cwikel et al., eds.), *Lecture Notes in Math.*, vol. 1302, Springer-Verlag, New York, 223–246.

[11] ———, (1990). A discrete transform and decompositions of distribution spaces. *J. Funct. Anal.* **93**, 34–170.

[12] Frazier, M., Jawerth, B., and Weiss, G., (1991). "Littlewood–Paley Theory and the Study of Function Spaces. *CBMS Reg. Conf. Ser. in Math.*, no. 79, Amer. Math. Soc., Providence, RI.

[13] Frazier, M. and Torres, R., (1993). The sampling theorem, φ-transform and Shannon wavelets for $\mathbf{R}$, $\mathbf{Z}$, $\mathbf{T}$, and $\mathbf{Z}_n$. *Wavelets: Mathematics and Applications* (Benedetto and Frazier, eds). CRC Press, Boca Raton, FL., 221–245.

[14] Kramer, H. P., (1959). A generalized sampling theorem. *J. Math. Phys.* **38**, 68–72.

[15] Lemarié, P. and Meyer, Y., (1986). Ondelettes et bases hilbertienne, *Rev. Mat. Iberoamericana* **2**, 1–18.

[16] Mallat S., (1989). Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbf{R})$. *Trans. Amer. Math. Soc.* **315**, 69–87.

[17] Meyer, Y., (1990). Ondelettes et Opérateurs I,II,III. Hermann, Paris.

[18] Peetre, J., (1975). On spaces of Triebel–Lizorkin type. *Ark. Mat.* **13**, 123–130.

[19] Ruskai, M. et al., Eds., (1992). *Wavelets and Their Applications.* Jones and Bartlett, Boston, MA.

[20] Stein, E. M. and Weiss G., (1971). *Fourier Analysis on Euclidean Spaces*. Princeton University Press, Princeton, NJ.

[21] Watson, G. N., (1944). *A Treatise on the Theory of Bessel Functions*, second ed. Cambridge University Press, Cambridge.

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