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Poisson Summation, the Ambiguity Function, and the Theory of Weyl–Heisenberg Frames

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ABSTRACT. *In the early 1960s research into radar signal synthesis produced important formulas describing the action of the two-dimensional Fourier transform on auto- and crossambiguity surfaces. When coupled with the Poisson Summation formula, these results become applicable to the theory of Weyl–Heisenberg systems, in the form of lattice sum formulas that relate the energy of the discrete crossambiguity function of two signals f and g over a lattice with the inner product of the discrete autoambiguity functions of f and g over a "complementary" lattice. These lattice sum formulas provide a framework for a new proof of a result of N. J. Munch characterizing tight frames and for establishing an important relationship between l^1-summability (condition A) of the discrete ambiguity function of g over a lattice and properties of the Weyl–Heisenberg system of g over the complementary lattice. This condition leads to formulas for upper frame bounds that appear simpler than those previously published and provide guidance in choosing lattice parameters that yield the most snug frame at a stipulated density of basis functions.*

1. Introduction

In the early 1960s, a major research effort was undertaken to establish the (narrowband) ambiguity function as a tool for radar signal synthesis [1, 10, 11, 12, 13, 14, 16]. This effort produced important formulas describing the action of the two-dimensional Fourier transform on auto- and crossambiguity surfaces. When coupled with Poisson summation, these formulas lead to equally important lattice sum formulas that can be applied to the study of Weyl–Heisenberg systems.

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1.1. Ambiguity Functions

For $f, g \in L^2(\mathbf{R})$, define the *crossambiguity function* of f and g by

$$\begin{aligned} A(f, g)(u, v) &= \int f(t) g^*(t - v) e^{-2\pi i u t} dt \\ &= \langle f, g_{u,v} \rangle, \qquad u, v \in \mathbf{R}, \end{aligned} \tag{1}$$

where

$$g_{u,v}(t) = g(t - v) e^{2\pi i u t}, \qquad u, v \in \mathbf{R}. \tag{2}$$

We denote $A(f, f)$ by $A(f)$ and call $A(f)$ the *autoambiguity* or simply the *ambiguity function* of f.

One goal of the above research was to provide a function-theoretic characterization of *ambiguity surfaces*

$$| A(f)(u, v) |^2, \qquad f \in L^2(\mathbf{R}). \tag{3}$$

The effort was unsuccessful but produced the following fundamental formula describing the action of the two-dimensional Fourier transform on crossambiguity surfaces.

A. Theorem

If $f, g \in L^2(\mathbf{R})$, then

$$\int \int | A(f, g)(u, v) |^2 \, e^{-2\pi i (xu + yv)} \, du \, dv = A(f)(y, -x) A^*(g)(y, -x), \qquad x, y \in \mathbf{R}. \tag{4}$$

We will also work with the following generalization.

B. Theorem

If $f_1, f_2, g_1, g_2 \in L^2(\mathbf{R})$, then

$$\begin{aligned} &\int \int A(f_1, f_2)(u, v) A^*(g_1, g_2)(u, v) e^{-2\pi i (xu + yv)} \, du \, dv \\ &= A(f_1, g_1)(y, -x) A^*(f_2, g_2)(y, -x), \qquad x, y \in \mathbf{R}. \end{aligned} \tag{5}$$

Setting $f = g$ in Theorem A, we have the *self-transform property* of ambiguity surfaces that states that 90° rotation ambiguity surfaces are invariant under two-dimensional Fourier transforms.

A *discrete ambiguity function* is formed by sampling an ambiguity function over the points of a lattice in the plane. In general, different lattices lead to discrete ambiguity functions having different norm-square energy. For signals f and g in Schwartz space, the Poisson summation formula applied to Theorem A results in the following discrete lattice sum formula relating the energy of the discrete crossambiguity function of f and g over a lattice with the inner product of the discrete ambiguity functions of f and g over a complementary lattice.

D1. For positive real numbers A and B,

$$\sum_m \sum_n |A(f,g)(mR,nS)|^2 = \frac{1}{RS} \sum_m \sum_n A(f)\left(\frac{n}{S}, \frac{m}{R}\right) A^*(g)\left(\frac{n}{S}, \frac{m}{R}\right). \tag{6}$$

Applying the Poisson summation formula to Theorem B leads to the following generalization. □

D2. For positive real numbers A and B,

$$\begin{aligned} &\sum_m \sum_n A(f_1,g_1)(mR,nS) A^*(f_2,g_2)(mR,nS) \\ &\quad = \frac{1}{RS} \sum_m \sum_n A(f_1,f_2)\left(\frac{n}{S}, \frac{m}{R}\right) A^*(g_1,g_2)\left(\frac{n}{S}, \frac{m}{R}\right). \end{aligned} \tag{7}$$

Setting $f = g$ in D1, we have that the energy of $A(f)$ over the lattice determined by R and S is equal to $\frac{1}{RS}$ times the energy of $A(f)$ over the complementary lattice determined by $\frac{1}{S}$ and $\frac{1}{R}$. □

1.2. Weyl–Heisenberg Systems

For $g \in L^2(\mathbf{R})$ and positive real numbers R and S, the *Weyl–Heisenberg* (W–H) *system* (g, R, S) is the set of signals

$$\{g_{mR,nS} : \quad m, n \in Z\}. \tag{8}$$

We call g the *analysis signal* and R and S the *lattice parameters* of the W–H system. A signal $f \in L^2(\mathbf{R})$ admits a (time-frequency) discrete representation over (g, R, S) by the set of all inner products

$$\langle f, g_{mR,nS} \rangle, \qquad m, n \in Z, \tag{9}$$

or equivalently by the values of the discrete crossambiguity function formed by sampling $A(f, g)$ over the lattice determined by R and S.

In general, there may not exist a numerically stable algorithm reconstructing f from the inner products in (9). In [4] conditions on (g, R, S) for the existence of such an algorithm are described using the language of frames.

Associate to (g, R, S) the operator A_g of $L^2(\mathbf{R})$ defined by

$$A_g f(m,n) = \langle f, g_{mR,nS} \rangle = A(f,g)(mR,nS), \qquad m,n \in Z, \quad f \in L^2(\mathbf{R}). \tag{10}$$

Set

$$\| A_g f \|^2 = \sum_m \sum_n |A(f,g)(mR,nS)|^2 . \tag{11}$$

We say that (g, R, S) has an *upper frame bound* if A_g is continuous. A real number $B < \infty$ is called an upper frame bound of (g, R, S), if

$$\| A_g f \|^2 \leq B \| f \|^2, \qquad f \in L^2(\mathbf{R}). \tag{12}$$

If A_g is continuous, the *adjoint* A_g^* is given by

$$A_g^*(\alpha) = \sum_m \sum_n \alpha(m, n) g_{mR,nS}, \qquad \alpha \in l^2(Z^2). \tag{13}$$

(g, R, S) is called a *W–H frame* if A_g is continuous and A_g^* is surjective. The surjectivity of A_g^* is equivalent to any one of the following conditions,

- $A_g^* A_g$ is boundedly invertible in $L^2(\mathbf{R})$.
- There exists $A > 0$ such that

$$A \| f \|^2 \leq \| A_g f \|^2, \qquad f \in L^2(\mathbf{R}).$$

We call A a *lower frame bound* of (g, R, S).

If (g, R, S) is a frame, then f can be reconstructed in $L^2(\mathbf{R})$ from the inner products given in (9) by the following argument. Set $T_g = A_g^* A_g$. In $L^2(\mathbf{R})$,

$$T_g f = \sum_m \sum_n \langle f, g_{mR,nS} \rangle g_{mR,nS}. \tag{14}$$

Since T_g is boundedly invertible in $L^2(\mathbf{R})$, we can define

$$h = T_g^{-1} g \in L^2(\mathbf{R}). \tag{15}$$

Then

$$h_{mR,nS} = T_g^{-1}(g_{mR,nS}) \tag{16}$$

and in $L^2(\mathbf{R})$

$$f = \sum_m \sum_n \langle f, g_{mR,nS} \rangle h_{mR,nS}. \tag{17}$$

A frame (g, R, S) is called a *tight frame* if for some A

$$\| A_g f \|^2 = A \| f \|^2, \qquad f \in L^2(\mathbf{R}).$$

We call A the *frame constant* of (g, R, S).

If (g, R, S) is a tight frame, then we have

$$f = A^{-1} \sum_m \sum_n \langle f, g_{mR,nS} \rangle g_{mR,nS}. \tag{18}$$

1.3. Main Results

In §2, we prove D1 and D2 for signals f and g in Schwartz space by direct application of Poisson summation (PS). Operator methods will then be called on to extend their validity to signals outside Schwartz space for application to Weyl–Heisenberg systems and frame theory.

Condition D1, when valid, provides a formula for $\| A_g f \|^2$ in terms of the inner product of the discrete autoambiguity functions of f and g over the complementary lattice determined by $\frac{1}{S}$ and $\frac{1}{R}$. D1 linearizes the impact of the discrete autoambiguity function of g on $\| A_g f \|_{l^2}^2$. In particular, applying the Schwartz inequality to the right-hand side of D1 factors out the lattice sum,

$$\sum_m \sum_n \mid A(g) \left(\frac{n}{S}, \frac{m}{R} \right) \mid, \tag{19}$$

in expressions for the upper bound. In §3, we will study the consequences of the convergence of this lattice sum on properties of (g, R, S) and derive a simple condition (condition B) for tight frames first proved in [7]. One consequence is that (g, R, S) is a tight frame if and only if $(g, \frac{1}{S}, \frac{1}{R})$ is orthogonal. The case of the Gaussian W–H system is considered in §4. New formulas for upper frame bounds will be derived that appear substantially simpler than those previously published. Based on [5], we will show that the lattice sum is the minimal upper frame bound when $g(t) = 2^{1/4} e^{-\pi t^2}$ and $\frac{1}{RS}$ is an even integer.

2. Proofs of D1 and D2

Denote n-dimensional *Schwartz space* by S_n. S_n is closed under addition, convolution, and multiplication and is a dense subspace in several normed spaces including $L^1(\mathbf{R})$ and $L^2(\mathbf{R})$ [9]. The Fourier transform is a linear isomorphism of S_n onto S_n.

The Poisson summation formula (PS) holds in every dimension n, but for our purposes, we require only the two-dimensional case.

Poisson Summation. For $\phi \in S_2$ and positive real numbers R and S, the expression

$$P(\phi)(x, y) = \sum_m \sum_n \phi(x + mR, y + nS) \tag{20}$$

absolutely and uniformly converges to a periodic C^∞ function with respect to the lattice determined by R and S and has absolutely converging Fourier series (FS)

$$\frac{1}{RS} \sum_m \sum_n \hat{\phi} \left(\frac{m}{R}, \frac{n}{S} \right) e^{2\pi i (\frac{m}{R} x + \frac{n}{S} y)}, \tag{21}$$

where $\hat{\phi}$ is the Fourier transform of ϕ. In particular, if $\phi \in S_2$, then $\hat{\phi} \in S_2$, and

$$\sum_m \sum_n \phi(mR, nS) = \frac{1}{RS} \sum_m \sum_n \hat{\phi}\left(\frac{m}{R}, \frac{n}{S}\right), \tag{22}$$

where both sides converge absolutely. Since S_2 is closed under multiplication, we also have

$$\sum_m \sum_n \mid \phi(mR, nS) \mid^r < \infty, \qquad \phi \in S_2, \tag{23}$$

for all integers $r > 0$.

For $\phi \in L^1(\mathbf{R}^2)$, $P(\phi)$ converges in $L^1(\mathbf{R}^2)$ to a periodic integrable function with respect to the lattice determined by R and S having Fourier coefficients

$$\left\{\hat{\phi}\left(\frac{m}{R}, \frac{n}{S}\right)\right\}, \qquad m, n \in Z, \tag{24}$$

but the corresponding FS does not in general converge [6].

Since $f, g \in S_1$ implies $A(f, g) \in S_2$, we can apply PS to $\mid A(f, g)(x, y) \mid^2$ and use Theorem A to prove the following result.

1. Theorem

If $f, g \in S_1$ and R and S are positive real numbers, then

$$P(f, g)(x, y) = \sum_m \sum_n \mid A(f, g)(x + mR, y + nS) \mid^2 \tag{25}$$

uniformly converges to a periodic C^∞ function with respect to the lattice determined by R and S and has absolutely converging FS*,*

$$\frac{1}{RS} \sum_m \sum_n A(f)\left(\frac{n}{S}, -\frac{m}{R}\right) A^*(g)\left(\frac{n}{S}, -\frac{m}{R}\right) e^{2\pi i(\frac{m}{R}x + \frac{n}{S}y)}. \tag{26}$$

1. Corollary

If $f, g \in S_1$, then D1 *holds.*

For $f_1, f_2, g_1, g_2 \in S_1$,

$$A(f_1, g_1)(x, y) A^*(f_2, g_2)(x, y) \in S_2, \tag{27}$$

and we can argue as above with Theorem B to prove the following result.

2. Corollary

If $f_1, f_2, g_1, g_2 \in S_1$, then D2 *holds.*

For application to W–H systems we must extend the validity of D1 and D2 to signals outside Schwartz space. Consider the W–H system (g, R, S) with associated operator A_g.

2. Theorem

If $g \in S_1$, then (g, R, S) has an upper frame bound

$$B = \frac{1}{RS}\sum_m \sum_n \mid A(g)\left(\frac{n}{S}, \frac{m}{R}\right) \mid . \tag{28}$$

Proof. For $f \in S_1$, we have by Corollary 1,

$$|| A_g f ||^2 = \frac{1}{RS}\sum_m \sum_n A(f)\left(\frac{n}{S}, \frac{m}{R}\right) A^*(g)\left(\frac{n}{S}, \frac{m}{R}\right). \tag{29}$$

Since $A(f)$ achieves its maximum $|| f ||^2$ at the origin,

$$|| A_g f ||^2 \leq B \, || f ||^2, \qquad f \in S_1. \tag{30}$$

Since S_1 is dense in $L^2(\mathbf{R})$, the inequality holds for all $f \in L^2(\mathbf{R})$, completing the proof of the theorem. $\square$

A direct computation shows that

$$P(f, g)(x, y) = || A_g f_{-x,-y} ||^2, \qquad x, y \in \mathbf{R}, \quad f, g \in L^2(\mathbf{R}), \tag{31}$$

which can be used to relate properties of A_g with properties of $P(f, g)$.

3. Theorem

If (g, R, S) has an upper frame bound $B < \infty$, then $P(f, g)$ is continuous and

$$P(f, g)(x, y) \leq B \, || f ||^2, \qquad x, y \in \mathbf{R}, \quad f \in L^2(\mathbf{R}). \tag{32}$$

Proof. Take $f \in L^2(\mathbf{R})$. Since A_g is continuous and the mapping

$$(x, y) \to f_{-x,-y} : \mathbf{R}^2 \to L^2(\mathbf{R})$$

is continuous, the mapping

$$P(f, g) : (x, y) \to \|\, A_g f_{-x,-y} \,\|^2 : \mathbf{R}^2 \to \mathbf{C}$$

is continuous and

$$P(f, g)(x, y) \leq B \,\|\, f_{-x,-y} \,\|^2 = B \,\|\, f \,\|^2,$$

completing the proof of the theorem. □

The same argument shows that if (g, R, S) is a frame with frame bounds $0 < A \leq B < \infty$, then

$$A \,\|\, f \,\|^2 \leq P(f, g)(x, y) \leq B \,\|\, f \,\|^2, \qquad x, y \in \mathbf{R}, \quad f \in L^2(\mathbf{R}). \tag{33}$$

In particular, if (g, R, S) is a tight frame, then

$$P(f, g)(x, y) = A \,\|\, f \,\|^2, \qquad x, y \in \mathbf{R}, \quad f \in L^2(\mathbf{R}). \tag{34}$$

Under the conditions of Theorem 3, the Fourier series of $P(f, g)$ will not, in general, converge, but we do have the following result.

4. Theorem

If $g \in S_1$ and $f \in L^2(\mathbf{R})$, then $P(f, g)$ is continuous and has absolutely converging Fourier series.

Proof. $P(f, g)$ is continuous by Theorems 2 and 3. The absolute convergence of the Fourier series follows from

$$\sum_m \sum_n \left| A(f)\left(\frac{n}{S}, \frac{m}{R}\right)\right| \left| A(g)\left(\frac{n}{S}, \frac{m}{R}\right)\right| \leq \|\, f \,\|^2 \sum_m \sum_n \left| A(g)\left(\frac{n}{S}, \frac{m}{R}\right)\right|,$$

completing the proof of the theorem. □

3. Corollary

D1 *holds for all $g \in S_1$ and $f \in L^2(\mathbf{R})$.*

For application to W–H systems we need to reverse the conditions on f and g. We use Theorem 4 and the formula

$$P(g, f)(-x, -y) = P(f, g)(x, y), \qquad x, y \in \mathbf{R}, \quad f, g \in L^2(\mathbf{R}), \tag{35}$$

to prove the following result.

4. Corollary

D1 *holds for all* $f \in S_1$ *and* $g \in L^2(\mathbf{R})$.

Arguing in the same way we have the following results.

5. Corollary

D2 *holds for all* $g_1, g_2 \in S_1$ *and* $f_1, f_2 \in L^2(\mathbf{R})$.

6. Corollary

D2 *holds for all* $f_1, f_2 \in S_1$ *and* $g_1, g_2 \in L^2(\mathbf{R})$.

3. Lattice Sum Conditions

We will study the effect on a W–H system (g, R, S) of imposing conditions on the discrete ambiguity function of g over the complementary lattice determined by $\frac{1}{S}$ and $\frac{1}{R}$.

3.1. l^1-Lattice Sum Condition

A W–H system (g, R, S) is said to satisfy condition A if

$$\sum_m \sum_n \left| A(g)\left(\frac{n}{S}, \frac{m}{R}\right) \right| < \infty. \tag{36}$$

5. Theorem

If (g, R, S) *satisfies condition A, then for all* $f \in L^2(\mathbf{R})$*,*

1. (g, R, S) has an upper frame bound

$$B = \frac{1}{RS} \sum_m \sum_n \left| A(g)\left(\frac{n}{S}, \frac{m}{R}\right) \right| .$$

2. $P(f, g)$ is continuous and has absolutely converging Fourier series.
3. D1 holds for f and g.

Proof. Corollary 4 implies that D1 holds for f and g, whenever $f \in S_1$. Arguing as in the proof of Theorem 2, (g, R, S) has an upper frame bound B. From Theorem 3 and the proof of Theorem 4, $P(f, g)$ is continuous and has absolutely converging Fourier series for all $f \in L^2(\mathbf{R})$. In particular, D1 holds for $f \in L^2(\mathbf{R})$ and g, completing the proof of the theorem. □

Two W–H systems (g_1, R, S) and (g_2, R, S) are said to satisfy condition A′ if

$$\sum_m \sum_n \mid A(g_1, g_2)\left(\frac{n}{S}, \frac{m}{R}\right) \mid < \infty. \tag{37}$$

Arguing as above from Corollary 6 and D2, we can prove the following result.

6. Theorem

If (g_1, R, S) *and* (g_2, R, S) *satisfy condition* A′*, then* D2 *holds for* f_1, g_1, f_2, g_2*, whenever* $f_1, f_2 \in L^2(\mathbf{R})$.

3.2. Tight Frame

A W–H system (g, R, S) is said to satisfy condition B if $A(g)(\frac{n}{S}, \frac{m}{R}) = 0$, unless $m = n = 0$. Based on certain results by M. Rieffel in [8], R. Howe and T. Steger (see [2]) show that a W–H system (g, R, S) cannot be a frame when $RS > 1$. *We will assume throughout that* $RS \leq 1$ and use this assumption to show the existence of a W–H system satisfying condition B.

Define

$$h(t) = \begin{cases} 1, & 0 \leq t < S, \\ 0, & otherwise. \end{cases}$$

Since $RS \leq 1$,

$$h(t)h\left(t - \frac{m}{R}\right) = 0 \quad unless \;\; m = 0. \tag{38}$$

Condition B follows from

$$A(h)\left(\frac{n}{S}, 0\right) = \int_0^S e^{-2\pi i \frac{n}{S} t}\, dt = \begin{cases} S, & n = 0, \\ 0, & otherwise. \end{cases}$$

We can now give a new proof of a result that first appeared in [7].

7. Theorem

(g, R, S) *is a tight frame if and only if* (g, R, S) *satisfies condition* B. *In this case, the frame constant is*

$$B = \frac{\| g \|^2}{RS}. \tag{39}$$

Proof. If (g, R, S) is a tight frame with frame constant A, then by the comments following Theorem 3, $P(g) \equiv P(g, g)$ is the constant function $A \| g \|^2$ and has Fourier coefficients

$$\frac{1}{RS} \mid A(g)\left(\frac{n}{S}, \frac{m}{R}\right) \mid^2 = \begin{cases} \frac{||g||^4}{RS}, & n = m = 0, \\ 0, & otherwise, \end{cases}$$

proving that g satisfies condition B.

Conversely if (g, R, S) satisfies condition B, then by Theorem 5, D1 holds for $f \in L^2(\mathbf{R})$ and g and we have

$$\begin{aligned} || A_g f ||^2 &= \frac{1}{RS} \sum_m \sum_n A(f)\left(\frac{n}{S}, \frac{m}{R}\right) A^*(g)\left(\frac{n}{S}, \frac{m}{R}\right) \\ &= \frac{|| g ||^2}{RS} || f ||^2, \end{aligned} \tag{40}$$

completing the proof of the theorem. □

Theorem 7 implies the following important corollary [15].

7. Corollary

(g, R, S) is a tight frame if and only if $(g, \frac{1}{S}, \frac{1}{R})$ is orthogonal.

A W–H system (g, R, S) is called *weakly linearly independent* if

$$\sum_n \sum_m a_{n,m} g_{n/S,m/R} = 0 \tag{41}$$

with $\sum_n \sum_m \mid a_{n,m} \mid^2 < \infty$ implies $a_{n,m} = 0$, for all $n, m \in Z$.

8. Theorem

If (g, R, S) satisfies condition A *and has dense linear span in $L^2(\mathbf{R})$, then (g, R, S) is weakly linearly independent.*

Proof. Suppose $\sum_n \sum_m a_{n,m} g_{n/S,m/R} = 0$ with $\sum_n \sum_m \mid a_{n,m} \mid^2 < \infty$. Take any tight frame (h, R, S). By Corollary 7, $(h, \frac{1}{S}, \frac{1}{R})$ is orthogonal, implying

$$f = \sum_n \sum_m b_{n,m} h_{n/S,m/R} \in L^2(\mathbf{R}),$$

where $b_{n,m} = a^*_{-n,-m} e^{-2\pi i n m/RS}$. By Theorem 5, D1 holds for f and g. Direct computation shows that for all $r, s \in \mathbf{Z}$

$$|| A_g f ||^2 = \frac{|| h ||^2}{RS} \sum_s \sum_r b_{s,r} \sum_n \sum_m b^*_{s-n,r-m} e^{2\pi i (s-n)m/RS} A^*(g)\left(\frac{n}{S}, \frac{m}{R}\right) \tag{42}$$

and that

$$\sum_n \sum_m b^*_{s-n,r-m} e^{2\pi i(s-n)m/RS} A^*(g)\left(\frac{n}{S}, \frac{m}{R}\right) = \sum_n \sum_m a_{n,m} e^{-2\pi inr/RS} A^*(g)\left(\frac{n+s}{S}, \frac{m+r}{R}\right)$$

$$= e^{-2\pi inr/RS} \left\langle \sum_n \sum_m a_{n,m} g_{n/S,m/R}, g_{-s/S,-r/R} \right\rangle$$

$$= 0.$$

Since the linear span of (g, R, S) is dense in $L^2(\mathbf{R})$, $f = 0$, completing the proof of the theorem. □

4. Frame Bound Calculation: Gaussian Window

By Theorem 5, if (g, R, S) satisfies condition A, then an upper frame bound is given by

$$\frac{1}{RS} \sum_m \sum_n \mid A(g)\left(\frac{n}{S}, \frac{m}{R}\right) \mid . \tag{43}$$

Consider the case of a Gaussian window of unit L^2 norm

$$g(t) = \left(\frac{1}{\pi\sigma^2}\right)^{\frac{1}{4}} e^{-t^2/2\sigma^2}. \tag{44}$$

A direct computation shows that

$$\mid A(g)(u, v) \mid = e^{-\frac{1}{2}(\frac{1}{2}(v/\sigma)^2 + 2\pi^2(\sigma u)^2)}. \tag{45}$$

If B is the minimal upper frame bound, then

$$B \leq \frac{1}{RS} \sum_n \sum_m e^{-\frac{1}{2}(\frac{1}{2}(nS/\sigma)^2 + 2\pi^2(mR\sigma)^2)}. \tag{46}$$

Setting $K = \frac{1}{RS}$, we can write

$$B \leq K \sum_m e^{-(\pi\sigma/KS)^2 m^2} \sum_n e^{-(S/2\sigma)^2 n^2}. \tag{47}$$

The following argument suggested by A. J. E. M. Janssen shows that equality holds in (47) whenever K is an even integer and $g(t) = 2^{1/4}e^{-\pi t^2}$, providing a basis for previously computed values for the Gaussian case [2, 3]. In these works, upper (and lower) frame bounds are computed by maximizing (and minimizing) expressions involving functions of the window g.

Set

$$f^{\delta}(t) = e^{-\pi\delta t^2}\sum_{n} e^{-\pi\delta n^2R^2}e^{2\pi i nRt} \in L^2(\mathbf{R}) \tag{48}$$

and place f^{δ} into the inequality

$$\frac{1}{\|\ f\ \|^2}\sum_{k}\sum_{l}\langle g, g_{k/S,l/R}\rangle\langle f_{k/S,l/R}, f\rangle \leq \sum_{k}\sum_{l} \mid \langle g, g_{k/S,l/R}\rangle \mid . \tag{49}$$

With $K = \frac{1}{RS}$ an even integer

$$\langle g, g_{k/S,l/R}\rangle = e^{-\frac{\pi}{2}(l^2/R^2+k^2/S^2)} = \mid \langle g, g_{k/S,l/R}\rangle \mid . \tag{50}$$

A direct computation shows that the left-hand-side of (49) can be written as

$$\frac{\sum_l e^{-\frac{\pi}{2}(1+\delta)l^2/R^2}\sum_k e^{-\frac{\pi}{2}k^2/S^2}\sum_m e^{-\frac{\pi}{2}\delta R^2m^2}(-1)^{ml}e^{-\frac{\pi}{2}\delta^{-1}(k/S+mR)^2}}{\sum_m e^{-\frac{\pi}{2}(\delta+\delta^{-1})R^2m^2}}. \tag{51}$$

As $\delta \downarrow 0$, the summation over m in the numerator vanishes unless $m = -\frac{k}{RS}$, an even integer. For the purpose of determining the limit in (51) as $\delta \downarrow 0$ we can replace the numerator by

$$\sum_{l} e^{-\frac{\pi}{2}(1+\delta)l^2/R^2}\sum_{k} e^{-\frac{\pi}{2}(1+\delta)k^2/S^2}. \tag{52}$$

It is now easy to see that this limit is

$$\sum_{l} e^{-\frac{\pi}{2}l^2/R^2}\sum_{k} e^{-\frac{\pi}{2}k^2/S^2}, \tag{53}$$

proving the claim.

In the general case, a lengthy argument [5] shows that the tightest bound in (47) results from choosing the delays of the two exponential terms to be equal, resulting in optimum S and R as follows:

$$S_0 = \sigma\sqrt{\frac{2\pi}{K}}, \tag{54}$$

$$R_0 = \frac{1}{\sigma\sqrt{2\pi K}}. \tag{55}$$

Then

$$B \leq B_0 = K\left(\sum_m e^{-\frac{\pi}{2}Km^2}\right)^2. \tag{56}$$

One may not be able to pick $S = S_0$ and $R = R_0$ in an application, in which case it is of interest to evaluate the bound corresponding to the mismatched case. For $K = 1$ (the Gabor case) computations show that the minimum B occurs at $\frac{S}{\sigma} = 2$. As K increases, the bound monotonically increases at fixed $\frac{S}{\sigma}$, and its optimum value approaches K from above, as predicted by (56).

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