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On Orthogonal Wavelets with the Oversampling Property

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ABSTRACT. *In this note, we consider orthogonal wavelets with the oversampling property. We prove that if an orthogonal scaling function with exponential decay has the oversampling property, then it has the sampling property (i.e., it takes values* 1 *at* 0 *and* 0 *at other integers); therefore, an orthogonal scaling function with compact support has the oversampling property if and only if it is the Haar function.*

1. Introduction

The sampling theorems in multiresolution spaces have been discussed recently by several researchers, such as Walter [1, 2], Aldroubi and Unser [3, 4], and Xia and Zhang [5]. These theorems are generalizations of the classical Shannon sampling theorem for band-limited signals. In [1] the following sampling representation was obtained. Assume the spaces $\{V_j\}$, $V_j \subset V_{j+1}$, form a multiresolution analysis (MRA) of $L^2(\mathbf{R})$ and that $\phi(t)$ is its associated scaling function; see §2 or [6]. Then for any $f \in V_0$

$$f(t) = \sum_n f(n)\chi(t-n),$$

where the Fourier transform of χ is defined as

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$$\hat{\chi}(\omega) = \frac{\hat{\phi}(\omega)}{\Phi(\omega)},$$

provided $\Phi(\omega) \stackrel{\Delta}{=} \sum_n \hat{\phi}(\omega + 2n\pi) \neq 0$ for any real ω. The same result can be found in Aldroubi–Unser [3, 4], where the sampling function $\chi(t)$ is described in the time domain.

In particular, if $\chi = \phi$, then for any $f \in V_0$

$$f(t) = \sum_n f(n)\phi(t-n). \tag{1.1}$$

It is clear that the condition $\chi = \phi$ is equivalent to $\phi(n) = \delta_{0,n}$, that is, $\phi(n) = 1$ for $n = 0$ and $\phi(n) = 0$ for nonzero integers. This property for a scaling function $\phi(t)$ is called the *sampling property* (also, $\phi(t)$ is called a cardinal scaling function). In this article, a scaling function is assumed to be an orthogonal scaling function (see §2), where the orthogonality is defined in the usual $L^2(\mathbf{R})$ sense, that is,

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\,dx, \qquad \text{for all } f, g \in L^2(\mathbf{R}).$$

For scaling functions with the sampling property, Xia and Zhang [5] showed that a scaling function with compact support has the sampling property if and only if it is the Haar function. Moreover, they presented a family of such scaling functions with exponential decay. However, many important families of orthogonal wavelets, such as Meyer wavelets and Daubechies wavelets, do not have the sampling property.

To weaken this sampling property on wavelets, Walter [2] proposed that instead of looking for a sampling function in V_0 we look for one in the dilation space V_1 and try to recover $f \in V_0$ by its values on the half integers. Mathematically, we want to find a scaling function $\phi(t)$ or a MRA $\{V_j\}$ so that for any $f \in V_0$

$$f(t) = \sum_n f(\frac{n}{2})\phi(2t-n). \tag{1.2}$$

The property (1.2) for a scaling function $\phi(t)$ is called the oversampling property with sampling rate $1/2$. Clearly, all scaling functions with the sampling property do satisfy (1.2) and therefore have the oversampling property. In [2] Walter proved that all Meyer type wavelets satisfy the oversampling property (1.2). In addition, he showed that under a certain condition a band-limited wavelet with the oversampling property (1.2) must be a Meyer type wavelet. However, many wavelet properties, such as compact supportness and exponential decay, have not been addressed in this context.

In this note, we consider the orthogonal scaling functions with the following general oversampling property: For a fixed integer $J \geq 0$ and any $f \in V_0$

$$f(t) = \sum_n f(\frac{n}{2^J})\phi(2^J t-n). \tag{1.3}$$

The property (1.3) for an orthogonal scaling function $\phi(t)$ is called the oversampling property with sampling rate 2^{-J}. Let $\mathcal{S}_J$ denote the set of all orthogonal scaling functions with the oversampling property (1.3). Thus, $\mathcal{S}_0$ consists of all orthogonal scaling functions with the sampling property and $\mathcal{S}_J \subset \mathcal{S}_{J+1}$ for $J = 0, 1, 2, \ldots$. The results in [2] also show that $\mathcal{S}_0 \neq \mathcal{S}_1$, i.e., the space of the orthogonal scaling functions with the sampling property is a proper subspace of that of the orthogonal scaling functions with the oversampling property. Let $\mathcal{S}_{\mathrm{e}}$ and $\mathcal{S}_{\mathrm{c}}$ denote all orthogonal scaling functions with exponential decay and compact support, respectively. In this note, we will prove that $\mathcal{S}_0 \cap \mathcal{S}_e = \mathcal{S}_J \cap \mathcal{S}_e$ and $\mathcal{S}_0 \cap \mathcal{S}_c = \mathcal{S}_J \cap \mathcal{S}_c = \{\text{the Haar function}\}$ for any $J \geq 0$. In other words, if an orthogonal scaling function with exponential decay has the oversampling property, then it has the sampling property; and if an orthogonal scaling function with compact support has the oversampling property, then it must be the Haar function $\chi_{[0,1)}(t)$, which is 1 when $0 \leq t < 1$ and 0 otherwise. These results imply that weakening the sampling property to the oversampling property for wavelets is not useful in generating more wavelet families with good decay properties.

2. Some Known Results in Wavelets

This section reviews some known results in multiresolution analysis (MRA). An *orthogonal multiresolution analysis* of $L^2(\mathbf{R})$ is a nested sequence of closed subspaces $\{V_j\}_{j\in\mathbf{Z}}$ of $L^2(\mathbf{R})$ such that

i. $\{\phi(t-n)\}$ is an orthogonal basis of V_0,
ii. $V_j \subset V_{j+1}$ for all $j \in \mathbf{Z}$,
iii. $f(t) \in V_j$ if and only if $f(2t) \in V_{j+1}$,
iv. $\overline{\bigcup_j V_j} = L^2(\mathbf{R})$ and $\bigcap_j V_j = \{0\}$,

where $\phi(t)$ is called an *orthogonal scaling function* associated with the MRA $\{V_j\}$. If $\psi(t)$ is an associated mother wavelet, then $\psi_{jk}(t) \triangleq 2^{j/2}\psi(2^j t - k),\ j, k \in \mathbf{Z}$, is an orthonormal basis of $L^2(\mathbf{R})$ (see [6]). Let $H(\omega)$ with $H(0) = 1$ be a lowpass filter with impulse response $\frac{1}{2}h_k$, and let $G(\omega)$ be the bandpass filter with impulse response $\frac{1}{2}g_k = (-1)^k \frac{1}{2}h_{1-k}$ so that H and G are the associated quadrature mirror filters. Then for any $\omega \in \mathbf{R}$

$$\hat{\phi}(\omega) = H(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}) = \prod_{k=1}^{\infty} H(2^{-k}\omega), \tag{2.1}$$

$$\hat{\psi}(\omega) = G(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}), \tag{2.2}$$

and

$$|H(\omega)|^2 + |G(\omega)|^2 = 1, \tag{2.3}$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ, i.e.,

$$\hat{\phi}(\omega) = \int_{-\infty}^{\infty} \phi(t) e^{-it\omega}\, dt.$$

Detailed introductions to wavelets may be found in [6–12].

For scaling functions with the sampling property (1.1),

$$\Phi_0(\omega) \stackrel{\Delta}{=} \Phi(\omega) = \sum_n \phi(n)e^{-in\omega} = \sum_n \hat{\phi}(\omega + 2n\pi) = 1 \quad \text{for } \omega \in \mathbf{R}. \tag{2.4}$$

The following properties were obtained in [5].

2.1. Proposition

An orthogonal scaling function $\phi(t)$ has the sampling property if and only if

$$H(\omega) = \frac{1}{2} + \frac{1}{2}\tilde{H}(2\omega)e^{i\omega}, \tag{2.5}$$

where $\tilde{H}(\omega)$ has the impulse response $\tilde{h}_k = h_{2k+1}$, $\tilde{H}(0) = 1$, and $|\tilde{H}(\omega)| \equiv 1$.

From this result, a family of orthogonal scaling functions with the sampling property and exponential decay was presented in [5].

2.2. Proposition

An orthogonal scaling function $\phi(t)$ with compact support has the sampling property if and only if it is the Haar function, i.e., $\phi(t) = \chi_{[0,1)}(t)$.

For scaling functions with the oversampling property (1.3), it was proved in [2] that (1.3) holds for all $f \in V_0$ if and only if

$$\hat{\phi}(\omega) = \Phi_J(\omega)\hat{\phi}(2^{-J}\omega), \tag{2.6}$$

where

$$\Phi_J(\omega) \stackrel{\Delta}{=} 2^{-J}\sum_k \phi(2^{-J}k)e^{-ik2^{-J}\omega} = \sum_k \hat{\phi}(\omega + 2^{J+1}k\pi). \tag{2.7}$$

3. Wavelets with the Oversampling Property

A wavelet $\psi(t)$ or a scaling function $\phi(t)$ is said to be of exponential decay if the lowpass filter $H(\omega)$ with impulse response h_k satisfies $|h_k| \le O(a^{|k|})$ for certain constant a with $0 < a < 1$. This implies that for a scaling function with exponential decay, the Laurent series

$$L(z) \stackrel{\Delta}{=} \frac{1}{2}\sum_k h_k z^k$$

is analytic on the open annulus $R_1 < |z| < R_2$ for some constants R_1 and R_2 with $0 < R_1 < 1$ and $1 < R_2 < \infty$. This suggests the following lemma.

3.1. Lemma

If a scaling function $\phi(t)$ has exponential decay, then its associated lowpass filter $H(\omega)$ has finitely many zeros in $[-\pi, \pi)$.

Proof. Assume that $H(\omega)$ has infinitely many zeros in $[-\pi, \pi)$. Then $L(z)$ has infinitely many zeros on the unit circle $|z| = 1$. Since $\phi(t)$ has exponential decay, $L(z)$ is analytic in a region D that contains the unit circle. Therefore, the condition that $L(z)$ has infinitely many zeros implies that $L(z) = 0$ for all $z \in D$, and therefore $H(\omega) = 0$ for all real ω. This contradicts the assumption that $H(\omega)$ is a lowpass filter. □

Now we give the main result concerning scaling functions with the oversampling property.

3.2. Theorem

If an orthogonal scaling function $\phi(t)$ has exponential decay and the oversampling property with sampling rate 2^{-J} for some nonnegative integer J, then $\phi(t)$ has the sampling property, i.e., $\phi(n) = \delta_{0,n}$.

Proof. From Lemma 1, $H(\omega)$ has only finitely many zeros in $[-\pi, \pi)$. Thus,

$$\hat{\phi}(\omega) = \prod_{j=1}^{\infty} H(2^{-j}\omega) \neq 0$$

almost everywhere in $[-2^J\pi, 2^J\pi]$. From (2.1) and (2.6),

$$\Phi_J(\omega) = \prod_{j=1}^{J} H(\frac{\omega}{2^j}) \quad \text{almost everywhere.} \tag{3.1}$$

From (1.3),

$$\phi(\frac{k}{2^J}) = \sum_n \phi(\frac{n}{2^J})\phi(k-n).$$

Taking the Fourier transform of both sides of this equation yields

$$\sum_k \phi(\frac{k}{2^J})e^{-ik\omega} = \Phi_0(\omega) \sum_n \phi(\frac{n}{2^J})e^{-in\omega}.$$

Utilizing (2.7) and (3.1) produces

$$\Phi_J(2^J\omega) = \Phi_0(\omega)\Phi_J(2^J\omega)$$

or, equivalently,

$$\prod_{j=1}^{J} H(2^{J-j}\omega) = \Phi_0(\omega)\prod_{j=1}^{J} H(2^{J-j}\omega).$$

Since $H(\omega)$ has only finitely many zeros in $[-\pi, \pi)$,

$$\prod_{j=1}^{J} H(2^{J-j}\omega) \neq 0 \quad \text{almost everywhere.}$$

Therefore, $\Phi_0(\omega) = 1$ almost everywhere in $\mathbf{R}$. That is, ϕ satisfies the sampling property. □

3.3. Corollary

An orthogonal scaling function $\phi(t)$ with compact support has the oversampling property with sampling rate 2^{-J} for certain nonnegative integer J if and only if it is the Haar function.

Proof. This is a direct consequence of Theorem 1 and Proposition 2. □

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