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Self-Similar Lattice Tilings

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ABSTRACT. *We study the general question of the existence of self-similar lattice tilings of Euclidean space. A necessary and sufficient geometric condition on the growth of the boundary of approximate tiles is reduced to a problem in Fourier analysis that is shown to have an elegant simple solution in dimension one. In dimension two we further prove the existence of connected self-similar lattice tilings for parabolic and elliptic dilations. These results apply to produce Haar wavelet bases and certain canonical number systems.*

1. Introduction

Let Γ be a lattice in $\mathbb{R}^m$, and let Q be a set whose translates by elements of Γ tessellate, or tile, $\mathbb{R}^m$. This sort of tesselation will be referred to as a lattice tiling. Given an expansive linear transformation A that induces an automorphism of Γ, we say that Q is (Γ, A)-self-similar if AQ is a union of translates of Q by elements of Γ. Then we can write $AQ = \bigcup_{k\in\mathcal{D}}(k + Q)$ and $\mathcal{D} \subset \Gamma$ is called the digit set for Q. For tiles that generate a multiresolution analysis, it is immediate that the digit set $\mathcal{D}$ must be a set of distinct coset representatives for the quotient group $\Gamma/A\Gamma$ [13]. Under this assumption on $\mathcal{D}$ a number of researchers have demonstrated the existence of a unique (Γ, A)-self-similar set Q with digit set $\mathcal{D}$ [3, 13]. In all but a small class of well-understood cases Q has revealed itself as a complicated set with fractal boundary.

In this paper we investigate conditions on Γ, A, and $\mathcal{D}$ under which the set Q will tile or lattice tile $\mathbb{R}^m$. This and related problems have been considered by other authors. Their work provides many examples of self-similar tilings, relates these tilings to several different areas of mathematics, and, in some instances, gives necessary and sufficient conditions for the existence of a (Γ, A)-self-similar tile Q. Although these conditions are interesting and enlightening, they are, in practice, difficult to verify in all but a few special cases; consequently, they have not proved to be suitable as the basis for a general theory of self-similar tilings. The lack of a theory is illustrated by the fact that even in dimension two it remains unknown whether for each transformation A there is a digit set $\mathcal{D}$ so that the Γ-translates of the associated self-similar set Q tile $\mathbb{R}^2$.

Our approach has several facets. First we show that there is always a subset $\Gamma' \subset \Gamma$ so that the Γ'-translates of Q tile $\mathbb{R}^m$. This leaves open the possibility that Q gives an aperiodic tiling of $\mathbb{R}^m$, which we conjecture is not the case. In terms of the data A, Γ, and $\mathcal{D}$ we construct a matrix C and show that the Γ-translates of Q tile $\mathbb{R}^m$ if and only if the spectral radius of C is smaller than the absolute value of the determinant of A. This result renders the question of whether a par-

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ticular Q is a lattice tile into a fairly straightforward calculation. A deeper study of the matrix C using Fourier analytic methods leads to a complete characterization of the self-similar tilings in dimension one. It is seen that the subset $\Gamma' \subset \Gamma$ is always a sublattice that is explicitly determined by the digits $\mathcal{D}$. In particular, when $0 \in \mathcal{D}$, the Γ-translates of Q tile $\mathbb{R}$ if and only if the elements of $\mathcal{D}$ are relatively prime. In dimension $n = 2$ the linear transformations can be classified as elliptic, parabolic, or hyperbolic by considering their projections into PSL $(2, \mathbb{C})$. Given a transformation A that is elliptic, parabolic, or rational hyperbolic (i.e., rational eigenvalues), we develop various methods that produce a digit set $\mathcal{D}$ for which the (Γ, A)-self-similar set Q tiles $\mathbb{R}^2$. Furthermore, for elliptic dilations we construct connected tiles. There is no method presented for dealing in a unified fashion with irrational hyperbolics. These results have applications to computational number theory and wavelet theory.

The problem of self-similar tilings has appeared in the literature in a variety of contexts. In the classical theory of tilings the related notion of a k-rep tile is considered and it is shown that these self-similar sets, when they exist, always "tile" $\mathbb{R}^m$ where "tile" is used in a fairly loose sense [14]. This approach is used to conclude that the unique (Γ, A)-self-similar set Q with digits $\mathcal{D}$ "tiles" $\mathbb{R}^m$ [3].

Another viewpoint comes from what has come to be known as the field of fractal geometry. Fractals are intriguing sets often characterized by their nonintegral Hausdorff dimension [5, 23]. The boundaries of many self-similar sets are fractals, and some authors have studied these and the more general fractal recurrent sets with a view to determining their Hausdorff dimensions [15]. This has produced a fairly well-developed theory based on a construction due to Dekking [10, 6]. Our initial approach is similar to theirs in that it gives necessary and sufficient conditions for the Γ-translates of Q to tile in terms of the matrix C that catalogues the growth of the boundary of objects approximating Q. This method was employed successfully by Gilbert [11] to compute the Hausdorff dimensions for a family of self-similar lattice tiles that arose from complex radix expansions [12].

A beautiful theory of self-similar tilings was put forth by Thurston [27]. His approach, although far more general in that it considers aperiodic self-similar tiling using several tiles, does not address the question of lattice invariance and, moreover, presupposes transformations A that are Euclidean similarities. A related approach is taken by Kenyon [17] whose work, although overlapping slightly with ours, does not pay much attention to the lattice issue.

Most recently, self-similar lattice tilings have been found to be of interest by investigators in the field of wavelets. In two independent papers [13, 22] the existence of a self-similar lattice tiling of $\mathbb{R}^m$ was shown to imply the existence of a generalized Haar basis for $L^2(\mathbb{R}^m)$. Such an orthonormal basis for $L^2(\mathbb{R}^m)$ is constructed from simple functions with support on a (Γ, A)-self-similar tile Q of $\mathbb{R}^m$ by composition with combinations of translations in Γ and dilations by A. Both papers present necessary and sufficient conditions for Q to tile based on a result by Cohen [7]. In another recent paper [26] the existence of a digit set $\mathcal{D}$ producing a self-similar tile Q is demonstrated for a large class of transformations A that are Euclidean similarities.

After the presentation [1] and dissemination of our result, the general theory of self-similar tilings has received much attention and several papers have appeared since then or are in print [4, 28]. The most remarkable results are due to Lagarias and Wang [19], who seem to have solved the classification and existence problem for self-similar tilings.

2. Definitions and Results

Let $\lambda(S)$ denote the Lebesgue measure of a measurable set $S \subseteq \mathbb{R}^m$. We write $S \simeq T$ when the measurable sets S and T are equal up to a set of measure zero.

Let Q be a measurable subset of $\mathbb{R}^m$, Γ a lattice in $\mathbb{R}^m$, and Γ' a subset of Γ. The Γ-*translates of Q tile* $\mathbb{R}^m$ (or Q is a Γ'-tile of $\mathbb{R}^m$) if the following two conditions are satisfied:

$$\bigcup_{k\in\Gamma'}(k+Q)=\mathbb{R}^m, \tag{1}$$

$$(k+Q)\cap(l+Q)\simeq\emptyset \quad \text{for all } k,l\in\Gamma',\, k\neq l. \tag{2}$$

If Γ' is a lattice, then we speak of a *lattice tiling* of $\mathbb{R}^m$ by Q.

Let A be a nonsingular linear transformation of $\mathbb{R}^m$ realized with respect to a basis of Γ as an integral matrix with $q=|\det A|$. Then $A\Gamma$ is a sublattice of Γ, and the quotient $\Gamma/A\Gamma$ is a finite group of order q. A set Q is (Γ, A)-*self-similar* if there is a set $\mathcal{D}=\{k_0,\ldots,k_{q-1}\}$ of distinct coset representatives for the group $\Gamma/A\Gamma$, such that

$$AQ=\bigcup_{i=0}^{q-1}(k_i+Q), \qquad \text{and} \qquad (k_i+Q)\cap(k_j+Q)\simeq\emptyset \quad \text{for } i\neq j. \tag{3}$$

$\mathcal{D}$ is called the *digit set*. References to Γ and A will be suppressed where there is no chance for misunderstanding. It should also be noted that, although we use the term self-similar, the transformation A need not be a Euclidean similarity. Sometimes sets satisfying (3) are referred to as self-affine or self-replicating.

The transformation A is called a *dilation* if all its eigenvalues have modulus greater than one.

Given a lattice Γ, a dilation A, and a set of digits $\mathcal{D}$ as the data, several approaches exist for producing the unique (Γ, A)-self-similar set Q with digit set $\mathcal{D}$ [3, 13]. We shall follow the point of view taken in [13]. Since A is a dilation, the sum $\sum_{j=1}^{\infty}A^{-j}\epsilon_j$ converges for any sequence of $\epsilon_j\in\mathcal{D}$. Then the compact set

$$Q(A,\mathcal{D})=\{x\mid x=\sum_{j=1}^{\infty}A^{-j}\epsilon_j,\ \epsilon_j\in\mathcal{D}\} \tag{4}$$

is the unique (Γ, A)-self-similar set with digit set $\mathcal{D}$ [13].

What follows is an outline of our main results.

2.1. Theorem

There is a subset $\Gamma'\subseteq\Gamma$ such that the Γ'-translates of $Q(A,\mathcal{D})$ tile $\mathbb{R}^m$.

The theorem is rather crude in that it makes no assertion regarding the structure of Γ'. It is not even clear whether Γ' is self-similar in the sense that for each $k\in\Gamma'$, $A(k+Q)$ is a union of Γ'-translates of Q although it is obvious from the definition (4) that $Q(A,\mathcal{D})$ is always a union of translates of $Q(A,\mathcal{D})$ of the form k_i+Ak with $k_i\in\mathcal{D}$.

The approach is very similar to that mentioned in [14], which is also followed by Bandt [3].

The rest of our work is concerned with understanding when Γ' is either a sublattice of Γ or Γ itself.

Fix a basis $e_1, \ldots, e_m$ for Γ, and set $\mathcal{T}_0 = \{\pm e_1, \ldots, \pm e_m\}$. Recursively define the sets

$$\mathcal{T}_n = \{k \in \Gamma \mid (Ak + \mathcal{D}) \cap (l + \mathcal{D}) \neq \emptyset \text{ for some } l \in \mathcal{T}_{n-1}\} \tag{5}$$

and set $\mathcal{T} = \bigcup_{n=1}^{\infty} \mathcal{T}_n$. As we shall see in §4, the $\mathcal{T}_n$ eventually stabilize, from which we infer that $\mathcal{T}$ is finite. Define a $|\mathcal{T}| \times |\mathcal{T}|$-matrix C, called the *contact matrix*, with entries

$$c_{k,l} = |(Ak + \mathcal{D}) \cap (l + \mathcal{D})| \quad \text{for } k, l \in \mathcal{T}. \tag{6}$$

We use $|S|$ to denote the cardinality of a set S.

2.2. Theorem

The Γ-translates of $Q(A, \mathcal{D})$ tile $\mathbb{R}^m$ if and only if the contact matrix has spectral radius less than $|\det A|$.

The entries of the contact matrix are easily computed from the basic data Γ, A, and $\mathcal{D}$. Thus Theorem 2.2 provides an algorithm, which can be implemented on a computer, for determining whether $Q(A, \mathcal{D})$ is a Γ-tile.

In dimension one we use methods from Fourier analysis to study the eigenvalues of the contact matrix. This leads to a complete understanding of the tiling properties of the sets $Q(A, \mathcal{D})$ as detailed in the following theorem.

2.3. Theorem

Let $A : \mathbb{R} \mapsto \mathbb{R}$ be the map $Ax = qx$ for $q \in \mathbb{Z}$, $|q| > 1$; and let $\mathcal{D} = \{k_0, \ldots, k_{q-1}\}$. Set $d = \gcd(k_1 - k_0, k_2 - k_0, \ldots, k_{q-1} - k_0)$, where $k_i \equiv i \pmod{q}$. Then the $d\mathbb{Z}$-translates of $Q(A, \mathcal{D})$ tile $\mathbb{R}$. In particular, $Q(A, \mathcal{D})$ is a $\mathbb{Z}$-tile if and only if the numbers $k_1 - k_0, k_2 - k_0, \ldots, k_{q-1} - k_0$ are relatively prime.

Although Theorem 2.2 is computationally efficient, it still leaves open the question of the existence of lattice tilings.

In dimension two we classify the dilation A as *elliptic, parabolic*, or *hyperbolic* if A has, respectively, no real eigenvalues, one real eigenvalue, or two real eigenvalues. In this setting the orientation-preserving Euclicean similarities are all elliptic. One further distinguishes a class of *rational* dilations, that is, those dilations with rational eigenvalues. All parabolic and some hyperbolic dilations are of this type.

Given the data Γ and A one would like to determine a digit set $\mathcal{D}$ so that $Q(A, \mathcal{D})$ is a Γ-tile. We give general methods for doing this in dimension two in all cases except that of the irrational hyperbolic dilations.

Let A be a rational dilation. Then A has an eigenvector $e_1 \in \Gamma$ of minimal norm. Choose $e_2 \in \Gamma$ so that e_1 and e_2 are a basis for Γ. The eigenvalue a is an integer that divides $q = |\det A|$. Set $c = q/a$, and let $L = \{l_0, l_1, l_2, \ldots\}$ be an arbitrary sequence of integers. Define the set

$$\mathcal{B}(L) = \{(l_j + k)e_1 + je_2 | j = 0, 1, \ldots, |c| - 1, k = 0, 1, \ldots, |a| - 1\}. \tag{7}$$

2.4. Theorem

Let A be a rational dilation of the lattice Γ. Then for any sequence of integers L, $\mathcal{D} = \mathcal{B}(L)$ is a digit set and the Γ-translates of $Q(A, \mathcal{D})$ tile $\mathbb{R}^2$.

In order to treat the elliptic case we need some notions of a topological nature. Fix a basis e_1, e_2 of Γ, and recall the definition of $\mathcal{T}_0 = \{\pm e_1, \pm e_2\}$. A set $S \subseteq \Gamma$ is $\mathcal{T}_0$-*connected* if for any $k, l \in S$ there is a sequence $k = h_0, h_1, \dots h_r = l$ of elements in S, called a *path*, so that $h_{j+1} - h_j \in \mathcal{T}_0$.

Also we say that $k \in \Gamma$, $k \neq 0$ *pairs faces of* the digit set $\mathcal{D}$, if there is an $e \in \mathcal{T}_0$ so that $(Ak + \mathcal{D}) \cap (e + \mathcal{D}) \neq \emptyset$.

2.5. Theorem

Let $\{e_1, e_2\}$ be a basis for Γ and $\mathcal{D}$ be a $\mathcal{T}_0$-connected set of digits so that e_1 and e_2 both pair faces of $\mathcal{D}$. Then $Q(A, \mathcal{D})$ is a connected set and the Γ-translates of $Q(A, \mathcal{D})$ tile $\mathbb{R}^2$.

The previous theorem gives sufficient conditions on the digit set $\mathcal{D}$ for $Q(A, \mathcal{D})$ to be a connected Γ-tile. This can be employed with many but not all irrational hyperbolic dilations. For elliptic dilations, however, there is a definite procedure for generating a digit set that satisfies the hypothesis of Theorem 2.5.

Let $a_1, a_2 \in \mathbb{R}^2$ be linearly independent. A half-open parallelogram is one of the sets $\{x = sa_1 + ta_2 | 0 \leq s < 1, 0 \leq t < 1\}$ or $\{x = sa_1 + ta_2 | 0 < s \leq 1, 0 \leq t < 1\}$.

2.6. Theorem

Let A be an elliptic dilation of the lattice Γ. Then there is a basis $\{e_1, e_2\}$ for Γ and a half-open parallelogram P spanned by Ae_1 and Ae_2 so that the set $\mathcal{D} = P \cap \Gamma$ is a digit set satisfying the hypothesis of Theorem 2.5; *thus, $Q(A, \mathcal{D})$ is a connected Γ-tile.*

The proof is constructive in that it explicitly produces the basis.

The methods for Theorem 2.6 are applicable to a much wider class of dilations including many hyperbolic dilations.

2.7. Theorem

Let A be a dilation matrix of the lattice $\mathbb{Z}^2$ with basis $e_1 = (1, 0)$, $e_2 = (0, 1)$. Assume that both $\|Ae_1\| \geq \sqrt{2}$ and $\|Ae_2\| \geq \sqrt{2}$ and that $\frac{\pi}{4} \leq \sphericalangle(Ae_1, Ae_2) \leq \frac{3\pi}{4}$. Then there is a half-open parallelogram P spanned by Ae_1 and Ae_2 so that $\mathcal{D} = P \cap \mathbb{Z}^2$ is a digit set and $Q(A, \mathcal{D})$ is a connected $\mathbb{Z}^2$-tile.

Examples and Illustrations.

(1) Let $\Gamma = \mathbb{Z}$, $\mathcal{D} = \{0, 1, 5\}$, and $Ax = 3x$. Then $Q(A, \mathcal{D})$ is a $\mathbb{Z}$-tile by Theorem 2.3. $Q(A, \mathcal{D}) \times [0, 1] = Q$ is illustrated in Figure 1. Furthermore, Q itself is a $(\mathbb{Z}^2, A)$-self-similar tile with $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and digit set $\mathcal{D} = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (5, 0), (5, 1), (5, 2)\}$. See also [25].

(2) Let $\Gamma = \mathbb{Z}^2, \mathcal{D} = \{(0, 0), (1, 0), (2, 0), (6, 1), (7, 1), (8, 1), (3, 2), (4, 2), (5, 2)\}$, and $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$. Then $\mathcal{D} = \mathcal{B}(L)$ as in (7) with $L = (0, 6, 3)$ and A is a parabolic dilation with eigenvector $(1, 0)$. $Q(A, \mathcal{D})$ is then a $\mathbb{Z}^2$-tile by an application of Theorem 2.4. See Figure 2.

FIGURE 1

FIGURE 2

(3) Let $\Gamma = \mathbb{Z}^2$, $\mathcal{D} = \{(j, 0) | j = 0, 1, \ldots a^2\}$, and $A = \begin{pmatrix} -a & -1 \\ 1 & -a \end{pmatrix}$, $a \in \mathbb{Z} \setminus \{0\}$. For a basis choose $e_1 = (1, 0)$ and $e_2 = (-a, -1)$. Then $\mathcal{D}$ is clearly a $\mathcal{T}_0$-connected subset of $\mathbb{Z}^2$. One easily verifies that e_1 and e_2 pair faces of $\mathcal{D}$. By Theorem 2.5, $Q(A, \mathcal{D})$ is a $\mathbb{Z}^2$-tile. Thus Theorem 2.5 contains Gilbert's result [12] as a special case.

FIGURE 3

(4) Let Γ be the hexagonal lattice with basis $e_1 = (1/2, \sqrt{3}/2)$, $e_2 = (1/2, -\sqrt{3}/2)$, $\mathcal{D} = \{(0,0), (1,0), (1/2, \sqrt{3}/2)\}$, and $A = \begin{pmatrix} 3/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 3/2 \end{pmatrix}$. One can check directly that $\mathcal{D}$ is the set of lattice points inside the half-open parallelogram P spanned by Ae_1 and Ae_2 and that they satisfy the hypothesis of Theorem 2.5. It can also be verified that the digit set $\mathcal{D}' = \{(0,0), (1,0), (2,0)\}$, although lying outside the context of Theorem 2.6, satisfies the hypothesis of Theorem 2.5. The (Γ, A)-self-similar tiles $Q(A, \mathcal{D})$ and $Q(A, \mathcal{D}')$ are illustrated in Figures 3 and 4.

(5) Let $\Gamma = \mathbb{Z}^2$, $\mathcal{D} = \{(1,1), (1,2), (2,1), (2,2), (2,3)\}$, and $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$. A is an irrational hyperbolic dilation. $Q(A, \mathcal{D})$ tiles as a consequence of either Theorem 2.5 or Theorem 2.7 working with the standard basis. See Figure 5.

Applications and a Conjecture.

We end with two applications: the first to a number-theoretic question, appearing in the theory of computation [18], and the second to wavelet theory.

(a) Number Systems. Let $\mathbb{Z}[i] = \{m + ni | m, n \in \mathbb{Z}\}$ be the lattice of Gaussian integers in $\mathbb{C}$. For any $q \in \mathbb{Z}[i]$, $q\mathbb{Z}[i]$ is a sublattice of $\mathbb{Z}[i]$. As always $\mathcal{D}$ will denote a set of distinct coset representatives in $\mathbb{Z}[i]$ for the group $\mathbb{Z}[i]/q\mathbb{Z}[i]$. Following Kátai and Szabó [16] we say that $(q, \mathcal{D})$ is a number system if each $\gamma \in \mathbb{Z}[i]$ has a representation of the form

$$\gamma = \epsilon_0 + \epsilon_1 q + \epsilon_2 q^2 + \cdots + \epsilon_n q^n \tag{8}$$

FIGURE 4

FIGURE 5

where each $\epsilon_j \in \mathcal{D}$, $j = 0, 1, \ldots, n$. In [16] $(q, \mathcal{D})$ is shown to be a number system when $q = -a \pm i, a > 0, \mathcal{D} = \{0, 1, \ldots, a^2\}$. Gilbert observed that these number systems could be used to construct $(A, \mathbb{Z}^2)$-self-similar tiles $Q(A, \mathcal{D})$ where $A = \begin{pmatrix} -a & -1 \\ 1 & -a \end{pmatrix}$, $a \in \mathbb{Z}$. Our methods provide a wealth of new examples.

2.8. Theorem

Given $q \in \mathbb{Z}[i]$ with $|q| > 1$ and $q \notin \{2, 1 \pm i\}$, there exists a set $\mathcal{D}$ of residue classes mod q, such that $(q, \mathcal{D})$ is a number system. Moreover, the representation (8) *is unique.*

Also, extending Theorem 2 of [16] we have

2.9. Theorem

Let $q \in \mathbb{Z}[i]$ satisfying $|q| > 1$ and $q \notin \{2, 1 \pm i\}$. Then there exists a set $\mathcal{D}$ of residue classes mod q, such that every $z \in \mathbb{C}$ can be written in the form

$$z = \epsilon_n q^n + \cdots + \epsilon_1 q + \epsilon_0 + \sum_{j=1}^{\infty} \epsilon'_j q^{-j}, \tag{9}$$

where $\epsilon_j, \epsilon'_j \in \mathcal{D}$. Moreover, for almost all $z \in \mathbb{C}$ the representation is unique.

In §9 we see that there does not exist a digit set $\mathcal{D}$ so that $(1 \pm i, \mathcal{D})$ is a number system.

(b) Wavelet Theory. The following theorem on orthonormal bases was proved in [13] under the assumption that there exist self-similar lattice tilings.

Let $U = (u_{i,j})_{i,j=0,\ldots,q-1}$ be a unitary $(q \times q)$-matrix, with constant first row, that is, $u_{0j} = q^{-1/2}$, $j = 0, \ldots, q-1$. Given a compact set $Q \subseteq \mathbb{R}^2$ and a finite set $\mathcal{D} = \{k_0, \ldots, k_{q-1}\} \subseteq \mathbb{R}^2$, define the functions $\psi_i(U, Q, \mathcal{D}) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ by

$$\psi_i(U, Q, \mathcal{D})(x) = \sum_{j=0}^{q-1} u_{ij} q^{1/2} \chi_Q(Ax - k_j) \quad \text{for } i = 1, \ldots, q-1. \tag{10}$$

2.10. Theorem

Let $\Gamma \subseteq \mathbb{R}^2$ be a lattice and A an elliptic, parabolic, or hyperbolic rational dilation on Γ. Then there is a set of digits $\mathcal{D} = \{k_0, \ldots, k_{q-1}\}$ so that $Q = Q(A, \mathcal{D})$ is a Γ-tile. Given any unitary $(q \times q)$-matrix U with constant first row and associated functions $\psi_i = \psi_i(U, Q, \mathcal{D})$, the collection of functions

$$q^{j/2} \psi_i(A^j x - k) \qquad j \in Z,\ k \in \Gamma,\ i = 1, \ldots, q-1,$$

is a complete orthonormal basis for $L^2(\mathbb{R}^2)$.

(c) A Conjecture. All results of this paper, especially the analysis in §5, lead us to the

Conjecture. Let $\mathcal{D}$ be a digit set for Γ and A, and let Γ' be the lattice generated by $\{A^j(k_l - k_0),\ j \geq 0, l = 1, \ldots, q-1\}$. Then the Γ'-translates of $Q(A, \mathcal{D})$ tile $\mathbb{R}^m$. $\square$

The rest of the paper is devoted to the proofs and is organized as follows. Section 3 gives some notation, some necessary facts from [13], and a proof that $Q(A, \mathcal{D})$ always tiles. In §4 we develop the theory of the contact matrix and prove the criterion for lattice tilings in terms of the matrix. Section 5 treats the one-dimensional case, §6 the rational dilations in $\mathbb{R}^2$, §7 connected tiles in $\mathbb{R}^2$, and in § 8 we prove the existence of lattice tilings in $\mathbb{R}^2$ for elliptic dilations. Section 9 contains the results on number systems.

3. $Q(A, \mathcal{D})$ Always Tiles

We first collect some notation and some facts needed in later sections.

Given Γ, A, and $\mathcal{D}$, let $e_1, \dots, e_m$ be a basis for Γ and Q_0 the parallelepiped spanned by the basis vectors. Define

$$Q_n = A^{-1}\left(\bigcup_{k\in\mathcal{D}}(k + Q_{n-1})\right). \tag{11}$$

These sets are approximations to $Q(A, \mathcal{D})$ that are central to the approach in [13] and continue to be of importance in our work. In the language of [5] Q is the attractor of an iterated function system and the Q_n arise by letting the system act on Q_0.

The important facts concerning this construction are detailed in the following lemma. Their proofs are either easy and left to the reader or may be found in [13].

Set

$$\mathcal{D}_n = \{k \in \Gamma | k = \sum_{j=0}^{n-1} A^j \epsilon_j \ , \quad \epsilon_j \in \mathcal{D}\} \ . \tag{12}$$

Given a set $S \subseteq \mathbb{R}^m$, we denote the difference set by $\Delta S = \{x = s - s' | s, s' \in S\}$.

3.1. Lemma

1. $|\mathcal{D}_n| = q^n$, *and for* $l \in \Gamma$, $l \neq 0$ *we have*

$$(A^n l + \mathcal{D}_n) \cap \mathcal{D}_n = \emptyset \quad \textit{and} \quad \bigcup_{k\in\Gamma}(A^n k + \mathcal{D}_n) = \Gamma.$$

2. $A^n Q_n = \bigcup_{k\in\mathcal{D}_n}(k + Q_0)$ *and* $A^n Q(A, \mathcal{D}) = \bigcup_{k\in\mathcal{D}_n}(k + Q(A, \mathcal{D}))$.
3. *The* Γ*-translates of* Q_n *tile* $\mathbb{R}^m$ *for all* $n \geq 0$.
4. Q_n *converges to* $Q(A, \mathcal{D})$ *in the compact-open topology.*
5. *The* Γ*-translates of* $Q(A, \mathcal{D})$ *cover* $\mathbb{R}^m$, $\bigcup_{k\in\Gamma}(k + Q(A, \mathcal{D})) = \mathbb{R}^m$.

For the remainder of the section fix a lattice Γ and a dilation A. It is easy to find a necessary and sufficient but rather unsatisfactory condition on the digits $\mathcal{D}$ to produce a Γ-tile $Q(A, \mathcal{D})$.

3.2. Proposition

Let $\mathcal{D}$ be a set of digits. Then $Q(A, \mathcal{D})$ is a Γ-tile if and only if $\bigcup_{n+1}^{\infty} \Delta\mathcal{D}_n = \Gamma$.

From this follows immediately a necessary condition for tiling that is easy to check:

3.3. Corollary

Let $\mathcal{D}$ be a digit set for which $Q(A, \mathcal{D})$ is a Γ-tile. Then the set $\{A^j(k_i - k_0),\ j \geq 0, i = 0, \ldots, q-1\}$ generates Γ.

Proof of the Proposition. Assume that $\bigcup_{n=1}^{\infty} \Delta\mathcal{D}_n = \Gamma$. Since each $k \in \Gamma$ can be written as $k = l - l'$, $l, l' \in \mathcal{D}_n$, for some $n \geq 1$, it suffices to show

$$(l + Q) \cap (l' + Q) \simeq \emptyset \quad \text{for all } l, l' \in \mathcal{D}_n \text{ and for all } n \geq 1. \tag{13}$$

For this we take measure on both sides of the equation $A^n Q = \bigcup_{l \in \mathcal{D}_n} l + Q$ and obtain

$$q^n \lambda(Q) = \lambda(A^n Q) = \lambda(\bigcup_{l \in \mathcal{D}_n} l + Q) \leq \sum_{l \in \mathcal{D}_n} \lambda(l + Q) = q^n \lambda(Q)$$

and thus the equality of $\lambda(\bigcup_{l \in \mathcal{D}_n} l + Q)$ and $\sum \lambda(l + Q)$. This is only possible if the translates $l + Q$, $l \in \mathcal{D}_n$, are mutually disjoint a.e., which was to be shown.

The converse follows from the following statement: *If $Q(A, \mathcal{D})$ is a Γ-tile, then for all $R > 0$ there exist $n, N > 0$ and $k \in \Gamma$ so that $B(A^N k, R) \cap \Gamma \subseteq \mathcal{D}_{N+n}$.* Here $k \in \Gamma$ and $n > 0$ can be chosen to be independent of R.

Denote the diameter of Q by δ. Observe that by Baire's theorem, since Γ-translates of Q cover $\mathbb{R}^m$, Q has nonempty interior . Therefore, there are $n \geq 0$ and $k \in \Gamma$ so that $k \in A^n Q^0$ and an $\varepsilon > 0$ so that $B(k, \varepsilon) \subseteq A^n Q^0$.

Since by assumption on A all sufficiently large powers of A^{-1} are contractive, for any $R > 0$ there exists an $N > 0$ such that

$$B(A^N k, R + \delta) \subseteq A^N B(k, \varepsilon) \subseteq A^{N+n} Q.$$

Clearly $l + Q \subseteq B(A^N k, R + \delta)$ for $l \in B(A^N k, R)$; therefore, we obtain the inclusions

$$\bigcup_{l \in B(A^N k, R) \cap \Gamma} (l + Q) \subseteq A^N B(k, \varepsilon) \subseteq A^{N+n} Q = \bigcup_{l \in \mathcal{D}_{N+n}} (l + Q).$$

We have assumed that Γ-translates of Q are mutually disjoint; hence,

$$B(A^N k, R) \cap \Gamma \subseteq \mathcal{D}_{N+n}$$

as desired. $\square$

Although easy counterexamples show that $Q(A, \mathcal{D})$ need not in general be a Γ-tile, it does always tile $\mathbb{R}^m$ with respect to some subset of Γ.

3.4. Theorem

Let $\mathcal{D}$ be a set of digits and $Q(A, \mathcal{D})$ be the resulting self-similar set. Then there exists a set $\Gamma' \subseteq \Gamma$ so that

$$\bigcup_{l \in \Gamma'} l + Q = \mathbb{R}^m \quad \textit{and} \quad (l + Q) \cap (l' + Q) \simeq \emptyset \qquad \textit{for} \;\; l \neq l', l, l' \in \Gamma'.$$

Proof. As in the proof of the previous proposition there is a lattice point $k \in \Gamma$ and an $n \geq 0$ so that $k \in A^n Q^0$. Set $R = A^n Q - k$. Notice that R now contains an open neighborhood of 0. Let U_j be the connected component of 0 in $A^j R$. With the notation $\bar{\mathcal{D}}_m = \{l - A^m k | l \in \mathcal{D}_{m+n}\}$ the dilates of R become $A^m R = \bigcup_{l \in \bar{\mathcal{D}}_m} l + Q$ where the translates $l + Q, l \in \bar{\mathcal{D}}_m$, are mutually disjoint by (13).

Set $X_{m,j} = \{l \in \bar{\mathcal{D}}_m : (l + Q) \cap U_j^0 \neq \emptyset\}$.

Suppose N_j are a sequence of infinite subsets of $\mathbb{N}$ with $N_{j+1} \subseteq N_j$ so that for $m, m' \in N_j$, $X_{m,j} = X_{m',j}$. Set $X_j = X_{m,j}$ for some, and hence all, $m \in N_j$. Since powers of A are expansive, we have $U_j \subseteq U_{j+1}$ for $j \in \mathbb{N}$, and therefore, $X_j \subseteq X_{j+1}$ for $j \in \mathbb{N}$.

Then $\Gamma' = \bigcup X_j \subseteq \Gamma$ has the desired properties. Since $\bigcup_{j \geq 0} U_j = \mathbb{R}^m$, we have $\bigcup_{l \in \Gamma'} l + Q = \mathbb{R}^m$. Also $(l + Q) \cap (l' + Q) = \emptyset, l \neq l', l, l' \in \Gamma'$ follows from the fact that the $X_{m,j}$ are subsets of $\bar{\mathcal{D}}_m$.

Finally we construct the infinite subsets $N_j \subseteq \mathbb{N}$ by induction.

Since there are only finitely many possibilities for covering the bounded set U_0 by translates, there exists an infinite subset $N_0 \subseteq \mathbb{N}$ so that for $m, m' \in N_0$ we have $X_{m,0} = X_{m',0}$.

Suppose that $N_0 \supset N_1 \supset \cdots \supset N_j$ are already defined. Define an equivalence relation $m \sim m'$ on N_j by $X_{m,j+1} = X_{m',j+1}, m, m' \in N_j$. By the same combinatorial argument as above there are only finitely many equivalence classes. At least one of these is an infinite subset of N_j. Choose one of these infinite classes and call it N_{j+1}. This finishes the proof. $\square$

4. A Criterion for Lattice Tilings

There is an important measure-theoretic counterpart to the disjointness condition (2) in the definition of a tiling. We already know by Lemma that the Γ-translates of the self-similar set $Q(A, \mathcal{D}) \subseteq \mathbb{R}^m$ cover $\mathbb{R}^m$. Let Q_0 be the parallelepiped spanned by a basis of Γ. Since a Γ-tile is a fundamental domain for $\mathbb{R}^m / \Gamma$ and the volume of a fundamental domain is an invariant, we have [13]:

4.1. Proposition

$\lambda(Q(A,\mathcal{D})) \geq \lambda(Q_0)$ and the Γ-translates of $Q(A,\mathcal{D})$ tile $\mathbb{R}^m$ if and only if $\lambda(Q(A,\mathcal{D})) = \lambda(Q_0)$.

The tiling criterion given by the proposition will be employed frequently.

4.2. Lemma

Let V be an invertible linear transformation on $\mathbb{R}^m$; and set $A' = VAV^{-1}$, $\Gamma' = V\Gamma$, $\mathcal{D}' = V\mathcal{D}$. Then $Q(A',\mathcal{D}') = VQ(A,\mathcal{D})$. Thus, $Q(A',\mathcal{D}')$ is a Γ'-tile if and only if $Q(A,\mathcal{D})$ is a Γ-tile.

If V maps a basis of Γ to the standard basis of $\mathbb{Z}^m$, then A' has integer entries and $\Gamma' = \mathbb{Z}^m$.

In the following we may, whenever it is convenient, assume that A is an integral matrix acting on $\Gamma = \mathbb{Z}^m$.

The proof of Theorem 2.2 will proceed in several parts.

Recall that the nth approximation of $Q(A,\mathcal{D})$ is $Q_n = \bigcup_{k\in\mathcal{D}_n} A^{-n}k + A^{-n}Q_0$ and let $\hat{Q}_n = A^nQ_n = \bigcup_{k\in\mathcal{D}_n} k + Q_0$. First we show that $\lambda(Q) = 1$ if and only if the boundaries of the approximating $\hat{Q}_n$ grow sufficiently slowly.

The set $\hat{Q}_n$ is a union of m-cubes with vertices at integral lattice points. Two cubes $k+Q_0$ and $l+Q_0$ with $l, m \in Z^m$ have a $(m-1)$-face in common if and only if $k-l \in \mathcal{T}_0$. The boundary of $\hat{Q}_n$, written $\partial\hat{Q}_n$, consists of those faces of the cubes Q_0+k, $k\in\mathcal{D}_n$, that are not common to two such cubes. Let $\sigma(\partial\hat{Q}_n)$ denote the number of faces on the boundary of $\hat{Q}_n$. We write $Q = Q(A,\mathcal{D})$.

4.3. Proposition

$\lambda(Q) = 1$ if and only if $\lim_{n\to\infty} q^{-n}\sigma(\partial\hat{Q}_n) = 0$.

For a subset S of $\mathbb{R}^m$ and $\varepsilon > 0$ let $N(S,\varepsilon)$ denote the neighborhood of S of radius ε. The collar about S of radius ε is the set $C(S,\varepsilon) = N(S,\varepsilon)\backslash S$.

We shall need the following fact, which is seen by considering the Jordan form of the dilation A : there exist positive constants C, λ, r with $\lambda > 1$ so that for all $x, y \in \mathbb{R}^m$ and integers $n \geq 0$

$$||A^{-n}x - A^{-n}y|| \leq C\lambda^{-n}n^r||x-y|| \tag{14}$$

and, therefore,

$$A^{-n}N(A^nS,\varepsilon) \subseteq N(S, C\lambda^{-n}n^r\varepsilon).$$

4.4. Lemma

There is a constant $a > 1$ such that for all $n \geq 0$

$$A^nQ \subseteq N(\hat{Q}_n, a) \tag{15}$$

and

$$\hat{Q}_n \subseteq N(A^n Q, a). \tag{16}$$

Proof. Choose $a > 0$ so that $Q \subseteq N(Q_0, a)$ and $Q_0 \subseteq N(Q, a)$. We argue by induction. Suppose $A^n Q \subseteq N(\hat{Q}_n, a)$. Then $A^{n+1}Q = A^n(\bigcup_{k\in\mathcal{D}} k + Q) = \bigcup_{k\in\mathcal{D}}(A^n k + A^n Q) \subseteq \bigcup_{k\in\mathcal{D}} A^n k + N(\hat{Q}_n, a) = \bigcup_{k\in\mathcal{D}} N(A^n k + \hat{Q}_n, a) = N(\hat{Q}_{n+1}, a)$. On the other hand, if $\hat{Q}_n \subseteq N(A^n Q, a)$, then $\hat{Q}_{n+1} \subseteq \bigcup_{k\in\mathcal{D}_{n+1}}(k + N(Q, a)) = N(A^{n+1}Q, a)$ using Lemma 3.1(2). □

Proof of Proposition 4.3. First assume that $\lim_{n\to\infty} q^{-n}\partial(\hat{Q}_n) = 0$. Let I_{m-1} be an $(m-1)$ face of the unit cube $Q_0 \subseteq \mathbb{R}^m$. The boundary of $\hat{Q}_n$ is a union of translates of $m-1$ faces of Q_0. Thus, by taking neighborhoods of all faces of $\partial\hat{Q}_n$ we obtain the estimate $\lambda(C(\hat{Q}_n, a)) \leq \lambda(N(I_{m-1}, a))\sigma(\partial\hat{Q}_n)$, and consequently $\lambda(A^{-n}C(\hat{Q}_n, a)) \leq \lambda(N(I_{m-1}, a))q^{-n}\sigma(\partial\hat{Q}_n)$ for any $a > 0$. Applying the hypothesis we deduce that

$$\lim_{n\to\infty} \lambda(A^{-n}C(\hat{Q}_n, a)) = 0. \tag{17}$$

Now we show that $\lambda(Q) = 1$. Choose a to be as in Lemma 4.4. We write

$$Q = A^{-n}[A^n Q \cap C(\hat{Q}_n, a)] \cup (Q \cap Q_n).$$

By (17) the measure of the first term on the right vanishes in the limit and we obtain $\lambda(Q) = \lim_{n\to\infty} \lambda(Q\cap Q_n)$. Using that $\lambda(Q) \geq 1$ and $\lambda(Q_n) = 1$ for all n, we get $1 \leq \lambda(Q) = \lim_{n\to\infty} \lambda(Q\cap Q_n) \leq 1$ and, consequently, $\lambda(Q) = 1$.

To prove the converse we assume that $\lambda(Q) = 1$. For each face I_{m-1} of $\partial\hat{Q}_n$ take a plate of height $\frac{1}{2}$ of the form $I_{m-1} \times [0, \frac{1}{2}]$ where the arc $p \times [0, \frac{1}{2}]$, for $p \in I_{m-1}$, is orthogonal to I_{m-1} and meets $\hat{Q}_n$ in the face I_{m-1}. The plate on I_{m-1} is contained in $N(I_{m-1}, a)$, and two plates can only meet in sets of lower dimension. Thus the union of these plates is contained in $C(\hat{Q}_n, a)$ and we have

$$\frac{1}{2}\sigma(\partial\hat{Q}_n) \leq \lambda(C(\hat{Q}_n, a)).$$

This implies

$$\frac{1}{2}q^{-n}\sigma(\partial\hat{Q}_n) \leq \lambda(A^{-n}C(\hat{Q}_n, a)). \tag{18}$$

From (16) we may infer that $C(\hat{Q}_n, a) = N(\hat{Q}_n, a)\backslash\hat{Q}_n \subseteq N(N(A^n Q, a), a)\backslash\hat{Q}_n \subseteq N(A^n Q, 2a)\backslash\hat{Q}_n$. Applying A^{-n} and taking the measure on both sides gives

$$\lambda(A^{-n}C(\hat{Q}_n, a)) \leq \lambda(A^{-n}N(A^n Q, 2a)) - \lambda(Q_n). \tag{19}$$

The estimate (14) tells us that there are constants λ, r, a' so that for all $n \geq 0$

$$Q \subseteq A^{-n} N(A^n Q, 2a) \subseteq N(Q, \lambda^{-n} n^r a').$$

Combining this with the estimates (18) and (19) yields

$$\begin{aligned}\frac{1}{2} q^{-n} \sigma(\partial \hat{Q}_n) &\leq \lambda(A^{-n} N(A^n Q, 2a)) - \lambda(Q_n) \\ &\leq \lambda(N(Q, \lambda^{-n} n^r a')) - \lambda(Q_n).\end{aligned}$$

Since $N(Q, \lambda^{-n} n^r a')$ is a neighborhood basis of the compact set Q, the regularity of Lebesgue measure implies that $\lim_{n\to\infty} \lambda(N(Q, \lambda^{-n} n^r a')) = \lambda(Q)$, which is 1 by hypothesis. As $\lambda(Q_n) = 1$ for all $n \geq 0$ we conclude that $\lim_{n\to\infty} q^{-n} \sigma(\partial \hat{Q}_n) = 0$. $\square$

We must now relate the growth of $\partial \hat{Q}_n$ to the contact matrix C. We begin with a digression in which we clarify the definition of the set of contact points $\mathcal{T}$ and prove its finiteness.

Let $\mathcal{T}_n^* = \{l \in \mathbb{Z}_0^m | (A^n l + \mathcal{D}_n) \cap (f + \mathcal{D}_n) \neq \emptyset$ for some $f \in \mathcal{T}_0\}$ where $\mathbb{Z}_0^m = \mathbb{Z}^m \setminus \{0\}$. Observing that as $\mathcal{D}_n$ is precisely the set of points of the form $\sum_{i=0}^{n-1} A^i \varepsilon_i,\ \ \varepsilon_i \in \mathcal{D}, l \in \mathcal{T}_n^*$ is characterized by stipulating that

$$A^n l = f + \sum_{i=0}^{n-1} A^i (\varepsilon_i - \varepsilon_i') \quad \text{for } f \in \mathcal{T}_0 \text{ and } \varepsilon_i, \varepsilon_i' \in \mathcal{D}. \tag{20}$$

4.5. Lemma

1. $\mathcal{T}_n = \mathcal{T}_n^*$.
2. *$\mathcal{T}$ is the smallest set such that $\mathcal{T}_0 \subset \mathcal{T}$ and $\mathcal{D} \cup (\mathcal{T} + \mathcal{D}) \subseteq \mathcal{D} \cup (A\mathcal{T} + \mathcal{D})$.*
3. *$\mathcal{D}_n \cup (\mathcal{T} + \mathcal{D}_n) \subseteq \mathcal{D}_n \cup (A^n \mathcal{T} + \mathcal{D}_n)$ for $n \geq 1$.*
4. *$\mathcal{T}$ is finite.*

Proof. **(1)** We argue by induction. Suppose that $k \in \mathcal{T}_n$ if and only if $A^n k$ can be written in the form (20). By definition $l \in \mathcal{T}_{n+1}$ if and only if $(Al + \mathcal{D}) \cap (k + \mathcal{D}) \neq \emptyset$ for some $k \in \mathcal{T}_n$ and, hence, if and only if $(A^{n+1} l + A^n \mathcal{D}) \cap (A^n k + A^n \mathcal{D}) \neq \emptyset$ for some $k \in \mathcal{T}_n$. Using the inductive hypothesis, this holds if and only if $(A^{n+1} l + A^n \mathcal{D}) \cap (f + \sum_{i=0}^{n-1} A^i (\varepsilon_i - \varepsilon_i') + A^n \mathcal{D}) \neq \emptyset$ for some $f \in \mathcal{T}_0,\ \varepsilon_i,\ \varepsilon_i' \in \mathcal{D}$ or, equivalently, $A^{n+1} l + A^n \varepsilon_n' = f + \sum_{i=0}^{n-1} A^i (\varepsilon_i - \varepsilon_i') + A^n \varepsilon_n$ for some $f \in \mathcal{T}_0,\ \varepsilon_i,\ \varepsilon_i' \in \mathcal{D}, i = 0, \ldots, n$. The last assertion is equivalent to $A^{n+1} l$ being of the form (20) and thus of l belonging to $\mathcal{T}_n^*$.

(2) Suppose $e \in \mathcal{T}$ is in $\mathcal{T}_n$ for some $n \geq 0$. Then, by definition of $\mathcal{T}_{n+1},\ e + \mathcal{D} \subseteq \mathcal{D} \cup (A\mathcal{T}_{n+1} + \mathcal{D}) \subseteq \mathcal{D} \cup (A\mathcal{T} + \mathcal{D})$. To show minimality of $\mathcal{T}$, let $\mathcal{T}'$ be another set with $\mathcal{T}_0 \subseteq \mathcal{T}'$ and $\mathcal{T}' + \mathcal{D} \subseteq \mathcal{D} \cup (A\mathcal{T}' + \mathcal{D})$. Assume that we know already that $\mathcal{T}_k \subseteq \mathcal{T}'$ for $k \leq n$, and keep in mind that the $A\mathbb{Z}^m$-translates of $\mathcal{D}$ are all disjoint. If $e \in \mathcal{T}_{n+1} \backslash \mathcal{T}'$, then for some $l \in \mathcal{T}_n$, $l + \mathcal{D} \cap Ae + \mathcal{D} \neq \emptyset$ and $l + \mathcal{D} \not\subseteq \mathcal{D} \cup (A\mathcal{T}' + \mathcal{D})$, a contradiction. Thus $\mathcal{T}_{n+1} \subset \mathcal{T}'$ and $\bigcup_{k \geq 0} \mathcal{T}_k \subseteq \mathcal{T}'$.

(3) This follows by induction. The first step was (2). Now suppose that the result holds for n. Then

$$\begin{aligned}
\mathcal{T} + \mathcal{D}_{n+1} &= \mathcal{T} + \mathcal{D}_n + A^n\mathcal{D} \\
&\subseteq (A^n\mathcal{T} + \mathcal{D}_n + A^n\mathcal{D}) \bigcup (\mathcal{D}_n + A^n\mathcal{D}) \\
&\subseteq (A^n(\mathcal{T} + \mathcal{D}) + \mathcal{D}_n) \bigcup \mathcal{D}_{n+1} \\
&\subseteq (A^n(A\mathcal{T} + \mathcal{D}) + \mathcal{D}_n) \bigcup \mathcal{D}_{n+1} \\
&= (A^{n+1}\mathcal{T} + \mathcal{D}_{n+1}) \bigcup \mathcal{D}_{n+1}.
\end{aligned}$$

(4) Choose a digit $k_0 \in \mathcal{D}$. By part (1), $l \in \mathcal{T}_n$ can be written in the form

$$\begin{aligned}
l &= [A^{-n} f + \sum_{i=0}^{n-1} A^{i-n}\varepsilon_i - \sum_{i=0}^{n-1} A^{i-n}\varepsilon_i'] \\
&= [A^{-n} f + (\sum_{i=0}^{n-1} A^{i-n}\varepsilon_i + \sum_{i=n}^{\infty} A^{-i}k_0) - (\sum_{i=0}^{n-1} A^{i-n}\varepsilon_i' + \sum_{i=n}^{\infty} A^{-i}k_0)]
\end{aligned}$$

for some $f \in \mathcal{T}_0$, $\varepsilon_i, \varepsilon_i' \in \mathcal{D}$. Notice that $A^{-n} f \in A^{-n} R$ where $R = [-1, 1]^m$ and that the two sums belong to Q. Thus for $n > 0$, $\mathcal{T}_n \subseteq \bigcup_{r=0}^{\infty} A^{-r} R + Q - Q$, which is a compact set. Consequently $\mathcal{T}$ is finite. $\square$

The next lemma provides a means for counting the number of faces on $\partial \hat{Q}_n$.

4.6. Lemma

$\sigma(\partial \hat{Q}_n) = \sum_{f \in \mathcal{T}_0, l \in \mathcal{T}} |A^n l + \mathcal{D}_n \cap f + \mathcal{D}_n|$.

Proof. Since the $A^n\mathbb{Z}^m$-translates of $\hat{Q}_n$ tile $\mathbb{R}^m$, the intersection $(A^n l + \hat{Q}_n) \cap \hat{Q}_n$, $l \in \mathbb{Z}^m$, is either empty or a subset of $\partial \hat{Q}_n$. Moreover, every $(m-1)$ face of $\partial \hat{Q}_n$ lies in exactly one such intersection. Hence we may write

$$\sigma(\partial \hat{Q}_n) = \sum_{l \in \mathbb{Z}_0^m} \sigma(A^n l + \hat{Q}_n \cap \hat{Q}_n) = \sum_{\substack{k, k' \in \mathcal{D}_n \\ l \in \mathbb{Z}_0^d}} \sigma(A^n l + k' + Q_0 \cap k + Q_0)$$

where the last equality is a consequence of the decomposition of $\hat{Q}_n$ into $\mathcal{D}_n$-translates of Q_0.

As discussed earlier, $\sigma(k + Q_0 \cap k' + Q_0) = 1$ if and only if $k - k' \in \mathcal{T}_0$ (and it takes the value zero otherwise). It follows that $\sigma(A^n l + k' + Q_0 \cap k + Q_0) = 1$ if and only if $A^n l + k' - k \in \mathcal{T}_0$ or, equivalently, $|A^n l + k' \cap f + k| = 1$ for some $f \in \mathcal{T}_0$.

Consequently, the above equality may be continued to give

$$\sigma(\partial\hat{Q}_n) = \sum_{\substack{k,k'\in\mathcal{D}_n\\ l\in\mathbb{Z}_0^m\\ f\in\mathcal{T}_0}} |A^n l + k' \cap f + k| = \sum_{\substack{f\in\mathcal{T}_0\\ l\in\mathbb{Z}_0^m}} |A^n l + \mathcal{D}_n \cap f + \mathcal{D}|. \tag{21}$$

That completes the proof, since by Lemma 4.5(3), $|A^n l + \mathcal{D}_n \cap f + \mathcal{D}_n| > 0$ if and only if $l \in \mathcal{T}$. $\square$

The last three lemmas point to a nice geometric interpretation of the sets $\mathcal{T}_n$. As was observed in the proof of Lemma 4.6, each Q_n will intersect finitely many of its $\mathbb{Z}^m$-translates; these are of the form $k + Q_n$ for $k \in \mathcal{T}_n$. Thus the set $\mathcal{T}$ records those translates that pair boundary faces of Q_n for some $n \geq 0$.

The information about intersecting translates can be given a quantitative form. Let E_n be the $|\mathcal{T}| \times |\mathcal{T}|$ matrix with entries $e_{lf}^{(n)} = |A^n l + \mathcal{D}_n \cap f + \mathcal{D}_n|$ where $f, l \in \mathcal{T}$. In these terms Lemma becomes $\sigma(\partial\hat{Q}_n) = \sum_{f\in\mathcal{T}_0, l\in\mathcal{T}} e_{lf}^{(n)}$.

The next lemma shows that the matrix C is rich enough to describe all of the matrices E_n and, thus, also the growth of $\partial\hat{Q}_n$.

4.7. Lemma

For $l, f \in \mathcal{T}, n > 0$, we have $|A^n l + \mathcal{D}_n \bigcap f + \mathcal{D}_n| = (C^n)_{lf}$, and consequently $\sigma(\partial\hat{Q}_n) = \sum_{f\in\mathcal{T}_0, l\in\mathcal{T}} (C^n)_{lf}$.

Proof. As $\mathcal{D}_n = \bigcup_{i=0}^{q-1}(A^{n-1}k_i + \mathcal{D}_{n-1})$ where the sets $A^{n-1}k_i + \mathcal{D}_{n-1}$ are mutually disjoint, we have the following

$$\begin{aligned}
&|(A^n l + \mathcal{D}_n) \cap (f + \mathcal{D}_n)| \\
&\quad = |\bigcup_{i=0}^{q-1}\bigcup_{j=0}^{q-1}(A^n l + A^{n-1}k_j + \mathcal{D}_{n-1}) \cap (f + A^{n-1}k_i + \mathcal{D}_{n-1})| \\
&\quad = \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}|(A^n l + A^{n-1}k_j + \mathcal{D}_{n-1}) \cap (f + A^{n-1}k_i + \mathcal{D}_{n-1})| \\
&\quad = \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}|(A^{n-1}(Al + k_j - k_i) + \mathcal{D}_{n-1}) \cap (f + \mathcal{D}_{n-1})|.
\end{aligned} \tag{22}$$

By Lemma 4.5

$$(A^{n-1}(Al + k_j - k_i) + \mathcal{D}_{n-1}) \cap (f + \mathcal{D}_{n-1}) \neq \emptyset$$

if and only if $Al + k_j - k_i \in \mathcal{T}_{n-1}$. For each $m \in \mathcal{T}$ the number of pairs (i, j) for which $Al + k_j - k_i = m$

is $|Al + \mathcal{D} \cap m + \mathcal{D}|$. Thus by collecting together such pairs (22) becomes

$$
\begin{aligned}
&|A^n l + \mathcal{D}_n \cap f + \mathcal{D}_n| \\
&\quad = \sum_{m \in \mathcal{T}} |Al + \mathcal{D} \cap m + \mathcal{D}| \ \ |A^{n-1}m + \mathcal{D}_{n-1} \cap f + \mathcal{D}_{n-1}| \\
&\quad = \sum_{m \in \mathcal{T}} c_{lm} e_{mf}^{(n-1)};
\end{aligned}
$$

in other words, $E_n = CE_{n-1}$. Repeating this argument gives $E_n = C^n$.

Interpreting Lemma 4.6 in terms of E_n we have

$$\sigma(\partial \hat{Q}_n) = \sum_{\substack{f \in \mathcal{T}_0 \\ l \in \mathcal{T}}} e_{lf}^{(n)} = \sum_{\substack{f \in \mathcal{T}_0 \\ l \in \mathcal{T}}} (C^n)_{lf}$$

as claimed. $\square$

We are now in a position to complete the proof of Theorem 2.2. Using the tiling criterion of Proposition 4.1, the theorem may be restated as

4.8. Theorem

The Γ-translates of $Q(A, \mathcal{D})$ tile $\mathbb{R}^m$ if and only if $r(C) < q$. Furthermore, if $\lambda(Q) \neq 1$, then $r(C) = q$.

Proof. If $r(C) < q$, then $\frac{1}{q}C$ is a contraction and consequently $\lim_{n\to\infty} q^{-n}C^n = 0$, the zero matrix. In particular, $q^{-n}\sum_{f\in\mathcal{T}_0, l\in\mathcal{T}}(C^n)_{lf} = q^{-n}\sigma(\partial\hat{Q}_n)$ converges to zero. By Proposition 4.3, $\lambda(Q) = 1$.

To see that $\lambda(Q) \neq 1$ implies $r(C) = q$, we observe that the matrix $q^{-1}C$ is substochastic; i.e., $q^{-1}\sum_{l\in\mathcal{T}} c_{lk} \leq 1$. This follows since $(Al + \mathcal{D}) \cap (Al' + \mathcal{D}) = \emptyset$ for $l \neq l'$, and consequently, $q^{-1}\sum_{l\in\mathcal{T}} c_{lk} = q^{-1}\sum_{l\in\mathcal{T}} |(Al+\mathcal{D})\cap(k+\mathcal{D})| = q^{-1}|[\bigcup_{l\in\mathcal{T}}(Al+\mathcal{D})]\cap(k+\mathcal{D})| \leq q^{-1}|k+\mathcal{D}| = 1$. Then as $q^{-1}C$ is a nonnegative matrix it follows that $r(q^{-1}C) \leq 1$ [20, Chapter 10]. Thus we always have $r(C) \leq q$.

Now we argue the converse. If $\lambda(Q) = 1$, then by Proposition $\lim_{n\mapsto\infty} q^{-n}\sum_{f\in\mathcal{T}_0, l\in\mathcal{T}}(C^n)_{lf} = \lim_{n\mapsto\infty} q^{-n}\sigma(\partial\hat{Q}_n) = 0$. Therefore, $\lim_{n\mapsto\infty} q^{-n}(C^n)_{lf} = 0$ for $l \in \mathcal{T}$, $f \in \mathcal{T}_0$. More generally, since $(A^n l + \mathcal{D}_n) \cap \mathcal{D}_n = \emptyset$, we have $(C^n)_{lk} = |(A^n l + \mathcal{D}_n) \cap (k + \mathcal{D}_n)| \leq \|k\|\sigma(\partial\hat{Q}_n)$ for *any* $k, l \in \mathcal{T}$. Here the number on the right side bounds the cardinality of the set of lattice points lying within a neighborhood of width $\|k\|$ of $\partial\hat{Q}_n$ inside $\hat{Q}_n$.

It follows that $\lim_{n\mapsto\infty} q^{-n}C^n = 0$. Thus C cannot have an eigenvector of eigenvalue q, and we conclude $r(C) < q$. $\square$

In the next section a more refined version of Theorem 4.8 is needed. We end this section by deriving the necessary variation.

4.9. Lemma

1. *$k \in \mathcal{T}$ if and only if $-k \in \mathcal{T}$ and $c_{-l,-k} = c_{lk}$.*
2. *Given a vector $u = (u_k)$, $k \in \mathcal{T}$, define $\bar{u} = (\bar{u}_k)$ by $\bar{u}_k = u_{-k}$. If u is an eigenvector of C with eigenvalue λ, then so is $\bar{u}$.*

Proof. **(1)** Using Lemma 4.5, $k \in \mathcal{T}$ if and only if $A^n k + \mathcal{D}_n \cap f + \mathcal{D}_n \neq \emptyset$ for some $n > 0$, $f \in \mathcal{T}_0$; or in other words, $A^n k + d_1 = f + d_2$ for some $n > 0$, $f \in \mathcal{T}_0$, and $d_1, d_2 \in \mathcal{D}_n$. Multiplying both sides by -1 and shifting the d_i gives $A^n(-k) + d_2 = -f + d_1$ or $A^n(-k) + \mathcal{D}_n \cap -f + \mathcal{D}_n \neq \emptyset$ for some $n > 0$, $f \in \mathcal{T}_0$. That proves the first assertion.

To see the relation between the entries of C write

$$c_{lk} = |Al + \mathcal{D} \cap k + \mathcal{D}| = \sum_{d_1, d_2 \in \mathcal{D}} |Al + d_1 \cap k + d_2|.$$

Then $Al + d_1 = k + d_2$ if and only if $A(-l) + d_2 = -k + d_1$, which implies that

$$c_{lk} = \sum_{d_1, d_2 \in \mathcal{D}} |A(-l) + d_2 \cap -k + d_1| = c_{-l-k}.$$

(2) $\sum_{k \in \mathcal{T}} c_{lk} \bar{u}_k = \sum_{k \in \mathcal{T}} c_{lk} u_{-k} = \sum_{k \in \mathcal{T}} c_{-l,-k} u_{-k} = \lambda u_{-l} = \lambda \bar{u}_l$. Thus $C\bar{u} = \lambda \bar{u}$. □

4.10. Proposition

$\lambda(Q) \neq 1$ if and only if there is a nonzero, nonnegative vector $v = (v_k)$, $k \in \mathcal{T}$, with $v_{-k} = v_k$ satisfying $Cv = qv$.

Proof. By Theorem 4.8 the existence of such an eigenvector v immediately implies $\lambda(Q) \neq 1$. To prove the converse we use the Frobenius Theorem. Since C is nonnegative, there is a nontrivial, nonnegative eigenvector u with eigenvalue $r(C) = q$. By Lemma 4.9 $\bar{u}$ is also an eigenvector with eigenvalue q. The vector $u + \bar{u}$ has the asserted properties. □

5. Self-Similar Lattice Tilings in Dimension 1

In order to obtain more information about the matrix C and the dependence of its spectrum on the choice of digits we shall investigate its Fourier transform.

Fix a dilation A of the lattice $\mathbb{Z}^m$ and a digit set $\mathcal{D} = \{k_0, \ldots, k_{q-1}\}$. Let Ω be the finite-dimensional subspace of $l^2(\mathbb{Z}^m)$ of sequences $\mathbf{a} = (a_k)$, $k \in \mathbb{Z}^m$, where $a_k = 0$ for $k \notin \mathcal{T}$. Interpret the contact matrix C as a linear operator on Ω by setting $(C\mathbf{a})(l) = \sum_{k \in \mathcal{T}} c_{lk} a_k$ for $l \in \mathcal{T}$ and zero otherwise.

Let $\mathcal{F} : L^2(\mathbb{T}^m) \to l^2(\mathbb{Z}^m)$ be the Fourier transform, that is

$$\hat{f}(k) = \mathcal{F} f(k) = \frac{1}{(2\pi)^m} \int_{[0,2\pi]^m} f(x) e^{-ik\cdot x}\, dx$$

for $f \in L^2(\mathbb{T}^m) = L^2([0, 2\pi]^m)$.

Let $\hat{\Omega} = \{f \in L^2(\mathbb{T}^m) |\ f(x) = \sum_{k\in\mathcal{T}} a_k e^{ik\cdot x}\}$, the finite-dimensional space of trigonometric polynomials supported on $\mathcal{T}$. One easily sees that $\mathcal{F}\hat{\Omega} = \Omega$. Define the linear operator $\hat{C}$ on $\hat{\Omega}$ by $\hat{C} = \mathcal{F}^{-1} C \mathcal{F}$. Since $\mathcal{F}$ is unitary, the operators C and $\hat{C}$ have identical spectra.

Define the function $\mathbf{m} \in L^2(\mathbb{T}^m)$ by $\mathbf{m}(x) = \frac{1}{q} \sum_{j=0}^{q-1} e^{ik_j\cdot x}$. Let $B = A^t$, and choose a complete set $\{l_j\}$ of representatives of the group $\mathbb{Z}^m / B\mathbb{Z}^m$ with $l_0 = 0$, in other words, $\bigcup_{i=0}^{q-1} (l_j + B\mathbb{Z}^m) = \mathbb{Z}^m$.

5.1. Lemma

1.
$$\frac{1}{q} \sum_{j=0}^{q-1} e^{2\pi i k\cdot B^{-1} l_j} = \begin{cases} 1 & \text{if } k \in A\mathbb{Z}^m, \\ 0 & \text{otherwise.} \end{cases}$$

2.
$$\sum_{j=0}^{q-1} |\mathbf{m}(x + 2\pi B^{-1} l_j)|^2 = 1 \text{ for all } x \in \mathbb{R}^m.$$

Proof. Define $\chi_j(k) = e^{2\pi i k\cdot B^{-1} l_j}$. If $k \in A\mathbb{Z}^m$, then $A^{-1}k \in \mathbb{Z}^m$ and $\chi_j(k) = e^{2\pi i k\cdot B^{-1} l_j} = e^{2\pi i A^{-1} k\cdot l_j} = 1$. This gives the first assertion of (1).

For $k \in \mathbb{Z}^m$ consider the finite subset $G_k = \{\chi_j(k) |\ j = 0, \ldots, q-1\}$ of the unit circle in $\mathbb{C}$. One easily verifies that G_k is a finite multiplicative, hence cyclic, subgroup of $\mathbb{C}^*$ and that $l_j \to \chi_j(k)$ defines a homomorphism of $\mathbb{Z}^m / B\mathbb{Z}^m$ onto G_k.

We claim that for $k \notin A\mathbb{Z}^m$, G_k is a nontrivial subgroup of $\mathbb{C}^*$. It will suffice to show that $\chi_j(k) \neq 1$ for some $0 \leq j < q$. If this were not true, then for all $j = 0, \ldots, q-1$ we would have $1 = \chi_j(k) = e^{2\pi i k\cdot B^{-1} l_j} = e^{2\pi i k\cdot B^{-1}(l_j + Bl)}$ for any $l \in \mathbb{Z}^m$. As every $l' \in \mathbb{Z}^m$ is of the form $l_j + Bl$ for some $0 \leq j < q$ and $l \in \mathbb{Z}^m$, $1 = e^{2\pi i k\cdot Bl'} = e^{2\pi i A^{-1} k\cdot l'}$ for all $l' \in \mathbb{Z}^m$, and hence $A^{-1}k \in \mathbb{Z}^m$, contrary to hypothesis.

As a homomorphic image of $\mathbb{Z}^m / B\mathbb{Z}^m$, G_k has order dividing q; therefore, its elements must satisfy the polynomial equation $0 = z^q - 1 = (z-1)(z^{q-1} + \cdots + 1)$. Choosing as generator of G_k, $z = \chi(k)$ say, the last term of the right gives $\sum_{j=0}^{q-1} \chi_j(k) = 0$ as required.

The proof of (2) is by computation:

$$\sum_{j=0}^{q-1} |\mathbf{m}(x + 2\pi B^{-1} l_j)|^2 = \frac{1}{q^2} \sum_{j,r,s=0}^{q-1} e^{i(k_r - k_s)\cdot(x + 2\pi B^{-1} l_j)}$$

$$= \frac{1}{q} \sum_{r,s=0}^{q-1} e^{i(k_r - k_s)\cdot x} \left(\frac{1}{q} \sum_{j=0}^{q-1} e^{2\pi i (k_r - k_s)\cdot B^{-1} lj} \right)$$

$$= \frac{1}{q} \sum_{r,s,=0}^{q-1} e^{i(k_r - k_s)\cdot x} \delta_{rs} = 1,$$

where we have used (1) and the fact that $k_r - k_s \in A\mathbb{Z}^m$ if and only if $r = s$. $\square$

The next lemma provides an intrinsic formulation of the operator $\hat{C}$. It is essentially due to Lawton [21]. Since we also need that $\hat{C}$ leaves $\hat{\Omega}$ invariant, we include its short proof.

5.2. Lemma

$\hat{C} f(x) = q \sum_{j=0}^{q-1} |\mathbf{m}(B^{-1}(x + 2\pi l_j)|^2 f(B^{-1}(x + 2\pi l_j))$, *where* $f \in \hat{\Omega}$.

Proof. Let $f(x) = \sum_{k\in\mathcal{T}} a_k e^{ikx}$. We show that with $\hat{C}$ defined as in the statement of the lemma, $C(\mathcal{F} f)(l) = \mathcal{F}(\hat{C} f)(l)$ for all $l \in \mathbb{Z}^m$.

Since $\mathcal{F} f(k) = a_k$ for $k \in \mathcal{T}$ and $\mathcal{F} f(k) = 0$ for $k \in \mathcal{T}$, $C(\mathcal{F} f)(l) = \sum_{k\in\mathcal{T}} c_{lk} a_k$ for $l \in \mathcal{T}$ and zero otherwise. On the other hand,

$$\hat{C} f(x) = \frac{1}{q} \sum_{j,r,s=0}^{q-1} e^{i(k_r - k_s)\cdot(B^{-1}x + 2\pi B^{-1} l_j)} \left(\sum_{k\in\mathcal{T}} a_k e^{ik\cdot(B^{-1}x + 2\pi B^{-1} l_j)} \right)$$

$$= \sum_{k\in\mathcal{T}} a_k \sum_{r,s=0}^{q-1} e^{i(k_r - k_s + k)\cdot B^{-1} x} \left(\frac{1}{q} \sum_{j=0}^{q-1} e^{2\pi i(k_r - k_s + k)\cdot B^{-1} l_j} \right).$$

By Lemma 5.1 the last expression in parenthesis is nonzero if and only if $k_r - k_s + k = Al$ for some $l \in \mathbb{Z}^m$, in other words, if and only if $Al + \mathcal{D} \cap k + \mathcal{D} \neq \emptyset$ for some $l \in \mathbb{Z}$. Since $k \in \mathcal{T}$, we also have $l \in \mathcal{T}$ by Lemma 4.5. Carrying out the summation over r and s gives

$$\hat{C} f(x) = \sum_{k\in\mathcal{T}} a_k \sum_{l\in\mathcal{T}} e^{iAl\cdot B^{-1}x} |Al + \mathcal{D} \cap k + \mathcal{D}| = \sum_{l,k\in\mathcal{T}} c_{lk} a_k e^{il\cdot x}.$$

Thus $(\mathcal{F}\hat{C} f)(l) = \sum_{k\in\mathcal{T}} c_{lk} a_k$ for $l \in \mathcal{T}$ and zero otherwise, as was to be proved. $\square$

Now it is possible to derive an equivalent formulation of Proposition 4.10 in terms of the operator $\hat{C}$.

5.3. Proposition

$\lambda(Q) \neq 1$ *if and only if there is a nonconstant, real-valued function* $f \in \hat{\Omega}$ *satisfying* $\hat{C} f = qf$.

Proof. Suppose $\lambda(Q) \neq 1$. Let $a \in \mathbb{R}^m$ be the vector given by Proposition with $Ca = qa$ and $a_k = a_{-k}$. Then $\mathcal{F}^{-1} a = \sum_{k\in\mathcal{T}} a_k e^{ik\cdot x}$ is a real-valued eigenfunction of $\hat{C}$ of eigenvalue q. By definition, $0 \notin \mathcal{T}$, and consequently $\mathcal{F}^{-1} a$ is nonconstant.

Now let $f \in \hat{\Omega}$ satisfy $\hat{C} f = qf$. Then $\mathcal{F} f$ is a vector in Ω with $C(\mathcal{F} f) = q \mathcal{F} f$. It follows from Theorem 4.8 that $\lambda(Q) \neq 1$. $\square$

So far we have considered the general case of a dilation A of $\mathbb{R}^m$. Throughout the remainder of this section our attention shall be restricted to the one dimensional case of the dilation $Ax = qx$ of $\mathbb{R}$, for $q \in \mathbb{Z}$, $|q| > 1$.

5.4. Lemma

Suppose that the operator $\hat{C}$ has a nonconstant, real-valued eigenfunction $f \in \hat{\Omega}$ of eigenvalue q. Then there are positive integers r and s with $0 < s < q^r - 1$ so that for $x = 2\pi s/(q^r - 1)$, $\mathbf{m}(x) = 1$.

Proof. Recall that f satisfies $\hat{C} f = q \sum_{j=0}^{q-1} |\mathbf{m}(\frac{x+2\pi j}{q})|^2 f(\frac{x+2\pi j}{q}) = q\, f$. In dimension one Lemma 5.1 takes the form

$$\sum_{j=0}^{q-1} \left| \mathbf{m}\left(y + \frac{2\pi j}{q} \right) \right|^2 = 1, \tag{23}$$

and therefore we can write

$$0 = \hat{C} f - qf = q \sum_{j=0}^{q-1} \left| \mathbf{m}\left(\frac{x + 2\pi j}{q} \right) \right|^2 \left[f\left(\frac{x + 2\pi j}{q} \right) - f(x) \right]. \tag{24}$$

We now analyze the structure of the set of absolute extreme points of f, which we refer to as extrema. Two extrema x, y are said to be of the same type if they are either both maxima or minima. As f is nonconstant and 2π-periodic, it has distinct maxima and minima. Thus we may choose an extremum x_0 of f so that $0 < x_0 < 2\pi$.

Take $y = x_0/q$. For some j_0 the term $\mathbf{m}(y + \frac{2\pi j_0}{q})$ in (23) is nonzero, and consequently we must have $f((x_0 + 2\pi j_0)/q) = f(x_0)$. Thus $x_1 \equiv ((x_0 + 2\pi j_0)/q) (\mathrm{mod}\ 2\pi)$ is an extremum of the same type as x_0 with $qx_1 \equiv x_0 \mod 2\pi$ and $0 < x_1 < 2\pi$. Repeating this argument produces a sequence of extrema $x_0, x_1, \ldots, x_r$ all of the same type with $qx_i \equiv x_{i-1}\ (\mathrm{mod}\ 2\pi)$ and $0 < x_i < 2\pi$.

Since f is a trigonometric polynomial, it has only a finite number of extrema $\mod 2\pi$. Thus for some r and i with $0 \leq i < r$, $x_r = x_i$. Let r be the smallest value for which this occurs. We claim that $x_r = x_0$. If not, then $i > 0$ and $x_{i-1} \equiv qx_i = qx_r \equiv x_{r-1}$. It follows that $x_{i-1} = x_{r-1y}$, contradicting the minimality of r. Therefore for all i, $q^r x_i \equiv x_i \mod 2\pi$. This may be written as $q^r x_i = x_i + 2\pi s$ for an integer s, depending on i, with $0 < s < q^r - 1$, or in other words, $x_i = 2\pi s/(q^r - 1)$.

We refer to the sequence $x_0, \ldots, x_r$ as a cycle of extrema. The proof will be completed by showing that $\mathbf{m}(x_1) = 1$. The same argument works with any x_i.

Let x^* be an element in the above cycle of extrema. Then for j not divisible by q, $x^* + \frac{2\pi j}{q}$ cannot be an extremum of the same type as x^*. If it were, then since $q(x^* + \frac{2\pi j}{q}) = qx^* \mod 2\pi$, the cycle of extrema could be constructed to contain both x^* and $x^* + \frac{2\pi j}{q}$. More precisely, if the original cycle has length r, then starting with $x_0 = x^*$ one may consistently set $x_r = x^* + \frac{2\pi j}{q}$ and continue defining the cycle, as above. Then $x^* + \frac{2\pi j}{q} \equiv q^R(x^* + \frac{2\pi j}{q}) \equiv q^R x^* \equiv x^* \mod 2\pi$, where R is the length of the cycle. This is clearly impossible.

By the previous argument $(x_0 + 2\pi j)/q$ for $j \neq j_0$ and $0 \leq j < q$ cannot be an extremum of the same type as x_0. Therefore, from (24) we see that for $j \neq j_0$, $\mathbf{m}((x_0 + 2\pi j)/q) = 0$ and by (23) conclude that $\mathbf{m}(x_1) = \mathbf{m}((x_0 + 2\pi j_0)/q) = 1$. $\square$

Remark. The analysis of the extrema of an eigenfunction of $\hat{C}$ is adapted from [9]. The operator $\hat{C}$ plays an important role in many recent results in wavelet theory, since its spectrum encodes information on the orthonality or smoothness of wavelet bases, see for instance [8].

Almost all steps in Lemma 5.4 can be extended to higher dimensions. However, since in higher dimensions the set of extrema of a trigonometric polynomial is in general nondiscrete, our analysis breaks down. Nevertheless we think that in some form the result carries over to arbitrary dimension. $\square$

We end this section with a proof of Theorem 2.3.

Proof of Theorem 2.3. We work with the tiling criterion of Proposition 4.1. Observe that there is no loss of generality in assuming that $k_0 = 0$. This follows by considering the construction based around the modified digit set $\mathcal{D}' = \{0, k_1 - k_0, \ldots, k_{q-1} - k_0\}$. Then $Q(A, \mathcal{D}') = \{x \in \mathbb{R} | x = \sum \varepsilon_j' q^{-j}, \varepsilon_j' \in \mathcal{D}'\} = \{x \in \mathbb{R} | x = \sum (\varepsilon_j - k_0) q^{-j}, \varepsilon_j \in \mathcal{D}\} = Q(A, \mathcal{D}) - (\sum_{j=1}^{\infty} k_0 q^{-j})$, and so $Q(A, \mathcal{D}')$ and $Q(A, \mathcal{D})$ have the same measure.

First suppose that $\gcd \mathcal{D} = d > 1$. The elements of $\mathcal{D}^* = \{0, k_1/d, \ldots, (k_q - 1)/d\}$ are distinct $\bmod q$, and therefore $\mathcal{D}^*$ is also a set of digits for the dilation q of $\mathbb{R}$. It is known from Proposition 4.1 that in general $\lambda(Q(A, \mathcal{D}^*) \geq 1$. Since $Q(A, \mathcal{D}) = dQ(A, \mathcal{D}^*)$, we obtain $\lambda(Q(A, \mathcal{D})) = d\lambda(Q(A, \mathcal{D}^*) \geq d > 1$.

To argue the converse assume that $\lambda(Q(A, \mathcal{D})) > 1$. Proposition 5.3 asserts the existence of a nonconstant, real-valued eigenfunction f of $\hat{C}$ with eigenvalue q. By Lemma there is a number $x = 2\pi s/(q^r - 1)$ for some positive integers r and s, $1 < s < q^r - 1$, for which $\mathbf{m}(x) = 1$. Writing this out gives $1 = \mathbf{m}(x) = \frac{1}{q} \sum_{j=0}^{q-1} e^{ik_j x}$. This is only possible if $e^{ik_j x} = 1$ for $j = 0, \ldots, q - 1$. We infer that $k_j x = k_j 2\pi s/(q^r - 1) \in 2\pi\mathbb{Z}$. Since $s < q^r - 1$, in lowest terms $s/(q^r - 1) = \frac{a}{d}$ with a, d relatively prime and $d > 1$. We conclude that $d | k_j$ for $j = 0, \ldots, q - 1$, and thus $\gcd(\mathcal{D}) \geq d > 1$.

The general case now follows immediately. $\square$

6. The Rational Case

To treat the case when A has two rational eigenvalues we first give explicit digit sets in the special case of triangular matrices A.

6.1. Proposition

Let $\Gamma = \mathbb{Z}^2$; let A be a dilation matrix of the form

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \qquad a, b, c \in \mathbb{Z},\ |a|, |c| > 1;$$

and let $l_0, l_1, l_2, \ldots, l_{|c|-1}$ be arbitrary integers. Set

$$\mathcal{D} = \left\{(l_j + k, j),\ \ j = 0, 1, 2, \ldots, |c| - 1,\ \ k = 0, 1, \ldots, |a| - 1\right\},$$

then the $\mathbb{Z}^2$-translates of $Q(A, \mathcal{D})$ tile $\mathbb{R}^2$.

Proof. It is easily checked that $\mathcal{D}$ is a set of representatives for $\mathbb{Z}^2/A\mathbb{Z}^2$. A computation yields

$$A^{-n} = \begin{pmatrix} a^{-n} & b_n a^{-n} c^{-n} \\ 0 & c^{-n} \end{pmatrix}, \text{ where } b_n = -b \sum_{j=0}^{n-1} a^j c^{n-1-j}.$$

Now every point $x = (\alpha, \beta) \in Q(A, \mathcal{D})$ has the coordinates

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \sum_{j-1}^{\infty} A^{-j} \varepsilon_j = \sum_{j=1}^{\infty} \begin{pmatrix} a^{-j} & b_j a^{-j} c^{-j} \\ 0 & c^{-j} \end{pmatrix} \begin{pmatrix} \varepsilon_j' \\ \varepsilon_j'' \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{j=1}^{\infty} (a^{-j}\varepsilon_j' + b_j a^{-j} c^{-j} \varepsilon_j'') \\ \sum_{j=1}^{\infty} \varepsilon_j'' c^{-j} \end{pmatrix}$$

for some sequence $\varepsilon_j = (\varepsilon_j', \varepsilon_j'') \in \mathcal{D}$.

Since $\varepsilon_j'' \in \{0, 1, \ldots, |c| - 1\}$, the second coordinate is just the c-adic expansion of some number β in an interval I_c of length 1, more precisely, $I_c = [0, 1]$ for $c > 0$ and $I_c = [\frac{c}{1-c}, \frac{1}{1-c}]$ for $c < 0$. Every $\beta \in I_c$ has a c-adic expansion that is unique for almost all β, more precisely, for β irrational.

For a fixed irrational β, with c-adic expansion $\sum_{j=1}^{\infty} \epsilon_j'' c^{-j}$, we look at the section of Q at height β :

$$Q(\beta) = \{\alpha \in \mathbb{R} : (\alpha, \beta) \in Q\}$$

$$= \{\alpha | \alpha = \sum_{j-1}^{\infty} (a^{-j}\varepsilon_j' + b_j a^{-j} c^{-j} \varepsilon_j''), \ \text{ where } \ l_{\varepsilon_{j''}} \le \varepsilon_j' \le l_{\varepsilon_{j''}} + |a| - 1\}.$$

Consequently the left end point of such a section

$$m(\beta) = \inf\{\alpha \in Q(\beta)\} = \sum_{j=1}^{\infty} (a^{-j} l_{\varepsilon_j''} + b_j a^{-j} c^{-j} \varepsilon_j'') \quad \text{ when } a > 1$$

is obtained by choosing the smallest possible digit for each $j \geq 1$. Similarly, the right end point is

$$M(\beta) = \sup\{\alpha \in Q(\beta)\} = \sum_{j=1}^{\infty}(a^{-j}(l_{\varepsilon''_j} + a - 1) + b_j a^{-j} c^{-j} \varepsilon''_j) \quad \text{when } a > 1.$$

When $a < 0$, the same considerations give

$$m(\beta) = \sum_{j=1}^{\infty}(a^{-2j} l_{\varepsilon''_{2j}} + \sum_{j=1}^{\infty}(a^{-2j+1}(l_{\varepsilon''_{2j-1}} + |a| - 1) + \sum_{j=1}^{\infty} b_j a^{-j} c^{-j} \varepsilon''_j)$$

and

$$M(\beta) = \sum_{j=1}^{\infty}(a^{-2j}(l_{\varepsilon''_{2j}} + |a| - 1) + \sum_{j=1}^{\infty}(a^{-2j+1} l_{\varepsilon''_{2j-1}} + \sum_{j=1}^{\infty} b_j a^{-j} c^{-j} \varepsilon''_j).$$

Since one term of the expansion for α is an a-adic expansion, for almost all β the section $Q(\beta)$ is the entire interval $[m(\beta), M(\beta)]$ that has the length $|Q(\beta)| = M(\beta) - m(\beta) = (a-1)\sum_{j=1}^{\infty} a^{-j} = 1$. Using Fubini's theorem, we obtain for the measure of Q :

$$\lambda(Q) = \int_{I_c} |Q(\beta)|\, d\beta = |I_c| = 1. \qquad \square$$

Remark. Using Theorem 2.3 the set of the second coordinates of the digits $\{0, 1, \ldots, |c| - 1\}$ can be replaced by $\{k_0, k_1, \ldots, k_{c-1}\}$, where $k_i \equiv i \pmod{c}$. If $\gcd(k_i) = d$, the projection of $Q(A, \mathcal{D})$ onto the second coordinate is a $d\mathbb{Z}$-tile for $\mathbb{R}$ and, by the same proof as above, $Q(A, \mathcal{D})$ is a tile with respect to the lattice $\mathbb{Z} \times d\mathbb{Z}$.

To prove the general case, let V be the nonsingular matrix that maps a given basis of Γ to the standard basis of $\mathbb{Z}^2$. Then $A' = VAV^{-1}$ is an integral matrix acting on $\mathbb{Z}^2$. Consequently the eigenvalues of A' are integers and its eigenvectors can be chosen in $\mathbb{Z}^2$. In particular, there exist $a, m, n \in \mathbb{Z}$ with $|a| > 1$ and $gcd(m, n) = 1$ such that $A'\begin{pmatrix} m \\ n \end{pmatrix} = a\begin{pmatrix} m \\ n \end{pmatrix}$. Choose $r, s \in \mathbb{Z}$, so that $mr - ns = 1$, and define $W = \begin{pmatrix} m & s \\ n & r \end{pmatrix}$. Then $W \in \mathrm{SL}(2, \mathbb{Z})$ and $W^{-1}A'W = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ where $b, c \in \mathbb{Z}$, $|a|, |c| > 1$. This observation and Proposition together prove Theorem 2.4. $\square$

7. Connected Lattice Tilings

Recall that given a lattice Γ with basis $\mathcal{T}_0 = \{\pm e_1, \pm e_2\}$ a set $S \subseteq \Gamma$ is $\mathcal{T}_0$-*connected* if for any $k, l \in S$ there is a sequence $k = h_0, h_1, \ldots, h_r = l$ of elements of S, called a *path* from k to l, so that $h_{j+1} - h_j \in \mathcal{T}_0$. One easily sees that a set S is $\mathcal{T}$-connected if and only if the interior of the set $\bigcup_{s \in S} s + Q_0$ is a connected subset of $\mathbb{R}^m$. A point $k \in \Gamma$, $k \neq 0$, *pairs faces* of Q_n if there is an $e \in \mathcal{T}_0$ such that $(A^n k + \mathcal{D}_n) \cap (e + \mathcal{D}_n) \neq \emptyset$. In the context of §4, k pairs faces of Q_n if and only if $k \in \mathcal{T}_n$. It can also be verified that k pairs faces of Q_n if and only if $Q_n \cap k + Q_n$ is a nonempty $(m-1)$-dimensional piecewise linear submanifold of $\mathbb{R}^m$. We shall make use of both viewpoints.

7.1. Lemma

Suppose that $\mathcal{D}$ is $\mathcal{T}_0$-connected and each $e \in \mathcal{T}_0$ pairs faces of Q_1. Then, for each integer $n > 0$, $\mathcal{D}_n$ is $\mathcal{T}_0$-connected and each $e \in \mathcal{T}_0$ pairs faces of Q_n.

Proof. We argue both conclusions by induction. Certainly for $\mathcal{D}_1 = \mathcal{D}$, the hypotheses coincide with the conclusions. Suppose that the conclusions hold for $\mathcal{D}_n$.

First we show that $\mathcal{D}_{n+1}$ is $\mathcal{T}_0$-connected. Given $k, l \in \mathcal{D}_{n+1}$ there are digits $k', l' \in \mathcal{D}_n$ such that $k \in Ak' + \mathcal{D}$ and $l \in Al' + \mathcal{D}$. Choose a path $k' = h_0, h_1, \dots, h_r = l'$ from k' to l' in $\mathcal{D}_n$. Since $h_{j+1} - h_j \in \mathcal{T}_0$ and each element of $\mathcal{T}_0$ pairs faces of Q_1, we can find $k_j \in \mathcal{D} + Ah_j$ and $l_j \in \mathcal{D} + Ah_{j+1}$ such that $k_j - l_j \in \mathcal{T}_0$. Setting $l_{-1} = k$ and $k_r = l$ we have k_j and l_{j-1}, $0 \leq j \leq r$, both belonging to the $\mathcal{T}_0$-connected set $\mathcal{D} + Ah_j$. Choose a path $l_{j-1} = h_{j,0}, h_{j,1}, \dots, h_{j,s_j} = k_j$ for $j = 0, \dots, r$. Then since $k_j - l_j \in \mathcal{T}_0$, we can concatenate these to form a path $k = h_{0,0}, h_{0,1}, \dots, h_{0,s_0}, h_{1,0}, h_{1,1}, \dots, h_{r,s_r} = l$ from k to l in $\mathcal{D}_{n+1}$. Thus $\mathcal{D}_{n+1}$ is $\mathcal{T}_0$-connected.

To see that each $e \in \mathcal{T}_0$ pairs faces of Q_{n+1}, recall from Lemma 4.7 that

$$|A^{n+1}e + \mathcal{D}_{n+1} \cap f + \mathcal{D}_{n+1}| = \sum_{e' \in \mathcal{T}} |Ae + \mathcal{D} \cap e' + \mathcal{D}||A^n e' + \mathcal{D}_n \cap f + \mathcal{D}_n|.$$

Given $e \in \mathcal{T}_0$ we know from the hypothesis that $|Ae + \mathcal{D} \cap e' + \mathcal{D}| \neq 0$ for some $e' \in \mathcal{T}_0$. By induction there is $f \in \mathcal{T}_0$ with $|A^n e' + \mathcal{D}_n \cap f + \mathcal{D}_n| \neq 0$. This implies $|A^{n+1}e + \mathcal{D}_{n+1} \cap f + \mathcal{D}_{n+1}| \neq 0$, in other words, e pairs sides of $\mathcal{D}_{n+1}$. □

For the remainder of this section we shall restrict attention to the case where Γ is a lattice in $\mathbb{R}^2$. In the next proposition we adopt a topological point of view.

Let F be a compact subset of $\mathbb{R}^2$ with piecewise linear boundary. We say that the set F is good if for any $x, y \in \partial F$ there exists a continuous, simple curve $\gamma : [0, 1] \to F$ with $\gamma(0) = x$, $\gamma(1) = y$, and $\gamma((0, 1)) \subset F^0$.

7.2. Proposition

Suppose that F is a good Γ-tile and that both e_1 and e_2 pair faces of F. If $k \in \Gamma$ also pairs faces, then $k \in \Delta\mathcal{T}_0$.

Observe that under the same assumptions as in Lemma 7.1 each Q_n is a good Γ-tile and e_1 and e_2 pair faces of Q_n. Thus we obtain the following:

7.3. Corollary

Suppose that $\mathcal{D}$ is $\mathcal{T}_0$-connected and that e_1 and e_2 pair faces of Q_1. Then $\mathcal{T} \subset \Delta_0\mathcal{T}_0$.

Let us mention that the hypothesis of the proposition is not sufficient to guarantee the conclusion for dimensions $m > 2$. A counterexample has been constructed by J. Tollefson.

Proof. Define $R \subseteq \mathbb{R}^2$ by $R = \bigcup_{k \in \Delta_0 \mathcal{T}_0}(k + F)$. We arrange $\Delta_0\mathcal{T}_0$ as follows: $l_0 = e_1$, $l_1 = e_1 + e_2, l_2 = e_2, l_3 = e_2 - e_1, l_4 = -e_1, l_5 = -e_1 - e_2, l_6 = -e_2, l_7 = -e_2 + e_1, l_8 = l_0$.

By the hypothesis there exist line segments $\alpha_j, \beta_j, \subseteq \partial F$, $j = 1, 2$, with $\alpha_j + e_j = \beta_j$. Setting $X = \alpha_1 \cup \beta_1 \cup \alpha_2 \cup \beta_2$, we may choose points $p_i \in (l_{i-1} + X^0) \cap (l_i + X^0)$, $i = 1, 2 \ldots, 7$, and with $p_0 = p_8 = p_1 - e_2$. Since F is a good Γ-tile, there are simple curves $\varphi_i : [\frac{i}{8}, \frac{i+1}{8}] \to l_i + F$ such that $\varphi_i(\frac{i}{8}) = p_i$, $\varphi_i(\frac{i+1}{8}) = p_{i+1}$ and $\varphi_i((\frac{i}{8}, \frac{i+1}{8})) \subseteq l_i + F^0$.

Then $\Phi : [0, 1] \to R$, defined by $\Phi(x) = \varphi_i(x)$ for $\frac{i}{8} \le x \le \frac{i+1}{8}$, $i = 0, 1, \ldots, 7$, is a simple, closed curve whose graph, also denoted by Φ, is contained in R^0. This property is equivalent to

$$\Phi \cap (k + F) = \emptyset \quad \text{for all} \quad k \notin \Delta_0\mathcal{T}_0. \tag{25}$$

It follows from the Jordan curve theorem that the complement of Φ in $\mathbb{R}^2$ consists of a bounded connected region B and an unbounded connected region U. By the construction of Φ the set F must lie either in B or in U.

Next we show $F \subseteq B$. Since Φ is dividing, it separates $e_1 + F$ into two sets P_1 and P_2, which are also good sets. Since $\Phi \subseteq R^0$ and $\beta_1, \beta_1 + e_1 \subset \partial R$, either of P_1 or P_2, say P_2, contains $\beta_1 + e_1$, and therefore $P_2 \cap (2e_1 + F) \neq \emptyset$. Thus P_2 lies in the same component of Φ as $2e_1 + F$. Now notice that $\bigcup_{n=2}^{\infty}(ne_1 + F)$ is a connected, unbounded set disjoint from Φ, and, therefore, is contained in U. In particular, $2e_1 + F \subseteq U$. Consequently, also $P_2 \subseteq U$ and $P_1 \subseteq B$. Therefore, the inclusion $F \subseteq B$ will follow from $F \cap P_1 \neq \emptyset$.

If $F \cap P_1 = \emptyset$, then also $\beta_1 \subseteq P_2$, and we can choose a point $x \in \beta_1 \subseteq P_2$ so that $y = x + e_1 \in \beta_1 + e_1 \subseteq P_2$. Therefore, there is a simple curve $\gamma_1 : [0, 1] \to P_2$ with $\gamma_1(0) = x$, $\gamma_1(1) = y$, and $\gamma_1((0, 1)) \subset P_2^0$. Set $\gamma_2 = \varphi_0 = \Phi|_{e_1+F}$, and notice that γ_1 and γ_2 are disjoint.

To see how this leads to a contradiction, consider the projections $\bar{\gamma}_1$ and $\bar{\gamma}_2$ of the two curves to the quotient torus $\mathbb{R}^2/\Gamma$. By construction both $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are simple, closed, disjoint, homotopically nontrivial curves on $\mathbb{R}^2/\Gamma$. But this can only happen when $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are freely homotopic curves. Since the covering group Γ is 0, $\bar{\gamma}_1$ and $\bar{\gamma}_2$ must be covered by the same transformation in Γ [24]. By construction $\bar{\gamma}_1$ is covered by e_1 and $\bar{\gamma}_2$ by e_2, which gives the contradiction. Thus we have proved $F \subseteq B$.

Now assume that $l \notin \Delta_0\mathcal{T}_0$ pairs faces of F. Then the set $F \bigcup (l + F)$ is connected and disjoint from Φ by (25). It must therefore lie in the bounded component B. On the other hand, $V = \cup_{m=0}^{\infty}(ml + F)$ is connected, disjoint from Φ, and unbounded. Thus $V \subseteq U$ and $F \subseteq V \subseteq U$ provides a contradiction. This finishes the proof of the proposition. □

7.4. Lemma

Suppose that $\mathcal{D}$ is $\mathcal{T}_0$-connected and each element of $\mathcal{T}_0$ pairs faces of Q_1. Then $\Delta\mathcal{D}_n \supset \Delta\mathcal{T}_0$ for $n \ge 4$.

We shall refer to elements of the set $\{e_1 + e_2, -e_1 - e_2, e_1 - e_2, -e_1 + e_2\} = \Delta_0\mathcal{T}_0 \backslash \mathcal{T}_0$ as *vertices*. A digit $k \in \mathcal{D}_n$ is said to be *surrounded* if for each $e \in \mathcal{T}_0$, $k + e \in \mathcal{D}_n$.

Proof. The argument will be divided into several steps. The hypothesis of Lemma 7.4 shall be assumed at all stages.

Step 1. $\mathcal{T}_0 \subseteq \Delta\mathcal{D}_2$.

Since $\mathcal{D}$ is $\mathcal{T}_0$-connected there is an element of $\mathcal{T}_0$ belonging to $\Delta\mathcal{D}$. Without loss of generality suppose that $e_1 \in \Delta\mathcal{D}$. If $e_2 \in \Delta\mathcal{D}$, then we are done, so further assume that $e_2 \notin \Delta\mathcal{D}$. Then the digits of $\mathcal{D}$ must all be multiples of e_1 plus some fixed constant.

For some $m_i, l_i \in \mathbb{Z}$, $i = 1, 2$, $Ae_i = m_i e_1 + l_i e_2$. From the hypothesis that e_1 and e_2 pair faces of Q_1 and the fact that $\mathcal{D}$ lies along an e_1-line we conclude that for $i = 1, 2$, l_i must take one of the values 0 or ± 1. If either l_1 or l_2 takes one of the values ± 1, then $e_2 \in \Delta\mathcal{D}_2$, as desired. If both l_1 and l_2 are zero, then $Ae_1 = m_1 e_1$ and $Ae_2 = m_2 e_1$. Since a dilation matrix is invertible, this last possibility cannot occur. We conclude that $\mathcal{T}_0 \subseteq \Delta\mathcal{D}_2$.

Step 2. *Suppose that $\mathcal{T}_0 \subseteq \Delta\mathcal{D}$. Then $\Delta\mathcal{D}$ contains a vertex.*

Since both $e_1, e_2 \in \Delta\mathcal{D}$, there must be a transitional $k \in \mathcal{D}$ and digits $l, l' \in \mathcal{D}$ so that $k - l = \pm e_1$ and $k - l' = \pm e_2$. Then $l - l' \in \Delta\mathcal{D}$ is a vertex.

Step 3. *Suppose that $\mathcal{T}_0 \subseteq \mathcal{D}$, $e_1 + e_2 \in \Delta\mathcal{D}$, and $e_1 - e_2 \notin \Delta\mathcal{D}$. Then $\mathcal{D}$ is a path $P = \{\underline{k} = h_0, h_1, \ldots, h_r = \overline{k}\}$ where $h_{i+1} - h_i = e_1$ or e_2.*

A digit $k \in \mathcal{D}$ can be written in e_1, e_2-coordinates as $k = me_1 + le_2$. Let min $\subseteq \mathcal{D}$ consist of those digits with minimal e_1-coordinate. Let $\underline{k} \in$ min have minimal e_2-coordinate in min. Similarly let max $\subseteq \mathcal{D}$ consist of those digits with maximal e_1-coordinate and choose $\overline{k} \in$ max with maximal e_2-coordinate. Since $\mathcal{D}$ is $\mathcal{T}_0$-connected, there is a shortest length path $P = \{\underline{k} = h_0, h_1, \ldots, h_r = \overline{k}\}$ from $\underline{k}$ to $\overline{k}$ in $\mathcal{D}$.

We now argue that $h_{i+1} - h_i = e_1$ or e_2. This is clear for $i = 0$ from the definition of $\underline{k}$. Suppose there is a first $j > 0$ for which this fails. Then for some $s, t \in \{1, 2\}$, $h_{j+1} - h_j = -e_s$ and $h_j - h_{j-1} = e_t$. The minimality of the length of P ensures that $t \neq s$. Thus, $h_{j-1} - h_{j+1} = (h_j - e_t) - (h_j - e_s) = e_s - e_t$. Since $k \in \Delta\mathcal{D}$ if and only if $-k \in \Delta\mathcal{D}$, this last conclusion is contrary to the hypothesis. Consequently $h_{i+1} - h_i = e_1$ or e_2.

The proof of Step 3 will be completed by showing that $P = \mathcal{D}$. Since $\mathcal{D}$ is $\mathcal{T}_0$-connected, there is a path from any $k \in \mathcal{D}$ to any point $h \in P$. It will therefore suffice to show that, assuming the existence of $h \in P$, $k \notin P$ with $k - h \in \mathcal{T}_0$ leads to a contradiction.

First consider the case where $h = \underline{k}$. By the minimality of $\underline{k}$ we must have $k = \underline{k} + e_i$ for $i = 1$ or 2. Also, $h_1 = \underline{k} + e_j$ for $j \neq i$. Then $k - h_1 = e_i - e_j \in \Delta\mathcal{D}$, contrary to the hypothesis. One can similarly argue the case $h = \overline{k}$.

Now suppose $0 < i < r$. Then there are $s, t \in \{1, 2\}$ such that $h_{i-1} - h_i = -e_s$ and $h_{i+1} - h_i = e_t$. This forces $k - h_i = u$ where $u = e_s$ or $-e_t$. If $u = e_s$, then $k - h_{i+1} = e_s - e_t \in \Delta\mathcal{D}$. If $u = -e_t$ then $h_{i-1} - k = e_t - e_s \in \Delta\mathcal{D}$. Both conclusions are contrary to the hypothesis. The proof of Step 3 is complete.

Step 4. *If $\mathcal{T}_0 \subseteq \Delta\mathcal{D}$, then $\Delta\mathcal{T}_0 \subseteq \Delta\mathcal{D}_2$.*

We argue by contradiction. By Step 2 $\Delta\mathcal{D}$ contains a vertex. Without loss of generality suppose that $e_1 + e_2 \in \Delta\mathcal{D} \subseteq \Delta\mathcal{D}_2$ but $e_1 - e_2 \notin \Delta\mathcal{D}_2$. By the previous step $\mathcal{D}$ is a path P_1 from $\underline{k}$ to $\overline{k}$ and $\mathcal{D}_2$ is also a path P_2 that contains a translate of P_1. This means that the only points where faces of Q_1 can be paired are the end points $\underline{k}$ and $\overline{k}$ of $P_1 = \mathcal{D}$. This leaves the following possibilities, where $s, t \in \{1, 2\}$:

I. $\overline{k} + e_s = Ae_1 + \underline{k}$ and $\overline{k} + e_t = Ae_2 + \underline{k}$ where (a) $s = t$ or (b) $s \neq t$;
II. $\overline{k} + e_s = Ae_1 + \underline{k}$ and $\underline{k} - e_t = Ae_2 + \overline{k}$ where (a) $s = t$ or (b) $s \neq t$;

and four similar conditions with $\underline{k}$ and $\overline{k}$ interchanged:

I(a) implies $Ae_1 = Ae_2$. Thus A is not invertible, which is a contradiction.

I(b) implies $A(e_1 - e_2) = e_s - e_t = \pm(e_1 - e_2)$. This gives a contradiction since a dilation matrix cannot have eigenvalues ± 1.

II(a) implies $\overline{k} = \overline{k} + e_s - e_t = Ae_1 + \underline{k} - e_t = A(e_1 + e_2) + \overline{k}$. Thus $A(e_1 + e_2) = 0$ which is impossible.

II(b) implies, as above, that $A(e_1 + e_2) = e_s - e_t$, $s \neq t$.

Since by assumption $e_1 + e_2 \in \Delta\mathcal{D}$, there exists an i, $1 \leq i \leq r - 1$, such that $h_{i+1} - h_{i-1} = e_1 + e_2$. Define $l = Ah_{i+1} + \overline{k}$ and $m = Ah_{i-1} + \overline{k}$. Both l and m belong to $\mathcal{D}_2$ and $l - m = A(e_1 + e_2) = e_s - e_t = \pm(e_1 - e_2) \in \Delta\mathcal{D}_2$, which contradicts the assumption.

The analogous conditions with $\underline{k}$ and $\overline{k}$ interchanged are argued similarly. That completes the proof of Step 4.

It is now an easy matter to finish the proof of Lemma . Using Step 1 we conclude that $\mathcal{T}_0 \subseteq \Delta\mathcal{D}_2$. Notice that $\mathcal{D}' = \mathcal{D}_2$ is a valid digit set for the dilation $B = A^2$ and that $\mathcal{D}'_n = \mathcal{D}_{2n}$. Then $\mathcal{D}'$ satisfies the hypothesis of Step 4, and consequently $\Delta\mathcal{D}_4 = \Delta\mathcal{D}'_2 \supset \Delta\mathcal{T}_0$ as asserted. □

We are now able to prove Theorem 2.5 on the existence of connected Γ-tiles in $\mathbb{R}^2$ under assumptions that are easy to verify.

Proof of Theorem 2.5. From Lemma 4.5(3) we know that $A^n k + \mathcal{D}_n \cap e + \mathcal{D}_n \neq \emptyset$ for some $e \in \mathcal{T}$ and some integer $n > 0$ implies $k \in \mathcal{T}$ or $k = 0$. Thus for $e \in \mathcal{T}$

$$e + \mathcal{D}_n = (\mathcal{D}_n \cap e + \mathcal{D}_n) \cup \bigcup_{l \in \mathcal{T}} (A^n l + \mathcal{D}_n \cap e + \mathcal{D}_n)$$

is a decomposition into disjoint sets. Taking cardinalities, we obtain

$$\begin{aligned} q^n = |e + \mathcal{D}_n| &= |\mathcal{D}_n \cap e + \mathcal{D}_n| + \sum_{l \in \mathcal{T}} |A^n l + \mathcal{D}_n \cap e + \mathcal{D}_n| \\ &= |\mathcal{D}_n \cap e + \mathcal{D}_n| + \sum_{l \in \mathcal{T}} (C^n)_{le} \end{aligned} \tag{26}$$

where in the last equality we have used the identification $|A^n l + \mathcal{D}_n \cap e + \mathcal{D}_n| = (C^n)_{le}$ from Lemma 4.7.

Now, under the assumptions stated, $\mathcal{T} \subseteq \Delta\mathcal{T}_0$ is a consequence of Corollary 7.3 and, for $n \geq 4$, $\Delta\mathcal{T}_0 \subset \Delta\mathcal{D}_n$ of Lemma 7.4. In particular, we have for $n \geq 4$

$$\mathcal{D}_n \cap (e + \mathcal{D}_n) \neq \emptyset \quad \text{for all} \quad e \in \mathcal{T}. \tag{27}$$

Using (26) we find for the column sums of C^n

$$\sum_{l \in \mathcal{T}} (C^n)_{le} < q^n \quad \text{for all} \quad e \in \mathcal{T}. \tag{28}$$

This means that for $n \geq 4$ the matrix $q^{-n} C^n$ is strictly substochastic, and therefore it must have spectral radius $r(q^{-n} C^n) < 1$. Consequently $r(C) < q$, and an application of Theorem 2.2 proves the claim that Q is a Γ-tile.

Since all Q_n are connected by Lemma 7.1 and connectedness is preserved under limits in the compact-open topology, the tile $Q(A, \mathcal{D})$ is also connected. $\square$

8. The Elliptic Case

In this section we consider elliptic dilation matrices A, that is, those with complex eigenvalues $\lambda \pm i\mu$, $\mu \neq 0$. Then A is a conjugate to a similarity and, since $\det A = \lambda^2 + \mu^2 > 0$, is orientation-preserving. Thus there exists a $C \in SL(2, \mathbb{R})$, such that $CAC^{-1} = \sqrt{q}O$, $q = \det A$, $O \in \mathrm{SO}(2)$. Moreover, using basic facts about the fundamental domain for $SL(2, \mathbb{Z})$ and its relation to lattices in $\mathbb{R}^2$ [2], we may choose C so that $C\Gamma$ has a basis of the form $e_1 = (1, 0)$, $e_2 = (x, y)$ with $-\frac{1}{2} < x \leq \frac{1}{2}, x^2 + y^2 \geq 1$, and $x \geq 0$ if $x^2 + y^2 = 1$. We shall refer to such a basis as a normal basis and a lattice Γ with a normal basis as a normal lattice. For the remainder of this section we assume without loss of generality that Γ is a normal lattice and that A is a multiple of an orthogonal matrix that leaves Γ invariant. The angle $\theta = \sphericalangle(e_1, e_2)$ between the basis vectors satisfies $\frac{\pi}{3} \leq \theta < \frac{2\pi}{3}$. If $\alpha = \sqrt{x^2 + y^2}$, then $x = \alpha \cos\theta$ and $-\frac{1}{2} < \alpha\cos\theta \leq \frac{1}{2}$.

For $a > 0$ let $\bar{a} = (a, 0)$ and set $v_+ = \bar{a} - x$ and $v_- = -\bar{a} - x$ for $x \in \mathbb{R}^2$. Let W be the set of all $x \in \mathbb{R}^2$ for which $\sphericalangle(v_+,\ v_-) \geq \frac{\pi}{3}$. Then we have

8.1. Lemma

$W \subseteq B(0, \sqrt{3}a)$.

Proof. Let $\theta_x = \sphericalangle(v_+,\ v_-)$. Given $x \in \mathbb{R}^2$ with $|x| = \tau a$, we show that $\cos\theta_x > \frac{1}{2}$ for $\tau > \sqrt{3}$. Then $\bar{a} \cdot x = \tau a^2 \cos\gamma$ for some angle γ and we obtain

$$\begin{aligned}\cos\theta_x &= \frac{(x - \bar{a}) \cdot (x + \bar{a})}{\|x - \bar{a}\|\, \|x + \bar{a}\|} = \frac{|x|^2 - a^2}{\sqrt{(|x|^2 + a^2)^2 - 4(\bar{a} \cdot x)^2}} \\ &= \frac{(\tau^2 - 1)a^2}{\sqrt{(\tau^2 + 1)^2 a^4 - 4\tau^2 a^4 \cos^2\gamma}} \geq \frac{\tau^2 - 1}{\tau^2 + 1} > \frac{1}{2} \quad \text{for} \quad \tau > \sqrt{3}. \qquad \square\end{aligned}$$

8.2. Lemma

Let e_1, e_2 be a normal basis for Γ, and let $p = p_1 e_1 + p_2 e_2 \in \Gamma$, with $p_1, p_2 \in \mathbb{Z}$. Then $\sphericalangle(-p + e_1, -p + e_2) \leq \frac{\pi}{3}$ for $p_1 < 0,\ \ p_2 < 0$ or $p_1 > 1,\ \ p_2 > 1$ and $\sphericalangle(-p, -p + e_1 + e_2) \leq \frac{\pi}{3}$ for $p_1 < 0,\ \ p_2 > 1$ or $p_1 > 1,\ \ p_2 < 0$. Equality can hold if and only $\sphericalangle(e_1,\ e_2) = \frac{2\pi}{3}$ or $\frac{\pi}{3}$ and $\|e_2\| = 1$, i.e., for the hexagonal lattice.

Proof. Let $\theta = \sphericalangle(e_1, e_2)$ and $\alpha = \|e_2\| \geq 1$. Then by assumption $|\cos\theta| \leq \frac{1}{2}$ and $|e_1 \cdot e_2| = \alpha|\cos\theta| \leq \alpha/2$. By the preceding lemma we only have to show that the distance from p

to the center $\frac{1}{2}(e_1+e_2)$ of the fundamental parallelogram spanned by e_1, e_2 is greater than $\frac{\sqrt{3}}{2}$ times the length of the diagonal of the parallelogram that is seen from p. Thus it will suffice to verify that

$$\|p - \frac{1}{2}(e_1+e_2)\| \geq \frac{\sqrt{3}}{2} \max(\|e_1 - e_2\|, \quad \|e_1+e_2\|) \tag{29}$$

for all $p \in \Gamma$ in the four defined quadrants. But for p as specified, $p - \frac{1}{2}(e_1+e_2) = r_1e_1 + r_2e_2$ with $|r_1| \geq \frac{3}{2}$ and $|r_2| \geq \frac{3}{2}$. Taking squares and computing out the norms, (29) is equivalent to the inequality

$$r_1^2 + r_2^2\alpha^2 + 2r_1r_2\alpha\cos\theta \geq \frac{3}{4}(1+\alpha^2+2\alpha|\cos\theta|).$$

Using $|r_1|, |r_2| \geq \frac{3}{2}$ and $|\cos\theta| \leq \frac{1}{2}$, the left side can be estimated as

$$\begin{aligned}(r_1^2 + r_2^2\alpha^2 + 2r_1r_2\alpha\cos\theta) &\geq r_1^2 + r_2^2\alpha^2 - |r_1r_2|\alpha \\ &= \frac{1}{2}(r_1^2+\alpha^2r_2^2) + \frac{1}{2}(|r_1| - \alpha|r_2|)^2 \\ &\geq \frac{1}{2}(r_1^2+\alpha^2r_2^2) \geq \frac{9}{8}(1+\alpha^2).\end{aligned} \tag{30}$$

It is now easily verified that

$$\frac{9}{8}(1+\alpha^2) \geq \frac{3}{4}(1+\alpha^2+\alpha) \geq \frac{3}{4}(1+\alpha^2+2\alpha|\cos\theta|)$$

as was to be proved.

Here equality holds if and only if $\alpha = 1, \theta = \frac{\pi}{3}$, and $r_1 = -r_2 = \pm\frac{3}{2}$, or $\theta = \frac{2\pi}{3}$, and $r_1 = r_2 = \pm\frac{3}{2}$. $\square$

8.3. Lemma

Suppose Γ has a normal basis e_1, e_2, with $\measuredangle(e_1, e_2) = \theta$ and $\|e_2\| = \alpha$. Let $a_1 = pe_1 + qe_2$, $a_2 = re_1 + se_2$ where $p, q, r, s \in \mathbb{Z}$; and set $\measuredangle(a_1, a_2) = \varphi$.

1. $a_1 = \frac{ps-qr}{r+s}(e_1 - e_2) + \frac{p+q}{r+s}a_2$.
2. *If $|q| \geq 3$ or $|q|, |p| \geq 2$, then $\|a_1\| > \frac{2}{\sqrt{3}}\|e_1 \pm e_2\|$.*
3. *$\varphi \leq \min(\theta, \pi - \theta)$ or $\varphi \geq \max(\theta, \pi - \theta)$ is equivalent to the inequality $|ps - qr|\alpha \leq \|a_1\| \ \|a_2\|$ with equality only if $\varphi = \theta$ or $\varphi = \pi - \theta$.*

Proof. (1) is a simple computation.

(2) follows from $\|a_1\|^2 = p^2 + 2pq\alpha\cos\theta + \alpha^2 q^2 \geq p^2 - |pq|\alpha + q^2\alpha^2 \geq \frac{1}{2}(p^2 + q^2\alpha^2) = C(p,q)$. If $|q| \geq 3$, then $C(p,q) \geq \frac{9}{2}\alpha^2$; if $|p|, |q| \geq 2$, then $C(p,q) \geq 2 + 2\alpha^2$. Therefore, in both cases $C(p,q) \geq \frac{4}{3}(1 + \alpha + \alpha^2) \geq \frac{4}{3}\|e_1 \pm e_2\|^2$, as in the proof of Lemma 8.2.

(3) follows from $\|a_1\|\,\|a_2\| \sin\varphi = \|a_1 \times a_2\| = |ps - qr|\,\|e_1 \times e_2\| = |ps - qr|\alpha \sin\theta$. $\square$

In the following we call a parallelogram spanned by two vectors $a_1, a_2 \in \mathbb{R}^2$ *half-open*, if it contains two sides that enclose the larger angle but not their opposite sides. That is, if $\measuredangle(a_1, a_2) \geq \frac{\pi}{2}$, then $P = \{z \in \mathbb{R}^2 : z = s_1 a_1 + s_2 a_2, 0 \leq s_1, s_2 < 1\}$ (or $0 < s_1, s_2 \leq 1$); if $\measuredangle(a_1, a_2) < \frac{\pi}{2}$, then $P = \{z \in \mathbb{R}^2 : z = s_1 a_1 + s_2 a_2,\ 0 < s_1 \leq 1,\ 0 \leq s_2 < 1\}$ (or $0 \leq s_1 < 1,\ 0 < s_2 \leq 1$).

8.4. Theorem

Let Γ be a normal lattice with normal basis e_1, e_2. Let $a_1, a_2 \in \Gamma$ be vectors such that $\measuredangle(a_1, a_2) = \measuredangle(e_1, e_2) = \theta$, and let P be the half-open parallelogram spanned by a_1, a_2. Set $\mathcal{E} = P \cap \Gamma$. Then one of the following holds:

1. *$\mathcal{E}$ is connected.*
2. *$e_2 = (x, y)$ with $x = \frac{-r}{2s}$, $0 < r < s$, $a_1 = \epsilon e_1$, and $a_2 = \epsilon(re_1 + se_2)$ with $\epsilon = \pm 1$; or $e_2 = (x, y)$ with $x = \frac{r}{2s}$, $0 < r < s$, $a_1 = \epsilon e_1$, and $a_2 = \epsilon(-re_1 + se_2)$, $\epsilon = \pm 1$; or either of the above with a_1 and a_2 interchanged. In this case $\mathcal{E}$ is connected if and only if $|r| = 1$.*
3. *$e_2 = (x, y)$ with $x = \frac{-\alpha^2 s}{2r}$, $\|e_2\| = \alpha \geq 1, r > \alpha^2 s > 0, a_1 = -\epsilon e_2$, and $a_2 = \epsilon(re_1 + se_2)$; or $e_2 = (x, y)$ with $x = \frac{\alpha^2 s}{2r}$, $\|e_2\| = \alpha \geq 1, r > \alpha^2 s$, $a_1 = -\epsilon e_2$, and $a_2 = \epsilon(-re_1 + se_2)$; or either of the above with a_1 and a_2 interchanged. $\mathcal{E}$ is connected if and only if $|s| = 1$.*

Although our proof of Theorem 8.4 is analytic and technical, we encourage the reader to draw the appropriate diagrams to assist his/her deliberations on the various cases. He/she will find that many of the calculations have a simple geometric analogue.

Proof. To prove connectedness it suffices to show that for distinct $k, l \in \mathcal{E}$ there exists $f \in \mathcal{T}_0$ such that $k + f \in \mathcal{E}$ and $d(k + f, l) < d(k, l)$, where $d(k, l)$ is the length of the shortest path from k to l. Then given a path $k = h_0, h_1, \ldots, h_n$ in $\mathcal{E}$ with $d(h_i, l) > d(h_{i+1}, l) > 0$, applying the above to h_n and l produces a point h_{n+1} in $\mathcal{E}$ closer to l. Repetition of this argument gives a path from k to l.

Let $k = r_1 e_1 + s_1 e_2$, $l = r_2 e_1 + s_2 e_2$ both belong to $\mathcal{E}$. If $r_1 = r_2$, and $s_2 > s_1$, then $k_i = r_1 e_1 + (s_1 + j)e_2$, $j = 0, 1, 2, \ldots, s_2 - s_1$, are all in $\mathcal{E}$ because P is convex. A similar statement holds if $s_1 = s_2$. Thus if either the e_1 or the e_2 coordinates of k and l agree, then they are joined by a path in $\mathcal{E}$. If $r_1 < r_2, s_1 < s_2$, then we have to show that either $k + e_1 \in \mathcal{E}$ or $k + e_2 \in \mathcal{E}$. If $r_1 > r_2, s_1 < s_2$, we have to show that either $k - e_1 \in \mathcal{E}$ or $k + e_2 \in \mathcal{E}$. We analyze the first case. The latter can be reduced to the former by a reflection about the y-axis (i.e., $e_1 \to e_1, e_2 \to (-x, y)$).

Arguing by contradiction, assume that $k + e_1 \notin \mathcal{E}$ and $k + e_2 \notin \mathcal{E}$. This implies that the line segment joining k to $k + e_1$ and the line segment joining k to $k + e_2$ must intersect different sides σ_1 and σ_2 of the parallelogram P (Figure 6). Neither of these segments can lie in part on a side of

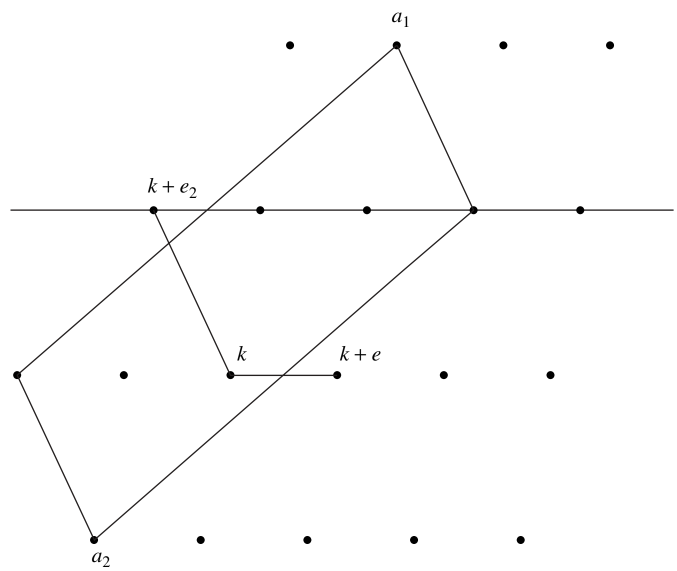

FIGURE 6

P because then, by definition of half-open, one of $k+e_1$ or $k+e_2$ will belong to $\mathcal{E}$. Several cases and subcases will be considered.

Case I. σ_1 and σ_2 are adjacent and intersect at a vertex v of P. Then $v = t_1e_1 + t_2e_2$ with either $t_1 \le r_1, t_2 \le s_1$ or $t_1 \ge r_2, t_2 \ge s_2$. Equality for the first pair implies that both of the segments lie on the boundary of the closed polygon; consequently, one of $k+e_1$ or $k+e_2$ must lie on a side of the half-open parallelogram, contrary to the assumption. Equality for the second pair is problematic only if $k+e_1 = l-e_2$ and $k+e_2 = l-e_1$, and then the above analysis produces a contradiction. Under all other circumstances Lemma 8.2 asserts that $\sphericalangle(\sigma_1, \sigma_2) < \sphericalangle(k+e_1-v,\ k+e_2-v) < \frac{\pi}{3}$, and this contradicts the assumption that $\sphericalangle(\sigma_1, \sigma_2) = \theta$ or $\pi - \theta$ with $\frac{\pi}{3} \le \theta,\ \pi - \theta \le \frac{2\pi}{3}$.

Case II. σ_1 and σ_2 are opposite sides of P and thus parallel to, say, a_2. This can only occur if the following propositions hold:

(1) a_2 is in the interior of the first or third quadrant with respect to basis e_1, e_2; that is, $a_2 = re_1 + se_2$ for $r, s > 0$ or $r, s < 0$. We argue for $r, s > 0$. The other case follows by reversing signs.

(2) The height of P perpendicular to a_2 is shorter than the length of the diagonal of the parallelogram spanned by the basis e_1, e_2, that is, $\|a_1\| \sin\theta < \|e_1 - e_2\|$. Since $\frac{\pi}{3} \le \theta < \frac{2\pi}{3}$, this implies $\|a_1\| < \frac{2}{\sqrt{3}}\|e_1 - e_2\|$.

(3) The projection of a_1 onto $e_1 - e_2$ along a_2 is shorter than the diagonal $\|e_1 - e_2\|$, that is, if $a_1 = \lambda(e_1 - e_2) + \mu a_2$, then $|\lambda| < 1$.

Now Lemma 8.3 asserts that for $a_1 = pe_1 + qe_2$ we must have $|q| \le 2$ and $|\lambda| = \frac{|ps-rq|}{r+s} < 1$. This allows us to reduce further the number of possible cases: $|ps - rq| < r + s$ entails $p, q \ge 0$ or $p, q \le 0$. If $p, q \ge 0$, then both a_1 and a_2 lie in the first quadrant with respect to the bases e_1, e_2 whence $\sphericalangle(a_1, a_2) < \sphericalangle(e_1, e_2) = \theta$, a contradiction. If $p, q \le 0$, then a_1 lies in the third quadrant. If we further assume that $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$, then clearly $\sphericalangle(a_1, a_2) > \pi - \theta \ge \theta$, again contradicting $\sphericalangle(a_1, a_2) = \theta$.

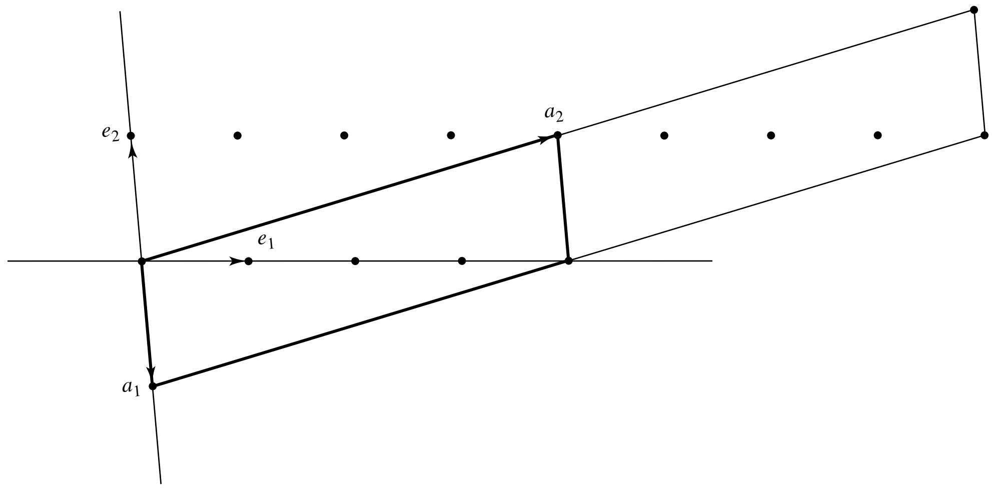

FIGURE 7

This leaves $\frac{\pi}{2} < \theta \leq \frac{2\pi}{3}$ and the possibilities $q = 0, -1$ and $p \leq 0$ or $q = -2$ and $p = 0, -1$. The restriction on p when $q = -2$ derives from Lemma 8.3(2).

We consider these subcases.

Case 1. If $q = 0$, $p < 0$, then Lemma (3) asserts $\|a_1\|\ \|a_2\| = |p|s\alpha$; whence, $\|a_2\| = s\alpha$. This implies $\|a_2\|^2 = r^2 + 2rs\alpha\cos\theta + s^2\alpha^2 = s^2\alpha^2$ or $2s\alpha\cos\theta + r = 0$. Then with $e_2 = (x, y)$, $x = \alpha\cos\theta = -\frac{r}{2s} < 0$; and since $x > -\frac{1}{2}$, we also have $r < s$. On the other hand, $|\lambda| = \frac{|ps-qr|}{r+s} < 1$ forces $-\frac{r}{s} - 1 < p < 0$ or $p = -1$. This is the first exception (2) stated in the theorem.

Case 2. $a_1 = pe_1 - e_2$, $p \leq 0$. Then $|\lambda| = \frac{|ps-rq|}{r+s} < 1$ implies the restriction $-\frac{2r}{s} - 1 < p \leq 0$ on the range of p.

If $p = 0$, then Lemma 8.3(3) implies $\|a_1\|\ \|a_2\| = |ps - rq|\alpha = r\alpha$ and consequently $\|a_2\| = r$. This entails $2r\cos\theta + \alpha s = 0$ or $x = \alpha\cos\theta = -s\alpha^2/2r$. Again since $x > -\frac{1}{2}$, we must have $r > \alpha^2 s$. This is the second exception (3) in the statement of the theorem.

If $s = 1$, then $\mathcal{E} = \{(j, 0),\ j = 1, \ldots, r-1\}$, which is obviously connected (Figure 7). Since it is not needed in the proof of Theorem 8.5, we leave it to the reader to prove that $\mathcal{E}$ connected implies $|s| = 1$.

Case 2 is completed by showing that $p < 0$ does not occur. In the context of Lemma 8.3(3) this follows by proving that

$$
\begin{aligned}
(ps + r)^2\alpha^2 &< \|a_1\|^2\ \|a_2\|^2 \\
&= (p^2 - 2p\alpha\cos\theta + \alpha^2)(r^2 + 2rs\alpha\cos\theta + s^2\alpha^2) \\
&= C(p, r, s) \qquad (31)
\end{aligned}
$$

for $p \leq -1$. Assume first that $p = -1$; then $C \geq \alpha^2(r^2 - rs + s^2\alpha^2) > \alpha^2(r - s)^2 = \alpha^2(ps + r)^2$, where we have used the normalization $|\alpha\cos\theta| = |x| \leq \frac{1}{2}$ (and will continue to).

Now suppose $r \geq s$. Observe that when $p = -\frac{r}{s}$, $\measuredangle(a_1, a_2) = \pi$. It follows that if $-\frac{2r}{s} - 1 < p \leq -\frac{r}{s}$, then $\measuredangle(a_1, a_2) > \measuredangle((-\frac{2r}{s} - 1)e_1 - e_2, a_2)$; consequently, it suffices to verify (31) for $p = -\frac{2r}{s} - 1$.

If $-\frac{r}{s} < p < 0$, then $\measuredangle(a_1, a_2) > \measuredangle(-e_1 - e_2, a_2)$ and by the above computation for $p = -1$ we have $\measuredangle(-e_1 - e_2, a_2) > \theta$. Therefore, when $r \geq s$ we need only consider the case $p = -\frac{2r}{s} - 1$:

$$\begin{aligned}
C &\geq (p^2 - |p| + \alpha^2)(r^2 - rs + s^2\alpha^2) \geq (p^2 - |p| + \alpha^2)\alpha^2 s^2 \\
&= \alpha^2((2r + s)^2 - 2rs - s^2 + s^2\alpha^2) = \alpha^2(4r^2 + 2sr + s^2\alpha^2) \\
&> \alpha^2(r + s)^2 = \alpha^2(ps + r)^2,
\end{aligned}$$

as asserted

The case $r < s$ remains to be considered.

If $\frac{2r}{s} < 1$, then $p = -1$, which was already dispatched above. The last possibility is $\frac{r}{s} < 1 < \frac{2r}{s}$, which entails $p = -2$ or -1. If $p = -2$, then $\measuredangle(a_1, a_2) > \measuredangle(-2e_1 - e_2, e_1 + e_2) = \varphi$. Then $\varphi \geq \max(\theta, \pi - \theta)$ or equivalently $\sin\varphi \leq \sin\theta$ follows from Lemma 8.3(3) and the easily verified inequality $\alpha^2 \leq \|2e_1 + e_2\| \ \|e_1 + e_2\|$.

We now turn to the third and final case.

Case 3. $q = -2$, $p = 0$, or $p = -1$. Then $|\lambda| = \frac{|ps - qr|}{r+s} < 1$ implies $r < s$ for $p = 0$ and $r < 2s$ for $p = -1$.

First consider $p = 0$. Since $r < s$, we have $\measuredangle(a_1, a_2) > \measuredangle(-2e_2, e_1 + e_2) = \pi - \measuredangle(2e_2, e_1 + e_2)$. Applying Lemma 8.3(3) to the obvious inequality $2\alpha \leq \|2e_2\| \ \|e_1 + e_2\|$, we conclude that $\sin\measuredangle(2e_2, e_1 + e_2) \leq \sin\theta$, and therefore $\measuredangle(a_1, a_2) > \theta$.

Finally, let $p = -1$. With $r < 2s$ we have $\measuredangle(a_1, a_2) > \measuredangle(-e_1 - 2e_2, 2e_1 + e_2) = \pi - \measuredangle(e_1 + 2e_2, 2e_1 + e_2)$, and $\measuredangle(e_1 + 2e_2, 2e_1 + e_2) \leq \frac{\pi}{3}$ follows immediately from Lemma 8.2.

This completes the proof of Theorem 8.4. □

Remark. The exceptional cases (2) or (3) of Theorem are characterized by one of a_1 or a_2 being of the form $\pm e_1$ or $\pm e_2$. When, in particular, we have a Euclidean similarity A and set $a_1 = Ae_1$ and $a_2 = Ae_2$, this becomes $A\mathcal{T}_0 \cap \mathcal{T}_0 \neq \emptyset$.

The next theorem settles the question of existence of self similar Γ-tilings in the case of elliptic dilation matrices.

8.5. Theorem

Let Γ be a normal lattice with normal basis e_1, e_2, and suppose that the dilation A is a Euclidean similarity and $A\Gamma \subseteq \Gamma$. Then there is a basis e_1', e_2' for Γ and a half-open parallelogram P spanned by Ae_1' and Ae_2' such that for $\mathcal{D} = P \cap \Gamma$ the Γ-translates of $Q(A, \mathcal{D})$ tile $\mathbb{R}^2$.

By conjugation we therefore obtain the existence of connected self-similar tiles for *all* elliptic dilation matrices and, hence, Theorem 2.6.

Proof. (a) $\mathcal{D}$ is a set of digits. This is clear, since P is a fundamental domain for $\mathbb{R}^2/A\Gamma$ and the kernel of the projection $p : \mathbb{R}^2/A\Gamma \to \mathbb{R}^2/\Gamma$ equals $\Gamma/A\Gamma \cong P \cap \Gamma$.

(b) Connectedness and face pairing. Either $A\mathcal{T}_0 \cap \mathcal{T}_0 = \emptyset$, and then $\mathcal{D}$ is connected according to Theorem 8.4(1); or $A\mathcal{T}_0 \cap \mathcal{T}_0 \neq \emptyset$, and we are in one of the exceptional cases (2), (3) of Theorem 8.4.

If $A\mathcal{T}_0 \cap \mathcal{T}_0 = \emptyset$, then e_1 and e_2 pair faces. To see this consider the parallelograms $P_1 = P \cup (Ae_1 + P)$ and $P_2 = P \cup (Ae_2 + P)$, which are spanned by $\{2Ae_1, Ae_2\}$ and $\{Ae_1, 2Ae_2\}$ respectively. Then $P_i \cap \Gamma = \mathcal{D} \cup (Ae_i + \mathcal{D})$, $i = 1, 2$, are $\mathcal{T}_0$-connected, again by Theorem 8.4. Thus there is a path from $k \in \mathcal{D}$ to $l_i \in Ae_i + \mathcal{D}$, $i = 1, 2$, and we can find $k_i \in \mathcal{D}$, $f_i \in \mathcal{T}_0$, $m_i \in Ae_i + \mathcal{D}$ so that $k_i + f_i = m_i$, $i = 1, 2$. This means that e_1 and e_2 pair faces. Thus in this case the choice $e_1' = e_1$ and $e_2' = e_2$ is appropriate.

(c) We now turn to the exceptional cases $A\mathcal{T}_0 \cap \mathcal{T}_0 \neq \emptyset$. Since $\|Ae_i\| = \sqrt{q}\|e_i\| > 1 = \|e_1\|$, case (2) in Theorem 8.4 is impossible for Euclidean similarities. However, $Ae_1 = \pm e_2$ does occur precisely when the matrix is of the form $A = \pm\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, where $e_2 = (x, y)$. Writing $Ae_2 = re_1 + se_2$ in terms of x, y gives $\pm(x^2 - y^2, 2xy) = (r + sx, sy)$, and from $|x| \leq 1/2$ we infer $|s| = |2x| \leq 1$. $s = 0$ is no problem because then P is a parallelogram aligned along e_1 and e_2, and the above argument prevails.

If $s = \pm 1$, then $\mathcal{D}$ is connected (for all choices of signs), e.g., $\mathcal{D} = \pm\{(j, 0), j = 0, \dots r - 1\}$ will do. However, e_2 does not pair faces since $Ae_2 + \mathcal{D} = \{(j+r)e_1 \pm e_2 \mid j = 0, \dots, r-1(\text{ or } -j = 0, \dots, r-1)\}$ and $(Ae_2 + \mathcal{D}) \cap (\mathcal{T}_0 + \mathcal{D}) = \emptyset$ (see Figure 7).

The situation is repaired by the choice of a different basis. To be specific, assume $Ae_1 = -e_2$ and $Ae_2 = re_1 + e_2$ with $r > 0$ and choose $e_1' = e_1, e_2' = e_1 + e_2$. The other possible choices of signs are treated in the same fashion. Then the digit set $P' \cap \Gamma$, where P' is the parallelogram spanned by $Ae_1' = -e_2$ and $Ae_2' = re_1$, remains $\mathcal{D} = \{(j, 0), j = 0, \dots, r - 1\}$. Then clearly $(Ae_1' + \mathcal{D}) \cap (-e_2' + \mathcal{D}) = (-e_2 + \mathcal{D}) \cap (-e_1 - e_2 + \mathcal{D}) \neq \emptyset$ and $(Ae_2' + \mathcal{D}) \cap (e_1' + \mathcal{D}) = (re_1 + \mathcal{D}) \cap (e_1 + \mathcal{D}) = \{re_1\} \neq \emptyset$ and e_1', e_2' pair faces.

This settles the exceptional cases and the proof of Theorem 8.5 is complete. □

The method developed in Theorem 8.5 has a much wider applicability than just the existence of self-similar Γ-tilings for elliptic dilations. Although we do not have a general existence proof of self-similar Γ-tilings for irrational hyperbolic dilations we can show the existence for a large class of such dilations. The conditions of the following theorem are extremely easy to check.

8.6. Theorem

Let A be an (integer-valued) dilation matrix acting on $\mathbb{Z}^2$ with basis $e_1 = (1, 0)$, $e_2 = (0, 1)$. Assume that both $\|Ae_1\| \geq \sqrt{2}$ and $\|Ae_2\| \geq \sqrt{2}$ and that $\frac{\pi}{4} \leq \measuredangle(Ae_1, Ae_2) \leq \frac{3\pi}{4}$. Let P be a half-open parallelogram spanned by Ae_1 and Ae_2. Then the set $\mathcal{D} = P \cap \mathbb{Z}^2$ produces a self-similar connected $\mathbb{Z}^2$ tile $Q(A, \mathcal{D})$.

The proof is similar to the proofs of Theorem 8.4 and Theorem 8.5. On the lattice $\mathbb{Z}^2$ many details are easier to check. Also the angle $\measuredangle(Ae_1, Ae_2)$ may vary more in this situation, and the cumbersome Case II in the proof of Theorem 8.4 does not occur since $\|Ae_i\| \geq \sqrt{2}$. We may therefore leave the modifications that are required to the reader. The basic ideas appear in the proof of Lemma 9.2.

9. Number Systems

We now give proofs of Theorems 2.8 and 2.9. The method follows the general elliptic case, although deviating slightly from the description of digit sets given in §8. Attention will be restricted to the lattice $\mathbb{Z}^2$ and dilations of the form $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $a, b \in \mathbb{Z}$, which are all Euclidean similarities on $\mathbb{Z}^2$. We work with the standard basis $\{e_1, e_2\}$.

Let $N(0, r)$ be the set of points $k \in \mathbb{Z}^2$ for which there is a path from k to 0 of length at most r. Let $\mathcal{D}$ be a connected digit set for the dilation A, and suppose that $\mathcal{T}_0$ pairs faces.

9.1. Lemma

If $\Delta\mathcal{T}_0 \subseteq \mathcal{D}$, then $N(0, n) \subseteq \mathcal{D}_n$.

Proof. We argue by induction. Suppose $N(0, n) \subseteq \mathcal{D}_n$. Given $k \in N(0, n+1)$, we show that $k \in \mathcal{D}_{n+1}$. First since $0 \in \mathcal{D}$, we have $\mathcal{D}_m \subseteq \mathcal{D}_n$ for all $m \leq n$. Every $k \in N(0, n+1)$ is of the form $k = h + e$ for $h \in N(0, n)$ and $e \in \mathcal{T}_0$ and therefore $k - e \in N(0, n) \subseteq \mathcal{D}_n$. Since the disjoint union of the $A^n\mathbb{Z}^2$-translates of $\mathcal{D}_n$ is $\mathbb{Z}^2$ (Lemma 3.1), we have $k \in A^n l + \mathcal{D}_n$ for some $l \in \mathbb{Z}^2$. Then from $k - e \in \mathcal{D}_n$ we conclude that l pairs faces of $\mathcal{D}_n$. By Lemma 7.3, $l \in \Delta\mathcal{T}_0 \subseteq \mathcal{D}$, and thus $k \in A^n\mathcal{D} + \mathcal{D}_n = \mathcal{D}_{n+1}$ as required. □

Let P be any of the half-open polygons with vertices $(\pm e_1 \pm e_2)/2$. P is a unit square at the origin with half of its boundary deleted (as defined in § 8). Then clearly $\mathcal{D} = AP \cap \mathbb{Z}^2$ is a set of digits for A.

9.2. Lemma

Let A be a Euclidean similarity, and set $\mathcal{D} = AP \cap \mathbb{Z}^2$. Then $\mathcal{D}$ is $\mathcal{T}_0$-connected and e_1 and e_2 both pair faces of $\mathcal{D}$. Thus by Theorem 7.1 *the $\mathbb{Z}^2$-translates of $Q(A, \mathcal{D})$ tile $\mathbb{R}^2$.*

Proof. The proof is similar to that of Theorem 8.4.

First we see that $\mathcal{D}$ is connected. Let $k = re_1 + se_2$ and $k' = r'e_1 + s'e_2$ be two points in $\mathcal{D}$. As in the proof of Theorem 8.4 there is no loss of generality in assuming that $r < r'$ and $s < s'$. Also, as before, we are reduced to showing that for one of the basis vectors e, $k + e \in \mathcal{D}$. If this were not true, then the line segments from k to $k + e_1$ and from k to $k + e_2$ would both meet distinct sides σ_1 and σ_2 of ∂P.

Suppose that σ_1 and σ_2 are opposite, parallel sides. Then it follows that both sides meet the diagonal joining $k + e_1$ to $k + e_2$, and therefore the sides of the square AP have length strictly less than $\|e_1 - e_2\| = \sqrt{2}$. This implies that $\text{area}(AP) < 2$, which is impossible for a Euclidean similarity. Thus σ_1 and σ_2 cannot be opposite sides of AP.

If σ_1 and σ_2 are adjacent, then they share a vertex v. From Lemma 8.2 we infer that $\theta = \measuredangle(k + e_1 - v, k + e_2 - v) \leq \pi/3$, which contradicts the fact that the angles of the square AP are $\pi/2$. It follows that $k + e$ belongs to $\mathcal{D}$ for some basis vector e.

Almost the identical argument shows that the set $\mathcal{D} \cup (Ae + \mathcal{D}) = [AP \cup (Ae + AP)] \cap \mathbb{Z}^2$ is connected for any basis vector e. It is immediate that e_1 and e_2 pair faces of $\mathcal{D}$. □

Remark. Let $a_1, a_2 \in \mathbb{Z}^2$, and let P be the half-open parallelogram spanned by a_1 and a_2. It is *not* true in general that connectedness of $P \cap \mathbb{Z}^2$ implies connectedness of $(x + P) \cap \mathbb{Z}^2$ for all $x \in \mathbb{R}^2$. Thus Lemma 9.2 does not follow directly from Theorem 8.4 or Theorem 8.6. □

9.3. Proposition

Let the dilation A be a Euclidean similarity, and set $\mathcal{D} = AP \cap \mathbb{Z}^2$. If $|\det A| > 8$, then $\bigcup_{n=1}^{\infty} \Delta\mathcal{D}_n = \mathbb{Z}^2$.

Proof. The proposition will follow from the preceding two lemmas by showing that $\Delta\mathcal{T}_0 \subseteq \mathcal{D}$. Since $A = \sqrt{q}O$, AP is a rotated copy of the square P_0 with vertices $(\pm\sqrt{q}/2, \pm\sqrt{q}/2)$. Thus $\mathrm{dist}(0, \partial AP) = \mathrm{dist}(0, \partial P_0) = \sqrt{q}/2$. Since $|\det A| = q > 8$, $\mathrm{dist}(0, \partial AP) = \sqrt{q}/2 > \sqrt{2} = \sup_{k\in\Delta\mathcal{T}_0} \|k\|$. Hence $\Delta\mathcal{T}_0 \subseteq AP \cap \mathbb{Z}^2$. $\square$

We now state Theorems 2.8 and 2.9 in their matrix version.

9.4. Theorem

Let A be a Euclidean similarity other than $\begin{pmatrix} 1 & \pm 1 \\ \mp 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, and set $\mathcal{D} = AP \cap \mathbb{Z}^2$. Then each $k \in \mathbb{Z}^2$ has a unique representation of the form

$$k = \sum_{j=0}^{n} A^j \epsilon_j \ \textit{with}\ \epsilon_j \in \mathcal{D} \ . \tag{32}$$

By uniqueness we mean that if also $k = \sum_{j=0}^{N} A^j \epsilon_j'$, $\epsilon_j' \in \mathcal{D}$, then $\epsilon_j = \epsilon_j'$ for $j = 0, \dots n$ and $\epsilon_j' = 0$ for $j > n$.

Proof. First let $|\det A| > 8$. Given $k \in \mathbb{Z}^2$, it may be inferred from Proposition 9.3 that there is an integer $n > 0$ such that $k \in \mathcal{D}_n$. By the definition of $\mathcal{D}_n$ (12), $k = \sum_{j=0}^{n} A^j \epsilon_j$ with $\epsilon_j \in \mathcal{D}$.

Since $\mathcal{D}_n \subseteq \mathcal{D}_{n+1}$ when $0 \in \mathcal{D}$, uniqueness is proven by showing that the representation of length n in $\mathcal{D}_n$ is unique for all n. We argue by induction. Certainly the representation for $k \in \mathcal{D} = \mathcal{D}_1$ is unique. Suppose this is true for $\mathcal{D}_n$. Since $\mathcal{D}_{n+1} = A^n\mathcal{D} + \mathcal{D}_n$ and $(A^n k + \mathcal{D}_n) \cap (A^n l + \mathcal{D}_n) = \emptyset$ for $k \neq l$, the uniqueness in $\mathcal{D}_{n+1}$ follows.

The remaining matrices $|\det A| \leq 8$ are now checked by hand. One observes that for all these matrices $\Delta\mathcal{T}_0 \subseteq \mathcal{D}_m$ for some $m \leq 5$. Then Proposition 9.3 is applicable to A^m with $\mathcal{D}_m$ as digit set. $\square$

9.5. Theorem

Let A be a Euclidean similarity other than $\begin{pmatrix} 1 & \pm 1 \\ \mp 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, and set $\mathcal{D} = AP \cap \mathbb{Z}^2$. Then each $x \in \mathbb{R}^2$ has a representation of the form

$$x = \sum_{j=1}^{\infty} A^{-j} \epsilon_j + \sum_{j=0}^{N} A^j \hat{\epsilon}_j \quad \textit{for}\ \epsilon_j, \hat{\epsilon}_j \in \mathcal{D}\ \textit{and some}\ N > 0. \tag{33}$$

Moreover, the representation is unique for almost all $x \in \mathbb{R}^2$.

Proof. By Lemma 9.2 the $\mathbb{Z}^2$-translates of $Q(A, \mathcal{D})$ tile $\mathbb{R}^2$. Thus each $x \in \mathbb{R}^2$ can be written in the form $x = \alpha + k$, where $k \in \mathbb{Z}^2, \alpha \in Q(A, \mathcal{D})$. Theorem 9.4 and the definition of $Q(A, \mathcal{D})$ give the existence of the desired representation. Moreover, the form $x = k + \alpha$ is unique except when $\alpha \in \partial Q(A, \mathcal{D})$.

Let $F = \bigcup_{k \in \mathbb{Z}^2}(k + \partial Q(A, \mathcal{D}))$, and set $X = (\bigcup_{n=0}^{\infty} A^{-n} F) \cap Q(A, \mathcal{D})$. Since $\lambda(\partial Q(A, \mathcal{D})) = 0$ by Proposition 4.3, we have $\lambda(X) = 0$.

Set $V = Q(A, \mathcal{D}) \setminus X$. To complete the proof we show that for $\alpha \in V$, $\alpha = \sum_{j=1}^{\infty} A^{-j}\epsilon_j = \sum_{j=1}^{\infty} A^{-j}\epsilon'_j$, $\epsilon_j, \epsilon'_j \in \mathcal{D}$, implies $\epsilon_j = \epsilon'_j$ for all $j \in \mathbb{Z}$.

Any two finite length expansions for α must be identical, since multiplying by a sufficiently large power of A puts us in the case treated by the previous theorem. Therefore, we may suppose that $\epsilon_j \neq 0$ for infinitely many j. Let n be the smallest number for which $\epsilon_n \neq \epsilon'_n$. Then $A^{-n}(\epsilon_n + \sum_{j=1}^{\infty} A^{-j}\epsilon_{j+n}) = \sum_{j=n}^{\infty} A^{-j}\epsilon_j = \sum_{j=n}^{\infty} A^{-j}\epsilon'_j = A^{-n}(\epsilon'_n + \sum_{j=1}^{\infty} A^{-j}\epsilon'_{j+n})$. Consequently, we have $\epsilon_n + \beta = \epsilon'_n + \beta'$ for $\beta, \beta' \in Q(A, \mathcal{D})$. As already observed, this can only occur if $\beta \in \partial Q(A, \mathcal{D})$. But $\beta \in \partial Q(A, \mathcal{D})$ implies $\alpha \in V$, contrary to the assumption. Thus every $\alpha \in V$ has a unique digit expansion. $\square$

The preceding theorems are now easily translated into the complex forms given in §2. $\mathbb{R}^2$ is replaced by $\mathbb{C}$ in the obvious way. $\mathbb{Z}^2$ becomes the lattice of Gaussian integers $\mathbb{Z}[i] = \{m + ni | m, n \in \mathbb{Z}\}$. If $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is a Euclidean similarity and $(x, y) \in \mathbb{R}^2$, then the vector $A(x, y)$ corresponds to $(a + ib)(x + iy) \in \mathbb{C}$.

Finally, we show that there does not exist a set of digits $\mathcal{D}$ for which $(1 + i, \mathcal{D})$ is a number system. In other words, writing $q = 1 + i$, for each digit set $\mathcal{D}$ there is a $\gamma \in \mathbb{Z}[i]$ that cannot be written in the form

$$\gamma = \epsilon_0 + \epsilon_1 q + \cdots + \epsilon_n q^n \tag{34}$$

for $\epsilon_j \in \mathcal{D}$. Working in complex notation with $\mathcal{D} = \{\sigma, \tau\}$ where $\tau \neq 0$, we show that $i\tau \notin \mathcal{D}_n$ for any integer $n \geq 0$. It follows from the identity $i\tau = \tau + qi\tau$ that $i\tau \equiv \tau \bmod q$. If $i\tau$ could be written in the form (34), then there would be a smallest value of n, necessarily $n > 1$ so that $i\tau \in \mathcal{D}_n$. Consequently, we could write $i\tau = \tau + q\beta$ for some $\beta \in \mathcal{D}_{n-1}$. But then we obtain $\beta = i\tau \in \mathcal{D}_{n-1}$, contradicting the choice of n.

A similar argument works for $q = 1 - i$.

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