The Journal of Fourier Analysis and Applications
Volume 1, Number 2, 1994

Survey Article

The Hausdorff–Young Theorems of Fourier Analysis and Their Impact

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ABSTRACT. This paper is devoted to a study of the Hausdorff–Young theorems from a historical perspective, beginning with the F. Riesz–Fischer theorem. Introduced by W. H. Young (1912), these theorems were considered and extended by F. Hausdorff (1923), F. Riesz (1923), E. C. Titchmarsh (1924), G. H. Hardy and J. E. Littlewood (1926), M. Riesz (1927), and O. Thorin (1939/48). Special emphasis is placed upon the development of the proofs of the two Hausdorff–Young inequalities and their impact upon Fourier analysis as a whole, in particular on the M. Riesz–Thorin convexity theorem and on the interpolation of operators. The golden thread connecting the various extensions and generalizations is the concept of logarithmic convexity, one that goes back to the work of J. Hadamard (1896), A. Liapounoff (1901), J. L. W. V. Jensen (1906), and O. Blumenthal (1907).

1. Introduction

The aim of this paper is to consider the work of William Henry Young and Felix Hausdorff concerning the so-called Hausdorff–Young theorems of 1912/23 and to study their impact on Fourier analysis as a whole, in particular on the M. Riesz–Thorin convexity theorem and interpolation theory for operators. Although the Hausdorff–Young theorems are dealt with in detail in practically every book on Fourier analysis and are called as such, with both names attached to them; however, they do not seem to have been considered from a historical perspective. They are Young's and also Hausdorff's most popular work in the broad area of Fourier analysis.

Math Subject Classifications. Primary 01A60, 42-03, 47-03, 52-03.
Keywords and Phrases. Hausdorff–Young theorems, Fourier analysis, Titchmarsh inequality, Riesz–Thorin convexity theorem, interpolation theory for operators, logarithmic convexity
Acknowledgments. The author thanks his colleagues, Franziska Fehér and Rolf Nessel, for their critical reading of the manuscript and helpful suggestions.

ISSN 1069-5869

In order to put these theorems in their proper setting, it is best of all to begin this study with the famous F. Riesz–E. Fischer theorem of 1907 and to see how both Will Young[1] and his wife Grace Chisholm Young looked at this theorem in their joint survey paper [YY] on the subject from as early as 1913.

In this respect it is of interest to note what their son, Laurence Young (1905) [57, p. 282], wrote concerning the papers of his parents: "My parents published, in addition to the *Theory of Sets of Points* (1906), two other mathematical books and 214 papers; 18 papers were my mother's and 13 were joint. This is partly because my mother wanted the credit to go to my father; it would be fairer to say that about a third of the material, and virtually all the writing up of the final versions, was due to my mother: this last was made necessary by the fact that my father had also to make a living. There was no such thing as a research grant."

Of Young's papers, some fifty are devoted to Fourier analysis (and orthogonal series). The first appeared in 1908, when Young was already in his 45th year; the last paper in 1924. Concerning Hausdorff, of his 42 publications only one is devoted to Fourier analysis.

Since papers treating the history of mathematics, specifically here the development of a basic theorem in the course of some 50 years, are usually read by a wide audience, a sincere attempt is made to make this treatment accessible for nonexperts also. In this respect this author has recently co-authored two papers on the work of de La Vallée Poussin (see [14, 15] as well as [12]).

2. The Riesz–Fischer Theorem

The Youngs' first major joint interest in Fourier analysis was the so-called Riesz–Fischer Theorem[2] of 1907, one of the corner stones of mathematical analysis and mathematical physics. It is stated by them as follows [YY, p. 52]:

If the squares of the coefficients of a trigonometrical series

$$\frac{1}{2}a_0 + \sum_{n=1}(a_n \cos nx + b_n \sin nx)$$

[1]**William Henry Young**, born 1863 in London, died 1942, studied in Cambridge. Only after his marriage 1896 with Grace Chisholm (1868–1944), who received her doctorate under F. Klein at Göttingen 1896, did Young begin to do mathematical research. Both worked together at Göttingen, Geneva, and Lausanne, where they established their permanent residence. Young was a professor in Calcutta (1913), Liverpool (1916), and Aberystwyth (1919–1923) and was president of the International Union of Mathematicians 1929–36.

Independently of Lebesgue (1902) he developed (1904) the new integral concept (he named it after Lebesgue). In 1911 he did preliminary work on P. Daniell's general integral concept (of 1918). Important contributions to the theory of Fourier and general orthogonal series as well as to the foundations of the differentiable calculus, particularly to differentiation of functions of several variables. See *J. London Math. Soc.* **17** (1942), 218–237; *Historia Math.* **2** (1975), 43–58; *J. London Math. Soc.* **19** (1944), 185–192.

[2]Concerning the reason why the "Riesz–Fischer" theorem is known as such, the Youngs write [YY, footnote, p. 52] that whereas Fischer presented his result to the Paris Académie des Sciences on May 13, 1907 (published in the Comptes Rendus [22]) and communicated it to the Mathematical Society of Brünn already on March 5, Riesz's paper had been laid before the Göttingen Gesellschaft der Wissenschaften by Prof. Hilbert on March 9 and published in the *Göttinger Nachrichten* [43], with that date attached. In this respect L. C. Young writes [57, p. 309]: *"Göttingen never forgave his [F. Riesz's] part in the Riesz–Fischer theorem, published after hearing the Seminar talk of Fischer [at Göttingen*], . . . ". Thus F. Riesz must have heard Fischer's talk at Göttingen before March 9, 1907. In contrast, according to Hilb–Riesz [31, p. 1212], Riesz's paper also appeared in the same Comptes Rendus volume [43, pp. 615–19] (thus, 400 pages earlier) and was presented to the Paris Academy on March 18; they speak of *"der von F. Riesz und E. Fischer nahezu gleichzeitig gefundene sogenannte Riesz–Fischersche Satz . . . "*.

form a convergent series $\sum_{n=0}(a_n^2 + b_n^2)$*, then the trigonometrical series is the Fourier series of a function whose square is summable.*

In the introduction to their paper the Youngs write "Though first stated scarcely more than five years ago, no fewer than seven different proofs have been supplied by continental writers, . . . Each proof has its own peculiar advantages and throws valuable light on the still extremely modern Theory of Functions of a Real Variable . . . . We have thought it therefore not superfluous to give an account of these proofs. We think that each proof will gain by being compared with its rivals. We have, moreover, occasionally taken the opportunity to introduce various simplifications of a character to bring out the principles underlying the reasoning employed . . . . Throughout, in fact, our main object has been to expose the variety of methods possible, rather than to obtain fresh results."

In fact, the Youngs present Frigyes Riesz's first proof ([43], of 1907), as well as a second one ([44], of 1910), the proof [22] by Ernst Fischer[3], a proof by Hellinger, found in Hilbert's paper [32, p. 195] of 1906, a proof due to H. Weyl [55] of 1909. There is finally a proof by M.Plancherel [42] of 1910; it is "not reproduced here, but utilized in the proofs of Weyl's theorem and other auxiliary theorems given in §§17–20." The paper concludes with Riesz's (first) proof of the extension of the Riesz–Fischer theorem from the system of trigonometrical to general orthogonal functions.

Concerning the Fischer approach, the Youngs write that Fischer regards the Riesz–Fischer theorem as a special case of the theorem established by him in the same paper, namely, that every Cauchy sequence $(f_n)_{n=1^\infty}$ in $L^2(a, b)$ converges to a function f in $L^2(a, b)$, i.e., that the space $L^2(a, b)$ is complete. The Riesz–Fischer theorem was indeed one of the first major applications of the Lebesgue integral, which had been discovered just five years earlier. William Young, who in 1904 found the definition of the integral independently of Lebesgue's work of 1902, introduced the phrase "Lebesgue integral" into the literature.

To indicate their mathematical style, let us cite Youngs's conclusion [YY, p. 87] to their 39-page paper:

"§34. Summing up the information we have obtained about the Fourier series of a function whose square is summable, we have the following result: *The Fourier series of a function whose square is summable need not converge to its function, but it does so, except at a set of content zero, when the terms are properly grouped together; moreover, the convergence is then of the Weyl type, i.e., we can, by excluding a set of content as small as we please, make the convergence uniform with respect to the remaining set; further, the complete succession of partial summations* $s_n(x)$ *is such that* $| f - s_n | \leq e$ *except at a set whose content vanishes when* $n \to \infty$*.*

No similar theorem has been obtained for functions whose $(1 + p)$th powers are summable, if $p \leq 1$."

G. H. Hardy (1877–1947), practically a contemporary of Young (and perhaps Britain's most influential mathematician of the period 1900–50) wrote in his obituary notice (*J. London Math. Soc.* **17** (1942), p. 228) on Young: ". . . and he (or he and his wife together) could write an excellent historical and critical résumé, with just the right spice of originality." As an example he gives the paper [YY].

[3]**Ernst Fischer**, born 1875 in Vienna, died 1954, studied in Vienna, receiving his doctorate 1899 under F. Mertens, the Habilitation 1904. Taught from 1902 until 1811 in Brünn, full professor 1911 in Erlangen, from 1920 to 1938 in Cologne when the Nazi race laws forced him to retire. Although especially well known due to the Riesz–Fischer theorem, his main work was on invariant theory and determinants. His work on algebra while at Erlangen influenced E. Noether. *Neue Deutsche Biographie* **5** (1961), p. 183.

3. The Hausdorff–Young Theorems and Their First Extensions

We now discuss one of Young's major contributions to the theory of Fourier series: basically the problem is how far can the Riesz–Fischer theorem be extended to powers of $f(x)$ other than two?

(A.) *If f is a function whose $(1+1/q)$th power is summable, where q is an odd integer, and a_n and b_n are its typical Fourier constants, then the sum*

$$\sum_{n=1} \left(a_n^{1+q} + b_n^{1+q}\right)$$

is convergent.

(B.) *If a trigonometrical series $a_0/2 + \sum_{n=1}(a_n \cos nx + b_n \sin nx)$ is such that*

$$\sum_{n=1} \left(a_n^{1+1/q} + b_n^{1+1/q}\right),$$

where q is an odd integer, is a convergent series, then the trigonometrical series is the Fourier series of a function $f(x)$ whose $(1+q)$th power is summable.

Part (A) is stated in slightly different words in [Y3, p. 79] (within a proof) and [Y2] and is proved in [Y2] and [Y1, p. 336] (using a result of [Y5]).

Part (B) is a direct citation from [Y3, p. 80] (except that the letter q is used instead of Young's p); the proof for the case $q = 3$ is to be found there. According to [Y3, p. 80], "In the general case the demonstration is precisely parallel." See especially §5 in this respect.

Part (A) can be regarded as a generalization either of Bessel's inequality to which it can be said to reduce when $q = 1$, namely,

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx,$$

or of Parseval's theorem, the case of equality here. Part (B) extends the Riesz–Fischer theorem.

Both parts are not converses of each other, except for $1 + q = 2$. Specifically, both parts become false if the numbers q and $1/q$ are interchanged. In fact, regarding (A), Carleman [16] constructed a continuous f (so that $f \in L_{2\pi}^r$ for all $r > 0$) such that $\sum_{n=1}^{\infty} \left(a_n^{1+q} + b_n^{1+q}\right) = \infty$ for all $q < 1$.

In the case of (B), Hardy and Littlewood [28] showed by means of an example that when $\sum_{n=1}^{\infty} \left(a_n^{1+1/q} + b_n^{1+1/q}\right)$ converges for a value of q that is < 1, the coefficients are not necessarily Fourier constants. Indeed, take the series $\sum_{n=1}^{\infty} n^{-1/2} \cos 2^n x$ (not a Fourier series) with $\sum_{n=1}^{\infty} \left(n^{-1/2}\right)^{1+1/q} < +\infty$ for any $q < 1$. See, for example, [33, Volume II, p. 227; 3, Volume I, p. 223].

It was some eleven years later before Felix Hausdorff[4] [H] seriously extended the two Young theorems. He answered the question of what happens if q is not necessarily an odd integer and put the theorems in their present known forms; he worked with trigonometrical series in their complex notation.

Now let p and p' be conjugate exponents, i.e., $1/p + 1/p' = 1$, $1 < p \le 2 \le p'$. Let f be real or complex, $c_k = (1/2\pi) \int_{-\pi}^{\pi} f(u) \exp(-iku)\, du$, $k \in \mathbf{Z}$, being the Fourier coefficients of f. Set

$$J_p(f) := \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \mid f(u) \mid^p \, du \right]^{1/p}, \qquad S_p(c) := \left[\sum_{k=-\infty}^{\infty} \mid c_k \mid^p \right]^{1/p}. \tag{3.1}$$

Hausdorff's version of the theorems, in slightly more modern terminology, reads:

(A$'$.) *If $f \in L^p_{2\pi}$ with $1 < p \le 2$, then $\sum_{k \in \mathbf{Z}} \mid c_k \mid^{p'}$ is convergent; in fact*

$$S_{p'}(c) \le J_p(f).$$

(B$'$.) *If $(c_k)_{-\infty}^{\infty}$ is any sequence of (complex) numbers such that $\sum_{k \in \mathbf{Z}} \mid c_k \mid^p$ is convergent, where $1 < p \le 2$, then the c_k are the Fourier coefficients of a function f such that $f \in L^{p'}_{2\pi}$; in fact*

$$J_{p'}(f) \le S_p(c).$$

In the Young notation $p = 1 + 1/q$, $p' = 1 + q$ $(q > 1)$. Young proved the theorems for q odd, $q = 2m - 1$, namely $p = 2m/2m - 1$, thus for the special sequence of values of p: 2, 4/3, 6/5, 8/7, ..., in other words, for $p' = 2, 4, 6, 8, \ldots$. He did not give the two explicit inequalities.

Just as in the case of the original Young theorems, both parts (A$'$) and (B$'$) become false, when $p > 2$; the examples mentioned there again apply. It can be shown (cf. [59, Volume II, p. 105]) that equality occurs in (A$'$) if and only if $f(x) = \text{const}\, e^{inx}$ for some $n \in \mathbf{Z}$ and in (B$'$) if and only if the c_n vanish for all but at most one element of $\mathbf{Z}$. (The "only if" assertion is not trivial and due to Hardy–Littlewood [29] (1926)).

Between the two parts of the Hausdorff–Young theorem there is a certain duality: (B$'$) is deduced from (A$'$) if the function f, depending on the variable x, is replaced by the function c

[4]**Felix Hausdorff**, born 1868 in Breslau, who committed suicide together with his wife before being deported to a concentration camp in 1942, studied in Leipzig, Freiburg, and Berlin. He received both his doctorate 1891 and the Habilitation 1895 in Leipzig. He was appointed associate professor in Leipzig 1902, 1910 in Bonn, and then full professor in Greifswald 1913. He returned to Bonn as a full professor 1921 where he stayed until his retirement in 1935. He began his work with astronomy, probability theory, and geometry. He was the author of 42 papers and 6 books (four of them under the pseudonym Paul Mongré). He wrote 3 papers on Hilbert's identities and bilinear forms (1909, 1919, 1927). His book *Grundzuege der Mengenlehre* (1914) made him world famous. Hausdorff space, Hausdorff moment problem, and Hausdorff summability are some basic concepts attached with his name. See *Jahresber. Deutsch. Math.-Verein.* **69** (1967), 51–76.

depending on the variable n, where integration is replaced by summation and vice versa. This is a major occurrence of such a duality in Fourier analysis.

In the Hausdorff–Young theorem the intrinsic assumption is that $p > 1$. However, it remains valid at $p = 1$; for then $p' = \infty$, and $\| f(x) \|_\infty =$ ess. $\sup_x | f(x) |$, and $\| c \|_\infty = \max_n | c_n |$.

Frigyes Riesz[5] [44] observed that the Hausdorff extensions of the Young theorems are also valid for any system of (complex) functions $\varphi_1(x), \varphi_2(x), \ldots$ that are orthogonal, normal, and uniformly bounded on some interval (a, b), i.e., $\int_a^b \varphi_n(u)\varphi_m(u)\, du = \delta_{n,m}$, $| f_n(x) | \leq M$, $n \in \mathbf{N}$. Riesz's extensions read:

(A″.) *If $f \in L^p(a, b)$ for $1 < p \leq 2$ and if c_k are the Fourier coefficients of f with respect to φ_k, i.e., $c_k = \int_a^b f(x)\overline{\varphi_k(x)}\, dx$, $k \in \mathbf{N}$, then $\sum_{k=1}^\infty | c_k |^{p'}$ is finite; in fact*

$$\left(\sum_{k=1}^\infty | c_k |^{p'} \right)^{1/p'} \leq M^{(2-p)/p} \left(\int_a^b | f(x) |^p\, dx \right)^{1/p}.$$

(B″.) *If $(c_k)_1^\infty$ is any sequence for which $\sum_1^\infty | c_k |^p$ is finite, then there is an $f \in L^{p'}(a, b)$ whose Fourier coefficients with respect to φ_k are c_k; in fact*

$$\left[\int_a^b | f(x) |^{p'}\, dx \right]^{1/p'} \leq M^{(2-p)/p} \left[\sum_{k=1}^\infty | c_k |^p \right]^{1/p}.$$

4. The Hausdorff–Young–Titchmarsh Inequality

Another mathematician who took up Young's work was Hardy's student E. C. Titchmarsh. Already on May 30, 1923, he presented his paper [51] to the London Mathematics Society, which gave the analogue of the Hausdorff formulation in the instance of Fourier transforms for the real line $\mathbf{R}$. Working in the spirit of Plancherel's approach [42, of 1910], Titchmarsh[6] himself regards the analogues as follows:

[5] **Frigyes Riesz**, born 1880 in Györ (Raab), died 1956, studied at the ETH Zürich, in Budapest and Göttingen. Doctorate 1902 in Budapest, then a *Gymnasium* teacher. Appointed associate professor in Klausenburg (=Koloszvar, Cluij) 1912, full professor there 1914, 1920 in Szeged, and 1946 in Budapest. At Szeged he built up the internationally known mathematical institute, and founded the Acta Sci. Math. Szeged in 1922. He was a member or corresponding member of the Academies of Hungary, Paris, Sweden, and Bavaria, and received honorary doctorates from Szeged, Budapest, and Paris. He was the author of some 95 papers on topology, real and analytic functions, ergodic theory, subharmonic functions, Riesz representation theorems, and compact operators. See B. Sz.-Nagy, J. Szabados (eds.): Functions, Series, Operators. *Colloquia Mathematica Societatis János Bolyai* 35, North-Holland, Amsterdam, Vol. I, 1983, pp. 37–48, 69–76.

[6] **Edward Charles Titchmarsh**, born 1899 in Newburry, died 1963, studied in Oxford under G. H. Hardy, receiving his B.A. 1922, later the M.A. Already 1929 professor of pure mathematics at Liverpool, 1931 in Oxford. FRS in 1931, honorary doctorate Sheffield 1953. Especially well known for his books on the Riemann Zeta function (1930, 1951), on the theory of functions (1932), on the theory of Fourier integrals (1937), and on eigenfunction expansions associated with second-order differential equations (1946). He was also the author of some 135 papers. See *J. London Math. Soc.* **39** (1964), p. 544–565.

Analogue of the Parseval Theorem: *If* $f \in L^p(o, \infty)$*,* $1 < p \leq 2$*, then the function*

$$F(x) = \left(\frac{2}{\pi}\right)^{1/2} \frac{d}{dx} \int_0^\infty \frac{\sin xu}{u} f(u)\, du$$

belongs to the space $L^{p'}(0, \infty)$.

Analogue of the Riesz–Fischer Theorem: *If* $f \in L^p(0, \infty)$*,* $1 < p \leq 2$*, then there is a function* $F \in L^{p'}(0, \infty)$ *such that*

$$f(x) = \left(\frac{2}{\pi}\right)^{1/2} \frac{d}{dx} \int_0^\infty \frac{\sin xu}{u} F(u)\, du.$$

The function F, the same in both parts, which has a meaning for almost all x, is the *Fourier cosine transform*.

Titchmarsh states that his results apply equally well to Fourier sine transforms and shows by means of an example that the results are false if $p > 2$. However, nowadays the inequality

$$\left(\int_0^\infty \mid F(x) \mid^{p'} dx\right)^{1/p'} \leq \frac{\pi}{2} \left(\int_0^\infty \mid f(x) \mid^p dx\right)^{1/p},$$

just mentioned by Titchmarsh [51, p. 298] in passing, is usually regarded as Titchmarsh's analogue of the Hausdorff–Young results. In modern terminology, the Fourier transform $f\hat{} = \mathcal{F}^p[f]$ of $f \in L^p(\mathbf{R})$, $1 < p \leq 2$, being defined by

$$\mathcal{F}^p[f](v) = \underset{\varrho \to \infty}{\overset{(p')}{\text{l.i.m.}}} \frac{1}{\sqrt{2\pi}} \int_{-\varrho}^{\varrho} f(u) e^{-ivu}\, du,$$

it reads: The Fourier transform $\mathcal{F}^p$ defines a bounded linear transformation of $L^p(\mathbf{R})$, $1 < p \leq 2$, into $L^{p'}(\mathbf{R})$, which contracts norms, i.e.,

$$\left(\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \mid \mathcal{F}^p[f](v) \mid^{p'} dv\right)^{1/p'} \leq \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \mid f(u) \mid^p du\right)^{1/p} \qquad (f \in L^p(\mathbf{R})). \quad (4.1)$$

By contrast with the situation for the circle group, where this inequality was noticed to be sharp, equality is never attained in (4.1) except when $f \equiv 0$. In this respect K. I. Babenko [2] (1961) obtained a sharp form of the Titchmarsh inequality, stating that

$$\| \mathcal{F}^p[f] \|_{p'} \leq \left(\frac{p^{1/p}}{p'^{\,1/p'}}\right)^{1/2} \| f \|_p \qquad (f \in L^p(\mathbf{R})) \quad (4.2)$$

is valid for the special values $p' = 2, 4, 6, \ldots$. Using the methods of entire functions, Babenko proved that equality occurs in (4.2) for the Gaussian functions $f(x) = \exp(-\alpha x^2)$, $\alpha > 0$. This inequality for the full range of values of p, $1 < p \le 2$, is due to W. Beckner [6, 1975]; he also studied the situation for the Fourier transform on $\mathbf{R}^n$, the best constant in (4.2) now being the nth power of the multiplicative $(p - p')$-factor there.

5. Concerning the Proofs

Let us look at Young's proofs in the particular case $p = (2m)/(2m-1)$, $m \in \mathbf{N}$. A. Zygmund [58, p. 191f] in his original edition of 1935 writes "This case is fairly easy and, what is more important, in certain interesting applications of the Hausdorff–Young theorem it suffices entirely." Concerning notation, let, for example, $f, g \in L^2_{2\pi}$ with $f(x) \sim \sum_{k=-\infty}^{\infty} a_k e^{ikx}$, $g(x) \sim \sum_{k=-\infty}^{\infty} b_k e^{ikx}$. Then $f \cdot g \in L^1_{2\pi}$ with $f(x) \cdot g(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$, where c_k is now the *resultant* or *convolution* of a_k and b_k,

$$c_k := \sum_{j=-\infty}^{\infty} a_j b_{k-j}.$$

Concerning the resultant of two arbitrary positive sequences (a_k), (b_k), playing around with Hölder's inequality, Young [YY, Y1, Y3] established the *"bemerkenswerte Ungleichung"* (in the words of F. Hausdorff [H, p. 166]):

If $r > 1$, $s > 1$, $\frac{1}{r} + \frac{1}{s} > 1$ (so that $rs < r + s$), then

$$\sum_k c_k^{rs/r+s-rs} \le \left(\sum_k a_k^r\right)^{s/r+s-rs} \cdot \left(\sum_k b_k^s\right)^{r/r+s-rs}. \tag{5.1}$$

More elegantly, if $\lambda > 0$, $\mu > 0$, and $\lambda + \mu < 1$, then, in the notation of (3.1.),

$$S_{1/(1-\lambda-\mu)}(c) \le S_{1/(1-\lambda)}(a) \cdot S_{1/(1-\mu)}(b). \tag{5.2}$$

Inequality (5.2) precisely in this form is to be found in Hausdorff [H] and Hardy–Littlewood–Polya [30, p. 199]. The latter remarks that Young did not the consider the question of equality; it can occur only if all the a_k, or all the b_k, or all a_k but one or all the b_k but one, are zero.

Now let us take Young's proof (following [27]) of the simplest case of (B′), namely $p = 2m/(2m-1)$ for $m = 3$, i.e., $p = 4/3$, supposing that $\sum_k | c_k |^{4/3} < +\infty$. Denoting the resultant of c_k with itself by $c_k^{(2)}$, then $c_k^{(2)}$ is the Fourier coefficient of f^2. Taking $\lambda = \mu = 1/4$ in (5.2) yields $\sum_k | c_k^{(2)} |^2 \le \left(\sum_k | c_k |^{4/3}\right)^3$. Hence, by the Riesz–Fischer theorem, $f^2 \in L^2_{2\pi}$, i.e., $f \in L^4_{2\pi}$, and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} | f^2 |^2 \, dx = \sum_k | c_k^{(2)} |^2 \le \left(\sum_k | c_k |^{4/3}\right)^3.$$

This is $J_4(f) \le S_{4/3}(c)$. Young established this result explicitly for $m = 3$ and 4. For general $p = 2m/(2m-1)$ one would first have to generalize inequality (5.1) to m-fold convolutions. This result is given in a clearer form in [30, Theorem 279]: If

$$c_k^{(m)} := \sum_{i_1+\cdots+i_m=k} a_{i_1} a_{i_2} \cdots a_{i_m},$$

then, unless all but one are zero,

$$\sum_k \mid c_k^{(m)} \mid^2 \le \left(\sum_k a_k^{2m/(2m-1)} \right)^{2m-1}. \tag{5.3}$$

If $c_k^{(m)}$ is in particular the m-fold convolution of c_k with itself, where $\left(\sum_k c_k^{2m/(2m-1)}\right) < \infty$, then $c_k^{(m)}$ is the Fourier coefficient of f^{2m}. Hence again by the Riesz–Fischer theorem, $f^m \in L^2_{2\pi}$ and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \mid f^m \mid^2 dx = \sum_k \mid c_k^{(m)} \mid^2 \le \left(\sum_k \left(c_k^{2m/(2m-1)} \right) \right)^{2m-1}.$$

This would deliver $J_{2m}(f) \le S_{2m/(2m-1)}(c)$.

Concerning the proof of (A′), Young's proof being somewhat difficult to follow, let us take Hausdorff's modification of it (see also [30, p. 203] for the latter). The basic idea is to deduce (A′) from (B′). Let

$$s_n(x) = \sum_{k=-n}^{n} a_n e^{ikx}, \qquad g(x) = \sum_{k=-n}^{n} b_k e^{ikx}$$

—the former the Fourier partial sum of f, the latter an arbitrary polynomial. If $f \in L^p_{2\pi}$, Parseval's theorem (see, e.g., Young [YS]) and Hölder's inequality yield

$$\mid \sum_{k=-n}^{n} a_k b_k \mid = \mid \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) g(-u)\, du \mid \le J_p(f) J_{p'}(g) \le J_p(f) S_{p,n}(b),$$

where $S_{p,n}(b) := \left(\sum_{k=-n}^{n} \mid b_k \mid^p\right)^{1/p}$, the last inequality being an application of (B′). Taking in particular $b_k = \mid a_k \mid^{p'-1}$ sgn $\bar{a_k}$, then $a_k b_k = \mid a_k \mid^{p'} = \mid b_k \mid^p$. Hence

$$S_{p,n}(b) = \left(\sum_{k=-n}^{n} \mid a_k \mid^{p'} \right)^{1/p} = S_{p',n}^{p'/p}(a)$$

or

$$S_{p'}^{p'}(a) = \sum_{k=-n}^{n} | a_k b_k | \le J_p(f) S_{p',n}^{p'/p}(a).$$

Thus $S_{p'}(a) \le J_p(f)$, so that $\sum_{k=-\infty}^{\infty} | a_k |^{p'}$ is convergent, proving (A′).

One could also establish (A′), independently of (B′), by using a dual of (5.3). It is to be found in [30, Theorem 284]: If f has period 2π and

$$f_m(x) := \frac{1}{(2\pi)^{m-1}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(u_1) \cdots f(u_{m-1}) f(x - u_1 - \cdots - u_{m-1})\, du_1 \cdots du_{m-1}, \quad (5.4)$$

then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} | f_m(x) |^2 \, dx \le \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)^{2m/(2m-1)} \, dx \right)^{2m-1}. \quad (5.5)$$

Note that $f_1(x) = f(x)$, $f_j(x) := (f * f_{j-1})(x) := (1/2\pi) \int_{-\pi}^{\pi} f_{j-1}(x-u) f(u)\, du$, $j = 2, 3, \dots$.

Supposing that $f \in L_{2\pi}^{2m/(2m-1)}$, noting that if c_k are the Fourier coefficients of f, those of f_j are c_k^j, it follows by Parseval's theorem that

$$\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)^{2m/(2m-1)} \, dx \right)^{2m-1} \ge \frac{1}{2\pi} \int_{-\pi}^{\pi} | f_m(x) |^2 \, dx = \sum_{k=-\infty}^{\infty} | c_k^m |^2 = \sum_{k=-\infty}^{\infty} | c_k |^{2m}.$$

This is $S_{2m}(c) \le J_{2m/(2m-1)}(f)$.

Similary as above one could now deduce (B′) from (A′). It is to be found in Zygmund [58, §9.121, p. 191]. Therefore, both parts are in reality equivalent (in the particular Young instance).

Note that inequality (5.5) is related to, respectively, an extension of the so-called Young inequality about convolutions. This states that if $f \in L_{2\pi}^p$ and $g \in L_{2\pi}^p$ where $p \ge 1$, $q \ge 1$, $1/r = 1p + 1/q - 1 \ge 0$, then $f * g \in L_{2\pi}^r$ and

$$\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} | (f * g)(u) |^r \, du \right\}^{1/r} \le \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} | f(u) |^p \, du \right\}^{1/p} \cdot \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} | g(u) |^q \, du \right\}^{1/q}.$$

The particular case $1/p + 1/q = 1$ is the classical Hölder inequality, with $f * g$ continuous. See, for example, Hilb–Riesz [31, p. 1211] in regard to the history.

Finally three citations in regard to the proofs of Hausdorff's extensions of the Young theorems, namely (A′) and (B′) of §3. It is the commentator's impression that the breakthrough in regard to these proofs is due to Hausdorff and not to F. Riesz.

In this respect it is of interest to cite F. Riesz [45] in his own words: "Herr Hausdorff hat über seine Resultate im September d.J. der Naturforschersammlung in Leipzig berichtet; bezüglich des Beweises verwies er auf seine in dieser Zeitschrift demnächst erscheinende Arbeit. Durch diese Mitteilung angeregt, suchte ich mir einen Beweis auszuarbeiten; dabei fand ich, daß jene

Resultate nicht nur für das trigonometrische Orthogonalsystem, sondern auch für jedes beschränkte Orthogonalsystem gelten.

Mein Beweis läuft teilweise, wie ich aus einer brieflichen Mitteilung des Herrn Hausdorff erfahre, dem seinigen ziemlich parallel; der wesentliche Unterschied, der mir den allgemeineren Schluß gestattet, besteht in folgendem: Hausdorff führt den Fall eines beliebigen dyadisch rationalen $q = r/2^s > 1$ (aus welchem ja die entsprechenden Ungleichungen für beliebige $q > 1$ durch Einschachtelung unmittelbar folgen) in einer endlichen Anzahl von Schritten auf den Youngschen Fall zurück, während ich (für beliebige $q > 1$) denselben Prozeß u n e n d l i c h o f t wiederhole und den allgemeinen Fall auf den fast trivialen Grenzfall $q \longrightarrow \infty$ zurückführe. Genauer gesagt ist dieser Grenzübergang nur der leitende Gedanke meines Beweises, tatsächlich ersetze ich ihn in §2 durch eine einfacher Abschätzung für große Werte von q."

G. H. Hardy and J. E. Littlewood [29, p. 160] observed that F. Riesz's "alternative proof [was] somewhat simpler than Hausdorff's," and that "We give a new proof in §3. The proof will be useful as an introduction to the more difficult analysis which follows. It has also two points of independent interest; it enables us to complete the solution of the minimal problem suggested in Hausdorff's analysis, and it involves what would appear to be the least possible amount of 'existence-theory' ." It is odd that their (section) heading of §3, called "A new proof of Hausdorff's Theorem, with determination of the minimal functions," does not contain the word "Young." This new proof covers six pages.

Finally let us see what Will's son Laurence C. Young [57, p. 300] said in the matter: "Hausdorff's part of the [Hausdorff–Young] inequality was that of a professional rather than a pioneer.". In this respect, Hardy, concerning whom L. C. Young writes [57, p. 288] "Hardy was always generous in reference to my father," expresses the following general opinion about Will Young's work [27]: "His style is better in his books than in his papers, which are sometimes rather rambling and diffuse—· · · A theorem will be proved, in varying degrees of generality, in a half a dozen different papers, with continual cross-references, and promises of further developments not always fulfilled." Nevertheless the originator of the fundamental Hausdorff–Young theorem is Young; the basic facts concerning the proof of his particular cases are also his. Hausdorff essentially put the matter in its present form.

The proof of the Hausdorff–Young theorem generally found in the early literature is Frigyes Riesz's modification. Examples are Hobson [33, II, pp. 600–606] and Kacmarz–Steinhaus [36, pp. 203–208]. However, already in 1926 Frigyes's brother Marcel Riesz[7] came up with a new approach in establishing inequalities of the type given by the Young theorems, first known under "M. Riesz's convexity theorems"; see below.

6. The Riesz–Thorin Convexity Theorem

Whereas J. L. W. V. Jensen [34] (1906) first recognized the importance of the class of convex functions and associated inequalities, according to Zygmund [58, p. 70] it is apparently

[7] **Marcel Riesz**, born 1886 in Györ, died 1969; after studying 1904–1910 in Budapest, Göttingen, and Paris, doctorate 1909 in Budapest, he accepted an invitation by G. Mittag–Leffler in 1911 to be a lecturer at Stockholm's Högskola. Full professor at Lund Univ. 1926–1952. He had honorary degrees from Copenhagen and Lund, was a member of the Swedish Academy, of one in Lund and Copenhagen. He was the author of 60 papers on summability theory of power series, trigonometric series, Dirichlet series, potential theory, wave progation, relativity theory, and elementary number theory. *Acta. Math.* **124** (1979), I–XI.

F. Hausdorff [33] who observed[8], using Hölder's inequality, that the ordinary sum and integral means $S_p(c)$, $J_p(f)$ of (3.1), which are nondecreasing, continuous functions of p, define *logarithmically (or multiplicatively) convex functions* of p for $p > 0$. Thus $p \log S_p(c)$ is convex, namely,

$$S_p^p(c) \leq \left(S_{p_1}^{p_1}(c)\right)^{1-t} \cdot \left(S_{p_2}^{p_2}(c)\right)^t$$

where $p = p_1(1-t) + p_2 t,\ p_i > 0,\ 0 < t < 1$. As a follow-up, Marcel Riesz [46] (1927), continuing work of Hilbert (1888/93), Schur (1911), Hellinger and Toeplitz (1910), and his brother F. Riesz [45] (1923), considered the bilinear form in the variables $x_i,\ y_j$

$$A(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j$$

with real or complex coefficients a_{ij}. Letting $M_{\alpha,\beta}$ be the maximum of $\mid A(x, y) \mid$ for

$$\left[\sum_{i=1}^{m} \mid x_i \mid^{1/\alpha}\right]^\alpha \leq 1, \qquad \left[\sum_{j=1}^{n} \mid y_j \mid^{1/\beta}\right]^\beta \leq 1 \qquad (\alpha, \beta > 0)$$

$$\max_i \mid x_i \mid \leq 1, \qquad \max_j \mid y_j \mid \leq 1 \qquad (\alpha = \beta = 0)$$

he showed that $M_{\alpha,\beta}$ is a logarithmically convex function of the variables α, β in the lower triangle $\triangle = \{(\alpha, \beta);\ 0 \leq \beta \leq \alpha \leq 1\}$ of the unit square $\square = \{(\alpha, \beta);\ 0 \leq \alpha, \beta \leq 1\}$. Thus, if $\alpha = \alpha_1(1-t) + \alpha_2 t$, $\beta = \beta_1(1-t) + \beta_2 t$, $0 < t < 1$, then

$$M_{\alpha,\beta} \leq M_{\alpha_1,\beta_1}^{1-t} M_{\alpha_2,\beta_x}^{t}.$$

[8] Although, for example, Roberts–Varberg [47], D. S. Mitrinovič [40], M. Kuczma [38] go into the history of convex functions, they do not in the case of log-convex functions. One of the first results in this respect seems to be Hadamard's three-circles theorem: Let f be a holomorphic function, regular in the annulus domain $r_1 \leq \mid z \mid \leq r_2$. If $M_i = M(r_i) = \max_{|z|=r_i} \mid f(z) \mid$ for $i = 1, 2, 3$ with $r_1 < r_2 < r_3$, then

$$M_2^{\log(r_2/r_1)} \leq M_1^{\log(r_3/r_2)} M_3^{\log(r_2/r_1)},$$

i.e., $\log M(r)$ is a convex function of $\log r$. According to G. H. Hardy [26], this result "was discovered independently by O. Blumenthal [8], G. Faber [19], and Hadamard [25]. The first statement of the theorem was due to Hadamard and the first proof to Blumenthal." The theorem is related to results of E. Fabry (1902) and F. Hartogs (1905). It was perhaps Hardy (1914) who first used the terminology "$\log M(r)$ is a convex function of $\log r$". Hardy–Littlewood–Polya [30, p. 27] (who speak or deal with log-convex functions on pp. 71, 72, 122, 139, 145, but also do not go into its history) observe that for the means $\mathcal{M}_p = \mathcal{M}_p(a) = \left(n^{-1} \sum_{k=1}^{n} a_k^p\right)^{1/p}$ A. Liapounoff ([39], 1901) showed that for $p_1 < p_2 < p_3$, $\mathcal{M}_{p_2}^{p_1} \leq \left(\mathcal{M}_{p_1}^{p_1}\right)^{(p_3-p_2)/(p_3-p_1)} \left(\mathcal{M}_{p_3}^{p_3}\right)^{(p_2-p_1)/(p_3-p_1)}$ or, for $p_2 = p_1 t + p_3(1-t)$, with $0 < t < 1$, that $\sum a^{p_2} \leq \left(\sum a^{p_1}\right)^t \left(\sum a^{p_3}\right)^{1-t}$, i.e., in the present terminology $\log \mathcal{M}_p^p(a)$ is a convex function of p. See also G. Julia [35], E. F. Beckenbach [4], E. F. Beckenbach–R. Bellman [5]. It seems that the textbook [9] by Bohr–Mollerup (1922) contains the first proof of the fact that the Gamma function $\Gamma(x)$ is log-convex for $x > 0$. The author would like to thank J. Korevaar (Amsterdam), H. Kairies (Clausthal-Zellerfeld), and A. M. Bruckner (Santa Barbara) for valuable suggestions concerning the references to Hadamard, Kuczma, and Beckenbach, respectively.

Note that R. E. A. C. Paley [41] gave an alternative proof of the convexity theorem as did also Will Young's son Laurence C. Young [56] (1939). M. Riesz cites the Hausdorff paper four times.

Now the Riesz convexity theorem does not extend to the upper triangle of □, at least for real variables x_i, y_j. But if these are allowed to be *complex*, Riesz's student Olof Thorin (born 1912) showed [49, 50] (1939/48) that convexity does hold on the whole square □. Moreover, Thorin expressed the result not in the finite-dimensional setting of matrix transformations and their associated bilinear functionals but in terms of linear operators. But the basic idea of an "interpolation of linear operators" is M. Riesz's. Thorin's proof depends crucially on the use of complex functions, in fact on Hadamard's three lines theorem, whose conclusion also involves a logarithmically convex function. At last to the theorem; it reads as follows (see, e.g., [58, 59, 18, 7]):

Let (R_1, μ_1), (R_2, μ_2) be two measure spaces, and let T be a linear operator defined on all μ_1-simple functions on R_1 and taking values in the μ_2-measurable functions on R_2. Suppose that T is simultaneously of strong type $(1/\alpha_1; 1/\beta_1)$ and $(1/\alpha_2; 1/\beta_2)$, that,

$$\|Tf\|_{1/\beta_1,\mu_2} \leq M_1\|f\|_{1/\alpha_1,\mu_1}, \qquad \|Tf\|_{1/\beta_2,\mu_2} \leq M_2\|f\|_{1/\alpha_1,\mu_1}, \tag{6.1}$$

the points (α_1, β_1), (α_2, β_2) belonging to the square $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$. Then T is also of strong type $(1/\alpha; 1/\beta)$ for all

$$\alpha = (1-t)\alpha_1 + t\alpha_2, \qquad \beta = (1-t)\beta_1 + t\beta_2 \qquad (0 < t < 1)$$

and

$$\| Tf \|_{1/\beta,\mu_2} \leq M_1^{1-t} M_2^t \| f \|_{1/\alpha,\mu_1} . \tag{6.2}$$

In particular, if $\alpha > 0$, the operator T can be uniquely extended to the whole space $L_{\mu_1}^{1/\alpha}$, preserving norm.

If M_i is the strong type $(1/\alpha_i; 1/\beta_i)$ norm of T (i.e., the least constant M_i for which (6.1) hold), then (6.2) may be rewritten as

$$M_t \leq M_1^{1-t} M_2^t$$

where $M_t = M_t(\alpha, \beta)$ is the strong type $(1/\alpha; 1/\beta)$ norm of T. Inequality (6.2) in this form states that $M_t(\alpha, \beta)$ is logarithmically convex in □. Moreover, the set of all points (α, β) such that T is of strong type $(1/\alpha; 1/\beta)$ is a convex set. Thus the concept of logarithmically convex functions is the basic element of proofs in the papers of F. Hausdorff (also F. Riesz), M. Riesz, O. Thorin. It was used, at least implicitly, by W. Young when establishing the inequalities (5.1) and (5.2), as can best be seen from Hausdorff's presentation of Young's arguments.

Let us see how the Riesz–Thorin theorem applies to the Hausdorff–Young theorem. On the one hand, by definition of the complex Fourier coefficient of $f \in L_{2\pi}^1$,

$$\| c \|_\infty = \max_{n\in\mathbf{Z}} | c_n | \leq \| f \|_1 = J_1(f),$$

and on the other hand, by Bessel's inequality,

$$\| c \|_2 \equiv S_2(c) \leq \| f \|_2 \,.$$

Thus the Fourier-coefficient transform $f \longrightarrow c$ is simultaneously of types $(1/\alpha_1, 1/\beta_1) = (1, \infty)$ and $(1/\alpha_2, 1/\beta_2) = (2, 2)$ with the operator norms $M_1, M_2 \leq 1$. Thus $\alpha = (1-t)/1 + t/2 = (2-t)/2$, $\beta = (1-t)/\infty + t/2 = t/2$ so that for $p = 1/\alpha = 2/(2-t)$, $q = p' = 1/\beta = 2/t$ one has $0 \leq t \leq 1$, $1 \leq p \leq 2$ and so by the Riesz–Thorin theorem this transform is also of type (p, p'), mapping $L^p_{2\pi}$ into $l^{p'}$, with norm at most 1, or

$$\| c \|_{p'} \leq 1^{1-t} \cdot 1^t \; \| f \|_p \,.$$

This is (A′) of the Hausdorff result; (B′) can be established by a similar argument. So both parts are equivalent (see, e.g., [59, Vol. II, p. 103].

The Riesz–Thorin convexity theorem was the forerunner of a variety of interpolation theorems for linear and sublinear operators, and also in the realm of general Banach spaces. Thus, certain weaker conditions at the "end points" $(p_i; q_i)$, $i = 0, 1$, are often sufficient to guarantee strong type $(p_t; q_t)$ for $0 < t < 1$; here $1/p_t = (1-t)/p_1 + t/p_2$, likewise for q_t. In this respect there is the Marcinkiewicz interpolation theorem. It can be applied, for example, to the Hilbert transform or the conjugate function in the periodic instance; although of strong type $(2, 2)$ it is only of *weak* type $(1, 1)$. Still the operator is of strong type (p, p), $1 < p < 2$.

The same applies, for example, to the Hardy–Littlewood maximal operator, to fractional integrals. Then there is, for example, the E. M. Stein and G. Weiss [48] interpolation theorem for "restricted" weak type operators.

Let (X_1, X_2) be a *compatible couple*, i.e., a couple of Banach spaces X_1, X_2, both spaces continuously embedded in some Hausdorff topological spaces $\mathcal{X}$. Now the spaces $X_1 \cap X_2$ and $X_1 + X_2 = \{f = f_1 + f_2; f_1 \in X_1, f_2 \in X_2\}$ are Banach spaces under the norms

$$\| f \|_{X_1 \cap X_2} = \max\{\| f \|_{X_1}, \| f \|_{X_2}\},$$

$$\| f \|_{X_1 + X_2} = \inf\{\| f_1 \|_{X_1} + \| f_2 \|_{X_2}; f = f_1 + f_2\}.$$

Then any B-space $X \subset \mathcal{X}$ satisfying $X_1 \cap X_2 \subset X \subset X_1 + X_2$ with continuous embeddings is said to be an *intermediate* space of (X_1, X_2). For example, if $1 \leq p_1, p_2 \leq \infty$, $1/p_t = (1-t)/p_1 + t/p_2$, $0 \leq t \leq 1$, then L^{p_t} is an intermediate space between L^{p_1} and L^{p_2}.

Let (X_1, X_2) and (Y_1, Y_2) be two compatible couples in $\mathcal{X}, \mathcal{Y}$, and let X, Y be intermediate spaces of (X_1, X_2) and (Y_1, Y_2), respectively. Then the spaces X, Y are said to have the *interpolation property* with respect to the given compatible couples if each sublinear operator $T : X_1 + X_2 \longrightarrow Y_1 + Y_2$ with $T\,|_{X_i} \in [X_i, Y_i]$, $i = 1, 2$, is a bounded operator from X into Y. In particular, X and Y are called *interpolation spaces of type* t, $0 \leq t \leq 1$, if for each admissable operator T there holds the convexity inequality

$$M \leq c M_1^{1-t} M_2^t$$

with some positive constant c, M being the norm of T on X to Y and M_i the norm of $T\,|_{X_i}$ on X_i to Y_i. According to the Riesz convexity theorem, L^{p_t} is also an interpolation space between L^{p_1} and L^{p_2}.

Given two compatible couples (X_1, X_2) and (Y_1, Y_2), some of the general problems in the theory of interpolation of operators include the *characterization* of *all* interpolation spaces between (X_1, X_2) and (Y_1, Y_2), the *construction* of such spaces, the development of different interpolation methods. This material is treated in Butzer–Berens [11], one of the first books devoted to the material, and in the more modern books such as H. Triebel [53], who emphasizes important applications to differential equations, which originally gave impetus to the theory; Bennett–Sharpley [7], which is an elegant introduction into the theory of linear operators; and Krein–Petunin–Semenov [37] as well as Brudnyi–Krugljak [10], both typical fine Russian works, perhaps more in the direction of good hard analysis than functional analysis. For interpolation methods within the context of rearrangement invariant Banach function spaces and weak-type interpolation, see for example, [20, 21] of F. Fehér.

Now everybody admits (see, e.g., Garding [23], Hardy et al [30], Aronszajn–Gagliardo [1]) that the "beginning of what we now call interpolation methods between Banach spaces was the convexity theorem of M. Riesz." The bases to this theorem were the two Hausdorff–Young inequalities, as we saw. The golden thread connecting the results from 1912 until the present was—in my judgement—the concept of logarithmic convexity.

7. Cited papers of W. H. and G. C. Young and of F. Hausdorff

[YY] Young, W. H. and Young, G. C (1913). On the theorem of Riesz–Fischer. *Quarterly J. Math*. **44**, 49–88.

[Y1] Young, W. H. (1912). On the multiplication of successions of Fourier constants. *Roy. Soc. Proc*. (*A*) **87**, 331–339.

[Y2] ——— (1912). Sur la généralisation du théorème de Parseval. *C. R. Acad. Sci. Paris* **155**, 30–33 (472–475).

[Y3] ——— (1912/13). On the determinations of the summability of a function by means of its Fourier constants. *Proc. London Math. Soc*. **12**, 71–88.

[Y4] ——— (1911). On the Fourier constants of a function. *Roy. Soc. Proc*. (*A*) **85**, 14–24.

[Y5] ——— (1911). On a class of parametric integrals and their application in the theory of Fourier series. *Roy. Soc. Proc*. (*A*) **85**, 401–414.

[H] Hausdorff, F. (1923). Eine Ausdehnung des Parsevalschen Satzes über Fourierreihen. *Math. Zeit*. **16**, 163–169.

References

[1] Aronszajn, N. and Gagliardo, E. (1965). Interpolation spaces and interpolation methods. *Ann. Mat. Pura Appl*. (*4*) **68**, 51–118.

[2] Babenko, K. I. (1961). An inequality in the theory of Fourier integrals. *Izv. Akad. Nauk SSSR Ser. Mat*. **25**, 531–542; English transl., *Amer. Math. Soc. Transl*. (*2*) **44**, 115–128.

[3] Bary, N. A. (1964). A Treatise on Trigonometrical Series. 2 Vols. Pergamon, New York.

[4] Beckenbach, E. F. (1948). Convex functions. *Bull. Amer. Math. Soc*. **54**, 439–460.

[5] Beckenbach, E. F., and Bellman, R. (1971). *Inequalities*. Springer, Berlin, Heidelberg, and New York.

[6] Beckner, W. (1975). Inequalities in Fourier analysis. *Ann. of Math*. **102**, 159–182.

[7] Bennett, C., and Sharpley, R. (1988). *Interpolation of Operators*. Academic Press, Boston.

[8] Blumenthal, O. (1907). Über ganze transzendente Funktionen. *Jahresber. Deutsch. Math.-Verein*. **16**, 97–109.

[9] Bohr, H., and Mollerup, J. (1922). *Loerebog i matematisk analyse*. III. Graenseprocesser, København.

[10] Brudnyi, Yu. A., and Krugljak, N. Ya. (1991). *Interpolation Functors and Interpolation Spaces, Vol*. I. North-Holland, Amsterdam.

[11] Butzer, P. L., and Berens, H. (1967). *Semi-Groups of Operators and Approximations*. Springer, Heidelberg and New York.

[12] Butzer, P. L., and Mawhin, J. *Collected Works of Charles Baron de la Vallée Poussin*. Bibliographisches Institut, Mannheim, in preparation.

[13] Butzer, P. L., and Nessel, R. J. (1971). *Fourier Analysis and Approximation*. Birkhäuser, Basel, and Academic Press, New York.

[14] ——— (1993). Aspects of de La Vallée Poussin's work in approximation and its influence. *Arch. Hist. Exact. Sci*. **46**, 67–95.

[15] Butzer, P. L., and Stens, R. L. (1992). Sampling theory for not necessarily band-limited functions: a historical overview. *SIAM Rev*. **34**, 40–53.

[16] Carleman, T. (1918). Über die Fourier Koeffizienten einer stetigen Funktion. *Acta Math*. **41**, 377–384.

[17] Cartwright, M. L. (1944). Grace Chisholm Young. *J. London Math. Soc*. (*1*) **19**, 185–192.

[18] Edwards, R. E. (1967). *Fourier Series. A Modern Introduction*, 2 Vols. Holt, Rinehart & Winston, New York.

[19] Faber, G. (1906). Über das Anwachsen analytischer Funktionen. *Math. Ann*. **63**, 549–51.

[20] Fehér, F. (1977). Interpolation und Indexbedingungen auf rearrangement-invarianten Funktionenräumen. I. Grundlagen und die K-Methode. *J. Funct. Anal.* **25**, 147–161.

[21] ——— (1980). A note on weak-type interpolation and function spaces. *Bull. London Math. Soc*. **12**, 443–451.

[22] Fischer, E. (1907). Sur la convergence en moyenne. *C. R. Acad. Sci. Paris* **144**, 1022–1024.

[23] Garding, L. (1988). Marcel Riesz in Memoriam. *Marcel Riesz, Collected Works*, pp. 1–9. Springer, Berlin.

[24] Guinness, I. Grattan (1975). Mathematical bibliography for W. H. and G. C. Young. *Historia Math*. **2**, 43–58.

[25] Hadamard, J. (1896). Sur les fonctions entières. *Bull. Soc. Math. France* **24**, 186–187.

[26] Hardy, G. H. (1914). The mean value of the modulus of an analytic function. *Proc. London Math. Soc*. **14**, 269–277.

[27] ——— (1942). William Henry Young. *J. London Math. Soc*. (*1*) **17**, 218–237.

[28] Hardy, G. H., and Littlewood, J. E. (1914). Some problems on diophantine approximation. *Acta Math*. **37**, 155–239.

[29] ——— (1926). Some new properties of Fourier constants. *Math. Ann*. **97**, 159–209.

[30] Hardy, G. H., Littlewood, J. E., and Pólya, G. (1934). *Inequalities*. Cambridge University Press, Cambridge (2nd ed., 1952).

[31] Hilb, E., and Riesz, M. (1923). Neuere Untersuchungen über trigonometrische Reihen. *Enz. Math. Wiss*. II, **C 10**, 1189–1228.

[32] Hilbert, D. (1906). Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, vierte Mitteilung. *Göttinger Nachr*. 157–227 (contains the Hellinger proof).

[33] Hobson, E. W. (1921, 1926). *The Theory of Functions of a Real Variable and the Theory of Fourier Series*. 2 Vols. Cambridge University Press, Cambridge (republ. Dover, 1957).

[34] Jensen, J. L. W. V. (1906). Sur les fonctions convexes et les inégalités entre les valeurs moyennes. *Acta. Math*. **30**, 175–193.

[35] Julia, G. (1930, 1932). *Principes Géométriques d'Analyse*. 2 Parts. Gauthiers-Villars, Paris.

[36] Kaczmarz, S., and Steinhaus, H. (1935). Theorie der Orthogonalreihen. *Monograf. Mat*. Warsaw (republ. Chelsea, New York, 1951).

[37] Krein, S. G., and Petunin, Ju. I., and Semenov, E. M. (1982). Interpolation of linear operators. *Transl. Math. Monogr*., Vol. **54**, American Mathematical Society, Providence, RI.

[38] Kuczma, M. (1985). An Introduction to the Theory of Functional Equations and Inequalities, Cauchy's Equation and Jensen's Inequality, Państwowe Wydawnictwo Naukowe. Waszawa-Kraków-Katowice.

[39] Liapounoff, A. (1901). Nouvelle forme du théorème sur la limite de probabilité. *Mem. l'Acad. St. Petersburg* (*VIII*) **12**, No. 5.

[40] Mitrinović, D. S. (1970). Analytical inequalities. *Grundlehren Math. Wiss*., Vol. 165, Springer, Berlin, Heidelberg, and New York.

[41] Paley, R. E. A. C. (1931). Some theorems on orthogonal functions. *Studia Math*. **3**, 226–238.

[42] Plancherel, M. (1910). Contribution à l'étude de la réprésentation d'une fonction arbitraire par des intégrals défines. *Rend. Palermo* **30**, 289–335.

[43] Riesz, F. (1907). Über orthogonale Funktionensysteme. *Göttinger Nachr*. 116–122; *C. R. Acad. Sci. Paris* **144**, 615–619, 734–736.

[44] ——— (1910). Untersuchungen über Systeme integrierbarer Funktionen. *Math. Ann*. **69**, 449–497.

[45] ——— (1923). Über eine Verallgemeinerung der Parsevalschen Formel. *Math. Zeit*. **18**, 117–124.

[46] Riesz, M. (1927). Sur les maxima des formes bilinéaires et sur les fonctionelles linéaires. *Acta Math*. **49**, 465–497.

[47] Roberts, A. W., and Varberg, D. E. (1973). *Convex Functions*. Academic Press, New York and London.

[48] Stein, E. M., and Weiss, G. (1971). *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, Princeton, NJ.

[49] Thorin, G. O. (1939). An extension of a convexity theorem due to M. Riesz. *Kungl. Fys. Saellskapet Lund Forhaendlinger* **8**, No. 14.

[50] ——— (1948). Convexity theorems generalizing those of M. Riesz and Hadamard, with some applications. Doctoral Dissertation, Lund (= *Comm. Sém. Math. Univ. Lund* **9**, 1–58).

[51] Titchmarsh, E. C. (1924). A contribution to the theory of Fourier transforms. *Proc. London Math. Soc*. (*2*) **23**, 279–289.

[52] ——— (1948). *An Introduction to the Theory of Fourier Integrals*. Oxford University Press, Oxford.

[53] Triebel, H. (1978). *Interpolation Theory, Function Spaces, Differential Operators*. North-Holland, Amsterdam.

[54] Valiron, G. (1932). Fonctions convexes et fonctions entières. *Bull. Soc. Math. France*. **60**, 278–287.

[55] Weyl, H. (1909). Über die Konvergenz von Reihen, die nach Orthogonalfunktionen fortschreiten. *Math. Ann*. **67**, 225–245.

[56] Young, L. C. (1939). On an inequality of Marcel Riesz. *Ann. of Math*. **40**, 567–574.

[57] Young, L. C. (1984). Mathematicians and their Times. (Notas Math., Vol. 76) North-Holland, Amsterdam.

[58] Zygmund, A. (1935). Trigonometrical Series. *Monograf. Mat.*, No. V, Warsaw.

[59] ——— (1959). *Trigonometrical Series*. 2 vols. Cambridge University Press, Cambridge.

Received January 4, 1994

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