The Journal of Fourier Analysis and Applications
Volume 7, Issue 1, 2001

Banach Spaces of Solutions of the Helmholtz Equation in the Plane

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Communicated by Carlos Kenig

ABSTRACT. The purpose of this article is to study the Hilbert space $\mathcal{W}^2$ consisting of all solutions of the Helmholtz equation $\Delta u + u = 0$ in $\mathbb{R}^2$ that are the image under the Fourier transform of L^2 densities in the unit circle. We characterize this space as a close subspace of the Hilbert space $\mathcal{H}^2$ of all functions belonging to $L^2(|x|^{-3}dx)$ jointly with their angular and radial derivatives, in the complement of the unit disk in $\mathbb{R}^2$. We calculate the reproducing kernel of $\mathcal{W}^2$ and study its reproducing properties in the corresponding spaces $\mathcal{H}^p$, for $p > 1$.

1. Introduction and Preliminaries

The purpose of this article is to study Banach spaces of solutions of the Helmholtz equation $\Delta u + u = 0$ in $\mathbb{R}^2$. One can generate solutions to this equation from L^1 densities in the unit circle T through the operator (see [3, p. 3])

$$Wf(s,t) = \int_0^{2\pi} e^{i(s\sin\theta + t\cos\theta)} f(\theta) \frac{d\theta}{2\pi}$$

where $f \in L^1(T)$. The function Wf is nothing else but the Fourier transform of the density f on T, considered as a tempered distribution in $\mathbb{R}^2$. Much work has been done in the study of the Fourier transform of distributions supported in a surface and the related problem of the restriction of the Fourier transform. In this article we describe $W(L^2(T))$ as a function space. In n dimensions, this problem was studied by Guo in [5], where he gave a necessary and a sufficient condition in terms of mixed norms, for a temperate distribution to be the Fourier transform of an L^2 density in the sphere in $\mathbb{R}^n$. In two dimensions, González-Casanova and Wolf proved in [3] that given $f \in L^2(T)$, the restriction $(Wf(\cdot, 0), \frac{\partial Wf}{\partial t}(\cdot, 0))$ determines Wf completely. Then they constructed a reproducing kernel for the space of these restrictions in a certain Hilbert space provided with a nonlocal inner product.

Math Subject Classifications. 46E15.
Keywords and Phrases. Helmholtz equation, reproducing kernel.
Acknowledgements and Notes. Josefina Alvarez: partially supported by Arts and Sciences Research Center 01-3-43903, New Mexico State University. Magali Folch-Gabayet and Salvador Pérez-Esteva: partially supported by PAPIIT-UNAM IN106997 and CONACyT 32408-E.

ISSN 1069-5869

In this article, we characterize the image $W(L^2(T))$ as a closed subspace of a Hilbert space $\mathcal{H}^2$. This space $\mathcal{H}^2$ consists of all functions belonging to $L^2(|x|^{-3}\,dx)$ jointly with their angular and radial derivatives, in the complement of the unit disk in $\mathbb{R}^2$. We also construct and study the reproducing kernel for $W(L^2(T))$. This work is done in Section 2. In Section 3 we consider the p-version, $\mathcal{H}^p$, of the space $\mathcal{H}^2$ and we study the class of solutions of the Helmholtz equation belonging to $\mathcal{H}^p$. These turn out to be Banach spaces and the projection $\mathcal{P}$ found in Section 2 provides a reproducing formula for every $1 < p < \infty$. We also show by means of an example that in general the operator $\mathcal{P}$ cannot be extended as a bounded operator in $\mathcal{H}^p$.

Throughout this article we will use the following notations and results: D will denote the open unit disk in $\mathbb{R}^2$, and $\Omega = \mathbb{R}^2 \setminus \overline{D}$. The conjugate exponent of $p > 1$ will be denoted by p'.

For every integer n, the Bessel function $J_n(r)$ can be defined (see [7, p. 20]) by

$$J_n(r) = \int_0^{2\pi} e^{i(r\sin\theta - n\theta)}\,\frac{d\theta}{2\pi} .$$

The Bessel functions satisfy the following functional relations (see [4])

$$\begin{aligned} 2J_n'(r) &= J_{n-1}(r) - J_{n+1}(r) , & (1.1)\\ -rJ_{n+1}(r) &= rJ_n'(r) - nJ_n(r) , & (1.2)\\ J_{-n}(r) &= (-1)^n J_n(r) , & (1.3)\\ \sum_{n\in\mathbb{Z}} J_n(r)^2 &= 1 . & (1.4) \end{aligned}$$

For $n \geq 0$ we have the estimate (see [7, p. 16])

$$|J_n(r)| \leq \frac{r^n}{n!2^n} e^{r^2/4} . \tag{1.5}$$

Hence, for $n \geq 1$ we obtain

$$\int_0^1 J_n^2(r)\,\frac{dr}{r^2} = o\left(\frac{1}{n!2^n}\right) .$$

Also for $n \geq 1$, we have [4, p. 715]

$$\int_0^\infty J_n^2(r)\,\frac{dr}{r^2} = \frac{1}{\pi}\frac{1}{n^2 - 1/4} .$$

Then, we conclude that for every $n \geq 1$,

$$\int_1^\infty J_n^2(r)\,\frac{dr}{r^2} = \frac{1}{\pi}\frac{1}{n^2 - 1/4} + o\left(\frac{1}{n!2^n}\right) . \tag{1.6}$$

With $r = \sqrt{s^2 + t^2}$ and $\theta = \arctan\frac{t}{s}$, we consider the operators

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{s}{r}\frac{\partial u}{\partial s} + \frac{t}{r}\frac{\partial u}{\partial t}\\ \frac{\partial u}{\partial \theta} &= -t\frac{\partial u}{\partial s} + s\frac{\partial u}{\partial t} , \end{aligned}$$

defined on $\mathcal{D}'(\mathbb{R}^2 \setminus \{0\})$. Given a smooth function u, then $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$ are the radial and angular derivatives of u, respectively.

Definition 1. (a) For $1 \le p < \infty$, we denote with $\mathcal{H}^p$ the space of all $u \in \mathcal{D}'(\Omega)$ such that u, $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta} \in L^1_{\text{loc}}(\Omega)$ and

$$\|u\|_{\mathcal{H}^p} = \left\{ \iint\limits_{\Omega} \left(|u|^p + \left|\frac{\partial u}{\partial r}\right|^p + \left|\frac{\partial u}{\partial \theta}\right|^p \right) \frac{dx}{r^3} \right\}^{1/p} < \infty \, .$$

(b) We will denote with $\mathcal{W}^p$ the space of all functions $u \in \mathcal{H}^p$ satisfying the Helmoltz equation in $\mathbb{R}^2$.

Remark 1. (1) For any bounded open set $U \subset \Omega$, the restriction operator is continuous from $\mathcal{H}^p$ into the standard Sobolev space $W^{p,1}(U)$. Using this fact, it is easy to see that $\mathcal{H}^p$ is a Banach space.

(2) One can represent the dual space of $\mathcal{H}^p$ as in the case of the Sobolev space $W^{p,1}(\Omega)$. In fact, every $L \in (\mathcal{H}^p)^*$ can be represented as

$$Lu = \iint\limits_{\Omega} \left(uF_1 + \frac{\partial u}{\partial r} F_2 + \frac{\partial u}{\partial \theta} F_3 \right) \frac{dx}{|x|^3} \, ,$$

where $F_i \in L^{p'}(\Omega, |x|^{-3}\, dx)$ and $\|L\| \approx \sum_{i=1}^{3} \|F_i\|_{L^{p'}(\Omega,|x|^{-3}dx)}$ and as we said before, p' is the conjugate exponent of p. The proof is the same as in the case of the space $W^{p,1}(\Omega)$.

(3) If u satisfies the Helmoltz equation in $\mathbb{R}^2$ and $u = 0$ in Ω, then u is identically zero (see for example (3.1) below). Hence $\mathcal{W}^p$ may be thought of as a space of functions in $\mathbb{R}^2$, with $\|\cdot\|_{\mathcal{H}^p}$ as a well defined norm.

(4) Since the measure $\frac{dx}{|x|^3}$ is finite in Ω, we have $\mathcal{H}^{p_1} \subset \mathcal{H}^{p_2}$, when $p_1 > p_2$. The same can be said about the space $\mathcal{W}^p$.

We end this section with the following estimates used in Section 3:

Lemma 1.

For each $\gamma > 2$, there exist $C > 0$ and $C_\gamma > 0$ such that

(1) $\iint\limits_{\Omega} \frac{dy}{(1+|x-y|)^{3/2}|y|^2} \le C \frac{\log|x|}{|x|^{3/2}}$, $x \in \Omega$,

(2) $\iint\limits_{\Omega} \frac{dy}{(1+|x-y|)^{3/2}|y|^\gamma} \le \frac{C_\gamma}{|x|^{3/2}}$, $x \in \Omega$.

Proof. The proof follows from elementary estimates on these integrals splitting
$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$, where
$\Omega_1 = \{y \in \Omega : |x - y| \le |x|/2\}$,
$\Omega_2 = \{y \in \Omega : 1 < |y| \le |x|/2\}$,
$\Omega_3 = \{y \in \Omega : |x|/2 \le |y| \le 3|x|/2, |x|/2 \le |x - y|\}$,
$\Omega_4 = \{y \in \Omega : 3|x|/2 \le |y|\}$. □

For $x, y \in \mathbb{R}^2$, $x \cdot y$ will denote the dot product of x and y. Also, if $y = (y_1, y_2)$ we will denote $iy = (-y_2, y_1)$.

Throughout this article c and C will denote generic positive constants that may change in each occurrence.

2. A Reproducing Kernel for the Space $\mathcal{W}^2$

For $n \in \mathbb{Z}$, let $e_n(\varphi) = e^{-in\varphi}$. We also consider the translation operator $f_\theta(\varphi) = f(\varphi - \theta)$ and the rotation operator T_θ, that is the complex multiplication by $e^{i\theta}$. Then

(1) $Wf_\theta = Wf \circ T_\theta$;

(2) Since $(e_n)_\theta = e^{in\theta} e_n$, we have $We_n(T_\theta(s,t)) = e^{in\theta} We_n(s,t)$.
In particular, if (r,θ) are the polar coordinates of $x \in \mathbb{R}^2$; then

(3) $We_n(x) = e^{in\theta} We_n(r,0) = e^{in\theta} J_n(r)$.

Hence, we can represent the functions in $W(L^2(T))$ in polar coordinates (r,θ), as series

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} a_n J_n(r) e^{in\theta}$$

with $\sum |a_n|^2 < \infty$.

According to (1.5), this series converges absolutely and uniformly on compact subsets of $\mathbb{R}^2$. Moreover, using the identity (1.1) repeatedly, one can see that the series can be differentiated term by term, with absolute and uniform convergence in compact subsets of $\mathbb{R}^2$.

Following [7, p. 522], we will call these series Neumann series.

Theorem 1.

The operator W is a topological isomorphism of $L^2(T)$ onto $\mathcal{W}^2$.

Proof. For every $n \in \mathbb{Z}$, let $F_n(re^{i\theta}) = J_n(r)e^{in\theta}$. The family $\{F_n\}_{n\in Z}$ is orthogonal in $\mathcal{W}^2$. Furthermore, using (1.1) and integrating in polar coordinates, we have

$$\begin{aligned} \|F_n\|^2_{\mathcal{H}^2} &= \iint_\Omega \left(|F_n(x)|^2 + \left|\frac{\partial F_n}{\partial \theta}(x)\right|^2 + \left|\frac{\partial F_n}{\partial r}(x)\right|^2 \right) \frac{dx}{r^3} \\ &= 2\pi \int_1^\infty \left(1+n^2\right) J_n^2(r)\,\frac{dr}{r^2} + \frac{\pi}{2}\int_1^\infty \left(J_{n-1}(r) - J_{n+1}(r)\right)^2 \frac{dr}{r^2}\ . \end{aligned}$$

By (1.6), there exists $c > 0$, such that for every $n \in \mathbb{Z}$, $n \neq 0$,

$$\|F_n\|_{\mathcal{H}^2} > c$$

and

$$\|F_n\|_{\mathcal{H}^2} = \sqrt{2} + O\left(1/n^2\right)\ . \tag{2.1}$$

Thus, using the orthogonality of $\{F_n\}_{n\in\mathbb{Z}}$, we can conclude that

$$c\,\|f\|_2 \le \|Wf\|_{\mathcal{H}^2} \le C\,\|f\|_2\ .$$

It remains to prove that W is onto.

Given $u \in \mathcal{W}^2$, the ellipticity of the Helmholtz operator implies, in particular, that u is smooth in $\mathbb{R}^2$. If we consider the Fourier series representation of $u(r,\theta)$,

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} A_n(r) e^{in\theta}\ ,$$

the smoothness of u implies that the coefficients

$$A_n(r) = \int_0^{2\pi} u(r,\theta) e^{-in\theta}\,\frac{d\theta}{2\pi} \tag{2.2}$$

satisfy the following:

For each $k, l \geq 0$, the series $\sum_{n\in\mathbb{Z}} n^k \left|\frac{d^l A_n}{dr^l}(r)\right|$ converges uniformly on compact subsets of $(0, \infty)$. Thus, the Helmholtz operator written in polar coordinates,

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + 1$$

can be applied term by term.

We obtain

$$\sum_{n\in\mathbb{Z}} \left(r^2 A_n''(r) + r A_n'(r) + \left(r^2 - n^2\right) A_n(r)\right) e^{in\theta} = 0\,.$$

The orthogonality of the functions $e_n(\theta)$, implies that for each $n \in \mathbb{Z}$, the function $A_n(r)$ satisfies the Bessel equation of order n,

$$r^2 A_n''(r) + r A_n'(r) + \left(r^2 - n^2\right) A_n(r) = 0\,.$$

Then, A_n can be written as a linear combination,

$$A_n(r) = a_n J_n(r) + b_n N_n(r)\,,$$

where $N_n(r)$ is the Neumann function of order n (see [4, p. 960]). Since N_n has a singularity at $r = 0$ and $A_n(r)$ is bounded, it follows that $b_n = 0$ for all $n \in \mathbb{Z}$. We claim that $\sum_{n\in\mathbb{Z}} |a_n|^2 \leq C\,\|u\|^2_{\mathcal{H}^2}$.

In fact, by (1.6),

$$\begin{aligned}
\sum_{0<|n|<N} |a_n|^2 &\leq C \sum_{0<|n|<N} \int_1^\infty |a_n n|^2\, J_n^2(r)\frac{dr}{r^2} \\
&= C \sum_{|n|<N} \lim_{R\to\infty} \int_1^{\mathrm{R}} |a_n n|^2\, J_n^2(r)\frac{dr}{r^2} \\
&\leq C \lim_{R\to\infty} \int_1^{\mathrm{R}} \sum_{n\in\mathbb{Z}} |a_n n|^2\, J_n^2(r)\frac{dr}{r^2} \\
&= C \iint\limits_{\Omega} \left|\frac{\partial u}{\partial \theta}(r,\theta)\right|^2 d\theta\, \frac{dr}{r^2} \\
&\leq C\,\|u\|^2_{\mathcal{H}^2} < \infty\,.
\end{aligned}$$

Similarly, using the Cauchy-Schwarz inequality in (2.2) with $n = 0$ and integrating both sides of the resulting inequality on $(1, \infty)$ with respect to $\frac{dr}{r^2}$, we obtain

$$|a_0|^2 \int_1^\infty J_0^2(r)\frac{dr}{r^2} \leq \iint\limits_{\Omega} |u(r,\theta)|^2\, \frac{d\theta}{2\pi}\,\frac{dr}{r^2}\,.$$

Since $0 < \int_1^\infty \frac{J_0^2(r)}{r^2} dr < \infty$, we have

$$|a_0|^2 \leq \left(2\pi \int_1^\infty \frac{J_0^2(r)}{r^2} dr\right)^{-1} \iint\limits_{\Omega} |u(r,\theta))|^2\, d\theta \frac{dr}{r^2}\,.$$

Finally,

$$\sum_{n\in\mathbb{Z}} |a_n|^2 \leq C\,\|u\|^2_{\mathcal{H}^2}\,.$$

We conclude that $f = \sum_{n\in\mathbb{Z}} a_n e_n$ belongs to $L^2(T)$ and $u = Wf$. $\square$

We will now construct the reproducing kernel for $\mathcal{W}^2$, as a subspace of the Hilbert space $\mathcal{H}^2$.

In the proof of Theorem 1, we observed that if we set $\beta_n = \|F_n\|_{\mathcal{H}^2}$, the family $\left\{\beta_n^{-1} F_n\right\}$ is an orthonormal basis for $\mathcal{W}^2$.

Let

$$\begin{aligned} \mathcal{K}(x, y) &= \sum_{n\in\mathbb{Z}} \frac{F_n(x)\overline{F_n(y)}}{\beta_n^2} \\ &= \sum_{n\in\mathbb{Z}} \frac{J_n(r)J_n(s)}{\beta_n^2} e^{in(\theta-\varphi)} . \end{aligned} \tag{2.3}$$

In (2.3) and in the rest of this article, (r, θ) and (s, φ) will denote the polar coordinates of points $x, y \in \mathbb{R}^2$, respectively.

By (1.5), the series (2.3) that defines $\mathcal{K}(x, y)$, converges absolutely and uniformly on compacts subsets of $\mathbb{R}^2 \times \mathbb{R}^2$.

Notice that by (1.3), we have

$$\beta_n = \beta_{-n} ,$$

and $\mathcal{K}(x, y)$ is real and symmetric.

Also, from (1.4), we have that $\mathcal{K}(x, \cdot) \in \mathcal{W}^2$ for each $x \in \mathbb{R}^2$, with the series

$$\mathcal{K}(x, \cdot) = \sum_{n\in\mathbb{Z}} \frac{F_n(x)}{\beta_n^2} \overline{F_n}$$

converging in $\mathcal{W}^2$.

The orthogonal projection of $\mathcal{H}^2$ onto $\mathcal{W}^2$ is given by

$$\mathcal{P}u = \sum_{n\in\mathbb{Z}} \left\langle u, \beta_n^{-1} F_n \right\rangle_{\mathcal{H}^2} \beta_n^{-1} F_n$$

with convergence in $\mathcal{W}^2$ and also pointwise.

For $x \in \mathbb{R}^2$ fixed we have,

$$\begin{aligned} \mathcal{P}u(x) &= \sum_{n\in\mathbb{Z}} \left\langle u, \beta_n^{-2}\overline{F_n(x)} F_n \right\rangle_{\mathcal{H}^2} = \left\langle u, \overline{\mathcal{K}(x, \cdot)} \right\rangle_{\mathcal{H}^2} \\ &= \iint_{\Omega} \left[\mathcal{K}(x, y)u(y) + \frac{\partial}{\partial s}\mathcal{K}(x, y)\frac{\partial}{\partial s}u(y) \right. \\ &\quad \left. + \frac{\partial}{\partial\varphi}\mathcal{K}(x, y)\frac{\partial}{\partial\varphi}u(y) \right] \frac{dy}{s^3} . \end{aligned} \tag{2.4}$$

We would like to write the kernel $\mathcal{K}(x, y)$ in a closed form. We will be able to do it up to composition with a topological isomorphism of $\mathcal{W}^2$. More precisely we can prove the following.

Lemma 2.

Let $\mathcal{M}$ be the Fourier multiplier operator defined by the sequence $\left(\beta_n^2\right)$. That is,

$$\mathcal{M}\left(\sum \alpha_n e^{in\theta}\right) = \sum \beta_n^2 \alpha_n e^{in\theta} ,$$

for any trigonometric polynomial $\sum \alpha_n e^{in\theta}$. Then, $\mathcal{M}$ is a topological isomorphism of $\mathcal{W}^2$ onto itself. Moreover, the kernel of the composition $\mathcal{M} \circ \mathcal{P}$ is the function $J_0(|x - y|)$.

Proof. Notice that for some constants $c, C > 0$, we have

$$c \le \beta_n^2 \le C\ .$$

Then it is clear that the action of $\mathcal{M}$ in $\mathcal{W}^2$ defined by

$$\mathcal{M}\left(J_n(r)e^{in\theta}\right) = \beta_n^2 J_n(r)e^{in\theta}$$

for every $n \in \mathbb{Z}$, is a topological isomorphism. In particular, we have that (see [4, p. 992])

$$\mathcal{M}\mathcal{K}(x, y) = \sum_{n\in\mathbb{Z}} J_n(r)J_n(s)e^{in(\theta-\varphi)} = J_0\left(|x-y|\right)$$

where $\mathcal{M}$ may be thought of as acting on θ or on φ, since $\beta_n = \beta_{-n}$ for all $n \in \mathbb{Z}$.

The kernel $J_0(|x-y|)$ has the same properties as $\mathcal{K}$. Hence it defines a continuous operator $\widetilde{\mathcal{P}}$ from $\mathcal{H}^2$ into itself, given by

$$\widetilde{\mathcal{P}}u(x) = \iint_\Omega \left[J_0(|x-y|)u(y) + \frac{\partial}{\partial s}J_0(|x-y|)\frac{\partial}{\partial s}u(y) + \frac{\partial}{\partial\varphi}J_0(|x-y|)\frac{\partial}{\partial\varphi}u(y)\right]\frac{dy}{s^3}\ .$$

It is easy to verify that the operators $\widetilde{\mathcal{P}}$ and $\mathcal{M}\circ\mathcal{P}$ coincide on the space $\mathcal{H}_0$ defined as the linear span of the set $\left\{A(r)e^{in\theta} : A \in C_c^\infty(0,\infty)\right\}_{n\in\mathbb{Z}}$. Since $\mathcal{H}_0$ is dense in $\mathcal{H}^2$, we conclude that $\widetilde{\mathcal{P}} = \mathcal{M}\circ\mathcal{P}$. □

3. The Space $\mathcal{W}^p$

Theorem 2.
Let $1 < p < \infty$. Then,

(1) $\mathcal{W}^p$ is a Banach space.
(2) The linear span of $\left\{J_n(r)e^{in\theta}\right\}_{n\in\mathbb{Z}}$ is dense in $\mathcal{W}^p$.

Proof. (1) Let $\Phi(x, y)$ be the fundamental solution of the Helmholtz equation in two dimensions [1, p. 341] and [2, p. 106]), given as

$$\Phi(x, y) = \frac{i}{4}\left(J_0\left(|x-y|\right) + iN_0\left(|x-y|\right)\right)\ .$$

Let D_R be the open disk centered at the origin with radius $R > 1$. If $v \in \mathcal{W}^p$ and $x \in D_R$, then for $l \in [2R, 3R]$ we can write (see [1] and [2])

$$v(x) = \int_{|y|=l}\left(\frac{\partial v}{\partial s}(y)\Phi(x, y) - \frac{\partial\Phi}{\partial s}(x, y)v(y)\right)d\sigma(y)\ , \tag{3.1}$$

where $d\sigma$ is the Lebesgue measure on $|y| = l$. Integrating both sides of the equation above with respect to l on $[2R, 3R]$, we obtain

$$v(x) = \frac{1}{R}\iint_{2R\le|y|\le3R}\left(\frac{\partial v}{\partial s}(y)\Phi(x, y) - \frac{\partial\Phi}{\partial s}(x, y)v(y)\right)dy\ . \tag{3.2}$$

Using the reproducing formula (3.2), we can now show that $\mathcal{W}^p$ is complete.

Let $\{u_n\}_{n\geq 0}$ be a sequence in $\mathcal{W}^p$ converging to $u \in \mathcal{H}^p$. The function $\Phi(x, y)$ is smooth in $\overline{D_R} \times \left(\overline{D_{3R}} \setminus D_{2R}\right)$. Moreover, for each $y \in \overline{D_{3R}} \setminus D_{2R}$, the functions $\Phi(\cdot, y)$ and $\frac{\partial \Phi}{\partial s}(\cdot, y)$ satisfy the Helmholtz equation. Thus, we can conclude that the limit function u satisfies (3.2) and it is a solution of the Helmholtz equation in D_R, for every $R > 1$. It follows that $u \in \mathcal{W}^p$, and we have proved that $\mathcal{W}^p$ is closed in $\mathcal{H}^p$.

(2) Let

$$K_N(\theta) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{in\theta}$$

be the Féjer kernel.

Given $u \in \mathcal{W}^p$, the proof of the surjectivity in Theorem 1 shows that we can write

$$u(r, \theta) = \sum_{n\in\mathbb{Z}} a_n J_n(r) e^{in\theta}$$

for some $a_n \in \mathbb{C}$, where the convergence is absolute and uniform in compact subsets of $\mathbb{R}^2$.

Consider now

$$\begin{aligned} u_N(r, \theta) &= K_N * (u(r, \cdot))(\theta) \\ &= 2\pi \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) a_n J_n(r) e^{in\theta} . \end{aligned}$$

Then we clearly have that $u_N \in \mathcal{W}^p$. The fast decay of $a_n J_n(r)$, uniform with respect to r in compact subsets of $(0, \infty)$, implies that u_N converges to u uniformly in compact subsets of $\mathbb{R}^2$ as well. Let

$$\begin{aligned} \Psi_N^p(r) &= \int_0^{2\pi} \left[|u_N(r, \theta) - u(r, \theta)|^p + \left| \frac{\partial\,(u_N - u)}{\partial r}\,(r, \theta) \right|^p \right. \\ &\quad \left. + \left| \frac{\partial\,(u_N - u)}{\partial \theta}\,(r, \theta) \right|^p \right] d\theta . \end{aligned}$$

Inserting $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ inside the convolution that defines $u_N(r, \theta)$, we obtain

$$\begin{aligned} \Psi_N^p(r) &\leq C \int_0^{2\pi} \left[|u(r, \theta)|^p + \left| \frac{\partial}{\partial r} u(r, \theta) \right|^p \right. \\ &\quad \left. + \left| \frac{\partial}{\partial \theta} u(r, \theta) \right|^p \right] d\theta . \end{aligned}$$

By the dominated convergence theorem it follows that

$$\lim_{N\to\infty} \int_1^{\infty} \Psi_N^p(r) \frac{dr}{r^2} = 0 .$$

That is, we have shown that u_N converges to u in $\mathcal{W}^p$. □

In general, the operator $\mathcal{P}$ cannot be extended to a continuous operator from $\mathcal{H}^p$ into $\mathcal{W}^p$. In fact, Proposition 2 below shows that $\mathcal{P}$ cannot be extended to a continuous operator from $\mathcal{H}^p$ into itself, for $p < 4/3$. However, the next two results will show that $\mathcal{P}$ is continuous from $\mathcal{H}^p$ into $L^p(\Omega, |x|^{-3}\,dx)$ and that it has a reproducing property, namely, given $u \in \mathcal{H}^p$ the function $u \in \mathcal{W}^p$ if and only if $\mathcal{P}u = u$.

Proposition 1.
The operator $\mathcal{P}$ has a continuous extension from $\mathcal{H}^p$ into $L^p(\Omega, |x|^{-3}\,dx)$.

Proof. As observed in (2.1), for $n \neq 0$

$$\beta_n^2 = 2 + O\left(1/n^2\right) ,$$

then

$$\beta_n^{-2} = 1/2 + O\left(1/n^2\right) .$$

Hence, the Fourier multiplier operators $\mathcal{M}$ and $\mathcal{M}^{-1}$ are continuous in $L^p(T)$ for every $1 < p < \infty$ (see for example [6, Prop. 4.1, Ch. V]). They also may be thought of as acting continuously on $L^p(\Omega, |x|^{-3}\,dx)$, starting with their natural action on $\mathcal{H}_0$, which is also dense in $L^p(\Omega, |x|^{-3}\,dx)$. In fact, by Fubini's theorem we can find constants $c, C > 0$ such that

$$\begin{aligned} c\left\|\sum_n A_n(r)e^{in\theta}\right\|_{L^p(\Omega,|x|^{-3}dx)} &\leq \left\|\sum_n \beta_n^2 A_n(r)e^{in\theta}\right\|_{L^p(\Omega,|x|^{-3}dx)} \\ &\leq C\left\|\sum_n A_n(r)e^{in\theta}\right\|_{L^p(\Omega,|x|^{-3}dx)} , \end{aligned}$$

for every finite sum $\sum_n A_n(r)e^{in\theta}$ in $\mathcal{H}_0$.

We observe that $\mathcal{H}_0$ is dense in $\mathcal{H}^p$ for $1 < p < \infty$. Then the claimed continuity of $\mathcal{P}$ will be proved once we prove this continuity for $\widetilde{\mathcal{P}} = \mathcal{M} \circ \mathcal{P}$, originally defined on $\mathcal{H}_0$. To this end we shall prove the continuity on $L^p(\Omega, |x|^{-3}\,dx)$ of the operators M_1, M_2 and M_3 with kernels $J_0(|x-y|)$, $\frac{\partial}{\partial s}J_0(|x-y|)$, and $\frac{\partial}{\partial \varphi}J_0(|x-y|)$, respectively. We have the following estimates

(a) $|J_0(|x-y|)| \leq \frac{C}{(1+|x-y|)^{1/2}}$;

(b) $\left|\frac{\partial}{\partial s}J_0(|x-y|)\right| = \left|J_1(|x-y|)\frac{(x-y)\cdot y}{|x-y||y|}\right| \leq \frac{C}{(1+|x-y|)^{1/2}}$; and

(c) $\left|\frac{\partial}{\partial \varphi}J_0(|x-y|)\right| = \left|J_1(|x-y|)\frac{(x-y)\cdot iy}{|x-y|}\right| = \left|J_1(|x-y|)\frac{x\cdot iy}{|x-y|}\right| \leq \frac{C|x||y|}{(1+|x-y|)^{3/2}}$.

To prove the above inequalities, one observes that

$$|J_n(r)| \leq C_n r^{-1/2}$$

for $r > 0$, and that the function $J_n(r)$ has a zero of order n at $r = 0$.

Proceeding as in the proof of Lemma 1, we see that the integral

$$\iint_\Omega \frac{1}{(1+|x-y|)^{1/2}}\frac{dy}{|y|^3}$$

is a bounded function of $x \in \Omega$.

Since the kernel $\frac{1}{(1+|x-y|)^{1/2}}$ is symmetric, we can use Schur's lemma to conclude that it defines a bounded operator on $L^p(\Omega, |x|^{-3}\,dx)$ for $1 \leq p \leq \infty$. As a consequence, M_1 and M_2 have the same continuity property. Finally, we consider the operator M_3. According to the estimate (c) above, it suffices to prove that the operator T_3 with kernel $\frac{|x||y|}{(1+|x-y|)^{3/2}}$ is a bounded operator from $L^p(\Omega, |x|^{-3}\,dx)$ into itself for $1 < p < \infty$.

We have the following estimates, for every $x \in \Omega$:

$$\begin{aligned} |T_3 f(x)|^p &= \left| \iint_\Omega \frac{|x|\,|y|}{(1+|x-y|)^{3/2}} f(y) \frac{dy}{|y|^3} \right|^p \\ &\le \left(\iint_\Omega \frac{|x|\,|y|}{(1+|x-y|)^{3/2}} \, |f(y)| \, \frac{dy}{|y|^3} \right)^p \\ &= |x|^p \left(\iint_\Omega \frac{|y|}{(1+|x-y|)^{3/2}} \, |f(y)| \, \frac{dy}{|y|^3} \right)^p . \end{aligned}$$

Applying Hölder's inequality to this last integral, we obtain

$$\le |x|^p \left(\iint_\Omega \frac{|y|}{(1+|x-y|)^{3/2}} \frac{dy}{|y|^3} \right)^{p/p'} \iint_\Omega \frac{|y|\,|f(y)|^p}{(1+|x-y|)^{3/2}} \frac{dy}{|y|^3} .$$

Using Lemma 1, we can estimate the above by

$$\le C\, |x|^p \left(\frac{\log |x|}{|x|^{3/2}} \right)^{p/p'} \iint_\Omega \frac{|f(y)|^p}{(1+|x-y|)^{3/2}} \frac{dy}{|y|^2} .$$

Then fixing $0 < \beta < 1/2$ we obtain from $\frac{\log|x|}{|x|^{3/2}} \le \frac{C}{|x|^{1+\beta}}$,

$$|T_3 f(x)|^p \le \frac{C}{|x|^{\beta(p-1)-1}} \iint_\Omega \frac{|f(y)|^p}{(1+|x-y|)^{3/2}} \frac{dy}{|y|^2} .$$

By Fubini's Theorem and Lemma 1, we finally have

$$\begin{aligned} \iint_\Omega |T_3 f(x)|^p \frac{dx}{|x|^3} &\le C \iint_\Omega |f(y)|^p \frac{dy}{|y|^2} \iint_\Omega \frac{dx}{(1+|x-y|)^{3/2}\, |x|^{\beta(p-1)+2}} \\ &\le C \iint_\Omega |f(y)|^p \frac{dy}{|y|^{2+3/2}} \\ &\le C \iint_\Omega |f(y)|^p \frac{dy}{|y|^3} . \end{aligned}$$

This proves the continuity of M_3 for any $p > 1$. □

Now we are ready to prove the reproducing property mentioned previously.

Theorem 3.

Given $u \in \mathcal{H}^p$, the function $u \in \mathcal{W}^p$ if and only if $\mathcal{P}u = u$.

Proof. Given $u \in \mathcal{H}^p$, we claim that the function $\mathcal{P}u(x)$ is well defined for every $x \in \mathbb{R}^2$, and it satisfies the Helmholtz equation on the plane. In fact, as shown by (2.4) , the operator $\mathcal{P}$ consists of three terms. Let us consider first

$$\iint_\Omega \mathcal{K}(x, y) u(y) \frac{dy}{s^3} = \iint_\Omega \left(\sum_{n \in \mathbb{Z}} \frac{J_n(r) J_n(s)}{\beta_n^2} e^{in(\theta - \varphi)} \right) u(y) \frac{dy}{s^3} . \tag{3.3}$$

Using the uniform estimate [5, Lemma 3.4]

$$|J_n(s)| \le C s^{-1/3} \ ,$$

valid for $s \ge 1$, we obtain that the norms $\|F_n\|_{L^{p'}(\Omega,|x|^{-3}dx)}$ are uniformly bounded. Then, the estimate (1.5) implies that for each $x \in \mathbb{R}^2$, the integrand in (3.3) belongs to $L^1(\Omega, |y|^{-3}\,dy)$. Moreover, its norm is bounded uniformly with respect to x in each compact subset of $\mathbb{R}^2$.

This argument can be applied to the other two terms in the representation of $\mathcal{P}u$ given by (2.4). Moreover, it is legitimate to take derivatives of $\mathcal{P}u$ of any order under the integral sign. Since $\mathcal{K}(\cdot, y)$ satisfies the Helmholtz equation in $\mathbb{R}^2$ for each $y \in \Omega$, so does $\mathcal{P}u$. Then, we can conclude that $u \in \mathcal{W}^p$ if we assume that $\mathcal{P}u = u$.

To prove the converse, we recall that $\mathcal{P}$ is a continuous projection from $\mathcal{H}^2$ onto $\mathcal{W}^2$. This implies that $\mathcal{P}u = u$ for each u in the linear span of $\left\{J_n(r)e^{in\theta}\right\}_{n\in\mathbb{Z}}$. According to Theorem 2, this linear span is dense in $\mathcal{W}^p$. Thus, Proposition 1 implies that $\mathcal{P}u = u$ for any $u \in \mathcal{W}^p$. □

Proposition 2.

The operator $\mathcal{P}$ cannot be extended to a bounded operator in $\mathcal{H}^p$ for any $1 < p < 4/3$.

Proof. As in the proof of Proposition 1, we can show that $\mathcal{M}$ and $\mathcal{M}^{-1}$ are continuous operators in $\mathcal{H}^p$, for $1 < p < \infty$, assuming that they are initially defined on the linear span of $\left\{J_n(r)e^{in\theta}\right\}_{n\in\mathbb{Z}}$. Thus, the operators $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ will have the same continuity properties on $\mathcal{H}^p$.

We observe that $C_c^\infty(\mathbb{R}^2)$ is dense in $\mathcal{H}^p$. Let $u \in C_c^\infty(\mathbb{R}^2)$. To estimate $\|\widetilde{\mathcal{P}}u\|_{\mathcal{H}^p}$ we have to calculate the norm in $L^p(\Omega, |x|^{-3}\,dx)$ of $\widetilde{\mathcal{P}}u$, $\frac{\partial}{\partial r}\widetilde{\mathcal{P}}u$ and $\frac{\partial}{\partial\theta}\widetilde{\mathcal{P}}u$.

According to the proof of Proposition 1, the kernels $J_0(|x-y|)$, $\frac{\partial}{\partial s}J_0(|x-y|)$ and $\frac{\partial}{\partial\varphi}J_0(|x-y|)$ define continuous operators on $L^p(\Omega, |x|^{-3}\,dx)$.

It follows that

$$\|\widetilde{\mathcal{P}}u\|_{L^p(\Omega,|x|^{-3}dx)} \le C\,\|u\|_{\mathcal{H}^p} \ .$$

The function $\frac{\partial}{\partial r}\widetilde{\mathcal{P}}u$ involves the kernels $\frac{\partial}{\partial r}J_0(|x-y|)$, $\frac{\partial^2}{\partial r\partial s}J_0(|x-y|)$ and $\frac{\partial^2}{\partial r\partial\varphi}J_0(|x-y|)$. These kernels are all bounded in modulus by a constant multiple of $\frac{|x||y|}{(1+|x-y|)^{3/2}}$, which is the kernel of the operator T_3 that we used in the proof of Proposition 1. Thus we have also

$$\left\|\frac{\partial}{\partial r}\widetilde{\mathcal{P}}u\right\|_{L^p(\Omega,|x|^{-3}dx)} \le C\,\|u\|_{\mathcal{H}^p} \ .$$

Finally, we need to consider the function $\frac{\partial}{\partial\theta}\widetilde{\mathcal{P}}u$. This function is again the sum of three terms

$$\begin{aligned}\frac{\partial}{\partial\theta}\widetilde{\mathcal{P}}u \quad &= \quad \iint\limits_{\Omega}\left[\frac{\partial}{\partial\theta}J_0(|x-y|)u(y) + \frac{\partial^2}{\partial\theta\partial s}J_0(|x-y|)\frac{\partial}{\partial s}u(y)\right. \\ &\qquad \left. + \frac{\partial}{\partial\theta\partial\varphi}J_0(|x-y|)\frac{\partial}{\partial\varphi}u(y)\right]\frac{dy}{s^3} \ . \end{aligned} \tag{3.4}$$

Repeating the argument above, we can prove that the kernels $\frac{\partial}{\partial\theta}J_0(|x-y|)$ and $\frac{\partial^2}{\partial\theta\partial s}J_0(|x-y|)$ define continuous operators on $L^p(\Omega, |x|^{-3}\,dx)$.

The last term in (3.4) is an integral operator evaluated in $\frac{\partial}{\partial\varphi}u$, with kernel

$$\begin{aligned}\frac{\partial^2}{\partial\theta\partial\varphi}J_0(|x-y|) \quad &= \quad \frac{\partial}{\partial\theta}\left[\frac{J_1(|x-y|)}{|x-y|}x\cdot iy\right] \\ &= \quad \frac{J_1(|x-y|)}{|x-y|}ix\cdot iy + \frac{J_2(|x-y|)}{|x-y|^2}\,(y\cdot ix)\,(x\cdot iy) \ .\end{aligned}$$

Once again, the term $\frac{J_1(|x-y|)}{|x-y|} ix \cdot iy$ can be handled by the above argument.

Recalling that $J_2(t)$ has a zero of order 2 at $t = 0$, the asymptotic expansion [4, p. 972]

$$J_2(t) = \sqrt{2/\pi t}\cos(t - 3\pi/4) + O\left(t^{-3/2}\right) ,$$

and the relations $(x - y) \cdot ix = x \cdot iy = (x - y) \cdot iy$, we can write

$$\frac{J_2(|x-y|)}{|x-y|^2}(y \cdot ix)(x \cdot iy) = L(x, y) + \sqrt{2/\pi}\frac{\cos(|x-y| - 3\pi/4)}{(1 + |x-y|)^{5/2}}(x \cdot iy)^2 ,$$

with

$$|L(x, y)| \le \frac{C\,|x|\,|y|}{(1 + |x-y|)^{3/2}} .$$

Hence, we obtain a representation of the integral operator

$$\iint_{\Omega} \frac{\partial}{\partial\theta\partial\varphi} J_0(|x-y|) f(y)\frac{dy}{s^3} = Rf(x) + Tf(x) ,$$

where R is a bounded operator in $L^p(\Omega, |x|^{-3}\,dx)$ and

$$Tf(x) = \iint_{\Omega} \frac{\cos(|x-y| - 3\pi/4)}{(1 + |x-y|)^{5/2}}(x \cdot iy)^2 f(y)\frac{dy}{|y|^3} .$$

The discontinuity of $\widetilde{\mathcal{P}}$ will come from the term $T(\frac{\partial u}{\partial\varphi})$ as we shall see.

To this end, we write $(x \cdot iy)^2 = |x|^2 |y|^2 \sin^2(\alpha(x, y))$, where $\alpha(x, y)$ is any choice of the angle between x and y.

For every positive integer m, let D_m be the disk of center $(0, 2\pi m)$ and fixed radius $\varepsilon \le \pi/8$. For $k \ge m$ let A_k be the region in the first quadrant between the circles centered at $(0, 2\pi m)$ with radii $2\pi k + \frac{3\pi}{4}$ and $2\pi k + \frac{7\pi}{8}$ respectively, below the diagonal and above the horizontal line that passes through $(0, 2\pi m)$. Let ψ be a nonnegative smooth function supported in the disk centered at the origin with radius ε such that $1/2 \le \psi(x) \le 1$ for $|x| \le \varepsilon/2$. Define

$$f_m(x) = \psi(x + (0, 2\pi m)) .$$

For every $y \in D_m$ and $x \in A_k$ we have that $ck \le |x| \le Ck$, $cm \le |y| \le Cm$ and $ck \le |x-y| \le Ck$ for some positive constants c and C. We also have that $\sin^2(\alpha(x, y)) \ge c$ and $\cos(|x-y| - \frac{3\pi}{4}) \ge c$, for c small enough.
Hence, for $x \in A_k$

$$\begin{aligned} |Tf_m(x)| &\ge c|x|^2 \iint_{D_m} \frac{f_m(y)}{|x-y|^{5/2}}\frac{dy}{|y|} \\ &\ge c|x|^2 k^{-5/2} m^{-1} . \end{aligned}$$

Then

$$\begin{aligned} \iint_{A_k} \frac{|Tf_m(x)|^p}{|x|^3}dx &\ge ck^{-5p/2}m^{-p} \iint_{A_k} |x|^{2p-3}dx \\ &\ge ck^{2p-3}k^{-5p/2}m^{-p}\,|A_k| \ge ck^{-p/2-2}m^{-p} . \end{aligned}$$

Adding on k we obtain the estimate

$$\|Tf_m\|_{L^p(\Omega, |x|^{-3}dx)} \ge cm^{-3/2-1/p} . \tag{3.5}$$

Now consider the Fourier series expansion of f_m in polar coordinates

$$f_m = \sum_{n\in\mathbb{Z}} A_n^{(m)}(r)e^{in\theta} \ .$$

It is easy to see that $|f_m|$ and $|\nabla f_m|$ are uniformly bounded in $\mathbb{R}^2$. Then, since the distance of D_m to the origin is of order m, it is not hard to see that for every $r > 1$,

$$\left|A_0^{(m)}(r)\right| + \left|A_0^{(m)\prime}(r)\right| \le \int_0^{2\pi} \left|f_m\left(re^{i\theta}\right)\right| \frac{d\theta}{2\pi} + \int_0^{2\pi} \left|\nabla f_m\left(re^{i\theta}\right)\right| \frac{d\theta}{2\pi} \le \frac{C}{m} \ .$$

Hence, the following estimates hold

$$\begin{aligned} \|f_m\|^p_{L^p(\Omega,|x|^{-3}dx)} + \|\nabla f_m\|^p_{L^p(\Omega,|x|^{-3}dx)} &= O\left(m^{-3}\right) \ , \\ \left\|A_0^{(m)}\right\|^p_{L^p(\Omega,|x|^{-3}dx)} + \left\|A_0^{(m)\prime}\right\|^p_{L^p(\Omega,|x|^{-3}dx)} &= O\left(m^{-3}\right) \ . \end{aligned} \tag{3.6}$$

Thus

$$\left\|\sum_{n\neq 0} A_n^{(m)}(r)e^{in\theta}\right\|^p_{L^p(\Omega,|x|^{-3}dx)} + \left\|\sum_{n\neq 0} A_n^{(m)\prime}(r)e^{in\theta}\right\|^p_{L^p(\Omega,|x|^{-3}dx)} = Cm^{-3} \ . \tag{3.7}$$

Finally, we define

$$u_m = m^{3/p}\sum_{n\neq 0}\frac{A_n^{(m)}(r)}{in}e^{in\theta} \ .$$

From (3.6), (3.7), and the L^p continuity of the Fourier multiplier (see [6, Prop. 4.1, Ch. V])

$$\lambda_n = \begin{cases} 1/n : n \neq 0 \\ 0 : n = 0 \end{cases}$$

it follows that

$$\|u_m\|_{\mathcal{H}^p} \le C \ . \tag{3.8}$$

Since for every $x \in \Omega$

$$\iint_\Omega \frac{\partial}{\partial\varphi} J_0\left(|x-y|\right) A_0^{(m)}(s)\frac{dy}{s^3} = 0 \ ,$$

then (3.5) and (3.8) imply that

$$\begin{aligned} &\left\|\frac{\partial}{\partial\theta}\iint_\Omega \frac{\partial}{\partial\varphi}J_0(|\cdot-y|)\frac{\partial}{\partial\varphi}u_m(y)\frac{dy}{s^3}\right\|_{L^p(\Omega,|x|^{-3}dx)} \\ &= \left\|\frac{\partial}{\partial\theta}\iint_\Omega \frac{\partial}{\partial\varphi}J_0(|\cdot-y|)m^{3/p}f_m(y)\frac{dy}{s^3}\right\|_{L^p(\Omega,|x|^{-3}dx)} \\ &= \left\|R\left(m^{3/p}f_m\right) + T\left(m^{3/p}f_m\right)\right\|_{L^p(\Omega,|x|^{-3}dx)} \\ &\ge m^{3/p}\|Tf_m\|_{L^p(\Omega,|x|^{-3}dx)} - \left\|R\left(m^{3/p}f_m\right)\right\|_{L^p(\Omega,|x|^{-3}dx)} \\ &\ge Cm^{3/p}\|Tf_m\|_{L^p(\Omega,|x|^{-3}dx)} \ge Cm^{2/p-3/2} \ , \end{aligned}$$

valid for m large enough, where we have used that R is a continuous operator so that $\left\| R\left(m^{3/p} f_m\right)\right\|_{L^p(\Omega, |x|^{-3}dx)}$ is a bounded sequence, and $2/p - 3/2 > 0$. Since $\{u_m\}_{m \geq 0}$ is a bounded sequence in $\mathcal{H}^p$, this completes the proof of the proposition. $\square$

Remark 2. Notice that by a standard duality argument, we can conclude that the operator T in the proof of Proposition 2 cannot be extended to a bounded operator in $L^p(\Omega, |x|^{-3}\,dx)$ for $|1/p - 1/2| < 1/4$. However, we were not able to conclude anything about the continuity of $\mathcal{P}$ on $\mathcal{H}^p$ for $p > 4$, or $p \in [4/3, 4] \setminus \{2\}$.

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Received August 31, 1999

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