

A Theory of Generalized Multiresolution Structure and Pseudoframes of Translates

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ABSTRACT. The notion of a Generalized Multiresolution Structure (GMS) in $L^2(\mathbf{R})$ is introduced. Basically, it consists of an increasing sequence of closed subspaces of $L^2(\mathbf{R})$, with a pseudoframe of translates at each level. Using their shift-invariance a frame-like decomposition is derived based on such a GMS. As a major new contribution the construction of affine frames for $L^2(\mathbf{R})$ based on an GMS is presented. A fast algorithm for the GMS-based affine frame decomposition and reconstruction using filter banks is presented as well.

1. Introduction

Multiresolution analysis (MRA) and wavelet theory have found many applications in vision, image processing, and multiscale signal representation, e.g., [18]. They have their roots in multirate systems in digital signal processing, e.g., [21], and multiscale representation in machine vision, e.g., [19, 22].

The concept of Frame Multiresolution Analysis (FMRA) as described in [2, 3] generalizes the notation of MRA by allowing non-exact affine frames. However, subspaces at different resolutions in an FMRA are still generated by a frame formed by translations of a single function, and dilates of it are used at different levels. By this property any FMRA is naturally associated with multirate systems having perfect reconstruction. On the other hand, FMRA theory can be seen as a contribution to the theory of multirate systems as it provides a narrow band decomposition structure for arbitrary signals.

This article is motivated from the observation that standard methods in (regular) sampling theory provide examples of multiresolution structures which are neither MRAs nor FMRA's. It also lead to new constructions of affine frames.

Our article aims at a detailed description and analysis of this new multiresolution structure, and providing a new construction of affine frames. It starts by briefly recalling the basic properties

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of frames and FMRA in Section 2. An example of a generalized multiresolution structure (GMS) is discussed in Section 3 from a sampling point of view. In order to analyze this GMS, we introduce in Section 4 a new notion of *pseudoframes of translates* for a closed subspace of the Hilbert space $L^2(\mathbf{R})$. Based on this concept the formal definition of a GMS is given in Section 5. Necessary and sufficient condition for the construction of pseudoframes of translates are derived. Subsequently construction methods for GMSs are also explained and illustrating examples are presented. Furthermore a construction that allows us to obtain affine frames associated with such a GMS is given in Section 6. In this context a technique which applies to more general subspace decompositions for nested multiresolution subspaces is introduced and utilized. As a valuable consequence, affine frames constructed in this way are naturally associated with a decomposition and reconstruction algorithm using filter banks. We elaborate on the corresponding fast algorithm and the multirate systems associated with the GMS in Section 7.

2. Frames and Frame Multiresolution Analysis

Let \mathcal{H} be a separable Hilbert space. We recall that a sequence $\{x_n : n \in \mathbf{Z}\} \subseteq \mathcal{H}$ is a *frame* for \mathcal{H} if there exist constants $A, B > 0$ such that

$$\forall x \in \mathcal{H}, \quad A\|x\|^2 \leq \sum_n |\langle x, x_n \rangle|^2 \leq B\|x\|^2. \quad (2.1)$$

A sequence $\{x_n\}$ is a *Bessel sequence* if (only) the upper inequality of (2.1) holds. If $\{x_n\}$ is a frame, there exists a dual frame $\{x_n^*\}$ such that

$$x = \sum_n \langle x, x_n^* \rangle x_n = \sum_n \langle x, x_n \rangle x_n^* \text{ in } \mathcal{H}.$$

See, e.g., [13, 14]. For more references on frames and relevant terminologies, we refer readers to, e.g., [1, 6, 7, 8, 9, 10, 11], and [23].

Throughout this article, $\hat{\theta}$ stands for the Fourier transform of θ : $\hat{\theta}(\gamma) = \int_{\mathbf{R}} \theta(t) e^{-2\pi i \gamma t} dt$, and $\tau_k f$ stands for integer translates: $\tau_k f(t) \equiv f(t - k)$.

The following characterization of Bessel sequences will be useful in the discussion of GMS.

Proposition 1 ([3, 12]).

Let $\phi \in L^2(\mathbf{R})$. Define

$$\Phi(\gamma) \equiv \sum_n \left| \hat{\phi}(\gamma + n) \right|^2, \quad (2.2)$$

Then $\{\tau_n \phi\}$ is a Bessel sequence in $\overline{\text{span}}\{\tau_n \phi\}$ (as well as in $L^2(\mathbf{R})$) if and only if there is a constant $M < \infty$ such that

$$\Phi(\gamma) \leq M \quad \text{a.e.} \quad (2.3)$$

The proof of Proposition 1 is straightforward. The following relationship is also useful in the study of Bessel sequences and frames.

Proposition 2 ([13]).

Two Bessel sequences $\{x_n\} \subseteq \mathcal{H}$ and $\{x_n^*\} \subseteq \mathcal{H}$ are dual frames to each other for \mathcal{H} if and only if

$$\forall x, y \in \mathcal{H}, \quad \langle x, y \rangle = \sum_n \langle x, x_n^* \rangle \langle x_n, y \rangle.$$

Indeed, the lower frame bound of one sequence is implied by the upper Bessel bound of the other.

Frame Multiresolution Analysis (FMRA)

An MRA is usually built upon an affine bounded unconditional basis. An FMRA is based on an affine frame other than a basis, see, [2, 3], and [12]. For a quick reference recall that a frame multiresolution analysis (FMRA) $\{V_j, \phi\}$ of $L^2(\mathbf{R})$ is a sequence of closed linear subspaces $V_j \subseteq L^2(\mathbf{R})$ and an element $\phi \in V_0$ for which the following hold:

1. $V_j \subseteq V_{j+1}$.
2. $\overline{\cup_j V_j} = L^2(\mathbf{R})$ and $\cap_j V_j = 0$,
3. $f(t) \in V_j$ if and only if $f(2t) \in V_{j+1}$,
4. $f \in V_0$ implies $\tau_k f \in V_0$, for all $k \in \mathbf{Z}$,
5. $\{\tau_k \phi : k \in \mathbf{Z}\}$ is a frame for the subspace V_0 .

One of the important ingredient for the characterization (and construction) of FMRA is a necessary and sufficient condition for frames of integer translates.

Indeed, let $\phi \in L^2(\mathbf{R})$ and let $V_0 \equiv \overline{\text{span}}\{\tau_k \phi : k \in \mathbf{Z}\}$ be a closed subspace of $L^2(\mathbf{R})$. Assume $\Phi \in L^\infty(\mathbf{T})$. Then the sequence $\{\tau_k \phi\}$ is a frame for V_0 if and only if there are positive constants A and B such that

$$A \leq \Phi(\gamma) \leq B \quad \text{a.e. on } \mathbf{T} \setminus \mathbf{N}, \quad (2.4)$$

where $\mathbf{N} \equiv \{\gamma \in \mathbf{T} : \Phi(\gamma) = 0\}$, and \mathbf{N} is defined up to sets of measure zero. For details on FMRA, we refer to [2, 3], and [12]. Related to this subject there are also articles, e.g., [4], and [20]. Evidently the usual MRAs is a (proper) subclass of FMRA. As we shall show next there are however even more general multiresolution structures.

3. An Example Beyond FMRA

Consider a classical example from sampling theory. For $\Omega \equiv [-\frac{1}{4}, \frac{1}{4})$, let $PW_{\frac{1}{4}}$ be the Paley-Wiener space with spectrum Ω . According to the classical Shannon sampling theorem for any function $\phi \in L^2(\mathbf{R})$ such that

$$\hat{\phi}(\gamma) = \begin{cases} 1 & -\frac{1}{4} \leq \gamma < \frac{1}{4} \\ \text{decaying to zero continuously} & \frac{1}{4} \leq |\gamma| < \frac{1}{2} \\ 0 & |\gamma| \geq \frac{1}{2} \end{cases}$$

and for $T = 1$ satisfying the Nyquist rate $2T \cdot \frac{1}{4} < 1$, we have an expansion for functions in $PW_{\frac{1}{4}}$:

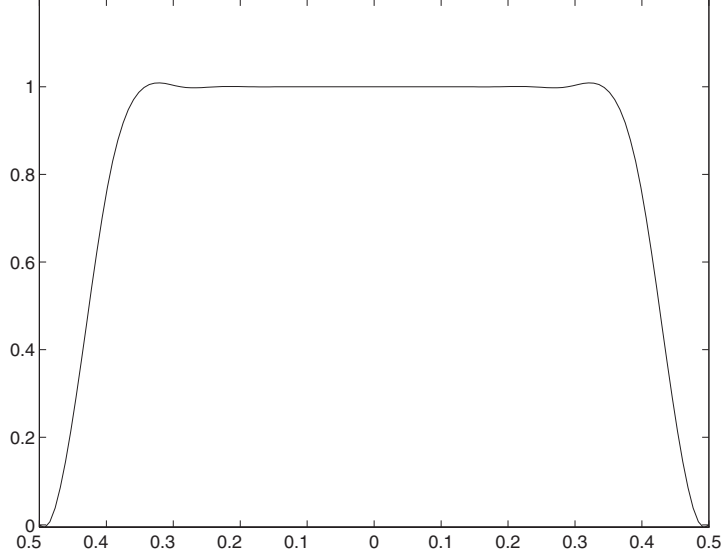
$$\forall f \in PW_{\frac{1}{4}}, \quad f(t) = \sum_n f(n)\phi(t-n). \quad (3.1)$$

Two observations about (3.1) arise, assuming $\alpha \leq \frac{1}{2}$, and $\hat{\phi} = 0$ a.e. on $\hat{\mathbf{R}} \setminus [-\alpha, \alpha]$ as shown in Figure 1.

- a. By the theory of FMRA [see condition (2.4)], $\{\phi(t-n)\}_n$ cannot be a frame for the subspace $\overline{\text{span}}\{\phi(t-n)\}$ since $\Phi(\gamma) \equiv \sum_k |\hat{\phi}(\gamma+k)|^2$ is a continuous function [3].
- b. $\{\phi(t-n)\}_n$ cannot be a frame for $PW_{\frac{1}{4}}$ either since $\phi \notin PW_{\frac{1}{4}}$.

Moreover, if one defines

$$V_0 \equiv PW_{\frac{1}{4}},$$

FIGURE 1 A function $\hat{\phi}$.

and

$$V_j \equiv PW_{2^j \frac{1}{4}},$$

then, for all $j \in \mathbf{Z}$, $V_j \subseteq V_{j+1}$, and $\overline{\cup V_j} = L^2(\mathbf{R})$ since the set of all band-limited functions is dense in $L^2(\mathbf{R})$, and that $\cap V_j = \{0\}$.

Evidently, $\{\phi, V_j\}$ generates a multiresolution structure which is not covered by those mentioned before. It is neither an MRA nor an FMRA.

By using functions $\phi \notin V_0$ to “generate” V_0 we have gained the liberty of additional smoothness of the sampling/reconstruction function $\hat{\phi}$, hence the faster decay of ϕ . This property represents a notable freedom worthwhile further exploration. More importantly, we will show that affine frames are naturally associated with GMSs, which can be constructed effectively based on a GMS.

In order to analyze and characterize such a GMS, we introduce a new notion of a *pseudoframes of translates*. Note that pseudoframes of translates are particular examples of the notion of *pseudoframes for subspaces* of separable Hilbert spaces studied in [15].

4. Pseudoframes of Translates

Definition 1. Let $\{\tau_k \phi\}$ and $\{\tau_k \phi^*\}$ ($k \in \mathbf{Z}$) be two sequences in \mathcal{H} . Let \mathcal{X} be a closed subspace of \mathcal{H} . We say $\{\tau_k \phi\}$ forms a *pseudoframe of translates* for \mathcal{X} with respect to $\{\tau_k \phi^*\}$ ($k \in \mathbf{Z}$) if

$$\forall x \in \mathcal{X}, \quad x = \sum_{k \in \mathbf{Z}} \langle x, \tau_k \phi^* \rangle \tau_k \phi. \quad (4.1)$$

It is important to note that ϕ and ϕ^* need not be contained in \mathcal{X} . The example of Section 3 and Example 1 are such cases. Consequently, the positions of $\{\tau_k \phi\}$ and $\{\tau_k \phi^*\}$ are not generally “commutable” [15], i.e., there exists $x \in \mathcal{X}$ such that

$$\sum_k \langle x, \tau_k \phi \rangle \tau_k \phi^* \neq \sum_k \langle x, \tau_k \phi^* \rangle \tau_k \phi = x.$$

However, in the context of the affine structure, the commutativity in the above sense is easily achievable. See Theorem 1 of the next section.

5. Generalized Multiresolution Structure

Definition 2. A *generalized multiresolution structure* (GMS) $\{V_j, \phi, \phi^*\}$ of $L^2(\mathbf{R})$ is an increasing sequence of closed linear subspaces $V_j \subseteq L^2(\mathbf{R})$ and two elements $\phi, \phi^* \in L^2(\mathbf{R})$ for which the following hold:

- (i) $\overline{\cup_j V_j} = L^2(\mathbf{R})$ and $\cap_j V_j = \{0\}$,
- (ii) $f(t) \in V_j$ if and only if $f(2t) \in V_{j+1}$,
- (iii) $f \in V_0$ implies $\tau_k f \in V_0$, for all $k \in \mathbf{Z}$,
- (iv) $\{\tau_k \phi : k \in \mathbf{Z}\}$ forms a pseudoframe of translates for V_0 with respect to $\{\tau_k \phi^* : k \in \mathbf{Z}\}$.

Remark: To comment on the generality of GMSs in Definition 2, we note that if $\{\tau_k \phi\}$ and $\{\tau_k \phi^*\}$ are a pair of frame and dual frame of V_0 , a GMS is simply an FMRA [2]; if $\{\tau_k \phi\}$ is an exact frame of V_0 and if $\phi^* \in V_0$, a GMS becomes an MRA, in which $\{\tau_k \phi^*\}$ is the biorthogonal sequence to $\{\tau_k \phi\}$ in V_0 . Note that even when $\{\tau_k \phi\}$ is an exact frame of V_0 in an GMS, it could be that $\phi^* \notin V_0$. This would still correspond to an GMS.

5.1 Construction of a GMS of Paley-Wiener Subspaces

The following theorem is a necessary and sufficient condition for the construction of pseudoframes of translates for Paley-Wiener subspaces.

Theorem 1.

Let $\phi \in L^2(\mathbf{R})$ be such that $|\hat{\phi}| > 0$ a.e. on a connected neighborhood of 0 in $[-\frac{1}{2}, \frac{1}{2})$, and $|\hat{\phi}| = 0$ a.e. otherwise. Define $\Omega \equiv \{\gamma \in \hat{R} : |\hat{\phi}| \geq c > 0\}$, and let $V_0 \equiv PW_\Omega = \{f \in L^2(\mathbf{R}) : \text{supp}(\hat{f}) \subseteq \Omega\}$. Then, for a $\phi^* \in L^2(\mathbf{R})$, $\{\tau_k \phi\}$ is a pseudoframe of translates for V_0 with respect to $\{\tau_k \phi^*\}$ if and only if

$$\hat{\phi} \overline{\hat{\phi}^*} \cdot \chi_\Omega = \chi_\Omega \quad \text{a.e.}, \quad (5.1)$$

where χ_Ω is the characteristic function on Ω . Moreover, if ϕ^* is also such that $|\hat{\phi}^*| > 0$ a.e. on a connected neighborhood of 0 in $[-\frac{1}{2}, \frac{1}{2})$, and $|\hat{\phi}^*| = 0$ a.e. otherwise, and that (5.1) holds, then $\{\tau_k \phi\}$ and $\{\tau_k \phi^*\}$ are a commutative pair of pseudoframes for \mathcal{X} , i.e.,

$$\forall x \in \mathcal{X}, \quad x = \sum_k \langle x, \tau_k \phi^* \rangle \tau_k \phi = \sum_k \langle x, \tau_k \phi \rangle \tau_k \phi^* .$$

Before we give a proof to Theorem 1, let us elaborate briefly on the commutativity issue of pseudoframes of translates.

Assume both $\{\tau_n \phi\}$ and $\{\tau_n \phi^*\}$ are Bessel sequences in \mathcal{H} . Define $U : \mathcal{H} \rightarrow l^2$ by

$$\forall x \in \mathcal{H}, \quad Ux = \{\langle x, \tau_n \phi \rangle\}, \quad (5.2)$$

and define $V : \mathcal{H} \rightarrow l^2$ by

$$\forall x \in \mathcal{H}, \quad Vx = \{\langle x, \tau_n \phi^* \rangle\}. \quad (5.3)$$

As a special case of [15], $\{\tau_n\phi\}$ is a pseudoframe of translates for \mathcal{X} w.r.t. $\{\tau_n\phi^*\}$ if and only if

$$V^*UP = P,$$

where P is the orthogonal projection onto \mathcal{X} . And $\{\tau_n\phi\}$ and $\{\tau_n\phi^*\}$ are a commutative pair of pseudoframes if and only if [15]

$$V^*UP = P = PU^*V, \quad (5.4)$$

where U^* and V^* are the unique bounded adjoints of U and V , respectively. The commutativity condition (5.4) is simply because the adjoint operation U^*V of V^*U interchanges the positions of $\{\tau_n\phi\}$ and $\{\tau_n\phi^*\}$ in equation (4.1).

We comment that while commutativity is achieved in our examples presented in Theorem 1, pseudoframes for subspaces are generally rather delicate depending on the spanning behavior of sequences $\{\tau_n\phi\}$ and $\{\tau_n\phi^*\}$, relative to the subspace \mathcal{X} . We refer readers to [15] for further discussions.

Proof of Theorem 1. For all $f \in PW_\Omega$, consider

$$\begin{aligned} & \left(\sum_n \langle f, \tau_n\phi^* \rangle \tau_n\phi \right)^\wedge \\ &= \sum_n \int_R \hat{f}(\lambda) \overline{\hat{\phi}^*(\lambda)} e^{2\pi i n \lambda} d\lambda \hat{\phi}(\gamma) e^{-2\pi i n \gamma} \\ &= \sum_n \int_0^1 \sum_k \hat{f}(\lambda + k) \overline{\hat{\phi}^*(\lambda + k)} e^{2\pi i n \lambda} d\lambda \hat{\phi}(\gamma) e^{-2\pi i n \gamma} \\ &= \hat{\phi}(\gamma) \sum_k \hat{f}(\gamma + k) \overline{\hat{\phi}^*(\gamma + k)} \\ &= \hat{f}(\gamma) \cdot \hat{\phi}(\gamma) \overline{\hat{\phi}^*(\gamma)}, \end{aligned}$$

where we have used the fact that $|\hat{\phi}| \neq 0$ only on $[-\frac{1}{2}, \frac{1}{2})$, and that

$$\sum_k \hat{f}(\gamma + k) \overline{\hat{\phi}^*(\gamma + k)}$$

is 1-periodic. Therefore,

$$\hat{\phi} \overline{\hat{\phi}^*} \cdot \chi_\Omega = \chi_\Omega \quad \text{a.e.}$$

is a necessary and sufficient condition for $\{\tau_n\phi\}$ to be a pseudoframe of translates for V_0 with respect to $\{\tau_n\phi^*\}$.

Direct calculation also shows that (5.4) is satisfied if $\hat{\phi}^*$ and $\hat{\phi}$ satisfy support conditions specified in the theorem. Hence, $\{\tau_n\phi\}$ and $\{\tau_n\phi^*\}$ are a commutative pair of pseudoframes for V_0 , \square

Proposition 3.

Let $\{\tau_k\phi\}$ be a pseudoframe of translates for V_0 with respect to $\{\tau_k\phi^*\}$. Define V_j by

$$V_j \equiv \left\{ f \in L^2(\mathbf{R}) : f \left(\frac{t}{2^j} \right) \in V_0 \right\}. \quad (5.5)$$

Then $\{\phi_{jk}\}_k$ is a pseudoframe (of translates) for V_j with respect to $\{\phi_{jk}^*\}_k$, where $\theta_{jk} \equiv \sqrt{2^j} \theta(2^j t - k)$.

We leave the verification of this proposition to readers.

Theorem 2.

Let $\phi, \phi^* \in L^2(\mathbf{R})$ have the properties specified in Theorem 1 such that the condition (5.1) is satisfied. Assume that V_j is defined by (5.5). Then $\{V_j, \phi, \phi^*\}$ forms a GMS.

Proof. There are three axioms to be verified in Definition 2, plus the property that $V_j \subseteq V_{j+1}$. The inclusion $V_j \subseteq V_{j+1}$ follows from the fact that V_j defined by (5.5) is equivalent to $PW_{(2^j\Omega)}$, and $PW_\Omega \subseteq PW_{2\Omega}$.

Condition (i) is satisfied because the set of all band-limited signals is dense in $L^2(\mathbf{R})$. On the other hand, the intersection of all band-limited signals is the trivial function.

Condition (ii) is an immediate consequence of (5.5).

For condition (iii) one may show that

$$\forall n \in \mathbf{Z}, \quad \sum_k \langle \tau_n f, \tau_k \phi^* \rangle \tau_k \phi = \tau_n f .$$

Or, it is a fact that $(\tau_n f)^\wedge$ has support in Ω for arbitrary $n \in \mathbf{Z}$. Therefore, $\tau_n f \in V_0$. \square

Example 1. Take ϕ be such that

$$\hat{\phi}(\gamma) = \begin{cases} 1 & \text{a.e. } -\frac{1}{4} \leq \gamma < \frac{1}{4} \\ 2 - 4|\gamma| & \text{a.e. } \frac{1}{4} \leq |\gamma| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Choose

$$\Omega \equiv \left\{ \gamma \in \hat{\mathbf{R}} : \left| \hat{\phi}(\gamma) \right| \geq 1 \right\} = \left[-\frac{1}{4}, \frac{1}{4} \right],$$

and define $V_0 = PW_\Omega$. Now, select $\phi^* \in L^2(\mathbf{R})$ such that

$$\hat{\phi}^*(\gamma) = \begin{cases} 1 & \text{a.e. } -\frac{1}{4} \leq \gamma < \frac{1}{4} \\ 3 - 8|\gamma| & \text{a.e. } \frac{1}{4} \leq |\gamma| \leq \frac{3}{8} \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Theorem 1, $\{\tau_k \phi\}$ and $\{\tau_k \phi^*\}$ form a pair of pseudoframes for $V_0 = PW_\Omega$ since $\hat{\phi} \cdot \hat{\phi}^* = 1$ a.e. on Ω . Further, define V_j as in (5.5), $\{V_j, \phi, \phi^*\}$ forms a generalized multiresolution structure for $L^2(\mathbf{R})$ by Theorem 2.

5.2 The Scaling Relationship Associated with a GMS

The familiar scaling relationships associated with MRAs between dilates of the function ϕ , as well as that of ϕ^* still hold in GMSs. Symbols H_0 and H_0^* are defined by $H_0 = \sum_n h_0(n) e^{-2\pi i n(\cdot)}$, and $H_0^* = \sum_n h_0^*(n) e^{-2\pi i n(\cdot)}$ for sequences h_0 and h_0^* wherever the sum is defined.

Proposition 4 ([12]).

Let $\{h_0(n)\}$ be such that $H_0(0) = \sqrt{2}$ and $H_0(\gamma) \neq 0$ in a neighborhood of 0. Assume also that $|H_0| \leq \sqrt{2}$. Then there exist $\phi \in L^2(\mathbf{R})$ such that

$$\phi(t) = \sqrt{2} \sum_n h_0(n) \phi(2t - n) . \quad (5.6)$$

The proof of Proposition 4 is very similar to Mallat's Lemma in [17]. See also [7, p. 175]. We hereby omit the proof.

Similarly, there exists a scaling relationship for ϕ^* under the same conditions as that of h_0 for a sequence h_0^* , as specified in Proposition 4:

$$\phi^*(t) = \sqrt{2} \sum_n h_0^*(n) \phi^*(2t - n). \quad (5.7)$$

In terms of the filters H_0 and H_0^* , Theorem 1 becomes the following:

Corollary 1.

Suppose H_0 and H_0^* generate ϕ and ϕ^* as in equations (5.6) and (5.7), respectively. Assume $\phi \in L^2(\mathbf{R})$ and $\phi^* \in L^2(\mathbf{R})$ have the properties specified in Theorem 1. Then $\{\tau_k \phi\}$ forms a pseudoframe of translates for V_0 with respect to $\{\tau_k \phi^*\}$ if and only if

$$H_0 \cdot \overline{H_0^*} \chi_{\Omega/2} = 2 \chi_{\Omega/2} \quad a.e. \quad (5.8)$$

Proof. Take the Fourier transform of equations (5.6) and (5.7), one has $\hat{\phi}(2\gamma) = \frac{1}{\sqrt{2}} H_0(\gamma) \hat{\phi}(\gamma)$ and $\hat{\phi}^*(2\gamma) = \frac{1}{\sqrt{2}} H_0^*(\gamma) \hat{\phi}^*(\gamma)$. Then (5.1) holds if and only if (5.8) holds. The result is immediate. \square

Therefore, the construction of GMSs may simply start from filters H_0, H_0^* satisfying (5.8) and the scaling equations (5.6) and (5.7).

An example of a pair of H_0 and H_0^* are given in Figures 2 and 3.

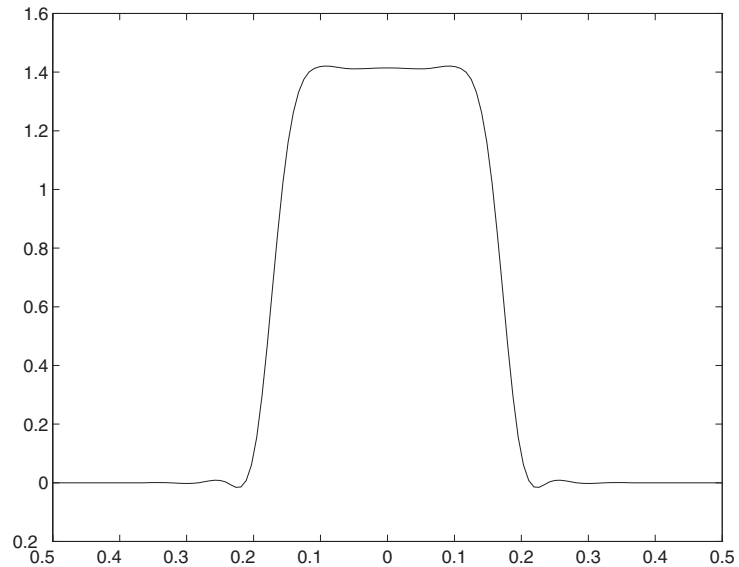
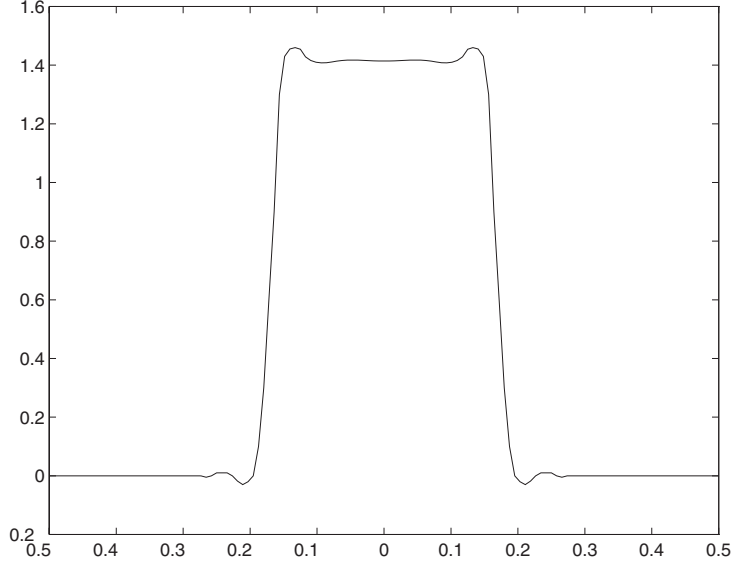


FIGURE 2 A filter H_0 generating ϕ .

FIGURE 3 A H_0^* generating ϕ^* .

6. Affine Frames for $L^2(\mathbf{R})$

As an important and major part of the theory of GMSs, we discuss in this section the construction of affine frames for $L^2(\mathbf{R})$ based on a GMS structure. This is not only an integrate part of the theory of GMSs, we also note that affine frames constructed via a GMS have a natural filter-bank-based fast decomposition and reconstruction algorithm, a favorable property that needs not hold for affine frames constructed some other ways. This can be of meaningful values.

6.1 The Decomposition of V_1

We shall denote the orthogonal complement of V_0 in V_1 by W_0 , as usual.

Due to the non-orthogonality and the unconventional behavior of pseudoframes, we need to further generalize the usual decomposition approach seen in conventional MRAs. In order to split a function f of V_1 into two functions (mostly) in V_0 and W_0 , respectively, we will construct an affine pseudoframe for V_1 , making use of the existing affine pseudoframe structure for V_0 . Conventional symbols, ψ and ψ^* , will be used as generating functions for W_0 (in a sense of pseudoframes of translates). Notice that ψ and ψ^* will still be “band-pass” functions. But they need not be contained in W_0 .

Definition 3. Let $\{V_j, \phi, \phi^*\}$ be a given GMS, and let ψ and ψ^* be two (band-pass) functions in $L^2(\mathbf{R})$. We say $\{\tau_n \phi, \tau_n \psi\}$ form a pseudoframe (of translates) for V_1 w.r.t. $\{\tau_n \phi^*, \tau_n \psi^*\}$ if

$$\forall f \in V_1, \quad f = \sum_n \langle f, \tau_n \phi^* \rangle \tau_n \phi + \sum_n \langle f, \tau_n \psi^* \rangle \tau_n \psi. \quad (6.1)$$

$\{\tau_n \phi^*, \tau_n \psi^*\}$ is called a *dual pseudoframe* to $\{\tau_n \phi, \tau_n \psi\}$ in the sense of (6.1).

Remark: Eventually, we shall give a condition for which the collection of $\{\psi_{j,k}\}$ and $\{\psi_{j,k}^*\}$ forms a pair of affine frames for $L^2(\mathbf{R})$. See Section 6.2.

To characterize the condition for which $\{\tau_n \phi, \tau_n \psi\}$ form an affine pseudoframe for V_1 w.r.t. $\{\tau_n \phi^*, \tau_n \psi^*\}$, we start from developing the “wavelet equations” with “band-pass” functions ψ and

ψ^* based on an GMS, namely,

$$\psi(t) = \sqrt{2} \sum_n h_1(n) \phi(2t - n) \text{ in } L^2(\mathbf{R}), \quad (6.2)$$

and

$$\psi^*(t) = \sqrt{2} \sum_n h_1^*(n) \phi^*(2t - n) \text{ in } L^2(\mathbf{R}). \quad (6.3)$$

In fact, with similar proof as in Proposition 4, we have the following.

Proposition 5.

Let $\{h_1(n)\}$ be such that $H_1(0) = 0$, and $H_1 \in L^\infty(\mathbf{T})$. Let $\phi \in L^2(\mathbf{R})$ and be defined by (5.6). Assume that $\{h_0(n)\}$ satisfies the conditions in Proposition 4. Then there exists $\psi \in L^2(\mathbf{R})$ generated from (6.2).

Proof. Define a function $\hat{\psi}$ by

$$\hat{\psi}(\gamma) = \frac{1}{\sqrt{2}} H_1\left(\frac{\gamma}{2}\right) \prod_{j=2}^{\infty} \frac{1}{\sqrt{2}} H_0\left(\frac{\gamma}{2^j}\right) = \frac{1}{\sqrt{2}} H_1\left(\frac{\gamma}{2}\right) \hat{\phi}\left(\frac{\gamma}{2}\right). \quad (6.4)$$

Since H_0 satisfies conditions in Proposition 4, ϕ is in $L^2(\mathbf{R})$. Therefore, because $H_1 \in L^\infty(\mathbf{T})$, ψ defined by (6.4) is in $L^2(\mathbf{R})$. It is now sufficient to use Parseval's Theorem and the inverse Fourier transform of (6.4) to obtain (6.2). \square

A similar condition and conclusion applies to (6.3) with respect to a sequence h_1^* .

Let $\chi_\Omega(\gamma)$ be the characteristic function of the interval Ω . We will also use the following 1-periodic function

$$\Lambda_\Omega(\gamma) \equiv \sum_k \chi_\Omega(\gamma + k). \quad (6.5)$$

Theorem 3.

Let Ω be the bandwidth of the subspace V_0 defined in Theorem 1. $\{\tau_n \phi, \tau_n \psi\}$ form a pseud-iframe of translates for V_1 w.r.t. $\{\tau_n \phi^*, \tau_n \psi^*\}$ if and only if there are G_0 and G_1 in $L^2(\mathbf{T})$ such that

$$\begin{aligned} \overline{G_0(\gamma)} H_0^*(\gamma) \Lambda_\Omega(\gamma) + \overline{G_1(\gamma)} H_1^*(\gamma) \Lambda_\Omega(\gamma) &= 2\Lambda_\Omega(\gamma) \quad a.e., \\ \overline{G_0(\gamma)} H_0^*\left(\gamma + \frac{1}{2}\right) \Lambda_\Omega(\gamma) + \overline{G_1(\gamma)} H_1^*\left(\gamma + \frac{1}{2}\right) \Lambda_\Omega(\gamma) &= 0 \quad a.e. \end{aligned} \quad (6.6)$$

Proof. We first note that since $\{\phi_{1m}^*\}_m$ is complete when restricted to V_1 , equation (6.1) holds if and only if

$$\forall m \in \mathbf{Z}, \quad \langle f, \phi_{1m}^* \rangle = \sum_n \langle f, \tau_n \phi^* \rangle \langle \tau_n \phi, \phi_{1m}^* \rangle + \sum_n \langle f, \tau_n \psi^* \rangle \langle \tau_n \psi, \phi_{1m}^* \rangle. \quad (6.7)$$

Define

$$c_0(n) = \langle f, \tau_n \phi^* \rangle, \quad c_1(n) = \langle f, \phi_{1n}^* \rangle, \quad d_0(n) = \langle f, \tau_n \psi^* \rangle,$$

and denote by

$$g_0(m - 2n) = \langle \tau_n \phi, \phi_{1m}^* \rangle, \quad g_1(m - 2n) = \langle \tau_n \psi, \phi_{1m}^* \rangle,$$

where the “ $m - 2n$ ” indexing is easily verifiable. Then (6.7) becomes

$$\forall m \in \mathbf{Z}, \quad c_1(m) = \sum_n c_0(n)g_0(m - 2n) + \sum_n d_0(n)g_1(m - 2n) .$$

Taking the Fourier Transform, we have

$$C_1(\gamma) = C_0(2\gamma)G_0(\gamma) + D_0(2\gamma)G_1(\gamma) , \quad (6.8)$$

where $C_i = \hat{c}_i$, $i = 1, 2$, and $D_0 = \hat{d}_0$, the Fourier series of c_0 , c_1 and d_0 , respectively. Note that (5.7) and (6.3) also imply

$$c_0(n) = \sum_m \overline{h_0^*(m - 2n)} c_1(m) ,$$

and

$$d_0(n) = \sum_m \overline{h_1^*(m - 2n)} c_1(m) .$$

And, their Fourier series are, respectively,

$$C_0(2\gamma) = \frac{1}{2} \left[C_1(\gamma) \overline{H_0^*(\gamma)} + C_1\left(\gamma + \frac{1}{2}\right) \overline{H_0^*\left(\gamma + \frac{1}{2}\right)} \right] , \quad (6.9)$$

and

$$D_0(2\gamma) = \frac{1}{2} \left[C_1(\gamma) \overline{H_1^*(\gamma)} + C_1\left(\gamma + \frac{1}{2}\right) \overline{H_1^*\left(\gamma + \frac{1}{2}\right)} \right] , \quad (6.10)$$

Combining (6.8), (6.9), and (6.10), we have

$$\begin{aligned} 2C_1(\gamma) &= G_0(\gamma) \left[C_1(\gamma) \overline{H_0^*(\gamma)} + C_1\left(\gamma + \frac{1}{2}\right) \overline{H_0^*\left(\gamma + \frac{1}{2}\right)} \right] \\ &\quad + G_1(\gamma) \left[\left[C_1(\gamma) \overline{H_1^*(\gamma)} + C_1\left(\gamma + \frac{1}{2}\right) \overline{H_1^*\left(\gamma + \frac{1}{2}\right)} \right] \overline{H_1^*\left(\gamma + \frac{1}{2}\right)} \right] \\ &= \left[G_0(\gamma) \overline{H_0^*(\gamma)} + G_1(\gamma) \overline{H_1^*(\gamma)} \right] C_1(\gamma) \\ &\quad + \left[G_0(\gamma) \overline{H_0^*\left(\gamma + \frac{1}{2}\right)} + G_1(\gamma) \overline{H_1^*\left(\gamma + \frac{1}{2}\right)} \right] C_1\left(\gamma + \frac{1}{2}\right) . \end{aligned}$$

This relationship holds for all $f \in V_1$. In particular, for those $f \in V_1$ such that $C_1(\gamma)C_1(\gamma + \frac{1}{2}) = 0$, the above relationship holds only if

$$\begin{aligned} G_0(\gamma) \overline{H_0^*(\gamma)} \Lambda_\Omega(\gamma) + G_1(\gamma) \overline{H_1^*(\gamma)} \Lambda_\Omega(\gamma) &= 2\Lambda_\Omega(\gamma) \quad \text{a.e. ,} \\ G_0(\gamma) \overline{H_0^*\left(\gamma + \frac{1}{2}\right)} \Lambda_\Omega(\gamma) + G_1(\gamma) \overline{H_1^*\left(\gamma + \frac{1}{2}\right)} \Lambda_\Omega(\gamma) &= 0 \quad \text{a.e.} \end{aligned}$$

The “if” part is clear. This establishes the result. \square

Corollary 2.

Let $\{\tau_n \phi, \tau_n \psi\}$ be an affine pseudoframe for V_1 w.r.t. $\{\tau_n \phi^*, \tau_n \psi^*\}$. Then, for each $j \in \mathbf{Z}$, $\{\phi_{jk}, \psi_{jk}\}_k$ form a pseudoframe (of translates) for V_{j+1} w.r.t. $\{\phi_{jk}^*, \psi_{jk}^*\}_k$, i.e.,

$$\forall f \in V_{j+1}, \quad f = \sum_k \langle f, \phi_{jk}^* \rangle \phi_{jk} + \sum_k \langle f, \psi_{jk}^* \rangle \psi_{jk} .$$

Proof. The proof uses the self-similarity property of GMSs and a change of variable technique.

\square

6.2 The Pyramid Decomposition Structure

Because a GMS possesses a nested multiresolution subspace structure, our further concern is to build for a given GMS a recursive decomposition scheme similar to that available for an MRA. While (6.1) is a two-stage decomposition and reconstruction formula, it does not provide by itself a recursive decomposition scheme in a GMS. Indeed for a general function $f \in V_1$ the ‘‘coarser’’ term of the decomposition is not generally in V_0 , i.e.,

$$\sum_n \langle f, \tau_n \phi^* \rangle \tau_n \phi \notin V_0 .$$

Further decompositions would not hold by using (6.1).

To enable recursive decompositions, it is necessary that we enforce a general decomposition architecture in a GMS.

Definition 4. Let $\{V_j, \phi, \phi^*\}$ be a given GMS. We say that the GMS has a *pyramid decomposition scheme* if there are band-pass functions $\psi, \psi^* \in L^2(\mathbf{R})$ such that

$$\forall f \in L^2(\mathbf{R}), \quad \sum_n \langle f, \phi_{1n}^* \rangle \phi_{1n} = \sum_n \langle f, \tau_n \phi^* \rangle \tau_n \phi + \sum_n \langle f, \tau_n \psi^* \rangle \tau_n \psi . \quad (6.11)$$

Remarks: *a.* We assume that the affine sequences involved in (6.11) are Bessel sequences. This can be easily achieved by Theorem 1. The primary requirements are that the 1-periodic functions Φ, Φ^*, Ψ , and Ψ^* are in $L^\infty(\mathbf{T})$, for which a good sufficient condition, for example, is that ϕ (etc.) belong to the Wiener amalgam space $W(L^2, l^1) \equiv \{f : \sum_k \|f \cdot 1_{[k, k+1]}\|_2 < \infty\}$.

b. (6.11) was easy in conventional MRAs or FMRAs. Because with the basis (or frame) structure, the right-hand-side of (6.11) consists of two projections onto V_0 and W_0 , respectively, and the left-hand-side is a projection onto V_1 [12]. This is no longer a simple case in GMSs.

c. (6.11) is also a more general decomposition scheme in light of subspace divisions in MRAs and FMRAs. While the basis/frame structure for V_0 are fixed by ϕ (in MRAs or FMRAs), the structure of W_0 can be set free. That is, the 3rd term of (6.11) needs not be a projection, and ψ needs not be in W_0 . The 3rd term could well be a pseudoframe expression. The freedom gained translates into the relaxed conditions for filter bank designs.

d. (6.11) and (6.1) are different. If only a two-stage decomposition is needed, (6.1) is a simpler condition to work with.

Theorem 4.

Let $\{V_j, \phi, \phi^*\}$ be a GMS. Assume that integer translates of each ϕ, ϕ^*, ψ , and ψ^* are all Bessel sequences in $L^2(\mathbf{R})$. Then, (6.11) holds if and only if

$$\begin{aligned} H_0(\gamma) \overline{H_0^*(\gamma)} \Phi(\gamma) + H_1(\gamma) \overline{H_1^*(\gamma)} \Phi(\gamma) &= 2\Phi(\gamma) \quad a.e. \\ H_0\left(\gamma + \frac{1}{2}\right) \overline{H_0^*(\gamma)} \Phi(\gamma) + H_1\left(\gamma + \frac{1}{2}\right) \overline{H_1^*(\gamma)} \Phi(\gamma) &= 0 \quad a.e. \end{aligned} \quad (6.12)$$

Proof. Define as in Theorem 3

$$c_0(n) = \langle f, \tau_n \phi^* \rangle, \quad c_1(n) = \langle f, \phi_{1n}^* \rangle, \quad d_0(n) = \langle f, \tau_n \psi^* \rangle . \quad (6.13)$$

Then, equation (6.11) becomes

$$\sum_n c_1(n) \phi_{1n} = \sum_n c_0(n) \tau_n \phi + \sum_n d_0(n) \tau_n \psi ,$$

whose Fourier transform is

$$\sqrt{2}C_1\left(\frac{\gamma}{2}\right)\hat{\phi}\left(\frac{\gamma}{2}\right) = C_0(\gamma)\hat{\phi}(\gamma) + D_0(\gamma)\hat{\psi}(\gamma).$$

Using the equations (5.6) and (6.2) and their Fourier transform, we have

$$\sqrt{2}C_1\left(\frac{\gamma}{2}\right)\hat{\phi}\left(\frac{\gamma}{2}\right) = C_0(\gamma)\sqrt{2}H_0\left(\frac{\gamma}{2}\right)\hat{\phi}\left(\frac{\gamma}{2}\right) + D_0(\gamma)\sqrt{2}H_1\left(\frac{\gamma}{2}\right)\hat{\phi}\left(\frac{\gamma}{2}\right)$$

or,

$$C_1(\gamma)\hat{\phi}(\gamma) = C_0(2\gamma)H_0(\gamma)\hat{\phi}(\gamma) + D_0(2\gamma)H_1(\gamma)\hat{\phi}(\gamma). \quad (6.14)$$

Note that C_0 and D_0 were computed in the proof of Theorem 3 in equations (6.9) and (6.10). Substituting (6.9) and (6.10) into (6.14), we have

$$\begin{aligned} 2C_1(\gamma)\hat{\phi}(\gamma) &= \left(C_1(\gamma)\overline{H_0^*(\gamma)} + C_1\left(\gamma + \frac{1}{2}\right)\overline{H_0^*\left(\gamma + \frac{1}{2}\right)} \right) H_0(\gamma)\hat{\phi}(\gamma) \\ &\quad + \left(C_1(\gamma)\overline{H_1^*(\gamma)} + C_1\left(\gamma + \frac{1}{2}\right)\overline{H_1^*\left(\gamma + \frac{1}{2}\right)} \right) H_1(\gamma)\hat{\phi}(\gamma) \\ &= \left(H_0(\gamma)\overline{H_0^*(\gamma)} + H_1(\gamma)\overline{H_1^*(\gamma)} \right) C_1(\gamma)\hat{\phi}(\gamma) \\ &\quad + \left(H_0(\gamma)\overline{H_0^*\left(\gamma + \frac{1}{2}\right)} + H_1(\gamma)\overline{H_1^*\left(\gamma + \frac{1}{2}\right)} \right) C_1\left(\gamma + \frac{1}{2}\right)\hat{\phi}(\gamma). \end{aligned} \quad (6.15)$$

Equation (6.15) is to hold for all $C_1 \in L^2(\mathbf{T})$. In particular, for C_1 (derived from a function $f \in L^2(\mathbf{R})$ by $C_1^\vee(n) = \langle f, \phi_{1n} \rangle$) such that $C_1(\gamma)C_1(\gamma + \frac{1}{2}) = 0$ a.e., equation (6.15) is true only if

$$\begin{aligned} H_0(\gamma)\overline{H_0^*(\gamma)}\hat{\phi}(\gamma) + H_1(\gamma)\overline{H_1^*(\gamma)}\hat{\phi}(\gamma) &= 2\hat{\phi}(\gamma) \\ H_0(\gamma)\overline{H_0^*\left(\gamma + \frac{1}{2}\right)}\hat{\phi}(\gamma) + H_1(\gamma)\overline{H_1^*\left(\gamma + \frac{1}{2}\right)}\hat{\phi}(\gamma) &= 0. \end{aligned} \quad (6.16)$$

The “if” part is obvious, i.e., (6.16) implies (6.15). We will only need to substitute $\gamma + k$ for γ in (6.16) and sum over $k \in \mathbf{Z}$ to obtain (6.12). \square

Remarks: *a.* In special cases, if the support of $\hat{\phi}$ is large enough or $\hat{\phi}$ is discontinuous (but $\phi \notin PW_\Omega$) so that $\Phi \geq A > 0$ on the support of Φ , then $\{\tau_n\phi\}$, while being a pseudoframe for $V_0 \equiv PW_\Omega$, has become a frame or basis for $\overline{\text{span}}\{\tau_n\phi\}$. In many such occasions, the GMS built upon the V_j s can be equivalent to a conventional FMRA or MRA based on subspaces $\overline{\text{span}}\{\phi_{jk}\}_k$ and related frames/basis structures. This is reflected in the condition (6.12) we just derived.

It is also important to note that even in cases when $\{\tau_n\phi\}$ becomes a frame for $\overline{\text{span}}\{\tau_n\phi\}$, the pseudoframe structure of GMSs will still include cases of multiresolution structures that are more general than conventional MRAs. The author has recently come to realize that a typical example can be found in [5], where everything can be well described using pseudoframes. We refer to [15] for detail discussions on pseudoframes.

b. Notice that since Φ can be a continuous function and may vanish in the context of pseudoframes, a sufficient condition for (6.16) to hold is clearly,

$$\begin{aligned} H_0(\gamma)\overline{H_0^*(\gamma)} + H_1(\gamma)\overline{H_1^*(\gamma)} &= 2 \quad \text{a.e. on } \text{supp}(\Phi) \\ H_0(\gamma)\overline{H_0^*\left(\gamma + \frac{1}{2}\right)} + H_1(\gamma)\overline{H_1^*\left(\gamma + \frac{1}{2}\right)} &= 0 \quad \text{a.e. on } \text{supp}(\Phi). \end{aligned} \quad (6.17)$$

The self-similarity of a GMS also passes on to equation (6.11).

Corollary 3.

Let $\{\tau_n\phi, \tau_n\psi\}$ be an affine pseudoframe for V_1 w.r.t. $\{\tau_n\phi^*, \tau_n\psi^*\}$ such that conditions in Theorem 4 and (6.12) holds. Then for all $j \in \mathbf{Z}$,

$$\forall f \in L^2(\mathbf{R}), \quad \sum_n \langle f, \phi_{j+1,n}^* \rangle \phi_{j+1,n} = \sum_n \langle f, \phi_{j,n}^* \rangle \phi_{j,n} + \sum_n \langle f, \psi_{j,n}^* \rangle \psi_{j,n}. \quad (6.18)$$

Consequently, a given function in any subspace V_j can be decomposed recursively using the “band-pass” functions $\{\psi_{jk}\}$.

Corollary 4.

Assume functions $\phi, \phi^*, \psi,$ and ψ^* in $L^2(\mathbf{R})$ are such that (6.11) holds. Then for any integers j and J with $j < J$,

$$\forall f \in V_J, \quad f = \sum_n \langle f, \phi_{jn}^* \rangle \phi_{jn} + \sum_{m=j}^{J-1} \sum_n \langle f, \psi_{mn}^* \rangle \psi_{mn}. \quad (6.19)$$

Proof.

$$\forall f \in V_J, \quad f = \sum_n \langle f, \phi_{J-1,n}^* \rangle \phi_{J-1,n} + \sum_n \langle f, \psi_{J-1,n}^* \rangle \psi_{J-1,n}. \quad (6.20)$$

Hence, (6.19) is the result of (6.18) applied to the first term of (6.20) recursively. \square

Theorem 5.

Let $\phi, \phi^*, \psi,$ and ψ^* be functions in $L^2(\mathbf{R})$ defined by (5.6), (5.7), (6.2), and (6.3), respectively. Assume that conditions in Theorem 4 are satisfied. Then, for all functions $f \in L^2(\mathbf{R})$,

$$\sum_n \langle f, \phi_{jn}^* \rangle \phi_{jn} = \sum_{m=-\infty}^{J-1} \sum_{n \in \mathbf{Z}} \langle f, \psi_{mn}^* \rangle \psi_{mn} \text{ in } L^2(\mathbf{R}). \quad (6.21)$$

Moreover,

$$\forall f \in L^2(\mathbf{R}), \quad f = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle f, \psi_{mn}^* \rangle \psi_{mn} \text{ in } L^2(\mathbf{R}). \quad (6.22)$$

Consequently, if $\{\psi_{mn}\}$ and $\{\psi_{mn}^*\}$ are also Bessel sequences, they are actually a pair of affine frames for $L^2(\mathbf{R})$.

Proof. a. Consider, for $M > 0$, the operator $T_M : \mathcal{H} \rightarrow \mathcal{H}$ such that

$$T_M f \equiv f_M \equiv \sum_{n \in \mathbf{Z}} \langle f, \phi_{-M,n}^* \rangle \phi_{-M,n}.$$

Then the operators T_M are well defined and uniformly bounded in the operator norm on \mathcal{H} . In order to show that $f_M \rightarrow 0$ as $M \rightarrow \infty$, it is therefore sufficient to show that, for all g in any dense subspace of band-limited functions in \mathcal{H} ,

$$\sum_{n \in \mathbf{Z}} \langle g, \phi_{-M,n}^* \rangle \phi_{-M,n} \rightarrow 0 \text{ as } M \rightarrow \infty.$$

In particular, we may choose the dense set of functions g whose Fourier transform have compact support, is continuous, and vanishes in a neighborhood of 0,

$$\begin{aligned} \left\| \sum_n \langle g, \phi_{-M,n}^* \rangle \phi_{-M,n} \right\|_2 &= \sup_{\|y\|=1} \sum_n \langle g, \phi_{-M,n}^* \rangle \langle \phi_{-M,n}, y \rangle \\ &\leq \sup_{\|y\|=1} \left(\sum_n |\langle g, \phi_{-M,n}^* \rangle|^2 \right)^{1/2} \left(\sum_n |\langle y, \phi_{-M,n} \rangle|^2 \right)^{1/2} \\ &\leq B^{1/2} \left(\sum_n |\langle g, \phi_{-M,n}^* \rangle|^2 \right)^{1/2}, \end{aligned}$$

where B is the Bessel bound of $\{\phi_{-M,n}\}_n$.

Standard calculation of the right-hand side shows

$$\begin{aligned} \sum_n |\langle g, \phi_{-M,n}^* \rangle|^2 &= \int \left(\sum_k \hat{g}(\gamma + 2^{-M}k) \overline{\hat{\phi}^*(2^M\gamma + k)} \right) \hat{\phi}^*(2^M\gamma) \overline{\hat{g}(\gamma)} d\gamma \\ &\leq \int \left(\sum_k |\hat{g}(\gamma + 2^{-M}k)|^2 \right)^{1/2} \left(\sum_k |\hat{\phi}^*(2^M\gamma + k)|^2 \right)^{1/2} \hat{\phi}^*(2^M\gamma) \overline{\hat{g}(\gamma)} d\gamma \\ &\leq B^{*1/2} \int \left(2^{-M} \sum_k |\hat{g}(\gamma + 2^{-M}k)|^2 \right)^{1/2} \cdot 2^{M/2} \hat{\phi}^*(2^M\gamma) \overline{\hat{g}(\gamma)} d\gamma, \end{aligned}$$

where B^* is the Bessel bound of $\{\phi_{-M,n}^*\}_n$. Following the lead of [16] and since \hat{g} is continuous with compact support, the term $2^{-M} \sum_k |\hat{g}(\gamma + 2^{-M}k)|^2 \leq C^2 < \infty$, being a Riemann sum to the finite

integral $\int |\hat{g}(\gamma + x)|^2 dx$. Furthermore, since \hat{g} vanishes in a neighborhood of 0, i.e., $\hat{g}(\gamma) = 0$ for all $|\gamma| < \delta_g$, we have

$$\begin{aligned} \sum_n |\langle g, \phi_{-M,n}^* \rangle|^2 &\leq B^{*1/2} C \int \left| 2^{M/2} \hat{\phi}^*(2^M\gamma) \overline{\hat{g}(\gamma)} \right| d\gamma \\ &\leq B^{*1/2} C \|g\|_2 \left(\int_{|\gamma| \geq \delta_g} \left| 2^{M/2} \hat{\phi}^*(2^M\gamma) \right|^2 \right)^{1/2}. \end{aligned}$$

Observe that the last integral at the right-hand side tends to 0 as $M \rightarrow \infty$. This proves the first part of the theorem since, by (6.19),

$$f_M = \sum_n \langle f, \phi_{J_n}^* \rangle \phi_{J_n} - \sum_{m=-M}^{J-1} \sum_{n \in \mathbf{Z}} \langle f, \psi_{mn}^* \rangle \psi_{mn}.$$

b. Now that $\overline{\cup V_j} = L^2(\mathbf{R})$, for any $f \in L^2(\mathbf{R})$ and any $\epsilon > 0$ there exists $J_0 = J_0(\epsilon) > 0$, and for any $J > J_0$ there exists $g \in V_{J_0} \subseteq V_J$ such that

$$g = \sum_n \langle g, \phi_{J_n}^* \rangle \phi_{J_n}.$$

Furthermore, for $K = BB^*$,

$$\|f - g\|_2 < \frac{\epsilon}{1 + K}.$$

Now, by (6.21), for all $J > J_0$,

$$\begin{aligned}
& \left\| f - \sum_{m=-\infty}^{J-1} \sum_{n \in \mathbf{Z}} \langle f, \psi_{mn}^* \rangle \psi_{mn} \right\|_2 \\
&= \left\| f - \sum_{n \in \mathbf{Z}} \langle f, \phi_{Jn}^* \rangle \phi_{Jn} \right\|_2 \\
&\leq \|f - g\|_2 + \left\| g - \sum_{n \in \mathbf{Z}} \langle f, \phi_{Jn}^* \rangle \phi_{Jn} \right\|_2 \\
&= \|f - g\|_2 + \left\| \sum_n \langle g - f, \phi_{Jn}^* \rangle \phi_{Jn} \right\|_2 \\
&\leq \|f - g\|_2 + K \|f - g\|_2 = \|f - g\|_2 (1 + K) < \epsilon .
\end{aligned}$$

The second part of the theorem is therefore established.

If $\{\psi_{mn}\}$ and $\{\psi_{mn}^*\}$ are Bessel sequences (which can be easily achieved since both ψ and ψ^* have a band-pass nature, i.e., satisfy $\hat{\psi}(0) = \hat{\psi}^*(0) = 0$), then equation (6.22) implies that both $\{\psi_{mn}\}$ and $\{\psi_{mn}^*\}$ will be affine frames due to Proposition 2. \square

7. Fast Affine Frame Decompositions

Generally speaking the numerical implementations of a frame decompositions may be time consuming due to the non-orthogonality of frames. In contrast, for the affine frame of $L^2(\mathbf{R})$ constructed under a GMS there is a naturally associated fast tree-structured algorithm, namely, the pyramid filter bank decomposition and reconstruction algorithm. This is a valuable feature of affine frames constructed via GMSs.

For any given signal f , and any small $\epsilon > 0$, there is a J and a signal $g \in V_j$ such that

$$\|f - g\| \leq \epsilon .$$

Without loss of generality, we assume that $f \in V_0$. Then from (6.19), we have, for a given $j > 0$,

$$\forall f \in V_0, \quad f = \sum_n \langle f, \tau_n \phi^* \rangle \tau_n \phi = \sum_n \langle f, \phi_{-jn}^* \rangle \phi_{-jn} + \sum_{m=-j}^{-1} \sum_n \langle f, \psi_{mn}^* \rangle \psi_{mn} . \quad (7.1)$$

For decomposition, define

$$c_m(n) = \langle f, \phi_{mn}^* \rangle, \quad d_m(n) = \langle f, \psi_{mn}^* \rangle, \quad \forall m \in \mathbf{Z} .$$

Using equations (5.7) and (6.3) we have the following decompositions:

$$c_{m-1}(n) = \sum_k h_0^*(k - 2n) c_m(k), \quad \forall m \in \mathbf{Z} . \quad (7.2)$$

$$d_{m-1}(n) = \sum_k h_1^*(k - 2n) c_m(k), \quad \forall m \in \mathbf{Z} . \quad (7.3)$$

For the reconstruction, assume that the decomposition is performed for $j > 0$ steps. From the inner product of (7.1) with $\tau_k \phi^*$ we have

$$c_0(k) = \langle f, \tau_k \phi^* \rangle = \sum_n c_{-j}(n) \langle \phi_{-jn}, \tau_k \phi^* \rangle + \sum_{m=-j}^{-1} \sum_n d_m(n) \langle \psi_{mn}, \tau_k \phi^* \rangle . \quad (7.4)$$

Now, define

$$r_j(k) = \langle \phi_{j0}, \tau_k \phi^* \rangle, \quad \forall j; \quad g_m(k) = \langle \psi_{m0}, \tau_k \phi^* \rangle, \quad \forall m .$$

Then

$$r_j = h_0 * r_{j-1}, \quad \forall j; \quad g_m = h_1 * g_{m-1}, \quad \forall m ,$$

and

$$\langle \phi_{jn}, \tau_k \phi^* \rangle = r_j(k - 2^j n), \quad (7.5)$$

and

$$\langle \psi_{mn}, \tau_k \phi^* \rangle = g_m(k - 2^m n). \quad (7.6)$$

Therefore, the reconstruction is provided by the combination of equations (7.4), (7.5), and (7.6), namely, the following filtering operation:

$$c_0(k) = \sum_n c_{-j}(n) r_j(k - 2^j n) + \sum_{m=-j}^{-1} \sum_n d_m(n) g_m(k - 2^m n). \quad (7.7)$$

To obtain the original signal f , one would do:

$$f = \sum_n c_0(n) \tau_n \phi .$$

Conclusion

We have introduced and studied the notion of a *Generalized Multiresolution Structure* more general than FMRA. The filtering mechanism behind the affine structure of a GMS can be constructed using fast decaying filters. This facilitates the design of narrow band multiresolution structure such as FMRA, e.g., [3], and adds to multiresolution analysis a broader constructive approach for generating affine frames. This new approach includes (but not limited to) a particular biorthogonal method in [5]. The study of GMSs thus constitutes another integrate part of the concept of multiresolution analysis.

As an important part of the theory of GMSs, there are constructible affine frames associated with GMSs. Systematic constructions of affine frames based on GMSs are presented. An immediate benefit of these affine frames is that there is an associated fast filter-bank-based decomposition and reconstruction algorithm. Our study of GMSs and the construction of affine frames based on a GMS uses a notion of *pseudoframes of translates* which plays a role as basis or frames in MRAs or FMRA – simple, flexible, and essential to the theory.

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