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TWO SINGULAR DYNAMICS OF THE NONLINEAR SCHRÖDINGER EQUATION ON A PLANE DOMAIN

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Abstract

We study the cubic, focusing nonlinear Schrödinger equation (NLS) posed on a bounded domain of $\mathbb{R}^2$ with Dirichlet boundary conditions. We describe two types of nonlinear evolutions. First we obtain solutions which blow up with a minimal L^2 norm in finite time at a fixed point of the interior of the domain. The argument can be performed equally well for the cubic NLS posed on the flat torus $\mathbb{T}^2$. In the case when the domain is a disc, we also prove that the Cauchy problem is ill posed in the following sense: the flow map is not uniformly continuous on bounded sets of the Sobolev space H^s, $s < 1/3$, contrary to what is known on the square (recall that the scale invariant Sobolev space for the cubic NLS in $2D$ is L^2).

1 Introduction and Statement of the Results

Let Ω be a bounded domain of $\mathbb{R}^2$. In this paper, we are studying the cubic, focusing nonlinear Schrödinger equation (NLS), posed on Ω

$$(i\partial_t + \Delta)u = -|u|^2 u\,, \quad \text{in } \mathbb{R}\times\Omega \tag{1.1}$$

with initial data

$$u(0,x) = u_0(x)\,, \quad x\in\Omega\,,$$

and Dirichlet boundary conditions

$$u(t,x) = 0\,, \quad (t,x)\in\mathbb{R}\times\partial\Omega\,.$$

In (1.1), t denotes the time variable, Δ is the Laplace operator acting on the spatial variables and u is a complex valued function, defined on $\mathbb{R}\times\Omega$.

When the NLS is posed on a bounded spatial domain the corresponding nonlinear dynamics is expected to differ from that of the NLS on $\mathbb{R}^d$. We refer to the recent book [Bo] and the references therein for results in the case of the flat torus (periodic boundary conditions) and to [BGT1,2] in the case of spheres or more general compact riemannian manifolds.

The equation (1.1) can be written in a Hamiltonian format with respect to the canonical coordinates $(u,\bar u)$

$$iu_t = \frac{\delta H}{\delta \bar u}, \quad i\bar u_t = -\frac{\delta H}{\delta u}.$$

where

$$H(u,\bar{u}) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{4}\int_\Omega |u|^4.$$

The quantity $H(u,\bar{u})$ is usually called the energy and is conserved by the time evolution (see e.g. [V] or [C, Theorem 4.5.1] for an H^1 existence result), as well as the L^2 norm. Our goal is to give two qualitative results for the Cauchy problem associated to (1.1). During the last two decades, there has been a lot of work on the corresponding issue in the case of data defined on $\mathbb{R}^2$ (or more generally on $\mathbb{R}^d$, $d \geq 1$, with other power nonlinearities), starting from the work of Ginibre–Velo [GV]. Notice that an important invariance of the NLS posed on $\mathbb{R}^d$ is the scaling one. We now summarize some known facts concerning the theory of local and global existence of solutions of the Cauchy problem associated to (1.1), posed on the whole plane $\mathbb{R}^2$.

1. Strong local well-posedness down to the critical (scaling) Sobolev regularity of the initial data via the Strichartz inequalities for the free evolution and the Picard iteration scheme (see [CW]).
2. Global well-posedness for initial data with L^2 norm smaller than the L^2 norm of the ground state (see (1.2) below) via the energy a priori bound and the sharp Gagliardo–Nirenberg interpolation inequality (see [We]).
3. Existence of smooth data developing singularities in finite time via the viriel identity (see [VlPT], [Z], [Gl]).
4. Classification of the blowing up solutions with L^2 norm equal to the L^2 norm of the ground state via the pseudo-conformal invariance of (1.1) (see [M2]).
5. Construction of solutions developing singularities in finite time as perturbations of explicit blowing-up solutions (see [M1], [BoW]).

It is natural to study the above issues in the context of (1.1). In this paper, we show that if Ω is a unit disc of $\mathbb{R}^2$ then the first of the above properties fails in the case of (1.1). It is interesting to mention that this is an effect of the boundary since when Ω is a square, the Cauchy problem associated to (1.1) is locally well-posed down to the critical Sobolev regularity of the initial data. This can be seen by an easy extension of the result of Bourgain on the flat torus $\mathbb{T}^2$ (see [Bo] and section 2 below).

Property 2 can be extended to the case of (1.1), posed on a plane domain (see [BrG], [C] and section 2 below). Concerning property 3, we refer to [K] where it is shown that under some geometric assumption on Ω, for smooth "large data" (for example data with negative energy) the solutions of (1.1) develop singularities in finite time. The proof uses again the classical viriel identity argument. The geometric assumption is imposed in order to evaluate the boundary terms. Whether property 4 has a reasonable extension to the case of domains is an important open problem. In this paper, we however construct minimal L^2 mass blowing up solutions with a precise description of the solutions near the blow-up time. As in [M1] and [BoW] our argument is perturbative. More precisely, we adapt to boundary problem (1.1) an idea elaborated by Ogawa–Tsutsumi [OT], in order to prove a similar result on the circle.

Now we state our results.

Theorem 1. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let $x_0 \in \Omega$ with $\psi \in C_0^\infty(\Omega)$, $\psi = 1$ near x_0. Denote by Q the unique positive radial solution of the nonlinear elliptic equation (Q is usually called ground state)*

$$(-\Delta_{\mathbb{R}^2} + 1)Q = |Q|^2 Q \tag{1.2}$$

(see [C]). Then there exist $\kappa > 0$, $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ there exist $T_\lambda > 0$ and a family $\{r_\lambda\}$ of functions defined on $[0, T_\lambda[\times\Omega$ satisfying

$$\left\| r_\lambda(t,\cdot) \right\|_{H^2(\Omega)} \le c e^{-\frac{\kappa}{\lambda(T_\lambda - t)}}, \quad t \in [0, T_\lambda[$$

such that

$$u_\lambda(t,x) = \frac{1}{\lambda(T_\lambda - t)} \psi(x) e^{\frac{i(4-\lambda^2(x-x_0)^2)}{4\lambda^2(T_\lambda - t)}} Q\left(\frac{x-x_0}{\lambda(T_\lambda - t)}\right) + r_\lambda(t,x)\,,$$
$$x \in \Omega\,,\ t \in [0, T_\lambda[$$

are solutions of (1.1), satisfying the Dirichlet boundary conditions, which blow up at x_0 in time T_λ in the energy space H^1 with blow up speed $(T_\lambda - t)^{-1}$. Moreover $\|u_\lambda(t,\cdot)\|_{L^2(\Omega)} = \|Q\|_{L^2(\mathbb{R}^2)}$.

REMARK 1.1. 1. Theorem 1 can be easily generalized to the same equation posed on the two dimensional torus $\mathbb{T}^2$.

2. Our construction could also be extended to the case of a large class of unbounded domains (for example the exterior of a smooth bounded domain).

We now come to our second result, which is a singular ansatz for the solution of (1.1) with particular high frequency data.

Theorem 2. *Let $D = \{x \in \mathbb{R}^2 : |x| < 1\}$ be the unit disc in $\mathbb{R}^2$ and Δ_D the Laplace operator on D with Dirichlet boundary conditions. Fix $\kappa > 0$ and $s \in]1/5, 1/3[$. Then there exists a sequence $\phi_n(x)$ of eigenfunctions of $-\Delta_D$ with corresponding eigenvalues ζ_n ($\lim_{n\to\infty} \zeta_n = +\infty$) such that*

$\|\phi_n\|_{H^s(D)} \approx 1$ *and equation* (1.1) *with Cauchy data* $\kappa\phi_n(x)$, *has, for* $n \gg 1$, *a unique global solution* $u_n(t,x)$ *which can be represented as*

$$u_n(t,x) = \kappa e^{-it(\zeta_n - \kappa^2\omega_n)}\big(\phi_n(x) + r_n(t,x)\big)\,,$$

where $\omega_n \approx n^{\frac{2}{3}-2s}$ *and* $r_n(t,x)$ *satisfies for any* $T>0$, n *large enough and* $t \in [0,T]$

$$\big\|r_n(t,\cdot)\big\|_{H^s(D)} \le C\, n^{-\delta}$$

where $\delta > 0$ *and* C *is independent of* n. *Moreover*

$$\|u_n\|_{L^\infty(\mathbb{R};H^s(D))} \le C|\kappa|\,.$$

As a consequence the Cauchy problem associated to (1.1) *is not locally well-posed with uniformly continuous flow map for data in any ball of* $H^s(D)$.

Remark 1.2. 1. Whether the Cauchy problem for (1.1) is locally well posed on $H^s(D)$ for $1/3 < s < 1/2$ is an interesting problem.

2. In [BGT2], we proved similar results for the NLS with spheres as spatial domain.

3. See Lemma 2.1 below for the precise notion of local well-posedness with uniformly continuous flow map (see also [KePV], [BGT2]).

4. Note that the scaling Sobolev regularity for (1.1) is $s = 0$.

5. Theorem 2 can be extended to the easier case of defocusing nonlinearity.

6. In [Bo], it is shown that the cubic NLS, posed on the torus $\mathbb{T}^2$ is locally well-posed with analytic flow map in $H^s(\mathbb{T}^2)$, $s > 0$. The result of Theorem 2 shows that the results of Bourgain in the boundaryless case of the flat torus cannot be extended to the case of the unit disc. This can be seen as an effect of the boundary to the initial value theory of NLS.

The rest of the paper is organized as follows. The next section is concerned with some well-posedness results for (1.1). Section 3 is devoted to the proof of Theorem 1. In section 4, we collect some properties of the Bessel functions, needed for the proof of Theorem 2. In section 5, we give the proof of Theorem 2. Finally in an appendix, we prove the results stated in section 4.

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2 Preliminary Well-posedness Results

Let Ω be a domain of $\mathbb{R}^2$ and $T > 0$. Consider the initial boundary value problem

$$\begin{cases} (i\partial_t + \Delta)u = -|u|^2 u\,, & \text{in } [-T,T]\times\Omega\,,\\ u(0,x) = u_0(x)\,, & x\in\Omega\,,\\ u(t,x) = 0\,, & (t,x)\in[-T,T]\times\partial\Omega\,. \end{cases} \tag{2.1}$$

We now state a basic local well-posedness result for (2.1).

LEMMA 2.1. *For any $R > 0$ there exists $T > 0$ such that, for every*

$$u_0 \in B_R := \left\{u_0 \in H^2(\Omega)\cap H_0^1(\Omega) : \|u_0\|_{H^2(\Omega)} < R\right\} \subset H^2(\Omega)\,,$$

the problem (2.1) *has a unique solution in $C([-T,T],H^2(\Omega))$. In addition the map $u_0 \to u$ is uniformly continuous from B_R to $C([-T,T],H^2(\Omega))$.*

Moreover, the following conservation laws hold

$$\big\|u(t,\cdot)\big\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)}\,,$$

and

$$\tfrac{1}{2}\big\|\nabla u(t,\cdot)\big\|^2_{L^2(\Omega)} - \tfrac{1}{4}\big\|u(t,\cdot)\big\|^4_{L^4(\Omega)} = \tfrac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega)} - \tfrac{1}{4}\|u_0\|^4_{L^4(\Omega)}\,,$$

where $t \in [0,T]$.

The proof of Lemma 2.1 is classical and will be omitted. We only give several comments.

REMARK 2.2. 1. As it was already pointed out in [BGT2] the uniform continuity of the flow map is typical for semilinear problems. This results from the fact that we solve (2.1) by performing the Picard iteration scheme which even gives the analyticity of the flow map.

2. The method of proof of Lemma 2.1 does not make use of the dispersive nature of (2.1). Well-posedness for (2.1) in Sobolev spaces of low order (smaller than 1) would probably require an appeal to Strichartz inequalities for the free evolution which are not known (see Remark 5.2 below).

It is a natural question to ask whether the local solutions obtained in Lemma 2.1 can be extended globally in time. We now state a classical result due to Brézis–Gallouet (see [BrG]) which gives an answer to that question for sufficiently small data u_0.

LEMMA 2.3. *For every $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ satisfying*

$$\|u_0\|_{L^2(\Omega)} < \|Q\|_{L^2(\mathbb{R}^2)} \tag{2.2}$$

the local solution of (2.1) *with initial data u_0 can be extended to the whole real line.*

Remark 2.4. 1. In the next section, we construct solutions of (2.1) with $\|u_0\|_{L^2(\Omega)} = \|Q\|_{L^2(\mathbb{R}^2)}$, developing singularities in finite time.

2. In [BrG], a stronger assumption on the L^2 mass of the initial data is imposed. But using the sharp Gagliardo–Nirenberg inequality (see [We]), it is not difficult to see that the assumption (2.2) suffices to ensure that the energy controls the H^1 norm.

3. Lemma 2.3 will be our starting point in the proof of Theorem 2.

4. Notice that Lemma 2.3 can be generalized to the case of H^1_0 Cauchy data (see [V], [C]). However the uniform continuity of the flow on bounded sets of H^1 is not clear in this case.

The result of Lemma 2.1 can be much improved if Ω is a square.

Lemma 2.5 (compare with Theorem 2). *Let Ω be a square and $s \in]0, 1/2[$. Then* (2.1) *is locally well-posed for data in $H^s(\Omega)$ with uniformly continuous flow map.*

Remark 2.6. 1. The flow map is actually analytic on $H^s(\Omega)$.

2. A similar result holds for $s \geq 1/2$ with a slight modification of the space for the initial data coming from the boundary condition (see for instance Lemma 2.1).

Proof of Lemma 2.5. We can suppose that $\Omega = (0,1) \times (0,1)$. Let $\mathbb{T}^2 = \mathbb{R}^2|(2\mathbb{Z})^2$ be a flat torus. We regard $\mathbb{T}^2$ as $(-1,1) \times (-1,1)$ with the usual identification on the boundary lines. Notice that $H^s(\Omega)$ can be identified to the subspace of $H^s(\mathbb{T}^2)$ characterized by the conditions

$$\begin{cases} u(x,y) = -u(-x,y)\,, \\ u(x,y) = -u(x,-y) \end{cases} \tag{2.3}$$

The equation (1.1) is invariant under the transformations $(x,y) \mapsto (-x,y)$ and $(x,y) \mapsto (x,-y)$ and therefore if the initial data of (1.1), posed on $\mathbb{T}^2$, satisfies the symmetry property (2.3) then so does the solution. Hence the flow of (2.1) on $H^s(\Omega)$ can be regarded as a restriction of the flow on $H^s(\mathbb{T}^2)$, established by Bourgain (see [Bo, Theorem 2.7]), to the functions of $H^s(\mathbb{T}^2)$ satisfying (2.3). □

3 Proof of Theorem 1

Let $T > 0$ and $\lambda > 0$. Recall that

$$\tilde{R}_\lambda(t,x) = \frac{1}{\lambda(T-t)} e^{\frac{i(4-\lambda^2(x-x_0)^2)}{4\lambda^2(T-t)}} Q\left(\frac{x-x_0}{\lambda(T-t)}\right), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2$$

is a solution of the cubic focusing NLS, posed on $\mathbb{R}^2$ which blows up at x_0 in time $t = T$ (see [C]). The function $\tilde{R}_\lambda(t,x)$ is a solution of (1.1) but has its support on the whole space $\mathbb{R}^2$. For that purpose, we set $R_\lambda(t,x) := \psi(x)\tilde{R}_\lambda(t,x)$. The function $R_\lambda(t,x)$ has its support in Ω but is not solution of (1.1) anymore. The proof of Theorem 1 consists of constructing a smooth correction $r_\lambda(t,x)$ such that with a suitable choice of T the function $R_\lambda(t,x) + r_\lambda(t,x)$ is again a solution of (1.1).

Write

$$(i\partial_t + \Delta)R_\lambda = -\psi|\tilde{R}_\lambda|^2\tilde{R}_\lambda + 2\nabla\psi\,\nabla\tilde{R}_\lambda + \Delta\psi\,\tilde{R}_\lambda\,.$$

We look for $v \in C([0,T[, H^2(\Omega)\cap H_0^1(\Omega))$ such that

$$\begin{cases}(i\partial_t+\Delta)v = -|R_\lambda+v|^2(R_\lambda+v)+\psi|\tilde{R}_\lambda|^2\tilde{R}_\lambda-2\nabla\psi\,\nabla\tilde{R}_\lambda-\Delta\psi\,\tilde{R}_\lambda\,,\\ v(t)\longrightarrow 0 \text{ as } t\longrightarrow T\,(t<T) \text{ in } H^2(\Omega)\cap H_0^1(\Omega)\,.\end{cases} \tag{3.1}$$

Set

$$\begin{aligned}-\,|R_\lambda+v|^2(R_\lambda+v)+\psi|\tilde{R}_\lambda|^2\tilde{R}_\lambda-2\nabla\psi\,\nabla\tilde{R}_\lambda-\Delta\psi\,\tilde{R}_\lambda&\\ = Q_0+Q_1(v)+Q_2(v)+Q_3(v)\,,&\end{aligned}$$

where

$$\begin{cases}Q_0 = \psi\big(1-|\psi|^2\big)|\tilde{R}_\lambda|^2\tilde{R}_\lambda - 2\nabla\psi\,\nabla\tilde{R}_\lambda - \Delta\psi\,\tilde{R}_\lambda\,,\\ Q_1(v) = -R_\lambda^2\bar{v} - 2|R_\lambda|^2 v\,,\\ Q_2(v) = -\bar{R}_\lambda v^2 - 2R_\lambda|v|^2,\\ Q_3(v) = -|v|^2 v\,.\end{cases}$$

Due to the properties of the ground state of the NLS on $\mathbb{R}^2$ (see [C]), there exists $\delta > 0$ such that

$$\big\|Q_0(t,\cdot)\big\|_{H^2(\Omega)} \le Ce^{-\frac{\delta}{\lambda(T-t)}}. \tag{3.2}$$

We shall look for solutions of (3.1) in the space

$$X_T = \big\{v \in C([0,T[, H^2(\Omega)\cap H_0^1(\Omega)) : \|v\|_{X_T} < +\infty\big\}\,,$$

where

$$\|v\|_{X_T} = \sup_{t\in[0,T[}\big\{e^{\frac{\delta}{2\lambda(T-t)}}\|v(t)\|_{L^2(\Omega)} + e^{\frac{\delta}{3\lambda(T-t)}}\|v(t)\|_{H^2(\Omega)}\big\}\,.$$

Theorem 1 is then a consequence of the following existence result for (3.1).

LEMMA 3.1. *There exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$ there exists $T > 0$ depending on λ such that* (3.1) *has a unique solution $v \in X_T$.*

Proof of Lemma 3.1. Let $S(t)$ be the unitary group which defines the free evolution of the Schrödinger equation on Ω with Dirichlet boundary conditions. As a consequence $S(t)$ acts as an isometry on $L^2(\Omega)$ and on

$H^2(\Omega)\cap H_0^1(\Omega)$. We shall use this fact frequently in the sequel of the proof. Set

$$\int_t^T S(t-\tau)Q_0(\tau)d\tau + \sum_{j=1}^{3}\int_t^T S(t-\tau)Q_j(v(\tau))d\tau := \sum_{j=0}^{3} I_j(t)\,. \qquad (3.3)$$

The goal is to solve the integral formulation of (3.1) by the contraction mapping principle in X_T, i.e. we need to show that the nonlinear operator defined by (3.3) acting on $v\in X_T$ is a contraction in X_T for sufficiently large λ and small T depending on λ.

We now estimate separately $\|\cdot\|_{X_T}$ norms of $I_j(t)$, $j=0,1,2,3$.

Estimate for $I_0(t)$. Due to (3.2), we have

$$\|I_0\|_{X_T}\leq CT\,. \qquad (3.4)$$

Estimate for $I_1(t)$. Recall that $Q_1(v)=-R_\lambda^2\bar{v}-2|R_\lambda|^2v$. We first bound $\|I_1(t)\|_{L^2}$. Write for $t\in[0,T[$

$$\begin{aligned}\|I_1(t)\|_{L^2} &\leq C\int_t^T \|R_\lambda(\tau)\|_{L^\infty}^2\|v(\tau)\|_{L^2}d\tau\\ &\leq C\left\{\int_t^T \frac{1}{(\lambda(T-\tau))^2}e^{-\frac{\delta}{2\lambda(T-\tau)}}d\tau\right\}\|v\|_{X_T}\\ &\leq \frac{C}{\lambda}e^{-\frac{\delta}{2\lambda(T-t)}}\|v\|_{X_T}\,.\end{aligned}$$

We next bound $\|I_1(t)\|_{H^2}$. Write for $t\in[0,T[$

$$\begin{aligned}\|I_1(t)\|_{H^2} &\leq C\int_t^T \big\|\nabla^2 Q_1(v(\tau))\big\|_{L^2}d\tau+\big\|I_1(t)\big\|_{L^2}\\ &\leq C\int_t^T \big\|\nabla^2(R_\lambda^2)(\tau)\big\|_{L^\infty}\big\|v(\tau)\big\|_{L^2}d\tau\\ &\quad + C\int_t^T \big\|\nabla(R_\lambda^2)(\tau)\big\|_{L^\infty}\big\|\nabla v(\tau)\big\|_{L^2}d\tau\\ &\quad + C\int_t^T \big\|R_\lambda^2(\tau)\big\|_{L^\infty}\big\|\nabla^2 v(\tau)\big\|_{L^2}d\tau+\frac{C}{\lambda}e^{-\frac{\delta}{2\lambda(T-t)}}\|v\|_{X_T}\,.\end{aligned}$$

Using the explicit formula for $\tilde{R}_\lambda(t,x)$, it is not difficult to derive the following bound for R_λ

$$\big\|\nabla^k((R_\lambda^2)(\tau))\big\|_{L^\infty}\leq \frac{C}{(\lambda(T-\tau))^{2+k}}(1+\lambda^k)\,,\quad k=0,1,2\,,\ \tau\in[0,T[\,. \qquad (3.5)$$

Now using the elementary interpolation inequality

$$\big\|\nabla v(\tau)\big\|_{L^2}\leq\big\|\nabla^2 v(\tau)\big\|_{L^2}^{1/2}\big\|v(\tau)\big\|_{L^2}^{1/2}\,,\quad v\in X_T\,,$$

and (3.5), we obtain for $\lambda \geq 1$ and $T \leq 1$,

$$\|I_1(t)\|_{H^2} \leq C\|v\|_{X_T}\left\{\frac{1}{\lambda}e^{-\frac{\delta}{2\lambda(T-t)}} + \int_t^T \frac{1+\lambda^2}{(\lambda(T-\tau))^4}d\tau \right.$$
$$+ \int_t^T \frac{1+\lambda}{(\lambda(T-\tau))^3}e^{-\frac{1}{2}(\frac{\delta}{2\lambda(T-\tau)}+\frac{\delta}{3\lambda(T-\tau)})}d\tau$$
$$\left. + \int_t^T \frac{1}{(\lambda(T-\tau))^2}e^{-\frac{\delta}{3\lambda(T-\tau)}}d\tau\right\}$$
$$\leq C\left(\frac{1}{\lambda}+\lambda^2 T\right)e^{-\frac{\delta}{3\lambda(T-t)}}\|v\|_{X_T}$$

Therefore

$$\|I_1\|_{X_T} \leq C\left\{\tfrac{1}{\lambda}+T^{1/2}\right\}\|v\|_{X_T}\,, \tag{3.6}$$

provided $\lambda^2 T^{1/2} \leq 1$, $\lambda \geq 1$.

Estimate for $I_2(t)$. We shall directly estimate $\|I_2(t)\|_{H^2}$. Recall that

$$Q_2(v) = -\bar{R}_\lambda v^2 - 2R_\lambda |v|^2.$$

Using again the explicit representation of $\tilde{R}_\lambda$, we get

$$\|R_\lambda(\tau)\|_{H^2} \leq \frac{C(1+\lambda^2)}{(\lambda(T-\tau))^2}\,. \tag{3.7}$$

Since $H^2(\Omega)$ is an algebra, we easily obtain via (3.7),

$$\|I_2(t)\|_{H^2} \leq C\int_t^T \|R_\lambda(\tau)\|_{H^2}\|v(\tau)\|_{H^2}^2 d\tau$$
$$\leq C\left\{\int_t^T \frac{1+\lambda^2}{(\lambda(T-\tau))^2}e^{-\frac{2\delta}{3\lambda(T-\tau)}}d\tau\right\}\|v\|_{X_T}^2$$
$$\leq CT(1+\lambda^2)e^{-\delta/2\lambda(T-t)}\|v\|_{X_T}^2$$

Hence

$$\|I_2\|_{X_T} \leq CT^{1/2}\|v\|_{X_T}^2\,, \tag{3.8}$$

provided $\lambda^2 T^{1/2} \leq 1$, $\lambda \geq 1$.

Estimate for $I_3(t)$. Recall that $Q_3(v) = -|v|^2 v$. We then have that

$$\|I_3(t)\|_{H^2} \leq C\int_t^T \|v(\tau)\|_{H^2}^3 d\tau$$
$$\leq \left\{\int_t^T e^{-3\frac{\delta}{3\lambda(T-\tau)}}d\tau\right\}\|v\|_{X_T}^3$$
$$\leq CTe^{-\frac{\delta}{3\lambda(T-t)}}\|v\|_{X_T}^3$$

Therefore

$$\|I_3\|_{X_T} \leq CT\|v\|_{X_T}^3. \tag{3.9}$$

In view of (3.4), (3.6), (3.8), (3.9), it is a standard procedure to solve the integral formulation of (3.1) by the contraction mapping principle in X_T for $\lambda \gg 1$, $T \ll 1$ and $\lambda^2 T^{1/2} \leq 1$. This completes the proof of Lemma 3.1. □

4 Some Properties of the Bessel Functions

In this section, we state some results concerning the Bessel functions of integer order. All these results are essentially well known in the literature (see [W]) but for the sake of completeness we shall give the needed proofs in an appendix of the present paper. Let n be a positive integer. The Bessel function of order n is defined as follows

$$J_n(z) = \Big(\frac{z}{2}\Big)^n \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{z}{2})^{2k}}{k!(n+k)!}$$

(see [S], [W]). The function $J_n(z)$ is analytic in $\mathbb{C}$ and satisfies the following second order ordinary differential equation

$$J_n''(z) + \tfrac{1}{z} J_n'(z) + \big(1 - \tfrac{n^2}{z^2}\big) J_n(z) = 0 \,. \tag{4.1}$$

Now we give estimates for $J_n(z)$ in different regimes for z.

Lemma 4.1. *Let β_1 and β_2 be two positive real numbers and $\gamma \in]0, 1/2[$. Then for*

$$\rho \in [1 - \beta_1 n^{-\gamma}, 1 + \beta_2 n^{-2/3},] \,, \quad n \gg 1 \,,$$

one has

$$J_n(n\rho) = \big(\tfrac{2}{n\rho}\big)^{1/3} \mathrm{Ai}\big(n^{2/3}(1-\rho)(2/\rho)^{1/3}\big) + r_n(\rho) \,,$$

where Ai *is the classical Airy function (see [H, p. 213, Definition 7.6.8]) and the following estimate holds uniformly in ρ,*

$$|r_n(\rho)| \leq C n^{-\gamma_1} \,, \quad \gamma_1 < 3\gamma - 1 \,.$$

Lemma 4.2. *Let $\beta > 0$ and $\gamma \in]0, 1/2[$. Then for any $k > 0$ there exists a positive constant c_k such that for $\rho \in [0, 1 - \beta n^{-\gamma}]$, $n \gg 1$, one has $|J_n(n\rho)| \leq c_k n^{-k}$.*

Let $z_{n1} < z_{n2} < z_{n3} < \ldots$ be the sequence of the positive zeros of $J_n(z)$. It is not difficult to see from equation (4.1) that $n < z_{n1}$ and that the zeros are simple. In the next lemma, we give more precise information about the the localization of z_{n1} and the gap between z_{n1} and z_{n2}.

Lemma 4.3. *Let $n \geq 1$. Then there exists a constant α, independent of n, such that $z_{n1} = n + \alpha n^{1/3} + \mathcal{O}(n^{\lambda})$, for any $\lambda > 1/6$. Moreover $|z_{n2} - z_{n1}| \geq C n^{1/3}$.*

5 Proof of Theorem 2

Let Δ_D be the Laplace operator on D with Dirichlet boundary conditions. By introducing polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ and using that

$$\partial_{x_1}^2 + \partial_{x_2}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\,, \tag{5.1}$$

it can be seen that $f_{nk}(r,\theta) := J_n(z_{nk}r)\exp(in\theta)$ is an orthogonal basis of $L^2(D)$ of eigenfunctions of $-\Delta_D$ with corresponding eigenvalue z_{nk}^2. For $s \in [0,1/2[$, we can express the norm in $H^s(D)$ in terms of f_{nk} and z_{nk}. Let $u \in L^2(D)$. Then there exists $c_{nk} \in \mathbb{C}$ such that $u = \sum_{n,k} c_{nk} f_{nk}$ and $u \in H^s(\Omega)$ if and only if

$$\|u\|_{H^s(D)} \approx \Big\{ \sum_{n,k} z_{nk}^{2s} |c_{nk}|^2 \Big\}^{1/2} < \infty\,. \tag{5.2}$$

Further, if $s \geq 1/2$, the space defined by (5.2) will be denoted by $H_0^s(D)$. By complex interpolation, elements of $H_0^{1/2}(D)$ can be extended by 0 to the whole of $\mathbb{R}^2$ as elements of $H^{1/2}(\mathbb{R}^2)$, and therefore, by the Sobolev embedding, $H_0^{1/2}(D) \subset L^4(D)$.

Moreover, given a positive integer n, for $u \in L^2$ one has

$$\big(\forall (x,\theta) \in D\times\mathbb{R},\ u(e^{i\theta}x) = e^{in\theta}u(x)\big) \Leftrightarrow u = \sum_{k\geq 1} c_k f_{nk} \tag{5.3}$$

Next, using the results of the previous section, we give the asymptotics for the L^p norms of f_{n1}.

Lemma 5.1. *Let $p \in [2,+\infty]$. Then $\|f_{n1}\|_{L^p(D)} \approx n^{-\frac{2}{3p}-\frac{1}{3}}$.*

Remark 5.2. As a consequence of Lemma 5.1, notice that usual Strichartz estimates on the disc fail. More precisely, by testing on eigenfunctions f_{n1}, one can show that for $s < 1/6$

$$\|e^{it\Delta_D}u_0\|_{L^4([0,1]\times D)} \leq C\|u_0\|_{H^s(D)}$$

cannot hold.

Proof. Write

$$\begin{aligned} \|f_{n1}\|_{L^p(D)}^p &= \int_0^1\int_0^{2\pi} \big|f_{n1}(r,\theta)\big|^p r\,d\theta\,dr \\ &= 2\pi\int_0^1 \big|J_n(z_{n1}r)\big|^p r\,dr\,. \end{aligned}$$

Next, we perform a change of variables $\rho = \frac{z_{n1}}{n}r$. Then

$$\|f_{n1}\|_{L^p(D)}^p = 2\pi\left(\tfrac{n}{z_{n1}}\right)^2 \int_0^{z_{n1}/n} \big|J_n(n\rho)\big|^p \rho\,d\rho\,.$$

In view of Lemma 4.3, we can write

$$\tfrac{z_{n1}}{n} = 1 + \alpha n^{-2/3} + \mathcal{O}(n^{-\lambda}),$$

for any $\lambda < 5/6$ and for some $\alpha > 0$. Thus, we can write

$$\|f_{n1}\|^p_{L^p(D)} \approx \int_0^{1-n^{-\gamma}} \big|J_n(n\rho)\big|^p \rho d\rho + \int_{1-n^{-\gamma}}^{1+\alpha n^{-2/3}+\mathcal{O}(n^{-\lambda})} \big|J_n(n\rho)\big|^p \rho d\rho,$$

where $\gamma < 1/2$. A use of Lemma 4.2 yields that for any $k > 0$ there exists c_k such that

$$\int_0^{1-n^{-\gamma}} \big|J_n(n\rho)\big|^p \rho d\rho \le c_k n^{-k}. \tag{5.4}$$

Then, with the notation of Lemma 4.1, we have

$$\int_{1-n^{-\gamma}}^{1+\alpha n^{-2/3}+\mathcal{O}(n^{-\lambda})} \big|r_n(\rho)\big|^p \rho d\rho \le C n^{-\gamma} n^{-p\gamma_1},$$

where $\gamma_1 < 3\gamma - 1$. Clearly $n^{-\gamma}n^{-p\gamma_1} \ll n^{-2/3}n^{-p/3}$ by taking $\gamma < 1/2$ sufficiently close to $1/2$ and γ_1 sufficiently close to $3\gamma - 1 \sim 1/2$. On the other hand a change of variables $\rho - 1 = -n^{-2/3}y$ leads to

$$\int_{1-n^{-\gamma}}^{1+\alpha n^{-2/3}+\mathcal{O}(n^{-\lambda})} \Big(\tfrac{2}{n\rho}\Big)^{p/3} \big|\mathrm{Ai}(n^{2/3}(1-\rho)(2/\rho)^{1/3})\big|^p \rho d\rho = c(n) n^{-2/3} n^{-p/3},$$

where

$$c(n) = 2^{\frac{p}{3}} \int_{-n^{\frac{2}{3}-\gamma}}^{\alpha+\mathcal{O}(n^{\frac{2}{3}-\lambda})} \frac{1}{(1+n^{-\frac{2}{3}}y)^{\frac{p}{3}}} \bigg|\mathrm{Ai}\bigg(\Big(\frac{2}{(1+n^{-\frac{2}{3}}y)^{\frac{1}{3}}}\Big)^{\frac{1}{3}} y\bigg)\bigg|^p (1+n^{-\frac{2}{3}}y) dy.$$

The proof of Lemma 5.1 is completed by observing that

$$c(n) \to 2^{p/3} \int_{-\infty}^{\alpha} \big|\mathrm{Ai}(2^{1/3}y)\big|^p dy.$$

using the Lebesgue theorem and the well-known bounds on the Airy function (see e.g. [H, §7.6, p. 215]). □

We now turn to the proof of Theorem 2. We can suppose that $\kappa > 0$ since the case $\kappa < 0$ can be treated in the same fashion. Set $\phi_n = n^{\frac{2}{3}-s} f_{n1}$. Due to Lemma 5.1 and Lemma 4.3, we have that $\|\phi_n\|_{H^s} \approx 1$. Set $\omega_n = \|\phi_n\|^4_{L^4}/\|\phi_n\|^2_{L^2}$. Another use of Lemma 5.1 gives $\omega_n \approx n^{\frac{2}{3}-2s}$. Since $\|\phi_n\|_{L^2} \approx n^{-s}$, Lemma 2.3 ensures that if $n \gg 1$ then there exists a unique global solution of (1.1) with Cauchy data $\kappa\phi_n(x)$, $x \in D$. Denote it by $u_n(t,x)$. Due to the L^2 conservation law, we obtain that $\|u_n(t,\cdot)\|_{L^2} \le C\kappa n^{-s}$. Then the energy conservation yields the bound $\|u_n(t,\cdot)\|_{H^1_0} \le C\kappa n^{1-s}$. Therefore, by interpolation, $\|u_n(t,\cdot)\|_{H^s(D)} \le C\kappa$.

According to (5.3), we can clearly write $|\phi_n|^2\phi_n = \omega_n\phi_n + r_n$, where $r_n = \sum_{k\geq 2} c_k f_{nk}$. Set

$$u_n(t,x) = \kappa \exp\big(-it(z_{n1}^2 - \kappa^2\omega_n)\big)\big(\phi_n(x) + w_n(t,x)\big)\,. \tag{5.5}$$

Then the equation satisfied by w_n is

$$\begin{cases} (i\partial_t+\Delta+z_{n1}^2-\kappa^2\omega_n)w_n = -\kappa^2\big(|\phi_n+w_n|^2(\phi_n+w_n)-|\phi_n|^2\phi_n+r_n\big) \\ \qquad w_n(0,x) = 0\,, \quad x \in D\,. \end{cases} \tag{5.6}$$

According to gauge invariance of NLS and (5.3) we now further decompose $w_n(t,x)$ as $w_n(t,x) = \lambda_n(t)\phi_n(x) + q_n(t,x)$, where

$$q_n(t,x) = \sum_{k\geq 2} c_k(t) f_{nk}(x)\,, \quad t\in\mathbb{R}\,,\ x\in D\,.$$

Since $w_n(0,x) = 0$, we have $\lambda_n(0) = 0$ and $q_n(0,x) = 0$. The L^2 conservation law can now be rewritten as follows,

$$\big|1+\lambda_n(t)\big|^2\|\phi_n\|_{L^2}^2 + \|q_n(t)\|_{L^2}^2 = \|\phi_n\|_{L^2}^2\,, \tag{5.7}$$

while the energy conservation becomes

$$\big|1+\lambda_n(t)\big|^2\|\nabla\phi_n\|_{L^2}^2+\|\nabla q_n(t)\|_{L^2}^2-\tfrac{1}{2\kappa^2}\|u_n(t)\|_{L^4}^4 = \|\nabla\phi_n\|_{L^2}^2-\tfrac{\kappa^2}{2}\|\phi_n\|_{L^4}^4\,. \tag{5.8}$$

Since ϕ_n is an eigenfunction of Δ_D, we obtain that $\|\nabla\phi_n\|_{L^2}^2 = z_{n1}^2\|\phi_n\|_{L^2}^2$. We now multiply (5.7) with $-z_{n1}^2$ and add it to (5.8). This yields

$$\begin{aligned}\big\|\nabla q_n(t)\big\|_{L^2}^2 - z_{n1}^2\big\|q_n(t)\big\|_{L^2}^2 &= \tfrac{1}{2\kappa^2}\big\|u_n(t)\big\|_{L^4}^4 - \tfrac{\kappa^2}{2}\|\phi_n\|_{L^4}^4 \\ &\leq \tfrac{1}{2\kappa^2}\big\|u_n(t)\big\|_{L^4}^4\,.\end{aligned}$$

In view of (5.7), we readily obtain that $|\lambda_n(t)| \leq 2$. Therefore we have the pointwise estimate

$$\big|u_n(t,x)\big|^4 \leq C\kappa^4\big(|\phi_n(x)|^4 + |q_n(t,x)|^4\big)\,, \quad t\in\mathbb{R}\,,\ x\in D\,. \tag{5.9}$$

Due to Lemma 4.3, For $k\geq 2$ one has

$$z_{nk}^2 - z_{n1}^2 \geq Cn^{1/3} z_{nk}\,. \tag{5.10}$$

Moreover,

$$\|\phi_n\|_{L^4}^4 = \omega_n\|\phi_n\|_{L^2}^2 \approx n^{\frac{2}{3}-4s}. \tag{5.11}$$

Now using (5.9), (5.10), (5.11), the definition of Sobolev spaces in terms of the spectral decomposition with respect to Δ_D and the Sobolev embedding $H_0^{1/2}(D)\subset L^4(D)$ give the estimates

$$\begin{aligned} n^{1/3}\big\|q_n(t,\cdot)\big\|_{H_0^{1/2}(D)}^2 &\leq C\kappa^2 n^{\frac{2}{3}-4s} + C\kappa^2\big\|q_n(t,\cdot)\big\|_{L^4(D)}^4 \\ &\leq C\kappa^2 n^{\frac{2}{3}-4s} + C\kappa^2\big\|q_n(t,\cdot)\big\|_{H_0^{1/2}(D)}^4\,.\end{aligned}$$

Since $q_n(0,\cdot)=0$, by a classical bootstrap argument we infer

$$\big\|q_n(t,\cdot)\big\|_{H_0^{1/2}(D)} \le C\kappa n^{\frac{1}{6}-2s}. \tag{5.12}$$

Using once again the Sobolev embedding $H_0^{1/2}(D)\subset L^4(D)$, we obtain that

$$\big\|q_n(t,\cdot)\big\|_{L^4(D)}^4 \le C\kappa^4 n^{\frac{2}{3}-8s}.$$

Coming back to (5.12) we also get the bound

$$\big\|q_n(t,\cdot)\big\|_{L^2(D)} \le C\kappa n^{-\frac{1}{3}-2s}. \tag{5.13}$$

An interpolation between (5.12) and (5.13) yields the estimate

$$\big\|q_n(t,\cdot)\big\|_{H^s(D)} \le C\kappa n^{-\frac{1}{3}-s}. \tag{5.14}$$

In the next lemma, we prove the needed bound for $|\lambda_n(t)|$.

Lemma 5.3. *There exist a constant $C>0$ and, for any $T>0$, a positive integer $N(T)$ such that, for any $n\ge N(T)$,*

$$\sup_{t\in[0,T]}\big|\lambda_n(t)\big| \le CTn^{\frac{1}{3}-3s}.$$

Proof of Lemma 5.3. We project the equation (5.6) on $\phi_n(x)$. Using (5.5), we obtain the following equation for $\lambda_n(t)$

$$\begin{cases} (i\partial_t-\omega_n\kappa^2)\lambda_n = -\frac{\kappa^2}{\|\phi_n\|_{L^2}^2}\big((|\phi_n+w_n|^2(\phi_n+w_n)\,|\,\phi_n)-(|\phi_n|^2\phi_n\,|\,\phi_n)\big)\\ \lambda_n(0)=0\,. \end{cases}$$

An easy computation shows that we can rewrite the right-hand side of the equation for λ_n as follows

$$-\frac{\kappa^2}{\|\phi_n\|_{L^2}^2}\bigg\{\int\big(2|\phi_n|^2w_n+\phi_n^2\bar w_n\big)\bar\phi_n + \int\big(2\,\mathrm{Re}(\bar\phi_n w_n)w_n\bar\phi_n+|w_n|^2|\phi_n|^2+|w_n|^2w_n\bar\phi_n\big)\bigg\}.$$

Using that $w_n(t,x)=\lambda_n(t)\phi_n(x)+q_n(t,x)$ and $|\phi_n(x)|^2\phi_n(x)=\omega_n\phi_n(x)+r_n(t,x)$, we can split the above expression into the sum of the following three terms:

$$L_1 = -(2\omega_n\kappa^2\lambda_n+\omega_n\kappa^2\bar\lambda_n)\,,$$

$$L_2 = \frac{\kappa^2}{\|\phi_n\|_{L^2}^2}\mathcal{O}\bigg(|\lambda_n|^2\int|\phi_n|^4+|\lambda_n|^3\int|\phi_n|^4\bigg),$$

$$L_3 = \frac{\kappa^2}{\|\phi_n\|_{L^2}^2}\mathcal{O}\bigg(\int|q_n|^3|\phi_n|+\int|q_n|^2|\phi_n|^2+|(q_n|r_n)|\bigg).$$

We now study the decay of the source term, L_3. Write

$$\frac{\int|q_n|^3|\phi_n|}{\|\phi_n\|_{L^2}^2} \le Cn^{2s}\||q_n|^3\|_{L^{4/3}}\|\phi_n\|_{L^4}$$

$$
\begin{aligned}
&\leq Cn^{2s}\|q_n\|_{L^4}^3\|\phi_n\|_{L^4}\\
&\leq Cn^{2s}\,n^{\frac12-6s}\,n^{\frac16-s}\\
&= Cn^{\frac23-5s}.
\end{aligned}
$$

Further we have

$$
\begin{aligned}
\frac{\int|q_n|^2|\phi_n|^2}{\|\phi_n\|_{L^2}^2} &\leq Cn^{2s}\|q_n\|_{L^2}^2\|\phi_n\|_{L^\infty}^2\\
&\leq Cn^{2s}\,n^{-\frac23-4s}\,n^{\frac23-2s}\\
&= Cn^{-4s}
\end{aligned}
$$

and moreover

$$
\begin{aligned}
\frac{|(q_n|r_n)|}{\|\phi_n\|_{L^2}^2} &\leq Cn^{2s}\|q_n\|_{L^2}\|r_n\|_{L^2}\\
&\leq Cn^{2s}\,n^{-\frac13-2s}\,\|\phi_n\|_{L^6}^3\\
&\leq Cn^{2s}\,n^{-\frac13-2s}\,n^{\frac23-3s}\\
&= Cn^{\frac13-3s}.
\end{aligned}
$$

For $s > 1/6$, the source term is bounded by $n^{\frac13-3s}$ and consequently the equation for $\lambda_n(t)$ can be written as

$$
\begin{cases} i\partial_t\lambda_n = -2\omega_n\kappa^2\,\mathrm{Re}(\lambda_n) + \mathcal{O}\big(\omega_n|\lambda_n|^2 + \omega_n|\lambda_n|^3 + n^{\frac13-3s}\big)\,, \\ \lambda_n(0) = 0\,. \end{cases} \tag{5.15}
$$

The L^2 conservation law yields

$$
1 - \big|1+\lambda_n(t)\big|^2 = \frac{\|q_n(t)\|_{L^2}^2}{\|\phi_n\|_{L^2}^2} = \mathcal{O}(n^{-\frac23-2s})
$$

and hence (5.15) now becomes

$$
\begin{cases} i\partial_t\lambda_n = \mathcal{O}\big(\omega_n|\lambda_n|^2 + \omega_n|\lambda_n|^3 + n^{\frac13-3s}\big)\,, \\ \lambda_n(0) = 0\,. \end{cases} \tag{5.16}
$$

Set $\gamma_n := n^{\frac13-3s}$ and $M_n(T) := \sup_{t\in[0,T]}\gamma_n^{-1}|\lambda_n(t)|$. Then by integrating (5.16), we obtain the estimate

$$
\begin{aligned}
M_n(T) &\leq CT\big(1 + \gamma_n\omega_n[M_n(T)]^2 + \gamma_n^2\omega_n[M_n(T)]^3\big)\\
&\leq CT\big(1 + n^{1-5s}([M_n(T)]^2 + n^{\frac13-3s}[M_n(T)]^3)\big)\,.
\end{aligned}
$$

Since $s > 1/5$ and $M(0) = 0$, we obtain that $M(T) \leq CT$ for n large enough with respect to T. Therefore

$$
|\lambda_n(t)| \leq CTn^{\frac13-3s}.
$$

This completes the proof of Lemma 5.3. □

It remains to prove that the flow map fails to be uniformly continuous on any ball of H^s. For that purpose, we fix $\kappa > 0$ and a sequence $\{\kappa_n\}$ tending to κ. Denote by $u_{\kappa,n}$ the solution of (1.1) with Cauchy data $\kappa\phi_n$ and by $u_{\kappa_n,n}$ the solution of (1.1) with data $\kappa_n\phi_n$. Clearly $(\kappa - \kappa_n)\phi_n$ tends to zero in $H^s(D)$ as n tends to infinity. Hence, if we suppose that the Cauchy problem associated to (1.1) (with $\Omega = D$) is locally well-posed with uniformly continuous flow map then the sequence

$$\big\|u_{\kappa,n}(t,\cdot) - u_{\kappa_n,n}(t,\cdot)\big\|_{H^s(D)}$$

should tend to zero as n tends to infinity. But due to the above considerations one easily gets the estimate

$$\big\|u_{\kappa,n}(t,\cdot) - u_{\kappa_n,n}(t,\cdot)\big\|_{H^s(D)} \geq C\big|\exp(it\omega_n(\kappa^2 - \kappa_n^2)) - 1\big| - Cn^{-\delta},$$

where $\delta > 0$. Therefore one should have

$$\lim_{n\to+\infty} \big|\exp(it\omega_n(\kappa^2 - \kappa_n^2)) - 1\big| = 0\,. \tag{5.17}$$

Since $\omega_n \approx n^{\frac{2}{3}-2s} \to \infty$, we easily obtain that with a suitable choice of $\{\kappa_n\}$, (5.17) fails for every $t \neq 0$. This completes the proof of Theorem 2. □

REMARK 5.4. Similarly to [BGT2], if one is interested only on the ill-posedness result, an adaptation of the proof of Theorem 2, involving the length of the time interval on which our ansatz is valid, can provide ill-posedness for s in the whole interval $[0, 1/3[$.

6 Appendix

In this appendix, we give the proofs of Lemmas 4.1, 4.2, 4.3.

Proof of Lemma 4.1. For a fixed $z \geq 0$, the function $J_n(z)$ can be seen as the n-th coefficient in the Fourier series expansion of the periodic smooth function $\exp(iz\sin\theta)$. Therefore

$$J_n(n\rho) = \frac{1}{2\pi}\int_{S^1} e^{in\phi_\rho(\theta)}d\theta\,, \tag{6.1}$$

where $\phi_\rho(\theta) := \rho\sin\theta - \theta$. Take a real number $\lambda < \gamma/2$. Let $\chi \in C_0^\infty(\mathbb{R})$ be real valued function such that $\chi(x) = 1$ for $|x| < 1$ and $\chi(x) = 0$ for $|x| > 2$. Then for $n \gg 1$, we define $\chi_n \in C^\infty(S^1)$ as $\chi_n(\theta) = \chi(n^\lambda\theta)$. Set $J_n(n\rho) = J_n^1(n\rho) + J_n^2(n\rho)$, where $J_n^1(n\rho) = (2\pi)^{-1}\int_{-\pi}^{\pi}\chi_n(\theta)\exp(in\phi_\rho(\theta))d\theta$. Since $\lambda < \gamma/2$, we can write, for $\theta \in \operatorname{supp}(1-\chi_n)$ and $|1-\rho| \leq \beta n^{-\gamma}$,

$$|\phi_\rho'(\theta)| = \big|\rho(\cos\theta - 1) + \rho - 1\big|$$

$$\geq 2\rho\sin^2\tfrac{\theta}{2} - |1-\rho|$$
$$\geq Cn^{-2\lambda}.$$

Note that $|\chi_n^{(j)}(\theta)| \leq Cn^{\lambda j}$, $j \in \mathbb{N}$. Hence the non-stationary phase estimate (see [H, p. 216, Theorem 7.7.1]) applied to $J_n^2(n\rho)$ gives $|J_n^2(n\rho)| \leq c_k n^{(4\lambda-1)k}$ for any k. Hence for $\lambda < 1/4$, we obtain that $|J_n^2(n\rho)|$ is rapidly decreasing for $n \gg 1$ uniformly on ρ. It remains to estimate $J_n^1(n\rho)$. Since for $|\theta| \leq Cn^{-\lambda}$, we have $\sin\theta = \theta - \theta^3/6 + r_n(\theta)$ with $|r_n(\theta)| \leq Cn^{-5\lambda}$, we obtain that

$$J_n^1(n\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\chi_n(\theta)e^{in((\rho-1)\theta-\frac{\rho\theta^3}{6})}d\theta + \tilde{r}_n\,, \tag{6.2}$$

where $|\tilde{r}_n| \leq Cn|r_n(\theta)|n^{-\lambda} \leq Cn^{1-6\lambda}$. Therefore $|\tilde{r}_n| \leq Cn^{-\gamma_1}$ for any $\gamma_1 < 3\gamma - 1$ by choosing $\lambda < 1/4$ sufficiently close to $\gamma/2$. Hence one only needs to evaluate the first term in the right-hand side of (6.2). Writing $\chi_n(\theta) = (\chi_n(\theta)-1)+1$, and observing that the expression $(\rho-1)\theta-\theta^3/6$, has no critical points θ such that $|\theta| \geq n^{-\lambda}$, another use of the non-stationary phase estimate gives for any $k \in \mathbb{N}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\chi_n(\theta)e^{in((\rho-1)\theta-\frac{\rho\theta^3}{6})}d\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{in((\rho-1)\theta-\rho\theta^3/6)}d\theta + \mathcal{O}(n^{-k})\,.$$

Now a simple change of variables provides the formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{in((\rho-1)\theta-\rho\theta^3/6)}d\theta = \left(\frac{2}{n\rho}\right)^{1/3}\mathrm{Ai}\big(n^{2/3}(1-\rho)(2/\rho)^{1/3}\big), \tag{6.3}$$

which completes the proof of Lemma 4.1. □

Proof of Lemma 4.2. Write $|\phi_\rho'(\theta)| = |1-\rho+\rho(1-\cos\theta)|$. Since $1-\rho \geq \beta n^{-\gamma}$, we have that $|\phi_\rho'(\theta)| \geq \beta n^{-\gamma}$. Hence an application of the non-stationary phase estimate to $J_n(n\rho)$ yields $|J_n(n\rho)| \leq c_k n^{-(1-2\gamma)k}$ which completes the proof of Lemma 4.2. □

Proof of Lemma 4.3. We first observe that due to Lemma 4.1, there is no zero of $J_n(n\rho)$ for $0 \leq \rho - 1 \ll n^{-2/3}$. Denote by $\tilde{\rho}_{n1} < \tilde{\rho}_{n2} < \cdots$ the zeros of

$$\mathrm{Ai}\big(n^{2/3}(1-\rho)(2/\rho)^{1/3}\big)\,.$$

Denote by z_k, $k = 1, 2$ the first zeros of $\mathrm{Ai}(-z)$. Then a straightforward computation shows that for $k = 1, 2$ one has

$$\tilde{\rho}_{nk} = 1 + \tfrac{z_k}{2^{1/3}}\,n^{-2/3} + r_{nk}\,, \tag{6.4}$$

where $|r_{nk}| \leq Cn^{-4/3}$. Let now $\rho_{n1} < \rho_{n2} < \cdots$ be the zeros of $J_n(n\rho)$, bigger than 1. Fix $\delta \in]0, 1/6[$. Let $\xi_{nk}^{\pm}$ solve the equations

$$n^{2/3}(1-\xi_{nk}^{\pm})(2/\xi_{nk}^{\pm})^{1/3} = -z_k \mp n^{-\delta}.$$

Note that the restriction on δ is imposed in order to keep the first term in the expansion of $J_n(n\rho)$ from Lemma 4.1 leading near $\xi^{\pm}_{nk}$. For $k = 1, 2$ one has

$$\xi^{\pm}_{nk} = 1 + \tfrac{z_k}{2^{1/3}}\, n^{-2/3} + \tilde{r}_{nk}$$

where $|\tilde{r}_{nk}| \le C n^{-\delta - \frac{2}{3}}$. For $n \gg 1$, the numbers $J_n(n\xi^{+}_{nk})$ and $J_n(n\xi^{-}_{nk})$ have different signs and hence there is at least one zero of $J_n(n\rho)$ between ξ^{-}_{nk} and ξ^{+}_{nk}. Using the method of proof of Lemma 4.1, we obtain the following representation for $J'_n(n\rho)$ showing that $J'_n(n\rho)$ has a constant sign on $[\xi^{-}_{nk}, \xi^{+}_{nk}]$.

$$J'_n(n\rho) = \Big(\tfrac{2}{n\rho}\Big)^{2/3} \mathrm{Ai}'\big(n^{2/3}(1-\rho)(2/\rho)^{2/3}\big) + \mathcal{O}(n^{-\gamma})\,, \quad \gamma < \tfrac{3}{4}$$

Hence, we conclude that the only zero of $J_n(n\rho)$ between ξ^{-}_{nk} and ξ^{+}_{nk} is ρ_{nk} and therefore

$$|\rho_{nk} - \tilde{\rho}_{nk}| \le C n^{-\delta - \frac{2}{3}}\,, \quad k = 1, 2\,. \tag{6.5}$$

Since $z_{nk} = n\rho_{nk}$, the proof of Lemma 4.3 is achieved thanks to (6.4) and (6.5) by choosing δ close to $1/6$. □

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