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PERIODIC SOLUTIONS OF THE ABELIAN HIGGS MODEL AND RIGID ROTATION OF VORTICES

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Abstract

The metric and potential energy on the reduced moduli space of self-dual vortices in the Abelian Higgs model on S^2 are computed in a certain limit, first identified by Bradlow. In this limit it is proved that the Higgs field is asymptotic to a standard holomorphic section. These results are then used to prove a theorem asserting the existence of time-periodic solutions of the Abelian Higgs model on $\mathbb{R} \times S^2$ which represent two vortices in rigid rotation about one another. The theorem answers affirmatively the question, raised by Jaffe and Taubes, of whether a balance between the inter-vortex attraction and the centrifugal repulsion provides for the existence of such solutions (as it does in the classical two body problem for point particles.) The starting point of the analysis is the adiabatic limit system, i.e. the Hamiltonian system defined by restricting the Abelian Higgs model to the moduli space of self-dual vortices. The Hamiltonian consists of a potential energy term and kinetic energy term which is given by the metric on the moduli space induced from L^2. It is shown under two assumptions on the metric and potential energy that the adiabatic limit system admits periodic solutions of the required type. Periodic solutions to the full system are then obtained by an application of the implicit function theorem. Explicit examples where the assumptions on the adiabatic limit system hold are provided by the computations of the metric and potential in the Bradlow limit.

1 Introduction

This paper is concerned with Abelian Higgs, or Ginzburg-Landau, vortices in the Abelian Higgs model on $\mathbb{R} \times S^2$; this is a hyperbolic system of equations for a complex function, or section, Φ (the Higgs field) and a connection (electromagnetic potential) A, with associated curvature or magnetic field $B_A = dA$. The main result is the construction of time periodic solutions of this system in which two vortices are in rigid rotation about one another (Theorem III and Theorem 3.4.1 in section 3.4). These solutions are obtained by continuation of periodic solutions to the adiabatic limit system (section 3.3). Information about this system is obtained from explicit algebraic formulae for the L^2 metric on the moduli space and for the interaction potential energy between two vortices in a certain limit (see Theorems I and II and sections 2.2, 2.3). In this introductory section the setting for the problem will be explained and the main theorems stated. Additional notation and background information is given in the appendix. The theorems which compute the metric and potential energy are proved in section 2, while the existence theorem for periodic solutions is proved in section 3.

A vortex is usually defined as a finite action critical point of the static energy:

$$V_{\lambda,\tau}(A,\Phi) = \frac{1}{2}\int_\Sigma \left(B_A^2 + |\nabla_A \Phi|^2 + \frac{\lambda}{4}(\tau - |\Phi|^2)^2 \right) d\mu_g \,, \tag{1.1}$$

where Σ is either an open set in $\mathbb{R}^2$ or a Riemann surface. Φ is a section of a one dimensional complex vector bundle E and A is a connection on E. Let $\mathcal{A}^k, H^k(E)$ be completions of the space of connections and sections with respect to the Sobolev norm $\|\cdot\|_k$ (see section 4.1); then $V_{\lambda,\tau}$ defines a smooth function:

$$V_{\lambda,\tau} : \mathcal{A}^k \times H^k(E) \to \mathbb{R}$$

for all integers $k \geq 1$. The gauge group, which acts on pairs (A,Φ) according to (4.1), leaves $V_{\lambda,\tau}$ invariant.

The case $\lambda = 1$, called the *self-dual* or *Bogomolny* limit, is particularly important. Computational studies ([JR]) indicated that when $\lambda = 1$ the vortices "do not interact", in the sense that there is no net force between them. A precise mathematical justification of this notion was provided by Taubes' existence theorem (see [JaT]), which proved the existence of a smooth manifold $\mathcal{M}$ of (gauge equivalence classes of) critical points representing nonlinear superpositions of N symmetric vortices. When $\lambda = 1$ if (A,Φ) is a minimiser then Φ has isolated zeros $\{Z_j\}_{j=1}^N$ where N can be any integer; the zeros of Φ are often identified with the centres of the vortices. The unordered set of points $\{Z_j\}$ may be chosen arbitrarily, and determine the solutions up to gauge equivalence, so that $\mathcal{M}$ may be identified with the symmetric product of N copies of the plane. Following Taubes' theorem the existence of these large families of minimisers was shown also to hold when Σ is a compact Riemann surface:

Theorem ([B],[G]). *For $\lambda = 1$ and $\tau \times$ Area$(\Sigma) > 4\pi N$ the minimum value of $V_{\lambda,\tau}$ on the space of smooth connections A and smooth sections Φ is attained on a non-empty set $\mathcal{S}$ whose quotient by the group of gauge* *transformations, $\mathcal{M}$, can be identified with the symmetric N-fold product of Σ.*

In the Sobolev setting defined in section 4.1 the manifold structure of $\mathcal{M}$ can be described in the standard way in terms of the elliptic operator defined by the Hessian of $V_{\lambda,\tau}$ (section 4.2). More recent studies have provided formulae for the volume of the moduli space for arbitrary Σ ([M1], [MN]). Bradlow found a new phenomenon in the compact case: a necessary and sufficient condition for existence of the multi-vortices with N zeros was that $\tau \times \mathrm{Area}(\Sigma) > 4\pi N$. We shall refer to the limit $\tau \to \tau_{cr} = 4\pi N/\mathrm{Area}(\Sigma)$ as the *Bradlow limit*. *Throughout this paper $N = 2$.* The behaviour of the volume of the moduli space in the Bradlow limit on higher genus surfaces is discussed in [N].

There are various dynamical models associated to the functional (1.1). In the present paper the Abelian Higgs model is the object of study. Write $\mathcal{A}$ for the smooth connections on E. The Abelian Higgs model is (subject to the usual proviso about gauge invariance) a hyperbolic system of equations for a map $t \mapsto (A(t), \Phi(t)) \in \mathcal{A} \times C^\infty(E)$; thus, at each time t, $(A(t), \Phi(t))$ are respectively, a connection and a section of E. The equations can be expressed, writing $\dot{}$ for $\frac{d}{dt}$, as

$$(\ddot{A}, \ddot{\Phi}) + V'_{\lambda,\tau}(A, \phi) = 0 \tag{AH}$$

together with the constraint equation

$$d^* \dot{A} + (i\Phi, \dot{\Phi}) = 0\,, \tag{C}$$

which is preserved by the evolution. This is the form of the equations in temporal gauge; they are written out more explicitly and generally below in (3.1).

Definition 1.1. *A T-periodic solution to* (AH) *consists of a smooth map $t \mapsto (A(t), \Phi(t)) \in \mathcal{A} \times C^\infty(E)$ which satisfies* (AH) *and with the property that for each t the configuration $(A(t+T), \Phi(t+T))$ is gauge equivalent to $(A(t), \Phi(t))$, i.e. there exists a smooth function χ such that $A(t+T) = A(t) + d\chi$ and $\Phi(t+T) = \Phi(t)e^{i\chi}$.*

It was suspected from numerical evidence ([JR]) that the vortices attract or repel according to whether λ is less or bigger than one. Given two points Z_1, Z_2 let (A, Φ) be a minimising multi-vortex in which Z_1, Z_2 are the zeros of Φ. In [St1] it was proved (in the case $\Sigma = \mathbb{R}^2$) that for λ close to 1 the integral

$$v_{\lambda,\tau}(Z_1, Z_2) = \frac{(\lambda - 1)}{8} \int_\Sigma \left(\tau - |\Phi|^2\right)^2 d\mu_g \tag{1.2}$$

determines a smooth function on $\mathcal{M}$ which acts as an interaction *potential energy* between vortices. The precise meaning attached to this latter statement in [St1] is as follows: the dynamics of vortices for $\lambda \approx 1$ is proved to be well approximated by a finite dimensional Hamiltonian system on $\mathcal{M}$ with potential energy term given by (1.2). This is called the adiabatic limit system; it is discussed further below and in section 3.3. This type of approximation had been conjectured by Manton to provide a description of the dynamics of BPS monopoles (see [M2], [U], [St2] for further information). Numerical computation of (1.2) ([Sh]) indicated that $v_{\lambda,\tau}$ is monotonically decreasing or increasing depending upon whether λ is greater or less than one. Theorem I provides an explicit formula which proves this for the case when Σ is a sphere in the Bradlow limit. Theorems I and II allow a proof, given in section 3.3, that the adiabatic limit problem admits periodic solutions which represent vortices in rigid rotation. The implicit function theorem is then employed in section 3.4 to yield periodic solutions to (AH) in the sense of Definition 1.1: see Theorem III.

To state these results it is convenient to introduce co-ordinates by stereographic projection. Take $\Sigma = S^2_R$ to be a sphere of radius R in $\mathbb{R}^3$ and apply the stereographic projection map to introduce a co-ordinate system $x = (x^1, x^2) \in \mathbb{R}^2$, with a corresponding holomorphic co-ordinate $z = x^1 + ix^2$, on S^2; the domain of the co-ordinate z is the whole sphere except a single point (the point at infinity). Recall that $N = 2$ so Φ is now a section of a degree 2 line bundle E; on the domain of z a natural (holomorphic non-unitary) frame for this bundle is $\partial/\partial z$. Let h be the metric on E given, with respect to this frame, by

$$\big|\Phi(z)\big|^2 = \frac{\overline{\Phi}(z)\Phi(z)}{(1+|z|^2)^2}\,. \tag{1.3}$$

(This is proportional to the Riemannian metric on the sphere given by its canonical embedding in $\mathbb{R}^3$.) In this instance ([B],[G]) the moduli space $\mathcal{M} = \mathbb{C}P^2$; let $\tilde{\mathcal{M}} \subset \mathcal{M}$ be the subset of $\mathcal{M}$ for which both Z_1, Z_2 are finite i.e. the zeros of Φ both lie in the domain of the co-ordinate z. It is proved in section 4.3 that

$$\begin{aligned} P &= Z_1 + Z_2 \\ Q &= Z_1 Z_2 \end{aligned} \tag{1.4}$$

form a system of holomorphic co-ordinates on $\tilde{\mathcal{M}}$ *even when* $Z_1 = Z_2$. Define the *reduced moduli space* $\tilde{\mathcal{M}}_0 \subset \mathcal{M}$ by $P = Z_1 + Z_2 = 0$: it is convenient to introduce co-ordinates ρ, θ by $Q = -\rho e^{i\theta}$, so that for $Q \neq 0$

this represents two vortices at $Z_1 = \sqrt{\rho}e^{i\theta/2}$, $Z_2 = -\sqrt{\rho}e^{i\theta/2}$. Define $\delta > 0$ by

$$\delta^2 = 4\pi(\tau R^2 - N)\,.$$

In terms of ρ, θ the *metric induced from* L^2 (the kinetic energy) on $\tilde{\mathcal{M}}_0$ takes the form (see section 2.3):

$$\tfrac{1}{2} f(\rho)(\dot\rho^2 + \rho^2\dot\theta^2)\,. \tag{1.5}$$

Periodic solutions will be obtained by perturbing periodic solutions to a finite-dimensional Hamiltonian system obtained in the adiabatic limit of (AH). This system is obtained by first restricting the action to $\mathcal{S}$, the space of finite-energy minimisers of the potential $V_{\lambda,\tau}$, and then dividing out by the action of the gauge group. This leads to a finite-dimensional Hamiltonian system which is then further reduced by factoring out the action of $SO(3)$ to yield a two-dimensional system with Hamiltonian,

$$H_{red}(\rho,\pi) = \frac{1}{2f(\rho)}\left(\pi^2 + \frac{J^2}{\rho^2}\right) + v_{\lambda,\tau}(\sqrt{\rho}, -\sqrt{\rho})\,. \tag{1.6}$$

Here ρ is as above, $\pi = f(\rho)\dot\rho$ is the corresponding momentum and $J = f(\rho)\rho^2\dot\theta$ is the angular momentum connected with rotation about the axis of symmetry of the two vortices. The next two theorems provide explicit information about the Hamiltonian (1.6) in the Bradlow limit.

Theorem I. *There exists a positive number δ_0 such that for $\delta \le \delta_0$ the interaction potential function $v_{\lambda,\tau}(\sqrt{\rho}, -\sqrt{\rho})$ defined in (1.2) satisfies*

$$v_{\lambda,\tau}(\sqrt{\rho}, -\sqrt{\rho}) = \frac{(\lambda - 1)}{8}\left(4\pi\tau N - \tau\delta^2 + \frac{3\delta^4(3 + 2\rho^2 + 3\rho^4)}{20\pi R^2(1+\rho^2)^2} + \delta^6 v_{rem}(\rho)\right)$$

where $\|v_{rem}\|_{C^2[0,K]} \le c = c(K,R)$.

Remark. The function $(3 + 2\rho^2 + 3\rho^4)/(1+\rho^2)^2$ is strictly decreasing for $0 < \rho < 1$. This implies that the potential energy is attractive for all such ρ for $\lambda < 1$ and δ sufficiently small. (Recall that $\rho = 1$ corresponds to two diametrically opposite points on the sphere, so this is the maximum possible separation.)

Theorem II. *There exists a positive number δ_1 such that for $\delta \le \delta_1$ the function f defined by* (1.5) *satisfies*

$$f(\rho) = \frac{2\delta^2(\rho^2 + 4\rho + 1)}{(1+\rho^2)^2(1+\rho)^2} + \delta^4 f_{rem}(\rho)\,.$$

where $\|f_{rem}\|_{C^2[0,K]} \le c = c(K,R)$.

Remarks. (i) The restriction $\rho \leq K$ in these theorems is not significant: it arises from the fact that the co-ordinate system becomes ill-defined at infinity. Using the co-ordinate $w = 1/z$ allows a description of $v_{\lambda,\tau}, f$ for large ρ which will be exactly the same as that just given.

(ii) In principle the metric and potential energy in the Bradlow limit could be computed for arbitrary N in the same way. A more general treatment of this problem will be given in a later publication; in this article only those aspects necessary to understand the existence of periodic solutions will be discussed.

It was conjectured in the book of Jaffe and Taubes that in the attractive case there may exist bound states of vortices rotating about one another in which the attraction is balanced by a centrifugal repulsion.

Theorem III. *For $\lambda < 1$ and $\delta, 1-\lambda$ sufficiently small there exist time periodic solutions of* (AH). *These solutions represent two vortices in rigid rotation about one another in the sense of Definition 3.2.1.*

Remarks. (i) This theorem is a consequence of Theorem 3.4.1, proved in section 3.4. The proof actually holds if $v_{\lambda,\tau}$ is attractive, regardless of whether λ is bigger or smaller than one; however in accordance with the discussion above it is conjectured that for $\lambda > 1$ it is repulsive.

(ii) There are two aspects to the question of existence of bound states of vortices of this type. Firstly there is the question of whether vortices behave like particles and in particular whether rotation produces a centrifugal force in some sense. Since vortices are solitons not particles this cannot be taken for granted; however the present result provides evidence that, at least in some cases, vortices do behave like particles. Secondly there is the aspect of radiation, which destabilises classical bound states such as the Rutherford atom. This issue is here bypassed by working on the sphere. It seems unlikely that bound states *involving rigid rotation of vortices* exist for $\Sigma = \mathbb{R}^2$. Solutions of the type discussed in [EGeSe] however may well exist. Various constraints on the existence of periodic solutions to nonlinear wave equations and generic non-existence results are given in the articles [C], [Si], [St3], [PySi], [SoWe1,2] and references therein.

(iii) The existence of rotating states in the Abelian Higgs model is a different phenomenon to their existence in the "non-linear Schrödinger type" dynamics ([M3]) where forces produce velocities in a perpendicular direction.

(iv) The idea of periodic solutions appearing as orbits of the rotation group has been successfully employed for two-dimensional skyrmions in

[PScZ], as well as being suggested for monopoles in [Mo].

2 The Bradlow Limit

In this section an analysis is presented of the limiting behaviour of multi-vortices in the Bradlow limit, and Theorems I and II are proved. Notation from section 4.1 will be used freely.

2.1 The Bogomolny equations. Some facts about the minimisers of $V_{\lambda,\tau}$ for $\lambda = 1$ are now collected for convenience; formulae are written for general N. Some further facts about these minimisers used in the paper are given in sections 4.2, 4.3 and 4.4. When $\lambda = 1$ $V_{\lambda,\tau}$ has many special properties and there is a good understanding of its critical points. It is known from [B] that if $\tau R^2 > N$ there is, up to gauge equivalence, a smooth $2N$ dimensional manifold $\mathcal{M}$ of minimisers of $V_{\lambda,\tau}$ called *multi-vortices*; these are pairs (A, Φ) which satisfy the Bogomolny equations (2.1). $\mathcal{M}$ is known as the moduli space and can be identified with the complex projective space $\mathbb{C}P^N$. The actual space of minimisers of $V_{\lambda,\tau}$ in $H^k(\Omega^1) \times H^k(E)$ will be denoted by $\mathcal{S}^k$; thus $\mathcal{M}$ is the orbit space of $\mathcal{S}^k$ under the action of the gauge group (see section 4.1), $\mathcal{M} \equiv \mathcal{S}^k/\mathcal{G}^{k+1}$. These spaces are independent of k for $k \geq 1$, which is why the index k is suppressed. There is a discussion of the manifold structure on these spaces in section 4.2, where their tangent spaces are identified with kernels of linear operators in the usual way.

An indication that $\lambda = 1$ is special is given by the Bogomolny decomposition formula:

$$V_{1,\tau}(A, \Phi) = \pi\tau N + = \frac{1}{2}\int_{S^2} 4|D_A^{(0,1)}\Phi|^2 + \left|B_A - \frac{1}{2}(\tau - |\Phi|^2)\right|^2 ,$$

which leads to the study of the Bogomolny equations:

$$\begin{aligned} D_A^{(0,1)}\Phi &= 0 \\ B_A - \tfrac{1}{2}\big(\tau - |\Phi|^2\big) &= 0 \end{aligned} \tag{2.1}$$

where $D_A^{(0,1)}\Phi = \frac{1}{2}((\partial_i - iA_1) + i(\partial_2 - iA_2))\Phi d\bar{z}$. Following Taubes' existence theorem for the plane ([JaT]) the Bogomolny equations have been studied on Riemann surfaces by a variety of methods ([B], [G]). One way, used in the first of these references, to obtain the solutions of (2.1) is to reduce them to the Kazdan-Warner equation (2.6), via the transformations (2.5).

Integration of the second of (2.1) together with the fact that

$$\int_{S^2} B_A d\mu_g = 2\pi N$$

implies that τ and the radius of the sphere are constrained by the inequality:

$$\tfrac{1}{2}\tau \mathrm{Area}(S^2) \geq 2\pi N\,, \tag{2.2}$$

with equality holding if and only if $|\Phi| \equiv 0$. Fix the radius of the sphere S^2 to be R then as τ approaches the critical value $\tau_{cr} = N/R^2$ it is clear that $\|\Phi\|_{L^2} \to 0$; this was called the Bradlow limit in the introduction. By a precise description of the approach to this limit formulae for the metric on the reduced moduli space and for the interaction potential energy between two vortices will be obtained in the next two sections.

2.2 Proof of Theorem I. To start with assume there is given a trivialisation of E over the domain of the co-ordinate z such that $|\Phi|^2 = h\overline{\Phi}\Phi$. Write the covariant derivative as $d_A = d - iA$, then the unitary connections are of the form

$$A_1 = \tfrac{i}{2}\partial_1 \ln h + \tilde{A}_1 \qquad A_2 = \tfrac{i}{2}\partial_2 \ln h + \tilde{A}_2$$

with $\tilde{A}_1, \tilde{A}_2$ real. First consider the zeroth order expansion: it is clear from the comments following (2.2) that in this limit $\Phi = 0$, and so by (2.1) the limiting connection a must satisfy

$$B_a = \tfrac{1}{2}\tau_{cr} = \tfrac{N}{2R^2}\,,$$

i.e. the limiting connection has constant curvature. Writing $h(z, \overline{z}) = 1/(1 + |z|^2)^2$ the corresponding connection a, which is unique up to gauge transformation, is given by

$$a_1 = \tfrac{1}{2}(\partial_2 \ln h + i\partial_1 \ln h) \qquad a_2 = \tfrac{1}{2}(-\partial_1 \ln h + i\partial_2 \ln h)\,.$$

In accordance with the remark above this is a unitary connection but is expressed relative to the non-unitary holomorphic frame provided by $\partial/\partial z$. Given h, (a_1, a_2) is the unique compatible unitary connection such that

$$D_a^{(0,1)} = \tfrac{\partial}{\partial \overline{z}}\,.$$

Now to zeroth order Φ is zero; however it is to be expected on account of the first equation of (2.1) that asymptotically it will approach a section of E holomorphic with respect to a. Such sections are determined (up to a constant) by a choice of an unordered pair of points on S^2; write these as $[Z] = [Z_1, Z_2]$. Assuming neither of these is the point at infinity in our co-ordinates the corresponding holomorphic section is written as

$$s_{[Z]} = c_{[Z]}(z - Z_1)(z - Z_2)\,. \tag{2.3}$$

Fix the normalization constant so that $\int |s_{[Z]}|^2 d\mu_g = 1$; this ensures that if $\phi = \delta s_{[Z]}$ then the integrated form of the second equation of (2.1) holds:

$$2\pi N = \frac{1}{2}\int \big(\tau - |\phi|^2\big) d\mu_g \,.$$

The case of interest is $Z_1 = \sqrt{\rho}$, $Z_2 = -\sqrt{\rho}$ in which case $c^{-2}_{[\sqrt{\rho},-\sqrt{\rho}]} = 4\pi R^2(1+\rho^2)/3$ and

$$s \equiv s_{[\sqrt{\rho},-\sqrt{\rho}]} = \frac{\sqrt{3}(z^2-\rho)}{2\sqrt{\pi}R\sqrt{(1+\rho^2)}} \,. \tag{2.4}$$

Now, following [JaT], [B], search for a solution in which Φ has zeros as $\sqrt{\rho}, -\sqrt{\rho}$ of the form

$$\begin{aligned} A_1(\delta) &= a_1 + \tfrac{1}{2}\partial_2 w \\ A_2(\delta) &= a_1 - \tfrac{1}{2}\partial_1 w \\ \Phi(\delta) &= \delta s e^{\frac{1}{2}w} \,, \end{aligned} \tag{2.5}$$

where w is a real-valued function on the sphere (depending also upon ρ). Substitution of this into (2.1) yields the single equation,

$$d^* dw + \delta^2 |s|^2 e^w = \frac{\delta^2}{4\pi R^2} \,. \tag{2.6}$$

This equation has been carefully analysed by Kazdan and Warner and admits a C^∞ solution in the present setting (see [KW, section 10].) Application of the maximum principle to this equation gives immediately the following upper bound:

$$|s|^2 e^w \le \frac{1}{4\pi R^2} \,.$$

Decompose w,

$$w = \overline{w} + \delta^2 \hat{w}$$

where $\overline{w}$ is the average of w over S^2 and $\delta^2\hat{w} = w - \overline{w}$. Then $\|d^* d\hat{w}\|_{L^2} \le c = c(R)$, and so since $\int_{S^2} \hat{w} = 0$:

$$\|\hat{w}\|_{W^{2,2}} \le c = c(R) \,. \tag{2.7}$$

Finally for every δ

$$e^{\overline{w}} \int |s|^2 e^{\hat{w}} = 1 \tag{2.8}$$

which together with the normalisation $\int |s|^2 = 1$ implies that

$$|\overline{w}| \le \delta^2 c(R) \,. \tag{2.9}$$

Differentiation of (2.6) with respect to ρ twice and estimating in the same way gives bounds

$$\|w\|_{W^{2,2}} + \|w_\rho\|_{W^{2,2}} + \|w_{\rho\rho}\|_{W^{2,2}} = O(\delta^2)\,. \tag{2.10}$$

The proof of the theorem is now completed by integrating

$$\frac{\lambda - 1}{8}\int_{S^2}\left(\tau - |s|^2\right)^2 d\mu_g\,,$$

with s given by (2.4); this may be done explicitly and leads to the formula given in the statement of Theorem I.

2.3 Proof of Theorem II. First of all as a general remark, in order to compute the L^2 metric on the reduced moduli space directly three steps are required:

(i) Obtain a complete set of solutions depending upon the parameters which are co-ordinates on the moduli space, and differentiate these with respect to the parameters; this provides elements of the kernel of the linearised operator.

(ii) Project these elements onto the subspace orthogonal to the gauge flow, i.e. at the point (a,ϕ) they must be projected onto $\operatorname{Ker}\delta^*_{(a,\phi)}$ (see (4.2)); call the resulting objects the zero modes.

(iii) Compute the L^2 inner products of all the zero modes obtained in (ii).

In the present situation (i) has been carried out in the limit $\delta \to 0$ in the proof of Theorem I. Step (ii) can here be done explicitly using the formula (see section 4.3 for explanations):

$$\frac{\dot\Phi}{\Phi} = \left(\dot Z_j \frac{\partial}{\partial Z_j}\right)\left(\ln|\Phi|^2\right)$$
$$(\dot A_1 - i\dot A_2) = i(\partial_1 - i\partial_2)\left(\dot Z_j \frac{\partial w}{\partial Z_j}\right)$$

which follows from [S]. Thus it remains to compute the L^2 inner products of the zero modes. Since the aim is only to compute the metric on the reduced moduli space $\tilde{\mathcal{M}}_0$ there is only one zero mode to consider, namely, that generated by variation of ρ,

$$\frac{\dot\Phi}{\Phi} = \left(\frac{\partial}{\partial\rho}\right)(\ln|\Phi|^2)$$
$$(\dot A_1 - i\dot A_2) = i(\partial_1 - i\partial_2)\left(\frac{\partial w}{\partial\rho}\right)\,.$$

where A, Φ are as in (2.5). Using the fact that $w = O(\delta^2)$ it is possible to evaluate the largest contribution to the L^2 norm of $(\dot A, \dot\Phi)$ which comes from $\dot{\Phi}$ only. This gives the following integral:

$$\int_0^\infty \int_{-\pi}^{+\pi} \frac{6\delta^2(\rho(1-r^4)+(\rho^2-1)r^2\cos 2\theta)^2}{\pi(1+\rho^2)^2(1+r^2)^4(r^4-2\rho r^2\cos 2\theta+\rho^2)} r dr d\theta \,.$$

Evaluating this integral explicitly gives the stated formula and the error is bounded in C^2 as stated by (2.10). □

3 The Abelian Higgs Model on $\mathbb{R} \times S^2$

3.1 The equations. Attention is now turned to the Abelian Higgs model (AH) on $\mathbb{R} \times S^2$ endowed with the Lorentzian metric

$$dt^2 - g\,,$$

where g is the metric induced on the sphere of radius R in $\mathbb{R}^3$ from the standard Euclidean structure. The bundle E extend to a bundle over $\mathbb{R}\times S^2$ which we denote $\tilde{E}$; $-iA_0 dt - iA_j dx^j$ is a connection on $\tilde{E}$ while Φ is a section of $\tilde{E}$. *Restricting to* S^2 the covariant derivative operator is $d_A = (d - iA) : \Omega^p(E) \to \Omega^{p+1}(E)$ and the associated curvature is written $-iF_A$ so that $F_A = dA$ in local co-ordinates; also as in section 2 $B_A = *F_A$, $*$ being the Hodge operator on S^2. The equations are obtained as the (formal) critical points of the action

$$S_{\lambda,\tau}(A,\Phi) = \int (T - V_{\lambda,\tau}) dt$$

with

$$T(A,\Phi) = \frac{1}{2}\int_{S^2} \left(|\partial_t A - dA_0|^2 + |\partial_t \Phi - iA_0\Phi|^2\right) d\mu_g$$

and $V_{\lambda,\tau}$ is as in (1.1). In these formulae the integration is taken with respect to the standard area two-form $d\mu_g = *1$ induced from g.

Let $j,k = 1,2$ and $\Lambda^2 = 4R^2/(1+|x|^2)^2$ so that $d\mu_g = \Lambda^2 dx^1 dx^2$. Writing the components of A as (A_0, A_1, A_2) the action can be written:

$$\int \Bigg(\sum_{j=1}^{2} (\dot{A}_j - \partial_j A_0)^2 + \Lambda^2 |\dot{\Phi} - iA_0\Phi|^2 - \frac{1}{\Lambda^2}(\partial_1 A_2 - \partial_2 A_1)^2 \\ - \sum_{j=1}^{2} \big|(\partial_j - iA_j)\Phi\big|^2 - \frac{\Lambda^2}{4}\big(1 - |\Phi|^2\big)^2 \Bigg) dx dt\,.$$

The equations are then (using a dot or ∂_t to denote the time derivative):

$$\begin{gathered}\partial_j \dot{A}_j - \Delta A_0 - \Lambda^2\big(i\Phi, (\partial_t - iA_0)\Phi\big) = 0\,,\\ (\partial_t - iA_0)^2\Phi - h^{-1}\Lambda^{-2}\sum_{j=1}^{2}(\partial_j - iA_j)\big(h(\partial_j - iA_j)\Phi\big) - \frac{\lambda}{2}\Phi\big(1-|\Phi|^2\big) = 0\,,\\ \ddot{A}_j - \partial_j\partial_t A_0 + \epsilon_{jk}\partial_k(\Lambda^{-2}B) - \big(i\Phi, (\partial_j - iA_j)\Phi\big) = 0\,.\end{gathered} \tag{3.1}$$

3.2 The ansatz for rigid rotation. The aim is to find time-periodic solutions which map out an orbit of the group S^1, acting as the subgroup of $SO(3)$ which leaves some axis fixed. This action of S^1 is written using the stereographic co-ordinate x as $x \mapsto R(\theta)x$, the rotation by θ about the origin. In order to define a notion of rigid rotation for vortex configurations it is necessary to lift this action to an action on the bundle E. There is not in general a canonical choice for this lifting; however in the present situation since E can be identified with the tangent bundle it is natural to use this identification to obtain a lifting. This leads to the following formula for the induced action of S^1 on a configuration:

$$\big(a_j(x), \phi(x)\big) \to \big(a_k(R(\theta)x)R_{kj}(\theta), \phi(R(\theta)x)e^{i\theta}\big)\,.$$

Other choices of the lifting would lead to actions which differ from this by a gauge transformation. In making an ansatz for a time-dependent configuration which represents vortex rotation this extra freedom may be absorbed into a time component of the connection $\tilde{A}_0(t,x) = A_0(R(\Omega t)x)$; if this term is non-zero the present solution will not be exactly time-periodic in temporal gauge, but will rather have the property that after time $2\pi/\Omega$ the configuration is *gauge equivalent* to the initial configuration. Thus in the present context we make the following definition:

DEFINITION 3.2.1. *An S^1 orbit of the rotation group in $\mathcal{A} \times C^\infty(E)$ at frequency ω consists of a smooth function $a_0 \in C^\infty(S^2)$ and a smooth map*

$$\begin{gathered}\mathbb{R} \to \mathcal{A} \times C^\infty(E)\\ t \mapsto (\tilde{A}_1 dx^1 + \tilde{A}_2 dx^2, \tilde{\Phi})\end{gathered}$$

of the form:

$$\begin{gathered}\tilde{A}_j(t,x) = A_k\big(R(\omega t)x\big)R_{kj}(\omega t) - \partial_j\chi(t,x)\\ \tilde{\Phi}(t,x) = \Phi\big(R(\omega t)x\big)e^{i\omega t}e^{i\chi(t,x)}\,,\end{gathered} \tag{3.2}$$

where

$$\chi(t,x) = \int_0^{\omega t} a_0\big(R(\omega s)x\big)ds\,,$$

and

$$\delta^*_{(\tilde{A},\tilde{\Phi})}\left(\tfrac{d\tilde{A}}{dt},\tfrac{d\tilde{\Phi}}{dt}\right)=0\,. \qquad (3.3)$$

Here $\delta^*_{(\tilde{A},\tilde{\Phi})}$ *is as in* (4.2), $A=A_1dx^1+A_2dx^2$ *is a smooth connection on* E *and* Φ *is a smooth section of* E.

Remark. The condition (3.3) is a gauge condition which is included in the definition for later convenience; it is always possible to choose a_0 so that (3.3) holds. It will transpire that condition (3.3) ensures that the orbit, as a time-dependent configuration, is in temporal gauge. The geometrical significance of this condition (section 4.1) is that it is equivalent to the requirement that $\frac{d}{dt}(\tilde{A},\tilde{\Phi})$ be L^2-orthogonal to the infinitesimal gauge transformations, the tangent space to the orbit of the gauge group, at $(\tilde{A},\tilde{\Phi})$. Using the Lie derivatives defined in section 4.1 the condition (3.3) can be rewritten:

$$\delta^*_{(A,\Phi)}(\mathcal{L}^{a_0}A,\mathcal{L}^{a_0}\Phi)=0\,. \qquad (3.4)$$

3.3 The adiabatic limit. The adiabatic limit of the Abelian Higgs model is the restriction of the system to the space of minimisers $\mathcal{S}^k$ defined in section 2.1. This then projects down to a well defined dynamical system on the moduli space $\mathcal{M}$. This can be reformulated as the statement that a one parameter family of pairs $t\mapsto(A,\Phi)(t)\in\mathcal{S}^k$ is a solution of the adiabatic limit of (AH) if it is critical point of the action function $S_{\lambda,\tau}$ *restricted to curves lying in* $\mathcal{S}^k$. Thus in deriving the Euler-Lagrange equation we make variations only in the direction of the tangent space of $\mathcal{S}^k$. This condition is particularly clear with the requirement that the curve satisfies the gauge condition (see 4.2):

$$\delta^*_{(A,\Phi)}(\dot{A},\dot{\phi})=0 \qquad (3.5)$$

(Combined with the constraint equation, this implies that $A_0=0$ (temporal gauge). It follows from (4.3) that given a differentiable curve in $\mathcal{S}^k$ it is always possible to obtain a gauge equivalent curve which satisfies (3.5).) Differentiation of (3.5) gives

$$\delta^*_{(A,\Phi)}(\ddot{A},\ddot{\Phi})=0\,. \qquad (3.6)$$

Using this gauge the condition that a curve $(A(t),\Phi(t))\in\mathcal{S}^k$ be a critical point of the action with respect to other curves in $\mathcal{S}^k$ now implies that the quantity

$$-\frac{\partial^2}{\partial t^2}(A,\Phi)+\left(0,\frac{\lambda-1}{2}\Phi\big(\tau-|\Phi|^2\big)\right)$$

be L^2-orthogonal to the tangent space of $\mathcal{S}^k$ at each time. But according to the discussion in section 4.2 the tangent space to $\mathcal{S}^k$ at (A,Φ) may be identified with the kernel of the Hessian $\mathrm{Hess}_{(A,\Phi)}$; this gives the following explicit formulation of the adiabatic limit:

$$\Big(-\frac{\partial}{\partial t^2}(A,\Phi)+\Big(0,\frac{\lambda-1}{2}\Phi\big(\tau-|\Phi|^2\big)\Big),n\Big)_{L^2}=0 \qquad \text{for all } n\in L^2\cap\operatorname{Ker}\tilde{L}_{(A,\Phi)}\,.$$

where $\tilde{L}_{(A,\Phi)}$ is the second order elliptic operator defined in (4.8). Using (3.6) and the definition of the operator $L_{(A,\Phi)}$ in (4.10) this condition can be expressed as

$$\Big(-\frac{\partial^2}{\partial t^2}(A,\Phi)+\Big(0,\frac{\lambda-1}{2}\Phi\big(\tau-|\Phi|^2\big)\Big),n\Big)_{L^2}=0 \qquad \text{for all } n\in L^2\cap\operatorname{Ker}L_{(A,\Phi)}\,. \tag{3.7}$$

Recall that the tangent space to $\mathcal{M}$ at the orbit of (A,Φ) can be identified with $\operatorname{Ker}L_{(A,\Phi)}$. From this it follows that the equations (3.7) are those of a Hamiltonian system (3.8) defined on the cotangent bundle to $\mathcal{M}$. Indeed the metric on $\mathcal{M}$ induced from L^2 described in Theorem II (see the remarks in section 2.3) determines a kinetic energy T_{ad}; the adiabatic Hamiltonian

$$H_{ad}=T_{ad}+v_{\lambda,\tau} \tag{3.8}$$

is then obtained by adding to T_{ad} the potential energy function $v_{\lambda,\tau}$ defined by:

$$\begin{aligned} v_{\lambda,\tau} &: M\to\mathbb{R}\\ v_{\lambda,\tau}([A,\Phi]) &= \frac{(\lambda-1)}{8}\int_{S^2}\big(\tau-|\Phi|^2\big)^2 d\mu_g\,. \end{aligned} \tag{3.9}$$

which is a smooth function on $\mathcal{M}$. Any integral curve of this Hamiltonian system determines a curve in $\mathcal{S}^k$ which satisfies (3.5), (3.7) and vice-versa.

Theorem 3.3.1. *For $\tau-\tau_{cr}$ sufficiently small and $\lambda<1$ there exists an S^1 orbit of the rotation group in $\mathcal{A}\times C^\infty(E)$ at frequency ω which satisfies (3.5), (3.7). This orbit is of the form*

$$\begin{aligned} \tilde{A}_j(t,x) &= a_k\big(R(\omega t)x\big)R_{kj}(\omega t)-\partial_j\chi(t,x)\\ \tilde{\Phi}(t,x) &= \phi\big(R(\omega t)x\big)e^{i\omega t}e^{i\chi(t,x)}\,. \end{aligned} \tag{3.10}$$

Here $\chi(t,x)=\int_0^{\omega t}a_0(R(\omega s)x)ds$ and $(a,\phi)\in\cap_{k\geq 1}\mathcal{S}^k$ is such that $\phi=0$ at $\pm\sqrt{\rho}$ for suitable $\rho>0$. The numbers ρ and $\omega=\omega(\rho)$ are determined in the succeeding lemma.

Proof. This is proved by applying symplectic reduction to obtain a reduced two-body Hamiltonian H_{red} which is shown to have periodic solutions in the Bradlow limit, discussed in section 2. The phase space of the Hamiltonian system (3.8) is the cotangent bundle, $T^*\mathcal{M}$ which is eight dimensional; however, the solutions in which we are interested can be obtained by restricting to the reduced moduli space $\tilde{\mathcal{M}}_0$ i.e. the submanifold defined by the condition $Z_1 + Z_2 = 0$. To write this explicitly use the co-ordinates (ρ, θ) on $\tilde{\mathcal{M}}_0$ described in the introduction. The momenta conjugate to ρ, θ are

$$\pi = f(\rho)\dot{\rho} \qquad J = f(\rho)\rho^2\dot{\theta}$$

in terms of which the reduced Hamiltonian takes the familiar "reduced two-body" form of (1.6):

$$H_{red}(\rho, \pi) = \tfrac{1}{2f(\rho)}\left(\pi^2 + \tfrac{J^2}{\rho^2}\right) + v_{\lambda,\tau}(\rho)\,.$$

(This Hamiltonian can be obtained by symplectic reduction with respect to the group $SO(3)$ which acts by rigid rotation on S^2 and hence on the system (AH) and its adiabatic limit. The momentum map is a smooth map from the cotangent bundle to the dual of the lie algebra of the symmetry group

$$\mu : T^*M \to so(3)^*$$

such that for $\xi \in so(3) = \mathbb{R}^3$ the function $\mu(\xi)$ is the conserved momentum associated to rotation about the axis $\xi \in \mathbb{R}^3$. Let e_1, e_2, e_3 be three orthonormal unit vectors, the third being in the direction of the axis passing through the mid-point of the two vortices, i.e. the origin of the stereographic co-ordinate system. Let $\mu_i = \mu(e_i)$ be the components of μ. Then we define the reduced Hamiltonian H_{red} system by restricting the Hamiltonian H_{ad} to $\mu^{-1}(0,0,J)/SO(3)$. The phase space for this reduced system is two dimensional.)

LEMMA 3.3.2. *Assume ρ is such that*

(Condition i) $v'_{\lambda,\tau}(\rho) < 0$,

(Condition ii) $\frac{d}{d\rho}\big(\rho^2 f(\rho)\big) > 0$.

Then there will exist a solution to the adiabatic limit problem (3.7) *which is an S^1 orbit of the rotation group with ρ constant and $J = J(\rho)$ determined by*

$$\frac{1}{2}J(\rho)^2\frac{d}{d\rho}\left(\frac{1}{\rho^2 f(\rho)}\right) + v'_{\lambda,\tau}(\rho) = 0$$

and at angular frequency $\dot{\theta} = \omega(\rho)$ determined by

$$\omega(\rho) = \frac{J(\rho)}{f(\rho)\rho^2}\,. \tag{3.11}$$

Now by reference to Theorem I if $\tau - \tau_{cr}$ is sufficiently small Condition (i) holds for $\lambda < 1$. Also Condition (ii) holds for small $\tau - \tau_{cr}$ by Theorem II. So let ρ be such that Lemma 3.3.2 holds and let $(a, \phi) \in \cap\mathcal{S}^k$ be a corresponding smooth minimising configuration, i.e. $\phi = 0$ at $\pm\sqrt{\rho}$. To complete the proof of the theorem define a_0 by the requirement (see (4.2))

$$\delta^*_{(a,\phi)}(\mathcal{L}^{a_0} a, \mathcal{L}^{a_0}\phi) = 0\,.$$

This leads to the equation

$$d^* da_0 + |\phi|^2 a_0 = \delta^*_{(a,\phi)}(\mathcal{L}a, \mathcal{L}\phi)$$

which has a smooth solution since $d^*d + |\phi|^2$ is a strictly positive self-adjoint elliptic operator with smooth coefficients. □

Remarks. (i) Theorem 3.3.1 is stated with the restriction that $\tau - \tau_{cr}$ is small because in this case Theorems I and II imply that Conditions (i) and (ii) in Lemma 3.3.2 are satisfied. It can be stated more generally that if ρ is such that these two conditions hold then the conclusions of Theorem 3.3.1 and also Theorem 3.4.1 are still valid.

(ii) Given such a periodic solution at frequency ω and with $\lambda = 1 - l$, rescaling provides a periodic orbit with frequency $\Omega = \epsilon\omega$ and $\lambda = 1 - \epsilon^2 l$.

3.4 Existence theorem for periodic solutions. In this section the main existence theorem is stated and proved. Terminology and notation will be used from Definitions 1.1 and 3.2.1 and the appendix.

Theorem 3.4.1. *Assume there exists an S^1 orbit of the rotation group at frequency ω_0 as in* (3.10) *which satisfies* (3.5), (3.7) *with $\lambda = 1 - l$, and such that $v'_{\lambda,\tau}(\rho) \neq 0$. Then there exists $\epsilon_* > 0$ and a function $\omega(\epsilon)$, satisfying $\omega(0) = \omega_0$ such that for $\epsilon < \epsilon_*$ there exists an S^1 orbit of the rotation group at frequency $\epsilon\omega(\epsilon)$ which is a smooth $2\pi/\epsilon\omega(\epsilon)$-periodic solution of* (AH) *with $\lambda = 1 - \epsilon^2 l$ (in the sense of Definition 1.1).*

Proof. Step one. Regard (3.10) as an approximate solution and search for an exact solution which is close to this. It is convenient to depart from temporal gauge for the time being so a time component to the connection

$\tilde{A}_0$ is also introduced:

$$\begin{aligned}\tilde{A}_0(t,x) &= A_0(R(\Omega t)x) = \epsilon\Omega\beta_0\big(R(\Omega t)x\big)\\ \tilde{A}_j(t,x) &= A_k\big(R(\Omega t)x\big)R_{kj}(\Omega t) - \partial_j\chi(t,x)\\ &= \big(a_k(R(\Omega t)x) + \epsilon^2\beta_k(R(\Omega t)x)\big)R_{kj}(\Omega t) - \partial_j\chi(t,x)\\ \tilde{\Phi}(t,x) &= \Phi\big(R(\Omega t)x\big)e^{i\chi(t,x)} = \big(\phi(R(\Omega t)x) + \epsilon^2\eta(R(\Omega t)x)\big)e^{i\chi(t,x)},\end{aligned} \tag{3.12}$$

with

$$\chi(t,x) = \int_0^{\omega t} a_0\big(R(\omega s)x\big)ds\,.$$

Due to gauge invariance it is possible to restrict (β,η) to satisfy the slice condition (see 4.2):

$$\delta^*_{(a,\phi)}(\beta,\eta) = d^*\beta + (i\phi,\eta) = 0\,.$$

Substitution of (3.10) into (3.5) and (3.7) gives (respectively):

$$\delta^*_{(a,\phi)}(\mathcal{L}^{a_0}a, \mathcal{L}^{a_0}\phi) = 0 \tag{3.13}$$

and

$$\left(-\omega^2\big(\mathcal{L}\mathcal{L}^{a_0}a, (\mathcal{L}^{a_0})^2\phi\big) + \left(0, \frac{\lambda-1}{2}\phi\big(\tau - |\phi|^2\big)\right), n\right)_{L^2} = 0$$

$$\text{for all } n \in L^2 \cap \operatorname{Ker} L_{(a,\phi)}\,. \tag{3.14}$$

As proved in Lemma 4.4.1 we may assume that (a,ϕ) is smooth and that

$$\phi(-x) = \phi(x) \qquad a(-x) = -a(x)\,. \tag{3.15}$$

The adiabatic limit describes slow motion, so apply the scaling described in Remark (ii) in section 3.3, and write:

$$\Omega(\epsilon) = \epsilon\omega(\epsilon) \qquad \lambda = 1 - \epsilon^2 l\,.$$

Substitution of the ansatz (3.12) into the equations and a direct calculation leads to

$$\begin{aligned}\epsilon^2\omega^2\big(\mathcal{L}\mathcal{L}^{(a_0+\epsilon\beta_0)}A, (\mathcal{L}^{(a_0+\epsilon\beta_0)})^2\Phi\big) + {V_{\lambda,\tau}}'(A,\Phi) = 0\\ d^*(\mathcal{L}^{(a_0+\epsilon\beta_0)}A) + \langle i\Phi, \mathcal{L}^{(a_0+\epsilon\beta_0)}\Phi\rangle = 0\,.\end{aligned} \tag{3.16}$$

Equations (3.16) are to be considered as a deformation of equations (3.13), (3.14); thus the implicit function theorem will be used to prove that a solution to (3.16) exists in the form (3.12) for small ϵ. The difficulty arises from the fact that the linearised operator $\tilde{L}_{(a,\phi)}$ has a cokernel, described in section 4.2. In Step two it is explained how to "remove" this cokernel; see Lemma 3.4.3.

Step two. The next lemma gives a set of equations (3.17) which can be solved and whose solutions generate periodic solutions of the Abelian Higgs model. The point about the new system is the addition of two new terms on the right-hand side which "fill out" the part of the cokernel of $\tilde{L}_{(a,\phi)}$ generated by gauge transformations and rotations.

LEMMA 3.4.2. *Assume ω, μ are real numbers, and $(a_0, \beta_0, A, \Phi, g) \in C^\infty \times C^\infty \times \mathcal{A} \times C^\infty(E) \times H^1$ are such that*

$$\begin{aligned} \epsilon^2\omega^2\big(\mathcal{L}\mathcal{L}^{(a_0+\epsilon\beta_0)}A, (\mathcal{L}^{(a_0+\epsilon\beta_0)})^2\Phi\big) + V'_{\lambda,\tau}(A,\Phi) \\ = \epsilon^2\mu(\mathcal{L}^{(a_0+\epsilon\beta_0)}A, \mathcal{L}^{(a_0+\epsilon\beta_0)}\Phi) + \epsilon^2\delta_{(A,\Phi)}g \\ d^*(\mathcal{L}^{(a_0+\epsilon\beta_0)}A) + \langle i\Phi, \mathcal{L}^{(a_0+\epsilon\beta_0)}\Phi\rangle = 0\,. \end{aligned} \tag{3.17}$$

where $\delta_{(A,\Phi)}$ is the differential operator defined in (4.2). *Then $(\tilde{A}_0, \tilde{A}_j, \tilde{\Phi})$ defined by* (3.12) *gives a smooth $2\pi/\epsilon\omega$ periodic solution of the Abelian Higgs model (in the sense of Definition 1.1) after passage to temporal gauge.*

Proof. By comparison with (3.16) it is necessary to prove that $\mu = 0$ and $g = 0$; the result then follows. Recall (see (4.3)) that the condition

$$\delta^*_{(A,\Phi)}(\beta,\eta) = d^*\beta + \langle i\Phi, \eta\rangle = 0$$

is equivalent to the requirement that $(\beta,\eta) \in C^\infty(\Omega^1 \times E)$ be L^2-orthogonal to $\delta_{(A,\Phi)}g$ for all $g \in H^1$. Now $SO(3)$ invariance of the metric on S^2 ensures that differentiation of the second equation of (3.17) leads to:

$$d^*(\mathcal{L}\mathcal{L}^{(a_0+\epsilon\beta_0)}A) + \big\langle i\Phi, (\mathcal{L}^{(a_0+\epsilon\beta_0)})^2\Phi\big\rangle = 0\,,$$

and so $\delta_{(A,\Phi)}g$ is orthogonal to the first terms on both sides of the first of equations (3.17). But gauge invariance of $V_{\lambda,\tau}$ ensures that $\delta_{(A,\Phi)}g$ is L^2-orthogonal to $V'_{\lambda,\tau}(A,\Phi)$ and hence $g = 0$. Next the fact that $\mu = 0$ follows from rotational invariance which implies that $\mathcal{L}^{(a_0+\epsilon\beta_0)}(A,\Phi)$ is orthogonal to the two remaining terms on the left hand side. Finally recall that it is always possible to go into temporal gauge by applying the time-dependent gauge transformation generated by $-\int_0^t A_0$. □

The introduction of μ, ν in this way will handle the subspace of the cokernel generated by the gauge transformations and by the action of the S^1 subgroup of $SO(3)$ which leaves the centre of the vortices fixed. To deal with the remaining two dimensional subspace of the kernel generated by $SO(3)$ it is expedient to work in the space of functions which have the same symmetry properties as the basic two vortex solution, as described in section 4.4 and (3.15). Indeed the action of $SO(3)$ in directions orthogonal to the S^1 subgroup breaks this symmetry. Therefore restricting to the symmetric subspace removes the remaining part of the cokernel. To make this explicit introduce the spaces

$$X_1^+ \equiv \{\beta_0 \in H^3 : \beta_0(-x) = \beta_0(x)\}$$
$$X_2^+ \equiv \{(\beta,\eta) \in H^2(\Omega^1) \times H^2(E) : \eta(-x) = \eta(x),\ \beta_j(-x) = -\beta_j(x)\}\,.$$

The $^+$ here designates the symmetry property of the functions; the same notation will be used below, thus, for example:

$$H^{1,+} \equiv \{\beta_0 \in H^1 : \beta_0(-x) = \beta_0(x) \text{ a.e.}\}$$
$$L^{2,+}(\Omega^1 \times E) \equiv \{(\beta,\eta) \in L^2(\Omega^1 \times E) : \eta(-x) = \eta(x)\,, \quad \beta_j(-x) = -\beta_j(x) \text{ a.e.}\}\,.$$

Another piece of notation used below is that if Y is a subspace of $L^2(\Omega^1 \times E)$ then

$$X_2^+ \cap Y^\perp = \Big\{(\beta,\eta) \in X_2^+ : \int \langle(\beta,\eta),n\rangle d\mu_g = 0 \ \forall\, n \in Y\Big\}\,,$$

and similarly for $L^{2,+} \cap Y^\perp$. Also as noted in section 4.4 the symmetry operation induces a decomposition

$$\operatorname{Ker} L_{(a,\phi)} = \operatorname{Ker} L^+_{(a,\phi)} \times \operatorname{Ker} L^-_{(a,\phi)}$$

into a subspace sharing the symmetry property of $L^{2,+}$ and one with the opposite symmetry.

Lemma 3.4.3. *Assume that (a,ϕ) satisfies the symmetry property* (3.15) *and $\tilde L_{(a,\phi)}$ is the Hessian defined in* (4.8)*.*

(i) *The operator $d^*d + |\phi|^2$ defines a bounded linear bijection from $H^{3,+}$ to $H^{1,+}$.*
(ii) *The operator $\tilde L_{(a,\phi)}$ defines a bounded linear bijection*

$$\tilde L_{(a,\phi)} : X_2^+ \cap \operatorname{Ker} \delta^*_{(a,\phi)} \cap (\operatorname{Ker} L^+_{(a,\phi)})^\perp \to L^{2,+}(\Omega^1 \times E) \cap \operatorname{Ker} \delta^*_{(a,\phi)} \cap (\operatorname{Ker} L^+_{(a,\phi)})^\perp\,.$$

Proof. The operator in (i) is a strictly positive second order elliptic operator with smooth coefficients and $|\phi|^2$ is even, so the first statement follows from standard arguments. For the second statement, notice that the operator $\tilde L$ is a continuous map $X_2^+ \to L^{2,+}$ by (3.15). Also by (4.11) $\tilde L_{(a,\phi)}$ maps $\operatorname{Ker} \delta^*_{(a,\phi)}$ to itself. But on $\operatorname{Ker} \delta^*_{(a,\phi)}$ it follows from (4.10) that $\tilde L_{(a,\phi)} = L_{(a,\phi)}$. Therefore the result follows from Lemma 4.2.1. □

Step three. The equations are now solved to zeroth order. Substitution of the ansatz (3.12) leads to the following two equations for $(\beta_0, \beta_1, \beta_2, \eta)$:

$$\epsilon^2\omega^2\big((\mathcal{L}\mathcal{L}^{a_0}\beta, \mathcal{L}^{a_0})^2\eta\big) + \tilde{L}_{(a,\phi)}(\beta,\eta) = -\omega^2\big((\mathcal{L}\mathcal{L}^{a_0}a, \mathcal{L}^{a_0})^2\phi\big) + \tfrac{l}{2}\phi\big(\tau - |\phi|^2\big)$$
$$+\epsilon\big(\omega d\mathcal{L}\beta_0, 2i\omega\beta_0\mathcal{L}\phi + i\phi\mathcal{L}\beta_0 + \epsilon\beta_0^2(\phi+\epsilon^2\eta)\big) \qquad (3.18)$$
$$-\epsilon^2\mathcal{N}_{(a,\phi)}(\beta,\eta) + \mu\mathcal{L}^{(a_0+\epsilon\beta_0)}(a+\epsilon^2\beta, \phi+\epsilon^2\eta) + \nu\delta_{(a+\epsilon^2\beta,\phi+\epsilon^2\eta)}g\,,$$

and

$$d^*d\beta_0 + |\phi|^2\beta_0 + \epsilon\omega\big(\langle i\eta, \mathcal{L}^{a_0}\phi\rangle + \langle i\phi, \mathcal{L}^{a_0}\eta\rangle\big) -$$
$$\epsilon^2\langle\phi, \beta_0\eta\rangle + \epsilon^3\langle i\eta, \mathcal{L}^{a_0}\eta\rangle - \epsilon^4\beta_0|\eta|^2 = 0\,. \quad (3.19)$$

In (3.18) $l = (\lambda-1)/\epsilon^2$ and

$$\epsilon^2\mathcal{N}_{(a,\phi)}(\beta,\eta) = \tfrac{1}{\epsilon^2}\big(V_{\lambda,\tau}{}'(a+\epsilon^2\beta, \phi+\epsilon^2\eta) - V_{1,\tau}{}'(a,\phi) - \epsilon^2\tilde{L}_{(a,\phi)}(\beta,\eta)\big)\,.$$

(Of course in this case $V_{1,\tau}{}'(a,\phi) = 0$.)

To start with, when $\epsilon = 0$, $\beta_0 = 0$ and (3.18) reduces to the equation:

$$\tilde{L}_{(a,\phi)}(\beta,\eta) = -\omega^2\big(\mathcal{L}\mathcal{L}^{a_0}a, (\mathcal{L}^{a_0})^2\phi\big) + l\phi\big(\tau - |\phi|^2\big)\,. \qquad (3.20)$$

Lemma 3.4.5. *There is a smooth solution $(\beta^{(0)}, \eta^{(0)})$ of equation (3.20) which is unique in*

$$X_2^+ \cap \operatorname{Ker}\delta^*_{(a,\phi)} \cap (\operatorname{Ker} L_{(a,\phi)})^\perp\,.$$

Proof. This follows by an application of statement (ii) of the lemma in Step two. Indeed the fact that the orbit (3.10) satisfies (3.5), (3.7) ensures that the right-hand side of (3.20) lies in the range of $\tilde{L}_{(a,\phi)}$ by (3.14). □

Step four. The implicit function theorem is now used to obtain solutions to (3.18), (3.19) for small ϵ. Regard (3.19) as an equation $F_1(\beta_0, \beta, \eta, \mu, \omega, \epsilon) = 0$ where $F_1 : H^{3,+} \times X_2^+ \times \mathbb{R}^3 \to H^{1,+}$; similarly regard (3.18) as an equation $F_2(\beta_0, g, \beta, \eta, \mu, \omega, \epsilon) = 0$ where $F_2 : H^{3,+} \times H^{1,+} \times X_2^+ \times \mathbb{R}^3 \to L^{2,+}$. The implicit function theorem is now applied to the equation $F = (F_1, F_2) = 0$ where F is a smooth map,

$$F : H^{3,+} \times X_2^+ \times \mathbb{R}^3 \to H^{1,+} \times L^{2,+}\,.$$

The following formulae give the derivatives of F_1, F_2 at ($\beta_0 = 0$, $g = 0$, $\beta = \beta^{(0)}$, $\eta = \eta^{(0)}$, $\mu = 0$, $\omega = \omega_0$, $\epsilon = 0$):

$$\begin{aligned} D_{\beta_0}F_1 &= d^*d + |\phi|^2 \\ D_{(\beta,\eta)}F_2 &= \tilde{L}_{(a,\phi)} \\ D_\mu F_2 &= (\mathcal{L}^{a_0}a, \mathcal{L}^{a_0}\phi) \\ D_g F_2 &= \delta_{(a,\phi)} \\ D_\omega F_2 &= \omega\big(\mathcal{L}\mathcal{L}^{a_0}a, (\mathcal{L}^{a_0})^2\phi\big)\,. \end{aligned} \qquad (3.21)$$

Notice first of all that (i) of Lemma 3.4.3 implies that $D_{\beta_0}F_1$ is an isomorphism $H^{3,+}\to H^{1,+}$. Next by (4.3)

$$D_gF_2(H^{1,+})\oplus \mathrm{Ker}\ \delta^*_{(a,\phi)}\big|_{L^{2,+}}=L^{2,+}\,.$$

Therefore by (ii) of Lemma 3.4.3 in order to apply the implicit function theorem it is left to prove that the range of the derivative contains $\mathrm{Ker}\,L^+_{(a,\phi)}$. As explained in section 4.4 this is a two dimensional subspace spanned by the two vectors

$$(\mathcal{L}^{a_0}a,\mathcal{L}^a\phi)\quad\text{and}\quad \Pi_{(a,\phi)}\tfrac{\partial}{\partial\rho}(a,\phi)$$

where $\Pi_{(a,\phi)}$ is the projection operator onto $\mathrm{Ker}\,\delta^*_{(a,\phi)}$ (which is well-defined by (4.3)). The first of these is just $D_\mu F_2$ so this is certainly in the range of the derivative of F. For the second notice that (3.7) implies that ω is determined by the equation,

$$\omega^2\int\Big\langle\Pi_{(a,\phi)}\tfrac{\partial}{\partial\rho}(a,\phi),\big(\mathcal{L}\mathcal{L}^{a_0}a,(\mathcal{L}^{a_0})^2\phi\big)\Big\rangle\,d\mu_g=\tfrac{d}{d\rho}v_{1-l,\tau}\,.$$

It follows from this that as long as $\frac{d}{d\rho}v_{1-l,\tau}\neq 0$

$$\int_{S^2}\Big\langle\Pi_{(a,\phi)}\tfrac{\partial}{\partial\rho}(a,\phi),D_\omega F_2\Big\rangle\,d\mu_g\neq 0\,.$$

Therefore the derivative of F is surjective and the implicit function theorem can be applied to prove existence of solutions for small ϵ. For small ϵ the system (3.18)–(3.19) is elliptic and all solutions in H^2 are smooth. Finally to obtain the result as stated it is necessary to go back to temporal gauge by application of the time-dependent gauge transformation $-\int_0^t A_0$.

4 Appendix

4.1 Notation and background information. Given a surface Σ with metric g and co-ordinates (x^1,x^2) we have a basis $\partial_{x^1},\partial_{x^2}$ for the tangent space, and dual basis dx^1,dx^2 for the cotangent space. The co-ordinates determine an expression for the metric at each point as a non-degenerate symmetric matrix $g_{ij}(x)$. The associated area measure is $d\mu_g=\sqrt{\det g}dx^1dx^2$. Write the inverse matrix of g_{ij} as g^{ij}; this determines the inner product on 1-forms. Given a function f the Hodge operator gives a 2-form $*f=fd\mu_g$; the inverse operator maps 2-forms to functions according to $*d\mu_g=1$, so that $(*)^2=1$ on functions and two-forms. On one-forms, on the other hand, the Hodge operator acts as a complex structure i.e. $(*)^2=-1$.

The p-forms are denoted Ω^p and for any vector bundle V (with norm) over S^2 write $H^k(V)$ for the Hilbert space obtained by completing the space of smooth sections with respect to the k^{th} Sobolev norm $\|\cdot\|_k$. The space of connections is an affine space modelled on Ω^1; we define the space of Sobolev connections by $\mathcal{A}^k = a + H^k(\Omega^1)$, where a is an arbitrary smooth connection. The group of gauge transformations can be identified in the present abelian context with real-valued functions χ which act according to

$$(A,\Phi) \mapsto (A + d\chi, \Phi e^{i\chi}) . \tag{4.1}$$

From this formula it follows that if we consider the group $\mathcal{G}_{k+1}$ defined by functions $\chi \in H^{k+1}$ then $\mathcal{G}_{k+1}$ acts smoothly on $\mathcal{A}^k \times H^k(E)$ for $k \geq 1$. The derivative of the action gives an action of the Lie algebra $\mathrm{Lie}\,\mathcal{G}^{k+1}$ which gives the tangent space to the orbit

$$T_{(a,\phi)}\big(\mathcal{G}^{k+1}\cdot(a,\phi)\big) = \big\{(d\chi, i\phi\chi) : \chi \in H^{k+1}\big\} .$$

It is convenient to define the differential operator

$$\begin{gathered}\delta_{(a,\phi)} : H^{k+1} \to H^k(\Omega^1) \times H^{k+1}(E)\\ \chi \mapsto (d\chi, i\phi\chi).\end{gathered}$$

and its adjoint:

$$\begin{gathered}\delta^*_{(a,\phi)} : H^k(\Omega^1) \times H^k(E) \to H^{k-1}\\ (\beta,\eta) \mapsto d^*\beta + (i\phi,\eta) .\end{gathered} \tag{4.2}$$

As usual there is an L^2-orthogonal decomposition of Helmholtz-Hodge type:

$$H^k(\Omega^1) \times H^k(E) = T_{(a,\phi)}\big(\mathcal{G}^{k+1}\cdot(a,\phi)\big) \oplus \mathrm{Ker}\,\delta^*_{(a,\phi)} . \tag{4.3}$$

Lie derivatives and "covariant Lie derivatives" are now defined, using the stereographic co-ordinates (x^1,x^2) as in the introduction. For a one-form $\beta \in \Omega^1$ the Lie derivative with respect to the vector field generated by rotation about the origin is given by:

$$\omega(\mathcal{L}\beta)_j = \tfrac{d}{dt}\big|_{t=0}\big(\beta_l(R(\omega t)x)R_{lj}(\omega t)\big) .$$

This gives an operator $\mathcal{L} : H^k(\Omega^1) \to H^{k-1}(\Omega^1)$. In the gauge theoretic setting this generalises as follows. Let

$$\chi(t,x) \equiv \int_0^{\omega t} A_0\big(R(\omega\tau)x\big)d\tau ,$$

then the corresponding Lie derivatives are defined by:

$$\begin{aligned}\omega\mathcal{L}^{A_0}\phi &= \tfrac{d}{dt}\big|_{t=0}\big(\phi(R(\omega t)x)e^{+i\chi(t,x)}\big) ,\\ \omega(\mathcal{L}^{A_0}a)_j &= \tfrac{d}{dt}\big|_{t=0}\big(a_k(R(\omega t)x)R_{kj}(\omega t) - \partial_j\chi(t,x)\big) .\end{aligned} \tag{4.4}$$

Since difference of any two connections is a one-form $\mathcal{L}^{A_0} : \mathcal{A}^k \to H^{k-1}(\Omega^1)$. Notice that we use the same notation $\mathcal{L}^{A_0}$ for the operators:

$$\mathcal{L}^{A_0} : H^k(E) \to H^{k-1}(E) \quad \mathcal{L}^{A_0} : \mathcal{A}^k \to H^{k-1}(\Omega^1)\,. \tag{4.5}$$

4.2 Static solutions. In this section some formulae for derivatives of the functional $V_{\lambda,\tau}$ and associated operators are given, together with some of their analytical properties. This is used to obtain the standard description of the manifold structure of $\mathcal{S}$ and $\mathcal{M}$. Differentiation of the Bogomolny equations at (a,ϕ) in the direction (β,η) defines a linear operator

$$\begin{aligned} \tilde{\mathcal{D}}_{(a,\phi)} &: H^k(\Omega^1) \times H^k(E) \to H^{k-1} \times H^{k-1}\big(\Omega^{0,1}(E)\big) \\ (\beta,\eta) &\mapsto \big(* d\beta + (\phi,\eta), D_A^{(0,1)}\eta - \tfrac{i}{2}(\beta_1 + i\beta_2)\phi d\bar{z}\big) \end{aligned} \tag{4.6}$$

where $D_A(0,1)$ is as defined in section 2.1. Gauge invariance of the Bogomolny equations implies that $\tilde{\mathcal{D}}_{(a,\phi)}$ has an infinite dimensional kernel which contains $T_{(a,\phi)}(\mathcal{G}^{k+1} \cdot (a,\phi))$. It will now be shown that orthogonal to $T_{(a,\phi)}(\mathcal{G}^{k+1} \cdot (a,\phi))$ the kernel of $\tilde{\mathcal{D}}_{(a,\phi)}$ has fixed dimension $2N$, and that $T_{(a,\phi)}\mathcal{S}^k$, the tangent space at (a,ϕ) to $\mathcal{S}^k$, is given by

$$T_{(a,\phi)}\mathcal{S}^k = \big\{(\beta,\eta) \in H^k(\Omega^1) \times H^k(E) : \tilde{\mathcal{D}}_{(a,\phi)}(\beta,\eta) = 0\big\}\,.$$

The first derivative of $V_{\lambda,\tau}$ determines a nonlinear differential operator, the Euler-Lagrange operator, which is a smooth function,

$$\begin{aligned} V'_{\lambda,\tau} &: \mathcal{A}^k \times H^k(E) \to H^{k-2}(\Omega^1) \times H^{k-2}(E) \\ (A,\Phi) &\mapsto \big(d^*dA + (i\Phi, d_A\Phi), d_A^* d_A\Phi - \tfrac{\lambda}{2}\Phi(\tau - |\Phi|^2)\big)\,. \end{aligned} \tag{4.7}$$

It should be mentioned that the adjoint operators d^*, d_A^* depend, respectively, on g and g,h. The second derivative of $V_{1,\tau}$ at (a,ϕ) defines a symmetric quadratic form $\mathrm{Hess}_{(a,\phi)}$, with associated symmetric operator $\tilde{L}_{(a,\phi)}$. Clearly

$$\mathrm{Hess}_{(a,\phi)}(\beta,\eta) = \int \big\langle(\beta,\eta), \tilde{L}_{(a,\phi)}(\beta,\eta)\big\rangle d\mu_g. = \int_{S^2} \big|\tilde{\mathcal{D}}_{(a,\phi)}(\beta,\eta)\big|^2 d\mu_g\,. \tag{4.8}$$

Since the Hessian has an infinite dimensional kernel the corresponding second order differential operator is not elliptic. If, however, following Taubes, we introduce a modified Hessian

$$\begin{aligned} \overline{\mathrm{Hess}}_{(a,\phi)}(\beta,\eta) &= \mathrm{Hess}_{(a,\phi)}(\beta,\eta) + \big\|\delta^*_{(a,\phi)}(\beta,\eta)\big\|^2_{L^2} \\ &= \int \big|\mathcal{D}_{(a,\phi)}(\beta,\eta)\big|^2 d\mu_g \\ &= \int \big\langle(\beta,\eta), L_{(a,\phi)}(\beta,\eta)\big\rangle d\mu_g\,. \end{aligned} \tag{4.9}$$

then $\mathcal{D}_{(a,\phi)}$ and $L_{(a,\phi)}$ are, respectively, first and second order elliptic operators given by the formulae

$$\mathcal{D}_{(a,\phi)} : H^k(\Omega^1) \times H^k(E) \to H^{k-1} \times H^{k-1} \times H^{k-1}\big(\Omega^{0,1}(E)\big)$$
$$(\beta,\eta) \mapsto \big(* \, d\beta + (\phi,\eta), d^*\beta + (i\phi,\eta), D_A^{(0,1)}\eta - \tfrac{i}{2}(\beta_1 + i\beta_2)\phi d\bar{z}\big)$$

and

$$\overline{\mathrm{Hess}}_{(a,\phi)}(\beta,\eta) = \int_{S^2} \Big(|d\beta|^2 + |d^*\beta|^2 + |\nabla_a\eta|^2 + |\phi|^2\big(|\beta|^2 + |\eta|^2\big) + 4\beta \cdot \langle i\nabla_a\phi,\eta\rangle - \tfrac{1}{2}\big(1 - |\phi|^2\big)\langle\phi,\eta\rangle^2\Big) d\mu_g \,. \quad (4.10)$$

Remark. It can be checked directly that $L_{(a,\phi)}$ preserves the decomposition (4.3); in fact on $T_{(a,\phi)}(\mathcal{G}^{k+1} \cdot (a,\phi))$ it acts as

$$L_{(a,\phi)}\big(\delta_{(a,\phi)}\chi = \delta_{(a,\phi)}(-\Delta + |\phi|^2)\chi\big) \,, \quad (4.11)$$

while on $\operatorname{Ker}\delta_{(a,\phi)}$ by definition $L_{(a,\phi)} = \tilde{L}_{(a,\phi)}$.

Lemma 4.2.1. *There exists $\gamma > 0$ such that, for all $(a,\phi) \in \mathcal{S}^1$ and $(\beta,\eta) \in H^1(\Omega^1) \times H^1(E)$ orthogonal in L^2 to $\operatorname{Ker} L_{(a,\phi)}$,*

$$\int_{S^2} \big\langle(\beta,\eta), L_{(a,\phi)}(\beta,\eta)\big\rangle d\mu_g \geq \gamma \big\|(\beta,\eta)\big\|^2_{H^1(\Omega^1)\times H^1(E)} \,. \quad (4.12)$$

$\operatorname{Ker} L_{(a,\phi)} = \operatorname{Ker}\mathcal{D}_{(a,\phi)}$ is a 2N dimensional vector space.

Proof. A direct calculation as in [St1], which goes back to E. Weinberg, shows that $\operatorname{Ker}\mathcal{D}^*_{(a,\phi)} = \{0\}$. On the other hand an examination of the principal term in the formula for $\mathcal{D}_{(a,\phi)}$ indicates that its index will be the same as that of the sum of twisted Dolbeault operators:

$$\big(\tfrac{\partial}{\partial\bar{z}} \oplus \tfrac{\partial}{\partial\bar{z}}\big) : \Omega^0(E) \times \Omega^{1,0} \to \Omega^{0,1}(E) \times \Omega^{1,1} \,,$$

which may be computed to be $2N$ (over $\mathbb{R}$). The inequality (4.12) is now proved by contradiction using (4.8) and the fact that $L_{(a,\phi)}$ is non-negative. The following lemma, which describes the manifold structure of $\mathcal{M}$ and $\mathcal{S}^k$, is now proved by standard methods (see [DSt] for details in the case $\Sigma = \mathbb{R}^2$):

Lemma 4.2.2. *For $k = 1, 2\ldots$ the space of minimisers $\mathcal{S}^k$ is a smooth submanifold of $\mathcal{A}^k \times H^k(E)$, with tangent space at (a,ϕ),*

$$T_{(a,\phi)}\mathcal{S}^k = T_{(a,\phi)}\big(\mathcal{G}^{k+1} \cdot (a,\phi)\big) \oplus \operatorname{Ker} L_{(a,\phi)}\big|_{H^k(\Omega^1)\times H^k(E)} \,.$$

The quotient spaces $\mathcal{S}^k/\mathcal{G}^{k+1} = \mathcal{M}$ are isomorphic smooth manifolds with tangent space at the orbit of (a,ϕ) identified with $\operatorname{Ker} L_{(a,\phi)}$.

Remark. As discussed in section 2.1 $\mathcal{M}$ is in fact complex projective space. Complex analytic co-ordinates are obtained in section 4.3.

4.3 Co-ordinates on $\mathcal{M}$. There is a 1-1 correspondence between points on $\mathcal{M}$ and unordered pairs of points on S^2. Choosing the stereographic co-ordinate $z = x^1 + ix^2$ let $\tilde{\mathcal{M}}$ be that part of $\mathcal{M}$ on which neither of the two points Z_1, Z_2 is the point at infinity.

Lemma 4.3.1.
$$P = Z_1 + Z_2 \qquad Q = Z_1 Z_2$$
form a holomorphic system of co-ordinates on $\tilde{\mathcal{M}}$.

Proof. Indeed for any $\delta > 0$ a representative of the equivalence class of solutions determined by $[Z_1, Z_2]$ is given by (2.5) with s as in (2.3). Thus $s = c_{[Z]}(z^2 - Pz + Q)$ so that w and hence (A, Φ) are smooth functions of P, Q. Now it is standard that the moduli space is a smooth manifold whose tangent space at the orbit of A, Φ can be identified with $\operatorname{Ker} \mathcal{D}_{(A,\Phi)}$, the operator defined in (4.6). By an index calculation
$$\dim_{\mathbb{R}} \operatorname{Ker} \mathcal{D}_{(A,\Phi)} = 2N = 4\,.$$
Using the stereographic co-ordinate $z = x^1 + ix^2$ the condition that $\operatorname{Ker} \mathcal{D}_{(A,\Phi)}(\dot{A}, \dot{\Phi}) = 0$ reads:
$$\begin{aligned} \partial_1 \dot{A}_2 - \partial_2 \dot{A}_1 &= -\Omega^2 (\Phi, \dot{\Phi}) \\ \partial_1 \dot{A}_1 + \partial_2 \dot{A}_2 &= \Omega^2 (i\Phi, \dot{\Phi}) \\ D_A^{(0,1)} \dot{\Phi} - \tfrac{i}{2} (\dot{A}_1 dx^1 + i \dot{A}_2 dx^2) \Phi &= 0\,. \end{aligned} \tag{4.13}$$
Notice that these equations define a complex vector space with complex structure
$$J : (\dot{\Phi}, \dot{A}_1, \dot{A}_2) \to (i\dot{\Phi}, -\dot{A}_2, \dot{A}_1).$$
Now, following [S], it is shown how to obtain solutions of (4.13); write $Z_j = X_j + iY_j$ for $j = 1, 2$. If it were not for the gauge condition (i.e. the second of equations (4.13) this would just be a matter of differentiating with respect to the parameters X_j, Y_j. A calculation however shows that that given a complex number $\dot{Z}_j$ an element $\dot{A}, \dot{\Phi}$ of $\operatorname{Ker} \mathcal{D}_{(A,\Phi)}$ can be obtained from the formula,
$$\begin{aligned} \frac{\dot{\Phi}}{\Phi} &= \left(\dot{Z}_j \frac{\partial}{\partial Z_j} \right) (\ln |\Phi|^2) \\ (\dot{A}_1 - i\dot{A}_2) &= i(\partial_1 - i\partial_2) \left(\dot{Z}_j \frac{\partial w}{\partial Z_j} \right) \end{aligned} \tag{4.14}$$

This follows using the fact that differentiation of (2.6) with respect to X_j gives

$$d^* d \tfrac{\partial w}{\partial X_j} + |\Phi|^2 \tfrac{\partial}{\partial X_j}\big(\ln|\Phi|^2\big) = 0 \tag{4.15}$$

and there is a similar equation with Y_j replacing X_j. Next it follows from the fact that the map $(Z_1, Z_2) \to (P, Q)$ is holomorphic that given $(\dot{P}, \dot{Q}) \in \mathbb{C}^2$

$$\begin{aligned} \tfrac{\dot{\Phi}}{\Phi} &= \Big(\dot{P}\tfrac{\partial}{\partial P} + \dot{Q}\tfrac{\partial}{\partial Q}\Big)\big(\ln|\Phi|^2\big) \\ (\dot{A}_1 - i\dot{A}_2) &= i(\partial_1 - i\partial_2)\Big(\dot{P}\tfrac{\partial w}{\partial P} + \dot{Q}\tfrac{\partial w}{\partial Q}\Big)\,. \end{aligned} \tag{4.16}$$

generates elements of $\operatorname{Ker}\mathcal{D}_{(A,\Phi)}$. It follows from (4.15) that $\partial w/\partial P \neq \partial w/\partial Q$, and therefore (4.16) provides a *complex* linear isomorphism $\mathbb{C}^2 \to \operatorname{Ker}\mathcal{D}_{(A,\Phi)}$ for all P, Q. Therefore since the moduli space is locally diffeomorphic to $\operatorname{Ker}\mathcal{D}_{(A,\Phi)}$ this proves that P, Q form holomorphic co-ordinates on $\tilde{\mathcal{M}}$. □

4.4 Symmetry properties of vortices. The following fact about the symmetry properties of the vortex solutions for $N = 2$ is needed.

LEMMA 4.4.1. *Choose co-ordinates by stereographic projection with the origin midway between the two zeros of* Φ*; then the solution to* (2.1) *generated according to* (2.5), (2.6) *has the symmetry properties:*

$$\phi(-x) = \phi(x)\,, \qquad a_j(-x) = -a_j(x)\,. \tag{4.17}$$

Proof. Firstly solutions of (2.6) are unique. Indeed let w_1, w_2 be two solutions then they are bounded by (2.7, 2.9), and therefore

$$(e^{w_1} - e^{w_2})(w_1 - w_2) \geq c(w_1 - w_2)^2\,.$$

Next it follows by subtracting the two equations and multiplying by $(w_1 - w_2)$ and then integrating that (using the same notation as in section 2.2)

$$\big\|d(w_1 - w_2)\big\|^2_{L^2} + c\delta^2 \int |s|^2 (w_1 - w_2)^2 \leq 0\,.$$

The operator $d^*d + c\delta^2|s|^2$ is a strictly positive operator since s vanishes at only two points. Therefore $w_1 = w_2$.

Now to prove the given symmetry property notice that since s is even it follows that if $w(x)$ is a solution then so is $w(-x)$. Therefore by uniqueness of the solution $w(x) = w(-x)$. □

With respect to the symmetry operation in (4.17) the kernel of $L_{(a,\phi)}$ admits an L^2-orthogonal decomposition into two two dimensional subspaces,

$$\operatorname{Ker} L_{(a,\phi)} = \operatorname{Ker} L^+_{(a,\phi)} \times \operatorname{Ker} L^-_{(a,\phi)}\,. \tag{4.18}$$

Indeed inspection of the formulae (4.14) shows that $\operatorname{Ker} L^+_{(a,\phi)}$ is spanned by the zero modes arising from varying the distance between the vortices and from the action of the S^1 subgroup which leaves fixed the origin. On the other hand $\operatorname{Ker} L^-_{(a,\phi)}$ arises from the (infinitesimal) action of $SO(3)$ in the directions orthogonal to that which fixes the origin. $\operatorname{Ker} L^+_{(a,\phi)}$ is spanned by $(\beta,\eta) \in \Omega^1 \times \Omega^0(E)$ which satisfy the same symmetry conditions as (4.17), while in $\operatorname{Ker} L^-_{(a,\phi)}$ the conditions are reversed,

$$\eta(-x) = -\eta(x)\,, \qquad \beta_j(-x) = \beta_j(x)\,. \tag{4.19}$$

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