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DETERMINANT BUNDLE, QUILLEN METRIC, AND PETERSSON-WEIL FORM ON MODULI SPACES

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1 Introduction

It is now known that a moduli space of projective manifolds with ample canonical class has a rich local structure. From a theorem of Aubin and Yau, a projective manifold – whose canonical class is ample – admits a unique Hermitian-Einstein metric. This Hermitian-Einstein metric induces a Hermitian structure on any associated space of harmonic forms, and in particular, on the first cohomology of the tangent sheaf. Given a moduli space of the above mentioned type, the Hermitian structure constructed on it using this Hermitian form on the first cohomology of the tangent sheaf turns out to be a Kähler structure. On a moduli space of compact hyperbolic Riemann surfaces the Kähler form coincides with the well-known Petersson-Weil form. This generalized Petersson-Weil form on a moduli space of smooth projective manifolds with ample canonical class can be realized as the curvature of the Hermitian connection on a certain determinant line bundle equipped with the Quillen metric [FuS]. If we restrict ourselves to the case of dimension one, namely the Riemann surfaces of genus g with $g \geq 2$, then this property of the Petersson-Weil form on the moduli space of Riemann surfaces of genus g was established earlier in [W] and [ZT1]. Further properties of the Petersson-Weil form on a moduli space of Riemann surfaces can be found in [ZT2] and [I].

An analytic construction of moduli spaces of Kähler-Einstein manifolds (also in the non-reduced category) can be based upon deformation theory: The automorphism group of the distinguished fiber acts on the base of a universal deformation, thus giving rise to an open subset of the moduli space. Unless just manifolds of dimension one are treated, the base of a universal deformation may be singular, in particular it may be non-reduced. In the latter case the space of infinitesimal deformations is larger than the tangent space of the reduced base space. However, the Petersson-Weil Hermitian metric is defined on the whole tangent space through the inner product of harmonic representatives of Kodaira-Spencer classes with respect to a

Kähler-Einstein metric on the respective fiber. These representatives arise in a natural way from the metric tensors on the fibers in the holomorphic family. The construction in [FuS] for singular reduced base spaces also applies in the non-reduced case: The differential structure of the central fiber is fixed and the singular base space embedded into some smooth space so that the given family of holomorphic structures can be extended to a differentiable family of almost complex structures. In this way all relevant differential operators can be extended to the ambient space. The implicit function theorem gives an extension of the Kähler-Einstein metric on the central fiber to a tensor on the total space of the whole family, which will be finally restricted to the (singular) holomorphic family. These considerations hold also in the situation of moduli spaces of Hermite-Einstein bundles.

In this paper we consider moduli spaces of pairs of the form (X, E), where E is a unitary flat bundle over a smooth projective manifold X whose canonical line bundle K_X is ample. Later, in the fourth section, we will consider projectively flat connections. With little modification our results can be carried over to the case of polarized Ricci-flat varieties.

The tangent space at a point of such a moduli space is given by the first cohomology of a certain holomorphic vector bundle over X known as the *Atiyah bundle*. Furthermore, such a moduli space has a natural Kähler structure, which will be called the Petersson-Weil metric. This Petersson-Weil Kähler form coincides with the natural Kähler form on the unitary representation space, if X is kept fixed. Also, if the representation is kept fixed locally, then the form coincides with the earlier mentioned Kähler form on a moduli of projective manifolds with ample canonical class. The precise statement is given in Lemma 2.15.

The main property established about this Petersson-Weil form is that it can be realized (even in the non-reduced case) as a "fiber-integral", i.e., fiberwise integral of a differential form over the universal space for X [Theorem 3.1]. As a consequence, the Kähler form (rather a rational multiple of it) coincides with the curvature form of the Hermitian connection on a certain determinant line bundle equipped with the Quillen metric. This is a generalization of the earlier mentioned theorem of [FuS]. When X is a fixed Riemann surface, then in his well-known paper [Q], D. Quillen proved that the curvature of the determinant line bundle over the representation space $\mathrm{Hom}(\pi_1(X), U(n))/U(n)$, equipped with the Quillen metric, coincides with the natural symplectic form on $\mathrm{Hom}(\pi_1(X), U(n))/U(n)$. We also recall that the earlier mentioned result of [W] and [ZT1] on the Petersson-Weil form on the moduli space of Riemann surfaces was also proved by establishing a fiber integral formula. If the base of a universal deformation is singular or non-reduced, it extends locally to a smooth ambient space as a Kähler form, since it possesses locally a $\partial\bar{\partial}$-potential. We note that there are some essential differences between the dimension one situation and the higher dimensional case.

H. Royden in [R] showed that the holomorphic sectional curvature of Teichmüller space $\mathcal{T}_g$ is strictly negative. Subsequently, Royden, Wolpert, [W], showed that the sectional curvature of $\mathcal{T}_g$ is strictly negative. In fact, Wolpert produced an explicit upper bound for the curvature, which was conjectured by Royden. Moreover, the curvature is strongly negative in the sense of Siu [S2]. The curvature of the Petersson-Weil metric on the Teichmüller space has been considered also in [J], [Tr].

In Theorem 4.1 the curvature of the Petersson-Weil Kähler metric on the moduli space of pairs (X, E) of the above type is computed. It turns out that when X is kept fixed, the holomorphic sectional curvature of the Petersson-Weil metric on the moduli space of unitary flat connections on X is nonnegative [Corollary 4.3].

In the second part of the paper (sections 5 and 6), we consider the situation where X is a smooth quasi-projective variety over $\mathbb{C}$. It is known that certain equivalence classes of unitary representation spaces for the fundamental group of $X_0 = X - D$, where D is a divisor with normal crossings in a projective manifold X, share many of the properties of the space of equivalence classes of unitary representation space for the fundamental group of a projective manifold. The representation spaces for the fundamental group of X_0 in question are obtained by fixing the conjugacy classes of the monodromies around the irreducible components of D. Such a space has a natural symplectic form, which was constructed in [BiGu] when X is a Riemann surface and in [BrF] for arbitrary dimension.

In sections 5 and 6 we actually consider the special case where all the local monodromies around the boundary components are of finite order. In such a situation, the symplectic form on the representation space constructed in [BiGu] and [BrF] has a rather simple description. Moreover the proof of the result, saying that the symplectic form is actually Kähler, is also immediate in this situation of monodromies of finite order. We express an integral multiple of this Kähler form as the curvature form of the Hermitian connection on a certain determinant line bundle equipped with the Quillen metric. This implies – in particular – that the cohomology class represented by the Kähler form is a rational one. Finally we compute the curvature of this Kähler form using Theorem 4.1.

W. Goldman in [G1] introduced a symplectic structure on the space $\mathcal{R}_X(G) = \mathrm{Hom}(\pi_1(X), G)/G$ consisting of equivalence classes of representation of the fundamental group of a compact oriented surface X into a reductive group G. If the genus of X is at-least two, then using the uniformization theorem, the Teichmüller space $\mathcal{T}(X)$ coincides with a component of $\mathcal{R}_X(PSL(2,\mathbb{R}))$. In [G1, Proposition 2.5] it has been proved that the restriction of the symplectic form on $\mathcal{R}_X(PSL(2,\mathbb{R}))$ to the component $\mathcal{T}(X)$ actually coincides with a multiple of the Petersson-Weil form on $\mathcal{T}(X)$.

Now, let $\mathcal{T}_g^n$ denote the Teichmüller space of genus g Riemann surfaces with n punctures. Assume that $2g-2+n > 0$, in order to to ensure that any punctured Riemann surface Y represented in $\mathcal{T}_g^n$ has the Poincaré metric. Using the Poincaré metric, the vector space of quadratic differentials on Y, with at-most simple poles at the punctures, gets a Hermitian structure. This Hermitian structure defines the Petersson-Weil form on $\mathcal{T}_g^n$. On the other hand, $\mathcal{T}_g^n$ coincides with a component of the subset of the representation space $\mathrm{Hom}(\pi_1(Y), PSL(2,\mathbb{R}))/PSL(2,\mathbb{R})$ defined by the condition that the monodromy around any puncture in Y is conjugate to the regular unipotent element in $PSL(2,\mathbb{R})$. Thus, by a construction in [BiGu], the space $\mathcal{T}_g^n$ gets a symplectic structure. Imitating the proof of [G1, Proposition 2.5] it is easy to establish that this symplectic form on $\mathcal{T}_g^n$ is again a multiple of the Petersson-Weil form.

Fixing an element $g \in \pi_1(X)$, and also a function f on G invariant under the adjoint action, a function F on the representation space $\mathcal{R}_X(G)$ is obtained. The Poisson bracket $\{F, F_1\}$ of two such functions F and F_1 has a very simple description, known as the *product formula* [G2]. It will be interesting to be able to develop an analog of the product formula expressing the Poisson bracket of two functions, obtained in the above fashion, on the space of equivalence classes of representations of the fundamental group of a higher dimensional Kähler manifold.

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2 Local Structure of a Moduli Space of Projective Manifolds Together with a Unitary Flat Vector Bundle

We denote by X a connected smooth projective variety over $\mathbb{C}$ with ample canonical line bundle K_X. The degree of a coherent sheaf on X will be defined using the polarization K_X. Let E be a stable vector bundle of rank r over X satisfying the condition

$$c_1(E) = 0 = c_2(E)\,. \tag{2.1}$$

Invoking a theorem due to Donaldson [Do], such a vector bundle E admits a unique unitary flat connection.

In [STo2], the existence of a moduli space of pairs (X, E) of the above type has been established. Furthermore, a Hermitian structure on the space of infinitesimal deformations of such a pair has been constructed. This Hermitian structure actually endows the moduli space with a Kähler structure [STo2, Theorem 2]. This Kähler structure is a generalization of the Petersson-Weil metric on the moduli space of compact hyperbolic Riemann surfaces. It may be noted that if the underlying complex manifold X is kept fixed, then there is a Petersson-Weil Kähler form on any moduli space of stable bundles over X, not necessarily satisfying the condition (2.1).

A word on the notation that will be used : the terminologies of a vector bundle and the corresponding coherent sheaf will be interchanged without making any distinction between them.

2a Infinitesimal deformations of a pair. Take a pair (X, E) of the earlier type. Let Σ_X denote the sheaf of infinitesimal deformations of the pair. Recall that

$$\Sigma_X \subset \mathrm{Diff}^1_X(E, E)$$

is a subbundle of the sheaf of differential operators of order one mapping sections of E to itself. The subbundle is defined by the property that the image of it by the symbol map coincides with the tangent bundle $TX \subset TX \otimes \mathrm{End}(E)$ consisting of vector of the type $v \otimes Id$. In other words, Σ_X fits into the following *Atiyah exact sequence*

$$0 \longrightarrow \mathrm{End}(E) \longrightarrow \Sigma_X(E) \xrightarrow{\sigma} TX \longrightarrow 0 \tag{2.2}$$

where σ is the symbol map. The space of all infinitesimal deformations of the pair (X, E) is parametrized by $H^1(X, \Sigma_X(E))$. The Kodaira-Spencer deformation map for the pair will be briefly recalled.

Take a connected analytic space S with a base point $0 \in S$, and a family

$$\begin{array}{ccc} E & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ 0 & \longrightarrow & S \end{array} \tag{2.3}$$

of pairs parametrized by S, together with a fixed identification of the pair over the base point 0 with the given pair (X, E). The map π in (2.3) is assumed to be proper and smooth with connected fibers $\mathcal{X}_s = \pi^{-1}(s)$, where $s \in S$. Furthermore, the restricted vector bundle $\mathcal{E}_s = \mathcal{E}|_{\mathcal{X}_s}$ is assumed to be stable for all $s \in S$. Taking the direct image, using π, of the exact sequence

$$0 \longrightarrow \Sigma_{\mathcal{X}/S} \longrightarrow \Sigma_{\mathcal{X}}(\mathcal{E}) \longrightarrow \pi^* TS \longrightarrow 0 \tag{2.4}$$

of sheaves on $\mathcal{X}$, where $\Sigma_{\mathcal{X}}(\mathcal{E})$ is the Atiyah bundle for the vector bundle $\mathcal{E}$ and $\Sigma_{\mathcal{X}/S}$ is the relative Atiyah bundle, the following connecting homomorphism

$$\pi_* \pi^* TS = TS \otimes_{\mathcal{O}_S} \pi_* \mathcal{O}_{\mathcal{X}} \longrightarrow R^1 \pi_* \Sigma_{\mathcal{X}/S}$$

is obtained. Let

$$\rho : TS \longrightarrow R^1 \pi_* \Sigma_{\mathcal{X}/S} \tag{2.5}$$

be the pre-composition of the above homomorphism with the homomorphism induced by the obvious inclusion of $\pi^{-1}(\mathcal{O}_S)$ in $\mathcal{O}_{\mathcal{X}}$. This homomorphism ρ is the Kodaira-Spencer map that we are seeking. The restriction

$$\rho(0) : T_0 S \longrightarrow H^1 = \big(X, \Sigma_X(E)\big) \, ,$$

of the above homomorphism of $\mathcal{O}_S$-modules to $0 \in S$, is the infinitesimal deformation map for the pair (X, E) in the family (2.3).

We do not assume that the dimensions of the vector spaces $H^1(\mathcal{X}_s, \Sigma_{\mathcal{X}_s})$, $s \in S$, are independent of s, thus allowing singular, and also not necessarily reduced, base spaces S of universal deformations of the pair (X, E).

The amplitude of the canonical line bundle K_X ensures that $H^0(X, TX) = 0$, and hence the exact sequence in (2.2) yields the following exact sequence of cohomologies

$$\begin{aligned} 0 \longrightarrow H^1\big(X, \mathrm{End}(E)\big) \longrightarrow H^1\big(X, \Sigma_X(E)\big) \longrightarrow \\ \longrightarrow H^1(X, TX) \stackrel{\mu}{\longrightarrow} H^2\big(X, \mathrm{End}(E)\big) \, . \end{aligned} \tag{2.6}$$

It will be shown that the above homomorphism μ vanishes. Furthermore, a natural splitting of the resulting surjection $H^1(X, \Sigma_X(E)) \to H^1(X, TX)$ in (2.6) will be constructed.

We already noted that since the vector bundle E is stable with vanishing Chern classes, it admits a unique unitary flat connection. This connection can be utilized in constructing a splitting of the sequence (2.2). More precisely, denoting the unitary flat connection on E by ∇, for a local section v of TX, the local differential operator D_v on E defined by

$$D_v(s) = \nabla_v s$$

is actually a lift of v to a local holomorphic section of $\Sigma_X(E)$, and, in particular, $\sigma(D_v) = v$. In other words, we have a splitting of the homomorphism σ in (2.2). Such a splitting amounts to a holomorphic decomposition

$$\Sigma_X(E) = \mathrm{End}(E) \oplus TX \tag{2.7}$$

with the property that the homomorphism σ in (2.2) is the natural projection onto the factor TX of the decomposition. This immediately implies that the homomorphism μ in (2.6) must vanish, and moreover, the inclusion

$$H^1(X, TX) \longrightarrow H^1\big(X, \Sigma_X(E)\big)$$

obtained using the decomposition in (2.7), is actually a splitting of the surjective homomorphism

$$H^1\big(X, \Sigma_X(E)\big) \longrightarrow H^1(X, TX)$$

constructed in (2.6).

The decomposition in (2.7) will be used in the construction of the Petersson-Weil form on the tangent space $H^1(X, \Sigma_X(E))$ to the moduli space of pairs at the point represented by (X, E).

2b Construction of the Petersson-Weil metric. Since K_X is ample, X admits a unique Kähler-Einstein metric, i.e., a Kähler metric with the property that the Ricci form is the negative of the Kähler form [A], [Y]. Let ω denote the Kähler-Einstein form on X.

The decomposition (2.7) gives the following decomposition

$$H^1\big(X, \Sigma_X(E)\big) = H^1\big(X, \mathrm{End}(E)\big) \oplus H^1(X, TX) \tag{2.8}$$

of cohomology.

The L^2-inner-product of the harmonic representatives of the elements of the cohomology group $H^1(X, TX)$, with respect to ω, gives a Hermitian form on $H^1(X, TX)$. *Let ω_X denote this nondegenerate Hermitian form on $H^1(X, TX)$.* For a family of Kähler-Einstein manifolds, the $(1,1)$-form on the parameter space, obtained by the above pointwise construction, is closed [STo2], [Koi].

The Hermitian form on $H^1(X, \mathrm{End}(E))$ is defined by

$$(\alpha, \beta) \longmapsto \int_X \mathrm{trace}(\alpha \wedge \bar{\beta}) \wedge \omega^{d-1} \tag{2.9}$$

where $d = \dim_{\mathbb{C}} X$, and α, β are the harmonic representatives with respect to the combination of the Kähler-Einstein metric and the natural unitary structure of $\mathrm{End}(E)$ (the vector bundle $\mathrm{End}(E)$ has a canonical Hermitian metric induced by a Hermitian-Einstein metric on E). It is known that the pairing in (2.9) is nondegenerate [Si2, Lemma 2.6].

Let ω_E denote the Hermitian form on $H^1(X, \mathrm{End}(E))$ defined by (2.9).

DEFINITION 2.10. The *Petersson-Weil form* on $H^1(X, \Sigma_X(E))$, denoted by ω^{pw}, is defined to be the Hermitian form $\omega_X + \omega_E$ with respect to the decomposition (2.8).

For a family of pairs as in (2.3), the pointwise construction of the Petersson-Weil form will give a $(1,1)$-form on the parameter space S. With a slight abuse of notation, this form on S will also be denoted by ω^{pw}.

LEMMA 2.11. *The $(1,1)$-form ω^{pw} on S is closed.*

Proof. The form ω^{pw} decomposes as $\omega_1 + \omega_2$, where ω_1 (respectively, ω_2) is obtained from the pointwise construction of ω_X (respectively, ω_E) as in Definition 2.10. We already noted that ω_1 is closed. So it suffices to prove that ω_2 is closed. Also, it is enough to prove the closedness of ω_2 under the assumption that S is simply connected.

Let r be the rank of the vector bundle E in (2.3).

Consider the space of equivalence classes of irreducible representations

$$\mathcal{R} := \frac{\mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), U(r))}{U(r)} \subset \frac{\mathrm{Hom}(\pi_1(X), U(r))}{U(r)} . \tag{2.12}$$

For any $\rho \in \mathcal{R}$, the Zariski tangent space $T_\rho \mathcal{R}$ is $H^1(X, Ad(\rho))$, where $Ad(\rho)$ is the local system on X associated to ρ for the adjoint action of $U(r)$ on its Lie algebra.

After identifying the unitary representation space $\mathcal{R}$ with the space of stable bundles over X satisfying the numerical condition (2.1), the Kähler form for the pairing in (2.9) translates into

$$(\hat{\alpha}, \hat{\beta}) \longmapsto \int_X \mathrm{trace}(\hat{\alpha} \wedge \hat{\beta}) \wedge \omega^{d-1} \tag{2.13}$$

where $\hat{\alpha}$ (respectively, $\hat{\beta}$) is the tangent vector of $\mathcal{R}$ corresponding to α (respectively, β). Note that $\mathrm{trace}(\hat{\alpha} \wedge \hat{\beta}) \in H^2(X, \mathbb{R})$ as $\hat{\alpha}, \hat{\beta} \in H^1(X, Ad(\rho))$.

The 2-form on $\mathcal{R}$ defined by (2.13), which we will denote by $\bar{\omega}$, is actually closed [Ko], [STo1].

If S is simply connected, the inclusion of $X = \pi^{-1}(0)$ in $\mathcal{X}$ induces an isomorphism of the corresponding fundamental groups. This implies that for any $s \in S$, the two representation spaces

$$\mathcal{R} \quad \text{and} \quad \frac{\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\pi^{-1}(s)), U(r))}{U(r)}$$

are canonically identified. In particular, we have a natural map

$$f : S \longrightarrow \mathcal{R}$$

for the family (2.3), which assigns to any $s \in S$ the element in $\mathcal{R}$ that corresponds to the stable vector bundle $\mathcal{E}|_{\mathcal{X}_s}$ over $\mathcal{X}_s$ by the above identification.

The 2-form ω_2 on S coincides with $f^*\bar{\omega}$, where $\bar{\omega}$ is defined in (2.13). Now the lemma follows from the closedness of the form $\bar{\omega}$. □

The above proof of Lemma 2.11 leads to the following observation on the local structure of the moduli space of pairs of the form (X, E).

Let $\mathcal{M}$ denote a moduli space of pairs of type (X, E), where K_X is ample and E is a stable vector bundle over X satisfying (2.1). Let $\mathcal{M}_1$ denote the moduli space of all X of the above type. So there is a forgetful surjective map

$$F : \mathcal{M} \longrightarrow \mathcal{M}_1$$

which sends the point of $\mathcal{M}$ representing the pair (X, E) to the point of $\mathcal{M}_1$ representing X. Clearly F is a holomorphic map.

For any simply connected subspace $U \subseteq \mathcal{M}_1$, the pre-image $F^{-1}(U)$ is naturally identified with $U \times \mathcal{R}$, where $\mathcal{R}$, as in the proof of Lemma 2.11, is the representation space for the fundamental group of a fixed variety

$$X \quad := F^{-1}(0) \subset F^{-1}(U)$$

This local product structure of $\mathcal{M}$ defines a foliation of $\mathcal{M}$ transversal to the fibers of the projection F. This foliation on $\mathcal{M}$ will be denoted by $\mathcal{F}$.

LEMMA 2.14.. *The leaves of the foliation $\mathcal{F}$ are locally holomorphic subspaces of $\mathcal{M}$.*

(We cannot say that a leaf is a holomorphic subspace since it may not be locally closed in $\mathcal{M}$.)

Proof of Lemma 2.14. Let $\bar{\pi} : \mathcal{X}_{\mathcal{M}_1} \to \mathcal{M}_1$ be the universal family over $\mathcal{M}_1$. Since the automorphism group of a projective manifold with ample canonical class is finite, the universal family exists over a finite cover of $\mathcal{M}_1$. Given the local nature of the statement of the lemma, we do not need to deal with this issue, and we will treat the family as if it is over $\mathcal{M}_1$.

Fix a point $\rho \in \mathcal{R}$. For a simply connected subspace $U \subseteq \mathcal{M}_1$, as above, using the isomorphism between $\pi_1(X)$ and $\pi_1(\bar{\pi}^{-1}(U))$, induced by the inclusion of X in $\bar{\pi}^{-1}(U)$, the representation ρ gives a unitary flat vector bundle $\mathcal{E}$ over $\bar{\pi}^{-1}(U)$.

Now set $S = U$ in (2.3), and also set $\mathcal{E}$ as above. The classifying map

$$f_S : U \longrightarrow F^{-1}(U)$$

for this family, which assigns to any $u \in U$ the element in $F^{-1}(u)$ represented by the vector bundle $\mathcal{E}_{\bar{\pi}^{-1}(u)}$, is the leaf of the foliation $\mathcal{F}$ passing through ρ. Since the map f_S is holomorphic, the lemma is established. □

We remark that Lemma 2.14 does not imply that $\mathcal{F}$ is a holomorphic foliation on $\mathcal{M}$. The foliation $\mathcal{F}$ would be holomorphic if and only if the C^∞ subbundle of the tangent bundle $T\mathcal{M}$ defining $\mathcal{F}$ is a holomorphic subbundle. In fact, $\mathcal{F}$ is not in general a holomorphic foliation – for example, when X is a Riemann surface of genus at-least 3.

The following lemma is immediate from the definition of the foliation $\mathcal{F}$ and the Petersson-Weil form.

Lemma 2.15. *The restriction of the Petersson-Weil form to a leaf of the foliation $\mathcal{F}$ is the pullback using the map F of the Petersson-Weil form on $\mathcal{M}_1$.*

Using the decomposition of the tangent space of $\mathcal{M}$ into the direct sum of tangents along fiber of F and tangents along $\mathcal{F}$, and invoking Lemma 2.14, we conclude that the almost complex structure of $\mathcal{M}_1$ and that of the moduli space of stable vector bundles together describe the almost complex structure of $\mathcal{M}$. This is done by considering the decomposition (2.8) and simply taking the direct sum of the almost complex structures of the two individual summands. Also, it is immediate that the two transversal foliations of $\mathcal{M}$ are orthogonal with respect to the Petersson-Weil form.

3 A Fiber Integral-formula and the Curvature of the Quillen Connection on a Determinant Bundle

In this section we will first establish a fiber-integral formula for the Petersson-Weil form. Using this it will be shown that the Petersson-Weil form on the moduli space $\mathcal{M}$ coincides with a constant scalar multiple of the curvature of the Hermitian connection on a certain determinant line bundle equipped with the Quillen metric. The determinant line bundle corresponds to a virtual Hermitian bundle on the total space of the family $F^*\mathcal{X}_{\mathcal{M}_1}$. The main theorem of [BGSo] constitutes the key input for the computation of the curvature.

3a A fiber-integral formula. Take a family as in (2.3), where the relative dimension of the projection π is d. Consider the Kähler-Einstein metric on the relative tangent bundle $T_{\mathcal{X}/S}$. The curvature form of the relative canonical line bundle $K_{\mathcal{X}/S}$ with the induced Hermitian metric has the following property: when it is restricted to a fiber of π, it coincides with the relative Kähler form. This curvature form on $\mathcal{X}$ will be denoted by $\omega_{\mathcal{X}}$. Let

$$\omega_{\mathcal{X}} = \sqrt{-1}(g_{\alpha\bar{\beta}}dz^{\alpha} \wedge \overline{dz^{\beta}} + g_{i\bar{\beta}}ds_i \wedge \overline{dz^{\beta}} + g_{\alpha\bar{j}}dz^{a} \wedge \overline{ds^{j}} + g_{i\bar{j}}ds_i \wedge \overline{ds_j})$$

be the local description of $\omega_{\mathcal{X}}$, where z_{α}, z_{β} are the vertical coordinates and z_i, z_j are the coordinates along S. The restriction of $\omega_{\mathcal{X}}$ to any X_s – which is the Kähler-Einstein form on X_s – will be denoted by $\omega_{\mathcal{X}_s}$.

Let h be a Hermitian metric on $\mathcal{E}$ such that the restriction to any $\mathcal{E}_s := \mathcal{E}|_{\pi^{-1}(s)}$ is a flat metric. The curvature of the corresponding Hermitian connection on $\mathcal{E}$ will be denoted by $\Omega_{\mathcal{E}}$.

Let ω^{pw} denote the Petersson-Weil form on the parameter space S for the family considered above.

Theorem 3.1. *The Petersson-Weil form ω^{pw} on S has the following expression:*

$$\omega^{\mathrm{pw}} = \frac{1}{2}\int_{\mathcal{X}/S} \mathrm{trace}(\Omega_{\mathcal{E}} \wedge \Omega_{\mathcal{E}}) \wedge \frac{\omega_{\mathcal{X}}^{d-1}}{(d-1)!} + \int_{\mathcal{X}/S} \frac{\omega_{\mathcal{X}}^{d+1}}{(d+1)!}$$

where $\int_{\mathcal{X}/S}$ is the integration along the fibers of π.

Proof. Before proving the theorem we make a

REMARK ABOUT THE NOTATION. We will use covariant derivatives of *differential forms* on the total space of $\mathcal{X}$ with values in $\mathrm{End}(\mathcal{E})$, or the trivial line bundle, and restrict these to the fibers of the projection π from $\mathcal{X}$ onto S. So there is no contribution of the respective Kähler metrics to the covariant derivatives. Pulling up of indices and the tensor $g^{\bar{\beta}\alpha}(z,s)$ always refer to the metrics $\omega_{\mathcal{X}_s}$. Since $\Omega_{\mathcal{E}}|_{\mathcal{X}_s}$ is equal to the curvature form $\Omega_{\mathcal{E}_s}$ of the given Hermitian metric on $\mathcal{E}_s$, this notation is consistent.

We begin the proof of the theorem by computing the pairing (2.9) explicitly. Let $\partial/\partial s_i \in T_{s_0}S$ be a tangent vector. Then a horizontal lift of $\partial/\partial s_i$ with respect to $\omega_{\mathcal{X}}$ is as follows:

$$v_i = \frac{\partial}{\partial s_i} + a_i^{\alpha}\frac{\partial}{\partial z^{\alpha}}$$

where $a_i^{\alpha} = -g^{\bar{\beta}\alpha}.g_{i\bar{\beta}}$ [FuS]. We apply the splitting (2.7) over the total space $\mathcal{X}$ and obtain $[\theta, v_i]$.

The $\overline{\partial}$-exterior derivative of this form restricted to the fiber X equals

$$(R_{i\bar\beta} + R_{\alpha\bar\beta}a_i^\alpha)dz^{\bar\beta} + A^\alpha_{i\bar\beta}\theta_\alpha dz^{\bar\beta}$$

where $A^\alpha_{i\bar\beta} = \partial a_i^\alpha/\partial z^{\bar\beta}$. With respect to the splitting (2.7), the End(E)-component equals

$$\mu_i = (R_{i\bar\beta} + R_{\alpha\bar\beta}a_i^\alpha)dz^{\bar\beta} = \Omega_{\mathcal{E}} \cup v_i|_X\,.$$

This formula also holds for deformations of pairs (X, E), where E is *projectively unitary flat*, i.e., $R_{\alpha,\bar\beta} = cg_{\alpha\bar\beta}Id_E$. Here in the flat case $c = 0$.

The TX-component is $A^\alpha_{i\bar\beta}\frac{\partial}{\partial z^\alpha}dz^{\bar\beta} \in \mathcal{A}^{0,1}(X, TX)$.

The above TX valued $(0,1)$-form coincides with the harmonic representative of the infinitesimal deformation of X corresponding to it [FuS].

As the next step in the proof we will show that μ_i is harmonic. This is carried out in the following lemma.

Lemma 3.2. *The form μ_i is harmonic.*

Proof of Lemma 3.2.

$$\begin{aligned}\overline{\partial}^*\mu_i &= g^{\bar\beta\alpha}\mu_{i;\alpha} = -g^{\bar\beta\alpha}R_{i\bar\beta;\alpha}\\ &= -g^{\bar\beta\alpha}R_{\alpha\bar\beta;\alpha} = 0\,.\end{aligned}$$

The last step follows from the fact that $R_{\alpha\bar\beta} = 0$. This completes the proof of the lemma. □

From the above lemma it follows that the $H^1(X, \mathrm{End}(E))$-contribution of (2.9) reads

$$\left(\frac{\partial}{\partial s^i}, \frac{\partial}{\partial s^j}\right)^{\mathrm{pw}}_{\mathcal{E}} = \int_X \mathrm{trace}(R_{i\bar\beta}R_{\alpha\bar\jmath})g^{\bar\beta\alpha}g\,dV\,.$$

Continuing with the proof of Theorem 3.1, we observe that since any $(\mathcal{E}_s, h_s)$ is flat,

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{X}/S}\mathrm{trace}(\Omega_{\mathcal{E}}\wedge\Omega_{\mathcal{E}})\wedge\frac{\omega_{\mathcal{X}}^{d-1}}{(d-1)!}\\ &\qquad = \left(\int_X \mathrm{trace}(R_{i\bar\beta}R_{\alpha\bar\jmath})g^{\bar\beta\alpha}g\,dV\right)\sqrt{-1}ds^i\wedge ds^{\bar\jmath} = \omega^{\mathrm{pw}}_{\mathcal{E}}\,.\end{aligned}$$

Next the $H^1(X, TX)$-contribution will be calculated (this is also done in [FuS]).

The component of $\omega_{\mathcal{X}}^{d+1}/(d+1)!$ which contributes to the fiber integral formula equals $\phi g dV\sqrt{-1}ds^i\wedge ds^{\bar\jmath}$, where $\phi = g_{i\bar\jmath} - g_{\bar\beta i}g_{\alpha\bar\jmath}g^{\bar\jmath\alpha}$. Now

$$\Box\phi + \phi = A^\alpha_{i\bar\beta}A^{\bar\delta}_{\bar\jmath\gamma}g_{\alpha\bar\delta}g^{\bar\beta\gamma}$$

so that

$$\int_X \phi g dV = \left(\frac{\partial}{\partial s^i}, \frac{\partial}{\partial s^j} \right)^{\mathrm{pw}}_{\mathcal{X}} = \int_X A^{\alpha}_{i\bar\beta} A^{\bar\delta}_{\bar j \gamma} g_{\alpha\bar\delta} g^{\bar\beta\gamma} g \, dV \, .$$

This completes the proof of the theorem. □

Given a complex space $\mathcal{Y}$, a *Kähler structure in the strong sense* is given by a local differential two form $\omega_{\mathcal{Y}}$ which possesses locally a $\partial\overline{\partial}$-potential of class C^∞ (with respect to the differentiable structure induced by local embeddings into open subsets of some $\mathbb{C}^N$). This form is required to induce the Hermitian metrics on all tangent spaces $T_y\mathcal{Y}$, $y \in \mathcal{Y}$. This means that $\omega_{\mathcal{Y}}$ is locally the restriction of a Kähler form on an ambient space. If $\mathcal{E}$ is a holomorphic vector bundle, a Hermitian metric of class C^∞ is defined in a similar strong sense.

Let $\mathcal{E} \to \mathcal{X} \to S$ be a universal deformation of a flat (respectively projectively flat) holomorphic vector bundle(E, h) over a Kähler-Einstein manifold (X, ω), where S is in general neither smooth nor reduced.

Let S be embedded into a polydisk $\mathcal{U} \subset H^1(X, \Sigma_X)$. Since our situation is local with respect to S, we can use a C^∞-trivialization $\mathcal{X} \to X \times S$ of $\mathcal{X} \to S$, and a holomorphic family J_s, $s \in \mathcal{U}$, of almost complex structures on X which are integrable over S. Furthermore, there is a family of semi-connections on E over $\mathcal{U}$, which satisfy the integrability condition over S and define the holomorphic structure of the vector bundle $\mathcal{E}$ over $\mathcal{X}$. The differential operators ∂ and $\overline{\partial}$ on $\mathcal{X}$ and the connection on $\mathcal{E}$ are induced by differential operators on the ambient space. A Kähler-Einstein form ω on the fiber $X = \mathcal{X}_s$ extends to a two form $\tilde\omega$ on $X \times \mathcal{U}$, whose restriction to $\mathcal{X}$ induces a Kähler form in the above strong sense. Primarily it will be positive definite on the fibers $\mathcal{X}_s$, but adding the pullback of a two form on $\mathcal{U}$, which is Kähler on S, will make it positive definite on $\mathcal{X}$. By adding a term $\sqrt{-1}\partial_s\overline{\partial}_s\phi$, where ∂_s and $\overline{\partial}_s$ are the respective operators on $\mathcal{X}_s$, $s \in S$, the term $\tilde\omega_{\mathcal{X}}|_{\mathcal{X}_s}$ is turned into a Kähler-Einstein metric (of the same constant Ricci curvature). This can be achieved by using the implicit function theorem and using the extended differential operators over $\mathcal{U}$ (cf. [FuS]). In this way we get a Kähler metric $\tilde{\tilde\omega}$ on $\mathcal{X}$ (with possibly singular structure).

In the same way a projective flat (or flat) metric on E over $\mathcal{X}$ is extended to a C^∞ metric on $\mathcal{E}$ using the implicit function theorem, giving the term (projectively) flat a meaning also over a singular base space.

In order to compute (covariant) derivatives of order p in holomorphic and order q in conjugate holomorphic directions say, at some point, we only need that the infinitesimal neighborhoods of the given point of order

$$r = \max(p, q)$$

is isomorphic to the r-th infinitesimal neighborhood of the corresponding target space (and not smaller). In particular, derivatives of order $(1,1)$ always exists.

Finally the metric $\tilde{\tilde{\omega}}$ will be modified. Its determinant defines a Hermitian metric g on the relative anticanonical bundle $K^{-1}_{\mathcal{X}/S}$, which is of class C^∞ in the above strong sense. The form $\sqrt{-1}\partial\overline{\partial} g = \omega_{\mathcal{X}}$ is such that the restriction $\omega_{\mathcal{X}}|_{\mathcal{X}_s}$ is the Kähler-Einstein form on $\mathcal{X}_s$, and the local $\partial\overline{\partial}$-potential is obviously given.

The following result follows from Theorem 3.1.

Corollary 3.3. *The Petersson-Weil form ω^{pw} is locally $\partial\overline{\partial}$-exact on S.*

Proof. Since the integrand in Theorem 3.1 has this property locally, the assertion in Corollary 3.3 follows from [V]. □

3b The determinant bundle on the moduli space. Let $\pi : \mathcal{X} \to S$ be a family of projective manifolds. As earlier, the relative canonical line bundle $K_{\mathcal{X}/S}$ will be assumed to be relatively ample.

Let $p : \mathcal{M} \to S$ be the relative moduli space of stable vector bundle of rank r with trivial determinant and vanishing Chern classes. Thus, for any $s \in S$, the fiber $p^{-1}(s)$ is naturally identified with the representation space

$$\frac{\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\pi^{-1}(s)), SU(r))}{SU(r)}\,.$$

It is known that in general there is no universal vector bundle over the fiber product $\mathcal{M} \times_S \mathcal{X}$. However, there is a natural vector bundle

$$Ad(\mathcal{M}) \longrightarrow \mathcal{M} \times_S \mathcal{X}$$

of rank $r^2 - 1$ such that for any $s \in S$, the restriction

$$Ad\big((p \times \pi)^{-1}(s)\big) := Ad(\mathcal{M})|_{(p\times\pi)^{-1}(s)}$$

to the fiber $(p \times \pi)^{-1}(s)$ is the universal adjoint bundle over $\mathcal{M}_s \times X_s$. The vector bundle $Ad(\mathcal{M})$ is constructed by simply patching the vector bundles given by the trace zero endomorphisms of the universal vector bundles over sufficiently small analytic subsets of $\mathcal{M}$. This construction works since by the projection formula, any two such universal vector bundles over $U \subseteq \mathcal{M}$ differ by the pullback of a line bundle over U.

Given any element of the K-group $K(\mathcal{M} \times_S \mathcal{X})$ of the fiber product $\mathcal{M} \times_S \mathcal{X}$, or equivalently, any formal expression of the type

$$\bar{E} = \sum_{i=1}^{m} a_i E_i$$

where $a_i \in \mathbb{Z}$ and E_i are vector bundles over $\mathcal{M} \times_S \mathcal{X}$, there is a determinant line bundle $\det(\bar{E})$ over $\mathcal{M}$ [KnMu], [BGSo]. The following identity is valid

$$\det(\bar{E}) = \bigotimes_{i=1}^{m} \det(E_i)^{\otimes a_i} ,$$

The fiber of $\det(E)$ over $m \in \mathcal{M}$ is naturally identified with the line

$$\bigotimes_j \bigwedge^{\mathrm{top}} H^j \big(\pi^{-1}(p(m)), E|_{\pi^{-1}(p(m))}\big)^{(-1)^{i+1}}$$

Let

$$\mathcal{L} := \det \big((\mathcal{O}^{r^2-1} - Ad(\mathcal{M})) \otimes (K_{\mathcal{X}/S} - K^{-1}_{\mathcal{X}/S})^{\otimes(d-1)} \big) \tag{3.4}$$

be the determinant line bundle over $\mathcal{M}$. The above line bundle $K_{\mathcal{X}/S}$ over $\mathcal{M} \times_S \mathcal{X}$ denotes, with a slight abuse of notation, the pullback of the relative canonical line bundle $K_{\mathcal{X}/S}$ over $\mathcal{X}$ using the projection onto the second factor.

The vector bundle $Ad(\mathcal{M})$ has a natural flat Hermitian structure induced by a flat Hermitian structures of stable vector bundles represented in the moduli space $\mathcal{M}$; the uniqueness of the Hermitian structure on $Ad(\mathcal{M})$ is ensured by the fact that any two flat Hermitian metrics on a stable vector bundle differ by a constant scalar. Also, the relative tangent bundle $T\mathcal{X}/S$ over $\mathcal{X}$ has a unique Kähler-Einstein metric. In fact, this metric is the restriction of the form $\omega_{\mathcal{X}}$ on the total space of $\mathcal{X}$. This, in turn, induces a Hermitian metric on $K_{\mathcal{X}/S}$.

Thus, invoking a general construction in [BGSo], we have a *Quillen metric* on the determinant line bundle $\mathcal{L}$ defined in (3.4). Let H denote this Hermitian metric on $\mathcal{L}$.

Our next aim is to compute the curvature of the Hermitian connection on $\mathcal{L}$ and compare it with the Petersson-Weil form defined in (2.10).

3c The curvature of the determinant bundle and the Petersson-Weil form. Using the decomposition of the Petersson-Weil form in Definition 2.10, the Petersson-Weil form ω^{pw} on $\mathcal{M}$ decomposes naturally as

$$\omega^{\mathrm{pw}} = \omega_{\mathcal{X}} + \omega_{\mathcal{E}} \, . \tag{3.5}$$

To calculate the curvature of the line bundle $\mathcal{L}$ defined in (3.4) using Theorem 0.1 of [BGSo], first note that there cannot be any contribution of the Todd class in the the formula (0.3) in Theorem 0.1, page 51, of [BGSo], for the case of $\mathcal{L}$ defined in (3.4). Indeed, the lowest degree term of the Chern character

$$Ch\big((\mathcal{O}^{r^2-1} - Ad(\mathcal{M})) \otimes (K_{\mathcal{X}/S} - K^{-1}_{\mathcal{X}/S})^{d-1}\big) = Ch\big((\mathcal{O}^{r^2-1} - Ad(\mathcal{M})) \otimes (K_{\mathcal{X}/S} - K^{-1}_{\mathcal{X}/S})\big)^{d-1}$$

is already of degree $d+1$ (i.e., the corresponding differential form is of degree $2d+2$).

The restriction of the Chern form $c_1(K_{\mathcal{X}/S})$ to a fiber $\mathcal{X}_s$ is the Kähler-Einstein form $\omega_{\mathcal{X}_s}$ on the fiber. Now, in view of Theorem 3.1, from Theorem 0.1 of [BGSo] it is immediate that

$$c_1(\mathcal{L}) = 4\pi r \omega_{\mathcal{E}} \tag{3.6}$$

where $\omega_{\mathcal{E}}$ is the component of ω^{pw} as in (3.5).

Let

$$\mathcal{L}' = \det\big((K_{\mathcal{X}/S} - K^{-1}_{\mathcal{X}/S})^{\otimes(d-1)}\big) \tag{3.7}$$

be the determinant line bundle over $\mathcal{M}$ equipped with the Quillen metric.

The Theorem 10.5, page 166, of [FuS] gives the following formula for the first Chern form of $\mathcal{L}'$:

$$c_1(\mathcal{L}') = \frac{2(d+1)!}{\pi}\omega_{\mathcal{X}} \tag{3.8}$$

(r in Theorem 10.5 of [FuS] is -1 in our situation).

Now combining (3.6) and (3.8) we conclude that

$$c\omega^{\mathrm{pw}} = \det\big(a((\mathcal{O}^{r^2-1} - Ad(\mathcal{M})) \otimes (K_{\mathcal{X}/S} - K^{-1}_{\mathcal{X}/S})^{\otimes(d-1)}) + b(K_{\mathcal{X}/S} - K^{-1}_{\mathcal{X}/S})^{\otimes(d-1)}\big)$$

where the positive numbers a, b and c can be fixed using the constants in (3.6) and (3.8).

4 Curvature of the Petersson-Weil Metric

Take a family of pairs as in (2.3) which is locally complete.

Our aim in this section is to prove the following theorem describing the curvature of the Petersson-Weil metric:

Theorem 4.1.

$$R^{\mathrm{pw},\mathcal{E}}_{i\bar{j}k\bar{l}} = \int_X tr(R_{i\bar{j}} \Box R_{k\bar{l}}) g\, dV + \int_X tr(R_{i\bar{l}} \Box R_{k\bar{j}}) g\, dV$$

$$= \int_X tr\big(G(\Lambda[\mu_i \wedge \mu_{\bar{j}}])\Lambda[\mu_k \wedge \mu_{\bar{l}}]\big) g\, dV + \int_X tr\big(G(\Lambda[\mu_i \wedge \mu_{\bar{l}}])\Lambda[\mu_k \wedge \mu_{\bar{j}}]\big) g\, dV$$

where $\Box$ *denotes the Laplacian for smooth sections of* $\mathrm{End}(E)$ *with non-negative eigenvalues, and* G *is the corresponding Green's operator.*

We will first establish a lemma which will be needed in the proof of Theorem 4.1.

Lemma 4.2.
$$\frac{\partial G^{\mathrm{pw},\mathcal{E}}_{i\bar{j}}(s)}{\partial s^k} = \int_{\mathcal{X}_s} tr(R_{i\bar{\beta};k}R_{\alpha\bar{j}})g^{\bar{\beta}\alpha}g\,dV\,.$$

In order to make computations we use Lie derivatives with respect to horizontal lifts of tangent vectors. The lemma follows from the Remark below, since $L_{v_k}\omega_{\mathcal{X}}$ has no nonzero component in fiber direction.

For differential forms on fibers $\mathcal{X}_s$ the following seven remarks are valid.

Remark (a). $L_{v_k}(R_{i\bar{\beta}}dz^{\bar{\beta}}) = R_{i\bar{\beta};\bar{k}}dz^{\bar{\beta}}$.

Proof.
$$L_{v_k}(R_{i\bar{\beta}}dz^{\bar{\beta}}) = \left[\frac{\partial}{\partial s^k} + a^\alpha_k\frac{\partial}{\partial z^\alpha}, R_{i\bar{\beta}}dz^{\bar{\beta}}\right] = (R_{i\bar{\beta};\bar{k}} + a^\alpha_k R_{i\bar{\beta};\alpha})dz^{\bar{\beta}}$$
$$= (R_{i\bar{\beta};\bar{k}} + a^\alpha_k R_{i\bar{\beta};\alpha})dz^{\bar{\beta}} = R_{i\bar{\beta};\bar{k}}dz^{\bar{\beta}}a\,.$$

Remark (b). $L_{v_{\bar{l}}}(R_{i\bar{\beta}}dz^{\bar{\beta}}) = \bar{\partial}(R_{i\bar{l}} + a^{\bar{\tau}}_{\bar{l}}R_{i\bar{\tau}})$.

Proof.
$$L_{v_{\bar{l}}}(R_{i\bar{\beta}}dz^{\bar{\beta}}) = (R_{i\bar{\beta};\bar{l}} + a^{\bar{\tau}}_{\bar{l}}R_{i\bar{\beta};\bar{\tau}} + R_{i\bar{\tau}}a^{\bar{\tau}}_{\bar{l};\bar{\beta}})dz^{\bar{\beta}} = (R_{i\bar{l}} + a^{\bar{\tau}}_{\bar{l}}R_{i\bar{\tau}})_{;\bar{\beta}}dz^{\bar{\beta}}\,.$$

Remark (c). $\overline{\partial}^*(R_{i\bar{\beta};k}dz^{\bar{\beta}}) = 0$.

Proof.
$$\overline{\partial}^*(R_{i\bar{\beta};k}dz^{\bar{\beta}}) = -g^{\bar{\beta}\alpha}R_{i\bar{\beta};k\alpha} = -g^{\bar{\beta}\alpha}R_{\alpha\bar{\beta};ik} = 0\,.$$

Remark (d). $\overline{\partial}(R_{i\bar{\beta}k}dz^{\bar{\beta}}) = [R_{i\bar{\beta}}, R_{k\bar{\delta}}]dz^{\bar{\beta}}\bigwedge dz^{\bar{\delta}}$.

Proof.
$$R_{i\bar{\beta};k\bar{\delta}} = R_{i\bar{\beta};\bar{\delta}k} + [R_{i\bar{\beta}}, R_{k\bar{\delta}}]$$
$$= R_{i\bar{\delta};\bar{\beta}k} - [R_{i\bar{\beta}}, R_{k\bar{\beta}}] = R_{i\bar{\delta};k\bar{\beta}} + [R_{i\bar{\delta}}, R_{k\bar{\beta}}] - [R_{i\bar{\beta}}, R_{k\bar{\delta}}]\,.$$

Remark (e). $\overline{\partial}^*([R_{i\bar{\beta}}, R_{k\bar{\delta}}]dz^{\bar{\beta}}\bigwedge dz^{\bar{\delta}}) = 0$.

Proof. $g^{\bar{\delta}\gamma}[R_{i\bar{\beta}}, R_{k\bar{\delta}}]_{;\gamma} = 0$, since $R_{\alpha\bar{\beta}} = 0 = R_{\gamma\bar{\delta}}$.

Remark (f). $R^{i\bar{\beta};k}dz^{\bar{\beta}} \in \mathcal{A}^{0,1}(X, \mathrm{End}(E))$ is harmonic.

Remark (g). $\Box R_{i\bar{j}} = g^{\bar{\beta}\alpha}[R_{\alpha\bar{j}}, R_{i\bar{\beta}}]$.

Proof.

$$R_{i\bar{j};\alpha\bar{\beta}} = R_{\alpha\bar{j};i\bar{\beta}} = R_{\alpha\bar{j};\bar{\beta}i} + [R_{\alpha\bar{j}}, R_{\bar{\beta}i}]$$
$$= R_{\alpha\bar{\beta},\bar{j}i} - [R_{\alpha\bar{j}}, R_{i\bar{\beta}}] = -[R_{\alpha\bar{j}}, R_{i\bar{\beta}}]$$

Hence Remark (g) follows.

We begin the proof of Theorem 4.1.

$$\frac{\partial^2 G^{\mathrm{pw},\mathcal{E}}_{i\bar{j}}}{\partial s^k \partial s^{\bar{l}}} = \int_X \mathrm{tr}\big(L_{v_{\bar{l}}}(R_{i\bar{\beta};k} dz^{\bar{\beta}}) \wedge R_{\alpha\bar{j}} dz^{\alpha}\big) \frac{\omega_X^{d-1}}{(d-1)!}$$
$$+ \int_X \mathrm{tr}\big(R_{i\bar{\beta};k} dz^{\beta} \wedge L_{v_{\bar{l}}}(R_{i\bar{j};k} dz^{\alpha})\big) \frac{\omega_{\mathcal{X}}^{d-1}}{(d-1)!}\,.$$

Let I_1 denote $\int_X \mathrm{tr}(L_{v_{\bar{l}}}(R_{i\bar{\beta}_j k} dz^{\bar{\beta}}) \wedge R_{\alpha\bar{j}} dz^{\alpha}) \frac{\omega_X^{d-1}}{(d-1)!}$ and let I_2 denote the expression $\int_X \mathrm{tr}(R_{i\bar{\beta};k} dz^{\beta} \wedge L_{V_{\bar{l}}}(R_{i\bar{j};k} dz^{\alpha})) \frac{\omega_{\mathcal{X}}^{d-1}}{(d-1)!}$.

One can show that $L_{v_{\bar{l}}}(R_{i\bar{\beta}_j k} dz^{\bar{\beta}})$ is cohomologous to

$$\big([R_{i\bar{l}}, R_{k\bar{\beta}}] + [R_{k\bar{l}}, R_{i\bar{\beta}}]\big) dz^{\bar{\beta}}$$

and

$$\mathrm{tr}\big(L_{v_{\bar{l}}}(R_{i\bar{\beta};k}) R_{\alpha\bar{j}}\big) = \mathrm{tr}\big(R_{i\bar{l}}[R_{k\bar{\beta}}, R_{\alpha\bar{j}}] + R_{k\bar{l}}[R_{i\bar{\beta}}, R_{\alpha\bar{j}}]\big)\,.$$

Now together with Remark (g),

$$I_1 = -\int_X \mathrm{tr}(R_{i\bar{l}} \Box R_{k\bar{j}} + R_{k\bar{l}} \Box R_{i\bar{j}}) g\, dV\,.$$

We will now compute I_2. From Remark (a),

$$I_2 = \int \mathrm{tr}(R_{i\bar{\beta}k} \cdot R_{\alpha\bar{j};\bar{l}}) g\, dV\,.$$

Using normal coordinates at $s = s_0$, i.e.,

$$\frac{\partial^2 G_{i\bar{j}}}{\partial s^k}\Big|_{s=s_0} = 0$$

on the tangent directions of $H^1(X, \mathrm{End}(E))$, Lemma 4.2 and Remark (f) imply that all

$$R_{i\bar{\beta};k} dz^{\bar{\beta}} = 0\,.$$

Hence $I_2 = 0$, and

$$R^{\mathrm{pw},\mathcal{E}}_{i\bar{j}k\bar{l}} = -\frac{\partial^2 G^{\mathrm{pw},\mathcal{E}}_{i\bar{j}}}{\partial s^k \partial s^{\bar{l}}} = -I_1\,.$$

The contribution of the variation of $\mathcal{X}_s$ to the curvature is as follows [S3]:

$$R^{\mathrm{pw},\mathcal{E}}_{i\bar{j}k\bar{l}} = -\int_X (\Box + 1)^{-1} (A_i \cdot A_{\bar{j}})(A_k \cdot A_{\bar{l}}) g\, dV$$

$$-\int_X (\Box+1)^{-1}(A_i\cdot A_{\bar l})(A_k\cdot A_{\bar j})g\,dV - \int_X (\Box-1)^{-1}(A_i\wedge A_k)(A_{\bar j}\wedge A_{\bar l})g\,dV\,.$$

COROLLARY 4.3. *For fixed X, the Petersson-Weil metric on the moduli space of unitary flat vector bundles has nonnegative holomorphic sectional curvature.*

The results established in sections 2, 3 and 4 on the moduli space of pairs of the type (X,E), where E is a unitary flat vector bundle on a projective manifold X, can be easily extended to the slightly more general case where E has only a projectively unitary flat structure, i.e., we replace (2.1) by the equation

$$\big(2rc_2(E)-(r-1)c_1(E)^2\big)[X]=0$$

for the Chern numbers of a stable vector bundle of rank r over a canonically polarized manifold X. A projectively unitary flat connection on E gives a C^∞-splitting of the Atiyah sequence (2.2). We use the splitting in order to put a Hermitian metric on $\Sigma_X(E)$, which gives rise to an isomorphism of vector spaces

$$\mathcal{A}^{p,q}\big(X,\Sigma_X(E)\big)\longrightarrow \mathcal{A}^{p,q}\big(X,\mathrm{End}(E)\big)\oplus\mathcal{A}^{p,q}(X,T_X)\,.$$

The image of a typical section $\kappa\in\mathcal{A}^{p,q}(X,\Sigma_X(E))$ by this decomposition will be denoted by (μ,w). A similar splitting exists over $\mathcal{X}$ for a family (2.3), where ω_X and h are defined as in section 3a.

We express the operators $\bar\partial$ and $\bar\partial^*$ on $\Sigma_X(E)$ (and $\Sigma_{\mathcal{X}}(\mathcal{E})$) in terms of the above isomorphism:

LEMMA 4.4.

$$\bar\partial\kappa=(\bar\partial\mu+\Omega\cup w,\bar\partial w)$$
$$\bar\partial^*\kappa=(\bar\partial^*\mu,L\mu+\bar\partial^* w)\,,$$

where Ω is the curvature form of E over X or of $\mathcal{E}$ respectively, and

$$L:\mathcal{A}^{p,q}\big(X,\mathrm{End}(E)\big)\to\mathcal{A}^{p-1,q}\big(X,\mathrm{End}(E)\big)$$

is the adjoint operator of

$$\mathcal{A}^{p-1,q}\big(X,\mathrm{End}(E)\big)\to\mathcal{A}^{p,q}\big(X,\mathrm{End}(E)\big)$$
$$w\mapsto\Omega\cup w\,.$$

Denoting by $\mathrm{End}_X(E)=\mathrm{End}^0_X(E)\oplus\mathcal{O}_X$, $\Sigma_X(E)=\Sigma^0_X(E)\oplus\mathcal{O}_X$ the decomposition into the trace-free part and the trace part with similar notation for $\mathcal{E}$ over $\mathcal{X}$, we observe that these decompositions are orthogonal for (p,q)-forms. As in section 3 we assign to a tangent vector $\partial/\partial s^i\in T_{s_0}S$ its horizontal lift, apply the C^∞-splitting of the Atiyah sequence (over $\mathcal{X}$), and take its $\bar{\partial}$-exterior derivative, restricted to the central fiber X. This yields a pair

$$(\mu_i, w_i) = \left((R_{i\bar{\beta}} + R_{\alpha\bar{\beta}} a_i^\alpha) dz^{\bar{\beta}}, A_{i\bar{\beta}}^\alpha \frac{\partial}{\partial z^\alpha} dz^{\bar{\beta}} \right).$$

The definition of the Petersson-Weil norm is

$$\left\| \frac{\partial}{\partial s_i} \right\|_{PW}^2 = \|\mu_i\|^2 + \|w_i\|^2$$

(cf. [STo2]). With respect to a decomposition

$$\mu_i = \mu_i^0 + \nu_i \cdot id_E ,$$

where μ_i^0 is the trace-free part and $\nu_i \in \mathcal{A}^{0,1}(X)$, we have

Lemma 4.5. *The forms μ_i, μ_i^0, and ν_i are harmonic. Furthermore $L\mu_i^0 = 0$, $\Omega \cup w_i = 0$. In particular $(\mu_i^0, w_i) \in \mathcal{A}^{0,1}(X, \Sigma_X^0(E))$ is harmonic.*

Proof. The harmonicity of μ_i, μ_i^0, and ν_i is straightforward (cf. Lemma 3.2). Using that E is projectively flat and trace$(\mu_i^0) = 0$ we get $L\mu_i^0 = 0$, and $\Omega \cup w_i = 0$ holds, since $A_{i\bar{\beta}\bar{\delta}} = A_{i\bar{\delta}\bar{\beta}}$.

In this way the Petersson-Weil inner product is induced by *harmonic representatives* of *Kodaira-Spencer classes* in

$$H^1\big(X, \Sigma_X^0(E)\big) \oplus H^1(X, \mathcal{O}_X) .$$

If a vector bundle E has a projectively unitary flat structure, the induced connection on the vector bundle End(E) is flat. So we have a map from the moduli space of pairs of the form (X, E), where E is projectively flat, to the moduli space of pairs of the form (X, F), where F is unitary flat and the top exterior product of F is the trivial line bundle with the trivial connection. This map between moduli spaces fails to be injective only on the Picard group part. More precisely, for two such stable vector bundles E and E' if End(E) is isomorphic to End(E'), then $E' = E \otimes L$, where L is a line bundle. It is rather straight-forward to check that this map between the moduli spaces is compatible with both the holomorphic structure and the Hermitian structure. This ensures the extension of the earlier results to the case of projectively flat vector bundles.

5 The Infinitesimal Structure of a Moduli Space of Parabolic Stable Bundles

Let X be a connected smooth projective variety of dimension d over the field of complex numbers. Let $D \subset X$ be a divisor with normal crossings.

This means that D is a reduced divisor, each irreducible component of D is smooth, and the components intersect transversally. Let

$$D = \sum_{i=1}^{l} D_i$$

be the decomposition of D into its irreducible components.

For each component of D we fix a conjugacy class in the unitary group $U(r)$. Denoting the conjugacy class associated to D_i, $1 \leq i \leq l$, by C_i, assume that all the eigenvalues of any element in C_i are roots of unity.

Let $X_0 := X - D$ be the complement of D. Consider the following irreducible unitary representation space for the fundamental group of X_0

$$\frac{\mathrm{Hom}^{\mathrm{irr}}(\pi_1(X_0), U(r))}{U(r)} .$$

The group $U(r)$ acts by conjugation on $\mathrm{Hom}^{\mathrm{irr}}(\pi_1(X_0), U(r))$, the space of irreducible $U(r)$ representations of $\pi_1(X_0)$. The quotient space for this action of $U(r)$ does not depend upon the choice of the base point in X_0 needed to define the fundamental group. Consider the subspace

$$\mathcal{R}_{\mathcal{P}} \subseteq \frac{\mathrm{Hom}^{\mathrm{irr}}(\pi_1(X_0), U(r))}{U(r)} \tag{5.1}$$

defined by the property that, for any $\rho \in \mathcal{R}_{\mathcal{P}}$, the monodromy along a loop in X_0 around any D_i and contractible in X is in the conjugacy class C_i associated to D_i.

The space $\mathcal{R}_{\mathcal{P}}$ gets identified with a moduli space of parabolic stable bundles on X with parabolic structure over D and with vanishing parabolic Chern classes [Bi2]. The parabolic data needed in the construction of a moduli space of parabolic stable bundles is supplied by the conjugacy classes C_i. We will briefly describe this identification of $\mathcal{R}_{\mathcal{P}}$ with a moduli space of parabolic bundles.

Fix a positive integer N such that any eigenvalue for any conjugacy class C_i is a N-th root of unity. (The existence of such a N is warranted by the assumption on C_i.) Take a finite Galois cover

$$p : Y \longrightarrow X \tag{5.2}$$

such that Y is a connected smooth variety, and the reduced divisor $p^{-1}(D)_{\mathrm{red}}$ is a normal crossing divisor with

$$p^{-1}(D_i) = k_i N p^{-1}(D_i)_{\mathrm{red}} \tag{5.3}$$

for all $i \in [1, l]$, where $k_i \in \mathbb{N}$. The existence of such a covering p is ensured by the "Covering Lemma" due to Y. Kawamata [KMaMat, p. 303, Theorem 1.1.1]. The Galois group for p will be denoted by G.

The equality (5.3) implies that for any $\rho \in \mathcal{R}_{\mathcal{P}}$, its pullback by p, namely $p^*\rho$, is actually an element of $\mathrm{Hom}(\pi_1(Y), U(r))/U(r)$. Equivalently, denoting the flat vector bundle on X_0 corresponding to ρ by $E(\rho)$, the flat connection on $p^*E(\rho)$ extends across $p^{-1}(D)$. This unitary flat vector bundle on Y corresponding to the representation $p^*\rho$ will be denoted by $E(p^*\rho)$.

The vector bundle $E(p^*\rho)$ on Y is naturally equipped with a lift of the action of G on Y. In other words, $E(p^*\rho)$ is G-linearized. Moreover, the G-linearized connection on $E(p^*\rho)$ is irreducible in the sense that there is no proper flat subbundle of $E(p^*\rho)$ of positive rank which is left invariant by the action of G on $E(p^*\rho)$.

Let $\mathcal{R}_{\mathcal{G}}$ denote the space of all isomorphism classes of pairs of the form (ν, f_ν) where

$$\nu \in \frac{\mathrm{Hom}(\pi_1(Y), U(r))}{U(r)}$$

and f_ν is a lift of the action of G on Y to the unitary flat vector bundle $E(\nu)$, satisfying the following two conditions:

1. For any $i \in [1, l]$, C_i is the conjugacy class of the action of a generator of the isotropy subgroup of a general point y of $p^{-1}(D_i)_{\mathrm{red}}$ on the fiber $E(\nu)_y$. The action on the fiber of the isotropy subgroup of a general point of an irreducible component of the ramification divisor for the covering map p, not contained in $p^{-1}(D)$, is the trivial action. Note that the isotropy subgroup of a general point of an irreducible component of the ramification divisor is a cyclic group.
2. There is no proper flat subbundle of $E(\nu)$ of positive rank which is left invariant by the action f_ν of G.

It can be checked that the map

$$\Gamma : \mathcal{R}_{\mathcal{P}} \longrightarrow \mathcal{R}_{\mathcal{G}} \tag{5.4}$$

defined by $\rho \mapsto E(p^*\rho)$, is a diffeomorphism of possibly singular spaces.

The space $\mathcal{R}_{\mathcal{G}}$ is identified with a moduli space of G-linearized stable vector bundles over Y [Si1, page 878, Theorem 1], [Do]. Hence $\mathcal{R}_{\mathcal{G}}$ has a natural structure of a quasi-projective variety over $\mathbb{C}$. With a slight abuse of notation, this moduli space of G-linearized stable vector bundles will also be denoted by $\mathcal{R}_{\mathcal{G}}$.

A bijective correspondence between a class of parabolic bundles over X and G-linearized bundles over Y constructed in [Bi1] gives an isomorphism between the moduli spaces

$$F : \mathcal{R}_{\mathcal{G}} \longrightarrow \mathcal{M}_{\mathcal{P}}\,. \tag{5.5}$$

Finally, the composition $F \circ \Gamma$, where Γ and F are constructed in (5.4) and (5.5) respectively, gives the identification between $\mathcal{M}_{\mathcal{P}}$ and $\mathcal{R}_{\mathcal{P}}$ that we are seeking. It can be checked that this identification does not depend upon the choice of the covering p. Thus the representation space $\mathcal{R}_{\mathcal{P}}$ has a natural structure of a quasi-projective variety.

We will now see that using the above maps Γ and F it is quite easy to obtain descriptions of the real tangent space of $\mathcal{R}_{\mathcal{P}}$, the holomorphic tangent space of $\mathcal{M}_{\mathcal{P}}$, and also the isomorphism between these two tangent spaces induced by $F \circ \Gamma$.

For any $\rho \in \mathcal{R}_{\mathcal{P}}$, let $Ad(\rho)$ denote the unitary local system on X_0 associated to ρ for the adjoint action of $U(r)$ on its Lie algebra $\mathfrak{u}(r)$. The local system $p^*Ad(\rho)$ on $p^{-1}(X_0)$ extends to Y. This extension will also be denoted by $p^*Ad(\rho)$. We note that the complex vector bundle for the local system $p^*Ad(\rho)$ is $\mathrm{End}(E(p^*\rho))$. The following identification of the real tangent space

$$T^{\mathbb{R}}_{p^*\rho}\mathcal{R}_{\mathcal{G}} = H^1\big(Y, p^*Ad(\rho)\big)^G \tag{5.6}$$

is valid; $H^1(Y, p^*Ad(\rho))^G$ is the space of G-invariants for the natural action of G on $H^1(Y, p^*Ad(\rho))$. Let

$$\tau : X_0 \longrightarrow X$$

be the inclusion. It is rather straight-forward to check that the equality

$$H^1\big(Y, p^*Ad(\rho)\big)^G = H^1\big(X, \tau_*Ad(\rho)\big) \tag{5.7}$$

is valid. Indeed, the obvious inclusion $\tau_*Ad(\rho) \to (p_*p^*Ad(\rho))^G$ can be easily seen to be actually an isomorphism. In view of this equality, the identification in (5.7) follows immediately after first equating the G-invariants of the both sides of the following isomorphism of G-modules

$$H^1\big(Y, p^*Ad(\rho)\big) = H^1\big(X, p_*p^*Ad(\rho)\big)$$

and then finally noting the natural isomorphism

$$H^1\big(X, p_*p^*Ad(\rho)\big)^G = H^1\big(X, (p_*p^*Ad(\rho))^G\big)\,.$$

The isomorphisms (5.6), (5.7) and the map Γ, constructed in (5.4), combine together to yield the following identification of the real tangent space:

$$T^{\mathbb{R}}_{\rho}\mathcal{R}_{\mathcal{P}} = H^1\big(X, \tau_*Ad(\rho)\big)\,. \tag{5.8}$$

The isomorphism (5.8) can also be found in [BrF] (Lemma 4.2) and in [T].

Let $E(\rho)$ denote the unitary flat bundle over X_0 associated to ρ for the standard representation of $U(r)$ on $\mathbb{C}^r$. The corresponding holomorphic vector bundle has a natural extension, known as the *Deligne extension*, as a holomorphic vector bundle over X. A description of this extension can be found in [D]. Let $\overline{E(\rho)}$ denote the holomorphic vector bundle over X which is the Deligne extension of $E(\rho)$. It follows easily from the construction of the Deligne extension that

$$\overline{E(\rho)} = \left(p_* E(p^*\rho)\right)^G . \tag{5.9}$$

Recall that the flat vector bundle $E(p^*\rho)$ over Y, which is the natural extension of the flat vector bundle $p^*E(\rho)$ on $Y - \tilde{D}$, has a natural structure of a G-linearized bundle.

This vector bundle $\overline{E(\rho)}$ has a natural parabolic structure [MeSe], [Bi2]. Let $\overline{E(\rho)}_*$ denote this parabolic bundle. The parabolic bundle $\overline{E(\rho)}_*$ is the image of ρ by the map $F \circ \Gamma$ constructed earlier.

For a description of the holomorphic tangent space, $T_{\overline{E(\rho)}_*}\mathcal{M}_{\mathcal{P}}$ consider the flat complex vector bundle $Ad(\rho) \otimes_{\mathbb{R}} \mathbb{C}$ over X_0. Let $\overline{Ad(\rho)}$ denote the Deligne extension of it. The sheaf of holomorphic sections of $\overline{Ad(\rho)}$ can be identified with the subsheaf of the sheaf of endomorphisms of $\overline{E(\rho)}$ that preserves the quasi-parabolic flag over D. Indeed, denoting this quasi-parabolic flag preserving subsheaf of $\mathrm{End}(\overline{E(\rho)})$ by $\mathrm{End}^1(\overline{E(\rho)})$, the natural map from $\overline{Ad(\rho)} = (p_*(p^*Ad(\rho) \otimes_{\mathbb{R}} \mathbb{C}))^G$ to $\mathrm{End}((p_*E(p^*\rho))^G)$ gives the required isomorphism between $\mathrm{End}^1(\overline{E(\rho)})$ and $\overline{Ad(\rho)}$.

Since the holomorphic tangent space $T_{\overline{E(\rho)}_*}\mathcal{M}_{\mathcal{P}}$ known to coincide with the vector space $H^1(X, \mathrm{End}^1(\overline{E(\rho)}))$ [MYo], [Yo], we now have

$$T_{\overline{E(\rho)}_*}\mathcal{M}_{\mathcal{P}} = H^1\left(X, \overline{Ad(\rho)}\right) .$$

To construct the $\mathbb{R}$-linear isomorphism between the real tangent space $T^{\mathbb{R}}_{\rho}\mathcal{R}_{\mathcal{P}}$ and the holomorphic tangent space $T_{\overline{E(\rho)}_*}\mathcal{M}_{\mathcal{P}}$ induced by $F \circ \Gamma$, we first consider the Hodge decomposition of the cohomology of the unitary local system $p^*Ad(\rho) \otimes_{\mathbb{R}} \mathbb{C}$ on Y, namely

$$H^1\left(Y, p^*Ad(\rho)\right) \otimes_{\mathbb{R}} \mathbb{C} = H^1\left(Y, \mathrm{End}(E(p^*\rho))\right) \oplus H^0\left(Y, \Omega^1_Y \otimes \mathrm{End}(E(p^*\rho))\right) . \tag{5.10}$$

The operation of taking complex conjugation in $H^1(Y, p^*Ad(\rho)) \otimes_{\mathbb{R}} \mathbb{C}$ identifies the subspace $H^1(Y, \mathrm{End}(E(p^*\rho)))$ with $H^0(Y, \Omega^1_Y \otimes \mathrm{End}(E(p^*\rho)))$. Now, since the decomposition in (5.10) is a decomposition of G-modules, and that $H^1(Y, \mathrm{End}(E(p^*\rho)))^G$ is naturally a direct summand in $H^1(Y, \mathrm{End}(E(p^*\rho)))$ the following isomorphism

$$H^1\left(Y, p^*Ad(\rho)\right)^G = H^1\left(Y, \mathrm{End}(E(p^*\rho))\right)^G \tag{5.11}$$

is obtained by considering the G-invariants of the two sides of (5.10).

The right-hand side of (5.11) can be identified as

$$H^1\big(Y, \mathrm{End}(E(p^*\rho))\big)^G = H^1\big(X, , p_*\mathrm{End}(E(p^*\rho))\big)^G$$
$$= H^1\big(X, (p_*\mathrm{End}(E(p^*\rho)))^G\big) = H^1\big(X, \overline{Ad(\rho)}\big)\,.$$

The last equality is obtained by substituting $\mathrm{End}(E)$ in place of E in the equality (5.9). Thus from (5.11) we have

$$H^1\big(Y, \mathrm{End}(E(p^*\rho))\big)^G = H^1\big(X, \overline{Ad(\rho)}\big)\,.$$

Now, in view of the the map Γ in (5.4) and the isomorphism (5.6), the above isomorphism gives the required isomorphism between $T^{\mathbb{R}}_\rho \mathcal{R}_{\mathcal{P}}$ and $T_{\overline{E(\rho)}_*}\mathcal{M}_{\mathcal{P}}$.

An element $\nu \in \mathcal{R}_{\mathcal{G}}$ is a smooth point of the moduli space if

$$H^2\big(Y, \mathrm{End}(E(p^*\rho))\big)^G = 0$$

(the proof of the above criterion for the smoothness of a point of the moduli space of G-linearized stable vector bundles is identical to the proof of the analogous smoothness criterion for a point of the moduli space of usual stable vector bundles). Thus we conclude that $\rho \in \mathcal{M}_{\mathcal{P}}$ is a smooth point if $H^2(X, \mathrm{End}^1(\overline{E(\rho)})) = 0$ [Yo].

Our next step will be to construct a Kähler form on the Zariski tangent space of $\mathcal{M}_{\mathcal{P}}$. For this purpose fix an ample line bundle L on X. Now consider the following antisymmetric bilinear paring on $H^1(X, \tau_* Ad(\rho))$:

$$(\alpha, \beta) \longmapsto \int_Y \mathrm{trace}(\alpha \cup \beta) \cup c_1(L)^{d-1} \in \mathbb{R}\,, \tag{5.12}$$

where $d = \dim_{\mathbb{C}} X$. Note that for the constant sheaf $\mathbb{R}$ on X_0, the direct image $\tau_*\mathbb{R}$ is the constant sheaf on X. Thus $\mathrm{trace}(\alpha \cup \beta) \in H^2(X, \mathbb{R})$.

Using the description of $T^{\mathbb{R}}\mathcal{R}_{\mathcal{P}}$ given in (5.8), the pairing (5.12) defines a 2-form on $\mathcal{R}_{\mathcal{P}}$. This 2-form will be denoted by Ω. The form Ω coincides with the one defined in [BrF] if the Kähler form used in [BrF] represents $c_1(L)$. We will show that Ω is a Kähler form on $\mathcal{R}_{\mathcal{P}}$.

Since p is a finite morphism, the line bundle

$$\tilde{L} := p^* L$$

on Y is ample. For any $\nu \in \mathcal{R}_{\mathcal{G}}$, the pairing on $H^1(Y, Ad(\nu))^G$ defined by

$$(\alpha, \beta) \longmapsto \int_Y \mathrm{trace}(\alpha \cup \beta) \cup c_1(\tilde{L})^{d-1} \tag{5.13}$$

is nondegenerate – a fact which is a consequence of the Lefschetz decomposition of cohomology of unitary local systems [Si2].

Since the pairing in (5.13) is G-invariant, it restricts to a nondegenerate pairing on the G-invariant part $H^1(Y, Ad(\nu))^G$ which has been identified with $T_\nu^{\mathbb{R}}\mathcal{R}_{\mathcal{G}}$. Let Ω' denote the nondegenerate 2-form on $\mathcal{R}_{\mathcal{G}}$ obtained this way. It is known that Ω' is a closed, or in other words, it is a Kähler form [Ko], [STo1].

Consider the pullback Kähler form $\Gamma^*\Omega'$ on $\mathcal{R}_{\mathcal{P}}$, where Γ is the map defined in (5.4). Comparing (5.12) and (5.13) it is immediate that

$$\Gamma^*\Omega' = \#G.\Omega \tag{5.14}$$

where $\#G$ denotes the cardinality of the group G, and Ω is the Kähler form on $\mathcal{R}_{\mathcal{P}}$ defined in (5.12). This shows that the symplectic form Ω must be Kähler.

With a slight abuse of notation, the Kähler form $(F^{-1})^*\Omega$ on $\mathcal{M}_{\mathcal{P}}$, where F is the map defined in (5.5), will also be denoted by Ω.

6 Properties of the Kähler Form on the Representation Space

6a The Kähler form as a curvature form. Let $\mathcal{D} : \mathcal{M}_{\mathcal{P}} \to \mathrm{Pic}(X)$ be the morphism defined by $E_* \mapsto \bigwedge^r E$, where E is the underlying vector bundle of the parabolic bundle E_*. Fix an element ζ in the image of $\mathcal{D}$. Let

$$\mathcal{N}_{\mathcal{P}} := \mathcal{D}^{-1}(\zeta) \subseteq \mathcal{M}_{\mathcal{P}} \tag{6.1}$$

be the subvariety of $\mathcal{M}_{\mathcal{P}}$.

In the identification Γ in (5.4), between the space of parabolic stable bundles and the representation space $\mathcal{R}_{\mathcal{G}}$, the subvariety $\mathcal{N}_{\mathcal{P}}$ corresponds to a subvariety of $\mathcal{R}_{\mathcal{G}}$ consists of all ν such that the composition $det \circ \nu$ is a fixed character $\bar{\zeta}$ of $\pi_1(Y)$. The homomorphism det denotes the character defined by taking determinants. This subvariety of $\mathcal{R}_{\mathcal{G}}$ will be denoted by $\mathcal{R}_{\mathcal{G}}^{\zeta}$.

Fix a unitary character $\xi \in \mathrm{Hom}(\pi_1(Y), U(1))$ which is a r-th root of $\bar{\zeta}$, i.e.,

$$\xi^r = \bar{\zeta}\,.$$

As in section 3b, let $\mathcal{M}$ denote the moduli space of usual semistable vector bundle on Y of rank r with trivial determinant and with vanishing Chern classes. Let

$$P' : \mathcal{R}_{\mathcal{G}} \longrightarrow \mathcal{M} \tag{6.2}$$

be the map defined by forgetting the lift of the action of G. In other words, P' sends (ν, f_ν) to ν. Consider the map Γ that was defined in (5.4). Finally, let

$$P := P' \circ \Gamma : \mathcal{N}_{\mathcal{P}} \longrightarrow \mathcal{M} \tag{6.3}$$

be the composition of morphisms P' and Γ.

LEMMA 6.4. *The pullback of the Petersson-Weil form on $\mathcal{N}_{\mathcal{P}}$, namely $P^*\omega^{\mathrm{pw}}$, coincides with $\#G.\Omega$, where Ω is defined in* (5.12).

Proof. In view of the identity (5.14) it suffices to show that the map P' defined in (6.2) pulls back the form ω^{pw} on $\mathcal{M}$ to the form Ω' on $\mathcal{R}_{\mathcal{G}}$. But this is immediate after comparing (2.9) with (5.12). □

Let $\mathcal{L}_P := P^*\mathcal{L}$ be the pulled-back Hermitian line bundle over $\mathcal{N}_{\mathcal{P}}$, where $\mathcal{L}$ is the Hermitian line bundle equipped with the Quillen metric that was defined in (3.4). The Lemma 6.4 and the equation (3.6) combine together to yield the following theorem:

Theorem 6.5. *The curvature form of the Hermitian line bundle $\mathcal{L}_P$ is* $4\#G\pi r\Omega$.

An immediate consequence of Theorem 6.5 is that the form Ω represents a rational cohomology class on $\mathcal{N}_{\mathcal{P}}$.

6b Curvature of the Kähler form. Let $\overline{\mathcal{M}}$ denote the moduli space of semistable vector bundles over Y of rank r and with vanishing Chern classes, but the top exterior product is not assumed to be fixed.

There is a forgetful map from the moduli space $\mathcal{R}_{\mathcal{G}}$, defined in section 5, to $\overline{\mathcal{M}}$ which sends any (ν, f_ν) to $E(\nu)$, the vector bundle associated to the representation ν, forgetting the G-linearization f_ν. Let Q' denote this forgetful map.

Now define

$$Q := Q' \circ \Gamma : \mathcal{R}_{\mathcal{P}} \longrightarrow \overline{\mathcal{M}}\,.$$

From the proof of Lemma 6.4 we have

LEMMA 6.6. *The pullback of the Petersson-Weil form on $\mathcal{R}_{\mathcal{P}}$, namely $Q^*\omega^{\mathrm{pw}}$, coincides with $\#G.\Omega$, where Ω is defined in* (5.12).

In particular, the map Q is an immersion. The following lemma gives some more information on the map Q.

LEMMA 6.7. *The image of the map Q is a totally geodesic subspace of $\overline{\mathcal{M}}$.*

Proof. The Galois group G acts naturally on $\overline{\mathcal{M}}$. More precisely, for any $g \in G$, the automorphism of $\overline{\mathcal{M}}$ defined by

$$E \longmapsto (g^{-1})^*E$$

is the action of g. Clearly the image of Q is contained in the subspace of $\overline{\mathcal{M}}$ on which G acts trivially.

From the equation (5.6) it follows that the differential of the morphism Q maps the tangent space $T_E\mathcal{R}_{\mathcal{P}}$ at $E \in \mathcal{R}_{\mathcal{P}}$ to the G-invariant part of the tangent space $T_{Q(E)}\overline{\mathcal{M}}$ at the point $Q(E) \in \overline{\mathcal{M}}$. In other words, the differential of the inclusion map of $\mathcal{R}_{\mathcal{P}}$ into the subspace of $\overline{\mathcal{M}}$ consisting of all points fixed by G is actually an isomorphism. This implies that the image of Q is the union of some components of the subspace of $\overline{\mathcal{M}}$ consisting of all points on which G acts trivially. This immediately yields that the image of Q must be totally geodesic in $\mathcal{M}$, since any irreducible component of the space of all G-fixed points must be a totally geodesic subspace of $\mathcal{M}$. □

Now, in view of Lemma 6.6 and Lemma 6.7, the formula for the curvature of $\overline{\mathcal{M}}$ in Theorem 4.1 immediately gives a similar formula for the curvature of $\mathcal{R}_{\mathcal{G}}$ with respect to the Kähler form Ω.

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