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RELATIVE VOLUME COMPARISON WITH INTEGRAL CURVATURE BOUNDS

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Abstract

In this paper we shall generalize the Bishop-Gromov relative volume comparison estimate to a situation where one only has an integral bound for the part of the Ricci curvature which lies below a given number. This will yield several compactness and pinching theorems.

1 Introduction

In this paper we shall be concerned with proving some results related to the work from [PeSW]. The techniques here are similar, but independent of the developments in [PeSW], still we suggest that the reader read at least the introduction to [PeSW]. Also our techniques are different from those used in [G] and [Y]. This paper can therefore be read without prior knowledge of those papers. We will present a new relative volume comparison estimate which generalizes the classical Bishop-Gromov comparison inequality. The consequences of this are manifold and hopefully far reaching.

To state our results we need some notation. On a Riemannian manifold M define the function $g : M \to [0,\infty)$ as $g(x) =$ the smallest eigenvalue for $\mathrm{Ric} : T_xM \to T_xM$. Now consider

$$k(\lambda, p) = \int_M \big(\max\{-g(x) + (n-1)\cdot\lambda, 0\}\big)^p d\,\mathrm{vol}$$
$$\bar{k}(\lambda, p) = \frac{1}{\mathrm{vol}\, M}\int_M \big(\max\{-g(x) + (n-1)\cdot\lambda, 0\}\big)^p d\,\mathrm{vol}\,,$$

the last quantity is the averaged amount of curvature below $(n-1)\lambda$. This quantity is in many ways more natural than the first. We of course have that $\mathrm{Ric}(M) \geq (n-1)\lambda$ iff $\bar{k}(\lambda, p) = 0$.

Our main result is

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Theorem 1.1. *Let $x \in M$, $\lambda \leq 0$, and $p > n/2$ be given, then there is a constant $C(n, p, \lambda, R)$ which is nondecreasing in R such that when $r < R$ we have*

$$\left(\frac{\operatorname{vol} B(x,R)}{v(n,\lambda,R)}\right)^{1/2p} - \left(\frac{\operatorname{vol} B(x,r)}{v(n,\lambda,r)}\right)^{1/2p} \leq C(n,p,\lambda,R) \cdot (k(\lambda,p))^{1/2p}.$$

Furthermore when $r = 0$ we obtain

$$\operatorname{vol} B(x,R) \leq \left(1 + C(n,p,\lambda,R) \cdot (k(\lambda,p))^{1/2p}\right)^{2p} v(n,\lambda,R) .$$

Note that when $\operatorname{Ric} \geq (n-1)\lambda$, i.e. $k(p,\lambda) = 0$, this gives the classical relative volume comparison estimate. Our proof of this volume estimate does not use the setup used in Gallot and Yang's work. In fact our proof is somewhat different and is more inspired by some of the new estimates obtained in [PeSW]. The absolute volume estimate is actually better than the one in [Y], as we recover the correct volume estimates when $k(\lambda, p) = 0$. In [Y] one only arrives at the correct volume bound when $k(0,p) = 0$. As a corollary we obtain the following volume doubling result:

Corollary 1.2. *Under the same conditions as the above theorem we have that for all $\alpha < 1$ there is an $\varepsilon = \varepsilon(n,p,\lambda,D,\alpha) > 0$ such that if M is a Riemannian manifold with $\operatorname{diam} M \leq D$ and $\bar{k}(\lambda,p) \leq \varepsilon$, then for all $x \in M$ and $r < D$ we have*

$$\alpha \cdot \frac{v(n,\lambda,r)}{v(n,\lambda,D)} \leq \frac{\operatorname{vol} B(x,r)}{\operatorname{vol} M}.$$

As an immediate consequence we have the following extension of Gromov's precompactness result:

Corollary 1.3. *Given an integer $n > 1$, $p > n/2$ and $\lambda \leq 0$, $D < \infty$, we can find $\varepsilon(n,p,\lambda,D)$ such that the class of closed Riemannian n-manifolds with*

$$\operatorname{diam} M \leq D,$$
$$\bar{k}(\lambda,p) \leq \varepsilon$$

is precompact in the Gromov-Hausdorff topology.

The classical relative volume comparison result has proven very useful in many contexts. So one would expect the above inequality to give some new results that are similar but more general. Here we shall concentrate on compactness results. In a future paper we will show how other finiteness result also generalize when we use our new volume estimate.

The relative volume estimate will enable us to obtain some very general compactness and pinching results, where in addition to assuming lower volume bounds and upper diameter bounds one has some sort of L^p curvature bounds. The reader who is unfamiliar with the standard language of convergence theory might wish to consult [Pe] for a complete survey of results and proofs of this area of geometry.

Theorem 1.4. *Given an integer $n \geq 2$, and numbers $p > n/2$, $\lambda \leq 0$, $v > 0$, $D < \infty$, $\Lambda < \infty$, one can find $\varepsilon = \varepsilon(n,p,\lambda,D) > 0$ such that the class of closed Riemannian n-manifolds with*

$$\begin{gathered} \operatorname{vol}(M) \geq v \\ \operatorname{diam}(M) \leq D \\ \int_M \|R\|^p \leq \Lambda \\ \tfrac{1}{\operatorname{vol} M} \cdot k(\lambda,p) \leq \varepsilon(n,p,\lambda,D) \end{gathered}$$

is precompact in the C^α, $\alpha < 2 - \frac{n}{p}$, topology.

Note that ε in the previous theorem does not depend on Λ or v. The smallness condition on $\bar{k}(\lambda,p)$ is inevitable as is shown by examples in both [G] and [Y]. In fact in [Y] there are examples which have arbitrarily high Betti numbers but satisfy

$$\begin{gathered} \operatorname{vol}(M) \geq v \\ \operatorname{diam}(M) \leq D \\ \int_M \|R\|^p \leq \Lambda \end{gathered}$$

for some constants v, D, Λ.

Recall that a Riemannian metric (M,g) has constant sectional curvature λ iff the Riemannian curvature $(0,4)$-tensor $R = \lambda g \circ g$, where $g \circ g$ is the Kulkarni-Nomizu product.

Corollary 1.5. *Given an integer $n \geq 2$, and numbers $p > n/2$, $\lambda \in \mathbb{R}$, $v > 0$, $D < \infty$, one can find $\varepsilon = \varepsilon(n,p,\lambda,D) > 0$ such that a closed Riemannian n-manifold (M,g) with*

$$\begin{gathered} \operatorname{vol}(M) \geq v\,, \\ \operatorname{diam}(M) \leq D\,, \\ \frac{1}{\operatorname{vol} M} \int_M \|R - \lambda \cdot g \circ g\|^p \leq \varepsilon(n,p,\lambda,D) \end{gathered}$$

is C^α, $\alpha < 2 - \frac{n}{p}$ close to a constant curvature metric on M.

Corollary 1.6. *Given an integer $n \geq 2$, and numbers $p > n/2$, $\lambda \in \mathbb{R}$, $v > 0$, and $\Lambda, D < \infty$, one can find $\varepsilon = \varepsilon(n, p, \lambda, D) > 0$ such that a closed Riemannian n-manifold (M, g) with*

$$\operatorname{vol}(M) \geq v\,,$$
$$\operatorname{diam}(M) \leq D\,,$$
$$\int_M \|R\|^p \leq \Lambda\,,$$
$$\frac{1}{\operatorname{vol} M}\int_M \big\|\operatorname{Ric} - \lambda\cdot(n-1)\cdot g\big\|^p \leq \varepsilon(n,p,\lambda,D)$$

is C^α, $\alpha < 2 - \frac{n}{p}$ close to an Einstein metric on M.

These two pinching results, and the compactness theorem as well, are to our knowledge the most general of their type. Namely, where, aside from curvature conditions, one only has global diameter and volume bounds. In section 3 we shall also discuss how one gets compactness theorems for complete manifolds, and also how one can generalize the curvature condition $\int_M \|R\|^p \leq \Lambda$ to a slightly different curvature condition. This will lead us to a very attractive Ricci curvature pinching theorem.

Note also that we can in all of the above results replace the two conditions

$$\operatorname{vol}(M) \geq v\,,$$
$$\int_M \|R\|^p \leq \Lambda\,,$$

by the single condition

$$\operatorname{inj}(M) \geq i_0\,,$$

as is done in [A]. Along these lines we can also study manifolds with almost maximal volume as is also done in [A]. Specifically we have

Theorem 1.7. *Given an integer $n \geq 2$, and numbers $p > n/2$, $\lambda \leq 0$, $r > 0$, $\Lambda < \infty$, one can find $\varepsilon = \varepsilon(n, p, \lambda\cdot r^2) > 0$ and $\delta = \delta(n, \lambda\cdot r^2) > 0$ such that the class of complete Riemannian n-manifolds with*

$$\operatorname{vol} B(x,r) \geq (1-\delta)\cdot v(n,\lambda,r) \text{ for all } x \in M\,,$$
$$\int_{B(x,r)} \|\operatorname{Ric}\|^p \leq \Lambda\,,$$
$$k(\lambda, p) \leq \varepsilon$$

is precompact in the C^α, $\alpha < 2 - \frac{n}{p}$, topology.

Note that ε in the previous theorem does not depend on Λ and that δ depends only on the dimension. As an immediate corollary we have a very nice L^p Ricci pinching result

COROLLARY 1.8. *Given an integer $n \geq 2$, and numbers $p > n/2$, $\lambda \in \mathbb{R}$, $r > 0$ and $\Lambda, D < \infty$, one can find $\varepsilon = \varepsilon(n, p, \lambda, r, D) > 0$ and $\delta = \delta(n, \lambda \cdot D^2) > 0$ such that a closed Riemannian n-manifold (M, g) with*

$$\operatorname{vol} B(x,r) \geq (1-\delta) \cdot v(n,\lambda,r) \text{ for all } x \in M\,,$$
$$\operatorname{diam}(M) \leq D\,,$$
$$\int_{B(x,r)} \left\| \operatorname{Ric} - (n-1)\cdot\lambda\cdot g \right\|^p \leq \varepsilon$$

is C^α, $\alpha < 2 - \frac{n}{p}$ close to an Einstein metric on M.

In the last section of this paper we also discuss how some of the $L^{n/2}$ compactness and pinching results of Gao and Anderson fit into this new framework.

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2 The Volume Estimate

We suppose that we are given a complete Riemannian n-manifold M and $x \in M$. Around x use exponential polar coordinates and write the volume element as $d\operatorname{vol} = \omega dt \wedge d\theta_{n-1}$, where $d\theta_{n-1}$ is the standard volume element on the unit sphere $S^{n-1}(1)$. As t increases ω becomes undefined but we can just declare it to be zero for those t. We know that $\omega' = h\omega$, where h is the mean curvature of the distance spheres around x, and h satisfies the differential inequality

$$h' + \frac{h^2}{n-1} \leq -\operatorname{Ric}(\partial_t, \partial_t)\ .$$

Here ∂_t is the unit gradient of the distance function $d(\cdot, x)$.

In the n-dimensional space form S^n_λ of constant curvature λ, we can similarly choose $y \in S^n_\lambda$ and write the volume element as $d\operatorname{vol} = \omega_\lambda dt \wedge d\theta_{n-1}$. We use $\lambda \leq 0$ in order that the polar coordinates are defined on all of $S^n_\lambda - \{y\}$ and also so that h_λ is non-negative everywhere. Now we have again $\omega_\lambda' = h_\lambda \omega_\lambda$ where this time the mean curvature satisfies

$$h_\lambda' + \frac{h_\lambda^2}{n-1} = -(n-1)\lambda .$$

Thus we get that

$$\begin{aligned} h_\lambda(t) = h_\lambda(t,\cdot) &= (n-1)\cdot \frac{\mathrm{sn}_\lambda'(t)}{\mathrm{sn}_\lambda(t)}, \\ \omega_\lambda(t) = \omega_\lambda(t,\cdot) &= \ \mathrm{sn}_\lambda^{n-1}(t) , \end{aligned}$$

where sn_λ is the unique solution to

$$\begin{aligned} \varphi'' + \lambda\varphi &= 0 , \\ \varphi(0) &= 0 , \\ \varphi'(0) &= 1 . \end{aligned}$$

Having similar coordinate systems on M and S_λ^n allows us to compare their volume forms and mean curvatures. Let us define $\psi = \psi(t,\cdot) = \max\{0, h(t,\cdot) - h_\lambda(t,\cdot)\}$ and declare that ψ is 0 whenever it becomes undefined.

LEMMA 2.1. *For the volume ratio* $\frac{\mathrm{vol}\, B(x,r)}{v(n,\lambda,r)}$ *we have that*

$$\begin{aligned} &\frac{d}{dr}\frac{\mathrm{vol}\, B(x,r)}{v(n,\lambda,r)} \\ &\quad \le C_1(n,\lambda,r)\left(\frac{\mathrm{vol}\, B(x,r)}{v(n,\lambda,r)}\right)^{1-\frac{1}{2p}} \left(\int_{B(x,r)} \psi^{2p} d\,\mathrm{vol}\right)^{\frac{1}{2p}} (v(n,\lambda,r))^{-\frac{1}{2p}} , \end{aligned}$$

where

$$\begin{aligned} C_1(n,\lambda,r) &= \max_{t\in[0,r]} \frac{t\cdot\omega_\lambda(t)}{\int_0^t \omega_\lambda(s)\, ds}, \\ C_1(n,\lambda,0) &= n . \end{aligned}$$

Proof. First observe that the fraction ω/ω_λ satisfies

$$\begin{aligned} \frac{d}{dt}\frac{\omega}{\omega_\lambda} &\le (h-h_\lambda)\frac{\omega}{\omega_\lambda} \\ &\le \psi\frac{\omega}{\omega_\lambda} . \end{aligned}$$

Note that away from the cut locus the first inequality is actually an equality. At the cut locus the singular part of the derivative of ω has negative measure, hence when the derivative is interpreted correctly we get inequality. This implies

$$\frac{d}{dr}\frac{\int_{S^{n-1}} \omega(r,\cdot)\, d\theta_{n-1}}{\int_{S^{n-1}} \omega_\lambda(r)\, d\theta_{n-1}} = \frac{1}{\mathrm{vol}\, S^{n-1}} \int_{S^{n-1}} \frac{d}{dr}\frac{\omega(r,\cdot)}{\omega_\lambda(r)} d\theta_{n-1}$$

$$\leq \frac{1}{\operatorname{vol} S^{n-1}} \int_{S^{n-1}} \psi \frac{\omega}{\omega_\lambda} d\theta_{n-1} .$$

Thus for $t \leq r$ we have

$$\frac{\int_{S^{n-1}} \omega\,(r,\cdot)\,d\theta_{n-1}}{\int_{S^{n-1}} \omega_\lambda\,(r,\cdot)\,d\theta_{n-1}} - \frac{\int_{S^{n-1}} \omega\,(t,\cdot)\,d\theta_{n-1}}{\int_{S^{n-1}} \omega_\lambda\,(t,\cdot)\,d\theta_{n-1}}$$
$$\leq \frac{1}{\operatorname{vol} S^{n-1}} \int_t^r \int_{S^{n-1}} \psi \frac{\omega}{\omega_\lambda} d\theta_{n-1} \wedge ds ,$$

which implies

$$\int_{S^{n-1}} \omega(r,\cdot) d\theta_{n-1} \cdot \int_{S^{n-1}} \omega_\lambda(t) d\theta_{n-1} - \int_{S^{n-1}} \omega_\lambda(r) d\theta_{n-1} \cdot \int_{S^{n-1}} \omega(t,\cdot) d\theta_{n-1}$$
$$\leq \frac{1}{\operatorname{vol} S^{n-1}} \left(\int_{S^{n-1}} \omega_\lambda\,(r)\,d\theta_{n-1} \right) \left(\int_{S^{n-1}} \omega_\lambda\,(t)\,d\theta_{n-1} \right)$$
$$\cdot \int_t^r \int_{S^{n-1}} \psi \frac{\omega}{\omega_\lambda} d\theta_{n-1} \wedge ds$$
$$\leq \left(\int_{S^{n-1}} \omega_\lambda\,(r)\,d\theta_{n-1} \right) \int_t^r \int_{S^{n-1}} \psi \omega d\theta_{n-1} \wedge ds$$
$$\leq \omega_\lambda\,(r) \operatorname{vol} S^{n-1} \int_0^r \int_{S^{n-1}} \psi \omega d\theta_{n-1} \wedge ds$$
$$\leq \omega_\lambda\,(r) \operatorname{vol} S^{n-1} \left(\int_0^r \int_{S^{n-1}} \psi^{2p} \omega d\theta_{n-1} \wedge ds \right)^{1/2p}$$
$$\cdot \left(\int_0^r \int_{S^{n-1}} \omega d\theta_{n-1} \wedge ds \right)^{1-\frac{1}{2p}}$$
$$\leq \operatorname{vol} S^{n-1} \omega_\lambda\,(r) \left(\operatorname{vol} B\,(x,r)\right)^{1-\frac{1}{2p}} \left(\int_M \psi^{2p} d \operatorname{vol} \right)^{1/2p} .$$

Using the volume elements from above we can write

$$\frac{\operatorname{vol} B\,(x,r)}{v\,(n,\lambda,r)} = \frac{\int_0^r \int_{S^{n-1}} \omega \wedge dt}{\int_0^r \int_{S^{n-1}} \omega_\lambda d\theta_{n-1} \wedge dt} .$$

Thus we have

$$\frac{d}{dr} \frac{\operatorname{vol} B\,(x,r)}{v\,(n,\lambda,r)} = \frac{\left(\int_{S^{n-1}} \omega\,(r,\cdot)\,d\theta_{n-1} \right) \cdot \left(\int_0^r \int_{S^{n-1}} \omega_\lambda\,(t)\,d\theta_{n-1} \wedge dt \right)}{\left(v\,(n,\lambda,r)\right)^2}$$
$$- \frac{\left(\int_{S^{n-1}} \omega_\lambda\,(r)\,d\theta_{n-1} \right) \cdot \left(\int_0^r \int_{S^{n-1}} \omega\,(t,\cdot)\,d\theta_{n-1} \wedge dt \right)}{\left(v\,(n,\lambda,r)\right)^2} .$$

Now observe that the numerator can be written as

$$\int_0^r \left(\int_{S^{n-1}} \omega\,(r,\cdot)\,d\theta_{n-1} \cdot \int_{S^{n-1}} \omega_\lambda\,(t)\,d\theta_{n-1} \right) dt$$

$$
\begin{aligned}
&-\int_0^r \left(\int_{S^{n-1}} \omega_\lambda(r)\, d\theta_{n-1} \cdot \int_{S^{n-1}} \omega(t,\cdot)\, d\theta_{n-1}\right) dt \\
&\le \int_0^r \left(\operatorname{vol} S^{n-1} \cdot \omega_\lambda(r) \cdot (\operatorname{vol} B(x,r))^{1-\frac{1}{2p}} \cdot \left(\int_M \psi^{2p} d\operatorname{vol}\right)^{1/2p}\right) dt \\
&= \operatorname{vol} S^{n-1} \cdot r \cdot \omega_\lambda(r) \cdot (\operatorname{vol} B(x,r))^{1-\frac{1}{2p}} \cdot \left(\int_M \psi^{2p} d\operatorname{vol}\right)^{1/2p}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\frac{d}{dr}\frac{\operatorname{vol} B(x,r)}{v(n,\lambda,r)} &\le \frac{\operatorname{vol} S^{n-1} \cdot r \cdot \omega_\lambda(r) \cdot (\operatorname{vol} B(x,r))^{1-\frac{1}{2p}} \cdot \left(\int_M \psi^{2p} d\operatorname{vol}\right)^{\frac{1}{2p}}}{(v(n,\lambda,r))^2} \\
&\le C_1(n,\lambda,r) \cdot \left(\frac{\operatorname{vol} B(x,r)}{v(n,\lambda,r)}\right)^{1-\frac{1}{2p}} \left(\int_M \psi^{2p} d\operatorname{vol}\right)^{\frac{1}{2p}} (v(n,\lambda,r))^{-\frac{1}{2p}}.
\end{aligned}
$$

Here we used that

$$
\frac{\operatorname{vol} S^{n-1} \cdot r \cdot \omega_\lambda(r)}{v(n,\lambda,r)} = \frac{r \cdot \omega_\lambda(r)}{\int_0^r \omega_\lambda(s)\, ds}
$$

converges to n as $r \to 0$ and can therefore be estimated by its maximum value $C_1(n,\lambda,r)$ on $[0,r]$. Note that when $\lambda < 0$ the constant $C_1 \to \infty$ if either $r \to \infty$ or $\lambda \to \infty$. On the other hand when $\lambda = 0$ we can use $C_1 = n$ for all r. □

We must now estimate $\int_M \psi^{2p} d\operatorname{vol}$ in terms of $k(p,\lambda)$. This estimate is inspired by our similar estimates in [PeSW, Section 3]. To reduce the problem first write

$$
\int_{B(x,r)} \psi^{2p} d\operatorname{vol} = \int_{S^{n-1}} \int_0^r \psi^{2p} \omega dt \wedge d\theta_{n-1}.
$$

Thus it suffices to estimate $\int_0^r \psi^{2p} \omega dt$. Now define $\rho = \rho(t,\cdot) = \max\{0, (n-1)\lambda - \operatorname{Ric}(\partial_t, \partial_t)\}$. If we are beyond where the coordinate system is defined then we use the default that ρ is zero. Also define

$$
\begin{aligned}
k(p,\lambda,r) &= \int_{B(x,r)} \rho^p d\operatorname{vol} = \int_{S^{n-1}} \int_0^r (\rho(t,\cdot))^p \omega dt \wedge d\theta_{n-1} \\
\bar{k}(p,\lambda,r) &= \sup_{x\in M} \frac{k(p,\lambda,r)}{\operatorname{vol} B(x,r)}.
\end{aligned}
$$

With this notation behind us it is now clear that

$$
\begin{aligned}
&\frac{d}{dr}\frac{\operatorname{vol} B(x,r)}{v(n,\lambda,r)} \\
&\quad\le C_3(n,p,\lambda,r) \cdot \left(\frac{\operatorname{vol} B(x,r)}{v(n,\lambda,r)}\right)^{1-\frac{1}{2p}} (k(p,\lambda,r))^{\frac{1}{2p}} (v(n,\lambda,r))^{-\frac{1}{2p}}.
\end{aligned}
$$

provided we can prove

LEMMA 2.2. *There is a constant $C_2(n,p)$ such that when $p > n/2$ we have*

$$\int_0^r \psi^{2p}\omega dt \le C_2(n,p)\int_0^r \rho^p \omega dt\,.$$

Proof. We have that ψ is absolutely continuous and satisfies

$$\psi' + \frac{\psi^2}{n-1} + 2\frac{\psi\cdot h_\lambda}{n-1} \le \rho\,.$$

Multiply through by $\psi^{2p-2}\omega$ and integrate to get

$$\int_0^r \psi'\psi^{2p-2}\omega dt + \frac{1}{n-1}\int_0^r \psi^{2p}\omega dt + \frac{2}{n-1}\int_0^r h_\lambda \psi^{2p-1}\omega dt \le \int_0^r \rho\cdot\psi^{2p-2}\omega dt\,.$$

Integration by parts yields

$$\begin{aligned}\int_0^r & \psi'\psi^{2p-2}\omega dt\\ &= \frac{1}{2p-1}\psi^{2p-1}\omega\Big|_0^r - \frac{1}{2p-1}\int_0^r \psi^{2p-1}h\omega dt\\ &\ge -\frac{1}{2p-1}\int_0^r \psi^{2p-1}h\omega dt\\ &\ge -\frac{1}{2p-1}\int_0^r \psi^{2p}\omega dt - \frac{1}{2p-1}\int_0^r \psi^{2p-1}h_\lambda\omega dt\,.\end{aligned}$$

Inserting this in the above inequality we obtain

$$\left(\frac{1}{n-1}-\frac{1}{2p-1}\right)\int_0^r \psi^{2p}\omega dt + \left(\frac{2}{n-1}-\frac{1}{2p-1}\right)\int_0^r h_\lambda\psi^{2p-1}\omega dt \le \int_0^r \rho\cdot\psi^{2p-2}\omega dt\,.$$

When $p > n/2$ we therefore obtain

$$\begin{aligned}\left(\frac{1}{n-1}-\frac{1}{2p-1}\right)\int_0^r \psi^{2p}\omega dt &\le \int_0^r \rho\cdot\psi^{2p-2}\omega dt\\ &\le \left(\int_0^r \rho^p\omega dt\right)^{\frac{1}{p}}\cdot\left(\int_0^r \psi^{2p}\omega dt\right)^{1-\frac{1}{p}}\,.\end{aligned}$$

By dividing through by $\left(\int_0^r \psi^{2p}\omega dt\right)^{1-\frac{1}{p}}$ we then get

$$\left(\frac{1}{n-1}-\frac{1}{2p-1}\right)\left(\int_0^r \psi^{2p}\omega dt\right)^{1/p} \le \left(\int_0^r \rho^p\omega dt\right)^{1/p}\,.$$

Or in other words

$$\left(\int_0^r \psi^{2p}\omega dt\right) \le \left(\frac{1}{n-1}-\frac{1}{2p-1}\right)^{-p}\left(\int_0^r \rho^p\omega dt\right)\,,$$

with $C_2(n,p) = \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)^{-p}$. We therefore arrive at the desired inequality. □

We can now prove

Lemma 2.3. *There is a constant $C(n,p,\lambda,R)$ which is nondecreasing in R such that when $r < R$ we have*

$$\left(\frac{\operatorname{vol} B\,(x,R)}{v\,(n,\lambda,R)}\right)^{1/2p} - \left(\frac{\operatorname{vol} B\,(x,r)}{v\,(n,\lambda,r)}\right)^{1/2p} \le C\,(n,p,\lambda,R)\cdot\left(k\,(p,\lambda,R)\right)^{1/2p}.$$

Furthermore, when $r = 0$ we obtain

$$\operatorname{vol} B\,(x,R) \le \left(1 + C(n,p,\lambda,R)\cdot\left(k\,(p,\lambda,R)\right)^{1/2p}\right)^{2p} v\,(n,\lambda,R).$$

Proof. From Lemma 2.1 we have a differential inequality of the type

$$y' \le \alpha\cdot y^{1-\frac{1}{2p}}\cdot f(x)\,,$$
$$y\,(0) = 1 \text{ and } y > 0\,.$$

Separation of variables and integration yields

$$2p\cdot y^{1/2p}\,(R) - 2p\cdot y^{1/2p}\,(r) \le \alpha\int_r^R f\,(x)\,dx\,.$$

So we can simply use

$$C = \frac{1}{2p}C_3\int_0^R \left(v\,(n,\lambda,t)\right)^{-1/2p}dt\,.$$

Furthermore, we can use the initial condition $y\,(0) = 1$ to get

$$y^{1/2p}\,(R) \le 1 + \frac{\alpha}{2p}\int_0^R f\,(x)\,dx\,.$$

The final observation to be made is that the integral

$$\int_0^R \left(v\,(n,\lambda,t)\right)^{-1/2p}dt$$

indeed converges when $p > n/2$. To see this recall that $v\,(n,\lambda,t) \approx t^n$ as $t \to 0$. So the integrand looks like $t^{-n/2p}$ as $t \to 0$. And this function is integrable when $-n/2p > -1$, or in other words when $p > n/2$. □

Observe that the condition $p > n/2$ is necessary both to get the estimate for $\int \psi^{2p}\omega dt$ and for the integrability of $\left(v\,(n,\lambda,t)\right)^{-1/2p}$. It seems lucky indeed that those two different conditions are the same.

Finally we can use the inequality $k\,(p,\lambda,r) \le k\,(p,\lambda)$ to obtain the relative volume comparison estimate mentioned in the introduction.

Another remark is in order at this point. We never really seriously use that $\lambda \le 0$, and indeed one can extend the estimates to hold for $\lambda > 0$.

However, if R or r becomes larger than $\pi/\sqrt{\lambda}$ then we will run into trouble. Also there are a few things to worry about when $r > \pi/2\sqrt{\lambda}$ as h_λ becomes negative there.

We can now show

Corollary 2.4. *Given an integer $n > 1$ and $p > n/2$, $\lambda \leq 0$, and $D > 0$, we can for all $\alpha < 1$ find $\varepsilon = \varepsilon(n,p,\lambda,D,\alpha) > 0$ such that any complete Riemannian n-manifold M with* $\operatorname{diam} M \leq D$ *and $\bar{k}(p,\lambda) \leq \varepsilon$ satisfies for all $x \in M$ and $r < D$*

$$\alpha \cdot \frac{v(n,\lambda,r)}{v(n,\lambda,D)} \leq \frac{\operatorname{vol} B(x,r)}{\operatorname{vol} M}\,.$$

Proof. Consider the inequality

$$\left(\frac{\operatorname{vol} M}{v(n,\lambda,D)}\right)^{1/2p} - \left(\frac{\operatorname{vol} B(x,r)}{v(n,\lambda,r)}\right)^{1/2p} \leq C(n,p,\lambda,D) \cdot (k(p,\lambda))^{1/2p}$$

and then cross multiply to get

$$\left(\frac{v(n,\lambda,r)}{v(n,\lambda,D)}\right)^{1/2p} - \left(\frac{\operatorname{vol} B(x,r)}{\operatorname{vol} M}\right)^{1/2p} \leq C(n,p,\lambda,D) \cdot (v(n,\lambda,r))^{1/2p} \cdot \left(\frac{k(p,\lambda)}{\operatorname{vol} M}\right)^{1/2p}\,.$$

Now choose ε so that

$$(C(n,p,\lambda,D))^{2p} \cdot \varepsilon \leq (1-\alpha) \cdot \frac{1}{v(n,\lambda,D)}\,.$$

Then we have that

$$\left(\frac{v(n,\lambda,r)}{v(n,\lambda,D)}\right)^{1/2p} - \left(\frac{\operatorname{vol} B(x,r)}{\operatorname{vol} M}\right)^{1/2p} \leq (1-\alpha) \cdot \left(\frac{v(n,\lambda,r)}{v(n,\lambda,D)}\right)^{1/2p}$$

as long as

$$\bar{k}(p,\lambda) = \frac{k(p,\lambda)}{\operatorname{vol} M} \leq \varepsilon\,.$$

This is the desired estimate. □

3 Pinching and Compactness

In the survey article [Pe] it is proven that all the corollaries in the introduction are immediate consequences of the compactness theorems we mention there. Observe that the compactness theorems can be applied after one gets smallness for $\bar{k}(p,\lambda)$ from

$$\bar{k}(p,-|\lambda|) \leq \frac{1}{\operatorname{vol} M} \int_M \left\| \operatorname{Ric} - \lambda \cdot (n-1) \cdot g \right\|^p.$$

The compactness theorems are also almost immediate consequences of the results in [A], [Y] and [Pe]. Namely, from the last two sources it follows that for any $v, r_0, D, \Lambda \in (0,\infty)$ the class satisfying

$$\begin{aligned} \operatorname{vol} B(x,r) \geq v \cdot r^n \text{ for } r \leq r_0 \,, \\ \operatorname{diam}(M) \leq D \,, \\ \int_M \|R\|^p \leq \Lambda \end{aligned}$$

is precompact in the C^α, $\alpha < 2 - \frac{n}{p}$, topology. Thus we must establish this local volume growth condition from the conditions

$$\begin{aligned} \operatorname{vol} M \geq v \,, \\ \operatorname{diam} M \leq D \,, \\ \bar{k}(p,\lambda) \leq \varepsilon \,. \end{aligned}$$

Using $\alpha = 1/2$ in Corollary 2.4 we can clearly find $\varepsilon(n,p,\lambda,D)$ such that the condition $\bar{k}(p,\lambda) \leq \varepsilon$ implies

$$\frac{1}{2} \cdot \frac{v(n,\lambda,r)}{v(n,\lambda,D)} \leq \frac{\operatorname{vol} B(x,r)}{\operatorname{vol} M} \,.$$

Using the volume estimate $\operatorname{vol} M \geq v$ we then obtain

$$\begin{aligned} \operatorname{vol} B(x,r) &\geq \frac{1}{2} \cdot v(n,\lambda,r) \cdot \frac{v}{v(n,\lambda,D)} \\ &\geq C(n,p,\lambda,v,D) \cdot r^n \text{ for } r \leq D. \end{aligned}$$

It should be observed that we also get a compactness theorem for complete manifolds (see e.g. [Pe]).

Theorem 3.1. *Given an integer $n \geq 2$, and numbers $p > n/2$, $\lambda \leq 0$, $r > 0$, $v > 0$, $\Lambda < \infty$, one can find $\varepsilon = \varepsilon(n,p,\lambda,r) > 0$ such that the class of complete Riemannian n-manifolds with*

$$\begin{aligned} \operatorname{vol} B(x,r) \geq v \text{ for all } x \in M \,, \\ \int_{B(x,r)} \|R\|^p \leq \Lambda \text{ for all } x \in M \,, \\ \bar{k}(\lambda,p,r) \leq \varepsilon(n,p,\lambda,r) \end{aligned}$$

is precompact in the C^α, $\alpha < 2 - \frac{n}{p}$, topology as well.

In [Pe] it is discussed how one can replace the $\int_{B(x,r)} \|R\|^p \leq \Lambda$ by the two conditions

$$\int_{B(x,r)} \|\mathrm{Ric}\|^p \leq \Lambda\,,$$
$$\int_{B(x,r)} \|R\|^{n/2} \leq \epsilon\,,$$

where ϵ is some small number depending on dimension r and the lower volume bound v for the balls $B\,(x,r)$. These are actually the conditions used in Yang's work. Using this we can re-state all of the above compactness and pinching theorems with the condition

$$\int_{B(x,r)} \|\mathrm{Ric}\|^p \leq \Lambda\,,$$
$$\int_{B(x,r)} \|R\|^{n/2} \leq \epsilon\,.$$

replacing $\int_{B(x,r)} \|R\|^p \leq \Lambda$. This gives us a particularly nice pinching result.

Theorem 3.2. *Given an integer $n \geq 2$ and numbers $p > n/2$, $\lambda \in \mathbb{R}$, $r > 0$, $v > 0$, $D < \infty$, we can find $\varepsilon\,(n,p,\lambda,D) > 0$ and $\epsilon\,(n,\lambda,r,v) > 0$ such that any closed Riemannian manifold (M,g) satisfying*

$$\mathrm{diam}\,(M) \leq D\,,$$
$$\mathrm{vol}\,B\,(x,r) \geq v \textit{ for all } x \in M\,,$$
$$\int_{B(x,r)} \|R\|^{n/2} \leq \epsilon \textit{ for all } x \in M\,,$$
$$\frac{1}{\mathrm{vol}\,M}\int_{B(x,r)} \|\mathrm{Ric} - (n-1)\cdot\lambda\cdot g\|^p \leq \varepsilon \textit{ for all } x \in M$$

is C^α, $\alpha < 2 - \frac{n}{p}$ close to an Einstein metric on M.

In [A, Theorem 2.6] and [AC] the authors study the situation when $\int_M \|R\|^{n/2}$ is merely bounded rather than small. The convergence and finiteness results obtained there clearly also admit generalization to the situation where, instead of having $|\mathrm{Ric}| \leq \Lambda$, we assume $\int_{B(x,r)} \|\mathrm{Ric}\|^p \leq \Lambda$ and smallness of $\bar{k}\,(p,\lambda,r)$.

The compactness and pinching results that use almost maximal volume hypotheses rest on the following key result proved in [A, Gap Lemma 3.1].

Theorem 3.3. *There is an $\eta\,(n) > 0$ such if M is a complete Ricci flat Riemannian n-manifold with the property that*

$$\mathrm{vol}\,B\,(p,r) \geq (1-\eta)\cdot v\,(n,0,r) \textit{ for all } r > 0$$

and some $p \in M$, then M is isometric to $\mathbb{R}^n$.

We now need to observe that if we rescale manifolds in the class

$$\operatorname{vol} B(x,r) \geq (1-\delta)\cdot v(n,\lambda,r) \ \text{ for all } x \in M\,,$$
$$\int_{B(x,r)} \|\mathrm{Ric}\|^p \leq \Lambda\,,$$
$$k(\lambda,p,r) \leq \varepsilon(n,p,\lambda,r)\ ,$$

then we get complete Ricci flat manifolds. The needed volume growth condition comes from our volume assumption together with the smallness of $k(p,\lambda)$ in the following way. We have assumed that

$$\operatorname{vol} B(x,r) \geq (1-\delta)\cdot v(n,\lambda,r) \ \text{ for all } x \in M\,,$$
$$k(\lambda,p,r) \leq \varepsilon(n,p,\lambda,r)$$

for some fixed r. If $s < r$ then our relative volume comparison result tells us that

$$\begin{aligned} 1-\delta &\leq \frac{\operatorname{vol} B(x,r)}{v(n,\lambda,r)} \\ &\leq \left(\left(\frac{\operatorname{vol} B(x,s)}{v(n,\lambda,s)}\right)^{1/2p} + C(n,p,\lambda,r)\cdot\varepsilon^{1/2p}\right)^{2p} \\ &\leq \frac{\operatorname{vol} B(x,s)}{v(n,\lambda,s)} + \left(1+\left(\frac{\operatorname{vol} B(x,s)}{v(n,\lambda,s)}\right)^{1/2p}\right)^{2p}\cdot C(n,p,\lambda,r)\cdot\varepsilon^{1/2p} \\ &\leq \frac{\operatorname{vol} B(x,s)}{v(n,\lambda,s)} + 3^{2p}\cdot C(n,p,\lambda,r)\cdot\varepsilon^{1/2p}\,, \end{aligned}$$

where we assumed that $C(n,p,\lambda,r)\cdot\varepsilon^{1/2p} \leq 1$ for the penultimate inequality and that

$$\left(\frac{\operatorname{vol} B(x,s)}{v(n,\lambda,s)}\right)^{1/2p} \leq 1 + C(n,p,\lambda,r)\cdot\varepsilon^{1/2p} \leq 2$$

for the last inequality. We can now choose

$$\delta(n) = \frac{\eta(n)}{2}$$

and in addition $\varepsilon = \varepsilon(n,p,\lambda,r)$ such that

$$3^{2p}\cdot C(n,p,\lambda,r)\cdot\varepsilon^{1/2p} \leq \frac{\eta(n)}{2}\,.$$

With these choices for ε and δ we get that, for all $s \leq r$,

$$1-\eta \leq \frac{\operatorname{vol} B(x,s)}{v(n,\lambda,s)}\,.$$

Therefore, if we rescale the metrics in the class by constants which go to infinity we obtain metrics that satisfy the hypotheses in Anderson's Gap Lemma.

We also get some new results on Sobolev constants. Recall that if we have an n-dimensional Riemannian manifold (M,g) then for $q \in [n,\infty]$ we can define

$$C^q(M) = \inf \frac{\mathrm{vol}_{n-1} H}{(\mathrm{vol}_n \Omega)^{1-\frac{1}{q}}},$$

where H runs over all closed hypersurfaces in M dividing M into two pieces and Ω is the piece with the smallest volume. In [G] it was shown that when $q > n$ one can bound $C^q(M)$ in terms $\mathrm{vol}\, M$, $\mathrm{diam}\, M$, and $k(q/2, \lambda)$. We obviously can't improve this to cover the case where $q = n$, however using the method in [Y] we obtain bounds for the classical Sobolev constant $C^n(M)$ in terms of $\mathrm{vol}\, M, \mathrm{diam}\, M$, and $k(p,\lambda)$, $p > n/2$. One should compare this to the Sobolev constant estimates obtained in [BPPe] where the authors instead of assuming a diameter bound consider the spectrum.

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