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LORD RAYLEIGH'S CONJECTURE FOR VIBRATING CLAMPED PLATES IN POSITIVELY CURVED SPACES

Alexandru Kristály

Dedicated to the memory of my professor and friend, Csaba Varga (1959–2021).

Abstract. We affirmatively solve the analogue of Lord Rayleigh's conjecture on Riemannian manifolds with positive Ricci curvature for *any* clamped plates in 2 and 3 dimensions, and for sufficiently *large* clamped plates in dimensions beyond 3. These results complement those from the flat (Ashbaugh and Benguria in Duke Math J 78(1):1–17, 1995; Nadirashvili in Arch Ration Mech Anal 129(1):1–10, 1995) and negatively curved (Kristály in Adv Math 367:107113, 2020) cases that are valid only in 2 and 3 dimensions, and at the same time also provide the first positive answer to Lord Rayleigh's conjecture in higher dimensions. The proofs rely on an Ashbaugh–Benguria–Nadirashvili–Talenti nodal-decomposition argument, on the Lévy–Gromov isoperimetric inequality, on fine properties of Gaussian hypergeometric functions and on sharp spectral gap estimates of fundamental tones for both small and large clamped spherical caps. Our results show that positive curvature enhances genuine differences between low- and high-dimensional settings, a tacitly accepted paradigm in the theory of vibrating clamped plates. In the limit case—when the Ricci curvature is non-negative we establish a Lord Rayleigh-type isoperimetric inequality that involves the asymptotic volume ratio of the non-compact complete Riemannian manifold; moreover, the inequality is strongly rigid in 2 and 3 dimensions, i.e., if equality holds for a given clamped plate then the manifold is isometric to the Euclidean space.

Contents

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1 Introduction

The paper is devoted to the analogue of Lord Rayleigh's conjecture, concerning the lowest principal frequency of vibrating clamped plates on positively curved spaces. Our results can be viewed as the concluding piece in the theory of clamped plates after the seminal works of Ashbaugh and Benguria [AB95] and Nadirashvili [Nad95] in Euclidean spaces, and the recent paper by the author [Kri20] on non-positively curved spaces, all valid in dimensions 2 and 3. To our surprise, positively curved spaces provide an appropriate geometric setting for the validity of Lord Rayleigh's conjecture not only in dimensions 2 and 3 for any clamped plate, but also in dimensions beyond 3 for sufficiently large domains. Before presenting our results in details, we shortly recall some historical milestones related to the subtleties in the theory of vibrating clamped plates.

1.1 Historical Aspects. The original problem appeared in 1877, when John William Strutt, 3rd Baron Rayleigh [Ray45] formulated, inter alia, two isoperimetric inequalities arising from mathematical physics; he claimed that the disc has the minimal principal frequency among either clamped plates or fixed membranes with a given area. Although Lord Rayleigh formulated his conjectures for planar domains, he surely had the feeling that the statement should be valid in any dimension as subsequent literature referred to these conjectures; in particular, both the "clamped plate" and "fixed membrane" notions are commonly used in any dimension.

In the 1920s, Lord Rayleigh's conjecture for the fixed membrane problem has been confirmed independently by Faber [Fab23] and Krahn [Kra25], by showing that the principal/first Dirichlet eigenvalue of the Laplace operator for any bounded open domain $\Omega \subset \mathbb{R}^n$ is not less than the corresponding Dirichlet eigenvalue of a ball $\Omega^\star \subset \mathbb{R}^n$ that has the same volume as Ω. Their arguments are based on the classical isoperimetric inequality in $\mathbb{R}^n$ combined with a Schwarz-type rearrangement. In the 1980s, Lord Rayleigh's conjecture for fixed membranes has been solved on Riemannian manifolds of positive Ricci curvature, see Bérard and Meyer [BM82], and on those Cartan–Hadamard manifolds (i.e., complete and simply connected Riemannian manifolds with non-positive sectional curvature) which verify the Cartan–Hadamard conjecture, see Chavel [Cha84]; all these arguments rest upon the sign-definite character of the first eigenfunction for the second-order fixed membrane problems.

Transferring simply the arguments from the fixed membrane problem to clamped plates can be elusive. To be more precise, the clamped plate problem can be formulated as

$$\begin{cases} \Delta^2 u = \Lambda_0 u & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded open domain, Δ^2 is the bi-Laplace operator, $\frac{\partial}{\partial \mathbf{n}}$ is the outward normal derivative on $\partial\Omega$, while the principal frequency (or fundamental tone) of the clamped plate Ω can be characterized variationally as

$$\Lambda_0 := \Lambda_0(\Omega) = \inf_{u \in W_0^{2,2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega (\Delta u)^2 \mathrm{d}x}{\displaystyle\int_\Omega u^2 \mathrm{d}x}. \tag{1.2}$$

Assuming that the first eigenfunction of (1.1) is of fixed sign for a domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$, Szegő [Sze50] proved in the early 1950s the validity of Lord Rayleigh's conjecture, i.e., $\Lambda_0(\Omega) \geq \Lambda_0(\Omega^\star)$, where $\Omega^\star$ is a ball in $\mathbb{R}^n$ with the same volume as Ω. Szegő's proof used standard symmetrization/rearrangement techniques and he implicitly expressed his hope that any clamped domain should produce principal eigenfunctions of fixed sign. However, his hope has been shattered soon, as Duffin [Duf53] (see also Coffman, Duffin and Shaffer [CDS79]) constructed clamped plates with sign-changing first eigenfunctions. Accordingly, Szegő's initial argument for proving Lord Rayleigh's conjecture for clamped plates failed, becoming a hard nut to crack though several decades; in fact, the main obstructions to follow the arguments from the fixed membrane problem are formed by both the lack of a maximum principle for the fourth-order clamped plate problem and the failure of a suitable rearrangement of the first eigenfuntion u_1 in (1.1) with a suitable estimate of $\int_\Omega (\Delta u_1)^2 \mathrm{d}x$. These phenomena are deeply analyzed in the monograph of Gazzola, Grunau and Sweers [GGS10].

A breakthrough idea has been arising from Talenti [Tal81] in the early 1980s, who decomposed the domain $\Omega \subset \mathbb{R}^n$ corresponding to the positive and negative parts of the first eigenfunction u_1 in (1.1). Using a Schwarz-type rearrangement of these domains/functions, he was able to control in a suitable manner the quantities $\int_\Omega (\Delta u_1)^2 \mathrm{d}x$ and $\int_\Omega u_1^2 \mathrm{d}x$ in (1.2), obtaining a two-ball minimization problem by means of which he provided the non-sharp estimate $\Lambda_0(\Omega) \geq t_n \Lambda_0(\Omega^\star)$ with $t_n \in \left[\frac{1}{2}, 1\right)$ for every $n \geq 2$ and $\lim_{n\to\infty} t_n = \frac{1}{2}$.

More than 115 years had to pass before Nadirashvili [Nad94, Nad95] announced the solution to the original (i.e., 2-dimensional) Lord Rayleigh's conjecture, by slightly modifying Talenti's argument. Inspired by Nadirashvili's achievement, Ashbaugh and Benguria [AB95] proved Lord Rayleigh's conjecture in dimensions 2 and 3, by using sharp estimates in Talenti's decomposition combined with fine properties of Bessel functions. We note that the conjecture is still open in higher dimensions; however, almost simultaneously with [AB95], Ashbaugh and Laugesen [AL96] provided an asymptotically sharp estimate, i.e., $\Lambda_0(\Omega) \geq w_n \Lambda_0(\Omega^\star)$ with $w_n \in [0.89, 1)$ for every $n \geq 4$ and $\lim_{n\to\infty} w_n = 1$. Recently, Chasman and Langford [CL16] proved a non-sharp isoperimetric inequality for clamped plates on Gaussian spaces, stating that $\Gamma_w(\Omega) \geq c\Gamma_w(\Omega^\star)$ for some $c = c(\Omega, n) \in (0, 1)$, where

$\Gamma_w(\Omega)$ and $\Gamma_w(\Omega^\star)$ are the fundamental tones of clamped plates with respect to the Gaussian density w.

Since the fixed membrane problem of curved spaces is fully described, see [BM82, Cha84], a similar question naturally arises also for clamped plates. Fixing a complete n-dimensional ($n \geq 2$) Riemannian manifold (M, g), we consider instead of (1.2) the fundamental tone

$$\Lambda_g(\Omega) := \inf_{u \in W_0^{2,2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega (\Delta_g u)^2 \mathrm{d}v_g}{\displaystyle\int_\Omega u^2 \mathrm{d}v_g}, \tag{1.3}$$

of $\Omega \subset M$, where Δ_g and $\mathrm{d}v_g$ are the Laplace–Beltrami operator and canonical measure on (M, g).

When (M, g) is a Cartan–Hadamard manifold with sectional curvature bounded from above by $-\kappa \leq 0$, the author [Kri20] proved the validity of Lord Rayleigh's conjecture in dimensions 2 and 3 for small clamped plates, i.e., $\Lambda_g(\Omega) \geq \Lambda_\kappa(\Omega^\star)$ holds for every domain $\Omega \subset M$ having volume $V_g(\Omega) = V_\kappa(\Omega^\star) \leq c_n/\kappa^{n/2}$ with $c_2 \approx 21.031$ and $c_3 \approx 1.721$, respectively; here $V_g(\Omega)$ and $V_\kappa(\Omega^\star)$ denote the volumes of Ω in (M, g) and the geodesic ball $\Omega^\star$ in the space form (N_κ^n, g_κ) of constant curvature $-\kappa$, respectively, while $\Lambda_\kappa(\Omega^\star)$ stands for the fundamental tone of $\Omega^\star$ in (N_κ^n, g_κ) corresponding to (1.3). In particular, the above result provides in the limit case $\kappa \to 0$ the main result of Ashbaugh and Benguria [AB95]. The proofs in [Kri20] are based on the generalized Cartan-Hadamard conjecture (see e.g. Kloeckner and Kuperberg [KK19]) and peculiar properties of the Gaussian hypergeometric function, both valid only in dimensions 2 and 3. Some non-sharp estimates of $\Lambda_g(\Omega)$ for clamped plates $\Omega \subset M$ are also provided in dimensions beyond 3 in the same geometrical setting.

Since a systematic study concerning the fundamental tone of clamped plates on positively curved Riemannian manifolds is unavailable, the objective of the present paper is to fill this gap by solving the analogue of Lord Rayleigh's conjecture (and proving further related results) in this geometric setting; it turns out that unexpected phenomena occur with respect to the non-positively curved framework that are presented in the next subsection.

1.2 Main Results. Let (M,g) be a compact n-dimensional ($n \geq 2$) Riemannian manifold (M,g) with Ricci curvature $\mathsf{Ric}_{(M,g)} \geq (n-1)\kappa > 0$, and consider the clamped plate problem

$$\begin{cases} \Delta_g^2 u = \Lambda u & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset M$ is a bounded open domain, Δ_g^2 stands for the biharmonic Laplace–Beltrami operator on (M,g) and $\frac{\partial}{\partial \mathbf{n}}$ is the outward normal derivative on $\partial\Omega$. The fundamental tone $\Lambda_g(\Omega)$ of the set $\Omega \subset M$ associated with (1.4) is variationally expressed by (1.3). As a model space, let $\mathbb{S}_\kappa^n \subset \mathbb{R}^{n+1}$ be the n-dimensional sphere with radius $1/\sqrt{\kappa}$ (i.e., with constant curvature $\kappa > 0$), endowed with its natural Riemannian metric g_κ; for simplicity of notation, we use $\Lambda_\kappa(\Omega)$ and $V_\kappa(\Omega)$ instead of the fundamental tone $\Lambda_{g_\kappa}(\Omega)$ and volume $V_{g_\kappa}(\Omega)$, respectively, of the open domain $\Omega \subset \mathbb{S}_\kappa^n$. Moreover, $d_\kappa(N,x)$ is the geodesic distance on $\mathbb{S}_\kappa^n$ between the North pole $N = (0,\ldots,0,1/\sqrt{\kappa})$ and $x \in \mathbb{S}_\kappa^n$.

A deeper understanding of the feature of fundamental tones for generic clamped plates on (M,g) motivates the investigation of some particular cases. Our first result provides a quite complete picture on the behavior of the first eigenfunctions on 2-dimensional spherical belts depending on their relative width. Given $\kappa > 0$, let

$$B_\kappa(r,R) = \{x \in \mathbb{S}_\kappa^2 : r < d_\kappa(N,x) < R\}$$

be the *spherical belt* with radii $0 < r < R < \pi/\sqrt{\kappa}$. We also recall the critical Coffman–Duffin–Schaffer constant $\mathsf{C}_{CDS} \approx 762.3264$ (see [CDS79]), whose origin will be explained in the proof of the following result.

Theorem 1.1 (Spherical belts). *Let $R > r > 0$. Then there exists $\kappa_0 \in (0, \pi^2/R^2)$ with the following properties.*

- (i) Narrow relative width: *if $R/r < \mathsf{C}_{CDS}$, then for every $\kappa \in (0,\kappa_0)$, the first eigenfunction of problem (1.4) on the spherical belt $\Omega := B_\kappa(r,R) \subset \mathbb{S}_\kappa^2$ is of fixed sign.*
- (ii) Wide relative width: *if $R/r > \mathsf{C}_{CDS}$, then for every $\kappa \in (0,\kappa_0)$, the first eigenfunctions of problem (1.4) on the spherical belt $\Omega := B_\kappa(r,R) \subset \mathbb{S}_\kappa^2$ are sign-changing, having a pair of azimuthally opposite nodal circular arcs.*

Theorem 1.1 sheds light on the possibility of the rippling behavior of the first eigenfunctions on positively curved manifolds, similarly to the flat case, see also Fig. 1b. The proof of Theorem 1.1 is based on fine properties of Gaussian hypergeometric functions and a careful asymptotic argument which traces us back to Euclidean annuli studied by Coffman, Duffin and Schaffer [CDS79].

Our second result contains precise asymptotic estimates of the fundamental tones for small and large clamped spherical caps, which will play a key role in the proof of Lord Rayleigh's conjecture. Given $\kappa > 0$ and $L \in (0, \pi/\sqrt{\kappa})$, the *spherical cap* is

$$C_\kappa^n(L) = \{x \in \mathbb{S}_\kappa^n : d_\kappa(N, x) < L\}.$$

The aforementioned asymptotic estimates are based on the first zeros of the transcendental equations

$$\frac{\pi}{2}\tan\left(\pi\sqrt{\frac{1}{4}+\mu}\right) - \Psi\left(\sqrt{\frac{1}{4}+\mu}+\frac{1}{2}\right) + \Re\Psi\left(\sqrt{\frac{1}{4}-\mu}+\frac{1}{2}\right) = 0, \quad \mu > 0, \tag{1.5}$$

where $\Psi := (\ln\Gamma)'$ is the Digamma function and $\Re z$ is the real part of $z \in \mathbb{C}$, and

$$\sqrt{\mu-1}\coth\left(\pi\sqrt{\mu-1}\right) - \sqrt{\mu+1}\cot\left(\pi\sqrt{\mu+1}\right) = 0, \quad \mu > 1. \tag{1.6}$$

For later use, if $\nu \geq 0$ is fixed, J_ν and I_ν stand for the Bessel and modified Bessel functions of the first kind, while $\mathfrak{h}_\nu$ and $\mathfrak{j}_\nu$ denote the first positive zeros of the cross-product $J_\nu' I_\nu - J_\nu I_\nu'$ and the Bessel function J_ν, respectively.

Theorem 1.2 (Spherical caps). *If $n \in \mathbb{N}_{\geq 2}$, $\kappa > 0$ and $L \in (0, \pi/\sqrt{\kappa})$ are fixed and $C_\kappa^n(L) \subset \mathbb{S}_\kappa^n$ is a spherical cap, then we have the following asymptotic estimates.*

(i) Small spherical caps:

$$\Lambda_\kappa(C_\kappa^n(L)) \sim \frac{\mathfrak{h}_{\frac{n}{2}-1}^4}{L^4} \quad as \quad L \to 0. \tag{1.7}$$

(ii) Large spherical caps:

$$\Lambda_\kappa(C_\kappa^n(L)) \sim \begin{cases} \mu_n^2\kappa^2 & if\, n \in \{2,3\}, \\ 0 & if\, n \geq 4, \end{cases} \quad as \quad L \to \frac{\pi}{\sqrt{\kappa}}, \tag{1.8}$$

where $\mu_2 \approx 0.9125$ and $\mu_3 \approx 1.0277$ are the smallest positive zeros to the transcendental equations (1.5) and (1.6), respectively.

As expected, in small scales, the fundamental tone has a Euclidean character (cf. relation (1.7)); indeed, for any $L > 0$, one has that $\Lambda_0(B_0(L)) = \mathfrak{h}_{\frac{n}{2}-1}^4/L^4$, where $\Lambda_0(\Omega)$ comes from (1.2) and $B_0(L) \subset \mathbb{R}^n$ is the n-dimensional Euclidean ball of radius $L > 0$ and center 0. On the other hand, for large spherical caps (cf. relation (1.8)), we surprisingly see an essential difference: while in high dimensions the fundamental tones are gapless, in low dimensions there are spectral gaps given by means of the first positive zeros to the transcendental equations (1.5) and (1.6), respectively. In fact, relation (1.8) provides another evidence to the tacitly accepted view concerning the difference between the low- and high-dimensional character of the fundamental tone for vibrating clamped plates. We also note that relation (1.8) specifies a gap in Cheng, Ichikawa and Mametsuka [CIM10, p. 676], who—regardless of dimension—claimed that the fundamental tone always tends to 0 for clamped plates converging to the whole sphere; note however that this statement is valid only for fixed membranes on the sphere, see e.g. Betz, Cámera and Gzyl [BCG83]. The accuracy of estimates (1.7) and (1.8) are shown in Tables 1 and 2, respectively.

Theorem 1.2 paves the way to prove Lord Rayleigh's conjecture on positively curved spaces for any clamped plate in 2 and 3 dimensions, as well as for sufficiently large domains in high dimensions. The precise statement of our main result reads as follows:

Theorem 1.3 (Lord Rayleigh's conjecture; positively curved spaces). *Let (M,g) be a compact n-dimensional Riemannian manifold with $\mathsf{Ric}_{(M,g)} \geq (n-1)\kappa > 0$ where $n \geq 2$. Then there exists $v_n \in [0,1)$ (not depending on $\kappa > 0$), with $v_2 = v_3 = 0$ and $v_n > 0$ for $n \geq 4$ such that if $\Omega \subset M$ is a smooth domain and $\Omega^\star \subset \mathbb{S}^n_\kappa$ is a spherical cap with $\frac{V_g(\Omega)}{V_g(M)} = \frac{V_\kappa(\Omega^\star)}{V_\kappa(\mathbb{S}^n_\kappa)} > v_n$, then*

$$\Lambda_g(\Omega) \geq \Lambda_\kappa(\Omega^\star). \tag{1.9}$$

Equality holds in (1.9) if and only if (M,g) is isometric to $(\mathbb{S}^n_\kappa, g_\kappa)$ and Ω is isometric to $\Omega^\star$. In addition, $v_\infty := \limsup_{n\to\infty} v_n < 1$.

Since $v_2 = v_3 = 0$, there are no restrictions on the size of clamped plates in 2 and 3 dimensions. However, our arguments work only for sufficiently large domains $\Omega \subset M$ in higher dimensions, satisfying $V_g(\Omega) > v_n V_g(M)$ with $v_n > 0$ for every $n \geq 4$. Nevertheless, Theorem 1.3 provides the first positive answer in any geometric setting to Lord Rayleigh's conjecture in *arbitrarily high dimensions*. We notice that $v_n \in (0,1)$, $n \geq 4$, is implicitly given as a solution to a highly nonlinear equation. However, we have $v_\infty = \limsup_{n\to\infty} v_n < 1$, which shows that clamped plates in dimensions beyond 3 need not be particularly close—in the sense of volume—to the whole manifold in order for the isoperimetric inequality (1.9) to hold. Numerical tests indicate that $v_n \leq 1/2$ for every $n \geq 4$, and $v_\infty = 1/2$, see Table 3. Clearly, the most optimistic scenario would be to have $v_n = 0$ for every $n \geq 4$, which definitely requires a new approach with respect to the one presented in our paper.

The first part of the proof of Theorem 1.3 is inspired by Talenti [Tal81] together with the subsequent refinements of Ashbaugh and Benguria [AB95] and Nadirashvili [Nad95]. Indeed, since the minimizer in (1.3) can be sign-changing (see e.g. Theorem 1.1), a suitable nodal-decomposition is performed that we combine with the Lévy–Gromov isoperimetric inequality, reducing the initial problem to a coupled minimization problem involving spherical caps on the model space $\mathbb{S}^n_\kappa$. Then, fine asymptotic properties of the fundamental tone for clamped spherical caps (cf. Theorem 1.2) combined with further features of Gaussian hypergeometric functions provide the proof of Theorem 1.3.

We conclude the paper with the limit case when $\kappa \to 0$, i.e., (M,g) is a complete non-compact n-dimensional Riemannian manifold with $\mathsf{Ric}_{(M,g)} \geq 0$. The quantity

$$\mathsf{AVR}_{(M,g)} = \lim_{r\to\infty} \frac{V_g(B_x(r))}{\omega_n r^n} \tag{1.10}$$

stands for the *asymptotic volume ratio* of (M,g); here $B_x(r)$ is the open metric ball on M with center $x\in M$ and radius $r>0$, and ω_n is the volume of the Euclidean unit ball in $\mathbb{R}^n$. By the Bishop–Gromov comparison theorem one has that $\mathsf{AVR}_{(M,g)}\leq 1$, and this number is independent of the choice of $x\in M$, thus it is a global geometric invariant of (M,g); moreover, $\mathsf{AVR}_{(M,g)}=1$ if and only if (M,g) is isometric to the usual Euclidean space $(\mathbb{R}^n,g_0)$.

Theorem 1.4 (Lord Rayleigh's conjecture; non-negatively curved spaces). *Let (M,g) be a complete non-compact n-dimensional ($n\geq 2$) Riemannian manifold with $\mathsf{Ric}_{(M,g)}\geq 0$ and $\mathsf{AVR}_{(M,g)}>0$. Then*

$$\Lambda_g(\Omega)\geq \mathsf{AVR}_{(M,g)}^{\frac{4}{n}}w_n\Lambda_0(\Omega^\star), \tag{1.11}$$

for every smooth bounded domain $\Omega\subset M$, where $\Omega^\star\subset\mathbb{R}^n$ is a ball with $V_g(\Omega)=V_0(\Omega^\star)$ and

$$w_n=\begin{cases}1 & \text{if } n\in\{2,3\},\\ 2^{\frac{4}{n}}\dfrac{\mathfrak{j}^4_{\frac{n}{2}-1}}{\mathfrak{h}^4_{\frac{n}{2}-1}} & \text{if } n\geq 4.\end{cases} \tag{1.12}$$

If $n\in\{2,3\}$, then equality holds in (1.11) for some $\Omega\subset M$ if and only if (M,g) is isometric to $(\mathbb{R}^n,g_0)$ and $\Omega\subset M$ is isometric to the ball $\Omega^\star\subset\mathbb{R}^n$. In addition, $w_\infty:=\lim\limits_{n\to\infty}w_n=1$.

As in the proof of Theorem 1.3, the conclusion of Theorem 1.4 follows from a similar nodal-decomposition, combined with a recent isoperimetric inequality, valid on any complete non-compact n-dimensional Riemannian manifold (M,g) with $\mathsf{Ric}_{(M,g)}\geq 0$ and $\mathsf{AVR}_{(M,g)}>0$, see Brendle [Bre21] and Balogh and Kristály [BK] which is proved by the ABP-method and the optimal mass transport theory, respectively. We also emphasize the strong rigidity character of inequality (1.11) for $n\in\{2,3\}$; indeed, if a particular domain $\Omega\subset M$ produces equality in (1.11), the whole manifold (M,g) turns out to be isometric to the Euclidean space $(\mathbb{R}^n,g_0)$, which follows by the characterization of the equality in the aforementioned isoperimetric inequality (see [Bre21, BK]); the rest immediately follows from Ashbaugh and Benguria [AB95] for $n\in\{2,3\}$, and Ashbaugh and Laugesen [AL96] for $n\geq 4$.

Structure of the paper. In Sect. 2 we state/prove those properties of the sphere $\mathbb{S}^n_\kappa$ and Gaussian hypergeometric functions that are used in the paper. In Sect. 3 we discuss the sign-changing character of the first eigenfunctions on spherical belts (see Theorem 1.1). In Sect. 4 we perform the Ashbaugh–Benguria–Nadirashvili–Talenti nodal-decomposition on positively curved manifolds, providing a sharp estimate of the fundamental tone of a clamped plate by a coupled minimization expression involving spherical caps. In Sect. 5 we prove sharp spectral gaps for small and large spherical caps (see Theorem 1.2). Lord Rayleigh's conjecture on positively curved Riemannian manifolds (see Theorem 1.3) is proved in Sect. 6, while the limit case (see Theorem 1.4) is discussed in Sect. 7. Finally, "Appendix A" contains well-known properties of special functions that are collected for an easier reading of the proofs.

2 Preliminaries

2.1 The Model Space $\mathbb{S}^n_\kappa$. Let $n \in \mathbb{N}_{\geq 2}$ and $\kappa > 0$. The set $\mathbb{S}^n_\kappa \subset \mathbb{R}^{n+1}$ is the n-dimensional sphere with radius $1/\sqrt{\kappa}$ (i.e., with constant curvature κ), endowed with its natural Riemannian metric g_κ. Let (θ, ξ) be the spherical coordinates on $\mathbb{S}^n_\kappa$ with respect to the North pole $N = (0, \dots, 0, 1/\sqrt{\kappa}) \in \mathbb{S}^n_\kappa$, where $\theta \in (0, \pi)$ represents the latitude measurement along a unit speed geodesic from N, while $\xi \in \mathbb{S}^{n-1}_1 =: \mathbb{S}^{n-1}$ is a parameter representing the choice of 'azimuthal' direction of the geodesic in $\mathbb{S}^n_\kappa$. The distance from $x = x(\theta, \xi) \in \mathbb{S}^n_\kappa$ to the North pole is $d_\kappa(N, x) = \theta/\sqrt{\kappa} \in (0, \pi/\sqrt{\kappa})$. The set

$$C^n_\kappa(R) = \{x \in \mathbb{S}^n_\kappa : d_\kappa(N, x) < R\}$$

denotes the n-dimensional spherical cap with center N and radius $R \in (0, \pi/\sqrt{\kappa})$. Its volume is

$$V_\kappa(C^n_\kappa(R)) = \int_{C^n_\kappa(R)} \mathrm{d}v_\kappa = n\omega_n \int_0^R \left(\frac{\sin(\sqrt{\kappa}\rho)}{\sqrt{\kappa}} \right)^{n-1} \mathrm{d}\rho, \tag{2.1}$$

where $\mathrm{d}v_\kappa$ is the canonical measure on $\mathbb{S}^n_\kappa$ and ω_n stands for the volume of the unit n-dimensional Euclidean ball. Performing a change of variables and using the relation (2.1), for every integrable function $h : [0, L] \to \mathbb{R}$ with $L \in [0, V_\kappa(\mathbb{S}^n_\kappa)]$ we have that

$$\int_0^L h(s)\mathrm{d}s = \int_{C^n_\kappa(R_L)} h(V_\kappa(C^n_\kappa(d_\kappa(N, x)))\mathrm{d}v_\kappa(x), \tag{2.2}$$

where $R_L \geq 0$ is the unique number for which $V_\kappa(C^n_\kappa(R_L)) = L$.

The spherical Laplacian on $\mathbb{S}^n_\kappa$ is

$$\Delta_\kappa w(x) := \Delta_{g_\kappa} w(x) = \kappa\,(\sin\theta)^{1-n} \frac{\partial}{\partial\theta}\left((\sin\theta)^{n-1} \frac{\partial w}{\partial\theta} \right) + \frac{\kappa}{\sin^2\theta} \Delta_\xi w, \tag{2.3}$$

where Δ_ξ is the Laplace–Beltrami operator on the usual $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$.

2.2 Properties of Gaussian Hypergeometric Functions. Let $a, b, c \in \mathbb{C}$ ($c \notin \mathbb{Z}_-$) and $(a)_m = a(a+1)\dots(a+m-1) = \frac{\Gamma(a+m)}{\Gamma(a)}$ be the Pochhammer symbol, $m \in \mathbb{N}$. The *Gaussian hypergeometric function* is

$${}_2F_1(a, b; c; z) = \sum_{m \geq 0} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}, \; |z| < 1, \tag{2.4}$$

and extended by analytic continuation elsewhere. The corresponding differential equation to $z \mapsto {}_2F_1(a, b; c; z)$ is

$$z(1-z)w''(z) + (c - (a+b+1)z)w'(z) - abw(z) = 0. \tag{2.5}$$

PROPOSITION 2.1. *If $t \mapsto \lambda(t)$ is a positive function and $\lim_{t\to 0}\lambda(t) = \ell > 0$, then for every $C_1, C_2, C \in \mathbb{R}$, $\mu > -1$ and $x > 0$ one has that*

$$\lim_{t\to 0} {}_2F_1\left(C_1 + \sqrt{C \pm \frac{\lambda^2(t)}{t}}, C_2 - \sqrt{C \pm \frac{\lambda^2(t)}{t}}; 1+\mu; \sin^2\left(\frac{\sqrt{t}x}{2}\right)\right)$$
$$= \Gamma(1+\mu)\left(\frac{2}{\ell x}\right)^{\mu} \begin{cases} J_\mu(\ell x) \text{ for ` + '}; \\ I_\mu(\ell x) \text{ for ` − '}. \end{cases}$$

Proof. By definition, for every $m \in \mathbb{N}$ we have that

$$\lim_{t\to 0} t^{\frac{m}{2}}\left(C_1 + \sqrt{C \pm \frac{\lambda^2(t)}{t}}\right)_m = (\pm 1)^{\frac{m}{2}}\ell^m$$

and

$$\lim_{t\to 0} t^{\frac{m}{2}}\left(C_2 - \sqrt{C \pm \frac{\lambda^2(t)}{t}}\right)_m = (-1)^m(\pm 1)^{\frac{m}{2}}\ell^m.$$

Therefore,

$$\begin{aligned} A^{\pm}_{\lambda,\mu}(x) &:= \lim_{t\to 0} {}_2F_1\left(C_1 + \sqrt{C \pm \frac{\lambda^2(t)}{t}}, C_2 - \sqrt{C \pm \frac{\lambda^2(t)}{t}}; 1+\mu; \sin^2\left(\frac{\sqrt{t}x}{2}\right)\right) \\ &= \lim_{t\to 0}\sum_{m=0}^{\infty} \frac{\left(C_1 + \sqrt{C \pm \frac{\lambda^2(t)}{t}}\right)_m \left(C_2 - \sqrt{C \pm \frac{\lambda^2(t)}{t}}\right)_m}{(1+\mu)_m} \frac{\sin^{2m}\left(\frac{\sqrt{t}x}{2}\right)}{m!} \\ &= \Gamma(1+\mu)\sum_{m=0}^{\infty} \frac{(-1)^m(\pm 1)^m \ell^{2m}}{\Gamma(1+m+\mu)m!}\left(\frac{x}{2}\right)^{2m} \\ &= \Gamma(1+\mu)\left(\frac{2}{\ell x}\right)^{\mu} \begin{cases} J_\mu(\ell x) \text{ for ` + '}; \\ I_\mu(\ell x) \text{ for ` − '}, \end{cases} \end{aligned}$$

which concludes the proof. □

For every $\mu \geq 0$ and $n \geq 2$, we consider the number

$$\Lambda_{\pm}(\mu) := \sqrt{\frac{(n-1)^2}{4} \pm \mu} \in \mathbb{C}, \tag{2.6}$$

and the specific Gaussian hypergeometric function

$$v_{\pm}(\mu, t) := {}_2F_1\left(\frac{1}{2} - \Lambda_{\pm}(\mu), \frac{1}{2} + \Lambda_{\pm}(\mu); \frac{n}{2}; t\right) = \sum_{m=0}^{\infty} \beta^{\pm}_m(\mu)t^m, \quad t \in (0,1), \tag{2.7}$$

where

$$\beta_m^{\pm}(\mu) := \frac{(\frac{1}{2} - \Lambda_{\pm}(\mu))_m(\frac{1}{2} + \Lambda_{\pm}(\mu))_m}{m!(\frac{n}{2})_m}, \ m \in \mathbb{N}. \tag{2.8}$$

We now collect those properties of the Gaussian hypergeometric functions that are important in our further investigations, most of which coming by well-known properties listed in Olver, Lozier, Boisvert and Clark [OLBC10] and recalled in "Appendix A".

Proposition 2.2. *If $n \in \mathbb{N}_{\geq 2}$, the following properties hold*:

(i) $v_{\pm}(0,t) = (1-t)^{\frac{n}{2}-1}$ *for every* $t \in (0,1)$;
(ii) $v_-(\mu,t) > 0$ *for every* $\mu > 0$ *and* $t \in (0,1)$;
(iii) *for every* $t \in (0,1)$ *the function* $\mu \mapsto v_+(\mu,t)$ *has infinitely many zeros in* $[0,\infty)$;
(iv) *for every* $\mu > 0$ *the number of zeros of the mapping* $t \mapsto v_+(\mu,t)$ *in* $(0,1)$ *is given by the* Klein-number $s_n^{\mu} := \lfloor \Lambda_+(\mu) - \frac{n-3}{2} \rfloor + k_n^{\mu} \geq 1$, *where* $\lfloor a \rfloor$ *stands for the greatest integer less than* $a > 0$ *and* $k_n^{\mu} \in \{0,1\}$ *(additionally,* $k_n^{\mu} = 0$ *for every* $n \geq 3$ *and* $\mu > 0$*)*;
(v) *for every* $\mu > 0$ *with* $\Lambda_{\pm}(\mu) - \frac{1}{2} \notin \mathbb{Z}$, *the function* $t \mapsto \frac{w_{\pm}(\mu,t)}{v_{\pm}(\mu,t)}$ *is decreasing between any two consecutive zeros of* $v_{\pm}(\mu,\cdot)$, *where*

$$w_{\pm}(\mu,t) = \sum_{m=0}^{\infty} \beta_m^{\pm}(\mu) \frac{\Psi(m+\frac{1}{2}+\Lambda_{\pm}(\mu)) - \Psi(m+\frac{1}{2}-\Lambda_{\pm}(\mu))}{\Lambda_{\pm}(\mu)} t^m, \quad t \in (0,1),$$

(with the convention that a limit is taken in $w_-(\mu,t)$ *whenever* $\Lambda_-(\mu) = 0$*)*;
(vi) $v_+\left(\mu, \frac{1}{2}\right) > 0 = v_+\left(n, \frac{1}{2}\right)$ *for every* $\mu \in [0,n)$.

Proof. (i) Using the Euler–Pfaff transformation (A.12), we have that

$$\begin{aligned} v_{\pm}(0,t) &= {}_2F_1\left(1-\frac{n}{2}, \frac{n}{2}; \frac{n}{2}; t\right) = (1-t)^{\frac{n}{2}-1} {}_2F_1\left(1-\frac{n}{2}, 0; \frac{n}{2}; \frac{t}{t-1}\right) \\ &= (1-t)^{\frac{n}{2}-1}, \ \forall t \in (0,1). \end{aligned}$$

(ii) Assume that $\mu > 0$ and let $A := \frac{1}{2} - \Lambda_-(\mu) \in \mathbb{C}$ and $B := \frac{1}{2} + \Lambda_-(\mu) \in \mathbb{C}$. First, if $\mu > \frac{n(n-2)}{4}$, then it follows that $(A)_m(B)_m > 0$ for every $m \in \mathbb{N}$. Therefore, by the definition (2.4) we obtain that $v_-(\mu,t) > 0$ for every $t \in (0,1)$. Now, if $0 < \mu \leq \frac{n(n-2)}{4}$, it turns out by (A.12) that

$$\begin{aligned} v_-(\mu,t) &= (1-t)^{\frac{n}{2}-A-B} {}_2F_1\left(\frac{n}{2}-A, \frac{n}{2}-B; \frac{n}{2}; t\right) \\ &= (1-t)^{\frac{n}{2}-1} {}_2F_1\left(\frac{n-1}{2}+\Lambda_-(\mu), \frac{n-1}{2}-\Lambda_-(\mu); \frac{n}{2}; t\right), \ \forall t \in (0,1). \end{aligned}$$

Since $0 < \mu \leq \frac{n(n-2)}{4}$, every parameter in the latter expression is real and positive, implying again that $v_-(\mu,t) > 0$ by means of (2.4).

(iii) By formula (A.10), the property that $\mu \mapsto v_+(\mu, t)$ has infinitely many zeros in $[0, \infty)$ for every fixed $t \in (0, 1)$, is a consequence of the result of MacDonald [Mac99]; see also Hobson [Hob31, p. 403–406] and Baginski [Bag90].

(iv) This property is attributed to Klein [Kle91] for $n \geq 3$ and Gormley [Gor37/38, p. 30] for $n = 2$.

(v) Let $\mu > 0$ with $\Lambda_\pm(\mu) - \frac{1}{2} \notin \mathbb{Z}$. Due to the explicit forms of $w_\pm$ and $v_\pm$, and basic properties of the Digamma function $\Psi = (\ln \Gamma)'$, the monotonicity of $t \mapsto \frac{w_\pm(\mu,t)}{v_\pm(\mu,t)}$ follows by direct computations that we illustrate in certain cases; similar arguments are provided by Yang, Chua and Wang [YCW15] and Holtz and Tyaglov [HT12].

Whenever $\Lambda_\pm(\mu) \neq 0$, we will also use the notation

$$\alpha_m^\pm(\mu) := \beta_m^\pm(\mu) \frac{\Psi(m + \frac{1}{2} + \Lambda_\pm(\mu)) - \Psi(m + \frac{1}{2} - \Lambda_\pm(\mu))}{\Lambda_\pm(\mu)}, \ m \in \mathbb{N},$$

whenever $\Lambda_\pm(\mu) \neq 0$. On account of relations (A.6)–(A.7), if $m_\mu^\pm := \left[\Re\Lambda_\pm(\mu) + \frac{1}{2}\right] \in \mathbb{N}$ (where $[a]$ is the integer part of $a \in \mathbb{R}$), then $\left(\frac{\alpha_m^\pm(\mu)}{\beta_m^\pm(\mu)}\right)_{m=\overline{0,m_\mu^\pm}}$ is increasing and $\left(\frac{\alpha_m^\pm(\mu)}{\beta_m^\pm(\mu)}\right)_{m\geq m_\mu^\pm}$ is decreasing, respectively.

For the case '$-$', by (ii) we know that $v_-(\mu, \cdot) > 0$ on $(0, 1)$. Let us consider

$\mu > \frac{1}{4}n(n-2)$ (for every $n \geq 2$); thus $m_\mu^- = 0$ and $\beta_m^-(\mu) > 0$ for every $m \geq 1$. Moreover, by the latter property it follows that $\left(\frac{\alpha_m^-(\mu)}{\beta_m^-(\mu)}\right)_{m\geq 0}$ is positive and decreasing. A direct calculation or the monotonicity result of Yang, Chua and Wang [YCW15] (see also Biernacki and Krzyz [BK55]) implies that the function $(0, 1) \ni t \mapsto \frac{w_-(\mu,t)}{v_-(\mu,t)}$ is also decreasing. Analogously, when $\frac{1}{4}n(n-2) \geq \mu > \frac{1}{4}(n+2)(n-4)$

(for every $n \geq 4$), it follows that $m_\mu^- = 1$, $\beta_0^-(\mu) = 1$ and $\beta_m^-(\mu) < 0$ for every $m \geq 1$, while $\alpha_m^-(\mu) < 0$ for every $m \geq 0$ (see (A.6)–(A.7)), and the sequence $\left(\frac{\alpha_m^-(\mu)}{\beta_m^-(\mu)}\right)_{m\geq 1}$ is decreasing. Therefore, $v_-'(\mu, \cdot) < 0$ and $v_-(\mu, \cdot) > 0$ (due to (ii)), and in a similar way as above, the function $\frac{w_-'(\mu,\cdot)}{v_-(\mu,\cdot)}$ is decreasing on $(0, 1)$. If $H_\mu(t) := \frac{w_-'(\mu,t)}{v_-'(\mu,t)} v_-(\mu, t) - w_-(\mu, t) = \left(\frac{w_-(\mu,t)}{v_-(\mu,t)}\right)' \frac{v_-^2(\mu,t)}{v_-'(\mu,t)}$, then it turns out that $H_\mu'(t) = \left(\frac{w_-'(\mu,t)}{v_-'(\mu,t)}\right)' v_-(\mu, t) < 0$ for every $t \in (0, 1)$, i.e., H_μ is decreasing on $(0, 1)$. Thus, one has $H_\mu(t) > \lim_{v \nearrow 1} H_\mu(v) = \frac{w_-'(\mu,1)}{v_-'(\mu,1)} v_-(\mu, 1) - w_-(\mu, 1) > 0$ for every $t \in (0, 1)$, which implies that $\left(\frac{w_-(\mu,\cdot)}{v_-(\mu,\cdot)}\right)' < 0$ on $(0, 1)$, concluding the proof. Generically, if $k \geq 1$ and $\frac{1}{4}(n+2k-2)(n-2k) \geq \mu > \frac{1}{4}(n+2k)(n-2k-2)$

(for every $n \geq 2k+2$), then $m_\mu^- = k$ and a similar argument as above implies that $t \mapsto \frac{w_-(\mu,t)}{v_-(\mu,t)}$ is decreasing on $(0, 1)$. When $\Lambda_-(\mu) = 0$, we consider the limit in w_-, replacing the expression $\frac{\Psi(m+\frac{1}{2}+\Lambda_-(\mu))-\Psi(m+\frac{1}{2}-\Lambda_-(\mu))}{\Lambda_-(\mu)}$ by $2\Psi\left(1, m + \frac{1}{2}\right)$ for all $m \in \mathbb{N}$, where $\Psi(1, x) := \frac{\mathrm{d}}{\mathrm{d}x}\Psi(x)$; thus $\alpha_m^-(\mu) = 2\beta_m^-(\mu)\Psi\left(1, m + \frac{1}{2}\right)$ and the rest is similar as above.

For the case ‘ + ’, let $t_1^\mu < \ldots < t_{s_n^\mu}^\mu$ be the zeros of $v_+(\mu,\cdot)$ in $(0,1)$, where $s_n^\mu \geq 1$ is the Klein-number by (iii). A similar procedure as before, or by adapting the argument from Holtz and Tyaglov [HT12] to our setting shows that $w_+'(\mu,\cdot)v_+(\mu,\cdot) - w_+(\mu,\cdot)v_+'(\mu,\cdot) < 0$ on $(0,1) \setminus \{t_1^\mu, \ldots, t_{s_n^\mu}^\mu\}$, concluding the claim.

(vi) By (A.12) and (2.4), one has that

$$v_+\left(n, \frac{1}{2}\right) = {}_2F_1\left(-\frac{n}{2}, \frac{n}{2}+1; \frac{n}{2}; \frac{1}{2}\right) = \frac{1}{2^{\frac{n}{2}-1}}\,{}_2F_1\left(n, -1; \frac{n}{2}; \frac{1}{2}\right) = 0.$$

Moreover, if $0 \leq \mu < n$, then $\Lambda_+(\mu) < \frac{n+1}{2}$, thus by (A.24) it follows that

$$v_+\left(\mu, \frac{1}{2}\right) = \frac{2^{1-\frac{n}{2}}\sqrt{\pi}\Gamma(\frac{n}{2})}{\Gamma\left(\frac{n+1}{4} + \frac{\Lambda_+(\mu)}{2}\right)\Gamma\left(\frac{n+1}{4} - \frac{\Lambda_+(\mu)}{2}\right)} > 0,$$

which concludes the proof. □

3 Clamped Spherical Belts: Proof of Theorem 1.1.

Let $R > r > 0$, $\lambda > 0$ and fix $\kappa \in \big(0, (\pi/R)^2\big)$. Particularizing (1.4), we consider the clamped plate problem on the spherical belt $B_\kappa(r,R) \subset \mathbb{S}_\kappa^2$, i.e.,

$$\begin{cases} \Delta_\kappa^2 w = \lambda^4 w & \text{in } B_\kappa(r,R), \\ w = \dfrac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial B_\kappa(r,R). \end{cases} \tag{P_κ}$$

Here, for any $x = x(\theta,\xi) \in \mathbb{S}_\kappa^2$, the spherical Laplacian on $\mathbb{S}_\kappa^2$ from (2.3) reduces to

$$\Delta_\kappa w(x) = \frac{\kappa}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w}{\partial\theta}\right) + \frac{\kappa}{\sin^2\theta}\frac{\partial^2 w}{\partial\xi^2}. \tag{3.1}$$

As the following subsections show. the proof of Theorem 1.1 is divided into three parts.

3.1 Narrow Spherical Belts: eigenfunctions of Fixed Sign. We first observe that, if $w : B_\kappa(r,R) \to \mathbb{R}$ is an eigenfunction of (P_κ), the same is true for its ξ-average

$$\widetilde{w}(x) = \frac{1}{2\pi}\int_0^{2\pi} w(x(\theta,\xi))\mathrm{d}\xi.$$

We also notice that $\widetilde{w}$ is azimuthally-invariant (or, spherical cap symmetric), i.e., $\widetilde{w}(x(\theta,\cdot))$ is constant for every fixed $\theta \in (\sqrt{\kappa}r, \sqrt{\kappa}R)$. In particular, by using (2.3), $\widetilde{w}$ is an eigenfunction of the ordinary differential equation

$$\begin{cases} \dfrac{\kappa^2}{\sin\theta}\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\dfrac{1}{\sin\theta}\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\dfrac{\mathrm{d}\widetilde{w}}{\mathrm{d}\theta}\right)\right)\right) = \lambda^4\widetilde{w}, \quad \theta \in (\sqrt{\kappa}r, \sqrt{\kappa}R), \\ \widetilde{w}(\sqrt{\kappa}r) = \dfrac{\mathrm{d}\widetilde{w}}{\mathrm{d}\theta}(\sqrt{\kappa}r) = \widetilde{w}(\sqrt{\kappa}R) = \dfrac{\mathrm{d}\widetilde{w}}{\mathrm{d}\theta}(\sqrt{\kappa}R) = 0; \end{cases} \tag{3.2}$$

moreover $\widetilde{w}$ is of fixed sign whenever it is not identically zero, see, e.g. Leighton and Nehari [LN58].

For further use, by applying (2.6) for $n=2$, we introduce the numbers

$$\gamma_\lambda^\pm(\kappa) := \Lambda_\pm\left(\frac{\lambda^2}{\kappa}\right) = \sqrt{\frac{1}{4} \pm \frac{\lambda^2}{\kappa}} \in \mathbb{C}, \tag{3.3}$$

while for $z \in \mathbb{C}$ and $\theta \in (\sqrt{\kappa}r, \sqrt{\kappa}R)$ we also define the Gaussian hypergeometric function

$$\mathcal{P}(z,\theta) := {}_2F_1\left(\frac{1}{2}+z, \frac{1}{2}-z; 1; \sin^2\left(\frac{\theta}{2}\right)\right)$$

as well as

$$\mathcal{Q}(z,\theta) := \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}+z\right)_m \left(\frac{1}{2}-z\right)_m}{(m!)^2} \sin^{2m}\left(\frac{\theta}{2}\right)\left(\Psi\left(\frac{1}{2}+z+m\right)\right.$$
$$\left. +\Psi\left(\frac{1}{2}-z+m\right) - 2\Psi(1+m)\right) + \mathcal{P}(z,\theta)\ln\left(\sin^2\left(\frac{\theta}{2}\right)\right),$$

whenever $\frac{1}{2} \pm z \neq 0, -1, -2, \ldots$, see Olver, Lozier, Boisvert and Clark [OLBC10, rel. (15.10.8)].

By the factorization $(\Delta_\kappa w - \lambda^2 w)(\Delta_\kappa w + \lambda^2 w) = \Delta_\kappa^2 w - \lambda^4 w$ and (3.1), we observe that the azimuthally-invariant function

$$w(x) = C_1\mathcal{P}(\gamma_\lambda^+(\kappa),\theta) + C_2\mathcal{Q}(\gamma_\lambda^+(\kappa),\theta) + C_3\mathcal{P}(\gamma_\lambda^-(\kappa),\theta) + C_4\mathcal{Q}(\gamma_\lambda^-(\kappa),\theta) \tag{3.4}$$

verifies the first equation of (3.2), where $x = x(\theta,\xi) \in B_\kappa(r,R)$ and the constants $\{C_i\}_{i=1}^4 \subset \mathbb{R}$ are not all zero; hereafter we consider the general case $\frac{1}{2} \pm \gamma_\lambda^\pm(\kappa) \neq 0, -1, -2, \ldots$, as the complementary cases are obtained by limits, see [OLBC10, rel. (15.10.9)–(15.10.10))]. If w satisfies the boundary conditions in (3.2), then w is of fixed sign (see [LN58]) and we obtain four equations in $\{C_i\}_{i=1}^4$. Since some of these constants are non-zero, we necessarily have that

$$\det\begin{bmatrix} \mathcal{P}(\gamma_\lambda^+(\kappa),\sqrt{\kappa}r) & \mathcal{Q}(\gamma_\lambda^+(\kappa),\sqrt{\kappa}r) & \mathcal{P}(\gamma_\lambda^-(\kappa),\sqrt{\kappa}r) & \mathcal{Q}(\gamma_\lambda^-(\kappa),\sqrt{\kappa}r) \\ \mathcal{P}'(\gamma_\lambda^+(\kappa),\sqrt{\kappa}r) & \mathcal{Q}'(\gamma_\lambda^+(\kappa),\sqrt{\kappa}r) & \mathcal{P}'(\gamma_\lambda^-(\kappa),\sqrt{\kappa}r) & \mathcal{Q}'(\gamma_\lambda^-(\kappa),\sqrt{\kappa}r) \\ \mathcal{P}(\gamma_\lambda^+(\kappa),\sqrt{\kappa}R) & \mathcal{Q}(\gamma_\lambda^+(\kappa),\sqrt{\kappa}R) & \mathcal{P}(\gamma_\lambda^-(\kappa),\sqrt{\kappa}R) & \mathcal{Q}(\gamma_\lambda^-(\kappa),\sqrt{\kappa}R) \\ \mathcal{P}'(\gamma_\lambda^+(\kappa),\sqrt{\kappa}R) & \mathcal{Q}'(\gamma_\lambda^+(\kappa),\sqrt{\kappa}R) & \mathcal{P}'(\gamma_\lambda^-(\kappa),\sqrt{\kappa}R) & \mathcal{Q}'(\gamma_\lambda^-(\kappa),\sqrt{\kappa}R) \end{bmatrix} = 0, \tag{3.5}$$

where $P'(z,\theta) := \frac{\partial}{\partial\theta}P(z,\theta)$ and $Q'(z,\theta) := \frac{\partial}{\partial\theta}Q(z,\theta)$.

Let $\lambda =: \lambda_{r,R}^{SP}(\kappa) > 0$ in (3.3) be the smallest positive zero of the equation (3.5); it follows that any eigenvalue that corresponds to an eigenfunction of fixed sign of (P_κ)

cannot be less than $\lambda_{r,R}^{SP}(\kappa)$. By analyticity, the function $\kappa \mapsto \lambda_{r,R}^{SP}(\kappa)$ is continuous and let

$$\lambda_0 := \lambda_{r,R}^{SP} = \lim_{\kappa \to 0} \lambda_{r,R}^{SP}(\kappa). \tag{3.6}$$

We show that, for every $x > 0$ one has the limit

$$\lim_{\kappa \to 0} \mathcal{P}(\gamma^{\pm}_{\lambda_{r,R}^{SP}(\kappa)}(\kappa), \sqrt{\kappa} x) = \begin{cases} J_0(\lambda_0 x) \text{ for ` + '}; \\ I_0(\lambda_0 x) \text{ for ` − '}, \end{cases} \tag{3.7}$$

and

$$\lim_{\kappa \to 0} \mathcal{Q}(\gamma^{\pm}_{\lambda_{r,R}^{SP}(\kappa)}(\kappa), \sqrt{\kappa} x) = \begin{cases} \pi Y_0(\lambda_0 x) \text{ for ` + '}; \\ -2K_0(\lambda_0 x) \text{ for ` − '}, \end{cases} \tag{3.8}$$

where Y_ν and K_ν are the Bessel and modified Bessel functions of the second kind ($\nu \geq 0$), respectively, see "Appendix A". We first observe that $\lambda_0 > 0$; otherwise the terms involving the function $\mathcal{Q}$ blow up whenever $\kappa \to 0$, by reaching the branch point 0 of both Y_0 and K_0.

Relation (3.7) immediately follows by Proposition 2.1 (with $\mu = 0$). To prove (3.8), by Proposition 2.1 (with $\mu = 0$) and relations (A.9) and (A.5) (with $n = 0$) we obtain that

$$\lim_{\kappa \to 0} \mathcal{Q}(\gamma^{+}_{\lambda_{r,R}^{SP}(\kappa)}(\kappa), \sqrt{\kappa} x) = \pi Y_0(\lambda_0 x).$$

Furthermore, by using the fact that $\ln \mathfrak{i} = \frac{\pi}{2}\mathfrak{i}$ (where $\mathfrak{i} = \sqrt{-1}$) and relations (A.2), (A.4) and (A.5), we also have that

$$\begin{aligned} \lim_{\kappa \to 0} \mathcal{Q}(\gamma^{-}_{\lambda_{r,R}^{SP}(\kappa)}(\kappa), \sqrt{\kappa} x) &= 2I_0(\lambda_0 x) \ln\left(\frac{\lambda_0 x}{2}\right) - 2\sum_{m=0}^{\infty} \frac{1}{(m!)^2} \left(\frac{\lambda_0 x}{2}\right)^{2m} \Psi(1+m) \\ &= -2K_0(\lambda_0 x), \end{aligned}$$

which concludes the proof of (3.8).

Due to relations (3.7)–(3.8), the analyticity of the aforementioned special functions, the limit argument in relation (3.5) and simple properties of the determinants imply that

$$\det \begin{bmatrix} J_0(\lambda_0 r) & Y_0(\lambda_0 r) & I_0(\lambda_0 r) & K_0(\lambda_0 r) \\ J_0'(\lambda_0 r) & Y_0'(\lambda_0 r) & I_0'(\lambda_0 r) & K_0'(\lambda_0 r) \\ J_0(\lambda_0 R) & Y_0(\lambda_0 R) & I_0(\lambda_0 R) & K_0(\lambda_0 R) \\ J_0'(\lambda_0 R) & Y_0'(\lambda_0 R) & I_0'(\lambda_0 R) & K_0'(\lambda_0 R) \end{bmatrix} = 0. \tag{3.9}$$

3.2 Wide Spherical Belts: sign-Changing Eigenfunctions.

For every $z \in \mathbb{C}$ with $\frac{1}{2} \pm z \neq 0, -1, -2, \ldots$, and $\theta \in (\sqrt{\kappa}r, \sqrt{\kappa}R)$ we consider the functions

$$\mathcal{F}(z,\theta) := {}_2F_1\left(\frac{3}{2}+z, \frac{3}{2}-z; 2; \sin^2\left(\frac{\theta}{2}\right)\right),$$

and

$$\begin{aligned}\mathcal{H}(z,\theta) := \mathcal{F}(z,\theta)\ln\left(\sin^2\left(\frac{\theta}{2}\right)\right) + \frac{1}{\frac{1}{4}-z^2}\sin^{-2}\left(\frac{\theta}{2}\right) \\ + \sum_{m=0}^{\infty}\frac{\left(\frac{3}{2}+z\right)_m\left(\frac{3}{2}-z\right)_m}{(m+1)!m!}\sin^{2m}\left(\frac{\theta}{2}\right)\cdot \\ \cdot\left(\Psi\left(\frac{3}{2}+z+m\right)+\Psi\left(\frac{3}{2}-z+m\right)-\Psi\left(1+m\right)-\Psi\left(2+m\right)\right).\end{aligned}$$

For every $x = x(\theta,\xi) \in B_\kappa(r,R)$, let

$$\begin{aligned}w(x) := \big(D_1\mathcal{F}(\gamma_\lambda^+(\kappa),\theta) + D_2\mathcal{H}(\gamma_\lambda^+(\kappa),\theta) + D_3\mathcal{F}(\gamma_\lambda^-(\kappa),\theta) \\ +D_4\mathcal{H}(\gamma_\lambda^-(\kappa),\theta)\big)\sin\theta\sin\xi,\end{aligned} \tag{3.10}$$

where the constants $\{D_i\}_{i=1}^4$ are not all zero and $\frac{1}{2} \pm \gamma_\lambda^\pm(\kappa) \neq 0, -1, -2, \ldots$; in the complementary cases we consider the limiting values. We notice that w changes its sign on $B_\kappa(r,R)$ as

$$w(x(\theta,\xi)) = -w(x(\theta,\pi+\xi));$$

in fact, w has two azimuthally opposite nodal circular arcs, corresponding to the values $\xi = 0$ and $\xi = \pi$, respectively.

Using the hypergeometric differential equation (15.10.1) and (15.10.8) from [OLBC10], relation (3.1) shows that w from (3.10) verifies pointwisely the first equation of (P_κ), i.e., $\Delta_\kappa^2 w = \lambda^4 w$ in $B_\kappa(r,R)$. Moreover, the factorized form of the latter equation—which is relevant only in the variable θ after simplifying by $\sin\xi$—is equivalent to

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\widetilde{w}}{\mathrm{d}\theta}\right) - \left(\frac{\kappa^2}{\sin^2\theta} \pm \lambda^2\right)\widetilde{w} = 0, \quad \theta \in (\sqrt{\kappa}r, \sqrt{\kappa}R). \tag{3.11}$$

In fact, the solutions $\widetilde{w}$ (both for '$+$' and '$-$') correspond to the four expressions involving the functions $\mathcal{F}$ and $\mathcal{H}$ in (3.10). For abbreviation, let

$$\widetilde{\mathcal{F}}_{\gamma_\lambda^\pm(\kappa)}(\theta) := \mathcal{F}(\gamma_\lambda^\pm(\kappa),\theta))\sin\theta \quad \text{and} \quad \widetilde{\mathcal{H}}_{\gamma_\lambda^\pm(\kappa)}(\theta) := \mathcal{H}(\gamma_\lambda^\pm(\kappa),\theta))\sin\theta.$$

The clamped boundary conditions $w = \frac{\partial w}{\partial \mathbf{n}} = 0$ on $\partial B_\kappa(r,R)$ provide four equations involving the constants $\{D_i\}_{i=1}^4$; since these constants are not all zero, we necessarily obtain that

$$\det \begin{bmatrix} \widetilde{\mathcal{F}}_{\gamma_\lambda^+(\kappa)}(\sqrt{\kappa}r) & \widetilde{\mathcal{H}}_{\gamma_\lambda^+(\kappa)}(\sqrt{\kappa}r) & \widetilde{\mathcal{F}}_{\gamma_\lambda^-(\kappa)}(\sqrt{\kappa}r) & \widetilde{\mathcal{H}}_{\gamma_\lambda^-(\kappa)}(\sqrt{\kappa}r) \\ \widetilde{\mathcal{F}}'_{\gamma_\lambda^+(\kappa)}(\sqrt{\kappa}r) & \widetilde{\mathcal{H}}'_{\gamma_\lambda^+(\kappa)}(\sqrt{\kappa}r) & \widetilde{\mathcal{F}}'_{\gamma_\lambda^-(\kappa)}(\sqrt{\kappa}r) & \widetilde{\mathcal{H}}'_{\gamma_\lambda^-(\kappa)}(\sqrt{\kappa}r) \\ \widetilde{\mathcal{F}}_{\gamma_\lambda^+(\kappa)}(\sqrt{\kappa}R) & \widetilde{\mathcal{H}}_{\gamma_\lambda^+(\kappa)}(\sqrt{\kappa}R) & \widetilde{\mathcal{F}}_{\gamma_\lambda^-(\kappa)}(\kappa R) & \widetilde{\mathcal{H}}_{\gamma_\lambda^-(\kappa)}(\sqrt{\kappa}R) \\ \widetilde{\mathcal{F}}'_{\gamma_\lambda^+(\kappa)}(\sqrt{\kappa}R) & \widetilde{\mathcal{H}}'_{\gamma_\lambda^+(\kappa)}(\sqrt{\kappa}R) & \widetilde{\mathcal{F}}'_{\gamma_\lambda^-(\kappa)}(\sqrt{\kappa}R) & \widetilde{\mathcal{H}}'_{\gamma_\lambda^-(\kappa)}(\sqrt{\kappa}R) \end{bmatrix} = 0. \tag{3.12}$$

Let $\lambda_{r,R}^{SC}(\kappa) > 0$ be the smallest positive zero of (3.12) and consider

$$\lambda_1 := \lambda_{r,R}^{SC} = \lim_{\kappa\to 0} \lambda_{r,R}^{SC}(\kappa). \tag{3.13}$$

Similarly as before, we have that $\lambda_1 > 0$ and, by using Proposition 2.1 (with $\mu = 1$) and (3.13), it follows that for every $x > 0$ we have

$$\lim_{\kappa\to 0} \frac{1}{\sqrt{\kappa}} \widetilde{\mathcal{F}}_{\gamma^{\pm}_{\lambda_{r,R}^{SC}(\kappa)}(\kappa)}(\sqrt{\kappa}x) = \frac{2}{\lambda_1} \begin{cases} J_1(\lambda_1 x) \text{ for ` + '}; \\ I_1(\lambda_1 x) \text{ for ` − '}. \end{cases} \tag{3.14}$$

Moreover, relations (A.9) and (A.5) (with $n = 1$) imply that for every $x > 0$ one has the limits

$$\lim_{\kappa\to 0} \frac{1}{\sqrt{\kappa}} \widetilde{\mathcal{H}}_{\gamma^{\pm}_{\lambda_{r,R}^{SC}(\kappa)}(\kappa)}(\sqrt{\kappa}x) = \begin{cases} \frac{2\pi}{\lambda_1} Y_1(\lambda_1 x) \text{ for ` + '}; \\ \frac{4}{\lambda_1} K_1(\lambda_1 x) \text{ for ` − '}. \end{cases} \tag{3.15}$$

Thus, by taking the limit $\kappa \to 0$ in (3.12) for $\lambda := \lambda_{r,R}^{SC}(\kappa)$ and by using basic properties of determinants, we obtain

$$\det \begin{bmatrix} J_1(\lambda_1 r) & Y_1(\lambda_1 r) & I_1(\lambda_1 r) & K_1(\lambda_1 r) \\ J_1'(\lambda_1 r) & Y_1'(\lambda_1 r) & I_1'(\lambda_1 r) & K_1'(\lambda_1 r) \\ J_1(\lambda_1 R) & Y_1(\lambda_1 R) & I_1(\lambda_1 R) & K_1(\lambda_1 R) \\ J_1'(\lambda_1 R) & Y_1'(\lambda_1 R) & I_1'(\lambda_1 R) & K_1'(\lambda_1 R) \end{bmatrix} = 0. \tag{3.16}$$

3.3 Threshold Spherical Belts. Let $0 < r < R$. Using (3.6) and (3.13), we recall that

$$\lambda_0 := \lambda_{r,R}^{SP} = \lim_{\kappa\to 0} \lambda_{r,R}^{SP}(\kappa) \quad \text{and} \quad \lambda_1 := \lambda_{r,R}^{SC} = \lim_{\kappa\to 0} \lambda_{r,R}^{SC}(\kappa), \tag{3.17}$$

are the smallest positive solutions to equations (3.9) and (3.16), respectively. The unit exterior radius case of the main result of Coffman, Duffin and Schaffer [CDS79, §4] on planar annuli asserts (up to a scaling) that

(i) $\lambda_0 < \lambda_1$ whenever $R/r < \mathsf{C}_{CDS}$; and
(ii) $\lambda_0 > \lambda_1$ whenever $R/r > \mathsf{C}_{CDS}$,

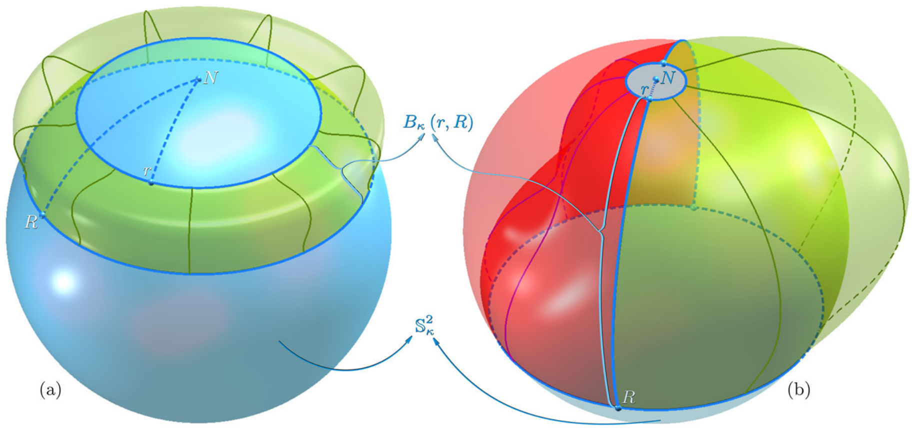

Figure 1: Clamped spherical belts $B_\kappa(r,R) \subset \mathbb{S}^2_\kappa$ of **a** narrow and **b** wide relative widths, in the case of which a first eigenfunction is of fixed sign and sign-changing, respectively. In both cases the zero altitude is represented by the light blue sphere $\mathbb{S}^2_\kappa$. The positive eigenfunction **a** is rendered with a transparent light green material, while the positive and negative values of the sign-changing eigenfunction **b** are illustrated by light green and dark red transparent materials, respectively. In the latter case, the eigenfunction is divided by two dark blue azimuthally opposite nodal circular arcs, while the preimages of the positive and negative parts are rendered with transparent dark green and light red materials, respectively. When $\kappa \to 0$, the relative threshold width R/r is given by the critical Coffmann–Duffin–Schaffer constant $\mathsf{C}_{CDS} \approx 762.3264$.

where C_{CDS} is the critical Coffman–Duffin–Schaffer constant. In fact, the value of the constant C_{CDS} is determined whenever $\lambda^{SP}_{r,R}$ and $\lambda^{SC}_{r,R}$ coincide, which is based on certain properties of Bessel functions, see also Coffman and Duffin [CD92]. In this critical case, one has that $\lambda^{SP}_{r,R} = \lambda^{SC}_{r,R} = \frac{\lambda_c^1}{r} = \frac{\lambda_c^2}{R}$ with $\lambda_c^1 \approx 0.00062557144$ and $\lambda_c^2 \approx 4.769102418$. Accordingly,

$$\mathsf{C}_{CDS} = \frac{R}{r} = \frac{\lambda_c^2}{\lambda_c^1} \approx 762.3264.$$

It remains to combine the latter relations with the limits (3.17) in order to conclude the existence of $\kappa_0 \in (0, \pi/R)$ such that for every $\kappa \in (0, \kappa_0)$:

(i) $\lambda^{SP}_{r,R}(\kappa) < \lambda^{SC}_{r,R}(\kappa)$ whenever $R/r < \mathsf{C}_{CDS}$, i.e., the first eigenfunction in (P_κ) is of fixed sign (see Fig. 1a); and
(ii) $\lambda^{SP}_{r,R}(\kappa) > \lambda^{SC}_{r,R}(\kappa)$ whenever $R/r > \mathsf{C}_{CDS}$, i.e., the first eigenfunction in (P_κ) is sign-changing having two azimuthally opposite nodal circular arcs (see Fig. 1b).

The proof is concluded. □

Remark 3.1. Let $\kappa > 0$, $R < \pi/\sqrt{\kappa}$ and consider the 2-dimensional spherical cap

$$C^2_\kappa(R) = \{x \in \mathbb{S}^2_\kappa : d_\kappa(N,x) < R\}.$$

Whenever $r \to 0$ (i.e., the interior radius of the spherical belt $B_\kappa(r,R)$ shrinks to the pole $N \in \mathbb{S}^2_\kappa$), the limit case of Theorem 1.1/(ii) states that for every $0 < R < \pi/\sqrt{\kappa}$, any first eigenfunction of the clamped problem (P_κ) on the punctured spherical cap $C^2_\kappa(R) \setminus \{N\} \subset \mathbb{S}^2_\kappa$ is always sign-changing; this scenario clearly holds for any value of κ. In this particular case of the punctured clamped spherical cap $C^2_\kappa(R) \setminus \{N\}$ the boundary condition for u reads as

$$u = \frac{\partial u}{\partial \mathbf{n}} = 0 \ \text{ on } \partial C^2_\kappa(R) \ \text{ and } \ u(N) = 0,$$

i.e., the tangential condition at the North pole N vanishes.

4 Reduction of Lord Rayleigh's Conjecture on Positively Curved Spaces

Let $n \in \mathbb{N}_{\geq 2}$ be fixed. In this section, we assume that (M, g) is an n-dimensional compact Riemannian manifold with Ricci curvature $\mathsf{Ric}_{(M,g)} \geq (n-1)\kappa > 0$. Rescaling the metric, we could consider $\mathsf{Ric}_{(M,g)} \geq n-1$ (and the unit sphere $\mathbb{S}^n := \mathbb{S}^n_1$), but as we have seen in Sect. 3 the presence of $\kappa > 0$ was crucial, thus for the sake of coherency we preserve the general form of the initial metric.

4.1 Ashbaugh–Benguria–Nadirashvili–Talenti Nodal-Decomposition.

If $S \subset M$ is a measurable set, its *normalized rearrangement* $S^\star \subset \mathbb{S}^n_\kappa$ is an open spherical cap centered at the North pole and

$$\frac{V_\kappa(S^\star)}{V_\kappa(\mathbb{S}^n_\kappa)} = \frac{V_g(S)}{V_g(M)}.$$

In a similar way, if $U : S \to (0,\infty)$ is a measurable function, we introduce its *normalized rearrangement* $U^\star : S^\star \to (0,\infty)$ such that

$$\frac{V_\kappa(\{x \in S^\star : U^\star(x) > t\})}{V_\kappa(\mathbb{S}^n_\kappa)} = \frac{V_g(\{x \in S : U(x) > t\})}{V_g(M)}, \ \forall t > 0. \tag{4.1}$$

By construction, $U^\star$ is a spherical cap symmetric function, i.e., $\xi \mapsto U^\star(x(\theta,\xi))$ is constant and

$$U^\star(x) = \sup\{t > 0 : x \in \{U > t\}^\star\}. \tag{4.2}$$

Lemma 4.1. *Let $U : S \to (0,\infty)$ be an integrable function and $U^\star : S^\star \to (0,\infty)$ be its normalized rearrangement. If $W \subseteq S$ is measurable and $W^* \subset \mathbb{S}^n_\kappa$ is its normalized rearrangement, then*

$$\frac{\displaystyle\int_W U \mathrm{d}v_g}{V_g(M)} \leq \frac{\displaystyle\int_{W^\star} U^\star \mathrm{d}v_\kappa}{V_\kappa(\mathbb{S}^n_\kappa)}. \tag{4.3}$$

In addition, if $W = S$, then we have equality in (4.3).

Proof. Relation (4.3) can be viewed as a normalized Hardy–Littlewood–Pólya inequality; although it can be verified by standard techniques, we provide its proof for completeness. Using the layer cake representation and the fact that $\chi_{W^\star} = \chi_W^\star$, where χ_Q denotes the indicator function of any non-empty set Q, it turns out by (4.1) that

$$\begin{aligned}
\int_W U\mathrm{d}v_g &= \int_S \chi_W(x)U(x)\mathrm{d}v_g(x) = \int_S \int_0^\infty \int_0^\infty \chi_{\{\chi_W>t\}}(x)\chi_{\{U>s\}}(x)\mathrm{d}t\mathrm{d}s\mathrm{d}v_g(x)\\
&= \int_0^\infty \int_0^\infty \int_S \chi_{\{\chi_W>t\}\cap\{U>s\}}(x)\mathrm{d}v_g(x)\mathrm{d}t\mathrm{d}s\\
&= \int_0^\infty \int_0^\infty V_g(\{\chi_W>t\}\cap\{U>s\})\mathrm{d}t\mathrm{d}s\\
&\leq \int_0^\infty \int_0^\infty \min\left\{V_g(\{\chi_W>t\}), V_g(\{U>s\})\right\}\mathrm{d}t\mathrm{d}s\\
&= \frac{V_g(M)}{V_\kappa(\mathbb{S}_\kappa^n)}\int_0^\infty \int_0^\infty \min\left\{V_\kappa(\{x\in S^\star : \chi_W^\star(x)>t\}),\right.\\
&\quad \left.\times\ V_\kappa(\{x\in S^\star : U^\star(x)>s\})\right\}\mathrm{d}t\mathrm{d}s\\
&= \frac{V_g(M)}{V_\kappa(\mathbb{S}_\kappa^n)}\int_0^\infty \int_0^\infty V_\kappa(\{x\in S^\star : \chi_W^\star(x)>t\}\cap\{x\in S^\star : U^\star(x)>s\})\mathrm{d}t\mathrm{d}s\\
&= \frac{V_g(M)}{V_\kappa(\mathbb{S}_\kappa^n)}\int_0^\infty \int_0^\infty \int_{S^\star} \chi_{\{\chi_W^\star>t\}\cap\{U^\star>s\}}(x)\mathrm{d}v_\kappa(x)\mathrm{d}t\mathrm{d}s\\
&= \frac{V_g(M)}{V_\kappa(\mathbb{S}_\kappa^n)}\int_{S^\star} \chi_{W^\star}(x)U^\star(x)\mathrm{d}v_\kappa(x)\\
&= \frac{V_g(M)}{V_\kappa(\mathbb{S}_\kappa^n)}\int_{W^\star} U^\star\mathrm{d}v_\kappa.
\end{aligned}$$

If $W = S$, equality clearly holds in the above relations, since the set $\{\chi_W > t\} = \{\chi_S > t\}$ is either empty or coincides with the whole S for every $t > 0$. □

Let $\Omega \subset M$ be a fixed open set. Since $\mathsf{Ric}_{(M,g)} \geq (n-1)\kappa > 0$, by the Bochner–Lichnerowicz–Weitzenböck formula it follows that the Sobolev space $H_0^2(\Omega) = W_0^{2,2}(\Omega)$ is a proper choice for problem (1.4), see Hebey [Heb99, Proposition 3.3]. Furthermore, on account of the compactness of (M, g) (by the Bonnet–Myers theorem) and basic properties of $W_0^{2,2}(\Omega)$, a similar argument as in Ashbaugh and Benguria [AB95, Appendix 2] shows that the infimum in (1.3), i.e., the fundamental tone

$$\Lambda_g(\Omega) = \inf_{u\in W_0^{2,2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega (\Delta_g u)^2\mathrm{d}v_g}{\displaystyle\int_\Omega u^2\mathrm{d}v_g},$$

is achieved. Let $u \in W_0^{2,2}(\Omega)$ be such a minimizer; elliptic regularity results guarantee that $u \in \mathcal{C}^\infty(\Omega)$.

Since u may change its sign in Ω (see e.g. Theorem 1.1), denote by $u_+ := \max(u, 0)$ and $u_- := -\min(u, 0)$ the positive and negative parts of u, respectively, and consider also the corresponding preimages

$$\Omega_+ := \{x \in \Omega : u_+(x) > 0\} \text{ and } \Omega_- := \{x \in \Omega : u_-(x) > 0\}.$$

In the rest of this section, we assume that

$$V_g(\Omega_+)V_g(\Omega_-) > 0,$$

otherwise the minimizer u is of fixed sign and the subsequent argument becomes considerable simpler. For further use, we also introduce the sets

$$\Omega_+^\Delta := \{x \in \Omega : (\Delta_g u)_+(x) > 0\} \text{ and } \Omega_-^\Delta := \{x \in \Omega : (\Delta_g u)_-(x) > 0\}.$$

Let $\Omega_\pm^\star \subset \mathbb{S}_\kappa^n$ and $\left(\Omega_\pm^\Delta\right)^\star \subset \mathbb{S}_\kappa^n$ be the normalized rearrangements of $\Omega_\pm \subset M$ and $\Omega_\pm^\Delta \subset M$, respectively. Since the subset of Ω where u is either constant or harmonic is negligible,

one has that

$$V_\kappa(\Omega_+^\star) + V_\kappa(\Omega_-^\star) = V_\kappa\left((\Omega_+^\Delta)^\star\right) + V_\kappa\left((\Omega_-^\Delta)^\star\right) = V_\kappa(\Omega^\star). \tag{4.4}$$

Let $u_\pm^\star : \Omega_\pm^\star \to (0,\infty)$ be the normalized rearrangement of $u_\pm : \Omega_\pm \to (0,\infty)$, i.e., for every $t > 0$,

$$\frac{V_\kappa(\{x \in \Omega_+^\star : u_+^\star(x) > t\})}{V_\kappa(\mathbb{S}_\kappa^n)} = \frac{V_g(\{x \in \Omega_+ : u_+(x) > t\})}{V_g(M)} =: j(t), \tag{4.5}$$

and

$$\frac{V_\kappa(\{x \in \Omega_-^\star : u_-^\star(x) > t\})}{V_\kappa(\mathbb{S}_\kappa^n)} = \frac{V_g(\{x \in \Omega_- : u_-(x) > t\})}{V_g(M)} =: h(t), \tag{4.6}$$

see Fig. 2.

Let $T_u^\pm := \sup_{x\in\Omega_\pm} u_\pm(x) \geq 0$; clearly, by definition, one has that $j(t) = 0$ for every $t \geq T_u^+$ and $h(t) = 0$ for every $t \geq T_u^-$.

In the same way as in (4.5) and (4.6), we introduce the normalized rearrangements $(\Delta_g u)_\pm^\star$ of $(\Delta_g u)_\pm : \Omega_\pm^\Delta \to (0,\infty)$. We extend $u_\pm^\star : \Omega_\pm^\star \to (0,\infty)$ and $(\Delta_g u)_\pm^\star : (\Omega_\pm^\Delta)^\star \to (0,\infty)$ by zero to the whole $\Omega^\star$ outside of $\Omega_\pm^\star$ and $\left(\Omega_\pm^\Delta\right)^\star$, respectively. Our further analysis is based on fine properties of the functions

$$\mathcal{J}(s) := (\Delta_g u)_-^*(s) - (\Delta_g u)_+^*(V_\kappa(\Omega^\star) - s)$$

and

$$\mathcal{H}(s) := -\mathcal{J}(V_\kappa(\Omega^\star) - s), \quad s \in [0, V_\kappa(\Omega^\star)],$$

where we will use the notation

$$(\Delta_g u)_\pm^*(s) := (\Delta_g u)_\pm^\star(x), \quad \text{whenever} \quad s = V_\kappa(C_\kappa^n(d_\kappa(N,x))),\ x \in \Omega^\star. \tag{4.7}$$

Some useful properties of the functions $\mathcal{J}$ and $\mathcal{H}$ are summarized in the sequel.

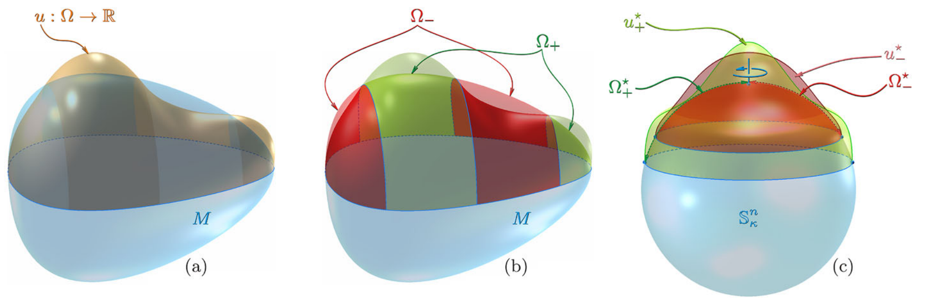

Figure 2: Illustration of the Ashbaugh–Benguria–Nadirashvili–Talenti nodal-decomposition argument. **a** A first eigenfunction $u : \Omega \to \mathbb{R}$ of the clamped plate problem (1.4) having nodal domains (the zero altitude is represented by the transparent light blue manifold M). **b** Rendering the segments of the null-measure set $u^{-1}(0)$ with dark blue lines, the image shows the nodal-decomposition of the open set $\Omega \subset M$ into the transparent light red and dark green preimages $\Omega_- = \{u < 0\}$ and $\Omega_+ = \{u > 0\}$, respectively. **c** The normalized rearrangements $u^\star_\pm : \Omega^\star_\pm \to (0, \infty)$ of $u_\pm : \Omega_\pm \to (0, \infty)$ on the spherical caps $\Omega^\star_\pm \subset \mathbb{S}^n_\kappa$, cf. relations (4.5) and (4.6).

LEMMA 4.2. *For every* $\sigma \in [0, V_\kappa(\Omega^\star)]$, *one has that:*

(i) $\displaystyle\int_0^\sigma \mathcal{J}(s)\mathrm{d}s \geq \int_0^{V_\kappa(\Omega^\star)} \mathcal{J}(s)\mathrm{d}s = 0;$

(ii) $\displaystyle\int_0^\sigma \mathcal{H}(s)\mathrm{d}s \geq \int_0^{V_\kappa(\Omega^\star)} \mathcal{H}(s)\mathrm{d}s = 0;$

(iii) $\displaystyle\int_{\Omega^\star_+} \mathcal{J}(V_\kappa(C^n_\kappa(d_\kappa(N,x))))\mathrm{d}v_\kappa(x) = \int_{\Omega^\star_-} \mathcal{H}(V_\kappa(C^n_\kappa(d_\kappa(N,x))))\mathrm{d}v_\kappa(x).$

Proof. By the divergence theorem and the boundary condition $\dfrac{\partial u}{\partial \mathbf{n}} = 0$ on $\partial\Omega$, it turns out that $\displaystyle\int_\Omega \Delta_g u \mathrm{d}v_g = 0$. Therefore, by applying Lemma 4.1 (with $S = W := \Omega^\Delta_\pm$ and $U := (\Delta_g u)_\pm$) and by using the change of variables (2.2), it turns out that

$$
\begin{aligned}
0 &= \frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)} \int_\Omega \Delta_g u \mathrm{d}v_g = -\frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)} \left(\int_{\Omega^\Delta_-} (\Delta_g u)_- \mathrm{d}v_g - \int_{\Omega^\Delta_+} (\Delta_g u)_+ \mathrm{d}v_g \right) \\
&= \int_{(\Omega^\Delta_-)^\star} (\Delta_g u)^\star_- \mathrm{d}v_\kappa - \int_{(\Omega^\Delta_+)^\star} (\Delta_g u)^\star_+ \mathrm{d}v_\kappa \\
&= \int_{\Omega^\star} (\Delta_g u)^\star_-(x) \mathrm{d}v_\kappa(x) - \int_{\Omega^\star} (\Delta_g u)^\star_+(x) \mathrm{d}v_\kappa(x) \\
&= \int_{\Omega^\star} (\Delta_g u)^*_-(V_\kappa(C^n_\kappa(d_\kappa(N,x))))\mathrm{d}v_\kappa(x) - \int_{\Omega^\star} (\Delta_g u)^*_+(V_\kappa(C^n_\kappa(d_\kappa(N,x))))\mathrm{d}v_\kappa(x) \\
&= \int_0^{V_\kappa(\Omega^\star)} \left((\Delta_g u)^*_-(s) - (\Delta_g u)^*_+(s) \right) \mathrm{d}s \\
&= \int_0^{V_\kappa(\Omega^\star)} \left((\Delta_g u)^*_-(s) - (\Delta_g u)^*_+(V_\kappa(\Omega^\star) - s) \right) \mathrm{d}s
\end{aligned}
$$

$$= \int_0^{V_\kappa(\Omega^\star)} \mathcal{J}(s)\mathrm{d}s.$$

In a similar way, one obtains that $\int_0^{V_\kappa(\Omega^\star)} \mathcal{H}(s)\mathrm{d}s = 0$. Now, by the definition of the normalized rearrangement, the functions $s \mapsto \mathcal{J}(s)$ and $s \mapsto \mathcal{H}(s)$ are decreasing on $[0, V_\kappa(\Omega^\star)]$; therefore

$$\int_0^\sigma \mathcal{J}(s)\mathrm{d}s \geq 0 \text{ and } \int_0^\sigma \mathcal{H}(s)\mathrm{d}s \geq 0, \quad \sigma \in [0, V_\kappa(\Omega^\star)], \tag{4.8}$$

which conclude the proof of (i) and (ii).

(iii) By the first part of the proof, relations (4.4) and (2.2) we have that

$$\begin{aligned} 0 &= \int_0^{V_\kappa(\Omega^\star)} \mathcal{J}(s)\mathrm{d}s = \int_0^{V_\kappa(\Omega^\star_+)} \mathcal{J}(s)\mathrm{d}s - \int_0^{V_\kappa(\Omega^\star_-)} \mathcal{H}(s)\mathrm{d}s \\ &= \int_{\Omega^\star_+} \mathcal{J}(V_\kappa(C^n_\kappa(d_\kappa(N,x))))\mathrm{d}v_\kappa(x) - \int_{\Omega^\star_-} \mathcal{H}(V_\kappa(C^n_\kappa(d_\kappa(N,x))))\mathrm{d}v_\kappa(x), \end{aligned}$$

which completes the proof. □

Besides Lemma 4.2, we need more precise estimates for the quantities in (4.8) as follows.

Proposition 4.1. *The following statements hold*:

(i) $\int_0^{V_\kappa(\mathbb{S}^n_\kappa)j(t)} \mathcal{J}(s)\mathrm{d}s \geq -\frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)} \int_{\{u>t\}} \Delta_g u(x)\mathrm{d}v_g(x)$ *for every* $t \in [0, T_u^+]$;

(ii) $\int_0^{V_\kappa(\mathbb{S}^n_\kappa)h(t)} \mathcal{H}(s)\mathrm{d}s \geq -\frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)} \int_{\{u<-t\}} \Delta_g u(x)\mathrm{d}v_g(x)$ *for every* $t \in [0, T_u^-]$;

(iii) *either* $(\Delta_g u)^*_-(s) = 0$ *or* $(\Delta_g u)^*_+(V_\kappa(\Omega^\star) - s) = 0$ *for every* $s \in [0, V_\kappa(\Omega^\star)]$.

Proof. (i) Let $t \in [0, T_u^+]$ be fixed; due to (4.5), one can find a unique number $a_t \geq 0$ such that $V_\kappa(C^n_\kappa(a_t)) = V_\kappa(\mathbb{S}^n_\kappa)j(t)$. Moreover, since $\{u > t\}^\star = \{u_+ > t\}^\star = C^n_\kappa(a_t)$, it follows that $\left(\{u > t\} \cap \Omega^\Delta_-\right)^\star \subseteq C^n_\kappa(a_t) \cap (\Omega^\Delta_-)^\star$. Using the latter relation together with the change of variables (2.2) and Lemma 4.1 (with $S := \Omega^\Delta_-$, $W := \{u > t\} \cap \Omega^\Delta_-$ and $U := (\Delta_g u)_-$), we have that

$$\begin{aligned} I &:= \int_0^{V_\kappa(\mathbb{S}^n_\kappa)j(t)} (\Delta_g u)^*_-(s)\mathrm{d}s = \int_0^{V_\kappa(C^n_\kappa(a_t))} (\Delta_g u)^*_-(s)\mathrm{d}s \\ &= \int_{C^n_\kappa(a_t)} (\Delta_g u)^*_-(V_\kappa(C^n_\kappa(d_\kappa(N,x))))\mathrm{d}v_\kappa(x) \\ &= \int_{C^n_\kappa(a_t)} (\Delta_g u)^\star_-(x)\mathrm{d}v_\kappa(x) = \int_{C^n_\kappa(a_t)\cap(\Omega^\Delta_-)^\star} (\Delta_g u)^\star_-(x)\mathrm{d}v_\kappa(x) \\ &\geq \int_{\left(\{u>t\}\cap\Omega^\Delta_-\right)^\star} (\Delta_g u)^\star_-(x)\mathrm{d}v_\kappa(x) \end{aligned}$$

$$\geq \frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)} \int_{\{u>t\}\cap\Omega^\Delta_-} (\Delta_g u)_-(x)\mathrm{d}v_g(x)$$
$$= \frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)} \int_{\{u>t\}} (\Delta_g u)_-(x)\mathrm{d}v_g(x). \tag{4.9}$$

Let $\widetilde{a}_t > 0$ be the unique value such that $V_\kappa(C^n_\kappa(\widetilde{a}_t)) = V_\kappa(\Omega^\star) - V_\kappa(\mathbb{S}^n_\kappa)j(t)$. In a similar manner as above, by a change of variables and (2.2), we infer that

$$\begin{aligned}
II &:= \int_0^{V_\kappa(\mathbb{S}^n_\kappa)j(t)} (\Delta_g u)^*_+(V_\kappa(\Omega^\star) - s)\mathrm{d}s \\
&= \int_0^{V_\kappa(\Omega^\star)} (\Delta_g u)^*_+(s)\mathrm{d}s - \int_0^{V_\kappa(\Omega^\star)-V_\kappa(\mathbb{S}^n_\kappa)j(t)} (\Delta_g u)^*_+(s)\mathrm{d}s \\
&= \int_{\Omega^\star} (\Delta_g u)^*_+(V_\kappa(C^n_\kappa(d_\kappa(N,x))))\mathrm{d}v_\kappa(x) - \int_{C^n_\kappa(\widetilde{a}_t)} (\Delta_g u)^*_+(V_\kappa(C^n_\kappa(d_\kappa(N,x))))\mathrm{d}v_\kappa(x) \\
&= \int_{(\Omega^\Delta_+)^\star} (\Delta_g u)^\star_+(x)\mathrm{d}v_\kappa(x) - \int_{C^n_\kappa(\widetilde{a}_t)\cap(\Omega^\Delta_+)^\star} (\Delta_g u)^\star_+(x)\mathrm{d}v_\kappa(x).
\end{aligned} \tag{4.10}$$

Applying Lemma 4.1 (with $S = W := \Omega^\Delta_+$ and $U := (\Delta_g u)_+$), one has that

$$\int_{(\Omega^\Delta_+)^\star} (\Delta_g u)^\star_+(x)\mathrm{d}v_\kappa(x) = \frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)} \int_{\Omega^\Delta_+} (\Delta_g u)_+(x)\mathrm{d}v_g(x). \tag{4.11}$$

Furthermore, if we consider the set $S_t = \{u \leq t\}$, it turns out that $S^\star_t = C^n_\kappa(\widetilde{a}_t)$; indeed, the latter fact follows from the definition of normalized rearrangement combined with relations $V_g(S_t) = V_g(\Omega) - V_g(M)j(t)$ and $V_\kappa(C^n_\kappa(\widetilde{a}_t)) = V_\kappa(\Omega^\star) - V_\kappa(\mathbb{S}^n_\kappa)j(t)$. Accordingly, due to Lemma 4.1 (with $S := \Omega^\Delta_+$, $W := S_t \cap \Omega^\Delta_+$ and $U := (\Delta_g u)_+$), it follows that

$$\begin{aligned}
\int_{C^n_\kappa(\widetilde{a}_t)\cap(\Omega^\Delta_+)^\star} (\Delta_g u)^\star_+(x)\mathrm{d}v_\kappa(x) &= \int_{S^\star_t\cap(\Omega^\Delta_+)^\star} (\Delta_g u)^\star_+(x)\mathrm{d}v_\kappa(x) \\
&\geq \int_{(S_t\cap\Omega^\Delta_+)^\star} (\Delta_g u)^\star_+(x)\mathrm{d}v_\kappa(x) \\
&\geq \frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)} \int_{S_t\cap\Omega^\Delta_+} (\Delta_g u)_+(x)\mathrm{d}v_g(x).
\end{aligned} \tag{4.12}$$

Using relations (4.10)–(4.12), it follows that

$$\begin{aligned}
II &\leq \frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)} \int_{\Omega^\Delta_+} (\Delta_g u)_+(x)\mathrm{d}v_g(x) - \frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)} \int_{S_t\cap\Omega^\Delta_+} (\Delta_g u)_+(x)\mathrm{d}v_g(x) \\
&= \frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)} \int_{\Omega^\Delta_+\setminus S_t} (\Delta_g u)_+(x)\mathrm{d}v_g(x) = \frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)} \int_{\Omega^\Delta_+\cap\{u>t\}} (\Delta_g u)_+(x)\mathrm{d}v_g(x) \\
&= \frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)} \int_{\{u>t\}} (\Delta_g u)_+(x)\mathrm{d}v_g(x).
\end{aligned} \tag{4.13}$$

The estimates (4.9) and (4.13) imply that

$$\begin{aligned}\int_0^{V_\kappa(\mathbb{S}^n_\kappa)j(t)} \mathcal{J}(s)\mathrm{d}s &= I - II \\ &\geq \frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)}\int_{\{u>t\}}(\Delta_g u)_-(x)\mathrm{d}v_g(x) - \frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)}\int_{\{u>t\}}(\Delta_g u)_+(x)\mathrm{d}v_g(x) \\ &= -\frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)}\int_{\{u>t\}}(\Delta_g u)(x)\mathrm{d}v_g(x),\end{aligned}$$

which is precisely the required inequality.

(ii) The proof is similar to (i).

(iii) Let us fix the parameter $s \in [0, V_\kappa(\Omega^\star)]$; if either $s = 0$ or $s = V_\kappa(\Omega^\star)$, the claim trivially holds. Otherwise, pick $x, \widetilde{x} \in \Omega^\star$ such that $s = V_\kappa(C^n_\kappa(d_\kappa(N, x)))$ and $V_\kappa(\Omega^\star) - s = V_\kappa(C^n_\kappa(d_\kappa(N, \widetilde{x})))$.

On one hand, if $x \notin (\Omega^\Delta_-)^\star$, then $(\Delta_g u)^*_-(s) = (\Delta_g u)^*_-(V_\kappa(C^n_\kappa(d_\kappa(N, x)))) = (\Delta_g u)^\star_-(x) = 0$. On the other hand, if $x \in (\Omega^\Delta_-)^\star$, then $s = V_\kappa(C^n_\kappa(d_\kappa(N, x))) < V_\kappa((\Omega^\Delta_-)^\star)$. Consequently,

$$V_\kappa(C^n_\kappa(d_\kappa(N, \widetilde{x}))) = V_\kappa(\Omega^\star) - s > V_\kappa(\Omega^\star) - V_\kappa((\Omega^\Delta_-)^\star) = V_\kappa((\Omega^\Delta_+)^\star),$$

i.e., $(\Omega^\Delta_+)^\star \subset C^n_\kappa(d_\kappa(N, \widetilde{x}))$ with strict inclusion. Therefore, one has $\widetilde{x} \notin (\Omega^\Delta_+)^\star$, i.e., $0 = (\Delta_g u)^\star_+(\widetilde{x}) = (\Delta_g u)^*_+(V_\kappa(C^n_\kappa(d_\kappa(N, \widetilde{x})))) = (\Delta_g u)^*_+(V_\kappa(\Omega^\star) - s)$, which concludes the proof. □

For further use, let $a, b \geq 0$ and $L > 0$ be such that

$$C^n_\kappa(a) = \Omega^\star_+, \quad C^n_\kappa(b) = \Omega^\star_- \quad \text{and} \quad C^n_\kappa(L) = \Omega^\star. \tag{4.14}$$

The main result of this subsection reads as follows.

Theorem 4.1. *The real functions*

$$U_a(x) := \frac{1}{n\omega_n}\int_{d_\kappa(N,x)}^a \left(\frac{\sin(\sqrt{\kappa}\rho)}{\sqrt{\kappa}}\right)^{1-n}\left(\int_0^{V_\kappa(C^n_\kappa(\rho))}\mathcal{J}(s)\mathrm{d}s\right)\mathrm{d}\rho, \ x \in \Omega^\star, \tag{4.15}$$

and

$$U_b(x) := \frac{1}{n\omega_n}\int_{d_\kappa(N,x)}^b \left(\frac{\sin(\sqrt{\kappa}\rho)}{\sqrt{\kappa}}\right)^{1-n}\left(\int_0^{V_\kappa(C^n_\kappa(\rho))}\mathcal{H}(s)\mathrm{d}s\right)\mathrm{d}\rho, \ x \in \Omega^\star, \tag{4.16}$$

satisfy the following statements:

(i) $\dfrac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)}\displaystyle\int_\Omega(\Delta_g u)^2\mathrm{d}v_g = \int_{C^n_\kappa(a)}(\Delta_\kappa U_a)^2\mathrm{d}v_\kappa + \int_{C^n_\kappa(b)}(\Delta_\kappa U_b)^2\mathrm{d}v_\kappa$;

(ii) $\dfrac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)}\displaystyle\int_\Omega u^2\mathrm{d}v_g \leq \int_{C^n_\kappa(a)} U_a^2\mathrm{d}v_\kappa + \int_{C^n_\kappa(b)} U_b^2\mathrm{d}v_\kappa$.

Proof. (i) By construction, both U_a and U_b in (4.15) and (4.16) are spherical cap symmetric functions; according to (2.3), a direct computation shows that U_a and U_b solve the problems

$$\begin{cases} -\Delta_\kappa U_a(x) &= \mathcal{J}(V_\kappa(C_\kappa^n((d_\kappa(N,x)))) \text{ in } C_\kappa^n(a)=\Omega_+^\star; \\ U_a &= 0 \qquad\qquad\qquad\qquad\quad \text{on } \partial C_\kappa^n(a), \end{cases} \tag{4.17}$$

and

$$\begin{cases} -\Delta_\kappa U_b(x) &= \mathcal{H}(V_\kappa(C_\kappa^n((d_\kappa(N,x)))) \text{ in } C_\kappa^n(b)=\Omega_-^\star; \\ U_b &= 0 \qquad\qquad\qquad\qquad\quad \text{on } \partial C_\kappa^n(b), \end{cases} \tag{4.18}$$

respectively. On one hand, by (4.17), (4.18), (2.2), the definition of $\mathcal{H}$ and by the fact that $V_\kappa(\Omega^\star)=V_\kappa(\Omega_+^\star)+V_\kappa(\Omega_-^\star)$, we obtain that

$$\begin{aligned} I &:= \int_{C_\kappa^n(a)}(\Delta_\kappa U_a)^2 \mathrm{d}v_\kappa + \int_{C_\kappa^n(b)}(\Delta_\kappa U_b)^2 \mathrm{d}v_\kappa \\ &= \int_{C_\kappa^n(a)} \mathcal{J}^2(V_\kappa(C_\kappa^n(d_\kappa(N,x))))\mathrm{d}v_\kappa(x) + \int_{C_\kappa^n(b)} \mathcal{H}^2(V_\kappa(C_\kappa^n(d_\kappa(N,x))))\mathrm{d}v_\kappa(x) \\ &= \int_0^{V_\kappa(\Omega_+^\star)} \mathcal{J}^2(s)\mathrm{d}s + \int_0^{V_\kappa(\Omega_-^\star)} \mathcal{H}^2(s)\mathrm{d}s \\ &= \int_0^{V_\kappa(\Omega^\star)} \mathcal{J}^2(s)\mathrm{d}s. \end{aligned}$$

On the other hand, by the definition of the function $\mathcal{J}$, Proposition 4.1/(iii), relation (2.2) and Lemma 4.1 (with $S=W:=\Omega_\pm^\Delta$ and $U:=(\Delta_g u)_\pm$), we have that

$$\begin{aligned} I &= \int_0^{V_\kappa(\Omega^\star)} \mathcal{J}^2(s)\mathrm{d}s \\ &= \int_0^{V_\kappa(\Omega^\star)} \Big[(\Delta_g u)_-^*(s)^2 + (\Delta_g u)_+^*(V_\kappa(\Omega^\star)-s)^2 \\ &\quad -2(\Delta_g u)_-^*(s)(\Delta_g u)_+^*(V_\kappa(\Omega^\star)-s)\Big]\,\mathrm{d}s \\ &= \int_0^{V_\kappa(\Omega^\star)} \Big[(\Delta_g u)_-^*(s)^2 + (\Delta_g u)_+^*(V_\kappa(\Omega^\star)-s)^2\Big]\,\mathrm{d}s \\ &= \int_0^{V_\kappa(\Omega^\star)} \Big[(\Delta_g u)_-^*(s)^2 + (\Delta_g u)_+^*(s)^2\Big]\,\mathrm{d}s \\ &= \int_{\Omega^\star} \Big[(\Delta_g u)_-^*(V_\kappa(C_\kappa^n(d_\kappa(N,x))))^2 + (\Delta_g u)_+^*(V_\kappa(C_\kappa^n(d_\kappa(N,x))))^2\Big]\,\mathrm{d}v_\kappa(x) \\ &= \int_{\Omega^\star} \Big[(\Delta_g u)_-^\star(x)^2 + (\Delta_g u)_+^\star(x)^2\Big]\,\mathrm{d}v_\kappa(x) \\ &= \int_{(\Omega_-^\Delta)^\star} (\Delta_g u)_-^\star(x)^2\mathrm{d}v_\kappa(x) + \int_{(\Omega_+^\Delta)^\star} (\Delta_g u)_+^\star(x)^2\mathrm{d}v_\kappa(x) \\ &= \frac{V_\kappa(\mathbb{S}_\kappa^n)}{V_g(M)}\int_{\Omega_-^\Delta}(\Delta_g u)_-^2(x)\mathrm{d}v_g(x) + \frac{V_\kappa(\mathbb{S}_\kappa^n)}{V_g(M)}\int_{\Omega_+^\Delta}(\Delta_g u)_+^2(x)\mathrm{d}v_g(x) \end{aligned}$$

$$= \frac{V_\kappa(\mathbb{S}^n_\kappa)}{V_g(M)} \int_\Omega (\Delta_g u)^2(x) \mathrm{d}v_g(x),$$

which concludes the proof of (i).

(ii) For every $t \in [0, T_u^+]$, with

$$T_u^+ = \sup_{x\in\Omega_+} u_+(x),$$

we introduce the sets

$$\Lambda_t := \partial(\{x \in \Omega : u_+(x) > t\}) \text{ and } \Lambda_t^\star := \partial(\{x \in \mathbb{S}^n_\kappa : u_+^\star(x) > t\}).$$

For any fixed $\varepsilon > 0$, Cauchy's inequality implies that

$$\begin{aligned}\left(\frac{1}{\varepsilon}\int_{t<u(x)\le t+\varepsilon} |\nabla_g u(x)|_g \mathrm{d}v_g(x)\right)^2 \\ \le \frac{V_g(u^{-1}((t,t+\varepsilon]))}{\varepsilon}\frac{1}{\varepsilon}\int_{t<u(x)\le t+\varepsilon} |\nabla_g u(x)|_g^2 \mathrm{d}v_g(x) \\ = V_g(M)\frac{j(t)-j(t+\varepsilon)}{\varepsilon}\frac{1}{\varepsilon}\int_{t<u(x)\le t+\varepsilon} |\nabla_g u(x)|_g^2 \mathrm{d}v_g(x).\end{aligned}$$

Since j is non-increasing, by letting $\varepsilon \to 0$, the co-area formula (see Chavel [Cha84, p.86]) and the latter inequality imply that

$$\mathcal{P}_g^2(\Lambda_t) \le -j'(t)V_g(M)\int_{\Lambda_t} |\nabla_g u| \mathrm{d}\mathcal{H}_{n-1} \text{ for a.e. } t \in [0, T_u^+],$$

where $\mathcal{P}_g(\Lambda_t)$ denotes the induced perimeter of the set $\{u_+ > t\} \subset M$, while $\mathcal{H}_{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure. Since the outward normal vector at $x \in \Lambda_t$ to $\{u > t\}$ is $-\frac{\nabla_g u(x)}{|\nabla_g u(x)|_g}$, the divergence theorem implies up to a negligible set that

$$\int_{\Lambda_t} |\nabla_g u|_g \mathrm{d}\mathcal{H}_{n-1} = -\int_{\{u_+>t\}} \Delta_g u \mathrm{d}v_g = -\int_{\{u>t\}} \Delta_g u \mathrm{d}v_g.$$

Therefore, for a.e. $t \in [0, T_u^+]$, we obtain the inequality

$$\mathcal{P}_g^2(\Lambda_t) \le V_g(M)j'(t)\int_{\{u>t\}} \Delta_g u \mathrm{d}v_g. \tag{4.19}$$

By relation (4.5), the Lévy–Gromov isoperimetric inequality implies that for every $t \in [0, T_u^+]$ one has

$$\frac{\mathcal{P}_\kappa(\Lambda_t^\star)}{V_\kappa(\mathbb{S}^n_\kappa)} \le \frac{\mathcal{P}_g(\Lambda_t)}{V_g(M)},\ \forall t \in [0, T_u^+], \tag{4.20}$$

where $\mathcal{P}_\kappa(\Lambda_t^\star)$ denotes the perimeter of the set $\{u_+^\star > t\} \subset \mathbb{S}^n_\kappa$.

Combining inequalities (4.19)–(4.20) with Proposition 4.1/(i), we have that

$$\mathcal{P}_\kappa^2(\Lambda_t^\star) \leq -V_\kappa(\mathbb{S}_\kappa^n)j'(t)\int_0^{V_\kappa(\mathbb{S}_\kappa^n)j(t)} \mathcal{J}(s)\mathrm{d}s \quad \text{for a.e. } t\in[0,T_u^+]. \tag{4.21}$$

We have seen in the proof of Proposition 4.1 that

$$\{u_+ > t\}^\star = C_\kappa^n(a_t) \quad \text{and} \quad V_\kappa(C_\kappa^n(a_t)) = V_\kappa(\mathbb{S}_\kappa^n)j(t)$$

for some $a_t \geq 0$; note that $t\mapsto a_t$ is decreasing on $[0,T_u^+]$. Since for sufficiently small $\varepsilon>0$ one has that

$$\{x :\in \mathbb{S}_\kappa^n : d_\kappa(x, C_\kappa^n(a_t)) < \varepsilon\} = C_\kappa^n(a_t+\varepsilon),$$

the Minkowski content of $C_\kappa^n(a_t)$ and (2.1) ensure that

$$V_\kappa(\mathbb{S}_\kappa^n)j'(t) = n\omega_n\left(\frac{\sin(\sqrt{\kappa}a_t)}{\sqrt{\kappa}}\right)^{n-1} a_t' = \mathcal{P}_\kappa(\Lambda_t^\star)a_t' \quad \text{for a.e. } t\in[0,T_u^+].$$

Combining this relation with (4.21), we infer that

$$1 \leq -\frac{1}{n\omega_n}\left(\frac{\sin(\sqrt{\kappa}a_t)}{\sqrt{\kappa}}\right)^{1-n} a_t' \int_0^{V_\kappa(C_\kappa^n(a_t))} \mathcal{J}(s)\mathrm{d}s \quad \text{for a.e. } t\in[0,T_u^+].$$

For every fixed $\eta\in[0,T_u^+]$, an integration of the latter inequality yields that

$$\eta \leq -\frac{1}{n\omega_n}\int_0^\eta \left(\frac{\sin(\sqrt{\kappa}a_t)}{\sqrt{\kappa}}\right)^{1-n} a_t' \left(\int_0^{V_\kappa(C_\kappa^n(a_t))} \mathcal{J}(s)\mathrm{d}s\right)\mathrm{d}t. \tag{4.22}$$

Arbitrary fixing the point $x\in C_\kappa^n(a) = \Omega_+^\star$, there exists a unique $\eta\in[0,T_u^+]$ such that $d_\kappa(N,x) = a_\eta$. By (4.2) and the monotonicity of $t\mapsto a_t$ on $[0,T_u^+]$, it follows that

$$\begin{aligned} u_+^\star(x) &= \sup\{t>0 : x\in\{u_+>t\}^\star\} = \sup\{t>0 : x\in C_\kappa^n(a_t)\} \\ &= \sup\{t>0 : d_\kappa(N,x) < a_t\} = \sup\{t>0 : a_\eta < a_t\} \\ &= \eta. \end{aligned}$$

Performing the change $\rho = a_t$ of variables at the right hand side of (4.22), and taking into account that $\lim_{t\to 0} a_t = a$ (with $a\geq 0$ by (4.14)), the latter inequality together with (4.15) implies that

$$u_+^\star \leq U_a \quad \text{in} \quad C_\kappa^n(a) = \Omega_+^\star; \tag{4.23}$$

see also Fig. 3. In a similar manner, we have that

$$u_-^\star \leq U_b \quad \text{in} \quad C_\kappa^n(b) = \Omega_-^\star, \tag{4.24}$$

by applying analogously to (4.19) that

$$\mathcal{P}_g^2(\Pi_t) \leq V_g(M)h'(t)\int_{\{u<-t\}} \Delta_g u \mathrm{d}v_g \quad \text{for a.e. } t\in[0,T_u^-], \tag{4.25}$$

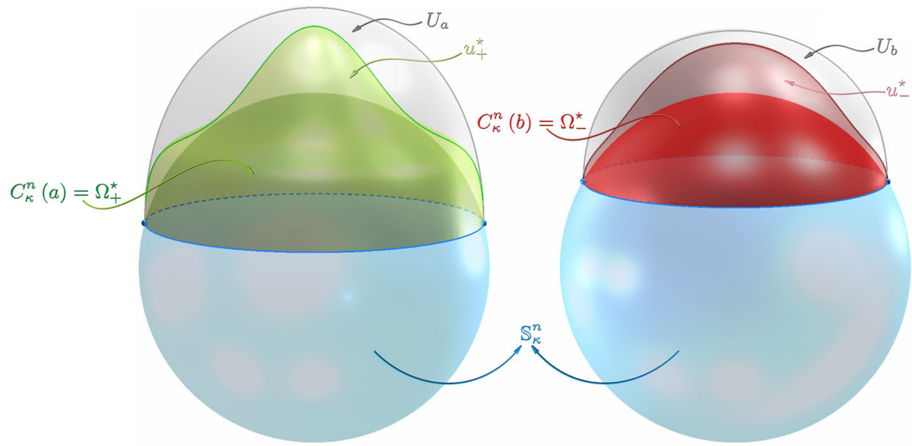

Figure 3: Spherical cap symmetric functions $U_a, u_+^\star : \Omega_+^\star \to (0,\infty)$ and $U_b, u_-^\star : \Omega_-^\star \to (0,\infty)$ which verify the inequalities (4.23)–(4.24). The functions $u_\pm^\star : \Omega_\pm^\star \to (0,\infty)$ arise from Fig. 2c, which are the normalized rearrangements of $u_\pm : \Omega_\pm \to (0,\infty)$ on the spherical caps $\Omega_\pm^\star \subset \mathbb{S}_\kappa^n$.

where $\Pi_t := \partial(\{x \in \Omega : -u(x) > t\})$.

Inequalities (4.23)–(4.24) and Lemma 4.1 (with $S = W := \Omega_\pm$ and $U := u_\pm^2$) imply that

$$\begin{aligned}\int_\Omega u^2 \mathrm{d}v_g &= \int_{\Omega_+} u_+^2 \mathrm{d}v_g + \int_{\Omega_-} u_-^2 \mathrm{d}v_g \\ &= \frac{V_g(M)}{V_\kappa(\mathbb{S}_\kappa^n)} \int_{\Omega_+^\star} (u_+^\star)^2 \mathrm{d}v_\kappa + \frac{V_g(M)}{V_\kappa(\mathbb{S}_\kappa^n)} \int_{\Omega_-^\star} (u_-^\star)^2 \mathrm{d}v_\kappa \\ &\le \frac{V_g(M)}{V_\kappa(\mathbb{S}_\kappa^n)} \left(\int_{C_\kappa^n(a)} U_a^2 \mathrm{d}v_\kappa + \int_{C_\kappa^n(b)} U_b^2 \mathrm{d}v_\kappa \right),\end{aligned}$$

which completes the proof. □

REMARK 4.1. We conclude this subsection with an observation concerning a joint boundary condition involving the functions U_a and U_b. In fact, we have that

$$(\sin(\sqrt{\kappa}a))^{n-1} \frac{\mathrm{d}U_a}{\mathrm{d}\theta}(\sqrt{\kappa}a) = (\sin(\sqrt{\kappa}b))^{n-1} \frac{\mathrm{d}U_b}{\mathrm{d}\theta}(\sqrt{\kappa}b), \tag{4.26}$$

where we explored the spherical cap symmetry of U_a and U_b, by identifying the point $x = x(\theta, \xi) \in \mathbb{S}_n^\kappa$ with the angle $\theta = \sqrt{\kappa} d_\kappa(N, x) \in (0, \pi)$. Applying Lemma 4.2/(iii) to problems (4.17)–(4.18), we first obtain that

$$\int_{C_\kappa^n(a)} \Delta_\kappa U_a(x) \mathrm{d}v_\kappa(x) = \int_{C_\kappa^n(b)} \Delta_\kappa U_b(x) \mathrm{d}v_\kappa(x). \tag{4.27}$$

Moreover, by (2.3) and the density of the measure $\mathrm{d}v_\kappa$ given by relation (2.1), the spherical cap symmetry of U_a and U_b implies the required relation (4.26).

4.2 Coupled Minimization on Spherical Caps. Given an open set $S \subset \mathbb{S}^n_\kappa$, we introduce the space

$$\mathcal{W}(S) := W_0^{1,2}(S) \cap W^{2,2}(S).$$

Based on (4.26), we will consider the boundary condition

$$(\sin(\sqrt{\kappa}a))^{n-1}\frac{\mathrm{d}u_a}{\mathrm{d}\theta}(\sqrt{\kappa}a) = (\sin(\sqrt{\kappa}b))^{n-1}\frac{\mathrm{d}u_b}{\mathrm{d}\theta}(\sqrt{\kappa}b), \tag{4.28}$$

for two spherical cap symmetric functions $u_a : C^n_\kappa(a) \to (0,\infty)$ and $u_b : C^n_\kappa(b) \to (0,\infty)$. For further use, let

$$\mathcal{W}_{a,b}(\Omega^\star) := \left\{ (u_a, u_b) \in \mathcal{W}(C^n_\kappa(a)) \times \mathcal{W}(C^n_\kappa(b)) : \begin{array}{l} u_a, u_b \text{ are spherical cap symmetric} \\ \text{functions that verify relation (4.28)} \end{array} \right\},$$

where

$$V_\kappa(C^n_\kappa(a)) + V_\kappa(C^n_\kappa(b)) = V_\kappa(\Omega^\star). \tag{4.29}$$

Using the aforementioned notations and Theorem 4.1, one can establish the connection between the fundamental tone of a set $\Omega \subset M$ and a coupled minimization problem on $\mathbb{S}^n_\kappa$; namely, if $a, b \geq 0$ are numbers such that (4.29) holds, then

$$\Gamma_g(\Omega) \geq \inf_{(u_a,u_b)\in\mathcal{W}_{a,b}(\Omega^\star)\setminus\{(0,0)\}} \frac{\displaystyle\int_{C^n_\kappa(a)} (\Delta_\kappa u_a)^2 \mathrm{d}v_\kappa + \int_{C^n_\kappa(b)} (\Delta_\kappa u_b)^2 \mathrm{d}v_\kappa}{\displaystyle\int_{C^n_\kappa(a)} u_a^2 \mathrm{d}v_\kappa + \int_{C^n_\kappa(b)} u_b^2 \mathrm{d}v_\kappa}. \tag{4.30}$$

For further use, we recall the Gaussian hypergeometric functions

$$\mathcal{F}_\pm(t,\lambda,\kappa,n) := {}_2F_1\left(\frac{1}{2} - \Lambda_\pm(\lambda), \frac{1}{2} + \Lambda_\pm(\lambda); \frac{n}{2}; t\right),\ t \in (0,1), \tag{4.31}$$

where

$$\Lambda_\pm(\lambda) = \Lambda_\pm(\lambda,\kappa,n) := \sqrt{\frac{(n-1)^2}{4} \pm \frac{\lambda^2}{\kappa}} \in \mathbb{C}. \tag{4.32}$$

Given $\lambda > 0$, we also need the cross-product of the Gaussian hypergeometric functions, i.e.,

$$\mathcal{K}_{\kappa,n}(t,\lambda) := \frac{\mathcal{F}'_-(t,\lambda,\kappa,n)}{\mathcal{F}_-(t,\lambda,\kappa,n)} - \frac{\mathcal{F}'_+(t,\lambda,\kappa,n)}{\mathcal{F}_+(t,\lambda,\kappa,n)}, \tag{4.33}$$

defined outside of the zeros of $\mathcal{F}_+(\cdot,\cdot,\kappa,n)$, see Proposition 2.2/(iii)–(iv); here we use the notation $\mathcal{F}'_\pm(t,\lambda,\kappa,n) := \frac{\partial}{\partial t}\mathcal{F}_\pm(t,\lambda,\kappa,n)$. Based on (4.30), a crucial result in our investigation can be formulated as follows.

Theorem 4.2. *Let (M, g) be a compact n-dimensional Riemannian manifold of Ricci curvature $\mathsf{Ric}_{(M,g)} \geq (n-1)\kappa > 0$, the open domain $\Omega \subset M$ with its normalized rearrangement $\Omega^\star \subset \mathbb{S}^n_\kappa$. If $a, b \geq 0$ are numbers such that (4.29) holds, then*

$$\Lambda_g(\Omega) \geq \Lambda(\kappa, n, a, b) =: \lambda_{\kappa,n}(\alpha, \beta)^4, \tag{4.34}$$

where

$$\alpha := \sin^2\left(\frac{\sqrt{\kappa}a}{2}\right), \ \beta := \sin^2\left(\frac{\sqrt{\kappa}b}{2}\right), \tag{4.35}$$

and $\lambda := \lambda_{\kappa,n}(\alpha, \beta) > 0$ is the smallest positive zero of the equation

$$(1-\alpha)^{\frac{n}{2}}\alpha^{\frac{n}{2}}\mathcal{K}_{\kappa,n}(\alpha, \lambda) + (1-\beta)^{\frac{n}{2}}\beta^{\frac{n}{2}}\mathcal{K}_{\kappa,n}(\beta, \lambda) = 0. \tag{4.36}$$

Proof. Let $a, b \geq 0$ be real numbers that verify (4.29). We note that the infimum in (4.30) is a minimum; this fact can be stated, by using a similar argument as in the flat and negatively curved cases studied by Ashbaugh and Benguria [AB95] and Kristály [Kri20], respectively. Accordingly, the value

$$\Lambda(\kappa, n, a, b) := \min_{(u_a,u_b)\in\mathcal{W}_{a,b}(\Omega^\star)\setminus\{(0,0)\}} \frac{\displaystyle\int_{C^n_\kappa(a)} (\Delta_\kappa u_a)^2 \mathrm{d}v_\kappa + \int_{C^n_\kappa(b)} (\Delta_\kappa u_b)^2 \mathrm{d}v_\kappa}{\displaystyle\int_{C^n_\kappa(a)} u_a^2 \mathrm{d}v_\kappa + \int_{C^n_\kappa(b)} u_b^2 \mathrm{d}v_\kappa} \tag{4.37}$$

is achieved by a pair of functions $(u_a, u_b) \in \mathcal{W}_{a,b}(\Omega^\star) \setminus \{(0,0)\}$. For simplicity of notation, let $\lambda := \lambda_{\kappa,n}(\alpha, \beta) > 0$ with $\lambda_{\kappa,n}(\alpha, \beta)^4 = \Lambda(\kappa, n, a, b)$. The Euler–Lagrange equation implies that

$$0 = \int_{C^n_\kappa(a)} (\Delta_\kappa u_a \Delta_\kappa \phi - \lambda^4 u_a \phi)\mathrm{d}v_\kappa + \int_{C^n_\kappa(b)} (\Delta_\kappa u_b \Delta_\kappa \psi - \lambda^4 u_b \psi)\mathrm{d}v_\kappa, \tag{4.38}$$

for every pair of functions $(\phi, \psi) \in \mathcal{W}_{a,b}(\Omega^\star)$; in particular, they verify the boundary condition

$$(\sin(\sqrt{\kappa}a))^{n-1}\frac{\mathrm{d}\phi}{\mathrm{d}\theta}(\sqrt{\kappa}a) = (\sin(\sqrt{\kappa}b))^{n-1}\frac{\mathrm{d}\psi}{\mathrm{d}\theta}(\sqrt{\kappa}b). \tag{4.39}$$

Using the particular form of the measure $\mathrm{d}v_\kappa$ in (2.1) and integrating by parts together with the fact that

$$\phi(\sqrt{\kappa}a) = \psi(\sqrt{\kappa}b) = 0,$$

we have that

$$\int_{C^n_\kappa(a)} \Delta_\kappa u_a \Delta_\kappa \phi \mathrm{d}v_\kappa = n\omega_n \kappa^{1-\frac{n}{2}} \Delta_\kappa u_a(\sqrt{\kappa}a)(\sin(\sqrt{\kappa}a))^{n-1} \times \frac{\mathrm{d}\phi}{\mathrm{d}\theta}(\sqrt{\kappa}a) + \int_{C^n_\kappa(a)} \Delta^2_\kappa u_a \phi \mathrm{d}v_\kappa$$

and

$$\int_{C_\kappa^n(b)} \Delta_\kappa u_b \Delta_\kappa \psi \mathrm{d}v_\kappa = n\omega_n \kappa^{1-\frac{n}{2}} \Delta_\kappa u_b(\sqrt{\kappa}b)(\sin(\sqrt{\kappa}b))^{n-1} \times \frac{\mathrm{d}\psi}{\mathrm{d}\theta}(\sqrt{\kappa}b) + \int_{C_\kappa^n(b)} \Delta_\kappa^2 u_b \psi \mathrm{d}v_\kappa.$$

Due to the latter relations, equation (4.38) transforms into

$$\begin{aligned} 0 = \ & n\omega_n \kappa^{1-\frac{n}{2}} \left(\Delta_\kappa u_a(\sqrt{\kappa}a)(\sin(\sqrt{\kappa}a))^{n-1} \frac{\mathrm{d}\phi}{\mathrm{d}\theta}(\sqrt{\kappa}a) \right. \\ & \left. + \Delta_\kappa u_b(\sqrt{\kappa}b)(\sin(\sqrt{\kappa}b))^{n-1} \frac{\mathrm{d}\psi}{\mathrm{d}\theta}(\sqrt{\kappa}b) \right) \\ & + \int_{C_\kappa^n(a)} (\Delta_\kappa^2 u_a - \lambda^4 u_a)\phi \mathrm{d}v_\kappa + \int_{C_\kappa^n(b)} (\Delta_\kappa^2 u_b - \lambda^4 u_b)\psi \mathrm{d}v_\kappa. \end{aligned} \tag{4.40}$$

In (4.40) we may choose either $\psi = 0$ and $\phi \in \mathcal{C}_0^2(C_\kappa^n(a))$, or $\phi = 0$ and $\psi \in \mathcal{C}_0^2(C_\kappa^n(b))$, obtaining that

$$\Delta_\kappa^2 u_a = \lambda^4 u_a \quad \text{in} \quad C_\kappa^n(a), \tag{4.41}$$

and

$$\Delta_\kappa^2 u_b = \lambda^4 u_b \quad \text{in} \quad C_\kappa^n(b), \tag{4.42}$$

respectively. Elliptic regularity theory shows that $u_a \in \mathcal{C}^\infty(C_\kappa^n(a))$ and $u_b \in \mathcal{C}^\infty(C_\kappa^n(b))$. Using again (4.40) and the boundary condition (4.39), it turns out that

$$\Delta_\kappa u_a(\sqrt{\kappa}a) + \Delta_\kappa u_b(\sqrt{\kappa}b) = 0. \tag{4.43}$$

The standard theory of ordinary differential equations shows that two of the four linearly independent solutions to the fourth-order equation (4.41) have singularities at the North pole $N \in \mathbb{S}_\kappa^n$; thus, the general non-singular solution to (4.41) has the form

$$\begin{aligned} u_a(x) &:= u_a(\theta) \\ &= \cos^{2-n}\left(\frac{\theta}{2}\right)\left[A_1\mathcal{F}_+\left(\sin^2\left(\frac{\theta}{2}\right), \lambda, \kappa, n\right) + A_2\mathcal{F}_-\left(\sin^2\left(\frac{\theta}{2}\right), \lambda, \kappa, n\right)\right], \end{aligned}$$

for every $x = x(\theta, \xi) \in C_\kappa^n(a)$ and some $A_1, A_2 \in \mathbb{R}$, where we have used the notations (4.31). In a similar way, the non-singular solution to (4.42) in general form is

$$\begin{aligned} u_b(x) &:= u_b(\theta) \\ &= \cos^{2-n}\left(\frac{\theta}{2}\right)\left[B_1\mathcal{F}_+\left(\sin^2\left(\frac{\theta}{2}\right), \lambda, \kappa, n\right) + B_2\mathcal{F}_-\left(\sin^2\left(\frac{\theta}{2}\right), \lambda, \kappa, n\right)\right], \end{aligned}$$

for every $x = x(\theta, \xi) \in C_\kappa^n(b)$ and some $B_1, B_2 \in \mathbb{R}$.

Since $(u_a, u_b) \in \mathcal{W}_{a,b}(\Omega^\star) \setminus \{(0,0)\}$, we have that $u_a(\sqrt{\kappa}a) = u_b(\sqrt{\kappa}b) = 0$; therefore, by (4.35) we infer that

$$A_1\mathcal{F}_+(\alpha,\lambda,\kappa,n) + A_2\mathcal{F}_-(\alpha,\lambda,\kappa,n) = 0 \tag{4.44}$$

and

$$B_1\mathcal{F}_+(\beta,\lambda,\kappa,n) + B_2\mathcal{F}_-(\beta,\lambda,\kappa,n) = 0. \tag{4.45}$$

Combining the boundary condition (4.28) with (4.44)–(4.45), it follows that

$$\begin{aligned} 0 &= (1-\alpha)\alpha^{\frac{n}{2}}\left(A_1\mathcal{F}'_+(\alpha,\lambda,\kappa,n) + A_2\mathcal{F}'_-(\alpha,\lambda,\kappa,n)\right) \\ &\quad - (1-\beta)\beta^{\frac{n}{2}}\left(B_1\mathcal{F}'_+(\beta,\lambda,\kappa,n) + B_2\mathcal{F}'_-(\beta,\lambda,\kappa,n)\right). \end{aligned} \tag{4.46}$$

Since we have the pointwise equality

$$\begin{aligned} &\Delta_\kappa\left(\cos^{2-n}\left(\frac{\theta}{2}\right)\mathcal{F}_\pm\left(\sin^2\left(\frac{\theta}{2}\right),\lambda,\kappa,n\right)\right) \\ &\quad = \mp\lambda^2\cos^{2-n}\left(\frac{\theta}{2}\right)\mathcal{F}_\pm\left(\sin^2\left(\frac{\theta}{2}\right),\lambda,\kappa,n\right), \ \theta\in(0,\pi), \end{aligned}$$

by (4.43) it follows that

$$\begin{aligned} 0 &= (1-\alpha)^{1-\frac{n}{2}}\left(-A_1\mathcal{F}_+(\alpha,\lambda,\kappa,n) + A_2\mathcal{F}_-(\alpha,\lambda,\kappa,n)\right) \\ &\quad + (1-\beta)^{1-\frac{n}{2}}\left(-B_1\mathcal{F}_+(\beta,\lambda,\kappa,n) + B_2\mathcal{F}_-(\beta,\lambda,\kappa,n)\right). \end{aligned} \tag{4.47}$$

Since A_1, A_2, B_1, B_2 cannot be simultaneously zero, by using the notation $\mathcal{F}_\pm(\cdot) := \mathcal{F}_\pm(\cdot,\lambda,\kappa,n)$, equations (4.44)–(4.47) necessarily imply that

$$\det\begin{bmatrix} \mathcal{F}_+(\alpha) & \mathcal{F}_-(\alpha) & 0 & 0 \\ 0 & 0 & \mathcal{F}_+(\beta) & \mathcal{F}_-(\beta) \\ (1-\alpha)\alpha^{\frac{n}{2}}\mathcal{F}'_+(\alpha) & (1-\alpha)\alpha^{\frac{n}{2}}\mathcal{F}'_-(\alpha) & -(1-\beta)\beta^{\frac{n}{2}}\mathcal{F}'_+(\beta) & -(1-\beta)\beta^{\frac{n}{2}}\mathcal{F}'_-(\beta) \\ -(1-\alpha)^{1-\frac{n}{2}}\mathcal{F}_+(\alpha) & (1-\alpha)^{1-\frac{n}{2}}\mathcal{F}_-(\alpha) & -(1-\beta)^{1-\frac{n}{2}}\mathcal{F}_+(\beta) & (1-\beta)^{1-\frac{n}{2}}\mathcal{F}_-(\beta) \end{bmatrix} = 0.$$

Expanding the determinant, we equivalently have that

$$(1-\alpha)^{\frac{n}{2}}\alpha^{\frac{n}{2}}\left(\frac{\mathcal{F}'_-(\alpha)}{\mathcal{F}_-(\alpha)} - \frac{\mathcal{F}'_+(\alpha)}{\mathcal{F}_+(\alpha)}\right) + (1-\beta)^{\frac{n}{2}}\beta^{\frac{n}{2}}\left(\frac{\mathcal{F}'_-(\beta)}{\mathcal{F}_-(\beta)} - \frac{\mathcal{F}'_+(\beta)}{\mathcal{F}_+(\beta)}\right) = 0,$$

which is precisely equation (4.36). □

5 Sharp Spectral Gaps on Clamped Spherical Caps: Proof of Theorem 1.2

In this section we prove Theorem 1.2, by establishing sharp growth estimates of the fundamental tone on the spherical cap $C^n_\kappa(L)$ in the two limit cases, i.e., when $L \to 0$ and $L \to \pi/\sqrt{\kappa}$, respectively. Before providing explicitly these estimates, we notice that the eigenfunctions on any spherical cap $C^n_\kappa(L)$ for the initial clamped problem (1.4) is of fixed sign, which follows by a Krein–Rutman argument and the sign-definite character of the solution to the Poisson-type biharmonic equation on $C^n_\kappa(L)$. By the proof of Theorem 4.2, these spherical cap symmetric eigenfunctions on $C^n_\kappa(L)$ are of the form

$$U_{c,L}(x) = c\cos^{2-n}\left(\frac{\theta}{2}\right)\left(\mathcal{F}_+\left(\sin^2\left(\frac{\theta}{2}\right), \lambda_L, \kappa, n\right) \right.$$
$$\left. -\frac{\mathcal{F}_+(\alpha_L, \lambda_L, \kappa, n)}{\mathcal{F}_-(\alpha_L, \lambda_L, \kappa, n)}\mathcal{F}_-\left(\sin^2\left(\frac{\theta}{2}\right), \lambda_L, \kappa, n\right)\right),$$

for every $x = x(\theta, \xi) \in C^n_\kappa(L)$ and $c \in \mathbb{R} \setminus \{0\}$, see Fig. 4, the value $\lambda_L := \Lambda_\kappa^{\frac{1}{4}}(C^n_\kappa(L))$ being the first positive zero of the equation

$$\mathcal{K}_{\kappa,n}(\alpha_L, \lambda) := \frac{\mathcal{F}_-'(\alpha_L, \lambda, \kappa, n)}{\mathcal{F}_-(\alpha_L, \lambda, \kappa, n)} - \frac{\mathcal{F}_+'(\alpha_L, \lambda, \kappa, n)}{\mathcal{F}_+(\alpha_L, \lambda, \kappa, n)} = 0, \tag{5.1}$$

where $\alpha_L := \sin^2\left(\frac{\sqrt{\kappa}L}{2}\right)$, while $\mathcal{F}_\pm$ is defined in (4.31).

5.1 Small Spherical Caps. In the infinitesimal case $L \ll 1$, we assume that

$$\lambda_L \sim \frac{C}{L} \quad \text{as} \quad L \to 0, \tag{5.2}$$

for some $C > 0$. By Proposition 2.1 (with settings $t = L^2$, $x = \sqrt{\kappa}$ and $\mu = \frac{n}{2} - 1$), on one hand, we obtain that

$$\lim_{L\to 0} \mathcal{F}_\pm(\alpha_L, \lambda_L, \kappa, n) = \Gamma\left(\frac{n}{2}\right)\left(\frac{2}{C}\right)^{\frac{n}{2}-1} \begin{cases} J_{\frac{n}{2}-1}(C) \text{ for ` + '}; \\ I_{\frac{n}{2}-1}(C) \text{ for ` − '}. \end{cases}$$

On the other hand, the differentiation formula (A.13) and Proposition 2.1 (with the choices $t = L^2$, $x = \sqrt{\kappa}$ and $\mu = \frac{n}{2}$) imply that

$$\lim_{L\to 0} L^2\mathcal{F}_\pm'(\alpha_L, \lambda_L, \kappa, n) = \Gamma\left(1 + \frac{n}{2}\right)\left(\frac{2}{C}\right)^{\frac{n}{2}} \begin{cases} J_{\frac{n}{2}}(C) \text{ for ` + '}; \\ -I_{\frac{n}{2}}(C) \text{ for ` − '}. \end{cases}$$

Since λ_L satisfies equation (5.1), the above limits immediately imply that $\frac{J_{\frac{n}{2}}(C)}{J_{\frac{n}{2}-1}(C)} + \frac{I_{\frac{n}{2}}(C)}{I_{\frac{n}{2}-1}(C)} = 0$. Due to (A.3), the latter equation is equivalent to

$$\frac{J_{\frac{n}{2}-1}'(C)}{J_{\frac{n}{2}-1}(C)} - \frac{I_{\frac{n}{2}-1}'(C)}{I_{\frac{n}{2}-1}(C)} = 0,$$

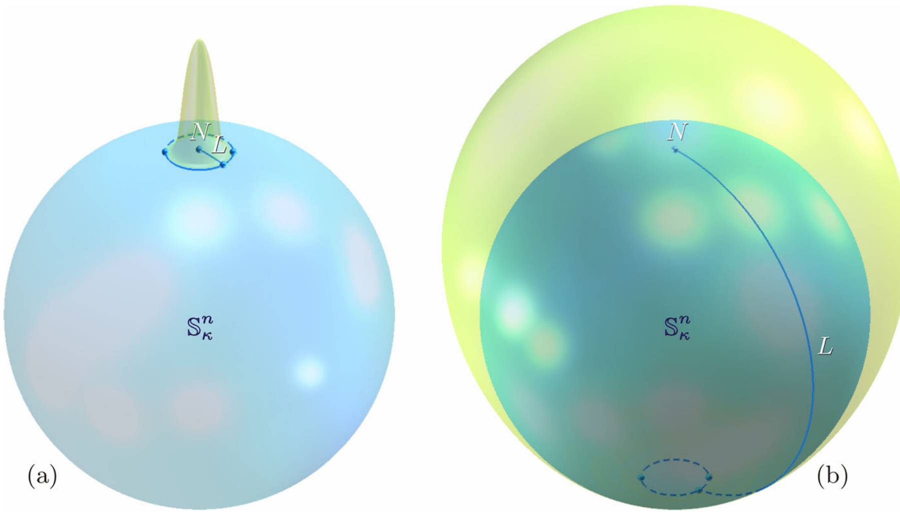

Figure 4: The shape of the first eigenfunction (light green) on **a** small and **b** large vibrating clamped spherical caps. The fundamental tone in **a** behaves as in the Euclidean case (cf. Sect. 5.1), while in **b** it is dimensional-dependent (cf. Sect. 5.2).

Table 1 Algebraic values and asymptotic estimates of the fourth root of the fundamental tone $\Lambda_\kappa(C^n_\kappa(L))$ for small clamped spherical caps in dimensions 2, 3 and 4 ($\kappa = 1$, for simplicity)

L	$n=2$		$n=3$		$n=4$	
	Algebraic value of $\Lambda_\kappa^{\frac{1}{4}}(C^2_\kappa(L))$	Asymptotic estimate of $\Lambda_\kappa^{\frac{1}{4}}(C^2_\kappa(L))$	Algebraic value of $\Lambda_\kappa^{\frac{1}{4}}(C^3_\kappa(L))$	Asymptotic estimate of $\Lambda_\kappa^{\frac{1}{4}}(C^3_\kappa(L))$	Algebraic value of $\Lambda_\kappa^{\frac{1}{4}}(C^4_\kappa(L))$	Asymptotic estimate of $\Lambda_\kappa^{\frac{1}{4}}(C^4_\kappa(L))$
0.4	7.9764	7.9906	9.7785	9.8165	11.4588	11.5272
0.03	106.5396	106.5406	130.8839	130.8867	153.6915	153.6966
0.002	1598.1102	1598.1103	1963.30097	1963.3011	2305.4496	2305.4501
0.0001	31962.205	31962.206	39266.0231	39266.0232	46108.9987	46108.9988
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$L \to 0$	$\to +\infty$	$\to +\infty$	$\to +\infty$	$\to +\infty$	$\to +\infty$	$\to +\infty$

The algebraic value $\Lambda_\kappa^{\frac{1}{4}}(C^n_\kappa(L))$ is the first positive zero of equation (5.1), while the asymptotic estimate is given by (1.7)

thus C coincides with the first positive zero $\mathfrak{h}_{\frac{n}{2}-1}$ of the cross-product of Bessel functions. According to (5.2), on has the estimate $\Lambda_\kappa(C^n_\kappa(L)) \sim \frac{\mathfrak{h}^4_{\frac{n}{2}-1}}{L^4}$ as $L \to 0$, which concludes the proof of (1.7).

The asymptotic estimate (1.7) shows that the fundamental tone $\Lambda_\kappa(C^n_\kappa(L))$ has an Euclidean character over small scales, since $\Lambda_0(B_0(L)) = \mathfrak{h}^4_{\frac{n}{2}-1}/L^4$ for every $L > 0$, see e.g. [AB95]. Table 1 provides an insight into the accuracy of the estimate (1.7) in a few dimensions.

5.2 Large Spherical Caps. In the sequel we will investigate the behavior of $\lambda_L > 0$ as $L \to \pi/\sqrt{\kappa}$, see (5.1), which has a dimension-depending character.

5.2.1 The case $n \geq 4$. On account of (A.13) and (5.1), the identity $\mathcal{K}_{\kappa,n}(\alpha_L, \lambda_L) = 0$ can be rewritten into the equivalent form

$$0 = \left(\frac{1}{4} - \Lambda_-^2(\lambda_L)\right) \frac{{}_2F_1\left(\frac{3}{2} - \Lambda_-(\lambda_L), \frac{3}{2} + \Lambda_-(\lambda_L); \frac{n+2}{2}; \sin^2\left(\frac{\sqrt{\kappa}L}{2}\right)\right)}{{}_2F_1\left(\frac{1}{2} - \Lambda_-(\lambda_L), \frac{1}{2} + \Lambda_-(\lambda_L); \frac{n}{2}; \sin^2\left(\frac{\sqrt{\kappa}L}{2}\right)\right)}$$
$$- \left(\frac{1}{4} - \Lambda_+^2(\lambda_L)\right) \frac{{}_2F_1\left(\frac{3}{2} - \Lambda_+(\lambda_L), \frac{3}{2} + \Lambda_+(\lambda_L); \frac{n+2}{2}; \sin^2\left(\frac{\sqrt{\kappa}L}{2}\right)\right)}{{}_2F_1\left(\frac{1}{2} - \Lambda_+(\lambda_L), \frac{1}{2} + \Lambda_+(\lambda_L); \frac{n}{2}; \sin^2\left(\frac{\sqrt{\kappa}L}{2}\right)\right)}, \quad (5.3)$$

where we have used the notation (4.32). We assume that $\lambda_L \to \lambda_0$ for some $\lambda_0 \geq 0$ as $L \to \pi/\sqrt{\kappa}$. If $n \geq 5$, we have that $\frac{n+2}{2} - \left(\frac{3}{2} - \Lambda_\pm(\lambda_L)\right) - \left(\frac{3}{2} + \Lambda_\pm(\lambda_L)\right) = \frac{n-4}{2} > 0$ and $\frac{n}{2} - \left(\frac{1}{2} - \Lambda_\pm(\lambda_L)\right) - \left(\frac{1}{2} + \Lambda_\pm(\lambda_L)\right) = \frac{n-2}{2} > 0$, thus the asymptotic formula (A.21) applied to (5.3) implies that $\lambda_0 = 0$. If $n = 4$, by using a similar argument as above, the asymptotic formulas (A.21) and (A.22) applied to (5.3) imply again that $\lambda_0 = 0$. Consequently, $\Lambda_\kappa(C_\kappa^n(L)) \to 0$ as $L \to \pi/\sqrt{\kappa}$ for every $n \geq 4$.

Remark 5.1. We note that when $n \in \{2, 3\}$, a similar argument as in the case $n \geq 4$ can be formally applied to (5.3) via the asymptotic formula (A.23); however, in both cases the asymptotic arguments lead us to an identity which looses any information on the behavior of $\Lambda_\kappa(C_\kappa^n(L))$ whenever $L \to \pi/\sqrt{\kappa}$. This phenomenon turns out to be unsurprising, since the low-dimensional cases behave in a different manner with respect to the higher dimensional counterparts.

5.2.2 The case $n = 3$. We first establish an elementary form of (5.1) which is valid only in the 3-dimensional case. By relation (15.4.16) of Olver, Lozier, Boisvert and Clark [OLBC10], we have that

$${}_2F_1\left(\frac{1}{2} - C, \frac{1}{2} + C; \frac{3}{2}; t\right) = \frac{\sin\left(2C\arcsin(\sqrt{t})\right)}{2C\sqrt{t}}, \ \forall C \in \mathbb{C} \setminus \{0\}, t \in (0,1). \quad (5.4)$$

Therefore, if we use the notations

$$\Lambda_\pm := \Lambda_\pm(\lambda, \kappa, 3) = \sqrt{1 \pm \frac{\lambda^2}{\kappa}} \ \text{ and } \ \widetilde{\Lambda}_- := \mathrm{i}\Lambda_-(\lambda, \kappa, 3) = \sqrt{\frac{\lambda^2}{\kappa} - 1},$$

see (4.32), one has that

$$\mathcal{F}_-(t, \lambda, \kappa, 3) = \begin{cases} \dfrac{\sinh\left(2\widetilde{\Lambda}_- \arcsin(\sqrt{t})\right)}{2\widetilde{\Lambda}_-\sqrt{t}} & \text{if } \lambda > \sqrt{\kappa}; \\ \dfrac{\arcsin\left(\sqrt{t}\right)}{\sqrt{t}} & \text{if } \lambda = \sqrt{\kappa}; \\ \dfrac{\sin\left(2\Lambda_- \arcsin(\sqrt{t})\right)}{2\Lambda_-\sqrt{t}} & \text{if } \lambda < \sqrt{\kappa}; \end{cases} \quad \text{and} \quad \mathcal{F}_+(t, \lambda, \kappa, 3) = \frac{\sin\left(2\Lambda_+ \arcsin(\sqrt{t})\right)}{2\Lambda_+\sqrt{t}}.$$

Accordingly, by using (4.33), we have for every $t \in (0,1)$ that

$$\mathcal{K}_{\kappa,3}(t,\lambda) = \frac{1}{\sqrt{t(1-t)}} \cdot \begin{cases} \widetilde{\Lambda}_- \coth\left(2\widetilde{\Lambda}_- \arcsin(\sqrt{t})\right) - \Lambda_+ \cot\left(2\Lambda_+ \arcsin(\sqrt{t})\right) & \text{if } \lambda > \sqrt{\kappa}; \\ \dfrac{1}{2\arcsin\left(\sqrt{t}\right)} - \sqrt{2}\cot\left(2\sqrt{2}\arcsin(\sqrt{t})\right) & \text{if } \lambda = \sqrt{\kappa}; \\ \Lambda_- \cot\left(2\Lambda_- \arcsin(\sqrt{t})\right) - \Lambda_+ \cot\left(2\Lambda_+ \arcsin(\sqrt{t})\right) & \text{if } \lambda < \sqrt{\kappa}. \end{cases} \tag{5.5}$$

We are ready to investigate the behavior of $\Lambda_\kappa(C_\kappa^3(L))$ as $L \to \pi/\sqrt{\kappa}$. By contradiction, we assume that $\lambda_L \leq \sqrt{\kappa}$, thus $\mu_L := \lambda_L^2/\kappa \in (0,1]$. First, when $\lambda_L < \sqrt{\kappa}$, by (5.5) the identity $\mathcal{K}_{\kappa,3}(\alpha_L,\lambda_L) = 0$ is equivalent to

$$\sqrt{1-\mu_L}\cot\left(\sqrt{1-\mu_L}\sqrt{\kappa}L\right) - \sqrt{1+\mu_L}\cot\left(\sqrt{1+\mu_L}\sqrt{\kappa}L\right) = 0. \tag{5.6}$$

We claim that (5.6) has no solution in μ_L for any $L \in (0, \pi/\sqrt{\kappa})$. On one hand, if $\sqrt{1+\mu_L}\sqrt{\kappa}L < \pi$, then by the monotonicity of $s \mapsto s\cot(s)$ on $(0,\pi)$, we have that

$$\sqrt{1-\mu_L}\cot\left(\sqrt{1-\mu_L}\sqrt{\kappa}L\right) > \sqrt{1+\mu_L}\cot\left(\sqrt{1+\mu_L}\sqrt{\kappa}L\right).$$

On the other hand, if $\sqrt{1+\mu_L}\sqrt{\kappa}L > \pi$, since $\mu_L < 1$ and $\sqrt{\kappa}L < \pi$, the monotonicity of $s \mapsto s\cot(s)$ on $(\pi, 2\pi)$ and on $(0,\pi)$, respectively, implies that

$$\begin{aligned}\sqrt{1+\mu_L}\sqrt{\kappa}L\cot\left(\sqrt{1+\mu_L}\sqrt{\kappa}L\right) &\geq \sqrt{2}\pi\cot\left(\sqrt{2}\pi\right) \approx 1.2272 \\ &> 1 = \lim_{s\to 0}\sqrt{1-\mu_L}s\cot\left(\sqrt{1-\mu_L}s\right) \\ &\geq \sqrt{1-\mu_L}\sqrt{\kappa}L\cot\left(\sqrt{1-\mu_L}\sqrt{\kappa}L\right).\end{aligned}$$

The above estimates conclude the claim together with the limit cases, i.e.,

- $\sqrt{1+\mu_L}\sqrt{\kappa}L = \pi$, when the left hand side of (5.6) blows up; and
- $\lambda_L = \sqrt{\kappa}$, when $\mathcal{K}_{\kappa,3}(\alpha_L,\lambda_L) = 0$ reduces (due to (5.5)) to an incompatible relation.

Consequently, the only possible case when $\mathcal{K}_{\kappa,3}(\alpha_L,\lambda_L) = 0$ might hold is when $\lambda_L > \sqrt{\kappa}$, obtaining by (5.5) that

$$\sqrt{\mu_L-1}\coth\left(\sqrt{\mu_L-1}\sqrt{\kappa}L\right) - \sqrt{1+\mu_L}\cot\left(\sqrt{1+\mu_L}\sqrt{\kappa}L\right) = 0,$$

where $\mu_L = \lambda_L^2/\kappa > 1$. Since $\sqrt{\kappa}L \to \pi$ and we may consider $\mu_L \to \mu$ for some $\mu \geq 1$, the latter equation reduces to

$$\sqrt{\mu-1}\coth\left(\pi\sqrt{\mu-1}\right) - \sqrt{1+\mu}\cot\left(\pi\sqrt{1+\mu}\right) = 0,$$

whose first positive zero is $\mu_3 := \mu \approx 1.0277$. Therefore,

$$\Lambda_\kappa(C_\kappa^3(L)) = \lambda_L^4 = \mu_L^2\kappa^2 \to \mu_3^2\kappa^2 \quad \text{as} \quad L \to \frac{\pi}{\sqrt{\kappa}}.$$

5.2.3 The case $n = 2$. In this special case, by using a simple relationship between the Gaussian hypergeometric and Legendre functions (see (A.10) for $\mu = 0$), the identity $\mathcal{K}_{\kappa,2}(\alpha_L, \lambda_L) = 0$ is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{\mathsf{P}^0_{\nu_-(\lambda_L)}(1-2t)}{\mathsf{P}^0_{\nu_+(\lambda_L)}(1-2t)}\Bigg|_{t=\alpha_L} = 0, \tag{5.7}$$

where $\nu_\pm(\lambda) = \Lambda_\pm(\lambda) - \frac{1}{2} = \sqrt{\frac{1}{4} \pm \frac{\lambda^2}{\kappa}} - \frac{1}{2}$. The derivation formula (A.15) transforms the equation (5.7) into

$$\begin{aligned} 0 = \nu_-(\lambda_L) &\left(\frac{\mathsf{P}^0_{\nu_-(\lambda_L)-1}\left(\cos\left(\sqrt{\kappa}L\right)\right)}{\mathsf{P}^0_{\nu_-(\lambda_L)}\left(\cos(\sqrt{\kappa}L)\right)} - \cos\left(\sqrt{\kappa}L\right) \right) \\ - \nu_+(\lambda_L) &\left(\frac{\mathsf{P}^0_{\nu_+(\lambda_L)-1}\left(\cos(\sqrt{\kappa}L)\right)}{\mathsf{P}^0_{\nu_+(\lambda_L)}\left(\cos(\sqrt{\kappa}L)\right)} - \cos\left(\sqrt{\kappa}L\right) \right). \end{aligned} \tag{5.8}$$

By (A.16) and (A.17), it turns out that for every fixed $s \in (-1, 1)$ the function $t \mapsto t\left(\frac{\mathsf{P}^0_{t-1}(s)}{\mathsf{P}^0_t(s)} - s\right)$ is increasing on $\left(-\frac{1}{2}, \frac{\sqrt{2}-1}{2}\right)$; in particular, one has the inequality

$$\begin{aligned} \nu_-(\lambda_L) &\left(\frac{\mathsf{P}^0_{\nu_-(\lambda_L)-1}\left(\cos(\sqrt{\kappa}L)\right)}{\mathsf{P}^0_{\nu_-(\lambda_L)}\left(\cos(\sqrt{\kappa}L)\right)} - \cos\left(\sqrt{\kappa}L\right) \right) \\ < \nu_+(\lambda_L) &\left(\frac{\mathsf{P}^0_{\nu_+(\lambda_L)-1}\left(\cos(\sqrt{\kappa}L)\right)}{\mathsf{P}^0_{\nu_+(\lambda_L)}\left(\cos(\sqrt{\kappa}L)\right)} - \cos\left(\sqrt{\kappa}L\right) \right) \end{aligned}$$

for every $L \in \left(0, \pi/\sqrt{\kappa}\right)$ and $\lambda_L \in \left(0, \sqrt{\kappa}/2\right)$. Thus, equation (5.8) has no solution whenever $(L, \mu_L) \in \left(0, \pi/\sqrt{\kappa}\right) \times \left(0, \sqrt{\kappa}/2\right)$. In particular, we necessarily have $\lambda_L \geq \sqrt{\kappa}/2$ for every $L \in \left(0, \pi/\sqrt{\kappa}\right)$, and $\nu_-(\lambda_L) \in \mathbb{C} \setminus \mathbb{R}$ when $\lambda_L > \sqrt{\kappa}/2$.

As -1 is a singularity in (5.8) whenever $L \to \pi/\sqrt{\kappa}$, the symmetrization formula (A.18) and the behavior at the singularity 1 of the Legendre functions (A.19) yield—after an asymptotic argument in (5.8)—that

$$\begin{aligned} &\sin((\nu_-(\lambda_0) - \nu_+(\lambda_0))\pi) + \frac{2}{\pi} \sin(\nu_-(\lambda_0)\pi) \sin(\nu_+(\lambda_0)\pi) \\ &\quad \times (\Psi(\nu_+(\lambda_0) + 1) - \Psi(\nu_-(\lambda_0) + 1)) = 0, \end{aligned}$$

where $\lambda_L \to \lambda_0$ as $L \to \pi/\sqrt{\kappa}$ for some $\lambda_0 \geq \sqrt{\kappa}/2$. We equivalently transform the latter equation into

$$\begin{aligned} &\tan\left(\Lambda_-(\lambda_0)\pi\right) - \tan\left(\Lambda_+(\lambda_0)\pi\right) + \frac{2}{\pi}\left(\Psi\left(\Lambda_+(\lambda_0) + \frac{1}{2}\right)\right. \\ &\left. -\Psi\left(\Lambda_-(\lambda_0) + \frac{1}{2}\right)\right) = 0. \end{aligned} \tag{5.9}$$

Table 2 Behavior of the fundamental tone $\Lambda_\kappa(C^n_\kappa(L))$ in dimensions $n \in \{2, \dots, 7\}$ for large spherical caps (i.e., $L \to \pi$ and $\kappa = 1$, for simplicity); μ_2 and μ_3 are the smallest positive zeros of equations (1.5) and (1.6), respectively

L	n					
	2	3	4	5	6	7
0.99π	0.8437	1.1091	0.8018	$2.32 \cdot 10^{-1}$	$2.91 \cdot 10^{-2}$	$2.35 \cdot 10^{-3}$
0.999π	0.8332	1.0612	0.4978	$2.39 \cdot 10^{-2}$	$2.96 \cdot 10^{-4}$	$2.36 \cdot 10^{-6}$
0.9999π	0.8328	1.05662	0.3605	$2.3 \cdot 10^{-3}$	$2.95 \cdot 10^{-7}$	$2.35 \cdot 10^{-9}$
0.99999π	0.83277	1.05661	0.2798	$2.16 \cdot 10^{-4}$	$2.4 \cdot 10^{-8}$	$1.72 \cdot 10^{-12}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\to \pi$	$\to \mu_2^2 \approx 0.83274$	$\to \mu_3^2 \approx 1.0561$	$\to 0$	$\to 0$	$\to 0$	$\to 0$

Due to (A.8), the imaginary part of (5.9) vanishes; thus, $\mu_2 := \lambda_0^2/\kappa \geq \frac{1}{4}$ is the first positive zero of

$$\frac{\pi}{2} \tan\left(\pi\sqrt{\frac{1}{4}+\mu}\right) - \Psi\left(\sqrt{\frac{1}{4}+\mu}+\frac{1}{2}\right) + \Re\Psi\left(\sqrt{\frac{1}{4}-\mu}+\frac{1}{2}\right) = 0, \quad \mu \geq \frac{1}{4}.$$

In fact, one has that $\mu_2 \approx 0.9125$, thus

$$\Lambda_\kappa(C^2_\kappa(L)) = \lambda_L^4 \to \lambda_0^4 = \mu_2^2\kappa^2 \quad \text{as} \quad L \to \frac{\pi}{\sqrt{\kappa}},$$

which concludes the proof of (1.8).

Table 2 presents the numerical behavior of the fundamental tone $\Lambda_\kappa(C^n_\kappa(L))$ in some dimensions whenever $L \to \pi/\sqrt{\kappa}$.

6 Lord Rayleigh's Conjecture: Proof of Theorem 1.3

This section is devoted to the proof of Lord Rayleigh's conjecture on positively curved spaces. Based on Theorem 4.2, first we further reduce the conjecture to the validity of an algebraic inequality (see Sect. 6.1), then we conclude the proof (see Sect. 6.2) by using the sharp growth estimates of the fundamental tone of spherical caps (see Sect. 5).

6.1 Reduction to Eigenvalue Comparison on 'half-caps'. We first need a monotonicity result that plays an important role in the proof of Theorem 1.3; the notations are the same as before.

Proposition 6.1. *If $\kappa > 0$, $n \in \mathbb{N}_{\geq 2}$ and $t \in (0,1)$ are fixed, the function $\lambda \mapsto \mathcal{K}_{\kappa,n}(t,\lambda)$ is increasing on $(0,\infty)$ between any two consecutive zeros of $\mathcal{F}_+(t,\cdot,\kappa,n)$. Moreover,* $\lim_{\lambda\to 0}\mathcal{K}_{\kappa,n}(t,\lambda) = 0$.

Proof. Let $\kappa > 0$ and $t \in (0,1)$ be fixed. For $n = 3$, the proof is trivial due to relation (5.5); indeed, a direct computation yields that $\lambda \mapsto \mathcal{K}_{\kappa,3}(t,\lambda)$ is increasing on $(0,\infty)$ between any two consecutive zeros of $\mathcal{F}_+(t,\cdot,\kappa,3)$, which have the explicit closed form

$$\mathfrak{f}_{\kappa,3,m}(t) = \sqrt{\kappa\left(\left(\frac{m\pi}{2\arcsin(\sqrt{t})}\right)^2 - 1\right)}, \quad m \in \mathbb{N}_{\geq 1}. \tag{6.1}$$

When $n \neq 3$, another approach is needed as no closed formula is available similar to (5.4) (and (5.5)). In fact, the strategy is to 'replace' the monotonicity of $\lambda \mapsto \mathcal{K}_{\kappa,n}(t,\lambda)$ with that of the ratio of hypergeometric-type functions with respect to the variable $t \in (0,1)$. In the sequel we consider those pairs (t,λ) in the open domain for which $\mathcal{F}_+(t,\lambda,\kappa,n) \neq 0$. In addition, we first assume that $\Lambda_\pm - \frac{1}{2} := \Lambda_\pm(\lambda,\kappa,n) - \frac{1}{2} \notin \mathbb{Z}$ and $\Lambda_- \neq 0$, see (4.32) for $\Lambda_\pm$. The analyticity of Gaussian hypergeometric functions together with relations (A.10) and (A.16) imply that

$$\begin{aligned}
\frac{\partial}{\partial\lambda}\mathcal{K}_{\kappa,n}(t,\lambda) &= \frac{\partial}{\partial\lambda}\left(\frac{\partial}{\partial t}\ln\frac{\mathcal{F}_-(t,\lambda,\kappa,n)}{\mathcal{F}_+(t,\lambda,\kappa,n)}\right) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial\lambda}\ln\frac{\mathcal{F}_-(t,\lambda,\kappa,n)}{\mathcal{F}_+(t,\lambda,\kappa,n)}\right)\\
&= \frac{\partial}{\partial t}\frac{\partial}{\partial\lambda}\ln\frac{\mathsf{P}^{1-\frac{n}{2}}_{\Lambda_--\frac{1}{2}}(1-2t)}{\mathsf{P}^{1-\frac{n}{2}}_{\Lambda_+-\frac{1}{2}}(1-2t)} = \frac{\partial}{\partial t}\frac{\frac{\partial}{\partial\lambda}\mathsf{P}^{1-\frac{n}{2}}_{\Lambda_--\frac{1}{2}}(1-2t)}{\mathsf{P}^{1-\frac{n}{2}}_{\Lambda_--\frac{1}{2}}(1-2t)} - \frac{\partial}{\partial t}\frac{\frac{\partial}{\partial\lambda}\mathsf{P}^{1-\frac{n}{2}}_{\Lambda_+-\frac{1}{2}}(1-2t)}{\mathsf{P}^{1-\frac{n}{2}}_{\Lambda_+-\frac{1}{2}}(1-2t)}\\
&= \frac{\lambda}{\kappa\pi}\left(\frac{1}{\Lambda_-}\frac{\partial}{\partial t}\frac{\mathsf{A}^{1-\frac{n}{2}}_{\Lambda_--\frac{1}{2}}(1-2t)}{\mathsf{P}^{1-\frac{n}{2}}_{\Lambda_--\frac{1}{2}}(1-2t)} + \frac{1}{\Lambda_+}\frac{\partial}{\partial t}\frac{\mathsf{A}^{1-\frac{n}{2}}_{\Lambda_+-\frac{1}{2}}(1-2t)}{\mathsf{P}^{1-\frac{n}{2}}_{\Lambda_+-\frac{1}{2}}(1-2t)}\right).
\end{aligned}$$

Therefore, it is enough to prove that the function $t \mapsto \frac{1}{\Lambda_\pm}\frac{\mathsf{A}^{1-\frac{n}{2}}_{\Lambda_\pm-\frac{1}{2}}(1-2t)}{\mathsf{P}^{1-\frac{n}{2}}_{\Lambda_\pm-\frac{1}{2}}(1-2t)}$, $t \in (0,1)$ is increasing in the aforementioned domain. Using the formula $\sin(\nu\pi)\Gamma(-\nu)\Gamma(\nu+1) = -\pi$ for every $\nu \in \mathbb{C}\setminus\mathbb{Z}$ (see (A.20) with suitable choices), relations (A.10) and (A.17) imply that

$$\frac{1}{\Lambda_\pm}\frac{\mathsf{A}^{1-\frac{n}{2}}_{\Lambda_\pm-\frac{1}{2}}(1-2t)}{\mathsf{P}^{1-\frac{n}{2}}_{\Lambda_\pm-\frac{1}{2}}(1-2t)} = -\pi\cdot\frac{\sum\limits_{m=0}^{\infty}\alpha_m^\pm t^m}{\sum\limits_{m=0}^{\infty}\beta_m^\pm t^m} =: -\pi\cdot\frac{v_\pm(t)}{w_\pm(t)},$$

where the coefficients are

$$\alpha_m^\pm = \beta_m^\pm\frac{\Psi(m+\frac{1}{2}+\Lambda_\pm) - \Psi(m+\frac{1}{2}-\Lambda_\pm)}{\Lambda_\pm} \quad \text{and}$$
$$\times \quad \beta_m^\pm = \frac{(\frac{1}{2}-\Lambda_\pm)_m(\frac{1}{2}+\Lambda_\pm)_m}{m!(\frac{n}{2})_m}, \ m \in \mathbb{N},$$

are well-defined. Consequently, the claimed monotonicity of $\lambda \mapsto \mathcal{K}_{\kappa,n}(t,\lambda)$ reduces to the decreasing character of $t \mapsto \frac{w_\pm(t)}{v_\pm(t)}$, $t \in (0,1)$, which follows from Proposition 2.2/(v). When $\Lambda_- = 0$, a limit is considered, by obtaining the equality $\alpha_m^\pm = 2\beta_m^\pm \Psi\left(1, m+\frac{1}{2}\right)$, and a similar proof applies as above. Finally, when $\Lambda_- - \frac{1}{2} \in \mathbb{Z}$ or $\Lambda_+ - \frac{1}{2} \in \mathbb{Z}$, another argument is needed, where the discussion is even simpler, since the corresponding series can be reduced to some polynomials.

The fact that $\lim_{\lambda\to 0} \mathcal{K}_{\kappa,n}(t,\lambda) = 0$ directly follows from (4.32)–(4.33), which ends the proof. □

From now on, we focus on the proof of Lord Rayleigh's conjecture for positively curved vibrating clamped plates. To this end, let (M,g) be a compact n-dimensional Riemannian manifold with $\mathsf{Ric}_{(M,g)} \geq (n-1)\kappa > 0$ and consider the non-empty smooth domain $\Omega \subset M$ with its normalized rearrangement $\Omega^\star \subset \mathbb{S}^n_\kappa$, i.e.,

$$\frac{V_g(\Omega)}{V_g(M)} = \frac{V_\kappa(\Omega^\star)}{V_\kappa(\mathbb{S}^n_\kappa)}. \tag{6.2}$$

Using the notation (4.37), Lord Rayleigh's conjecture is proved once we show that

$$\Lambda(\kappa,n,a,b) \geq \Lambda(\kappa,n,0,L) \tag{6.3}$$

for every $a,b \geq 0$ with $V_\kappa(C^n_\kappa(a)) + V_\kappa(C^n_\kappa(b)) = V_\kappa(\Omega^\star) = V_\kappa(C^n_\kappa(L))$, see (4.29). Indeed, if (6.3) holds, by Theorem 4.2 we have that

$$\Lambda_g(\Omega) \geq \Lambda(\kappa,n,a,b) \geq \Lambda(\kappa,n,0,L) = \Lambda_\kappa(C^n_\kappa(L)) = \Lambda_\kappa(\Omega^\star), \tag{6.4}$$

which is precisely the required inequality (1.9).

According to the statement of Theorem 4.2, inequality (6.3) is equivalent to

$$\lambda_{\kappa,n}(\alpha,\beta) \geq \lambda_{\kappa,n}(0,\alpha_L), \tag{6.5}$$

where $\lambda_{\kappa,n}(\alpha,\beta) > 0$ is the smallest positive zero of the equation (4.36), $\alpha_L = \sin^2\left(\frac{\sqrt{\kappa}L}{2}\right)$, while $\alpha,\beta \in (0,1)$ are arbitrarily chosen such that

$$\int_0^{\frac{2}{\sqrt{\kappa}}\sin^{-1}(\sqrt{\alpha})} \sin(\sqrt{\kappa}\rho)^{n-1}\mathrm{d}\rho + \int_0^{\frac{2}{\sqrt{\kappa}}\sin^{-1}(\sqrt{\beta})} \sin(\sqrt{\kappa}\rho)^{n-1}\mathrm{d}\rho = \int_0^L \sin(\sqrt{\kappa}\rho)^{n-1}\mathrm{d}\rho, \tag{6.6}$$

see (4.29) and (4.35).

Without loss of generality, we may choose $\alpha \leq \beta$ that verify (6.6). In view of Proposition 6.1, the function

$$\mathcal{S}_{\kappa,n}(\alpha,\beta,\lambda) = (1-\alpha)^{\frac{n}{2}}\alpha^{\frac{n}{2}}\mathcal{K}_{\kappa,n}(\alpha,\lambda) + (1-\beta)^{\frac{n}{2}}\beta^{\frac{n}{2}}\mathcal{K}_{\kappa,n}(\beta,\lambda)$$

inherits the properties of $\mathcal{K}_{\kappa,n}$, i.e., $\lambda \mapsto \mathcal{S}_{\kappa,n}(\alpha,\beta,\lambda)$ is increasing on $(0,\infty)$ between any two consecutive poles of $\mathcal{S}_{\kappa,n}(\alpha,\beta,\cdot)$ and $\lim_{\lambda\to 0}\mathcal{S}_{\kappa,n}(\alpha,\beta,\lambda) = 0$. In particular, if we denote the sequence of zeros of the Gaussian hypergeometric function

$\mathcal{F}_+(t,\cdot,\kappa,n)$ by $(\mathfrak{f}_{\kappa,n,m}(t))_m$ (cf. Proposition 2.2/(iii)), it turns out that the first positive zero $\lambda_{\kappa,n}(\alpha,\beta)$ of $\mathcal{S}_{\kappa,n}(\alpha,\beta,\cdot)$ will be situated between $\mathfrak{f}_{\kappa,n,1}(\beta)$ and $\mathfrak{f}_{\kappa,n,1}(\alpha)$; more precisely,

$$\mathfrak{f}_{\kappa,n,1}(\beta) \leq \lambda_{\kappa,n}(\alpha,\beta) \leq \min\{\mathfrak{f}_{\kappa,n,1}(\alpha), \mathfrak{f}_{\kappa,n,2}(\beta)\}. \tag{6.7}$$

We associate with $L \in (\pi/\sqrt{\kappa})$ (arising from $V_\kappa(C^n_\kappa(L)) = V_\kappa(\Omega^\star)$) the *half-cap radius* $L_0 \in \left(0, \frac{\pi}{2\sqrt{\kappa}}\right)$ defined by

$$2V_\kappa(C^n_\kappa(L_0)) = V_\kappa(C^n_\kappa(L)), \tag{6.8}$$

and we also introduce the notation $\alpha_{L_0} := \sin^2\left(\frac{\sqrt{\kappa}L_0}{2}\right)$. Letting $\alpha \to \alpha_{L_0}$ and $\beta \to \alpha_{L_0}$ in (6.7), we obtain that

$$\lambda_{\kappa,n}(\alpha_{L_0},\alpha_{L_0}) = \mathfrak{f}_{\kappa,n,1}(\alpha_{L_0}), \tag{6.9}$$

which corresponds to $a = b = L_0$. Due to (6.9), a *necessary* condition for the validity of (6.5) is

$$\mathfrak{f}_{\kappa,n,1}(\alpha_{L_0}) \geq \lambda_{\kappa,n}(0,\alpha_L). \tag{6.10}$$

Postponing the study of (6.10) (see Sect. 6.2), we show in the sequel that (6.10) is also *sufficient* to prove Rayleigh's conjecture that will be done in two steps.

First, if strict inequality occurs in (6.10), by using a continuity argument in the transcendental equation $\mathcal{S}_{\kappa,n}(\alpha,\beta,\lambda) = 0$ together with equality (6.9), it follows that there is $\alpha > 0$ sufficiently close to α_{L_0} such that $\lambda_{\kappa,n}(\alpha,\beta(\alpha)) > \lambda_{\kappa,n}(0,\alpha_L)$, where $\beta = \beta(\alpha)$ is from (6.6). Quantitatively, the last statement implies that one can find the unique minimal $\alpha_0 \in (0,\alpha_{L_0})$ such that $\lambda_{\kappa,n}(\alpha,\beta(\alpha)) > \lambda_{\kappa,n}(0,\alpha_L)$, holds for $\alpha \in [\alpha_0,\alpha_{L_0}]$, so (6.5) is verified. In particular, $\beta_0 = \beta(\alpha_0)$ verifies the condition $\mathfrak{f}_{\kappa,n,1}(\beta_0) = \lambda_{\kappa,n}(0,\alpha_L)$, where β_0 is a pole of the function $\mathcal{S}_{\kappa,n}(\alpha_0,\cdot,\lambda_{\kappa,n}(0,\alpha_L))$.

Second, by definition, we have that $\mathcal{S}_{\kappa,n}(0,\alpha_L,\lambda_{\kappa,n}(0,\alpha_L)) = 0$ and the construction of $\alpha_0 > 0$ and $\beta_0 = \beta(\alpha_0) > 0$ implies that $\lim_{\alpha\nearrow\alpha_0} \mathcal{S}_{\kappa,n}(\alpha,\beta(\alpha),\lambda_{\kappa,n}(0,\alpha_L)) = -\infty$. In fact, one has that $\mathcal{S}_{\kappa,n}(\alpha,\beta(\alpha),\lambda_{\kappa,n}(0,\alpha_L)) < 0$ for every $\alpha \in (0,\alpha_0)$. Indeed, since by (6.6) we have that

$$[\alpha(1-\alpha)]^{\frac{n}{2}-1} + \beta'(\alpha)[\beta(\alpha)(1-\beta(\alpha))]^{\frac{n}{2}-1} = 0,$$

a similar computation as in Karp and Sitnik [KS09] shows that $\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{S}_{\kappa,n}(\alpha,\beta(\alpha),\lambda_{\kappa,n}(0,\alpha_L)) < 0$ for every $\alpha \in (0,\alpha_0)$. Thus, the function $\alpha \mapsto \mathcal{S}_{\kappa,n}(\alpha,\beta(\alpha),\lambda_{\kappa,n}(0,\alpha_L))$ is decreasing on $(0,\alpha_0)$ and

$$\mathcal{S}_{\kappa,n}(\alpha,\beta(\alpha),\lambda_{\kappa,n}(0,\alpha_L)) < \mathcal{S}_{\kappa,n}(0,\alpha_L,\lambda_{\kappa,n}(0,\alpha_L)) = 0,\ \forall\alpha \in (0,\alpha_0). \tag{6.11}$$

If we assume by contradiction that there exists $\alpha \in (0,\alpha_0)$ such that $\lambda_{\kappa,n}(\alpha,\beta(\alpha)) < \lambda_{\kappa,n}(0,\alpha_L)$, by the property (6.11) and the fact that $\lambda \mapsto \mathcal{S}_{\kappa,n}(\alpha,\beta,\lambda)$ is increasing between any two consecutive poles of $\mathcal{S}_{\kappa,n}(\alpha,\beta,\cdot)$, it turns out that

$$0 > \mathcal{S}_{\kappa,n}(\alpha,\beta(\alpha),\lambda_{\kappa,n}(0,\alpha_L)) \geq \mathcal{S}_{\kappa,n}(\alpha,\beta(\alpha),\lambda_{\kappa,n}(\alpha,\beta(\alpha))) = 0,$$

which is a contradiction. Therefore, $\lambda_{\kappa,n}(\alpha,\beta(\alpha)) \geq \lambda_{\kappa,n}(0,\alpha_L)$ for every $\alpha \in (0,\alpha_0)$, which ends the proof of (6.5).

In conclusion, it remains to investigate the validity of inequality (6.10) which turns out to depend both on the *dimension* $n \geq 2$ and the *size* $L > 0$. In this way we can decide the validity of Lord Rayleigh's conjecture for any non-empty open smooth set $\Omega \subset M$ that verifies the equality $\Omega^\star = C^n_\kappa(L)$ and condition (6.2). The next subsection is devoted to this study.

6.2 Validity of the Conjecture: Dimension and Domain Dependence. Using the notations from Sect. 6.1, for every $\kappa > 0$ and $n \in \mathbb{N}_{\geq 2}$ we introduce the set

$$\mathcal{H}_{\kappa,n} := \left\{ L \in \left(0, \frac{\pi}{\sqrt{\kappa}}\right) : \mathfrak{f}_{\kappa,n,1}(\alpha_{L_0}) < \lambda_{\kappa,n}(0,\alpha_L) \right\},$$

where $\alpha_L = \sin^2\left(\frac{\sqrt{\kappa}L}{2}\right)$, $\alpha_{L_0} = \sin^2\left(\frac{\sqrt{\kappa}L_0}{2}\right)$ and the value $L_0 > 0$ is the half-cap radius associated with $L > 0$, see (6.8). The following result is crucial.

Proposition 6.2. *If $\kappa > 0$ and $n \in \mathbb{N}_{\geq 2}$, the following statements hold*:

(i) $\mathcal{H}_{\kappa,n} \neq \varnothing$ *for every* $n \geq 4$;
(ii) $\mathcal{H}_{\kappa,2} = \mathcal{H}_{\kappa,3} = \varnothing$.

Proof. (i) We first claim that $\mathcal{H}_{\kappa,n}$ does not contain elements close to $\pi/\sqrt{\kappa}$ for every $n \geq 2$. Indeed, for the half-cap radius L_0 associated with $L > 0$ we have that $L_0 \to \frac{\pi}{2\sqrt{\kappa}}$ whenever $L \to \frac{\pi}{\sqrt{\kappa}}$. Therefore, $\alpha_{L_0} = \sin^2\left(\frac{\sqrt{\kappa}L_0}{2}\right) \to \sin^2\left(\frac{\pi}{4}\right) = \frac{1}{2}$, and according to Proposition 2.2/(vi) we have that

$$\mathfrak{f}_{\kappa,n,1}(\alpha_{L_0}) \to \sqrt{n\kappa} \quad \text{as} \quad L \to \frac{\pi}{\sqrt{\kappa}}. \tag{6.12}$$

On the other hand, Theorem 1.2/(ii) yields that

$$\lambda_{\kappa,n}(0,\alpha_L) = \Lambda_\kappa^{\frac{1}{4}}(C^n_\kappa(L)) \sim \begin{cases} \sqrt{\mu_n\kappa} & \text{if } n \in \{2,3\}, \\ 0 & \text{if } n \geq 4, \end{cases} \quad \text{as} \quad L \to \frac{\pi}{\sqrt{\kappa}}, \tag{6.13}$$

where $\mu_2 \approx 0.9125$ and $\mu_3 \approx 1.0277$. Therefore, $\mathfrak{f}_{\kappa,n,1}(\alpha_{L_0}) > \lambda_{\kappa,n}(0,\alpha_L)$ if L is sufficiently close to $\pi/\sqrt{\kappa}$, which concludes the claim.

Again by (6.8), the half-cap radius L_0 associated with $L > 0$ verifies the asymptotic property $2L_0^n \sim L^n$ whenever $L \ll 1$. Moreover, a similar argument as in Sect. 5.1 yields that $\mathfrak{f}_{\kappa,n,1}(\alpha_{L_0}) \sim C/L_0$ as $L \to 0$ with $J_{\frac{n}{2}-1}(C) = 0$, i.e., $C = \mathfrak{j}_{\frac{n}{2}-1}$. Thus, by Theorem 1.2/(i) it follows that

$$\liminf_{L\to 0} \frac{\mathfrak{f}_{\kappa,n,1}(\alpha_{L_0})}{\lambda_{\kappa,n}(0,\alpha_L)} = 2^{\frac{1}{n}} \frac{\mathfrak{j}_{\frac{n}{2}-1}}{\mathfrak{h}_{\frac{n}{2}-1}}. \tag{6.14}$$

We observe that $2^{\frac{1}{n}}\frac{\mathfrak{j}_{\frac{n}{2}-1}}{\mathfrak{h}_{\frac{n}{2}-1}} < 1$ if and only if $n \geq 4$. In particular, for every $n \geq 4$, we obtain that $\mathfrak{f}_{\kappa,n,1}(\alpha_{L_0}) < \lambda_{\kappa,n}(0,\alpha_L)$ for sufficiently small $L > 0$, proving that $\mathcal{H}_{\kappa,n} \neq \varnothing$; this concludes the proof of (i).

(ii) The previous arguments show that for $n \in \{2,3\}$ the set $\mathcal{H}_{\kappa,n}$ does not contain elements in the vicinity of 0 or close to $\pi/\sqrt{\kappa}$. We will prove that this property persists to the whole interval $(0,\pi/\sqrt{\kappa})$ whenever $n \in \{2,3\}$. To do this, it is enough to prove that the equation $\mathfrak{f}_{\kappa,n,1}(\alpha_{L_0}) = \lambda_{\kappa,n}(0,\alpha_L)$ is not solvable in $L \in (0,\pi/\sqrt{\kappa})$ for $n \in \{2,3\}$, where $\alpha_L = \sin^2\left(\frac{\sqrt{\kappa}L}{2}\right)$, $\alpha_{L_0} = \sin^2\left(\frac{\sqrt{\kappa}L_0}{2}\right)$ and $L_0 > 0$ is the half-cap radius associated with $L > 0$. Our proof is dimension-dependent.

The case $n = 3$. Using (6.1) and $\alpha_{L_0} = \sin^2\left(\frac{\sqrt{\kappa}L_0}{2}\right)$, we recall that

$$\mathfrak{f}_{\kappa,3,1}(\alpha_{L_0}) = \sqrt{\kappa\left(\frac{\pi^2}{\kappa L_0^2} - 1\right)}. \tag{6.15}$$

Due to relations (6.15) and (5.5), the solvability of $\mathfrak{f}_{\kappa,3,1}(\alpha_{L_0}) = \lambda_{\kappa,3}(0,\alpha_L)$ reduces to the equation $\mathcal{K}_{\kappa,3}(\alpha_L, \mathfrak{f}_{\kappa,3,1}(\alpha_{L_0})) = 0$, i.e.,

$$\sqrt{\frac{\pi^2}{\kappa L_0^2} - 2}\cdot\coth\left(\sqrt{\kappa}L\sqrt{\frac{\pi^2}{\kappa L_0^2} - 2}\right) - \frac{\pi}{\sqrt{\kappa}L_0}\cdot\cot\left(\frac{\pi L}{L_0}\right) = 0, \tag{6.16}$$

where L and L_0 verify (6.8), i.e.,

$$2\left(L_0 - \frac{\sin(2\sqrt{\kappa}L_0)}{2\sqrt{\kappa}}\right) = L - \frac{\sin(2\sqrt{\kappa}L)}{2\sqrt{\kappa}}. \tag{6.17}$$

Our aim is to prove that equalities (6.16) and (6.17) are incompatible, which will imply $\mathcal{H}_{\kappa,3} = \varnothing$. If $x := \sqrt{\kappa}L_0$ and $y := \sqrt{\kappa}L$, the implicit function theorem together with (6.17) implies the existence of a (unique) differentiable function $p : (0,\pi/2) \to (0,\pi)$ such that $P(x,p(x)) = 0$ for every $x \in (0,\pi/2)$, where $P : (0,\pi/2)\times(0,\pi) \to \mathbb{R}$ is

$$P(x,y) := 2\left(2x - \sin(2x)\right) - 2y + \sin(2y).$$

Let us also consider $Q : (0,\pi/2)\times(0,\infty)\setminus S_Q \to \mathbb{R}$ defined by

$$Q(x,y) := \sqrt{\pi^2 - 2x^2}\cdot\coth\left(y\sqrt{\frac{\pi^2}{x^2} - 2}\right) - \pi\cot\left(\frac{\pi y}{x}\right),$$

where $S_Q := \{(x,y) \in (0,\pi/2)\times(0,\infty) : y/x \in \mathbb{N}\}$. In order to prove the claim, it is enough to show that $p(x) > q(x)$ for every $x \in (0,\pi/2)$, where $q : (0,\pi/2) \to (0,\pi)$ is the smallest differentiable function such that $Q(x,q(x)) = 0$ for every $x \in (0,\pi/2)$, see (6.16). The idea of the proof is to separate p and q by *piece-wise linear* functions; this fact is motivated by the monotonicity property of $x \mapsto Q(x,cx)$ whenever $c > 0$, which reduces to the monotonicity of $y \mapsto y\coth(y)$ on $(0,\infty)$. The separation argument can be described as follows.

Since $p(x) \sim 2^{\frac{1}{3}}x =: r_1(x)$ as $x \to 0$, it turns out that $\lim_{x\to 0} Q(x, p(x)) = \pi \coth\left(2^{\frac{1}{3}}\pi\right) - \pi \cot\left(2^{\frac{1}{3}}\pi\right) \approx 0.192 > 0$, thus $Q(x, p(x)) \neq 0$ for small values of x; in fact, this property can be also deduced by (6.14). On one hand, we observe that $p(x) > r_1(x)$ for every $x \in (0, \pi/2)$. Indeed, an elementary argument shows that $P(x, r_1(x)) > 0$ for every $x \in (0, \pi/2)$ and $p(\pi/2) = \pi > 2^{-\frac{2}{3}}\pi = r_1(\pi/2)$. Thus, if there exists $\widetilde{x} \in (0, \pi/2)$ such that $p(\widetilde{x}) \leq r_1(\widetilde{x})$, by a continuity reason there exists $\overline{x} \in [\widetilde{x}, \pi/2)$ such that $p(\overline{x}) = r_1(\overline{x})$, and $0 = P(\overline{x}, p(\overline{x})) = P(\overline{x}, r_1(\overline{x})) > 0$, which is a contradiction. On the other hand, we have that the function $x \mapsto Q(x, r_1(x))$ is decreasing on $(0, \pi/2)$ having its unique zero at $x_1 \approx 0.767$. If $c \approx 1.249$ is the first positive zero of the equation $\coth(c\pi) = \cot(c\pi)$, it follows that $\lim_{x\to 0} Q(x, cx) = \pi \coth(c\pi) - \pi \cot(c\pi) = 0$; taking into account that $c < 2^{\frac{1}{3}} \approx 1.259$ and $Q(x, q(x)) = 0$, the minimality property of q implies that $q(x) < r_1(x)$ for small values of $x > 0$. By construction, it follows that $q(x_1) = r_1(x_1) = 2^{\frac{1}{3}}x_1 \approx 0.967$. Thus, if there exists $\widetilde{x} \in (0, x_1)$ with $q(\widetilde{x}) \geq r_1(\widetilde{x})$, one can find $\overline{x} \in (0, \widetilde{x}]$ such that $q(\overline{x}) = r_1(\overline{x})$, so $0 = Q(\overline{x}, q(\overline{x})) = Q(\overline{x}, r_1(\overline{x})) > 0$, which is a contradiction. Therefore, $p(x) > r_1(x) > q(x)$ for every $x \in (0, x_1)$. Clearly, we also have $p(x_1) > r_1(x_1) = q(x_1)$.

The function r_1 cannot separate p and q beyond the value x_1. Therefore, we consider $r_2(x) := \frac{p(x_1)}{x_1}x$ for every $x \in [x_1, \pi/2)$, and a similar argument as above shows that $p(x) > r_2(x) > q(x)$ for every $x \in (x_1, x_2)$, where $x_2 \approx 1.462$ is the intersection point of r_2 and q. Moreover, $p(x_2) > r_2(x_2) = q(x_2)$. Finally, the function $r_3(x) := \frac{p(x_2)}{x_2}x$, $x \in [x_2, \pi/2)$, has the property that $p(x) > r_3(x) > q(x)$ for every $x \in (x_2, \pi/2)$.

Summing up, we have $p(x) > q(x)$ for every $x \in (0, \pi/2)$, which proves that $\mathcal{H}_{\kappa,3} = \varnothing$.

The case $n = 2$. Due to (6.8), in the case $n = 2$ we have that

$$2\alpha_{L_0} = 2\sin^2\left(\frac{\sqrt{\kappa}L_0}{2}\right) = \frac{\kappa}{2\pi}V_\kappa(C_\kappa^2(L_0))$$
$$= \frac{\kappa}{4\pi}V_\kappa(C_\kappa^2(L)) = \sin^2\left(\frac{\sqrt{\kappa}L}{2}\right) = \alpha_L. \tag{6.18}$$

Thus the question reduces to the non-solvability of $\mathfrak{f}_{\kappa,2,1}(t/2) = \lambda_{\kappa,2}(0, t)$ in $t \in (0, 1)$. On one hand, by (6.12)–(6.14) we know that this equation cannot be solved for values t close to 0 and 1. On the other hand, due to (A.10), (A.11) and (A.14) (since $n = 2$), we obtain that

$$\mathcal{F}_\pm(t, \lambda, \kappa, 2) = \mathsf{P}^0_{-\frac{1}{2}+\Lambda_\pm}(1 - 2t) \quad \text{and} \quad \mathcal{F}'_\pm(t, \lambda, \kappa, 2)$$
$$= \frac{1}{\sqrt{(1-t)t}}\mathsf{P}^1_{-\frac{1}{2}+\Lambda_\pm}(1 - 2t), \; t \in (0, 1),$$

where $\Lambda_\pm = \sqrt{\frac{1}{4} \pm \frac{\lambda^2}{\kappa}}$; see also Zhurina and Karmazina [ZK66]. In particular, $\mathcal{K}_{\kappa,2}(t, \lambda) = 0$ can be transformed into an equation containing only the associated Legendre functions $\mathsf{P}^r_{-\frac{1}{2}+\Lambda_\pm}$ of integer orders $r \in \{0, 1\}$. Thus, the tables of zeros with respect to the degree $\nu = -\frac{1}{2} + \Lambda_\pm$, see Bauer [Bau86] and Bauer and Reiss [BR72] (and also Zhang and Jin [ZJ96, Chapter 4]) imply that

$$\mathfrak{f}_{\kappa,2,1}\left(\frac{t}{2}\right) - \lambda_{\kappa,2}(0,t) > \frac{\sqrt{\kappa}}{5},\ \forall t \in (0,1),\ \kappa > 0.$$

Therefore, the latter estimate and relation (6.18) imply that $\mathcal{H}_{\kappa,2} = \varnothing$. □

Remark 6.1. Numerical tests show that the function $L \mapsto \frac{\mathfrak{f}_{\kappa,n,1}(\alpha_{L_0})}{\lambda_{\kappa,n}(0,\alpha_L)}$ is increasing on $(0, \pi/\sqrt{\kappa})$ for every $n \in \mathbb{N}_{\geq 2}$ and $\kappa > 0$, where L_0 is the half-cap radius associated with $L > 0$. If this statement indeed holds, we can present an alternative proof for Proposition 6.2/(ii). Indeed, by the assumed monotonicity and (6.14) we would have for every $L \in (0, \pi/\sqrt{\kappa})$ that

$$\begin{aligned}\frac{\mathfrak{f}_{\kappa,n,1}(\alpha_{L_0})}{\lambda_{\kappa,n}(0,\alpha_L)} &\geq \liminf_{L\to 0} \frac{\mathfrak{f}_{\kappa,n,1}(\alpha_{L_0})}{\lambda_{\kappa,n}(0,\alpha_L)} \\ &= 2^{\frac{1}{n}} \frac{\mathfrak{j}_{\frac{n}{2}-1}}{\mathfrak{h}_{\frac{n}{2}-1}} = \begin{cases} 2^{\frac{1}{2}} \frac{\mathfrak{j}_0}{\mathfrak{h}_0} \approx 2^{\frac{1}{2}} \frac{2.4048}{3.1962} \approx 1.064 > 1 \text{ if } n = 2, \\ 2^{\frac{1}{3}} \frac{\mathfrak{j}_{\frac{1}{2}}}{\mathfrak{h}_{\frac{1}{2}}} \approx 2^{\frac{1}{3}} \frac{\pi}{3.9266} \approx 1.008 > 1 \text{ if } n = 3, \end{cases}\end{aligned}$$

thus $\mathcal{H}_{\kappa,2} = \mathcal{H}_{\kappa,3} = \varnothing$.

Proof of Theorem 1.3. Let (M,g) be a compact n-dimensional Riemannian manifold with $\mathsf{Ric}_{(M,g)} \geq (n-1)\kappa > 0$, consider a smooth domain $\Omega \subset M$ and let $\Omega^\star \subset \mathbb{S}^n_\kappa$ be a spherical cap for which the conditions

$$\frac{V_g(\Omega)}{V_g(M)} = \frac{V_\kappa(\Omega^\star)}{V_\kappa(\mathbb{S}^n_\kappa)}$$

and $V_\kappa(\Omega^\star) = V_\kappa(C^n_\kappa(L))$ are satisfied for some $L \in (0, \pi/\sqrt{\kappa})$.

On one hand, Proposition 6.2/(ii) implies that inequality (6.10) is verified for $n \in \{2,3\}$ and any $L \in (0, \pi/\sqrt{\kappa})$; in particular, the (sufficiency) argument at the end of the previous subsection shows that Lord Rayleigh's conjecture is true, i.e., (1.9) is valid for any domain $\Omega \subset M$ whenever $n \in \{2,3\}$.

On the other hand, when $n \geq 4$, it turns out by the proof of Proposition 6.2/(i) that Lord Rayleigh's conjecture holds true on (M,g) for any domain $\Omega \subset M$ with $V_g(\Omega) > v_{n,\kappa} V_g(M)$, where

$$v_{n,\kappa} = \frac{V_\kappa(C^n_\kappa(L_{n,\kappa}))}{V_\kappa(\mathbb{S}^n_\kappa)} \in (0,1) \ \text{ and } \ L_{n,\kappa} := \sup \mathcal{H}_{\kappa,n} \in \left(0, \frac{\pi}{\sqrt{\kappa}}\right). \tag{6.19}$$

In addition, note that for every $\alpha \in (0,1)$, the expressions $f_{\kappa,n,1}(\alpha)/\sqrt{k}$ and $\lambda_{\kappa,n}(0,\alpha)$ $/\sqrt{\kappa}$ are κ-independent. Therefore, $L_{n,\kappa} = L_n/\sqrt{\kappa}$ for some κ-independent value $L_n \in (0,\pi)$, and a simple computation shows that $v_n := v_{n,\kappa}$ does not depend on $\kappa > 0$.

Table 3 Values of $L_{n,\kappa}$ and v_n from (6.19) in certain dimensions $n \geq 2$

n	2 & 3	4	5	6	7	10	50	100	200	500	1000
$L_{n,\kappa}$	0	$\frac{0.27\pi}{\sqrt{\kappa}}$	$\frac{0.355\pi}{\sqrt{\kappa}}$	$\frac{0.394\pi}{\sqrt{\kappa}}$	$\frac{0.417\pi}{\sqrt{\kappa}}$	$\frac{0.448\pi}{\sqrt{\kappa}}$	$\frac{0.487\pi}{\sqrt{\kappa}}$	$\frac{0.492\pi}{\sqrt{\kappa}}$	$\frac{0.495\pi}{\sqrt{\kappa}}$	$\frac{0.4969\pi}{\sqrt{\kappa}}$	$\frac{0.4981\pi}{\sqrt{\kappa}}$
v_n	0	0.0763	0.1616	0.2146	0.2515	0.3067	0.3869	0.401	0.4122	0.4138	0.4252

Assume that equality holds in (1.9) for some $\Omega \subset M$. By the proof of Theorem 4.1 (see the verification of inequalities (4.23) and (4.24)), we should have equality in the Lévy–Gromov inequality (4.20) for a.e. admissible $t > 0$. In particular, this equality implies that (M, g) is isometric to $(\mathbb{S}^n_\kappa, g_\kappa)$, and the sets $\{x \in \Omega_\pm : u_\pm(x) > t\}$ and $\{x \in \Omega^\star_\pm : u^\star_\pm(x) > t\}$ are isometric for a.e. $t \in [0, T_u^\pm]$. Since only one set of Ω_+ and Ω_- remains (say Ω_+, cf. subsection Sect. 6.1), it turns out that $\Omega = \Omega_+ \subset M$ is isometric to the spherical cap $\Omega^\star = \Omega^\star_+ \subset \mathbb{S}^n_\kappa$. The converse statement trivially holds.

Finally, let $L^0_{n,\kappa}$ be the half-cap radius associated with $L_{n,\kappa} > 0$. If we assume that $v_\infty := \limsup_{n\to\infty} v_n = 1$, which is equivalent to $\limsup_{n\to\infty} L_{n,\kappa} = \pi/\sqrt{\kappa}$, a similar reasoning as in (6.12)–(6.13) implies that

$$0 = \limsup_{n\to\infty} \lambda_{\kappa,n}(0, \alpha_{L_{n,\kappa}}) = \limsup_{n\to\infty} \mathfrak{f}_{\kappa,n,1}(\alpha_{L^0_{n,\kappa}}) = \limsup_{n\to\infty} \sqrt{n\kappa} = +\infty,$$

which is a contradiction. Therefore, $v_\infty = \limsup_{n\to\infty} v_n < 1$, which concludes the proof. $\square$

Remark 6.2. Theorem 1.3 states in particular that in high-dimensions Lord Rayleigh's conjecture is true for large clamped plates on any compact Riemannian manifold (M, g) of positive Ricci curvature. Note that the inequality $v_\infty < 1$ implies that these clamped plates need not be very 'close' to the whole manifold; quantitatively, the arguments are valid for clamped plates $\Omega \subset M$ with $\frac{V_g(\Omega)}{V_g(M)} \in (v_\infty, 1]$. Numerical tests show that $\limsup_{n\to\infty} L_{n,\kappa} = \pi/(2\sqrt{\kappa})$, thus $v_\infty = 1/2$, see Table 3, which would imply that clamped plates $\Omega \subset M$ with at least 'half-measure' of M verify Lord Rayleigh's conjecture.

7 Curvature Limit in Lord Rayleigh's Conjecture: Proof of Theorem 1.4

Let $\Omega \subset M$ be a bounded open set in a complete non-compact n-dimensional Riemannian manifold (M, g) with $\mathsf{Ric}_{(M,g)} \geq 0$ and $\mathsf{AVR}_{(M,g)} > 0$, see (1.10). Due to the boundedness of Ω, the injectivity radius is positive on Ω, see Klingenberg [Kli95, Proposition 2.1.10], thus the Sobolev space $H^2_0(\Omega) = W^{2,2}_0(\Omega)$ is an appropriate function space for the clamped problem on Ω, see Hebey [Heb99, Proposition 3.3]. Moreover, the fundamental tone $\Lambda_g(\Omega)$ defined by (1.3) is achieved by a minimizer $u \in W^{2,2}_0(\Omega)$; in fact, $u \in \mathcal{C}^\infty(\Omega)$.

The spirit of the proof of Theorem 1.4 is similar to the one presented in Sect. 4.1, by performing an Ashbaugh–Benguria–Nadirashvili–Talenti nodal-decomposition on (M, g) combined with an appropriate comparison argument that involves the asymptotic volume ratio $\mathsf{AVR}_{(M,g)} > 0$. We outline the proof, by emphasizing the differences with respect to the arguments used in Sect. 4.1.

Let $u_+ := \max(u, 0)$ and $u_- := -\min(u, 0)$ be the positive and negative parts of u, respectively, and consider their preimages $\Omega_+ := \{x \in \Omega : u_+(x) > 0\}$ and $\Omega_- := \{x \in \Omega : u_-(x) > 0\}$ as well. Assume that $V_g(\Omega_+)V_g(\Omega_-) > 0$; otherwise the proof reduces to the sign-definite case. Let $u^\star_\pm : \mathbb{R}^n \to [0, \infty)$ be the Euclidean radial rearrangements of $u_\pm : \Omega_\pm \to [0, \infty)$, i.e., for every $t > 0$,

$$V_0(\{x \in \mathbb{R}^n : u^\star_+(x) > t\}) = V_g(\{x \in \Omega : u_+(x) > t\}) =: j_0(t), \tag{7.1}$$

$$V_0(\{x \in \mathbb{R}^n : u^\star_-(x) > t\}) = V_g(\{x \in \Omega : u_-(x) > t\}) =: h_0(t). \tag{7.2}$$

The functions $u^\star_\pm$ are well-defined and radially symmetric having the property that

$$\{x \in \mathbb{R}^n : u^\star_+(x) > t\} = B_0(r_t) \text{ and } \{x \in \mathbb{R}^n : u^\star_-(x) > t\} = B_0(\rho_t), \tag{7.3}$$

for some $r_t > 0$ and $\rho_t > 0$ with $V_0(B_0(r_t)) = j_0(t)$ and $V_0(B_0(\rho_t)) = h_0(t)$, respectively. For further use, let $a, b \geq 0$ be such that $V_0(B_0(a)) = V_g(\Omega_+)$ and $V_0(B_0(b)) = V_g(\Omega_-)$. In particular, $a^n + b^n = L^n$ where $L > 0$ is given by the condition $V_0(B_0(L)) = V_g(\Omega)$.

We introduce the functions

$$\mathcal{J}_0(s) := (\Delta_g u)^*_-(s) - (\Delta_g u)^*_+(V_0(B_0(L)) - s) \text{ and}$$
$$\mathcal{H}_0(s) := -\mathcal{J}_0(V_0(B_0(L)) - s), \quad s \in [0, V_0(B_0(L))],$$

where

$$(\Delta_g u)^*_\pm(s) := (\Delta_g u)^\star_\pm(x) \text{ with } s = \omega_n|x|^n, \ x \in B_0(L); \tag{7.4}$$

here $(\Delta_g u)^\star_\pm$ are the Euclidean radial rearrangements of $(\Delta_g u)_\pm$. Similarly as in Lemma 4.2, we can prove that

$$\int_0^\sigma \mathcal{J}_0(s)\mathrm{d}s \geq \int_0^{V_0(B_0(L))} \mathcal{J}_0(s)\mathrm{d}s = 0,$$
$$\int_0^\sigma \mathcal{H}_0(s)\mathrm{d}s \geq \int_0^{V_0(B_0(L))} \mathcal{H}_0(s)\mathrm{d}s = 0, \ \forall \sigma \in [0, V_0(B_0(L))],$$

and

$$\int_{B_0(a)} \mathcal{J}_0(\omega_n|x|^n)\mathrm{d}x = \int_{B_0(b)} \mathcal{H}_0(\omega_n|x|^n)\mathrm{d}x. \tag{7.5}$$

Furthermore, analogously to Proposition 4.1, we have that

$$\int_0^{j_0(t)} \mathcal{J}_0(s)\mathrm{d}s \geq -\int_{\{u>t\}} \Delta_g u(x)\mathrm{d}v_g(x), \ \forall t \in [0, T_u^+], \tag{7.6}$$

and

$$\int_0^{h_0(t)} \mathcal{H}_0(s)\mathrm{d}s \geq -\int_{\{u<-t\}} \Delta_g u(x)\mathrm{d}v_g(x),\ \forall t \in [0, T_u^-], \tag{7.7}$$

where $T_u^\pm := \sup_{x\in\Omega_\pm} u_\pm(x) \geq 0$, and either $(\Delta_g u)_-^*(s) = 0$ or $(\Delta_g u)_+^*(V_0(B_0(L)) - s) = 0$ for every $s \in [0, V_0(B_0(L))]$.

The analogue of Theorem 4.1 can be stated as follows.

Proposition 7.1. *The real functions*

$$w_a(x) := \frac{1}{n\omega_n}\int_{|x|}^a \rho^{1-n}\left(\int_0^{\omega\rho^n} \mathcal{J}_0(s)\mathrm{d}s\right)\mathrm{d}\rho,\ x \in \Omega^\star, \tag{7.8}$$

and

$$w_b(x) := \frac{1}{n\omega_n}\int_{|x|}^b \rho^{1-n}\left(\int_0^{\omega\rho^n} \mathcal{H}_0(s)\mathrm{d}s\right)\mathrm{d}\rho,\ x \in \Omega^\star, \tag{7.9}$$

satisfy the following statements:

(i) $\int_\Omega (\Delta_g u)^2\mathrm{d}v_g = \int_{B_0(a)} (\Delta w_a)^2\mathrm{d}x + \int_{B_0(b)} (\Delta w_b)^2\mathrm{d}x$;

(ii) $\mathsf{AVR}_{(M,g)}^{\frac{4}{n}} \int_\Omega u^2\mathrm{d}v_g \leq \int_{B_0(a)} w_a^2\mathrm{d}x + \int_{B_0(b)} w_b^2\mathrm{d}x.$

Proof. One can easily verify that w_a and w_b are solutions to the Dirichlet problems

$$\begin{cases} -\Delta w_a(x) = \mathcal{J}_0(\omega_n|x|^n) & \text{in } B_0(a); \\ w_a = 0 & \text{on } \partial B_0(a); \end{cases} \quad \text{and} \quad \begin{cases} -\Delta w_b(x) = \mathcal{H}_0(\omega_n|x|^n) & \text{in } B_0(b); \\ w_b = 0 & \text{on } \partial B_0(b), \end{cases} \tag{7.10}$$

respectively. Since (i) is similar to the proof of Theorem 4.1/(i), we focus on property (ii). To complete this part, we recall the following sharp isoperimetric inequality on (M, g) (see Brendle [Bre21], and Balogh and Kristály [BK]): for every bounded open subset $\Omega \subset M$ with smooth boundary $\partial\Omega$, one has that

$$\mathcal{P}_g(\partial\Omega) \geq n\omega_n^{\frac{1}{n}}\ \mathsf{AVR}_{(M,g)}^{\frac{1}{n}} V_g(\Omega)^{\frac{n-1}{n}}, \tag{7.11}$$

where $\mathcal{P}_g(\partial\Omega)$ is the perimeter of $\partial\Omega$; moreover, equality holds in (7.11) if and only if $\mathsf{AVR}_{(M,g)} = 1$, i.e., (M, g) is isometric to $(\mathbb{R}^n, g_0)$ and $\Omega \subset M$ is isometric to a ball of volume $V_g(\Omega)$.[1]

We confine our argument only to w_a; the case of w_b works similarly. We consider the sets

$$\Lambda_t^\star := \partial(\{x \in \mathbb{R}^n : u_+^\star(x) > t\}),\quad \Lambda_t := \partial(\{x \in \Omega : u_+(x) > t\}),\quad t \in [0, T_u^+].$$

[1] Inequality (7.11) has been proven first by Agostiniani, Fogagnolo and Mazzieri [AFM20] by using Huisken's mean curvature flows; note however that their argument works only in dimension 3. Later on, Fogagnolo and Mazzieri [FM20] extended their arguments up to dimension 7.

Due to (7.1) and (7.11), and since the balls in $(\mathbb{R}^n, g_0)$ are the isoperimetric sets, we obtain that

$$\mathcal{P}_g(\Lambda_t) \geq n\omega_n^{\frac{1}{n}} \ \mathsf{AVR}_{(M,g)}^{\frac{1}{n}} j_0(t)^{\frac{n-1}{n}} = \mathsf{AVR}_{(M,g)}^{\frac{1}{n}} \mathcal{P}_0(\Lambda_t^\star), \quad t \in [0, T_u^+], \tag{7.12}$$

where $\mathcal{P}_0(\Lambda_t^\star)$ denotes the Euclidean perimeter of the sphere $\Lambda_t^\star \subset \mathbb{R}^n$.

A similar argument as in the proof of (4.19) implies that

$$\mathcal{P}_g^2(\Lambda_t) \leq j_0'(t) \int_{\{u>t\}} \Delta_g u \mathrm{d}v_g \ \text{ for a.e. } t \in [0, T_u^+]. \tag{7.13}$$

Inequality (7.13) together with (7.6) yields that

$$\mathsf{AVR}_{(M,g)}^{\frac{2}{n}} \mathcal{P}_0(\Lambda_t^\star)^2 \leq -j_0'(t) \int_0^{j_0(t)} \mathcal{J}_0(s)\mathrm{d}s \ \text{ for a.e. } t \in [0, T_u^+].$$

Since $j_0(t) = V_0(B_0(r_t)) = \omega_n r_t^n$, see (7.3), it turns out that $j_0'(t) = \mathcal{P}_0(\Lambda_t^\star) r_t' = n\omega_n r_t^{n-1} r_t'$ for a.e. $t \in [0, T_u^+]$. Therefore, we obtain the inequality

$$\mathsf{AVR}_{(M,g)}^{\frac{2}{n}} n\omega_n \leq -r_t^{1-n} r_t' \int_0^{\omega_n r_t^n} \mathcal{J}_0(s)\mathrm{d}s \ \text{ for a.e. } t \in [0, T_u^+]. \tag{7.14}$$

If $x \in B_0(a)$ is arbitrarily fixed, one can find a unique value $\eta \in [0, T_u^+]$ such that $|x| = a_\eta$; moreover, by construction, $u_+^\star(x) = \eta$. Accordingly, by integrating the inequality (7.14) on $[0, \eta]$ and performing a change of variables, one can conclude that

$$\mathsf{AVR}_{(M,g)}^{\frac{2}{n}} u_+^\star(x) \leq \frac{1}{n\omega_n} \int_{|x|}^a \rho^{1-n} \left(\int_0^{\omega\rho^n} \mathcal{J}_0(s)\mathrm{d}s \right) \mathrm{d}\rho = w_a(x).$$

In a similar way, by using (7.3) and (7.7), we obtain that

$$\mathsf{AVR}_{(M,g)}^{\frac{2}{n}} u_-^\star(x) \leq \frac{1}{n\omega_n} \int_{|x|}^b \rho^{1-n} \left(\int_0^{\omega\rho^n} \mathcal{H}_0(s)\mathrm{d}s \right) \mathrm{d}\rho \equiv w_b(x).$$

Thus, we infer that

$$\begin{aligned} \int_\Omega u^2 \mathrm{d}v_g &= \int_{\Omega_+} u_+^2 \mathrm{d}v_g + \int_{\Omega_-} u_-^2 \mathrm{d}v_g = \int_{B_0(a)} (u_+^\star)^2 \mathrm{d}x + \int_{B_0(b)} (u_-^\star)^2 \mathrm{d}x \\ &\leq \mathsf{AVR}_{(M,g)}^{-\frac{4}{n}} \left(\int_{B_0(a)} w_a^2 \mathrm{d}x + \int_{B_0(b)} w_b^2 \mathrm{d}x \right), \end{aligned}$$

which concludes the proof of (ii). ☐

Proof of Theorem 1.4. In view of Proposition 7.1, one has that

$$\Lambda_g(\Omega) = \min_{u \in W_0^{2,2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega (\Delta_g u)^2 \mathrm{d}v_g}{\displaystyle\int_\Omega u^2 \mathrm{d}v_g}$$

$$\geq \mathsf{AVR}_{(M,g)}^{\frac{4}{n}} \min_{(u_a,u_b)\neq(0,0)} \frac{\displaystyle\int_{B_0(a)} (\Delta u_a)^2 \mathrm{d}x + \int_{B_0(b)} (\Delta u_b)^2 \mathrm{d}x}{\displaystyle\int_{B_0(a)} u_a^2 \mathrm{d}x + \int_{B_0(b)} u_b^2 \mathrm{d}x}, \tag{7.15}$$

where $a^n + b^n = L^n$ (with $V_g(\Omega) = \omega_n L^n$) and (u_a, u_b) is taken over of all pairs of radially symmetric functions with $u_a \in W_0^{1,2}(B_0(a)) \cap W^{2,2}(B_0(a))$ and $u_b \in W_0^{1,2}(B_0(b)) \cap W^{2,2}(B_0(b))$, verifying the boundary condition

$$u_a'(a)a^{n-1} = u_b'(b)b^{n-1};$$

indeed, the latter condition is implied by (7.5) and the Dirichlet problems (7.10).

As we can observe, the optimization problem in (7.15) is precisely the one that appears in Ashbaugh and Benguria [AB95, p. 6]. Therefore, the following cases are distinguished.

- The case $n \in \{2,3\}$: by [AB95, Theorem 1], we obtain that

$$\Lambda_g(\Omega) \geq \mathsf{AVR}_{(M,g)}^{\frac{4}{n}} \Lambda_0(B_0(L)), \tag{7.16}$$

  which occurs when $a = L$ and $b = 0$ (or vice-versa) in (7.15).
- The case $n \geq 4$: by Ashbaugh and Laugesen [AL96, Theorem 4], it follows that

$$\Lambda_g(\Omega) \geq \mathsf{AVR}_{(M,g)}^{\frac{4}{n}} w_n \Lambda_0(B_0(L)),$$

  where

$$w_n = 2^{\frac{4}{n}} \frac{\mathfrak{j}_{\frac{n}{2}-1}^4}{\mathfrak{h}_{\frac{n}{2}-1}^4} < 1,$$

  which appears when $a = b = 2^{-\frac{1}{n}} L$ in (7.15). By [AL96], we also have that $\lim_{n\to\infty} w_n = 1$.

Let $\Omega \subset M$ be an open bounded set such that equality holds in (7.16). In particular, the Ashbaugh–Benguria–Nadirashvili–Talenti nodal-decomposition argument implies that the minimizer u has a constant sign (say, $u > 0$ in Ω, since, e.g. $b = 0$ and $a = L$); moreover, one necessarily has equality also in (7.12) for a.e. $t \in [0, T_u^+]$. Therefore, (M, g) is isometric to $(\mathbb{R}^n, g_0)$ and $\Lambda_t \subset \Omega$ is isometric to the ball $\{x \in \mathbb{R}^n : u_+^\star(x) > t\}$ for a.e. $t \in [0, T_u^+]$, cf. (7.11). In particular, Ω is also isometric to $B_0(L)$, which concludes the proof. The converse statement holds trivially. $\square$

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Appendix A: Special functions

This section briefly lists some basic properties of those special functions that were used throughout the paper; these properties can also be found in Olver, Lozier, Boisvert and Clark [OLBC10].

Let $\mu > -1$ be fixed. The Bessel and modified Bessel functions of the first kind are defined as

$$J_\mu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\mu+1)} \left(\frac{x}{2}\right)^{2m+\mu}, \ x \in \mathbb{R}, \tag{A.1}$$

and

$$I_\mu(x) = \mathrm{i}^{-\mu} J_\mu(\mathrm{i}x) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\mu+1)} \left(\frac{x}{2}\right)^{2m+\mu}, \ x \in \mathbb{R}, \tag{A.2}$$

respectively, see [OLBC10, §10]. The following recurrence relations hold for the Bessel functions and their derivatives; namely,

$$J_\mu'(x) = -J_{\mu+1}(x) + \frac{\mu}{x} J_\mu(x) \ \text{ and } I_\mu'(x) = I_{\mu+1}(x) + \frac{\mu}{x} I_\mu(x), \ x > 0. \tag{A.3}$$

For $\mu \notin \mathbb{Z}$, the Bessel and modified Bessel functions of the second kind are defined as

$$Y_\mu(x) = \frac{J_\mu(x)\cos(\mu\pi) - J_{-\mu}(x)}{\sin(\mu\pi)} \ \text{ and } \ K_\mu(x) = \frac{\pi}{2} \frac{I_{-\mu}(x) - I_\mu(x)}{\sin(\mu\pi)},$$

respectively, see [OLBC10, rels. (10.2.3) and (10.27.4)], while in the case of any integer order n, we have that

$$Y_n(x) = \lim_{\mu\to n} Y_\mu(x) \quad \text{and} \quad K_n(x) = \lim_{\mu\to n} K_\mu(x).$$

We also have that

$$Y_\mu(\mathrm{i}z) = e^{\frac{(\mu+1)\mathrm{i}\pi}{2}} I_\mu(z) - \frac{2}{\pi} e^{-\frac{\mu \mathrm{i}\pi}{2}} K_\mu(z), \quad z \in \mathbb{C}. \tag{A.4}$$

An alternative, more explicit representation for Y_n $(n \in \mathbb{N})$ is

$$\begin{aligned} Y_n(z) = & -\frac{(\frac{z}{2})^{-n}}{\pi} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{z}{2}\right)^{2m} + \frac{2}{\pi} J_n(z) \ln\frac{z}{2} \\ & -\frac{(\frac{z}{2})^{n}}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!} (\Psi(m+1) + \Psi(n+m+1)) \left(\frac{z}{2}\right)^{2m}, \ z \in \mathbb{C}, \end{aligned} \tag{A.5}$$

where $\Psi = (\ln\Gamma)'$ is the Digamma function. The function Ψ has the pointwise properties

$$\Psi(z+1) - \Psi(z) = \frac{1}{z}, \ z \neq 0, -1, -2 \ldots, \tag{A.6}$$

$$\Psi(1-z) - \Psi(z) = \pi\cot(\pi z), \ z \neq 0, \pm1, \pm2, \ldots, \tag{A.7}$$

$$\Im\Psi\left(\frac{1}{2} + \mathrm{i}y\right) = \frac{\pi}{2}\tanh(y\pi), \ y \in \mathbb{R}, \tag{A.8}$$

see [OLBC10, rels. (5.5.2), (5.5.4) and (5.4.17)], where $\Im z$ denotes the imaginary part of $z \in \mathbb{C}$, and the asymptotic property

$$\Psi(z) \sim \ln z - \frac{1}{2z} - \sum_{m=1}^{\infty} \frac{B_{2m}}{2mz^{2m}} \text{ as } z \to \infty, |\mathrm{ph} z| < \pi, \tag{A.9}$$

where $\{B_{2m}\}_{m\in\mathbb{N}_{\geq 1}}$ are the Bernoulli numbers, see [OLBC10, rel. (5.11.2)].
If $t \in (-1,1)$, $\mu \in (-\infty, 0]$ and $\nu \in \mathbb{C}$ with $\nu(1+\nu) \in \mathbb{R}$, the Legendre (called also Ferrers) and Gaussian hypergeometric functions of the first kind are connected by the relation

$$\mathsf{P}_\nu^\mu(t) = \frac{1}{\Gamma(1-\mu)} \left(\frac{1+t}{1-t}\right)^{\frac{\mu}{2}} {}_2F_1\left(1+\nu, -\nu; 1-\mu; \frac{1-t}{2}\right) \in \mathbb{R}, \tag{A.10}$$

see [OLBC10, rel. (14.3.1)]. The inversion formula for $\mu = m \in \mathbb{N}$ gives that

$$\mathsf{P}_\nu^{-m}(t) = (-1)^m \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} \mathsf{P}_\nu^m(t), \ t \in (-1,1), \tag{A.11}$$

see [OLBC10, rel. (14.3.5)] Using [OLBC10, rel. (15.10.11)], we recall the Euler–Pfaff transformations

$${}_2F_1(a,b;c;z) = (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right)$$

$$= (1-z)^{c-a-b} {}_2F_1\left(c-a, c-b; c; z\right), \quad |z| < 1. \tag{A.12}$$

The differentiation formula gives that

$$\frac{\mathrm{d}}{\mathrm{d}z} {}_2F_1(a,b;c;z) = \frac{ab}{c} {}_2F_1(a+1, b+1; c+1; z), \tag{A.13}$$

or

$$c(1-z)\frac{\mathrm{d}}{\mathrm{d}z} {}_2F_1(a,b;c;z) = (c-a)(c-b) {}_2F_1(a,b;c+1;z) \\ + c(a+b-c) {}_2F_1(a,b;c;z). \tag{A.14}$$

Moreover, the derivation formula for the Legendre function reads as

$$(1-t^2)\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{P}_\nu^\mu(t) = (\mu+\nu)\mathsf{P}_{\nu-1}^\mu(t) - \nu t\mathsf{P}_\nu^\mu(t), \quad t \in (-1,1), \tag{A.15}$$

while the derivatives with respect to the degree of the Legendre function is

$$\frac{\partial}{\partial\nu}\mathsf{P}_\nu^\mu(t) = \pi\cot(\nu\pi)\mathsf{P}_\nu^\mu(t) - \frac{1}{\pi}\mathsf{A}_\nu^\mu(t), \quad t \in (-1,1), \tag{A.16}$$

see [OLBC10, rels. (14.10.5) and (14.11.1)], where

$$\mathsf{A}_\nu^\mu(t) = \sin(\nu\pi)\left(\frac{1+t}{1-t}\right)^{\frac{\mu}{2}} \sum_{m=0}^\infty \frac{\Gamma(m-\nu)\Gamma(m+\nu+1)\left(\Psi(m+\nu+1) - \Psi(m-\nu)\right)}{m!\Gamma(m-\mu+1)} \\ \times \left(\frac{1-t}{2}\right)^m. \tag{A.17}$$

The symmetrization formula has the form

$$\mathsf{P}_\nu^0(-t) = \cos(\nu\pi)\mathsf{P}_\nu^0(t) - \frac{2}{\pi}\sin(\nu\pi)\mathsf{Q}_\nu^0(t), \quad t \in (-1,1), \tag{A.18}$$

see [OLBC10, rel. (14.9.10)], where Q_ν^0 is the associated Legendre function; moreover,

$$\mathsf{P}_\nu^0(1) = 1 \text{ and } \mathsf{Q}_\nu^0(t) = -\frac{1}{2}\ln(1-t) + \frac{\ln 2}{2} - \gamma - \Psi(\nu+1) \\ + \mathcal{O}((1-t)\ln(1-t)) \text{ as } t \nearrow 1, \tag{A.19}$$

see [OLBC10, rel. (14.8.3)], where $\nu \neq -1, -2, \ldots$ and $\gamma \approx 0.5772$ is the Euler constant. We also have that

$$\cosh(z\pi)\Gamma\left(\frac{1}{2} + \mathrm{i}z\right)\Gamma\left(\frac{1}{2} - \mathrm{i}z\right) = \pi, \quad z \in \mathbb{C}, \tag{A.20}$$

see [OLBC10, rel. (5.4.4)].

The behavior of the Gaussian hypergeometric functions at the singularity 1 is described as

$$\lim_{z\nearrow 1} {}_2F_1(a,b;c;z) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \quad \text{whenever } \Re(c-a-b) > 0; \tag{A.21}$$

$$\lim_{z\nearrow 1} \frac{{}_2F_1(a,b;c;z)}{-\ln(1-z)} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}, \quad \text{whenever } c = a+b; \tag{A.22}$$

$$\lim_{z\nearrow 1}\frac{{}_2F_1(a,b;c;z)}{(1-z)^{c-a-b}}=\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)},\quad \text{whenever}\ \ \Re(c-a-b)<0, \tag{A.23}$$

see [OLBC10, §15.4]. Moreover, by [OLBC10, rel. (15.4.30)] we have that

$$ {}_2F_1\left(a,1-a;c;\frac{1}{2}\right)=\frac{2^{1-c}\sqrt{\pi}\Gamma(c)}{\Gamma(\frac{a+c}{2})\Gamma(\frac{c-a+1}{2})}. \tag{A.24}$$

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A. Kristály
Department of Economics, Babes-Bolyai University, Cluj-Napoca, Romania.
and
Institute of Applied Mathematics, Óbuda University, Budapest, Hungary.
alex.kristaly@econ.ubbcluj.ro
kristaly.alexandru@uni-obuda.hu

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