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IRREDUCIBILITY OF THE FERMI VARIETY FOR DISCRETE PERIODIC SCHRÖDINGER OPERATORS AND EMBEDDED EIGENVALUES

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Abstract. Let H_0 be a discrete periodic Schrödinger operator on $\ell^2(\mathbb{Z}^d)$:

$$H_0 = -\Delta + V,$$

where Δ is the discrete Laplacian and $V : \mathbb{Z}^d \to \mathbb{C}$ is periodic. We prove that for any $d \geq 3$, the Fermi variety at every energy level is irreducible (modulo periodicity). For $d = 2$, we prove that the Fermi variety at every energy level except for the average of the potential is irreducible (modulo periodicity) and the Fermi variety at the average of the potential has at most two irreducible components (modulo periodicity). This is sharp since for $d = 2$ and a constant potential V, the Fermi variety at V-level has exactly two irreducible components (modulo periodicity). We also prove that the Bloch variety is irreducible (modulo periodicity) for any $d \geq 2$. As applications, we prove that when V is a real-valued periodic function, the level set of any extrema of any spectral band functions, spectral band edges in particular, has dimension at most $d-2$ for any $d \geq 3$, and finite cardinality for $d = 2$. We also show that $H = -\Delta + V + v$ does not have any embedded eigenvalues provided that v decays super-exponentially

1 Introduction and Main Results

Periodic elliptic operators have been studied intensively in both mathematics and physics, in particular for their role in solid state theory. One of the difficult and unsolved problems is the (ir)reducibility of Bloch and Fermi varieties [3–5, 17, 19, 20, 30, 42, 56, 58]. Besides its importance in algebraic geometry, the (ir)reducibility is crucial in the study of spectral properties of periodic elliptic operators, e.g., the structure of spectral band edges and the existence of embedded eigenvalues under a suitable decaying perturbation of the potential [1, 22, 37, 38, 57]. We refer readers to a survey [34] for the history and most recent developments.

Keywords and phrases: Analytic variety, Algebraic variety, Fermi variety, Bloch variety, Irreducibility, extrema, band function, Band edge, Embedded eigenvalue, Unique continuation, Landis' conjecture, Periodic Schrödinger operator

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In this paper, we will concentrate on discrete periodic Schrödinger operators on $\mathbb{Z}^d$. Given $q_i \in \mathbb{Z}_+$, $i = 1, 2, \ldots, d$, let $\Gamma = q_1\mathbb{Z}\oplus q_2\mathbb{Z}\oplus\cdots\oplus q_d\mathbb{Z}$. We say that a function $V : \mathbb{Z}^d \to \mathbb{C}$ is Γ-periodic (or just periodic) if for any $\gamma \in \Gamma$, $V(n+\gamma) = V(n)$.

Let Δ be the discrete Laplacian on $\ell^2(\mathbb{Z}^d)$, namely

$$(\Delta u)(n) = \sum_{||n'-n||_1=1} u(n'),$$

where $n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$, $n' = (n'_1, n'_2, \ldots, n'_d) \in \mathbb{Z}^d$ and

$$||n'-n||_1 = \sum_{i=1}^{d} |n_i - n'_i|.$$

We consider the discrete Schrödinger operator on $\ell^2(\mathbb{Z}^d)$,

$$H_0 = -\Delta + V. \tag{1}$$

In this paper, we always assume the greatest common factor of $q_1, q_2, \ldots, q_d$ is 1, V is periodic and H_0 is the discrete periodic Schrödinger operator given by (1).

Let $\{\boldsymbol{e}_j\}$, $j = 1, 2, \ldots d$, be the standard basis in $\mathbb{Z}^d$:

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \mathbf{e}_d = (0, 0, \ldots, 0, 1).$$

Definition 1. *The Bloch variety $B(V)$ of $-\Delta+V$ consists of all pairs $(k, \lambda) \in \mathbb{C}^{d+1}$ for which there exists a non-zero solution of the equation*

$$(-\Delta u)(n) + V(n)u(n) = \lambda u(n), n \in \mathbb{Z}^d, \tag{2}$$

satisfying the so called Floquet-Bloch boundary condition

$$u(n + q_j\boldsymbol{e}_j) = e^{2\pi i k_j} u(n), j = 1, 2, \ldots, d, \text{ and } n \in \mathbb{Z}^d, \tag{3}$$

where $k = (k_1, k_2, \ldots, k_d) \in \mathbb{C}^d$.

Definition 2. *Given $\lambda \in \mathbb{C}$, the Fermi surface (variety) $F_\lambda(V)$ is defined as the level set of the Bloch variety:*

$$F_\lambda(V) = \{k : (k, \lambda) \in B(V)\}.$$

Our main interest in the present paper is the irreducibility of Bloch and Fermi varieties as analytic sets.

Definition 3. *A subset $\Omega \subset \mathbb{C}^k$ is called an analytic set if for any $x \in \Omega$, there is a neighborhood $U \subset \mathbb{C}^k$ of x, and analytic functions $f_1, f_2, \ldots, f_p$ in U such that*

$$\Omega \cap U = \{y \in U : f_1(y) = 0, f_2(y) = 0, \ldots, f_p(y) = 0\}.$$

Definition 4. *An analytic set Ω is said to be irreducible if it can not be represented as the union of two non-empty proper analytic subsets.*

It is widely believed that the Bloch/Fermi variety (modulo periodicity) is always irreducible for periodic Schrödinger operators (1), which has been formulated as conjectures:

Conjecture 1 [34, Conjecture 5.17]. The Bloch variety $B(V)$ is irreducible (modulo periodicity).

Conjecture 2 [34, Conjecture 5.35][37, Conjecture 12]. Let $d \geq 2$. Then $F_\lambda(V)/\mathbb{Z}^d$ is irreducible, possibly except for finitely many $\lambda \in \mathbb{C}$.

We remark that in Conjecture 1, the irreducibility of Bloch variety modulo periodicity means for any two irreducible components Ω_1 and Ω_2 of $B(V)$, there exists $k \in \mathbb{Z}^d$ such that $\Omega_1 = (k, 0) + \Omega_2$. In Conjecture 2, for fixed λ, $F_\lambda(V)/\mathbb{Z}^d$ is irreducible means for any two irreducible components Ω_1 and Ω_2 of $F_\lambda(V)$, there exists $k \in \mathbb{Z}^d$ such that $\Omega_1 = k + \Omega_2$.

Conjectures 1 and 2 have been mentioned in many articles [3–5, 20, 30, 38]. It seems extremely hard to prove them, even for "generic" periodic potentials. See Conjecture 13 in [37] for a "generic" version of Conjecture 2.

In this paper, we will first prove both conjectures. For any $d \geq 3$, we prove that the Fermi variety at every level is irreducible (modulo periodicity). For $d = 2$, we prove that the Fermi variety at every level except for the average of the potential is irreducible (modulo periodicity). We also prove that the Bloch variety is irreducible (modulo periodicity) for any $d \geq 2$.

Theorem 1.1. *Let $d \geq 3$. Then the Fermi variety $F_\lambda(V)/\mathbb{Z}^d$ is irreducible for any $\lambda \in \mathbb{C}$.*

Denote by $[V]$ the average of V over one periodicity cell, namely

$$[V] = \frac{1}{q_1 q_2 \cdots q_d} \sum_{\substack{0 \leq n_1 \leq q_1 - 1 \\ \cdots\cdots \\ 0 \leq n_d \leq q_d - 1}} V(n_1, n_2, \ldots, n_d).$$

Theorem 1.2. *Let $d = 2$. Then the Fermi variety $F_\lambda(V)/\mathbb{Z}^2$ is irreducible for any $\lambda \in \mathbb{C}$ except maybe for $\lambda = [V]$. Moreover, if $F_{[V]}(V)/\mathbb{Z}^2$ is reducible, it has exactly two irreducible components.*

Theorem 1.3. *Let $d \geq 2$. Then the Bloch variety $B(V)$ is irreducible (modulo periodicity).*

Remark 1. (1) The special situation with the Fermi variety at the average level in Theorem 1.2 is not surprising. When $d = 2$, for a constant function V, $F_{[V]}(V)/\mathbb{Z}^2$ has two irreducible components.

(2) We should mention that in Theorems 1.1, 1.2, and 1.3, V is allowed to be any complex-valued periodic function.

(3) It is easy to show that Conjecture 1 holds for $d = 1$. See p.18 in [20] for a proof.

Significant progress in proving those Conjectures has been made for $d = 2, 3$. When $d = 2$, Theorem 1.3 was proved by Bättig [2]. In [20], Gieseker, Knörrer and Trubowitz proved that $F_\lambda(V)/\mathbb{Z}^2$ is irreducible except for finitely many values of λ. When $d = 3$, Theorem 1.1 has been proved by Bättig [4].

For continuous (rather than discrete) periodic Schrödinger operators, Knörrer and Trubowitz proved that the Bloch variety is irreducible (modulo periodicity) when $d = 2$ [30].

When the periodic potential is separable, Bättig, Knörrer and Trubowitz proved that the Fermi variety at any level is irreducible (modulo periodicity) for $d = 3$ [5].

In [2–5, 20, 30], proofs heavily depend on the construction of toroidal and directional compactifications of Fermi and Bloch varieties.

A novel approach will be introduced in this paper. Instead of compactifications, we focus on studying the Laurent polynomial $\mathcal{P}$ arising from the eigen-equation (2) and (3) after changing the variables. We develop an approach to study the irreducibility of a class of Laurent polynomials. Firstly, we show that the closure of the zero set of every factor of the Laurent polynomial $\mathcal{P}$ must contain either $z_1 = z_2 = \cdots = z_d = 0$ or $z_1 = z_2 = \cdots = z_{d-1} = 0, z_d = \infty$. Secondly, we prove that "asymptotics" of the Laurent polynomial at $z_1 = z_2 = \cdots = z_d = 0$ and $z_1 = z_2 = \cdots = z_{d-1} = 0, z_d = \infty$ are irreducible. This allows us to conclude that the Laurent polynomial $\mathcal{P}$ has at most two non-trivial factors. Finally, we use degree arguments to show that the only case that $\mathcal{P}$ has two factors is $d = 2$ and $\lambda = [V]$, which completes the proof. We mention that the irreducibility of the Laurent polynomial allows a difference of monomials (see Definition 6), same issue applies to the calculations of "asymptotics". This creates an extra difficulty in the degree arguments. We introduce a polynomial $\mathcal{P}_1$ based on the Laurent polynomial $\mathcal{P}$ multiplying by a proper monomial. Delicately playing between the polynomial $\mathcal{P}_1$ and the Laurent polynomial $\mathcal{P}$ is another significant ingredient to make the whole proof work.

Although the proof is written for Laurent polynomials coming from the Fermi variety of discrete periodic Schrödinger operators, it works for a larger class of Laurent polynomials. Some ideas developed in the proof have been extended to study the irreducibility of the Bloch variety in more general settings [14].

Irreducibility is a powerful tool to study many aspects of the spectral theory of periodic operators. Let $Q = q_1 q_2 \dots q_d$. Assume that V is a real valued periodic potential. Thus $H_0 = -\Delta + V$ is a self-adjoint operator on $\ell^2(\mathbb{Z}^d)$ and its spectrum

$$\sigma(H_0) = \bigcup_{m=1}^{Q} [a_m, b_m] \tag{4}$$

is the union of spectral bands $[a_m, b_m]$, $m = 1, 2, \dots, Q$, which is the range of a band function $\lambda_m(k)$, $k \in \mathbb{R}^d$. See Section 3 for the precise definition of $\lambda_m(k)$.

The structure of extrema of band functions plays a significant role in many problems, such as homogenization theory, Green's function asymptotics and Liouville type theorems. We refer readers to [9, 12, 16, 33, 34] and references therein for more details.

It is well known and widely believed that generically the band functions are Morse functions. The following conjecture gives a precise description.

Conjecture 3 [34, Conjecture 5.25] [36, Conjecture 5.1][12, Conjecture 5]. Generically (with respect to the potentials and other free parameters of the operator), the extrema of band functions

(1) are attained by a single band;
(2) are isolated;
(3) are nondegenerate, i.e., have nondegenerate Hessians.

The statement (1) of Conjecture 3 was proved in [29]. Some progress has been made towards Conjecture 3 at the bottom of the spectrum [27] or small potentials [9]. Recently, a celebrated work of Filonov and Kachkovskiy [16] proves that for a wide class (not "generic") of 2D periodic elliptic operators (continuous version), the global extrema of all spectral band functions are isolated.

As an application of the irreducibility[1] (Theorem 1.2) and Theorem 2.5 in Section 2, we are able to prove a stronger version (work for all extrema) of Filonov and Kachkovskiy's results [16] in the discrete settings. The advantage for discrete cases is that the Fermi variety is algebraic in Floquet variables $e^{2\pi i k_j}$, $j = 1, 2, \ldots, d$ which allows us to use Bézout's theorem to do the proof.

Theorem 1.4. *Let $d = 2$. Let λ_* be an extremum of $\lambda_m(k)$, $k \in [0,1)^2, m = 1, 2, \ldots, Q$. Then the level set*

$$\{k \in [0,1)^2 : \lambda_m(k) = \lambda_*\} \tag{5}$$

has cardinality at most $4(q_1 + q_2)^2$.

In particular, Theorem 1.4 shows that any extremum of any band function can only be attained at finitely many points, which is a stronger version (not "generic") than the statement (2) of Conjecture 3.

It is worth pointing out that Theorem 1.4 may not hold for discrete periodic Schrödinger operators on a diatomic lattice in $\mathbb{Z}^2$ [16].

Theorem 1.5. *Let $d \geq 3$. Let λ_* be an extremum of $\lambda_m(k)$, $k \in [0,1)^d, m = 1, 2, \ldots, Q$. Then the level set*

$$\{k \in [0,1)^d : \lambda_m(k) = \lambda_*\}$$

has dimension at most $d - 2$.

Since the edge of each spectral band is an extremum of the band function, immediately we have the following two corollaries.

[1] Indeed, a much weaker assumption is sufficient for our arguments. See Remark 10.

Corollary 1.6. *Let $d = 2$. Then both level sets*

$$\{k \in [0,1)^2 : \lambda_m(k) = a_m\} \text{ and } \{k \in [0,1)^2 : \lambda_m(k) = b_m\}$$

have cardinality at most $4(q_1+q_2)^2$.

Corollary 1.7. *Let $d \geq 3$. Then both level sets*

$$\{k \in [0,1)^d : \lambda_m(k) = a_m\} \text{ and } \{k \in [0,1)^d : \lambda_m(k) = b_m\}$$

have dimension at most $d-2$.

Remark 2. The statements in Theorem 1.5 and Corollary 1.7 are sharp for periodic Schrödinger operators on a particular lattice in $\mathbb{Z}^d$ [54].

The results of Corollary 1.6 without the explicit bound of the cardinality and Corollary 1.7 were announced by I. Kachkovskiy [24] during a seminar talk at TAMU, as a part of a joint work with N. Filonov [15]. During Kachkovskiy's talk, we realized that we could provide the approach to study the upper bound of dimensions of level sets of extrema based on the Fermi variety. In private communication, we were made aware that the proof from [15] extends to Theorem 1.4 without the explicit bound of the cardinality and Theorem 1.5. However, their approach is very different and is based on the arguments from [16].

We are going to talk about another application. Let us introduce a perturbed periodic operator:

$$H = H_0 + v = -\Delta + V + v, \tag{6}$$

where $v : \mathbb{Z}^d \to \mathbb{C}$ is a decaying function.

The (ir)reducibility of the Fermi variety is closely related to the existence of eigenvalues embedded into spectral bands of perturbed periodic operators [37, 38]. We postpone the full set up and background to Section 2, and formulate one main theorem before closing this section. Based on the irreducibility (Theorems 1.1 and 1.2), the arguments in [37], and a unique continuation result for the discrete Laplacian on $\mathbb{Z}^d$, we are able to prove that

Theorem 1.8. *Assume that V is real and periodic. If there exist constants $C > 0$ and $\gamma > 1$ such that the complex-valued function $v : \mathbb{Z}^d \to \mathbb{C}$ satisfies*

$$|v(n)| \leq Ce^{-|n|^\gamma}, \tag{7}$$

then $H = -\Delta + V + v$ does not have any embedded eigenvalues, i.e., for any $\lambda \in \bigcup_{m=1}^{Q}(a_m, b_m)$, λ is not an eigenvalue of H.

Finally, we mention that the irreducibility results established in this paper provide opportunities to explore more applications [44–46].

2 Main Results

Definition 5. *Let* $\mathbb{C}^\star = \mathbb{C}\backslash\{0\}$ *and* $z = (z_1, z_2, \ldots, z_d)$. *The Floquet variety is defined as*

$$\mathcal{F}_\lambda(V) = \{z \in (\mathbb{C}^\star)^d : z_j = e^{2\pi i k_j}, j = 1, 2, \ldots, d, k \in F_\lambda(V)\}. \tag{8}$$

In other words, $z \in (\mathbb{C}^\star)^d \in \mathcal{F}_\lambda(V)$ if the equation

$$(-\Delta u)(n) + V(n)u(n) = \lambda u(n), n \in \mathbb{Z} \tag{9}$$

with the boundary condition

$$u(n + q_j \mathbf{e}_j) = z_j u(n), j = 1, 2, \ldots, d, \text{ and } n \in \mathbb{Z}^d \tag{10}$$

has a non-trivial function. Introduce a fundamental domain W for Γ:

$$W = \{n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d : 0 \le n_j \le q_j - 1, j = 1, 2, \ldots, d\}.$$

By writing out $-\Delta + V$ as acting on the Q dimensional space $\{u(n), n \in W\}$, the eigen-equation (9) and (10) ((2) and (3)) translates into the eigenvalue problem for a $Q \times Q$ matrix $\mathcal{D}(z)$ ($D(k)$). Let $\mathcal{P}(z, \lambda)$ ($P(k, \lambda)$) be the determinant of $\mathcal{D}(z) - \lambda I$ ($D(k) - \lambda I$). We should mention that $\mathcal{D}(z)$ ($D(k)$) and $\mathcal{P}(z, \lambda)$ ($P(k, \lambda)$) depend on the potential V. Since the potential is fixed, we drop the dependence during the proof.

From the notations above, one has that

$$F_\lambda(V) = \{k \in \mathbb{C}^d : P(k, \lambda) = 0\}, \mathcal{F}_\lambda(V) = \{z \in (\mathbb{C}^\star)^d : \mathcal{P}(z, \lambda) = 0\}. \tag{11}$$

It is easy to see that $\mathcal{P}(z, \lambda)$ is a polynomial in the variables λ and

$$z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_d, z_d^{-1}.$$

In other words $\mathcal{P}(z, \lambda)$ is a Laurent polynomial of $z_1, z_2, \ldots, z_d$ and polynomial in λ. Therefore, the Floquet variety $\mathcal{F}_\lambda(V)$ is an algebraic set[2]. It implies that both $B(V)$ and $F_\lambda(V)$ are (principal) analytic sets. Since the identity (3) is unchanged under the shift: $k \to k + \mathbb{Z}^d$, it is natural to study $F_\lambda(V)/\mathbb{Z}^d$.

In our proof, we focus on studying the Floquet variety $\mathcal{F}_\lambda(V)$ to benefit from its algebraicity.

A Laurent polynomial of a single term is called monomial, i.e., $Cz_1^{a_1} z_2^{a_2} \ldots z_k^{a_k}$, where $a_j \in \mathbb{Z}$, $j = 1, 2, \ldots, k$, and C is a non-zero constant.

Definition 6. *We say that a Laurent polynomial* $h(z_1, z_2, \ldots, z_k)$ *is irreducible if it can not be factorized non-trivially, that is, there are no non-monomial Laurent polynomials* $f(z_1, z_2, \ldots, z_k)$ *and* $g(z_1, z_2, \ldots, z_k)$ *such that* $h = fg$.

[2] Usually, an algebraic set is defined as common zeros of a collection of polynomials. Here, we call $X \subset (\mathbb{C}^\star)^d$ an algebraic set even though X is the zeros of a Laurent polynomial.

REMARK 3. When h is a polynomial, the definition of irreducibility in Definition 6 differs the traditional one[3] (because of the monomial). For example, the polynomial $z^2 + z$ is irreducible according to Definition 6. This will not create any trouble since all polynomials arising from this paper do not have factors z_j, $j = 1, 2, \ldots, k$.

Based on the above notations and definitions, we have the following simple facts.

PROPOSITION 2.1. *Fix $\lambda \in \mathbb{C}$. We have*

(1) *The Fermi variety/surface $F_\lambda(V)/\mathbb{Z}^d$ is irreducible if and only if $\mathcal{F}_\lambda(V)$ is irreducible;*
(2) *If the Laurent polynomial $\mathcal{P}(z, \lambda)$ (as a function of z) is irreducible, then $\mathcal{F}_\lambda(V)$ is irreducible.*

Theorem 2.2. *Let $d \geq 3$. Then for any $\lambda \in \mathbb{C}$, the Laurent polynomial $\mathcal{P}(z, \lambda)$ (as a function of z) is irreducible.*

Theorem 2.3. *Let $d = 2$. Then the Laurent polynomial $\mathcal{P}(z, \lambda)$ (as a function of z) is irreducible for any $\lambda \in \mathbb{C}$ except maybe for $\lambda = [V]$, where $[V]$ is the average of V over one periodicity cell. Moreover, if $\mathcal{P}(z, [V])$ is reducible, $\mathcal{P}(z, [V])$ has exactly two distinct non-trivial irreducible factors (each factor has multiplicity one).*

By Theorems 2.2 and 2.3, and some basic properties of $\mathcal{P}$, we immediately obtain

Theorem 2.4. *Let $d \geq 2$. Then the Laurent polynomial $\mathcal{P}(z, \lambda)$ (as a function of both z and λ) is irreducible.*

REMARK 4. (1) By (11) and Prop.2.1, Theorems 1.1, 1.2 and 1.3 follow from Theorems 2.2, 2.3 and 2.4.
(2) Denote by $\mathbf{0}$ the zero Γ-periodic potential. From (41) below, one can see that if $\mathcal{P}(z, [V])$ is reducible ($d = 2$), then $F_{[V]}(V) = F_0(\mathbf{0})$.

REMARK 5. Reducible Fermi surfaces are known to occur for periodic graph operators, even at all energy levels, e.g., [17, 57].

Our next topic is about the extrema of band functions.

Theorem 2.5. *Assume that V is a real valued periodic potential. Let λ_* be an extremum of a band function $\lambda_m(k)$, for some $m = 1, 2, \ldots, Q$. Then we have*

$$\{k \in \mathbb{R}^d : \lambda_m(k) = \lambda_*\} \subset \{k \in \mathbb{R}^d : P(k, \lambda_*) = 0, |\nabla_k P(k, \lambda_*)| = 0\}, \tag{12}$$

where ∇ is the gradient.

[3] A polynomial h is called irreducible if there are no non-constant polynomials f and g such that $h = fg$.

Recall that a point x of an analytic set Ω is called a regular point if there is a neighborhood U of x such that $U \cap \Omega$ is an analytic manifold. Any other point is called a singular point.

By Theorems 2.2 and 2.3, one has that for any fixed λ, $\mathcal{P}(z,\lambda)$ $(P(k,\lambda))$ is a minimal defining function (see p.27 in [8] for the precise definition) of $\mathcal{F}_\lambda(V)$ $(F_\lambda(V))$. Therefore, Theorem 2.5 implies (see p.27 in [8])

Corollary 2.6. *Let λ_* be an extremum of a band function $\lambda_m(k)$, $k \in \mathbb{R}^d$, for some $m = 1, 2, \ldots, Q$. Then $\{k \in \mathbb{R}^d : \lambda_m(k) = \lambda_*\}$ is a subset of singular points of the Fermi variety $F_{\lambda_*}(V)$.*

The last topic we are going to discuss is the existence of embedded eigenvalues for perturbed discrete periodic operators (6).

For $d = 1$, the existence/absence of embedded eigenvalues has been understood very well [28, 40, 43, 47, 51, 55]. Problems of the existence of embedded eigenvalues in higher dimensions are a lot more complicated. The techniques of the generalized Prüfer transformation and oscillated integrals developed for $d = 1$ are not available.

In [37], Kuchment and Vainberg introduced a new approach to study the embedded eigenvalue problem for perturbed periodic operators. It employs the analytic structure of the Fermi variety, unique continuation results, and techniques of several complex variables theory.

Condition 1: Given $\lambda \in \bigcup(a_m, b_m)$, we say that λ satisfies Condition 1 if any irreducible component of the Fermi variety $F_\lambda(V)$ contains an open analytic hypersurface of dimension $d-1$ in $\mathbb{R}^d$.

Theorem 2.7 [37]. *Let $d = 2, 3$, and H_0 and H be continuous versions of (1) and (6) respectively. Assume that there exist constants $C > 0$ and $\gamma > 4/3$ such that*

$$|v(x)| \leq Ce^{-|x|^\gamma}.$$

Assume Condition 1 for some $\lambda \in \bigcup(a_m, b_m)$. Then this λ can not be an eigenvalue of $H = -\Delta + V + v$.

For λ in the interior of a spectral band, the irreducibility of the Fermi variety $F_\lambda(V)$ implies Condition 1 for this λ. See Lemma 8.1. The restriction on $d = 2, 3$ and the critical exponent $4/3$ arise from a quantitative unique continuation result. Suppose u is a solution of

$$-\Delta u + \tilde{V}u = 0 \text{ in } \mathbb{R}^d,$$

where $|\tilde{V}| \leq C$, $|u| \leq C$ and $u(0) = 1$. From the unique continuation principle, u cannot vanish identically on any open set. The quantitative result states [6]

$$\inf_{|x_0|=R} \sup_{|x-x_0|\leq 1} |u(x)| \geq e^{-CR^{4/3}\log R}. \tag{13}$$

A similar version of (13) was established in [50] (also see Remark 2.6 in [18]), namely, there is no non-trivial solution of $(-\Delta + \tilde{V})u = 0$ such that

$$|u(x)| \leq e^{-C|x|^{4/3}} \text{ for any } C > 0. \tag{14}$$

For complex potentials $\tilde{V}$, the critical exponent $4/3$ in (13) and (14) is optimal in view of the Meshkov's example [50]. It has been conjectured (referred to as Landis' conjecture, which is still open for $d \geq 3$) that the critical exponent is 1 for real potentials. See [11, 26, 48] and references therein for the recent progress of the Landis' conjecture. However, the unique continuation principle for discrete Laplacians is well known not to hold (see e.g., [23, 39]). This issue turns out to be the obstruction to generalize Kuchment–Vainberg's approach to discrete periodic Schrödinger operators [35].

Fortunately, we realize that a weak unique continuation result is sufficient for Kuchment–Vainberg's arguments in [37]. Such a unique continuation result is not difficult to establish for discrete Schrödinger operators on $\mathbb{Z}^d$. Actually, the critical component can be improved from "4/3" to "1". Therefore, we are able to establish the discrete version of Theorem 2.7 for any dimension.

Theorem 2.8. *Assume V is a real valued periodic function. Let $d \geq 2$, H_0 and H be given by (1) and (6) respectively. Assume that there exist constants $C > 0$ and $\gamma > 1$ such that*

$$|v(n)| \leq Ce^{-|n|^{\gamma}}. \tag{15}$$

Assume Condition 1 for some $\lambda \in \bigcup_{m=1}^{Q}(a_m, b_m)$. Then this λ can not be an eigenvalue of $H = -\Delta + V + v$.

Remark 6.
- It is well known that for general periodic graphs even compactly supported solutions can exist (see e.g. [39]).
- It is known that a compactly supported perturbation of the operator on a graph might have an embedded eigenvalue. If this case happens, under the assumption on irreducibility of the Fermi variety, Kuchment and Vainberg proved that the corresponding eigenfunction is compactly supported (invalid the unique continuation) [38]. Shipman provided examples of periodic graph operators with unbounded support eigenfunctions for embedded eigenvalues (the Fermi variety is reducible at every energy level) [57].

Assume that V is zero, which can be viewed as a Γ-periodic function for any Γ. Denote by $[a_m, b_m]$, $m = 1, 2, \ldots, Q$, the spectral bands of $-\Delta$. Clearly,

$$\bigcup_{m=1}^{Q} [a_m, b_m] = \sigma(-\Delta) = [-2d, 2d].$$

Lemma 2.9 [21, Lemmas 1.2 and 1.3]. *Let $d \geq 2$. Then*

- *for any* $\lambda \in (-2d, 2d) \setminus \{0\}$*,* $\lambda \in (a_m, b_m)$ *for some* $1 \leq m \leq Q$*,*
- *if at least one of* q_j*'s is odd, then* $0 \in (a_m, b_m)$ *for some* $1 \leq m \leq Q$*.*

For $d = 2$, Lemma 2.9 was also proved in [13]. Based on Lemma 2.9, Han and Jitomirskaya proved the discrete Bethe-Sommerfeld conjecture [21]. See [10, 53] for the continuous Bethe-Sommerfeld conjecture.

Theorem 1.8 and Lemma 2.9 imply

Corollary 2.10. *Assume that there exist some* $C > 0$ *and* $\gamma > 1$ *such that*

$$|v(n)| \leq Ce^{-|n|^\gamma}.$$

Then $\sigma_p(-\Delta + v) \cap (-2d, 2d) = \emptyset$*.*

Remark 7. Under a stronger assumption that v has compact support, Isozaki and Morioka proved that $\sigma_p(-\Delta + v) \cap (-2d, 2d) = \emptyset$ [22].

The rest of this paper is organized as follows. The proof of Theorems 2.2, 2.3 and 2.4 is entirely self-contained. We recall the discrete Floquet-Bloch transform in Section 3. In Section 4, we do preparations for proofs. Section 5 is devoted to proving Theorems 2.2, 2.3 and 2.4. Sections 6 and 7 are devoted to proving Theorems 2.5 and 2.8 respectively. In Section 8, we prove Theorems 1.4, 1.5 and 1.8.

3 Discrete Floquet-Bloch Transform

In this section, we recall the standard discrete Floquet-Bloch transform. We refer readers to [31, 34] for details.

Let

$$\bar{W} = \left\{0, \frac{1}{q_1}, \frac{2}{q_1}, \ldots, \frac{q_1 - 1}{q_1}\right\} \times \cdots \times \left\{0, \frac{1}{q_d}, \frac{2}{q_d}, \ldots, \frac{q_d - 1}{q_d}\right\} \subset [0, 1]^d.$$

Define the discrete Fourier transform $\hat{V}(l)$ for $l \in \bar{W}$ by

$$\hat{V}(l) = \frac{1}{Q} \sum_{n \in W} V(n) e^{-2\pi i l \cdot n},$$

where $l \cdot n = \sum_{j=1}^d l_j n_j$ for $l = (l_1, l_2, \ldots, l_d) \in \bar{W}$ and $n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$. For convenience, we extend $\hat{V}(l)$ to $\bar{W} + \mathbb{Z}^d$ periodically, namely for any $l \equiv \tilde{l} \mod \mathbb{Z}^d$,

$$\hat{V}(l) = \hat{V}(\tilde{l}).$$

The inverse of the discrete Fourier transform is given by

$$V(n) = \sum_{l \in \bar{W}} \hat{V}(l) e^{2\pi i l \cdot n}.$$

For a function $u \in \ell^2(\mathbb{Z}^d)$, its Fourier transform $\mathscr{F}(u) = \hat{u} : \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{C}$ is given by

$$\hat{u}(x) = \sum_{n \in \mathbb{Z}^d} u(n) e^{-2\pi i n \cdot x}.$$

For any periodic function V and any $u \in \ell^2(\mathbb{Z}^d)$, one has

$$\widehat{Vu}(x) = \sum_{l \in \bar{W}} \hat{V}(l) \hat{u}(x - l).$$

We remark that $\hat{u}$ is the Fourier transform for $u \in \ell^2(\mathbb{Z}^d)$ and $\hat{V}$ is the discrete Fourier transform for $V(n)$, $n \in W$. Let

$$\mathcal{B} = \prod_{j=1}^{d} \Big[0, \frac{1}{q_j}\Big).$$

Let $L^2(\mathcal{B} \times \bar{W})$ be all functions with the finite norm given by

$$||f||_{L^2(\mathcal{B} \times \bar{W})} = \sum_{l \in \bar{W}} \int_{\mathcal{B}} |f(x,l)|^2 dx.$$

Define the unitary map $U : \ell^2(\mathbb{Z}^d) \to L^2(\mathcal{B} \times \bar{W})$ by

$$(U(u))(x,l) = \hat{u}(x + l)$$

for $x = (x_1, x_2, \ldots, x_d) \in \mathcal{B}$ and $l \in \bar{W}$. For fixed $x \in \mathcal{B}$, define the operator $\tilde{H}_0(x)$ on $\ell^2(\bar{W})$:

$$(\tilde{H}_0(x)u)(l) = \left(\sum_{j=1}^{d} -2\cos(2\pi(l_j + x_j)) u(l) \right) + \sum_{j \in \bar{W}} \hat{V}(l - j) u(j), \tag{16}$$

where $l = (l_1, l_2, \ldots, l_d) \in \bar{W}$. Let $\hat{H}_0 : L^2(\mathcal{B} \times \bar{W}) \to L^2(\mathcal{B} \times \bar{W})$ be given by

$$(\hat{H}_0 u)(x,l) = \left(\sum_{j=1}^{d} -2\cos(2\pi(l_j + x_j)) u(x,l) \right) + \sum_{j \in \bar{W}} \hat{V}(l - j) u(x,j). \tag{17}$$

The following two Lemmas are well known.

LEMMA 3.1. *Let $H_0 = -\Delta + V$. Let $\hat{H}_0$ be given by (17). Then*

$$\hat{H}_0 = U H_0 U^{-1}. \tag{18}$$

Proof. Straightforward computations. □

Given $x \in \mathbb{R}^d$, let $\mathscr{F}^x$ be the Floquet-Bloch transform on $\ell^2(W)$: for any vector on W, $\{u(n)\}_{n\in W}$,

$$[\mathscr{F}^x u](n') = \frac{1}{\sqrt{Q}} \sum_{n\in W} e^{-2\pi i \sum_{j=1}^d \left(\frac{n'_j}{q_j}+x_j\right) n_j} u(n), \quad n' \in W.$$

Let $\tilde{D}(x)$ be the $Q \times Q$ matrix given by $D(q_1x_1, q_2x_2, \ldots, x_dq_d)$.

Lemma 3.2. *The operator $\tilde{H}_0(x)$ given by (16) is unitarily equivalent to $\tilde{D}(x)$.*

Proof. By (2) and (3), $\tilde{D}(x)$ is the restriction of $-\Delta + V$ to W with boundary conditions:

$$u(n + q_j\mathbf{e}_j) = e^{2\pi i q_j x_j} u(n), j = 1, 2, \ldots, d, n \in \mathbb{Z}^d. \tag{19}$$

Let $T : \ell^2(\bar{W}) \to \ell^2(W)$ given by $T(l_1, l_2, \ldots, l_d) = (q_1l_1, q_2l_2, \ldots, q_dl_d)$, where $(l_1, l_2, \ldots, l_d) \in \bar{W}$. Direct computations imply that

$$\tilde{H}_0(x) = T\mathscr{F}^x\tilde{D}(x)(\mathscr{F}^x)^*T^{-1} = T\mathscr{F}^x\tilde{D}(x)(\mathscr{F}^x)^{-1}T^{-1}. \qquad \square$$

Assume V is real. For each $k \in [0, 1)^d$, it is easy to see that $D(k)$ has $Q = q_1q_2 \ldots q_d$ eigenvalues. Order them in non-decreasing order

$$\lambda_1(k) \leq \lambda_2(k) \leq \cdots \leq \lambda_Q(k).$$

We call $\lambda_m(k)$ the m-th (spectral) band function, $m = 1, 2, \ldots, Q$. Then we have

Lemma 3.3.

$$[a_m, b_m] = \left[\min_{k\in[0,1)^d} \lambda_m(k), \max_{k\in[0,1)^d} \lambda_m(k)\right]$$

and $a_m < b_m$, $m = 1, 2, \ldots, Q$.

4 Preparations

For readers' convenience, we collect some notations and define a few new notations here, which will be constantly used in the proofs.

(1) $\mathcal{D}(z)$ is the $Q \times Q$ matrix arising from the eigen-equation (9) and (10).
(2) $z_j = e^{2\pi i k_j}$ and $k_j = q_jx_j$, $j = 1, 2, \ldots, d$. $\tilde{D}(x) = D(k) = \mathcal{D}(z)$. $\tilde{\mathcal{D}}(z) = \mathcal{D}(z_1^{q_1}, z_2^{q_2}, \ldots, z_d^{q_d})$.
(3) $\mathcal{P}(z, \lambda) = \det(\mathcal{D}(z)-\lambda I)$, $\tilde{\mathcal{P}}(z, \lambda) = \det(\tilde{\mathcal{D}}(z)-\lambda I)$, $P(k, \lambda) = \det(D(k)-\lambda I)$, $\tilde{P}(x, \lambda) = \det(\tilde{D}(x) - \lambda I)$.

(4) Let

$$\rho^j_{n_j} = e^{2\pi i \frac{n_j}{q_j}},$$

where $0 \leq n_j \leq q_j - 1$, $j = 1, 2, \ldots, d$. Denote by μ_{q_j} the multiplicative group of q_j roots of unity, $j = 1, 2, \ldots, d$. Let $\mu = \mu_{q_1} \times \mu_{q_2} \times \cdots \times \mu_{q_d}$.
For any $\rho = (\rho^1, \rho^2, \ldots, \rho^d) \in \mu$, we can define a natural action on $\mathbb{C}^d$

$$\rho \cdot (z_1, z_2, \ldots, z_d) = \left(\rho^1 z_1, \rho^2 z_2, \ldots, \rho^d z_d\right).$$

(5) For a polynomial $f(z)$, denote by $\deg(f)$ the degree of f.
(6) Let $\mathcal{P}_1(z, \lambda) = (-1)^Q z_1^{\frac{Q}{q_1}} z_2^{\frac{Q}{q_2}} \ldots z_d^{\frac{Q}{q_d}} \mathcal{P}(z, \lambda)$.
(7) For any $a = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d$, let $z^a = z_1^{a_1} z_2^{a_2} \ldots z_d^{a_d}$.

The following lemma is standard.

LEMMA 4.1. *Let $n = (n_1, n_2, \ldots, n_d) \in W$ and $n' = (n'_1, n'_2, \ldots, n'_d) \in W$. Then $\tilde{\mathcal{D}}(z)$ is unitarily equivalent to $A + B$, where A is a diagonal matrix with entries*

$$A(n; n') = -\left(\left(\sum_{j=1}^{d}\left(\rho^j_{n_j} z_j + \frac{1}{\rho^j_{n_j} z_j}\right)\right) - \lambda\right)\delta_{n,n'} \tag{20}$$

and B

$$B(n; n') = \hat{V}\left(\frac{n_1 - n'_1}{q_1}, \frac{n_2 - n'_2}{q_2}, \ldots, \frac{n_d - n'_d}{q_d}\right).$$

In particular,

$$\tilde{\mathcal{P}}(z, \lambda) = \det(A + B).$$

Proof. Recall that $x_j = \frac{k_j}{q_j}$, $z_j = e^{2\pi i k_j}$, $j = 1, 2, \ldots, d$. Lemma 4.1 follows from Lemma 3.2 and (16). □

We note that B is independent of $z_1, z_2, \ldots, z_d$ and λ.

Here are some simple facts about $\mathcal{P}$, $\tilde{\mathcal{P}}$ and $\mathcal{P}_1$.

(1) $\mathcal{P}(z, \lambda)$ is symmetric with respect to z_j and z_j^{-1}, $j = 1, 2, \ldots, d$.
(2) $\mathcal{P}(z, \lambda)$ is a polynomial in the variables $z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_d, z_d^{-1}$ and λ with highest degree terms (up to a $\pm$ sign) $z_1^{\frac{Q}{q_1}}, z_1^{-\frac{Q}{q_1}}, z_2^{\frac{Q}{q_2}}, z_2^{-\frac{Q}{q_2}} \ldots, z_d^{\frac{Q}{q_d}}, z_d^{-\frac{Q}{q_d}}$ and λ^Q.
(3) $\tilde{\mathcal{P}}(z, \lambda)$ is a polynomial in the variables $z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_d, z_d^{-1}$ and λ with highest degree terms (up to a $\pm$ sign) $z_1^Q, z_1^{-Q}, z_2^Q, z_2^{-Q}, \ldots, z_d^Q, z_d^{-Q}$ and λ^Q.

(4) $\mathcal{P}_1(z,\lambda)$ is a polynomial of z and λ. $\mathcal{P}_1(z,\lambda)$ can not have a factor z_j, $j = 1,2,\dots,d$, namely

$$z_j \nmid \mathcal{P}_1(z,\lambda), j = 1,2,\dots,d. \tag{21}$$

Therefore, the Laurent polynomial $\mathcal{P}(z,\lambda)$ is irreducible (as a function of z) if and only if the polynomial $\mathcal{P}_1(z,\lambda)$ (as a function of z) is irreducible in the traditional way, namely, there are no non-constant polynomials $f(z)$ and $g(z)$ such that $\mathcal{P}_1(z,\lambda) = f(z)g(z)$.

5 Proof of Theorems 2.2 , 2.3 and 2.4

Let

$$\tilde{h}_1(z) = z_1^Q z_2^Q \dots z_d^Q \prod_{\substack{0\leq n_j\leq q_j-1\\ 1\leq j\leq q}} \left(\sum_{j=1}^d \frac{1}{\rho^j_{n_j} z_j} \right), \tag{22}$$

and

$$\tilde{h}_2(z) = z_1^Q z_2^Q \dots z_{d-1}^Q z_d^{-Q} \prod_{\substack{0\leq n_j\leq q_j-1\\ 1\leq j\leq q}} \left(\rho^d_{n_d} z_d + \sum_{j=1}^{d-1} \frac{1}{\rho^j_{n_j} z_j} \right). \tag{23}$$

One can see that $\tilde{h}_1(z)$ is a polynomial in variables $z_1,\dots,z_{d-1},z_d$ and $\tilde{h}_2(z)$ is a polynomial in variables $z_1,\dots,z_{d-1},z_d^{-1}$.

Since both $\tilde{h}_1(z)$ and $\tilde{h}_2(z)$ are unchanged under the action of the group μ, we have that there exist $h_1(z)$ (a polynomial of $z_1,\dots,z_{d-1},z_d$) and $h_2(z)$ (a polynomial of $z_1,\dots,z_{d-1},z_d^{-1}$) such that

$$\tilde{h}_1(z_1,z_2,\dots,z_d) = h_1(z_1^{q_1}, z_2^{q_2},\dots,z_d^{q_d}), \tag{24}$$

and

$$\tilde{h}_2(z_1,z_2,\dots,z_d) = h_2(z_1^{q_1}, z_2^{q_2},\dots,z_d^{q_d}). \tag{25}$$

Lemma 5.1. *Both $h_1(z)$ and $h_2(z)$ are irreducible.*

Proof. Without loss of generality, we only show that $h_1(z)$ is irreducible. Suppose the statement is not true. Then there are two non-constant polynomials $f(z)$ and $g(z)$ such that $h_1(z) = f(z)g(z)$. Let

$$\tilde{f}(z) = f(z_1^{q_1}, z_2^{q_2},\dots,z_d^{q_d}), \tilde{g}(z) = g(z_1^{q_1}, z_2^{q_2},\dots,z_d^{q_d}).$$

Therefore,

$$\tilde{f}(z)\tilde{g}(z) = z_1^Q z_2^Q \dots z_d^Q \prod_{\substack{0\leq n_j\leq q_j-1\\ 1\leq j\leq q}} \left(\sum_{j=1}^d \frac{1}{\rho^j_{n_j} z_j} \right). \tag{26}$$

By the assumption that the greatest common factor of $q_1, q_2, \ldots, q_d$ is 1, we have for any n_j, n_j' with $0 \leq n_j, n_j' \leq q_j - 1$ and $(n_1, n_2, \ldots, n_d) \neq (n_1', n_2', \ldots, n_d')$,

$$\left\{ z \in (\mathbb{C}^\star)^d : \sum_{j=1}^{d} \frac{1}{\rho_{n_j}^j z_j} = 0 \right\} \neq \left\{ z \in (\mathbb{C}^\star)^d : \sum_{j=1}^{d} \frac{1}{\rho_{n_j'}^j z_j} = 0 \right\}. \tag{27}$$

By the fact that both $\tilde{f}(z)$ and $\tilde{g}(z)$ are unchanged under the action μ, and (27), we have that if $\tilde{f}(z)$ (or $\tilde{g}(z)$) has one factor $\left(\sum_{j=1}^{d} \frac{1}{\rho_{n_j}^j z_j}\right)$, then $\tilde{f}(z)$ (or $\tilde{g}(z)$) will have a factor $\prod_{\substack{0 \leq n_j \leq q_j - 1 \\ 1 \leq j \leq q}} \left(\sum_{j=1}^{d} \frac{1}{\rho_{n_j}^j z_j}\right)$. This contradicts (26). □

Lemma 5.2. *For any $\lambda \in \mathbb{C}$, the polynomial $\mathcal{P}_1(z, \lambda)$ (as a function of z) has at most two non-trivial factors (count multiplicity). In the case that $\mathcal{P}_1(z, \lambda)$ has two non-trivial factors, namely $\mathcal{P}_1(z, \lambda) = f(z)g(z)$, we have that (maybe exchange f and g)*

- *the closure[4] of $Z_1 = \{z \in (\mathbb{C}^\star)^d : f(z) = 0\}$ contains $z_1 = z_2 = \cdots = z_d = 0$,*
- *the closure of $Z_2 = \{z \in (\mathbb{C}^\star)^d : g(z) = 0\}$ contains $z_1 = z_2 = \cdots = z_{d-1} = 0, z_d^{-1} = 0$[5].*

Proof. Let $f(z)$ be a factor of polynomial $\mathcal{P}_1(z, \lambda)$ and

$$Z_f = \{z \in (\mathbb{C}^\star)^d : f(z) = 0\}.$$

Let

$$\tilde{f}(z) = f(z_1^{q_1}, z_2^{q_2}, \ldots, z_d^{q_d}).$$

Solving the equation $\det(A + B) = 0$ and by (20), we have that if $z_1 = z_0^2$, $z_2 = z_3 = \cdots = z_{d-1} = z_0$ and $z_0 \to 0$, then $z_d \to 0$ or $z_d^{-1} \to 0$. This implies that letting $z_1 = z_0^2$, $z_2 = z_3 = \cdots = z_{d-1} = z_0$ and $z_0 \to 0$, and solving the equation $f(z) = 0$, we must have either $z_d \to 0$ or $z_d^{-1} \to 0$. Therefore, the closure of Z_f contains either $z_1 = z_2 = \cdots = z_d = 0$ or $z_1 = z_2 = \cdots = z_{d-1} = 0, z_d^{-1} = 0$.

Take $z_1 = z_2 = \cdots = z_d = 0$ into consideration first. Let A and B be given by Lemma 4.1. Then the off-diagonal entries of $-z_1 z_2 \ldots z_d(A + B)$ are all divisible by $z_1 z_2 \ldots z_d$, while the diagonal entries are

$$\left(z_1 z_2 \ldots z_d \left(\sum_{j=1}^{d} \frac{1}{\rho_{n_j}^j z_j} \right) + \text{ functions divisible by } z_1 z_2 \ldots z_d \right), \tag{28}$$

where $0 \leq n_j \leq q_j - 1$. This shows the component of lowest degree of $\det(-z_1 z_2 \ldots z_d(A + B))$ with respect to variables $z_1, z_2, \ldots, z_d$, is

[4] The closure is taken in $(\mathbb{C} \cup \{\infty\})^d$.

[5] $z_d^{-1} = 0$ means $z_d = \infty$. In the proof, we view z_d^{-1} as a new variable when $z_d = \infty$.

$$\tilde{h}_1(z) = z_1^Q z_2^Q \dots z_d^Q \prod_{\substack{0\leq n_j\leq q_j-1 \\ 1\leq j\leq q}} \left(\sum_{j=1}^{d} \frac{1}{\rho_{n_j}^j z_j} \right). \tag{29}$$

Claim 1: by the fact that $h_1(z)$ is irreducible by Lemma 5.1, one has that there exists at most one factor $f(z)$ of $\mathcal{P}_1(z,\lambda)$ such that the closure of $\{z \in (\mathbb{C}^\star)^d : f(z) = 0\}$ contains $z_1 = z_2 = \cdots = z_d = 0$. Claim 1 immediately follows from some basic facts of algebraic geometry. For convenience, we include an elementary proof in the "Appendix".

Similarly, the component of lowest degree of $\det(-z_1 z_2 \dots z_{d-1} z_d^{-1}(A+B))$ with respect to variables $z_1, z_2, \dots, z_{d-1}, z_d^{-1}$ is

$$\tilde{h}_2(z) = z_1^Q z_2^Q \dots z_{d-1}^Q z_d^{-Q} \prod_{\substack{0\leq n_j\leq q_j-1 \\ 1\leq j\leq q}} \left(\rho_{n_d}^d z_d + \sum_{j=1}^{d-1} \frac{1}{\rho_{n_j}^j z_j} \right). \tag{30}$$

Since $h_2(z)$ is irreducible by Lemma 5.1, by a similar argument of the proof of Claim 1, one has that there exists at most one factor $f(z)$ of $\mathcal{P}_1(z,\lambda)$ such that the closure of $\{z \in (\mathbb{C}^\star)^d : f(z) = 0\}$ contains $z_1 = z_2 = \cdots = z_{d-1} = 0, z_d^{-1} = 0$. Therefore, $\mathcal{P}_1(z,\lambda)$ has at most two non-trivial factors. When $\mathcal{P}_1(z,\lambda)$ actually has two factors, by the above analysis, the statements in Lemma 5.2 hold. □

Remark 8. When $d = 2$, Gieseker, Knörrer and Trubowitz proved that the Fermi variety $F_\lambda(V)/\mathbb{Z}^2$ has at most two irreducible components for any λ [20, Corollary 4.1]. Even for $d = 2$, our approach is different. We show that the closure of the zero set of every factor of $\mathcal{P}_1$ must contain either $z_1 = z_2 = \cdots = z_d = 0$ or $z_1 = z_2 = \cdots = z_{d-1} = z_d^{-1} = 0$ by solving algebraic equations on properly choosing curves.

We are ready to prove Theorems 2.3 and 2.2.

Proof of Theorem 2.3. Without loss of generality, assume $[V] = 0$. Assume $\mathcal{P}(z,\lambda)$ is reducible for some $\lambda \in \mathbb{C}$. By Lemma 5.2, there are two non-constant polynomials $f(z)$ and $g(z)$ such that none of them has a factor z_1 or z_2 (by (21)), and

$$\mathcal{P}_1(z,\lambda) = (-1)^{q_1 q_2} z_1^{q_2} z_2^{q_1} \mathcal{P}(z_1, z_2, \lambda) = f(z_1,z_2) g(z_1,z_2). \tag{31}$$

Moreover, the closure of $\{z \in (\mathbb{C}^\star)^2 : f(z) = 0\}$ contains $z_1 = z_2 = 0$ and the closure of $\{z \in (\mathbb{C}^\star)^2 : g(z) = 0\}$ contains $z_1 = 0, z_2^{-1} = 0$.

Let

$$\tilde{f}(z) = \tilde{f}(z_1,z_2) = f\left(z_1^{q_1}, z_2^{q_2}\right), \tilde{g}(z) = \tilde{g}(z_1,z_2) = g\left(z_1^{q_1}, z_2^{q_2}\right).$$

Therefore, $\tilde{f}(z)$ and $\tilde{g}(z)$ are also polynomials and

$$\tilde{f}(z)\tilde{g}(z) = (-1)^{q_1q_2} z_1^{q_1q_2} z_2^{q_1q_2} \tilde{\mathcal{P}}(z_1, z_2, \lambda) = \det(-z_1z_2A - z_1z_2B). \tag{32}$$

By (29) and (30), we have there exists a non-zero constant K such that

$$\tilde{f}(z) = \left(\sum_{i=1}^{p} c_i z_1^{a_i} z_2^{b_i}\right) + K \prod_{\substack{0\leq n_1\leq q_1-1\\ 0\leq n_2\leq q_2-1}} \left(\frac{z_2}{\rho^1_{n_1}} + \frac{z_1}{\rho^2_{n_2}}\right), \tag{33}$$

where $a_i + b_i \geq q_1q_2 + 1$, and

$$\tilde{g}(z) = z_2^k \left[\left(\sum_{i=1}^{\tilde{p}} \tilde{c}_i z_1^{\tilde{a}_i} z_2^{-\tilde{b}_i}\right) + \prod_{\substack{0\leq n_1\leq q_1-1\\ 0\leq n_2\leq q_2-1}} \left(\frac{1}{z_2\rho^1_{n_1}} + z_1\rho^2_{n_2}\right)\right], \tag{34}$$

where $\tilde{a}_i + \tilde{b}_i \geq q_1q_2 + 1$ and $k = \max_{1\leq i\leq \tilde{p}}\{q_1q_2, \tilde{b}_i\}$ (this ensures that $g(z)$ is a polynomial and $g(z)$ does not have a factor z_2).

The matrix z_1z_2A is given by

$$-\left(\rho^1_{n_1} z_1^2 z_2 + \frac{z_2}{\rho^1_{n_1}} + \frac{z_1}{\rho^2_{n_2}} + \rho^2_{n_2} z_2^2 z_1 + \lambda z_1 z_2\right) \delta_{n_1,n_1'}\delta_{n_2,n_2'}$$

and all the entries of z_1z_2B only have a factor z_1z_2. Therefore, by (32),

$$\deg(\tilde{f}) + \deg(\tilde{g}) = \deg(\tilde{f}\tilde{g}) = \deg(\det(-z_1z_2A - z_1z_2B)) \leq 3q_1q_2. \tag{35}$$

By (33), one has if $c_i = 0$, $i = 1, 2, \ldots p$,

$$\deg(\tilde{f}) = q_1q_2, \tag{36}$$

and if one of c_i, $i = 1, 2, \ldots p$, is nonzero,

$$\deg(\tilde{f}) \geq q_1q_2 + 1. \tag{37}$$

By (34), one has

$$\deg(\tilde{g}) \geq k + q_1q_2. \tag{38}$$

By (35)–(38) and the fact that $k = \max_{1\leq i\leq \tilde{p}}\{q_1q_2, \tilde{b}_i\} \geq q_1q_2$, we must have $k = q_1q_2$, $\tilde{b}_i \leq q_1q_2$ and $c_i = 0$, $i = 1, 2, \ldots, p$. Therefore,

$$\tilde{f}(z) = K \prod_{\substack{0\leq n_1\leq q_1-1\\ 0\leq n_2\leq q_2-1}} \left(\frac{z_2}{\rho^1_{n_1}} + \frac{z_1}{\rho^2_{n_2}}\right). \tag{39}$$

Reformulate (32), (34) and (39) as,

$$\frac{1}{z_2^{2q_1q_2}}\tilde{f}(z)\tilde{g}(z) = (-1)^{q_1q_2}\det\left[\frac{z_1}{z_2}(A+B)\right],$$
$$\frac{1}{z_2^{q_1q_2}}\tilde{f}(z) = K\prod_{\substack{0\le n_1\le q_1-1\\ 0\le n_2\le q_2-1}}\left(\frac{1}{\rho^1_{n_1}}+\frac{z_1}{z_2\rho^2_{n_2}}\right),$$

and

$$\frac{1}{z_2^{q_1q_2}}\tilde{g}(z) = \left[\left(\sum_{i=1}^{\tilde{p}}\tilde{c}_i z_1^{\tilde{a}_i}z_2^{-\tilde{b}_i}\right)+\prod_{\substack{0\le n_1\le q_1-1\\ 0\le n_2\le q_2-1}}\left(\frac{1}{z_2\rho^1_{n_1}}+\rho^2_{n_2}z_1\right)\right],$$

where $\tilde{a}_i+\tilde{b}_i\ge q_1q_2+1$ and $\tilde{b}_i\le q_1q_2$.

The matrix $\frac{z_1}{z_2}A$ is

$$-\left(\rho^1_{n_1}\frac{z_1^2}{z_2}+\frac{1}{z_2\rho^1_{n_1}}+\frac{z_1}{\rho^2_{n_2}z_2^2}+\rho^2_{n_2}z_1+\lambda\frac{z_1}{z_2}\right)\delta_{n_1,n_1'}\delta_{n_2,n_2'}$$

and every entry of $\frac{z_1}{z_2}B$ only has a factor $\frac{z_1}{z_2}$.

Since $z_1^{\tilde{a}_i}z_2^{-\tilde{b}_i}\prod_{\substack{0\le n_1\le q_1-1\\ 0\le n_2\le q_2-1}}\left(\frac{1}{\rho^1_{n_1}}+\frac{z_1}{z_2\rho^2_{n_2}}\right)$ with $\tilde{a}_i+\tilde{b}_i\ge q_1q_2+1$ will contribute to $z_1^iz_2^{-j}$ with $i+j\ge 3q_1q_2+1$ and $\det(\frac{z_1}{z_2}(A+B))$ can only have $z_1^{\tilde{i}}z_2^{-\tilde{j}}$ with $\tilde{i}+\tilde{j}\le 3q_1q_2$, a degree argument (regard z_2^{-1} as a new variable) leads to $\tilde{c}_i=0$, $i=1,2,\dots,\tilde{p}$. Therefore,

$$\tilde{g}(z) = \prod_{\substack{0\le n_1\le q_1-1\\ 0\le n_2\le q_2-1}}\left(\frac{1}{\rho^1_{n_1}}+\rho^2_{n_2}z_1z_2\right). \tag{40}$$

We conclude that we prove that if $\mathcal{P}_1(z,\lambda)$ is reducible, then by (32), (39) and (40), there exists a constant $K\neq 0$ such that

$$\det(-A-B) = \frac{K}{z_1^{q_1q_2}z_2^{q_1q_2}}\prod_{\substack{0\le n_1\le q_1-1\\ 0\le n_2\le q_2-1}}\left(\frac{z_2}{\rho^1_{n_1}}+\frac{z_1}{\rho^2_{n_2}}\right)\prod_{\substack{0\le n_1\le q_1-1\\ 0\le n_2\le q_2-1}}\left(\frac{1}{\rho^1_{n_1}}+\rho^2_{n_2}z_1z_2\right). \tag{41}$$

We will prove that if (41) holds, then $\lambda=0$.

Let

$$\begin{aligned}t_{n_1,n_2}(z_1,z_2) &= \rho^1_{n_1}z_1+\frac{1}{\rho^1_{n_1}z_1}+\rho^2_{n_2}z_2+\frac{1}{\rho^2_{n_2}z_2}\\ &= \left(\rho^1_{n_1}z_1+\rho^2_{n_2}z_2\right)\left(1+\frac{1}{\rho^1_{n_1}\rho^2_{n_2}z_1z_2}\right).\end{aligned}$$

Then $t_{n_1,n_2}(z_1,z_2)+\lambda$ is the (n_1,n_2)-th diagonal entry of A.

Let $z_1=-z_2$. By (41), one has

$$\det(A+B)\equiv 0. \tag{42}$$

and

$$t_{0,0}(z_1,z_2)\equiv 0. \tag{43}$$

Since q_1 and q_2 are coprime, for any $(n_1,n_2)\in W\backslash(0,0)$,

$$\rho^1_{n_1}z_1-\rho^2_{n_2}z_1\neq 0, \text{ for } z_1\neq 0, \tag{44}$$

and hence t_{n_1,n_2} is not a zero function. Check the term of highest degree of $z_1(z_2)$ in $\det(A+B)$. By (20), (43) and (44), the term of highest degree (up to a nonzero constant factor) is

$$\lambda z_1^{q_1q_2-1}. \tag{45}$$

By (42) and (45), $\lambda=0$. We complete the proof of the first part of Theorem 2.3. The second part follows from (41). □

Proof of Theorem 2.2. The proof is similar to that of Theorem 2.3. Without loss of generality, assume $[V]=0$. Assume that $\mathcal{P}(z,\lambda)$ is reducible. Then there are two non-constant polynomials $f(z)$ and $g(z)$ such that none of them has a factor z_j, $j=1,2,\ldots,Q$, and

$$(-1)^Q z_1^{\frac{Q}{q_1}}z_2^{\frac{Q}{q_2}}\ldots z_d^{\frac{Q}{q_d}}\mathcal{P}(z,\lambda)=f(z)g(z). \tag{46}$$

Let

$$\tilde f(z)=f(z_1^{q_1},z_2^{q_2},\ldots,z_d^{q_d}),\tilde g(z)=g(z_1^{q_1},z_2^{q_2},\ldots,z_d^{q_d}).$$

Therefore, $\tilde f(z)$ and $\tilde g(z)$ are also polynomials and

$$\begin{aligned}\tilde f(z)\tilde g(z)&=(-1)^Q z_1^Q z_2^Q\ldots z_d^Q\tilde{\mathcal{P}}(z,\lambda)\\&=\det(-z_1z_2\ldots z_d(A+B)).\end{aligned} \tag{47}$$

Moreover, the closure of $\{z\in(\mathbb{C}^\star)^d: f(z)=0\}$ contains $z_1=z_2=\cdots=z_d=0$ and the closure of $\{z\in(\mathbb{C}^\star)^d: g(z)=0\}$ contains $z_1=z_2=\cdots=z_{d-1}=0$ and $z_d^{-1}=0$.

By (29) and (30), we have for some non-zero constant K,

$$\tilde f(z)=\left(\sum_{i=1}^p c_iz^{a_i}\right)+K\tilde h_1(z), \tag{48}$$

where $||a_i||_1\geq(d-1)Q+1$, and

$$\tilde{g}(z) = z_d^k \left[\left(\sum_{i=1}^{\tilde{p}} \tilde{c}_i \tilde{z}^{\tilde{a}_i} z_d^{-\tilde{b}_i} \right) + \tilde{h}_2(z) \right], \tag{49}$$

where $\tilde{z} = (z_1, z_2, \ldots, z_{d-1})$, $||\tilde{a}_i||_1 + \tilde{b}_i \geq (d-1)Q + 1$ and $k = \max_{1 \leq i \leq \tilde{p}}\{Q, \tilde{b}_i\}$.

By (48), one has

$$\deg(\tilde{f}) \geq \deg(\tilde{h}_1) = (d-1)Q. \tag{50}$$

By (49),

$$\deg(\tilde{g}) \geq \deg(z_d^k \tilde{h}_2(z)) \geq \deg(z_d^Q \tilde{h}_2(z)) = dQ. \tag{51}$$

By (50), (51) and (47), one has

$$\deg(\det(z_1 z_2 \ldots z_d(A+B))) = \deg(\tilde{f}\tilde{g}) \geq (2d-1)Q.$$

This is impossible since $\deg(\det(z_1 z_2 \ldots z_d(A+B))) \leq (d+1)Q$. □

Proof of Theorem 2.4. Assume $\mathcal{P}(z,\lambda)$ is irreducible. Then there exist two non-trivial factors $f_j(z,\lambda)$, Laurent polynomial in z and polynomial in λ, $j = 1, 2$, such that $\mathcal{P}(z,\lambda) = f_1(z,\lambda) f_2(z,\lambda)$. Rewrite $f_j(z,\lambda)$, $j = 1, 2$, as

$$f_j(z,\lambda) = \sum_{a \in A_j} t_j^a(\lambda) z^a,$$

where $t_j^a(\lambda)$ is a polynomial of λ and A_j is a proper finite subset of $\mathbb{Z}^d$. Let λ be large enough so that for any $j = 1, 2$ and $a \in A_j$, $t_j^a(\lambda) \neq 0$.

By Theorems 1.1 and 1.2, $\mathcal{P}(z,\lambda)$ (as a function of variables z) is irreducible for any large enough λ. Therefore, we must have that for any large enough λ, either $f_1(z,\lambda)$ or $f_2(z,\lambda)$ is a monomial of z. Then we conclude that either the cardinality of A_1 is one or the cardinality of A_2 is one. Without loss of generality assume that $f_1(z,\lambda) = t_1^{a_0}(\lambda) z^{a_0}$ for some $a_0 \in \mathbb{Z}^d$. Since f_1 is non-monomial, one has that $t_1^{a_0}(\lambda)$ is non-constant. Let $\lambda_0 \in \mathbb{C}$ be such that $t_1^{a_0}(\lambda_0) = 0$. Then we have $\mathcal{P}(z,\lambda_0) = 0$ for any z. Recall that the highest degree term (up to a $\pm$ sign) of z_1 in $\mathcal{P}(z,\lambda_0)$ is $z_1^{\frac{Q}{q_1}}$ (Fact (2) at the end of Section 4). We obtain the contradiction. □

6 Proof of Theorem 2.8

Theorem 6.1 [37, Lemma 17]. *Let Z be the set of all zeros of an entire function $\zeta(k)$ in $\mathbb{C}^d$ and $\cup Z_j$ be its irreducible components. Assume that the real part $Z_{j,\mathbb{R}} = Z_j \cap \mathbb{R}^d$ of each Z_j contains a submanifold of real dimension $d-1$. Let also $g(k)$ be* *an entire function in $\mathbb{C}^d$ with values in a Hilbert space $\mathcal{H}$ such that on the real space $\mathbb{R}^d$ the ratio*

$$f(k)=\frac{g(k)}{\zeta(k)}$$

belongs to $L^2_{loc}(\mathbb{R}^d,\mathcal{H})$. Then $f(k)$ extends to an entire function with values in $\mathcal{H}$.

The following lemma is well known, we include a proof here for completeness.

LEMMA 6.2. *Let $\hat{f}\in L^2(\mathbb{T}^d)$ and $\{f_n\}$ be its Fourier series, namely, for $n\in\mathbb{Z}^d$,*

$$f_n=\int_{\mathbb{T}^d}\hat{f}(x)e^{-2\pi i n\cdot x}dx.$$

Then the following statements are true:

(i) *If $\hat{f}$ is an entire function and $|\hat{f}(z)|\leq Ce^{C|z|^r}$ for some $C>0$ and $r>1$, then for any $0<w<\frac{r}{r-1}$,*

$$|f_n|\leq e^{-|n|^w},$$

for large enough n.

(ii) *If $|f_n|\leq Ce^{-C^{-1}|n|^r}$ for some $C>0$ and $r>1$, then $\hat{f}$ is an entire function and there exists a constant C_1 (depending on C and dimension d) such that*

$$|\hat{f}(z)|\leq e^{C_1|z|^{\frac{r}{r-1}}},$$

for large enough $|z|$.

Proof. Fix any large $n=(n_1,n_2,\ldots,n_d)\in\mathbb{Z}^d$. Without loss of generality, assume $n_1>0$ and $n_1=\max\{|n_1|,|n_2|,\ldots,|n_d|\}$. Then for any $\tilde{w}<\frac{1}{r-1}$,

$$\begin{aligned}|f_n|&=\left|\int_{\mathbb{T}^d}\hat{f}(x)e^{-2\pi i n\cdot x}dx\right|\\&=\left|\int_{\mathbb{T}^{d-1}}e^{-2\pi i(n_2x_2+\cdots n_dx_d)}dx_2\cdots dx_d\int_{\substack{z_1=x-in_1^{\tilde{w}}\\ x\in\mathbb{T}}}\hat{f}(z)e^{-2\pi i n_1z_1}dz_1\right|\\&\leq Ce^{Cn_1^{r\tilde{w}}}e^{-2\pi n_1^{1+\tilde{w}}}\\&\leq e^{-n_1^{1+\tilde{w}}},\end{aligned}$$

for large $|n|$. This proves (i).

Obviously,

$$\hat{f}(z)=\sum_{n\in\mathbb{Z}^d}f_ne^{2\pi i n\cdot z}.$$

Then one has

$$\begin{aligned}|\hat{f}(z)| &\leq \sum_{n\in\mathbb{Z}^d} Ce^{-C^{-1}|n|^r}e^{C|n||z|}\\ &\leq \sum_{l=1}^{\infty} Cl^d e^{-C^{-1}l^r}e^{Cl|z|}\\ &\leq e^{C|z|^{\frac{r}{r-1}}},\end{aligned}$$

for any large z. This completes the proof of (ii). □

LEMMA 6.3. *Let f and g be entire functions on $\mathbb{C}^d$. Assume that for some $C_1 > 0, \rho > 0$,*

$$|f(z)| \leq C_1 e^{C_1|z|^\rho}, |g(z)| \leq C_1 e^{C_1|z|^\rho}. \tag{52}$$

Assume that $h = g/f$ is also an entire function on $\mathbb{C}^d$. Then there exists a constant C such that

$$|h(z)| \leq Ce^{C|z|^\rho}.$$

REMARK 9. Lemma 6.3 is well known, e.g., see Theorem 5 of Section 11.3 in [41] for $d = 1$ and p.37 in [32] for $d \geq 2$.

The following Lemma can be obtained by a straightforward computation. For example, see p.49 in Bourgain–Klein [7] or Lyubarskii–Malinnikova [49].

LEMMA 6.4. *Let $\tilde{V} : \mathbb{Z}^d \to \mathbb{C}$ be bounded. Assume that u is a non-trivial solution of*

$$(-\Delta + \tilde{V})u = 0.$$

Then for some constant $C > 0$,

$$\sup_{|n|=R} (|u(n)| + |u(n-1)|) \geq e^{-CR}.$$

We are ready to prove Theorem 2.8.

Proof of Theorem 2.8. Suppose there exists $\lambda \in (a_m, b_m)$ such that $\lambda \in \sigma_p(H)$. Then there exists a non-zero function $u \in \ell^2(\mathbb{Z}^d)$ such that

$$-\Delta u + Vu + vu = \lambda u,$$

or

$$(H_0 - \lambda I)u = -vu. \tag{53}$$

Denote by the function on the right hand side by $\psi(n)$:

$$\psi(n) = -v(n)u(n), n \in \mathbb{Z}^d.$$

Applying U on both sides of (53), one has

$$((\hat{H}_0 - \lambda I)\hat{u})(x,l) = \hat{\psi}(x,l), \tag{54}$$

where $\hat{u}(x,l) \in L^2(\mathcal{B} \times \bar{W})$. For any fixed x, we regard both $\hat{u}(x,\cdot)$ and $\hat{\psi}(x,\cdot)$ as vectors on $\bar{W}$. Therefore, for any $x \in \mathcal{B}$,

$$(\tilde{H}_0(x) - \lambda I)\hat{u}(x,\cdot) = \hat{\psi}(x,\cdot). \tag{55}$$

By the assumption (15) and Lemma 6.2, we have that for any $l \in \bar{W}$,

$$|\hat{\psi}(x,l)| \leq Ce^{C|x|^{\frac{\gamma}{\gamma-1}}}. \tag{56}$$

From Lemma 3.2 ($\tilde{H}_0(x)$ is unitarily equivalent to $\tilde{D}(x)$), one can see that $\det(\tilde{H}_0(x) - \lambda I) = \tilde{P}(x,\lambda)$. By the Cramer's rule, we have

$$(\tilde{H}_0(x) - \lambda I)^{-1} = \frac{\tilde{S}(x,\lambda)}{\tilde{P}(x,\lambda)},$$

where $\tilde{S}(x,\lambda)$ is the adjoint matrix of $\tilde{H}_0(x) - \lambda I$. This concludes that

$$\hat{u}(x,\cdot) = \frac{\tilde{S}(x,\lambda)\hat{\psi}(x,\cdot)}{\tilde{P}(x,\lambda)}.$$

When λ satisfies Condition 1, by (11), one can see that $\zeta(x) = \tilde{P}(x,\lambda)$ satisfies the assumption of Theorem 6.1. Since $\hat{u}(x,l) \in L^2(\mathcal{B} \times \bar{W})$, namely for any fixed $l \in \bar{W}$, $\hat{u}(x,l) \in L^2(\mathcal{B})$, by Theorem 6.1, one has that $\hat{u}(x,l)$ is an entire function in the variable x for any $l \in \bar{W}$. Since all non-constant entries (in variables x) of $\tilde{H}_0(x) - \lambda I$ are consisted of $e^{2\pi i x_j}$ and $e^{-2\pi i x_j}$, we have that

$$||\tilde{S}(x,\lambda)|| \leq Ce^{C|x|}, |\tilde{P}(x,\lambda)| \leq Ce^{C|x|}. \tag{57}$$

By (56) and (57), one has that $\tilde{P}(x,\lambda)$ satisfies (52) with $\rho = 1$ and for any $l \in \bar{W}$, $(\tilde{S}\hat{\psi})(x,l)$ satisfies (52) with $\rho = \frac{\gamma}{\gamma-1}$. By Lemma 6.3, we have that for any $l \in \bar{W}$,

$$|\hat{u}(x,l)| \leq Ce^{C|x|^{\frac{\gamma}{\gamma-1}}}.$$

By Lemma 6.2, we have that for any w with $w < \gamma$,

$$|u(n)| \leq Ce^{-|n|^w}.$$

This is contradicted to Lemma 6.4. □

7 Proof of Theorem 2.5

Proof. (Proof of Theorem 2.5) Clearly, $(k, \lambda = \lambda_j(k))$, $j = 1, 2, \ldots, Q$, is one branch of solutions of equation

$$P(k, \lambda) = \mathcal{P}(e^{2\pi i k_1}, e^{2\pi i k_2}, \ldots, e^{2\pi i k_d}, \lambda) = 0, \tag{58}$$

and

$$P(k, \lambda) = \prod_{j=1}^{Q} (\lambda_j(k) - \lambda). \tag{59}$$

Assume that $k_0 = (k_0^1, k_0^2, \ldots, k_0^d)$ satisfies $\lambda_m(k_0) = \lambda_*$. Considering the matrix $D(k_0)$, let $m_1 \geq 1$ be the multiplicity of its eigenvalue λ_*.

Case 1: $m_1 = 1$. It means $\lambda = \lambda_*$ is a single root of $P(k_0, \lambda) = 0$. Then $\partial_\lambda P(k_0, \lambda)|_{\lambda=\lambda_*} \neq 0$. By the implicit function theorem, $\lambda_m(k)$ is an analytic function in a neighborhood of k_0. Since $\lambda_* = \lambda_m(k_0)$ is an extremum, one has

$$\nabla_k \lambda_m(k)|_{k=k_0} = (0, 0, \ldots, 0). \tag{60}$$

Rewrite (59) as

$$P(k, \lambda_*) = (\lambda_m(k) - \lambda_*) T(k), \tag{61}$$

where $T(k)$ is analytic in a neighborhood of k_0. By (60) and (61), we have

$$\nabla_k P(k, \lambda_*)|_{k=k_0} = (0, 0, \ldots, 0). \tag{62}$$

Case 2: $m_1 \geq 2$.

We will show that (62) still holds in this case. Without loss of generality, we only prove that

$$\partial_{k_1} P(k, \lambda_*)|_{k=k_0} = 0. \tag{63}$$

In order to prove (63), it suffices to show that

$$\partial_{k_1} P(k_1, k_0^2, \ldots, k_0^d, \lambda_*)|_{k_1=k_0^1} = 0. \tag{64}$$

By the Kato-Rellich perturbation theory [25], there exists $\tilde{\lambda}_l(k_1)$, $l = 1, 2, \ldots, m_1$, such that in a neighborhood of k_0^1, $\tilde{\lambda}_l(k_1)$ is analytic, $\tilde{\lambda}_l(k_0^1) = \lambda_*$ and $\tilde{\lambda}_l(k_1)$ is an eigenvalue of $D(k_1, k_0^2, \ldots, k_0^d)$, $l = 1, 2, \ldots, m_1$. Moreover,

$$P(k_1, k_2^0, \ldots, k_d^0, \lambda_*) = T(k_1) \prod_{l=1}^{m_1} (\tilde{\lambda}_l(k_1) - \lambda_*), \tag{65}$$

where $T(k_1)$ is analytic in a neighborhood of k_0^1. Now (64) follows from (65). We complete the proof. □

8 Proof of Theorems 1.4, 1.5 and 1.8

Proof of Theorem 1.4. By Lemma 5.2, the polynomial $z_1^{q_2}z_2^{q_1}\mathcal{P}(z,\lambda)$ (as a function of z_1 and z_2) is square-free for any λ. By Bézout's theorem, we have that

$$\#\{z \in (\mathbb{C}^\star)^2 : \mathcal{P}(z,\lambda_*) = 0, |\nabla_z \mathcal{P}(z,\lambda_*)| = 0\} \leq 4(q_1+q_2)^2,$$

and hence

$$\#\{k \in [0,1)^2 : P(k,\lambda_*) = 0, |\nabla_k P(k,\lambda_*)| = 0\} \leq 4(q_1+q_2)^2, \tag{66}$$

Now Theorem 1.4 follows from (12) and (66). □

Proof of Theorem 1.5. By Lemma 5.2, $z_1^{\frac{Q}{q_1}} z_2^{\frac{Q}{q_2}} \ldots z_d^{\frac{Q}{q_d}} \mathcal{P}(z,\lambda_*)$ is square-free, then by the basic fact of analytic sets (e.g., Corollary 4 in p.69 [52]), the analytic set $\{z \in (\mathbb{C}^\star)^d : \mathcal{P}(z,\lambda_*) = 0, |\nabla_z\mathcal{P}(z,\lambda_*)| = 0\}$ has (complex) dimension at most $d-2$. Since the real dimension of a real analytic set is always smaller than or equal to the complex dimension (e.g., p.63 in [52]), one has that $\{k \in [0,1)^d : P(k,\lambda_*) = 0, |\nabla_k P(k,\lambda_*)| = 0\}$ has dimension at most $d-2$. Now Theorem 1.5 follows from (12). □

Remark 10. In the proof of Theorems 1.4 and 1.5, we only use the fact that the polynomial $z_1^{\frac{Q}{q_1}} z_2^{\frac{Q}{q_2}} \ldots z_d^{\frac{Q}{q_d}} \mathcal{P}(z,\lambda_*)$ (as a function of z) is square-free.

Lemma 8.1 [38, Lemma 4]. *Let $d \geq 2$. Assume $\lambda \in (a_m, b_m)$ for some m. Then the Fermi variety $F_\lambda(V)$ contains an open analytic hypersurface of dimension $d-1$ in $\mathbb{R}^d$.*

Proof of Theorem 1.8. For $d = 1$, $H_0 + v$ does not have embedded eigenvalues if $v(n) = \frac{o(1)}{|n|}$ as $n \to \infty$ [47]. Therefore, it suffices to prove Theorem 1.8 for $d \geq 2$.

By Lemma 8.1, if $\lambda \in \cup(a_m, b_m)$ and $F_\lambda(V)$ is irreducible, then λ satisfies Condition 1. For $d = 2$, if $F_\lambda(V)$ is reducible, by Theorem 1.2, $\lambda = [V]$. By (41), $\lambda = [V]$ satisfies Condition 1. For $d \geq 3$, by Theorem 1.1, the Condition 1 holds for every $\lambda \in \cup(a_m, b_m)$. Now Theorem 1.8 follows from Theorem 2.8. □

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Appendix A. Proof of Claim 1

Proof. Otherwise, $\mathcal{P}_1(z,\lambda)$ has two non-trivial polynomial factors $f(z)$ and $g(z)$ such that both $\{z\in\mathbb{C}^d : f(z)=0\}$ and $\{z\in\mathbb{C}^d : g(z)=0\}$ contain $z_1=z_2=\cdots=z_d=0$. Let

$$\tilde{f}(z)=f\big(z_1^{q_1},z_2^{q_2},\ldots,z_d^{q_d}\big),\tilde{g}(z)=g\big(z_1^{q_1},z_2^{q_2},\ldots,z_d^{q_d}\big).$$

Let $\tilde{f}_1(z)$ ($\tilde{g}_1(z)$) be the component of the lowest degree of $\tilde{f}(z)$ ($\tilde{g}(z)$). Since both $\{z\in\mathbb{C}^d : f(z)=0\}$ and $\{z\in\mathbb{C}^d : g(z)=0\}$ contain $z_1=z_2=\cdots=z_d=0$, one has that $\tilde{f}_1(z)$ and $\tilde{g}_1(z)$ are non-constant.
Since both $\tilde{f}(z)$ and $\tilde{g}(z)$ are polynomials of $z_1^{q_1},z_2^{q_2},\ldots,z_d^{q_d}$, we have $\tilde{f}_1(z)$ and $\tilde{g}_1(z)$ are also polynomials of $z_1^{q_1},z_2^{q_2},\ldots,z_d^{q_d}$ and hence there exist $f_1(z)$ and $g_1(z)$ such that

$$\tilde{f}_1(z)=f_1\big(z_1^{q_1},z_2^{q_2},\ldots,z_d^{q_d}\big),\tilde{g}_1(z)=g_1\big(z_1^{q_1},z_2^{q_2},\ldots,z_d^{q_d}\big).$$

By (28) and (29), one has

$$\tilde{f}_1(z)\tilde{g}_1(z)=\tilde{h}_1(z)$$

and hence

$$f_1(z)g_1(z)=h_1(z).$$

This is impossible since $h_1(z)$ is irreducible. □

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