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THE SHARP UPPER BOUND FOR THE AREA OF THE NODAL SETS OF DIRICHLET LAPLACE EIGENFUNCTIONS

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Abstract. Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1 boundary and let u_λ be a Dirichlet Laplace eigenfunction in Ω with eigenvalue λ. We show that the $(n-1)$-dimensional Hausdorff measure of the zero set of u_λ does not exceed $C(\Omega)\sqrt{\lambda}$. This result is new even for the case of domains with C^∞-smooth boundary.

1 Introduction

Let Δ_M be the Laplace operator on an n-dimensional smooth compact Riemannian manifold and let u_λ be an eigenfunction of $-\Delta_M$ with the eigenvalue λ, i.e., $\Delta_M u_\lambda + \lambda u_\lambda = 0$. Denote by $Z(u_\lambda) = \{u_\lambda = 0\}$ the zero set of u_λ. S. T. Yau [21] conjectured that the surface area of the zero set of u_λ satisfies the following inequalities

$$c\sqrt{\lambda} \le \mathcal{H}^{n-1}(Z(u_\lambda)) \le C\sqrt{\lambda},$$

where the constants c, C depend on M. This conjecture was proved by Donnelly and Fefferman in [6] under the assumption that the metric is real analytic. The lower bound and a polynomial in λ upper bound were obtained recently by the first author in [16] and [15] respectively.

In this article we consider the case of eigenfunctions of the Euclidean Laplace operator on a bounded domain with sufficiently regular boundary and the Dirichlet boundary condition. One of our results is the following.

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1 boundary and let u_λ be an eigenfunction of the Laplace operator with the Dirichlet boundary condition, $\Delta u_\lambda + \lambda u_\lambda = 0$ and $u_\lambda|_{\partial\Omega} = 0$. Then*

$$\mathcal{H}^{n-1}(Z(u_\lambda)) \le C\sqrt{\lambda}, \tag{1}$$

where C depends only on Ω.

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The lower bound

$$\mathcal{H}^{n-1}(Z(u_\lambda)) \geq c\sqrt{\lambda},$$

for sufficiently large λ, follows from the results of Donnelly and Fefferman in [6] combined with Lemma 10 below. We remark that this bound also holds for any solution of the equation $\Delta u_\lambda + \lambda u_\lambda = 0$ and the boundary condition plays no role. This follows from the fact that the zero set is $C\lambda^{-1/2}$ dense and a non-trivial result of [16]. The inequality (1) was also proved by Donnelly and Fefferman in [7] for the case of real analytic boundary $\partial\Omega$. Their result was generalized to eigenfunctions of elliptic operators with real analytic coefficients by Kukavica [9]. Similar estimates were recently obtained by Lin and Zhu [14] for eigenfunctions of the bi-Laplace operator with various boundary conditions under the assumption that the boundary is real analytic. Also, the polynomial (in the eigenvalue) upper bounds for the area of the zero set of the Dirichlet, Neumann, and Robin eigenfunctions in smooth bounded domains in $\mathbb{R}^n$ were proved by Zhu in [22].

Our proof of Theorem 1 is based on the results of Donnelly and Fefferman and the ideas developed in [15–17]. In particular, we reduce the statement of the theorem to an estimate of the size of the nodal set of a harmonic function with controlled doubling index (the doubling index in defined in Section 3 below). The novelty of the current work is the treatment of domains with non-analytic boundaries. More precisely, we work with Lipschitz domains in the Euclidean space and assume that (locally) the Lipschitz constant is small enough; the precise definition and the formulation of the main result are given in the next section. This class of domains was recently considered by Tolsa [20] in a different problem.

The rest of the article is organized in the following way. In Section 3 we first discuss the doubling index of harmonic functions and its (weak) monotonicity properties near the boundary of Lipschitz domains with small Lipschitz constant, and then we formulate the main estimate for the size of the zero set of harmonic functions in terms of the doubling index, see Theorem 2 below. Two auxiliary results are contained in Section 4, where the low regularity of the boundary requires some careful considerations. We prove Theorem 2 for harmonic functions in Section 5, and explain how Theorem 1 follows from Theorem 2 in Section 6.

2 Preliminaries

2.1 Smoothness of the boundary. Some of the tools used in the current paper should be compared to those in [20], where the following *boundary uniqueness conjecture* is studied.

Let h be a bounded harmonic function in a Lipschitz domain Ω. Assume that h vanishes on a relatively open set $U \subset \partial\Omega$ and ∇h vanishes on a subset of U of positive surface measure. Then $h = 0$.

Recently Xavier Tolsa verified the conjecture for Lipschitz domains with small Lipschitz constant, see [20]. We use the following definition.

Definition 1. Let Ω be a domain in $\mathbb{R}^d$, $\tau \in (0,1)$, and let $B = B(x,r)$ be a ball centred on $\partial\Omega$. We say that $\partial\Omega$ is τ-Lipschitz in B if there is an isometry $T : \mathbb{R}^d \to \mathbb{R}^d$ and a function $f : B^{d-1}(0,r) \to \mathbb{R}$ such that $T(0) = x$, f is a Lipschitz function with the Lipschitz constant bounded by τ, $f(0) = 0$, and

$$\Omega \cap B = T\left(\{(y', y'') \in B^d(0,r) \subset \mathbb{R}^{d-1} \times \mathbb{R} : y'' > f(y')\}\right).$$

In this case we write $\partial\Omega \cap B \in Lip(\tau)$.

Remark 1. Our considerations are mostly local. When considering the part of the boundary $\partial\Omega \cap B \in Lip(\tau)$, we choose local coordinates in $B = B(x,r)$, $x \in \partial\Omega$, so that the isometry T in the definition is the identity. We denote by e_d the unit vector in the direction of the last coordinate, so that $x + \varepsilon e_d \in \Omega$ for $0 < \varepsilon < r$.

Remark 2. Note that if $\partial\Omega \cap B \in Lip(\tau)$ and $B_1 \subset B$ is a ball centred on $\partial\Omega$, then $\partial\Omega \cap B_1 \in Lip(\tau)$. Also, rescaling does not change the Lipschitz constant. So if $\partial\Omega \cap B \in Lip(\tau)$ and $x \in \partial\Omega$ is the centre of B, then, denoting $\Omega_c = \{x + c(y-x) : y \in \Omega\}$ and $B_c = cB = \{x + c(y-x) : y \in B\}$ for some $c > 0$, we have $\partial\Omega_c \cap B_c \in Lip(\tau)$.

Definition 2. We say that Ω is a Lipschitz domain with local Lipschitz constant τ if there exists $r > 0$ such that $\partial\Omega \cap B(x,r) \in Lip(\tau)$ for any $x \in \partial\Omega$.

Clearly, any bounded C^1 domain is a domain with local Lipschitz constant τ for any positive τ. So Theorem 1 follows from the next result.

Theorem 1′. *For each n, there exists $\tau_n > 0$ such that the following statement holds. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with local Lipschitz constant τ_n and let u_λ be an eigenfunction of the Laplace operator in Ω with the Dirichlet boundary condition, $\Delta u_\lambda + \lambda u_\lambda = 0$ and $u_\lambda|_{\partial\Omega} = 0$. Then*

$$\mathcal{H}^{n-1}(Z(u_\lambda)) \le C\sqrt{\lambda},$$

where C depends only on Ω.

The constant C depends only on the parameter r for Ω in the definition of a Lipschitz domain with local Lipschitz constant τ_n, on the diameter of Ω, and on the dimension n. In what follows we assume that the dimension of the ambient Euclidean space is fixed so usually we will not emphasize the dependence of our constants on it.

The rest of the article is devoted to a proof of Theorem 1′. We start with the following property of Lipschitz domains.

Lemma 1. *Suppose that $\partial\Omega \cap B \in Lip(\tau)$, $\tau < 1/4$, where $B = B(x,r)$ and $x \in \partial\Omega$. We choose coordinates as in Remark 1. Let $x_0 \in \overline{\Omega} \cap \frac{1}{4}B$, and let $x_1 = x_0 + \tau r e_d$. Then $\Omega \cap B(x_1, r/2)$ is star-shaped with respect to x_1.*

A version of this lemma can be found in [10]. We provide a proof for the convenience of the reader.

Proof. Let $x_1 = (x_1', x_1'')$. Suppose that $x_2 = (x_2', x_2'') \in \Omega \cap B(x_1, r/2)$. Let now $x_3 = (x_3', x_3'')$ be a point on the interval (x_1, x_2). Clearly $x_3 \in B(x_1, r/2)$. We want to check that $x_3'' > f(x_3')$.

Let $x_3 = ax_1 + (1-a)x_2$, $a \in (0,1)$. We have $x_1'' \geq f(x_1') + \tau r$ and $x_2'' > f(x_2')$. Therefore, we obtain

$$x_3'' = ax_1'' + (1-a)x_2'' > af(x_1') + a\tau r + (1-a)f(x_2').$$

Then, since $f(x_1') \geq f(x_2') - \tau|x_1' - x_2'| > f(x_2') - \tau r/2$, we have

$$x_3'' > f(x_2') + a\tau r - \frac{a\tau r}{2} = f(x_2') + \frac{a\tau r}{2} > f(x_3'),$$

where the last inequality holds since $|x_3' - x_2'| = a|x_1' - x_2'| < ar/2$ and f is τ-Lipschitz. □

2.2 Some observations. In this section we recall some results about harmonic functions.

Suppose that h is a harmonic function in Ω, $h \in C(\overline{\Omega})$, and $h = 0$ on $\partial\Omega \cap B$, where $B = B(x, r)$ and $x \in \overline{\Omega}$. We define the function v in B by $v = h^2$ in $\Omega \cap B$, and $v = 0$ in $B \backslash \Omega$. Then v is subharmonic in B and the mean-value theorem implies that for any $y \in B(x, r/2) \cap \Omega$,

$$h^2(y) \leq \frac{1}{|B(y, r/2)|} \int_{B(y,r/2)\cap\Omega} h^2 \leq \frac{1}{|B(y, r/2)|} \int_{B(x,r)\cap\Omega} h^2, \tag{2}$$

where $|E|$ is the d-dimensional Lebesgue measure of the set E.

Another known fact that we use is the following quantitative version of the Cauchy uniqueness theorem.

Lemma 2. *Let B_+ be the half-ball,*

$$B_+ = \{(x', x'') \in \mathbb{R}^{d-1} \times \mathbb{R} : |x'|^2 + (x'')^2 < 1, x'' > 0\}.$$

There exist $\gamma \in (0,1)$ and $C > 0$ such that if h is harmonic in B_+, $h \in C^1(\overline{B}_+)$ and satisfies the inequalities $|h| \leq 1, |\nabla h| \leq 1$ in B_+ and $|h| \leq \varepsilon$, $|\partial_d h| \leq \varepsilon$ on $\Gamma = \{(x', x'') \in \overline{B}_+, x'' = 0\}$, $\varepsilon \leq 1$, then

$$|h(x)| \leq C\varepsilon^\gamma \quad \textit{when} \quad x \in \frac{1}{3}B_+ = \left\{(x', x'') : |x'|^2 + (x'')^2 < \frac{1}{9}, x'' > 0\right\}.$$

The reader can find a proof of a similar statement in [13] and a general result on second order elliptic PDEs in Lipschitz domains in [5]. A simple proof is also given in Section A.3 for the convenience of the reader.

3 The Doubling Index

3.1 The doubling index inside the domain. Let $h \in C(\overline{\Omega})$ be a non-zero harmonic function in a domain $\Omega \subset \mathbb{R}^d$. For each $x \in \overline{\Omega}$ and $r > 0$, we define

$$H_h(x,r) = \int_{B(x,r)\cap\Omega} h^2 \quad \text{and} \quad N_h(x,r) = \log \frac{H_h(x,2r)}{H_h(x,r)}, \tag{3}$$

and, with some abuse of language, we call $N_h(x,r)$ the doubling index of h in $B = B(x,r)$.

Assume first that $B(x,2R) \subset \Omega$, then

$$N_h(x,r) \le N_h(x,R), \quad \text{when} \quad r < R. \tag{4}$$

An elementary proof can be obtained by decomposing h into spherical harmonics, see, e.g., [19]. This is a simple and useful result, its various versions go back to the works of Landis [11, 12], Agmon [1], and Almgren [2].

Suppose that $B(x,4r) \subset \Omega$. Then we rewrite the inequality $N_h(x,r) \le N_h(x,2r)$ as

$$\left(\int_{B(x,2r)} h^2\right)^2 \le \int_{B(x,r)} h^2 \int_{B(x,4r)} h^2. \tag{5}$$

Similarly to (2), for any $y \in B(x,3r/2)$, we have

$$h^2(y) \le \frac{1}{|B(y,r/2)|}\int_{B(y,r/2)} h^2 \le \frac{1}{|B(y,r/2)|}\int_{B(x,2r)} h^2.$$

Finally, applying (5) and using the trivial bound of the L^2 norms by the L^∞ norms, we obtain

$$\sup_{B(x,3r/2)} |h| \le 2^d (\sup_{B(x,r)} |h|)^{1/2} (\sup_{B(x,4r)} |h|)^{1/2}. \tag{6}$$

3.2 The doubling index on the boundary. We need a version of the monotonicity formula (4) and the three ball inequality (6) near a part of the boundary on which the harmonic function vanishes. First, we recall a lemma that is proven in [10].

LEMMA 3 (Kukavica, Nyström). *Let Ω be a domain in $\mathbb{R}^d$ and let B_1 be a ball centred on $\partial\Omega$ such that $\partial\Omega \cap B_1$ is C^3 smooth. Let also $x \in \Omega$ be such that $\Omega \cap B(x,R)$ is star-shaped with respect to x, $B(x,R) \subset B_1$. Suppose that $h \in C(\overline{\Omega})$ is a non-zero harmonic function in Ω and $h = 0$ on $\partial\Omega \cap B_1$. Then*

$$\log \frac{H_h(x,r_2)}{H_h(x,r_1)} \le \frac{\log(r_2/r_1)}{\log(r_3/r_2)} \log \frac{H_h(x,r_3)}{H_h(x,r_2)}, \tag{7}$$

when $0 < r_1 < r_2 < r_3 < R$.

The assumption that the boundary of Ω is C^3 smooth implies that $h \in C^2(\overline{\Omega} \cap B_1)$, so every integration by parts in [10] can be easily justified.

Now we prove the following almost monotonicity property of the doubling index in Lipschitz domains.

LEMMA 4. *Let Ω be a domain in $\mathbb{R}^d$. For any $\varepsilon > 0$, there exists $\tau_\varepsilon > 0$ such that if $\tau < \tau_\varepsilon$, $\partial\Omega \cap B \in Lip(\tau)$, where $B = B(x, R)$, $x \in \partial\Omega$, and $h \in C(\overline{\Omega})$ is a non-zero harmonic function in Ω, $h = 0$ on $\partial\Omega \cap B$, then*

$$N_h(x_0, r) \leq (1+\varepsilon) N_h(x_0, 2r), \tag{8}$$

for any $x_0 \in \overline{\Omega} \cap \frac{1}{4}B$ and $r < R/16$.

We remark that a stronger result holds when the boundary of the domain is smooth. For example for the case of a $C^{1,Dini}$ domain the inequality (8) can be replaced by $N_h(x_0, r_1) \leq (1+\varepsilon) N(x_0, r_2)$ when $r_1 < r_2 < R/8$, see [10][1]. Thus for $C^{1,Dini}$ domains, we know that the doubling index over balls centred on $\partial\Omega \cap B$ stays uniformly bounded. We do not know if this is still true for Lipschitz domains. For the case of the domains with small Lipschitz constant, we can conclude only that the doubling index $N_h(x_0, r)$ does not grow faster than r^{-a} with some small positive a as $r \to 0$, which is sufficient for our purposes.

Proof. First we assume that $\partial\Omega \cap B$ is a graph of a C^3-smooth function. Let e_d be as in Remark 1 and let $x_1 = x_0 + 16\tau r e_d$. We assume that $\tau < 1/16$. Then by Lemma 1 we see that $B(x_1, 8r) \cap \Omega$ is star-shaped with respect to x_1. We apply (7) and obtain

$$\begin{aligned} N_h(x_0, r) &= \log \frac{H_h(x_0, 2r)}{H_h(x_0, r)} \leq \log \frac{H_h(x_1, (2+16\tau)r)}{H_h(x_1, (1-16\tau)r)} \\ &\leq \frac{\log((2+16\tau)/(1-16\tau))}{\log((4-16\tau)/(2+16\tau))} \log \frac{H_h(x_1, (4-16\tau)r)}{H_h(x_1, (2+16\tau)r)} \\ &\leq (1+O(\tau)) \log \frac{H_h(x_0, 4r)}{H_h(x_0, 2r)} = (1+\varepsilon) N_h(x_0, 2r), \end{aligned}$$

when τ is small enough.

We want to drop the assumption that $\partial\Omega \cap B$ is C^3 smooth. We fix the ball B and assume that $\partial\Omega \cap B$ is given by the graph of a Lipschitz function $f : B^{d-1}(0, R) \to \mathbb{R}$ with the Lipschitz constant bounded by τ. In this coordinate system the ball B is identified with $B^d(0, R)$

Let φ be a mollifier supported in the unit ball of $\mathbb{R}^{d-1}$ and let, as usual, $\varphi_\delta(x) = \delta^{-(d-1)} \varphi\left(\frac{x}{\delta}\right)$. We define $f_n = f * \varphi_{R/n} + \tau R/n$. Then $\{f_n\}$ is a sequence of C^3 smooth functions such that

$$f_n : B^{d-1}(0, (1-1/n)R) \to \mathbb{R}, \quad f(y') < f_n(y') < f(y') + 2\tau R/n,$$

[1] We refer the reader also to the preceding works [4] and [3] for related results.

and the Lipschitz constant of f_n is also bounded by τ. We also define

$$\Omega_n = \{y = (y', y'') \in B^d(0, (1 - 1/n)R) \subset \mathbb{R}^{d-1} \times \mathbb{R} : y'' > f_n(y')\}.$$

Clearly, $\Omega_n \subset B^d(0, (1 - 1/n)R) \cap \Omega$. Let

$$\Gamma_n = \{y = (y', y'') \in B^d(0, (1 - 1/n)R) : y'' = f_n(y')\}.$$

First, we see that $\delta_n = \sup_{\Gamma_n} |h|$ converge to zero as $n \to \infty$ since h is uniformly continuous on $\overline{\Omega} \cap \overline{B}$, $h = 0$ on $\partial\Omega$, and $\mathrm{dist}(y, \partial\Omega) < 2\tau R/n$ when $y \in \Gamma_n$.

Next, we consider the harmonic function h_n in Ω_n such that on $\partial\Omega_n$

$$h_n(x) = \begin{cases} h(x) - \delta_n, & \text{if } h(x) > \delta_n, \\ 0, & \text{if } |h(x)| \le \delta_n, \\ h(x) + \delta_n, & \text{if } h(x) < -\delta_n. \end{cases}$$

Clearly we have $h_n \in C(\overline{\Omega}_n)$, $h_n = 0$ on Γ_n, and, by the maximum principle, $|h - h_n| \le \delta_n$ in Ω_n. Thus $h_n \to h$ uniformly on compact subsets of $B \cap \Omega$.

We fix $x_0 \in \Omega \cap \frac{1}{4}B$ and $r \in (0, R/16)$. Then $x_0 \in \Omega_n \cap B(0, (1 - 1/n)R/4)$ and $r < (1 - 1/n)R/16$ for n large enough. Also, $|h_n| \le \max_{\overline{\Omega} \cap \overline{B}} |h|$ and $|(\Omega \cap B(x, R)) \backslash \Omega_n| \to 0$ as $n \to \infty$. Then we have

$$N_h(x_0, r) = \frac{\int_{B(x_0,2r)\cap\Omega} h^2}{\int_{B(x,r)\cap\Omega} h^2} = \lim_{n\to\infty} \frac{\int_{B(x_0,2r)\cap\Omega_n} h_n^2}{\int_{B(x_0,r)\cap\Omega_n} h_n^2} = \lim_{n\to\infty} N_{h_n}(x_0, r).$$

The inequality (8) is now obtained as the limit of the corresponding inequalities for h_n. Finally the required inequality (8) for $x_0 \in \partial\Omega \cap \frac{1}{4}B$ follows by taking the limit as $\varepsilon \to 0+$ of the corresponding inequalities for $x_0 + \varepsilon e_d$. □

COROLLARY 1. *Let $\partial\Omega \cap B \in Lip(\tau)$, $\tau < \tau_\varepsilon$, $B = B(x, R)$, $x \in \partial\Omega$, and let $x_1, x_2 \in \overline{\Omega} \cap \frac{1}{4}B$ with $|x_1 - x_2| < r/4$ and $r < R/8$. If $h \in C(\overline{\Omega})$ is a non-zero harmonic function in Ω such that $h = 0$ on $\partial\Omega \cap B$, then*

$$N_h(x_1, r/2) \le 3(1 + \varepsilon)^2 N_h(x_2, r).$$

Proof. Note that $B(x_1, r) \subset B(x_2, 2r)$ and $B(x_1, r/2) \supset B(x_2, r/4)$. Thus we obtain

$$N_h(x_1, r/2) \le \log \frac{\int_{B(x_2,2r)\cap\Omega} h^2}{\int_{B(x_2,r/4)\cap\Omega} h^2} = N_h(x_2, r/4) + N_h(x_2, r/2) + N_h(x_2, r).$$

Now Lemma 4 implies the required estimate. □

3.3 Three ball inequality. We apply the monotonicity lemma a number of times. First, we claim that it implies a version of the three ball theorem for harmonic functions vanishing on some part of the boundary.

LEMMA 5. *Let Ω be a domain in $\mathbb{R}^d$ and let B be a ball centred on $\partial\Omega$. We assume that $\partial\Omega \cap B \in Lip(\tau)$, where τ is small enough. Then for any function $h \in C(\overline{\Omega})$ harmonic in Ω and vanishing on $\partial\Omega \cap B$, we have*

$$\sup_{\frac{3}{2}B_0\cap\Omega} |h| \le 3^d (\sup_{B_0\cap\Omega} |h|)^{1/3} (\sup_{4B_0\cap\Omega} |h|)^{2/3},$$

for any ball B_0 with the centre in $\overline{\Omega} \cap \frac{1}{4}B$ and such that $16B_0 \subset B$.

Proof. Assume that $\tau < \tau_1$ given by Lemma 4 for the case $\varepsilon = 1$. Let $B_0 = B(x_0, r)$. We apply Lemma 4. Taking the exponentials, we obtain

$$\int_{2B_0\cap\Omega} h^2 \le \left(\int_{B_0\cap\Omega} h^2\right)^{1/3} \left(\int_{4B_0\cap\Omega} h^2\right)^{2/3}.$$

Then (2) and the trivial bound of the L^2-norm by the L^∞-norm imply that for any $y \in \frac{3}{2}B_0 \cap \Omega$,

$$h^2(y) \le \frac{1}{|B(y, r/2)|} \int_{2B_0\cap\Omega} h^2 \le 8^d \left(\sup_{B_0\cap\Omega} |h|\right)^{2/3} \left(\sup_{4B_0\cap\Omega} |h|\right)^{4/3}.$$

□

3.4 The maximal doubling index. Let Ω be a domain in $\mathbb{R}^d$ and let $\partial\Omega \cap B \in Lip(\tau)$ where B is centred on $\partial\Omega$. We consider a closed cube $Q \subset \frac{1}{32}B$ such that $Q \cap \Omega \neq \varnothing$. Assume that a non-zero function $h \in C(\overline{\Omega})$ is harmonic in Ω and vanishes on $\partial\Omega \cap B$ and let $\ell = \operatorname{diam}(Q)$. We define the maximal doubling index of h in Q by

$$N_h^*(Q) = \sup_{x \in Q\cap\overline{\Omega}, \frac{\ell}{2} \le r \le \ell} N_h(x, r). \tag{9}$$

Clearly the function $(x, r) \mapsto N_h(x, r)$ is continuous on $(Q \cap \overline{\Omega}) \times [\ell/2, \ell]$. Therefore the supremum above is finite.

Lemma 4 on the monotonicity of the doubling index implies that if $\varepsilon > 0$ and $\tau < \tau_\varepsilon$, then for any cube $Q_1 \subset Q \subset \frac{1}{32}B$ and $Q_1 \cap \Omega \neq q\varnothing$, we have

$$N_h^*(Q_1) \le \left(\frac{2s(Q)}{s(Q_1)}\right)^{2\varepsilon} N_h^*(Q),$$

where $s(Q)$ is the side length of the cube Q; we have used the inequality $\log_2(1+\varepsilon) \le 2\varepsilon$.

3.5 A version of the main result for harmonic functions. Let Ω be a domain in $\mathbb{R}^d$ and let $h \in C(\overline{\Omega})$ be a non-zero harmonic function in Ω. We assume that $h = 0$ on the part $\partial\Omega \cap B$ of the boundary, where B is a ball centred on $\partial\Omega$ and $\partial\Omega \cap B \in Lip(\tau)$. Our aim is to estimate the $(d-1)$-dimensional measure of the zero set of h using the doubling index of h. We define the zero set of h by

$$Z(h) = \{x \in \Omega : h(x) = 0\},$$

so that the boundary points are not included into the zero set.

Theorem 2. *Let $\Omega \subset \mathbb{R}^d$, let $x \in \partial\Omega$ and let $r > 0$ be such that $\partial\Omega \cap B(x, 128r) \in Lip(\tau)$, where τ is small enough. Then there exists $C = C(d)$ such that*

$$\mathcal{H}^{d-1}(Z(h) \cap B(x,r)) \le C(N_h(x, 4r) + 1)r^{d-1},$$

for any non-zero function $h \in C(\overline{\Omega})$ that is harmonic in Ω and satisfies $h = 0$ on $\partial\Omega \cap B(x, 128r)$.

Theorem 2 is proved in Section 5.2. We then deduce Theorem 1' in Section 6.2, where we consider the harmonic extension of the eigenfunction and use Lemma 10 below to estimate the doubling index of the extension by a multiple of the square root of the eigenvalue.

Theorem 2 allows us to estimate the area of the zero set of a harmonic function near the part of the boundary where the function vanishes. We remark also that the estimate for the zero set inside the domain was proved by Donnelly and Fefferman in [6].

Lemma 6 (Donnelly, Fefferman). *Let h be a non-zero harmonic function in $\Omega \subset \mathbb{R}^d$. There exists C such that*

$$\mathcal{H}^{d-1}(Z(h) \cap B) \le C(N_h(x, 4r) + 1)r^{d-1},$$

for any ball $B = B(x, r)$ that satisfies $\overline{B}(x, 8r) \subset \Omega$.

The proof follows from the argument in [6], some versions of this result can be also found in [13] and [8]. We outline some steps of the proof for the interested reader in the Appendix, see A.1.

4 Two Auxiliary Lemmas

4.1 A standard construction. In this section we give two versions of the Hyperplane Lemma. We suggest that the reader compares the statements to the one of [15, Lemma 4.1]. Both statements refer to the following construction.

Assume that $\Omega \subset \mathbb{R}^d$ and $\partial\Omega \cap B \in Lip(\tau)$, where B is a ball centred on $\partial\Omega$ and $\tau \in (0, (16\sqrt{d})^{-1})$. We fix a coordinate system as in Remark 1. Let Q be a cube centred at $x_Q = (x'_Q, x''_Q) \in \partial\Omega \cap B$ whose sides are parallel to the axes of this coordinate system and such that $Q \subset B$. As above, the side length of Q is denoted

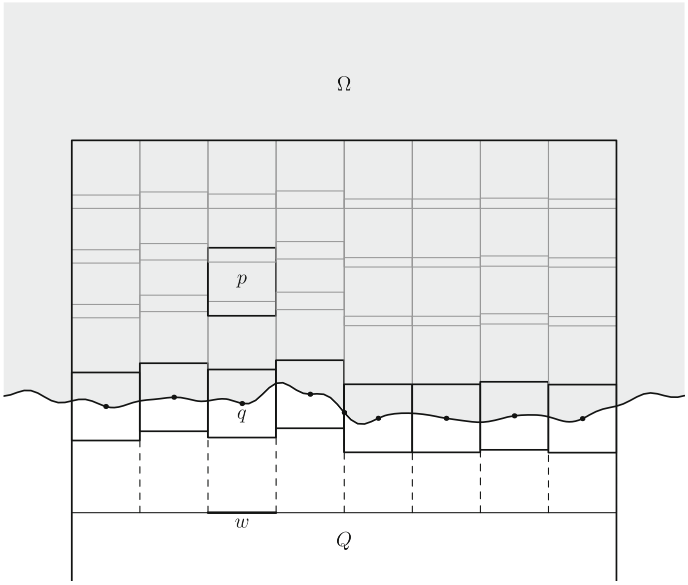

Figure 1: The standard construction.

by $s(Q)$. Our choice of τ implies that $\partial\Omega$ does not intersect the two faces of the cube Q which are orthogonal to e_d, moreover, $\partial\Omega \cap Q$ is contained in the middle part $\{(x', x'') \in Q : |x'' - x''_Q| < s(Q)/4\}$ of Q.

Let $k \geq 3$. We partition the projection $\pi(Q)$ of Q to the hyperplane $\mathbb{R}^{d-1} \times \{0\}$ into $2^{k(d-1)}$ small equal cubes w with the side length $s(w) = 2^{-k}s(Q)$ in the usual way so that any two distinct small cubes have no common inner points. For each small cube w, there is a uniquely defined d-dimensional cube q such that $\pi(q) = w$ and the centre of q lies on $\partial\Omega \cap Q$. Furthermore, we cover $(\pi^{-1}(w) \cap (\Omega \cap Q))\backslash q$ by at most 2^k cubes p such that $p \subset Q$, p, $\pi(p) = w$, p has no common inner points with q, and $s(p) = s(q) = 2^{-k}s(Q)$, cubes p may overlap. See Figure 1.

We denote the set of all boundary cubes q by $\mathcal{B}_k(Q)$ and the set of all inner cubes p by $\mathcal{I}_k(Q)$. Note that for each $p \in \mathcal{I}_k(Q)$, we have $\operatorname{dist}(p, \partial\Omega) > cs(p)$ for some absolute constant c. We call the triple $(Q, \mathcal{B}_k(Q), \mathcal{I}_k(Q))$ the standard construction. After we fix a coordinate system, our standard construction depends on the choice of the cube Q and the parameter k, the family $\mathcal{B}_k(Q)$ of the boundary cubes is defined uniquely and we may fix some choice for the inner cubes $\mathcal{I}_k(Q)$.

4.2 The first hyperplane lemma. In the first lemma we assume that the maximal doubling index $N_h^*(Q)$ is large enough.

LEMMA 7. *There exist constants $k_0 \geq 3$ and $N_0 \geq 1$ such that for any integer $k \geq k_0$, there exists $\tau(k) > 0$ for which the following statement holds. Suppose that Ω is a domain in $\mathbb{R}^d$, $\partial\Omega \cap B \in Lip(\tau)$, $\tau < \tau(k)$, and $Q \subset \frac{1}{64}B$ is a cube as above centred on $\partial\Omega$. Then for any function $h \in C(\overline{\Omega})$ harmonic in Ω, with $h = 0$ on $B \cap \partial\Omega$, and $N_h^*(Q) > N_0$, there exists a cube $q \in \mathcal{B}_k(Q)$ such that $N_h^*(q) \leq N_h^*(Q)/2$.*

Proof. Let x_Q be the centre of the cube Q and let $B_1 = B(x_Q, \ell)$, where $\ell = \mathrm{diam}(Q)$. We have $B_1 \subset B$ and define $M^2 = \int_{B_1\cap\Omega} h^2$.

Denote $N = N_h^*(Q)$ and suppose that the inequality $N_h^*(q) > N/2$ holds for each cube $q \in \mathcal{B}_k(Q)$. Then for each such q, there exist $y_q \in q \cap \overline{\Omega}$ and $r_q \in (2^{-k-1}\ell, 2^{-k}\ell)$ such that $N_h(y_q, r_q) > N/2$. Suppose that

$$\tau < \tau_\varepsilon, \quad \text{and} \quad (1+\varepsilon)^k < 2, \tag{10}$$

where we use the notation of Lemma 4. Then the almost monotonicity of the doubling index, Lemma 4, implies $N_h(y_q, 2^m r_q) > N/4$ when $0 \le m \le k$.

Assuming that $k \ge 20$, we apply the estimate of the doubling index $k-4$ times and use that $B(y_q, \ell/2) \subset B_1$ to obtain

$$\int_{B(y_q,2^{-k+2}\ell)\cap\Omega} h^2 \le \int_{B(y_q,8r_q)\cap\Omega} h^2 \le e^{-N(k-4)/4} \int_{B(y_q,2^{k-1}r_q)\cap\Omega} h^2$$
$$\le e^{-N(k-4)/4} \int_{B(y_q,\ell/2)\cap\Omega} h^2 \le e^{-Nk/5} M^2.$$

Next, we note that the integral estimate above implies a pointwise estimate in a smaller ball by (2). We have

$$\sup_{B(y_q,2^{-k+1}\ell)\cap\Omega} h^2 \le C 2^{dk} \ell^{-d} \int_{B(y_q,2^{-k+2}\ell)\cap\Omega} h^2 \le C 2^{dk} \ell^{-d} e^{-Nk/5} M^2, \tag{11}$$

where $C = C(d)$.

As above, we assume also that $\tau < (16\sqrt{d})^{-1}$. For each cube $q \in \mathcal{B}_k(Q)$, denote by q^+ its upper quarter, where "up" is in the direction of e_d. Then $q^+ \subset \Omega$ and $\mathrm{dist}(q^+, \partial\Omega) \ge 2^{-k} s(Q)/10$. For $y \in q^+$, the standard Cauchy estimate implies

$$|\nabla h(y)| \le C 2^k \ell^{-1} \sup_{B(y,2^{-k}s(Q)/10)} |h|.$$

We note that $B(y, 2^{-k}s(Q)/10) \subset B(y_q, 2^{-k+1}\ell) \cap \Omega$. Then combining the above inequality with (11), we obtain

$$\sup_{q^+} |\nabla h| \le C 2^k \ell^{-1} \sup_{B(y_q,2^{-k+1}\ell)\cap\Omega} |h| \le C 2^{k(d+2)/2} \ell^{-(d+2)/2} e^{-Nk/10} M. \tag{12}$$

Let $B_0 = B(x_Q + 3\cdot 2^{-k-3} s(Q) e_d, s(Q)/2)$ and let

$$B_{0,+} = \{x = (x', x'') \in B_0 : x'' \ge x''_Q + 3\cdot 2^{-k-3} s(Q)\}$$

be the upper half of B_0. We denote by Γ_0 the flat part of the boundary of $B_{0,+}$. We note that $2B_0 \subset B_1$. Assuming that $\tau < 2^{-k-3}$, we have $\mathrm{dist}(B_{0,+}, \partial\Omega) \ge 2^{-k-2} s(Q)$. Then using (2) and the Cauchy estimate, we get

$$\sup_{B_0\cap\Omega} |h| \le C\ell^{-d/2} M, \quad \sup_{B_{0,+}} |\nabla h| \le C 2^k \ell^{-d/2-1} M.$$

Also, by (11) and (12), we have

$$\sup_{\Gamma_0} |h| \le C 2^{kd/2} \ell^{-d/2} e^{-Nk/10} M, \quad \sup_{\Gamma_0} |\nabla h| \le C 2^{kd/2+k} \ell^{-d/2-1} e^{-Nk/10} M,$$

since $\Gamma_0 \subset \bigcup_{q \in \mathcal{B}_k(Q)} q^+$.

Applying Lemma 2 to $B_{0,+}$, we get

$$\sup_{\frac{1}{3}B_{0,+}} |h| \le C 2^{\gamma kd/2+k} \ell^{-d/2} e^{-\gamma Nk/10} M.$$

Let $y_Q = x_Q + s(Q) e_d/12$ and let m be the least integer such that $2^m > 16\sqrt{d}$. Then $B_2 = B(y_Q, 2^{-m}\ell) \subset \frac{1}{3} B_{0,+}$ when k is large enough (we remark that $B_{0,+}$ depends on k). Integrating the last inequality over B_2 and using that $vol(B_2) \le C\ell^d$, we obtain

$$\int_{B_2} h^2 \le C 2^{\gamma kd + 2k} e^{-\gamma Nk/5} M^2.$$

Finally, we compare the last integral to $\int_{B_1 \cap \Omega} h^2 = M^2$. Note that $B_1 \subset B(y_Q, 2\ell) = 2^{m+1} B_2$. By the almost monotonicity of the doubling index, recalling that $\tau < \tau_\varepsilon < \tau_1$, we have

$$2^{m+1} N_h(y_Q, \ell) \ge \sum_{j=0}^{m} N_h(y_Q, 2^{-j}\ell) = \log \frac{\int_{B(y_Q, 2\ell) \cap \Omega} h^2}{\int_{B_2} h^2}$$
$$\ge \log \frac{\int_{B_1 \cap \Omega} h^2}{\int_{B_2} h^2} \ge \gamma Nk/5 - \gamma kd - 2k - C.$$

Since $N_h(y_Q, \ell) \le N_h^*(Q) = N$, we get $2^{m+1} N \ge \gamma Nk/5 - \gamma kd - 2k - C$. Taking k large enough we may achieve $\gamma k/5 > 2^{m+2}$. Then the inequality above implies

$$N \le \frac{10(\gamma kd + 2k + C)}{\gamma k} \le 10\left(d + (2 + C)\gamma^{-1}\right).$$

Taking $N_0 = 10\left(d + (2 + C)\gamma^{-1}\right)$, we obtain a contradiction for $N > N_0$. We also choose $\varepsilon = \varepsilon(k)$ such that $(1 + \varepsilon)^k < 2$ and finally choose $\tau(k) = \min\{\tau_\varepsilon, 2^{-k-3}, (16\sqrt{d})^{-1}\}$. □

4.3 The second hyperplane lemma: cubes without zeros. For cubes with the maximal doubling index bounded by N_0, we use the following version of the above statement. The reader may compare it to Corollary in Section 3.4 of [18].

LEMMA 8. *For any $N > 0$ there exist $\tau(N)$ and $k(N)$ such that the following statement holds. Suppose that Ω is a domain in $\mathbb{R}^d$, $\partial\Omega \cap B \in Lip(\tau)$, $\tau < \tau(N)$, and $Q \subset \frac{1}{64} B$ is a cube centred on $\partial\Omega$. Let also $h \in C(\overline{\Omega})$ be a non-zero function harmonic in Ω, with $h = 0$ on $B \cap \partial\Omega$ and $N_h^*(Q) \le N$. Then for any $k \ge k(N)$, there exists $q \in \mathcal{B}_k(Q)$ such that $Z(h) \cap q = \varnothing$.*

We remark that in this version both τ and k depend on N. First, we prove the following version of the lemma for a half ball.

LEMMA 9. *Let B be the unit ball in $\mathbb{R}^d$ and let B_+ be the half ball,*

$$B_+ = \{y = (y', y'') \in \mathbb{R}^{d-1} \times \mathbb{R} : |y'|^2 + y''^2 < 1, y'' > 0\}.$$

Let g be a function harmonic in B_+, $g \in C(\overline{B}_+)$, $g = 0$ on $\overline{B}_+ \cap \{y'' = 0\}$, and

$$\sup_{\frac{1}{4}B_+} |g| = 1.$$

For any $N > 0$, there exist $\rho = \rho(N) \in (0, 1/16)$ and $c_0 = c_0(N) > 0$ such that if $N_g(0, 1/4) \le N$, then there is $x' \in \mathbb{R}^{d-1}$ with $|x'| < 1/16$ such that

$$|g(y)| \ge c_0 y'', \quad \textit{for any} \quad y = (y', y'') \in B((x', 0), \rho) \cap B_+.$$

Proof. Let B_- be the reflexion of the half-ball B_+ with respect to the hyperplane $y'' = 0$. Then g can be extended to a harmonic function in B by $g(y', y'') = -g(y', -y'')$ when $(y', y'') \in B_-$. We denote this extension by g as well. The normalization $\sup_{\frac{1}{4}B_+} |g| = 1$ and the standard Cauchy estimate imply that every partial derivative of g is uniformly bounded in $B(0, 1/8)$.

Let $\delta = \max_{x' \in \mathbb{R}^{d-1}, |x'| \le 1/16} |\nabla g(x', 0)|$. Lemma 2, applied to the half ball $\frac{1}{16}B_+$ implies that

$$\sup_{B(0,\frac{1}{64})} |g| \le C\delta^\gamma.$$

Then $\int_{B(0,\frac{1}{64})} g^2 \le C\delta^{2\gamma}$ and $\int_{B(0,\frac{1}{2})} g^2 \ge c \sup_{B(0,\frac{1}{4})} g^2 = c$. On the other hand,

$$\log \frac{\int_{B(0,\frac{1}{2})} g^2}{\int_{B(0,\frac{1}{64})} g^2} \le 5N_g(0, \frac{1}{4}) \le 5N.$$

We have used that the doubling index of g in B_+ and of the extension are the same for balls centred at the origin and that the doubling index inside the domain is monotone by (4). We conclude that $\delta \ge ce^{-3N\gamma^{-1}}$.

Let $x'_* \in \mathbb{R}^{d-1}$, $|x'_*| \le 1/16$, be such that $|\nabla g(x'_*, 0)| = \delta$. Clearly we have $|\nabla g(x'_*, 0)| = |\partial_d g(x'_*, 0)|$ and we may assume that $\partial_d g(x'_*, 0) = \delta$, otherwise we consider the function $-g$. Then $\partial_d g(x) > \delta/2$ when $\operatorname{dist}(x, (x'_*, 0)) < \rho = \min\{c_0\delta, 1/16\}$, where c_0 depends on the constant upper bound for the second derivatives of g in $B(0, 1/8)$. Therefore

$$g(y) \ge \delta y''/2 \ge ce^{-3N\gamma^{-1}} y'',$$

when $y = (y', y'') \in B((x'_*, 0), \rho)$. □

Proof of Lemma 8. Now we deduce Lemma 8 from Lemma 9. By rescaling, see Remark 2, we can achieve that $s(Q)=4$. We may also assume that

$$\sup_{B(x_Q,3)\cap\Omega}|h|=1. \tag{13}$$

Let $x_1=x_Q-3\tau e_d$, $B_1=B(x_1,1)$, and let $B_{1,+}$ be the upper half of B_1. Let also $B_2=2B_1$. First, we consider the harmonic function g_0 such that $g_0=1$ on the upper half of the sphere ∂B_2 and $g_0=-1$ on the lower half of ∂B_2. We denote as usual $x_1''=x_1\cdot e_d$. Clearly $g_0=0$ on

$$\Gamma_0=\{x=(x',x'')\in B_2: x''=x_1''\}$$

and $g_0\geq 0$ on $B_{2,+}$. We note that Γ_0 does not intersect $\overline{\Omega}$. Then $|h|\leq g_0$ on $\Omega\cap B_2\subset B_{2,+}$ by the maximum principle. We also have $g_0(x)\leq C_1(x''-x_1'')$ when $x=(x',x'')\in B_{1,+}$, since $g_0=0$ on Γ_0 and g_0 has bounded derivatives in B_1. Therefore $|h(x)|\leq g_0(x)\leq C_1(x''-x_1'')$ when $x=(x',x'')\in\Omega\cap B_1$.

Let now g be the harmonic function in $B_{1,+}$ such that $g=h$ on $\partial B_{1,+}\cap\Omega$ and $g=0$ on $\partial B_{1,+}\setminus\Omega$. We have $|g(x)|\leq C_1(x''-x_1'')$ in $B_{1,+}$ by the above estimate on h and the maximum principle. We consider the difference $g-h$. We have $g=h$ on $\Omega\cap\partial B_1$ and $|g-h|=|g|\leq 4C_1\tau$ on $\partial\Omega\cap B_1$. Then, by the maximum principle, $|g-h|\leq 4C_1\tau$ in $\Omega\cap B_1$. We extend h by zero to $B_{1,+}\setminus\Omega$. Then $|g-h|\leq 4C_1\tau$ in $B_{1,+}$.

Let m be the integer such that $2\sqrt{d}\leq 2^m<4\sqrt{d}$, clearly $m\geq 1$. Then the estimate $N_h^*(Q)\leq N$ implies $N_h(x_Q,2^m)\leq N$. We choose ε such that $(1+\varepsilon)^{m+3}\leq 2$ and assume that $\tau<\tau_\varepsilon$ using the notation of Lemma 4. Then $N_h(x_Q,2^j)\leq 2N$ when $-3\leq j\leq m$. We use (13) and (2) to conclude that

$$\int_{B(x_Q,\frac18)\cap\Omega}h^2\geq e^{-10N}\int_{B(x_Q,4)\cap\Omega}h^2\geq ce^{-10N}.$$

Suppose that $\tau<\frac{1}{24}$. Then $B(x_Q,\frac18)\cap\Omega\subset\frac14 B_{1,+}$ and we have

$$\left(\int_{\frac14 B_{1,+}}g^2\right)^{1/2}\geq\left(\int_{\frac14 B_{1,+}}h^2\right)^{1/2}-C_2\tau\geq\left(\int_{B(x_Q,\frac18)\cap\Omega}h^2\right)^{1/2}-C_2\tau.$$

Assuming that $\tau(N)$ is small enough, we conclude that

$$\int_{\frac14 B_{1,+}}g^2\geq c_1e^{-10N}. \tag{14}$$

We also have $\sup_{\frac12 B_{1,+}}|g|\leq\sup_{B_1\cap\Omega}|h|\leq 1$ by (13). Then

$$N_g(x_1,\frac14)=\log\frac{\int_{\frac12 B_{1,+}}g^2}{\int_{\frac14 B_{1,+}}g^2}\leq C(N+1).$$

We note that (14) implies $\sup_{\frac{1}{4}B_{1,+}}|g| \geq ce^{-5N}$. Then, by Lemma 9, there exist $x_* \in \Gamma_0 \cap \frac{1}{16}B_1$, $c_2 = c_2(N) > 0$, and $\rho = \rho(C(N+1))$ such that

$$|g(x)| \geq c_2(x'' - x_1'') \quad \text{for} \quad x = (x', x'') \in B(x_*, \rho) \cap B_{1,+}.$$

We may assume that $g > 0$ in $B(x_*, \rho) \cap B_{1,+}$, otherwise we consider $-h$ in place of h. Then we obtain

$$h(x) \geq g(x) - 4C_1\tau \geq c_2(x'' - x_1'') - 4C_1\tau \quad \text{in} \quad B(x_*, \rho) \cap \Omega.$$

We note that ρ does not depend on τ and for τ small enough we have $B(x_*, \frac{\rho}{4}) \cap \partial\Omega \neq \varnothing$. We also have $B(x_*, \frac{\rho}{2}) \subset Q$.

Our goal is to show that $h > 0$ on $B(x_*, \frac{\rho}{2}) \cap \Omega$. Let $y_* = (y_*', y_*'') \in B(x_*, \frac{\rho}{2}) \cap \partial\Omega$. We note that

$$h(x) \geq c_2(x'' - y_*'') - c_3\tau \quad \text{in} \quad B(x_*, \rho) \cap \Omega, \tag{15}$$

where $c_3 = 4C + 4c_2$. We consider the harmonic function

$$h_*(x) = \frac{1}{d\rho}\left((d-1)(x'' - y_*'')^2 - |x' - y_*'|^2\right),$$

where $x = (x', x'')$. We claim that $h(x) \geq c_2 h_*(x)$, when $x \in B\left(y_*, \frac{\rho}{2}\right) \cap \Omega$ and τ is small enough.

First, we note that $h_*(x) \leq 0$ if $|x'' - y_*''| \leq (d-1)^{-1/2}|x' - y_*'|$ and therefore $h_* \leq 0$ on $\partial\Omega \cap B(y_*, \frac{\rho}{2})$ when τ is small enough, while $h = 0$ on $\partial\Omega \cap B(y_*, \frac{\rho}{2})$. On $\partial B(y_*, \frac{\rho}{2}) \cap \Omega$ we have

$$h_*(x) = \frac{(x'' - y_*'')^2}{\rho} - \frac{\rho}{4d}.$$

Comparing (15) to the last identity and denoting $t = x'' - y_*''$, we reduce the inequality $h \geq c_2 h_*$ on $\partial B(y_*, \frac{\rho}{2}) \cap \Omega$ to the following one:

$$c_2 t - c_3\tau \geq c_2\left(\frac{t^2}{\rho} - \frac{\rho}{4d}\right),$$

when $t \in (-\tau\rho/2, \rho/2)$ and τ is small enough. It suffices to check the inequality for $t = -\tau\rho/2$ and $t = \rho/2$. For $t = -\tau\rho/2$ we obtain the inequality

$$\frac{c_2\rho}{4d} \geq \tau\left(\frac{c_2\rho}{2} + \frac{c_2\tau\rho}{4} + c_3\right),$$

which holds when τ is small enough. On the other hand, for $t = \rho/2$, the inequality is reduced to

$$c_2\rho\left(\frac{1}{4} + \frac{1}{4d}\right) \geq c_3\tau.$$

This one is also satisfied for small τ.

Thus, by the maximum principle, $h \geq c_2 h_*$ in $B(y_*, \rho/2) \cap \Omega$. In particular, $h(y_*', y'') \geq c_2 h_*(y_*', y'') > 0$ when $y_*'' < y'' < \rho/2$. Therefore $h > 0$ on $B(x_*, \frac{\rho}{2}) \cap \Omega$.

Finally, since $B(x_*, \rho/2)$ contains a ball of radius $\rho/4$ centred on $\partial\Omega$, if k is large enough, there is $q \in \mathcal{B}_k(Q)$ such that $q \subset B(x_*, \frac{\rho}{2})$ and then $Z(h) \cap q = \varnothing$. □

5 Proof of Theorem 2

Let N_0 be as in Lemma 7 and let Ω, $B = B(x,r)$, and h be as in the statement of Theorem 2. We remind that the maximal doubling index $N_h^*(Q)$ of h in a cube Q was defined by (9). For the rest of the proof we modify the maximal doubling index and write $N_h^{**}(Q) = \max\{N_h^*(Q), N_0/2\}$. Then Lemmas 7 and 8 imply that there is k such that for τ small enough, if $Q \subset 2B$ and $(Q, \mathcal{B}_k(Q), \mathcal{I}_k(Q))$ is a standard construction, then there is a cube $q_0 \in \mathcal{B}_k(Q)$ such that

$$\text{either} \quad \text{(i) } N_h^{**}(q_0) < N_h^{**}(Q)/2 \quad \text{or} \quad \text{(ii) } Z(h) \cap q_0 = \varnothing. \tag{16}$$

5.1 Reduction to one cube. Let Q be a cube as above. We claim that

$$\mathcal{H}^{d-1}(Z(h) \cap Q) \le C N_h^{**}(Q) s(Q)^{d-1}. \tag{17}$$

Assume first that (17) holds. We show that Theorem 2 follows. We need to switch from cubes to balls and from the maximal doubling index to the doubling index at a single point.

To this end, we cover the ball $B(x,r)$ with cubes $Q_j \subset B(x,2r)$ such that $\mathrm{diam}(Q_j) = r/10$ and either $\mathrm{dist}(Q_j, \partial\Omega) > s(Q_j)/10$ (inner cubes) or Q_j satisfies the assumptions in the main construction (boundary cubes). We may assume that there are not more than $C = C(d)$ of such cubes.

First, for each cube $Q = Q_j$ in this cover, we have $Q \cap B(x,r) \ne \varnothing$, and we compare $N_h^*(Q)$ to $N_h(x,4r)$. There exists $y \in Q \cap \overline{\Omega}$ and $r_y \in [r/20, r/10]$ such that $N_h^*(Q) = N_h(y, r_y)$. Assuming that $\tau < \tau_1$ in the notation of Lemma 4, we get $N_h(y, 32r_y) \ge 2^{-5} N_h^*(Q)$. We have $\mathrm{dist}(x,y) \le \frac{11}{10} r$ and

$$N_h(y, 32r_y) = \log \frac{\int_{B(y,64r_y)\cap\Omega} h^2}{\int_{B(y,32r_y)\cap\Omega} h^2} \le \log \frac{\int_{B(x,8r)\cap\Omega} h^2}{\int_{B(x,r/2)\cap\Omega} h^2} \le 16 N_h(x,4r)$$

by Lemma 4. Hence, $N_h^*(Q) \le 2^9 N_h(x,4r)$ and $N_h^{**}(Q) \le C(N_h(x,4r)+1)$.

Each inner cube $Q \subset \Omega$ can be covered by at most C balls b with centres in Q and with radii $s(Q)/100$. Then $8\overline{b} \subset \Omega$. Moreover, if $b = B(y, s(Q)/100)$, we have $N_h(y, s(Q)/25) \le C N_h^*(Q)$ by Lemma 4 again. Then we use Lemma 6 to estimate the area of the zero set of h in each of the balls b and obtain

$$\begin{aligned} \mathcal{H}^{d-1}(Z(h) \cap b) &\le C(N_h(y, s(Q)/25) + 1) r^{d-1} \\ &\le C'(N_h^*(Q)+1) r^{d-1} \le C''(N_h(x,4r)+1) r^{d-1}. \end{aligned}$$

For the boundary cubes, we use the inequality (17). Thus for every Q_j, we obtain

$$\mathcal{H}^{d-1}(Z(h) \cap Q_j) \le C(N_h(x,4r)+1) s(Q_j)^{d-1}.$$

Summing these inequalities over all cubes, we obtain the required estimate. It remains to prove (17).

5.2 Proof of (17). We fix a compact set $K \subset \Omega$ and prove that

$$\mathcal{H}^{d-1}(Z(h) \cap Q \cap K) \le C_0 N_h^{**}(Q) s(Q)^{d-1}, \tag{18}$$

where $Q \subset 2B$ is a cube as in the standard construction and C_0 is independent of K. Then (17) follows.

First, note that (18) holds for all cubes Q small enough, since $Q \cap K = \varnothing$ for such cubes. We prove (18) by induction on the size of Q, going from small cubes to larger ones. Assume that it holds for cubes with $s(Q) < s$, we want to prove it for cubes with $s(Q) < 2^k s$, where k is as in (16).

We consider the standard construction $(Q, \mathcal{B}_k(Q), \mathcal{I}_k(Q))$. Each inner cube $q \in \mathcal{I}_k(Q)$ can be covered by balls b centred in q with radii $s(q)/100$ and such that $8\bar{b} \subset \Omega$, so that the number of balls is bounded by a dimensional constant. For each such ball $b = B(y, s(q)/100)$, applying Lemma 4, we get $N_h(y, s(q)/25) \le C(k) N_h^*(Q)$ when τ is small enough. Then by Lemma 6, we have

$$\sum_{q \in \mathcal{I}_k(Q)} \mathcal{H}^{d-1}(Z(h) \cap q) \le C(N_h^*(Q) + 1) s(Q)^{d-1} \le C_1 N_h^{**}(Q) s(Q)^{d-1}, \tag{19}$$

where C and C_1 depend on k.

For all other boundary cubes q, we have $N_h^{**}(q) \le (1+\varepsilon)^k N_h^{**}(Q)$. Also (16) implies that there is a cube $q_0 \in \mathcal{B}_k(Q)$ such that either $N_h^{**}(q_0) \le N_h^{**}(Q)/2$ or $Z(h) \cap q_0 = \varnothing$. We apply the induction assumption to each boundary cube and obtain

$$\begin{aligned}
&\mathcal{H}^{d-1}(Z(h) \cap K \cap (\cup_{q \in \mathcal{B}_k(Q)} q)) \\
&\quad \le \sum_{q \in \mathcal{B}_k(Q), q \ne q_0} \mathcal{H}^{d-1}(Z(h) \cap K \cap q) + \mathcal{H}^{d-1}(Z(h) \cap K \cap q_0) \\
&\quad \le \sum_{q \in \mathcal{B}_k(Q), q \ne q_0} C_0 N_h^{**}(q) s(q)^{d-1} + \frac{C_0}{2} N_h^{**}(Q) s(q_0)^{d-1} \\
&\quad \le \left(\frac{2^{k(d-1)} - 1}{2^{k(d-1)}} (1+\varepsilon)^k + \frac{1}{2} \cdot \frac{1}{2^{k(d-1)}} \right) C_0 N_h^{**}(Q) s(Q)^{d-1}.
\end{aligned}$$

Finally, we choose ε small and C_0 large enough so that

$$C_1 + \left(\frac{2^{k(d-1)} - 1}{2^{k(d-1)}} (1+\varepsilon)^k + \frac{1}{2} \cdot \frac{1}{2^{k(d-1)}} \right) C_0 < C_0.$$

Note that C_0 does not depend on K. Then, assuming that τ is small enough and taking into account (19), we obtain

$$\mathcal{H}^{d-1}(Z(h) \cap K \cap Q) \le C_0 N_h^{**}(Q) s(Q)^{d-1}.$$

This concludes the induction step and the proof of (17).

6 Dirichlet Laplace Eigenfunctions

6.1 Harmonic extension and an estimate of the doubling index. Let $\Omega_0 \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let u_λ be an eigenfunction of the Dirichlet Laplace operator, $u_\lambda \in W_0^{1,2}(\Omega_0)$, $\Delta u_\lambda + \lambda u_\lambda = 0$. Then $u_\lambda \in C(\overline{\Omega}_0)$. This fact is well-known, we provide a proof in the Appendix below, see Section A.2.

We consider the harmonic extension of u_λ to the domain $\Omega = \Omega_0 \times \mathbb{R} \subset \mathbb{R}^{n+1}$, given by

$$h(x,t) = u_\lambda(x) e^{\sqrt{\lambda} t}.$$

Then $h \in C(\overline{\Omega})$ and, clearly, $Z(h) = Z(u_\lambda) \times \mathbb{R}$, where the zero sets are sets inside the domains Ω and Ω_0 respectively. We need the following estimate of the doubling index of this harmonic extension.

LEMMA 10. *Let Ω_0 be a bounded domain in $\mathbb{R}^n$ with a sufficiently small local Lipschitz constant τ. Let $r_0 > 0$ be such that $\partial\Omega_0 \cap B(x, r_0) \in Lip(\tau)$ for any $x \in \partial\Omega_0$. Then for any $r \in (0, r_0/16)$, there exists $C = C(r, \Omega_0) > 0$ such that for any Dirichlet Laplace eigenfunction u_λ, the corresponding harmonic extension $h(x,t) = u_\lambda(x)e^{\sqrt{\lambda}t}$ satisfies $N_h(y, r) \le C\sqrt{\lambda}$ when $y = (x,t) \in \overline{\Omega}$.*

This result is similar to the results of Donnelly and Fefferman, [6, 7], who considered eigenfunctions on compact manifolds and on domains with C^∞-smooth boundaries and obtained the above estimate of the doubling index for eigenfunctions. However, in contrast to the previous results, the doubling index is allowed to blow up as $r \to 0$ in the above lemma. The statement of the lemma follows by application of Lemma 5 and inequality (6) to a chain of balls, the argument is similar to the one in [18, Section 2.4]. For the convenience of the reader, we provide the details below.

Proof. We consider any $y = (x,t) \in \overline{\Omega}$ and let $y_0 = (x, 0)$. Since $h(x, t+s) = e^{\sqrt{\lambda}t} h(x,s)$, we have $N_h(y, r) = N_h(y_0, r)$. So it is enough to estimate the doubling index of h in the balls centred on $\overline{\Omega}_0 \times \{0\}$.

We fix $r \in (0, r_0/16)$ and let $\mathcal{S} \in \overline{\Omega}_0$ be a finite $r/8$-net for $\overline{\Omega}_0$, i.e., $\overline{\Omega}_0 \subset \bigcup_{p \in \mathcal{S}} B(p, r/8)$. Let $B_* = B(y, r)$ be a ball of radius r centred at $y = (y_*, 0) \in \overline{\Omega}_0 \times \{0\}$. Assume that $\max_{\Omega_0} |u_\lambda| = |u_\lambda(x_0)| = 1$. We consider a path $\gamma : [0,1] \to \overline{\Omega}_0$ from y_* to x_0 such that $\gamma((0,1)) \subset \Omega_0$. Now we construct a chain of balls $\{B_j\}_{j=0}^J$. Let $B_0 = B(y_*, r/2)$. Assuming that $B_j = B(y_j, r/2)$ is constructed, we define

$$s_j = \sup\{s \in [0,1] : |\gamma(s) - y_j| \le r/8\}.$$

If $s_j < 1$, we have $|\gamma(s_j) - y_j| = r/8$ and we choose $y_{j+1} \in \mathcal{S}$ such that $|y_{j+1} - \gamma(s_j)| < r/8$. If $s_j = 1$, we define $y_{j+1} = y_J = x_0$ and stop the chain. We have $|y_j - y_{j+1}| < r/4$ and define $B_{j+1} = B(y_{j+1}, r/2)$. We note that $s_{j+1} > s_j$ when $0 \le j < J-1$ and that $y_{j+1} \in \mathcal{S} \setminus \{y_0, \dots, y_j\}$ when $0 \le j < J-1$. We also have $B_{j+1} \subset \frac{3}{2} B_j$. The resulting chain is finite, moreover, the number of balls in the chain is bounded by the number of elements in $\mathcal{S}$ plus two.

Let now $\widetilde{B}_j = B((y_j,0),r/2)$ be the corresponding ball in $\mathbb{R}^{n+1}$. Then $\sup_{4\widetilde{B}_j\cap\Omega}|h| \le e^{2\sqrt{\lambda}r}$. If $4\tilde{B}_j \subset \Omega$, then (6) gives

$$\begin{aligned}\sup_{\frac{3}{2}\widetilde{B}_j}|h| &\le 2^{n+1}(\sup_{\widetilde{B}_j}|h|)^{1/2}(\sup_{4\widetilde{B}_j}|h|)^{1/2}\\ &\le 3^{n+1}(\sup_{\widetilde{B}_j}|h|)^{1/3}(\sup_{4\widetilde{B}_j}|h|)^{2/3} \le 3^{n+1}e^{4\sqrt{\lambda}r/3}(\sup_{\widetilde{B}_j}|h|)^{1/3}.\end{aligned}$$

Otherwise we have $\operatorname{dist}(y_j,\partial\Omega_0) < 2r < r_0/8$. In this case, there is a ball $\tilde{B}$ of radius r_0 centred on $\partial\Omega_0\times\{0\}$ such that $(y_j,0)\in\overline{\Omega}\cap\frac{1}{4}\tilde{B}$ and $16\tilde{B}_j\subset\tilde{B}$. Then Lemma 5, applied to the ball $\tilde{B}_j$, implies that

$$\sup_{\frac{3}{2}\tilde{B}_j\cap\Omega}|h| \le 3^{n+1}(\sup_{\tilde{B}_j\cap\Omega}|h|)^{1/3}(\sup_{4\tilde{B}_j\cap\Omega}|h|)^{2/3} \le 3^{n+1}e^{4\sqrt{\lambda}r/3}(\sup_{\tilde{B}_j\cap\Omega}|h|)^{1/3}.$$

Therefore, we obtain for each j,

$$\sup_{\widetilde{B}_j\cap\Omega}|h| \ge 3^{-3(n+1)}(\sup_{\frac{3}{2}\widetilde{B}_j\cap\Omega}|h|)^3e^{-4\sqrt{\lambda}r} \ge 3^{-3(n+1)}(\sup_{\widetilde{B}_{j+1}\cap\Omega}|h|)^3e^{-4\sqrt{\lambda}r}.$$

We also have $\sup_{\widetilde{B}_J\cap\Omega}|h| = e^{\sqrt{\lambda}r/2}$. Combining the above inequalities, we get

$$\sup_{\widetilde{B}_0\cap\Omega}|h| \ge c_1e^{-C_2\sqrt{\lambda}},$$

where c_1 and C_2 depend on r and J but not on λ. We can choose the $r/8$-net $\mathcal{S}$ so that the number of points in $\mathcal{S}$ depends only on $\operatorname{diam}(\Omega_0)$, r, and the dimension. Thus we conclude that the constants in the last inequality depend only on r, the diameter of Ω_0, and n.

Finally, applying (2), we obtain

$$\begin{aligned}N_h(y,r) &= \log\frac{\int_{4\widetilde{B}_0\cap\Omega}h^2}{\int_{2\widetilde{B}_0\cap\Omega}h^2}\\ &\le \log\frac{\sup_{4\widetilde{B}_0\cap\Omega}|h|^2}{\sup_{\widetilde{B}_0\cap\Omega}|h|^2}+C\\ &\le (4r+2C_2)\sqrt{\lambda}+C \le C\sqrt{\lambda},\end{aligned}$$

where $C = C(\Omega_0,r)$. We remark that $\lambda \ge \lambda_1(\Omega_0) > 0$, where $\lambda_1(\Omega_0)$ is the first Dirichlet Laplace eigenvalue in Ω_0. Moreover, if B^* is a ball of radius $\operatorname{diam}(\Omega_0)$ then $\lambda_1(\Omega_0) \ge \lambda_1(B^*)$. Thus the constant C in the conclusion of this Lemma depends only on r, $\operatorname{diam}(\Omega_0)$, and n. □

6.2 Proof of Theorem 1′. Let $\Omega_0 \subset \mathbb{R}^n$ be a bounded domain with a sufficiently small local Lipschitz constant τ. Let also $r_0 > 0$ be such that $\partial\Omega_0 \cap B(x, r_0) \in Lip(\tau)$ for every $x \in \partial\Omega_0$. We consider the domain $\Omega = \Omega_0 \times \mathbb{R} \subset \mathbb{R}^{n+1}$ and let $\Omega_1 = \Omega_0 \times [-1, 1]$. For each $x \in \partial\Omega \times [-1, 1]$ we consider a ball centred at x of radius $2^{-9}r_0$. These balls cover the closed $2^{-10}r_0$-neighborhood of the set $\partial\Omega \times [-1, 1]$. We can choose a disjoint collection of these balls b_j such that the balls $B_j = 4b_j$ cover the same closed neighborhood of $\partial\Omega \times [-1, 1]$. Then for each point of $\Omega_1 \setminus \cup_j B_j$, we choose a ball b centred at the point of radius $2^{-15}r_0$, so that $32b \subset \Omega$. Once again, we find a finite sub-collection of disjoint balls b'_k such that $B'_k = 4b'_k$ cover $\Omega_1 \setminus \cup_j B_j$. We note that $8B'_k \subset \Omega$. We fix this covering of Ω_1 and remark that radii of all balls depend only on r_0 and the number of balls depends on r_0, the diameter of Ω_0, and n.

Let now u_λ be a Dirichlet Laplace eigenfunction in Ω_0: $\Delta u_\lambda + \lambda u_\lambda = 0$ in Ω_0 and $u_\lambda = 0$ on $\partial\Omega_0$. We consider its harmonic extension $h(x, t) = e^{\sqrt{\lambda}t}u_\lambda(x)$. Then $h \in C(\overline{\Omega})$ is non-zero, and $h = 0$ on $\partial\Omega$. Let $C_0 = \max\{C(2^{-5}r_0, \Omega_0), C(2^{-11}r_0, \Omega_0)\}$, where $C(r, \Omega_0)$ is as in Lemma 10. Then for $B(x, r) \in \{B_j\} \cup \{B'_k\}$, we have $N_h(x, 4r) \le C_0\sqrt{\lambda}$. Finally, we apply Theorem 2 to each of the balls B_j and Lemma 6 to each of the inner balls B'_k. We conclude that

$$\begin{aligned}\mathcal{H}^n(Z(h) \cap \Omega_1) &\le \sum_j \mathcal{H}^n(Z(h) \cap B_j) + \sum_k \mathcal{H}^n(Z(h) \cap B'_k) \\ &\le C(C_0\sqrt{\lambda} + 1)\left(\sum_j r(B_j)^n + \sum_k r(B'_k)^n\right) \le C_1\sqrt{\lambda}.\end{aligned}$$

Then $\mathcal{H}^{n-1}(Z(u_\lambda) \cap \Omega_0) \le C_1\sqrt{\lambda}$, which finishes the proof of Theorem 1′.

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Appendix: Proofs of Some Auxiliary Results

A.1 Estimates for the zero set of harmonic functions inside the domain. We outline some steps of the proof of Lemma 6. First the harmonic function h is extended to a holomorphic function H on a domain in $\mathbb{C}^d$, see Lemma 7.2 in [6]. Our situation is particularly simple, since we only consider the standard Laplace operator on Euclidean domains. For this case the holomorphic extension is given by the complexification of the Poisson kernel. The Poisson kernel in a ball $B(x,r)\subset\mathbb{R}^d$ is given by

$$P_r(z,y)=c_d\frac{r^2-|z-x|^2}{r|z-y|^d},\quad |z-x|<r,\ |y-x|=r.$$

For any $y\in\partial B(x,r)$, the function $z=(z_1,\dots,z_d)\mapsto\sum_j(z_j-y_j)^2$ maps the complex ball $B_{\mathbb{C}}(x,r/\sqrt{2})\subset\mathbb{C}^d$ of radius $r/\sqrt{2}$ centred at $x\in\mathbb{R}^d\subset\mathbb{C}^d$ to the half-plane $\Re\xi>0$. Then the Poisson kernel has the holomorphic extension to $B_{\mathbb{C}}(x,r/\sqrt{2})$. Moreover, for any $a<1/\sqrt{2}$,

$$|P_r(z,y)|\le C(a)r^{-(d-1)},\quad z\in B_{\mathbb{C}}(x,r_0),\ r_0\le ar.$$

We consider a ball $B=B(x,8r)$ such that $\overline{B}\subset\Omega$. Then there exists a holomorphic extension $H(z)$ of h defined on a ball $B_{\mathbb{C}}(x,3r)$,

$$H(z)=\int_{\partial B(x,6r)}P_{6r}(z,y)h(y)d\sigma(y),$$

such that $|H(z)|\le C\max_{\overline{B}(x,6r)}|h|$. Then

$$\sup_{B_{\mathbb{C}}(x,3r)}|H(z)|\le C'r^{-d/2}\left(\int_{B(x,8r)}h^2\right)^{1/2}.$$

Now we can cover the set $Z(h)\cap B(x,r)$ by a finite number of balls with centrs in $B(x,r)$ of radii $r/20$ so that the number of the balls is bounded by a constant depending on the dimension only. Let $B(y,r/20)$ be one of such balls. By a version of Corollary 1 for the doubling index inside the domain, we have $N_h(y,2r)\le 3N$, where $N=N_h(x,4r)$, and, therefore, $N_h(y,r_1)\le 3N$ when $r_1<2r$. Thus

$$\sup_{B(y,\frac{r}{16})}h^2\ge cr^{-d}\int_{B(y,\frac{r}{16})}h^2\ge cr^{-d}e^{-15N}\int_{B(y,2r)}h^2\ge cr^{-d}e^{-15N}\int_{B(x,r)}h^2.$$

Therefore,

$$\sup_{B(y,\frac{r}{10})}|H|\ge\sup_{B(y,\frac{r}{16})}|h|\ge cr^{-d/2}e^{-7.5N}\left(\int_{B(x,r)}h^2\right)^{1/2}.$$

Combining the inequalities above, we obtain

$$\frac{\sup_{B_{\mathbb{C}}(y,2r)}|H|}{\sup_{B(y,r/10)}|H|} \le \frac{\sup_{B_{\mathbb{C}}(x,3r)}|H|}{\sup_{B(y,r/10)}|H|} \le Ce^{7.5N}\left(\frac{\int_{B(x,8r)}h^2}{\int_{B(x,r)}h^2}\right)^{1/2} \le Ce^{9N}.$$

Finally an estimate for the size of the zero set of a holomorphic function, Proposition 6.7 in [6], implies that

$$\mathcal{H}^{d-1}(Z(h)\cap B(y,r/20)) \le C(N_h(x,4r)+1).$$

We sum these inequalities over all balls $B(y,r/20)$ to obtain the required estimate for $\mathcal{H}^{d-1}(Z(h)\cap B(x,r))$.

A.2 Continuity of eigenfunctions in Lipschitz domains. First we prove the following regularity result.

LEMMA 11. *Let Ω be a domain in $\mathbb{R}^d$ and let h be a harmonic function in Ω. Suppose that B is a ball centred on $\partial\Omega$ and that there exists a sequence of functions $\{h_n\}$, $h_n \in C_0^\infty(\mathbb{R}^d)$ with the support of h_n contained in Ω, such that $h_n \to h$ and $\nabla h_n \to \nabla h$ in $L^2(B\cap\Omega)$. Assume also that $\partial\Omega\cap B \in Lip(\tau)$ and define $h=0$ on $\partial\Omega\cap B$. Then $h \in C(\overline{\Omega}\cap\frac{1}{2}B)$.*

Proof. We define the function

$$v = \begin{cases} h^2 \text{ in } \Omega\cap B, \\ 0 \text{ in } B\backslash\Omega. \end{cases}$$

Then $v \in L^1(B)$. Let $\varphi \in C_0^\infty(B)$. We have

$$\begin{aligned}\int_B v\Delta\varphi &= \lim_{n\to\infty}\int_B h_n^2\Delta\varphi \\ &= -2\lim_{n\to\infty}\int_B h_n\nabla h_n\cdot\nabla\varphi = -2\int_{B\cap\Omega} h\nabla h\cdot\nabla\varphi. \end{aligned} \tag{20}$$

On the other hand, since h is harmonic in Ω, we obtain

$$0 = \int_\Omega \nabla h\cdot\nabla(h_n\varphi) = \int_\Omega h_n\nabla h\cdot\nabla\varphi + \int_\Omega \varphi\nabla h\cdot\nabla h_n.$$

Taking the limit as $n\to\infty$, we get

$$\int_\Omega h\nabla h\cdot\nabla\varphi = -\int_\Omega \varphi|\nabla h|^2.$$

Combining the last identity and (20) gives

$$\int_B v\Delta\varphi = 2\int_{B\cap\Omega}|\nabla h|^2\varphi.$$

In particular, v is subharmonic in B in the weak sense: If $\varphi \geq 0$, $\varphi \in C_0^\infty(B)$, then $\int_B v\Delta\varphi \geq 0$. If α is a standard mollifier, $\alpha_\delta(x) = \delta^{-d}\alpha(\delta^{-1}x)$, and $v_\varepsilon = v * \alpha_{\varepsilon r}$, where r is the radius of B. Then v_ε is subharmonic in $(1-\varepsilon)B$ and $v_\varepsilon \to v$ in $L^1(B)$ and almost everywhere. In particular, v satisfies the mean value inequality at each of its Lebesgue points. Clearly any $y \in \Omega \cap B$ is a Lebesgue point of v as $v = h^2$ in $\Omega \cap B$ and $h \in C(\Omega)$. So for any $y \in \Omega \cap B$ and any ball $B_1 \subset B$ centred at y we have

$$v(y) \leq \frac{1}{|B_1|}\int_{B_1} v.$$

In particular,

$$\sup_{\frac{2}{3}B\cap\Omega} h^2 \leq \frac{3^d}{|B|}\int_{B\cap\Omega} h^2 < \infty.$$

Suppose that $x_1 \in \partial\Omega \cap \frac{1}{2}B$. There exists a cone $\mathcal{C}$ with the vertex at x_1 such that $\mathcal{C} \cap (\Omega \cap B) = \varnothing$ and the aperture of $\mathcal{C}$ does not depend on x_1 (it depends on τ only). We use the following simple fact. If $y_1 \in \mathbb{R}^d$ and $\rho > 2\,\mathrm{dist}(x_1, y_1)$, then

$$|B(y_1,\rho) \cap \mathcal{C}| \geq \alpha |B(y_1,\rho)|,$$

for some $\alpha = \alpha(\tau) \in (0,1)$.
Let $m_k = \sup_{B(x_1,3^{-k}r)\cap\Omega} |h|$ for $k \geq 2$. We know that $m_k < \infty$. Let $y \in B(x_1, 3^{-k}r) \cap \Omega$, $k \geq 3$. By the mean value inequality applied to v, we obtain

$$v(y) \leq \frac{1}{|B(y, 2\cdot 3^{-k}r)|}\int_{B(y,2\cdot 3^{-k}r)} v \leq (1-\alpha)m_{k-1}^2.$$

Thus $\sup_{B(x_1,3^{-k}r)\cap\Omega} |h| \leq (1-\alpha)^{(k-2)/2} \sup_{\frac{2}{3}B\cap\Omega} |h|$. We conclude that

$$\lim_{y\to x_1, y\in\Omega} h(y) = 0. \qquad \square$$

We remark that the argument above implies that h is Hölder continuous in $\overline{\Omega} \cap B$ and there exist $C > 0$ and $\beta \in (0,1)$ such that

$$|h(y)| \leq C\mathrm{dist}(y,\partial\Omega)^\beta r^{-\beta} \sup_{\Omega\cap\frac{2}{3}B} |h|, \quad y \in \Omega \cap \frac{1}{2}B.$$

Corollary 2. *Let $\Omega_0 \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let u_λ be Laplace Dirichlet eigenfunction in Ω_0. Then u_λ extended by zero to $\partial\Omega_0$ is continuous on $\overline{\Omega}_0$.*

Proof. We have $u_\lambda \in W_0^{1,2}(\Omega_0) \cap C^\infty(\Omega_0)$ and $\Delta u_\lambda + \lambda u_\lambda = 0$ in Ω_0. We consider the harmonic function $h(x,t) = e^{\sqrt{\lambda}t}u_\lambda(x)$ in $\Omega = \Omega_0 \times \mathbb{R}$. We note that for any B centred on $\partial\Omega$, h satisfies the assumptions of Lemma 11. Then h is continuous in $\overline{\Omega}$ and vanishes on $\partial\Omega$. This implies that $u_\lambda \in C(\overline{\Omega}_0)$ and $u_\lambda = 0$ on $\partial\Omega_0$. $\square$

A.3 Quantitative Cauchy uniqueness. We give an elementary proof of Lemma 2 in this section for the convenience of the reader.

Let $G(x,y) = -c_d|x-y|^{2-d}$ be the fundamental solution of the Laplace equation in $\mathbb{R}^d$ when $d \geq 3$ (similar computations can be done with $G(x,y) = c_2 \log|x-y|$ for $d=2$). We write $\partial B_+ = \Gamma \cup \Sigma$, where Γ is the flat part of the boundary and $\Sigma = \partial B_+ \backslash \Gamma$. We denote by n the outer normal to ∂B_+. Then for $x \in B_+$, the Green formula implies

$$\begin{aligned} h(x) &= \int_{\partial B_+} \left[\frac{\partial G}{\partial n}(x,y)h(y) - G(x,y)\frac{\partial h}{\partial n}(y) \right] dy \\ &= \int_{\Gamma} \left[\frac{\partial G}{\partial n}(x,y)h(y) - G(x,y)\frac{\partial h}{\partial n}(y) \right] dy \\ &\quad + \int_{\Sigma} \left[\frac{\partial G}{\partial n}(x,y)h(y) - G(x,y)\frac{\partial h}{\partial n}(y) \right] dy \\ &= h_1(x) + h_2(x). \end{aligned}$$

The functions h_1 and h_2 are defined in the complements of Γ and Σ respectively and are harmonic in the corresponding domains. Moreover, for $x \notin \overline{B}_+$, applying the Green formula to the functions h and $G(x,\cdot)$ in B_+, we obtain $h_1(x) + h_2(x) = 0$.

First, we estimate the value of h_1 at some point $x = (x', x'') \in B\backslash\Gamma \subset \mathbb{R}^{d-1} \times \mathbb{R}$. We divide the integral into two

$$h_1(x) = \int_\Gamma \frac{\partial G}{\partial n}(x,y)h(y)dy - \int_\Gamma G(x,y)\frac{\partial h}{\partial n}(y)dy = I_1(x) + I_2(x).$$

Since $|\partial h/\partial n| < \varepsilon$ on Γ, the second integral is bounded by

$$|I_2(x)| \leq c_d \varepsilon \int_{B^{d-1}(x',2)} |x'-y'|^{2-d} dy' \leq C\varepsilon.$$

To estimate the first term, we note that for $y \in \Gamma$,

$$\frac{\partial G}{\partial n}(x,y) = c'_d x'' |x-y|^{-d},$$

and thereby

$$\int_\Gamma \left| \frac{\partial G}{\partial n}(x,y) \right| dy \leq c'_d \int_{\mathbb{R}^{d-1}} \frac{|x''|}{(x''^2 + |x'-y'|^2)^{d/2}} dy' = c''_d.$$

Using that $|h(y)| < \varepsilon$ on Γ, we conclude that $|I_1(x)| < C\varepsilon$ in $B\backslash\Gamma$. Therefore $|h_1(x)| \leq C\varepsilon$ in $B\backslash\Gamma$. Since $h_1(x) + h_2(x) = 0$ when $x \in \mathbb{R}^d \backslash \overline{B}_+$, and $|h_1 + h_2| = |h| \leq 1$ in B_+, we obtain that $h_2(x)$ satisfies

$$|h_2(x)| < C\varepsilon \ \text{ in } B_- = B\backslash\overline{B}_+ \quad \text{and} \quad |h_2(x)| \leq 1 + C\varepsilon \ \text{ in } B_+.$$

Now we apply the three sphere inequality (6). We note that h_2 is harmonic in B. First we take $x = (0, -1/5)$ and $r = 1/5$ and obtain

$$\sup_{B(0,1/10)} |h_2| \le \sup_{B(x,3/10)} |h_2| \le 2^d (\sup_{B(x,1/5)} |h_2|)^{1/2} (\sup_{B(x,4/5)} |h_2|)^{1/2} \le C\varepsilon^{1/2}.$$

Next, we apply inequality (6) to the ball centred at the origin with $r = 1/10$. We obtain

$$\sup_{B(0,3/20)} |h_2| \le C\varepsilon^{1/4}.$$

Iterating two more times, by applying the same inequality to the balls centred at the origin and $r = 3/20$ and, finally, $r = 9/40$, and noticing that $27/80 > 1/3$, while $9/10 < 1$, we conclude that

$$\sup_{\frac{1}{3}B} |h_2| \le C\varepsilon^{1/16}.$$

Finally, combining the last inequality with the bound $|h_1| \le C\varepsilon$ in B_+, we get the required estimate $|h| \le C\varepsilon^\gamma$ in $\frac{1}{3}B_+$.

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