Geom. Funct. Anal. Vol. 28 (2018) 645–726
https://doi.org/10.1007/s00039-018-0453-z
Published online June 11, 2018

GAFA Geometric And Functional Analysis

METRIC INEQUALITIES WITH SCALAR CURVATURE

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Abstract. We establish several inequalities for manifolds with *positive scalar curvature* and, more generally, for the scalar curvature bounded from below. In so far as geometry is concerned these inequalities appear as generalisations of the classical bounds on the *distances between conjugates points* in surfaces with *positive sectional curvatures*. The techniques of our proofs is based on the Schoen–Yau *descent method via minimal hypersurfaces,* while the overall logic of our arguments is inspired by and closely related to the *torus splitting* argument in Novikov's proof of the *topological invariance of the rational Pontryagin classes.*

Contents

1 Formulation of the Key Inequalities

Our point of departure is the following inequality for *torical bands* which are smooth manifolds homeomorphic to *tori times intervals.*

$[\circledcirc_\pm]$ **Torical $\frac{2\pi}{n}$-Inequality.** Let V be an n-dimensional torical band, $V = \mathbb{T}^{n-1} \times [-1, +1]$, where the boundary is

$$\partial(V) = \partial_- \cup \partial_+ = \partial_-(V) \cup \partial_+(V) = (\mathbb{T}^{n-1} \times \{-1\}) \cup (\mathbb{T}^{n-1} \times \{+1\}).$$

Let g be a smooth Riemannian metric on V, where the scalar curvature is bounded from below by a positive constant $\sigma > 0$,

$$Sc(g) \geq \sigma > 0.$$

Then the distance between the two boundary components of V satisfies

$$\left[\circledcirc_\pm \leq 2\pi\sqrt{\frac{n-1}{\sigma n}}\right] \qquad dist_\pm = dist_g(\partial_-(V), \partial_+(V)) \leq 2\pi\sqrt{\frac{n-1}{\sigma n}} \left(< \frac{2\pi}{\sqrt{\sigma}}\right).$$

On Normalisation of Sc. We use the customary normalisation of the scalar curvature, where the unit spheres satisfy

$$Sc(S^n) = n(n-1).$$

Thus, by scaling, the inequality $\left[dist_\pm \leq 2\pi\sqrt{\frac{n-1}{\sigma n}}\right]$ for a non specified $\sigma > 0$ reduces to that for $Sc(V) \geq n(n-1)$, where it reads

$$\left[\circledcirc_\pm \leq \frac{2\pi}{n}\right] \qquad dist_\pm = dist_g(\partial_-(V), \partial_+(V)) \leq \frac{2\pi}{n}$$

In particular,
all torical bands in the unit sphere satisfy

$$dist(\partial_-, \partial+) \leq \frac{2\pi}{n}.$$

This is obvious for $n = 2$, where $[\circledcirc_\pm \leq \frac{2\pi}{2}]$ is sharp as well as obvious. One also expects a two line proof of a stronger inequality for all n, but to my surprise, I was unable to directly prove even the corresponding inequality for *principal curvatures* of $(n-1)$-tori embedded to S^n , where this inequality is formulated below in terms of *focal coradii* as follows.

Normal Tubes, Normal Bands and $rad^{\odot}(Y)$. The *normal focal radius* of a smooth submanifold Y in a Riemannian manifold X, denoted $rad^{\odot}(Y) = rad^{\odot}(Y \subset X)$ is the maximal r such that the normal exponential map

$$\exp : T_{\perp}(Y) = T(X)|Y \ominus T(Y) \to X$$

is one-to-one[1] on the subset of vectors $\nu \in T_{\perp}(Y)$, such that $\|\nu\| < r$.

In other words, this is the maximal r such that *the normal r-tube* around Y, called *normal r-band* if $codim(Y) = 1$, that is the open r-neighbourhood $U_r(Y) \subset X$ for $r = rad^{\odot}$, normally projects[2] to Y and fibers $U_r(Y)$ into r-balls of dimension $dim(X) - dim(Y)$.

Examples. (a) The normal focal radii and the geodesic curvatures of sub-spheres

$$S^m(\rho) = S^m_s(\rho) \subset S^n = S^n(1) \subset \mathbb{R}^{n+1},\ \rho \le 1,$$

centred at ponts $s \in S^n$ are

$$r = rad^{\odot}_{S^n}(S^m(\rho)) = \arcsin\rho \text{ and } curv_{S^n}(S^m(\rho)) = \frac{\sqrt{1-\rho^2}}{\rho} = \tan r.$$

(b) The Clifford torus $\mathbb{T}^n_{Cl} \subset S^{2n-1} \subset (\mathbb{R}^2)^n$, that is the product of n circles of radii $\frac{1}{\sqrt{n}}$ in the plane, satisfies:

$$rad^{\odot}(\mathbb{T}^n_{Cl}) = \arcsin\frac{1}{\sqrt{n}}.$$

Conjecturally, $\mathbb{T}^n_{Cl}$ has maximal $rad^{\odot}$ among all n-tori smoothly embedded to S^{2n-1}.

Normal Radius Inequality for $\mathbb{T}^{n-1} \subset S^n$. *If a smooth hypersurface Y in the unit n-sphere is homeomorphic to the $(n-1)$-torus, then*

$$\left[\odot \le \frac{\pi}{n}\right]. \qquad\qquad rad^{\odot}(Y) \le \frac{\pi}{n}.$$

This inequality—this will become clear later on—is non-sharp.

Conjecturally, the sharp constant must be asymptotic for $n \to \infty$ to

$$\frac{const}{n^{\alpha}} \text{ for some } \alpha > 1\ .$$

On Sharpness of $\left[\circledcirc_{\pm} \le \frac{2\pi}{n}\right]$. This inequality agrees with the obvious one in the 2-sphere (where the conventionally defined scalar curvature equals twice the

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[1] It would be more in the spirit of "focal" to require the normal exponential map to be *locally* one-to-one, but this, probably, makes no difference in the present context for $X = S^n$.

[2] This projection sends each $x \in U_r(Y)$ to the unique(!) nearest point in Y.

sectional curvature) where the widths of the bands between concentric circles as well as the distances between opposite sides of (all) quadrilaterals are bounded by $\frac{2\pi}{2} = \pi = diam(S^2)$ and these inequalities become sharp for doubly punctured spheres and for quadrilaterals which degenerate to *geodesic digons* joining opposite points in S^2.

And if $n \geq 2$, we shall see in the next section that the extremal bands, where $dist_\pm = \frac{2\pi}{n}$, also have *constant scalar curvatures* and their opposite sides *collapse to points*, but they *do not have constant sectional curvatures* for $n > 2$ anymore.

Quadratic Decay Theorem. Let X be a complete Riemannian manifold, and let

$$\min_{B(R)} Sc(X)$$

denote the minimum of the scalar curvature (function) of X on the ball $B(R) = B_{x_0}(R) \subset X$ for some centre point $x_0 \in X$.

If X is homeomorphic to $\mathbb{T}^{n-2} \times \mathbb{R}^2$, then there exists a constant $R_0 = R_0(X, x_0)$, such that

$$\left[\asymp \frac{4\pi^2}{R^2}\right] \qquad \min_{B(R)} Sc(X) \leq \frac{4\pi^2}{(R-R_0)^2} \quad \textit{for all } R \geq R_0.$$

Outline of the Proof. Let $X_0 \subset X$ corresponds to the torus $\mathbb{T}^{n-2} \times \{0\} \subset \mathbb{T}^{n-2} \times \mathbb{R}^2$ under the homeomorphism $\mathbb{T}^{n-2} \times \mathbb{R}^2 \leftrightarrow X$ and let $R_0 = diam_X(X_0)$. Then the $(R-R_0)$-neighbourhood $U_{R-R_0}(X_0) \subset X$ is contained in the ball $B_{x_0}(R)$ for $x_0 \in X_0$.

If $U_{R-R_0}(X_0)$ is homeomorphic to $\mathbb{T}^{n-1} \times (-1, +1)$, then $[\asymp \frac{4\pi^2}{R^2}]$ follows from the torical $\frac{2\pi}{n}$-inequality and if the topology of $U_{R-R_0}(X_0)$ is more complicated, then we apply a generalisation of the $\frac{2\pi}{n}$-inequality from section 4.

On Uniformly Positive Scalar Curvature. The obvious corollary to $[\asymp \frac{4\pi^2}{R^2}]$ is
non-existence of complete metrics with $Sc \geq \sigma > 0$ on $\mathbb{T}^{n-2} \times \mathbb{R}^2$.

Notice that there are similar results for other manifolds X proven with *Dirac operators twisted with suitable "almost flat" bundles over X* [GL83, HPS15].

However, for all I know, one can't rule out metrics with uniformly positive scalar curvature on $\mathbb{T}^{n-2} \times \mathbb{R}^2$ with the present day Dirac operator methods.[3]

Examples of Metrics on $\mathbb{T}^{n-2} \times \mathbb{R}^2$ with Quadratic Decay of Scalar Curvature. Let $g = dt^2 + \varphi(t)^2 d\theta^2$, $t \in [0, \infty), \theta \in [0, 2\pi]$, be a radial (rotationally symmetric) metrics on $\mathbb{R}^2$. Then

$$Sc(g)(t) = -\frac{2\varphi''(t)}{\varphi(t)};$$

thus, the metrics

$$g_{fl} + dt^2 + t^{2\alpha} d\theta^2$$

on $\mathbb{R}^2 \times \mathbb{T}^{n-2}$, where g_{fl} are flat on $\mathbb{T}^{n-2}$ and where $0 < \alpha < 1$, do the job.

[3] Bernhard Hanke indicated to me how one can do this, see section 7.

1.1 On Generalisations and Proofs: Admissions and and Acknowledgements. Simple generalisations of everything we stated so far is proven in section 2.

Then, in the following sections, we formulate and prove further *generalisations and refinements* of these. Also we indicate additional *applications* and articulate several *conjectures.*

Our approach is based on the Schoen–Yau *dimension descent argument* [SY79b, SY17] accompanied by *torical symmetrization* [GL83] and/or *symmetrizartion by reflection* [Gro14a].

However, some of our arguments have certain limitations which are indicated below.

1. Problem with Singularities. Applications of minimal hypersurfaces $Y \subset X$ to $Sc \geq 0$ depends on the regularity of these Y which is known to hold for all Y if $n = dim(X) \leq 7$ and for generic ones for $n = 8$ by a Nathan Smale theorem [Sma93] More recently, Lokhamp [Loh16] and Schoen and Yau [SY17] suggested ways of bypassing the singularity problem.

As far as I understand, the regularity results by Schoen and Yau in [SY17], such as theorem 4.6, suffice for the needs of the present paper and this is, probably, true about the corresponding results by Lokhamp. But since I have not studied these papers in depth, I can vouch for the validity of our proofs only for $n \leq 8$, where the singularity problem does not exist.

2. Doubling, Reflecting and Smoothing. Some results concerning closed Riemannian manifolds X with $Sc \geq 0$ generalise to manifolds X with boundary and, accordingly, to bands of the form $X \times [-1, 1]$.

For instance if the boundary $\partial = \partial X$ is *mean curvature convex*, i.e. if $mn.curv(\partial) > 0$, then X admits a full fledged theory of minimal hypersurfaces $Y \subset X$ with (free) boundary $\partial Y \subset \partial X$ and the Schoen–Yau descent method applies.

Alternatively, one may take the *double*

$$\tilde{X} = X \cup_\partial X$$

and show (see [GL80a], and section 11.4) that

the natural continuous (but not necessarily smooth, not even C^1) Riemannian metric $\tilde{g} = g\&g$ on $\tilde{X}$ can be C^0-approximated by smooth C^∞-smooth metrics $\tilde{g}_\varepsilon$ with no decrease of the scalar curvature of $\tilde{g}$. (The scalar curvature of $\tilde{g}$ is unambiguously defined away from ∂, which naturally embeds as a hypersurface in $\tilde{X}$.)

Thus

all results for closed manifolds X, including those obtained with the Dirac operator methods, extends to manifolds with mean curvature convex boundaries.

(Analysis of *extremal cases*, e.g. showing that every metric g on $X = \mathbb{T}^{n-1} \times [0, 1]$, where $mn.curv_g(\partial(X)) = 0$, is *Riemannian flat*, needs an additional care.)

Next, let X, be a manifold with corners, such, for instance, as the Cartesian product of several manifolds with boundaries, such as the n-cube

$$\square^n = \underbrace{[0,1], \times \cdots \times [0,1]}_{n},$$

or an V diffeomorphic to $\square^n$.

Schoen–Yau descent, probably, applies to such X, where one may encounter complications with singularities of minimal hypersurfaces, $Y \subset V$ with $\partial Y \subset \partial X$, at the corners of X on the boundary ∂X.[4]

There is an alternative approach applicable to those X, which serve as fundamental domains of *refection groups* Γ acting on manifolds $\tilde{X}$ without boundaries.

For instance, if X is *diffeomorphic* to the n-cube $[0,1]^n$, then $\tilde{X}$ is *naturally homeomorphic* to the Euclidean space $\mathbb{R}^n$ acted on by the group Γ (isomorphic to $\mathbb{Z}^n \rtimes \mathbb{Z}_2^n$) generated by reflections of $\mathbb{R}^n$ in the hyperplanes $\{x_i = 0\}$ and $\{x_i = 1\}$, $i = 1, \ldots, n$, in $\mathbb{R}^n$.

This $\tilde{X}$ carries a unique path metric $\tilde{g}$, which is equal to the original Riemannian metric g on $X \subset \tilde{X}$, where X is embedded to $\tilde{X}$ as a fundamental domain.

In general, unlike the case where X is isometric to $[0,1]^n$, rather than only diffeomorphic to it, the metric $\tilde{g}$ is non-smooth at the boundary $\partial X \subset \tilde{X}$ (as well as at the boundaries of the Γ-translates of X). In fact, this metric is continuous (but not, in general, C^1-smooth) at the smooth points of ∂X which correspond to the interior points in the $(n-1)$-faces of $[0,1]^n$, and it is only piecewise continuous at the edges.

In fact the metric $\tilde{g}$ is Riemannian continuous if and only the dihedral angles between the $(n-1)$-faces of X along the "edges" (corresponding to the $(n-2)$-faces of $[0,1]^n$ equal $\frac{\pi}{2}$ and then $\tilde{g}$ is C^1 (only) if the $(n-1)$-faces are totally geodesic with respect to this metric.

It follows from the Approximation/Reflection Lemma in section 4.9 in [Gro14a] that

if the the faces of X are mean curvature convex and if the dihedral angles between these faces along the corners in X are bounded by $\frac{\pi}{2}$, then $\tilde{X}$ carries smooth(!) Γ-invariant metrics $\tilde{g}_\varepsilon$, $\varepsilon \to 0$, which are, in some week sense converge to the original metric g on $X \subset \tilde{X}$ and such the scalar curvatures of $\tilde{g}_\varepsilon$ are bounded from below by $Sc(g)$ on $X \subset \tilde{X}$.

Granted this, many (all?) properties of closed (and complete) manifolds with $Sc \geq \sigma$ generalise to manifolds with suitable corners, e.g.

non-existence of cubical (i.e. diffeomorphic to $[0,1]^n$*) domains in Riemannian manifolds* X *with* $Sc(X) \geq 0$ *(e.g. in* $X = \mathbb{R}^n$*), such that the* $(n-1)$*-faces of these domains have* $mn.curv > 0$ *and the dihedral angles between these faces are* $\leq \frac{\pi}{2}$*, see [Gro14a]).*

[4] Possibly, an adequate regularity of minimising hypersurfaces at the corner points is known to some people, but I could not locate such result in the literature.

However—this was pointed out by the referee of the present paper—the way it was written in the original version of this paper was unsatisfactory.

We discuss in section 11.8 what should be done about it.

3. Errors indicated by the Referee. The anonymous referee to this paper pointed out inadequacy of the first version of our treatment of smoothing multiple corners and of minimal hypersurfaces in non-compact manifolds.

2 Bounds on Widths of Over-torical and Related Riemannian Bands

A *band* is a manifold V with two distinguished disjoint non-empty subsets in the boundary $\partial(V)$, denoted

$$\partial_- = \partial_-(V) \subset \partial V \text{ and } \partial_+ = \partial_+(V) \subset \partial V.$$

A band is called *proper* if $\partial_\pm$ are unions of connected components of ∂V and

$$\partial_- \cup \partial_+ = \partial V.$$

Band maps $V \to \underline{V}$ are those continuous ones which respect these $\pm$-boundaries, $\partial_\pm \to \underline{\partial}_\pm$.

If V is endowed with a Riemannian metric then the *width of a band* is the distance between ∂_- and ∂_+, that is the infimum of length of curves in V between ∂_- and ∂_+.

A compact proper orientable band is called *over-torical* if it admits a band map to the toric band,

$$f : V \to \underline{V} = \mathbb{T}^{n-1} \times [-1, +1],\ n = dim(V),$$

with *non-zero degree.*

Another way to put it is by saying that the relative fundamental class $[V] \in H^n(V, \partial V; \mathbb{Q})$ decomposes to the product

$$[V] = h_1 \smile \cdots \smile h_{n-1} \smile h_n$$

where h_i, $i = 1, \ldots, n-1$, are (absolute) 1-dimensional cohomology classes, $h_i \in H^1(V; \mathbb{Q})$, and $h_n \subset H^1(V, \partial V; \mathbb{Q})$ is the (relative) class, of the differential of a function $V \to [-1, +1]$ such that $\partial_\pm \mapsto \pm 1$.

If V is non-orientable, then *overtorical* means that an orientable finite cover of V is overtorical.

Torical Symmetrization. *There exists a quasi-functorial symmetrizartion "operator" from Riemannian over-torical bands to torical ones*

$$\mathsf{Sym} : V \rightsquigarrow \underline{V}$$

where $\underline{V}$ *admits a free isometric action of the torus* $\mathbb{T}^{n-1}$ *and such that*

$$width(\underline{V}) \geq width(V)$$

and

$$Sc(V) > \sigma \Rightarrow Sc(\underline{V}) > \sigma.$$

Proof. This is proven in a slightly different form in [GL83] for $n \leq 7$ by induction as it is explained below.

(Earlier, such symmetrization for $n = 3$ was used by Fisher-Colbrie and Schoen [FS80], while the proof for $n = 8$ is essentially the same as for $n \leq 7$ due to Nathan's Smale generic regularity theorem.)

Induction Step. Let V_k be a $\mathbb{T}^k$ invariant Riemannian band, $k = 0, \dots, n-2$, which admits a $\mathbb{T}^k$-equivariant band map to the torical band

$$f_k : V_k \to \mathbb{T}^{n-1} \times [-1, +1]$$

where $\mathbb{T}^k$ acts on $\mathbb{T}^{n-1} \times [-1, +1]$ via the standard (coordinate) embedding $\mathbb{T}^k \subset T^{n-1}$ and such that $deg(f_k) \neq 0$.

Let $Y_k \subset V_k$ be a volume minimising hypersurface which is homologous to the f_k-pullback of

$$\mathbb{T}^{n-2} \times [-1, +1] \subset \mathbb{T}^{n-1} \times [-1, +1]$$

for the torus $\mathbb{T}^{k+1} \supset \mathbb{T}^k$, where "homologous" refers to the relative group $H_{n-1}(V_k; \partial V_k = \partial_- \cup \partial_+)$.

It is easy to see that this Y_k is $\mathbb{T}^k$-invariant and that the lowest eigenfunction $\phi(y)$ of the second variation operator L on Y_k,

$$L = -\Delta + \frac{1}{2}(Sc(Y_k) - Sc(V_k|Y_k) - ||curv_{V_k}(Y_k)||^2),$$

is also $\mathbb{T}^k$-invariant. (Here, Δ is the Laplacian on Y_k, that is $\sum_i \frac{\partial^2}{dy_i^2}$ and $curv_X(Y)$ denotes the second fundamental form of $Y_k \subset V_k$.)

Then we let $V_{k+1} = Y_k \times \mathbb{T}^1$ with the metric $dy^2 + \phi^2 dt^2$, where a simple computation shows that if the scalar curvature of V_k restricted to Y_k is $\geq \sigma$, then the scalar curvature of V_{k+1} is also bounded from below by σ.

It is also clear that V_{k+1} admits a $\mathbb{T}^{k+1}$-equivariant band map of degree $= deg(f_k)$ to the torical band and that $width(V_{k+1}) \geq width(V_k)$.

Thus, the inductive step is completed and the existence of torical symmetrization follows. (See [GL83] for details). □

Remark on Singularities. $\mathbb{T}^k$-invariant minimal hypersurfaces in V_k correspond to hypersurfaces in the quotient manifolds $V_k/\mathbb{T}^k$, which are minimal with respect to the quotient metrics with obvious conformal weights. Then theorem 4.6 in [SY17] says, in effect, that even if some hypersurfaces Y_k were singular, say for $n - k \geq 8$, the final $\mathbb{T}^{n-1}$-symmetric $V_{n-1} = \underline{V}$ are non-singular.

(Schoen and Yau formulate their theorem for closed manifolds but the needed regularity for manifolds V with boundaries trivially reduces to that for doubles of V.)

$\frac{2\pi}{n}$**-Inequality for Over-Torical Bands.** *Overtorical bands with scalar curvatures* $\geq n(n-1)(=Sc(S^n))$ *satisfy*

$$\left[dist_\pm \leq \frac{2\pi}{n}\right] \qquad width(V) \leq \frac{2\pi}{n}.$$

Proof. Torical symmetrization reduces the general case to that of $\mathbb{T}^{n-1}$-invariant metrics g, on torical bands, where

$$g = dt^2 + \sum_i \varphi_i(t)^2 d\tau_i^2,\ i = 2, 3, \dots, n.$$

Then one easily computes

$$Sc(g)(t, \tau_2, \dots, \tau_n) = -2\sum_i \frac{\varphi''(t)}{\varphi_i(t)} - 2\sum_{i<j} \frac{\varphi_i'(t)}{\varphi_i(t)} \frac{\varphi_j'(t)}{\varphi_j(t)}$$

and shows that the the longest t-interval where this function remains defined for $Sc(g) \geq \sigma > 0$ is achieved with $\varphi_2 = \dots = \varphi_n = \varphi$, where the proof follows by simple computation on p. 401 in [GL83] which is reproduced below in the description of optimal (maximal) torical bands with $Sc \geq \sigma$. □

Proof of Propositions from Section 1. The inequality $\left[dist_\pm \leq \frac{2\pi}{n}\right]$ implies everything we have stated so far, where in the case of the quadratic decay theorem one needs to observe that the domains $U_{R-R_0}(X_0) \subset X$ (defined following the statement of this theorem) are, in an obvious sense, *open overtorical bands* to which the above $\frac{2\pi}{n}$-Inequality applies.

Notice at this point that this argument automatically delivers the following

Generalisation of The Quadratic Decay Theorem. *If a complete orientable Riemannian n-manifold X admits a proper continuous map $X \to \mathbb{T}^{n-2} \times \mathbb{R}^2$ of non-zero degree, then the minima of the scalar curvature of X over concentric R-balls in X satisfy*

$$\left[\asymp \frac{4\pi^2}{R^2}\right]^* \qquad \min_{B(R)} Sc(X) \leq \frac{4\pi^2}{(R-R_0)^2} \text{ for some } R_0 \geq 0 \text{ and all } R \geq R_0.$$

Optimality of $\frac{2\pi}{n}$. *Every smooth manifold $V = Y \times [-1,1]$ admits a Riemannian metric $g = g_\varepsilon$ with $Sc(g) \geq n(n-1)$ and the g-distance between the two boundary components $Y \times \{-1\}$ and $Y \times \{1\}$ in V equal $2\pi/n - \varepsilon$ for a given $\varepsilon > 0$.*

For instance, if $Y = S^1$, then the *spherical suspension* $V = \Theta(Y)$ serves this purpose for all $\varepsilon > 0$.

More generally, given a Riemannian metric g_0 on Y and a real function $\varphi(t)$, let $g = dt^2 + \varphi(t)^2 g_0$ be the metric on $Y \times [-l, l]$. If g_0 is flat then

$$◑ \qquad \sigma = Sc(g) = -2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\frac{\varphi'^2}{\varphi^2},$$

or

$$\frac{\sigma}{n-1} = -2\left(\frac{\varphi''}{\varphi} + \frac{\varphi'^2}{\varphi^2}\right) - n\frac{\varphi'^2}{\varphi^2},$$

that is

$$-2f' - nf^2 = \frac{\sigma}{n-1} \text{ for } f = \frac{\varphi'}{\varphi}.$$

Now let $\sigma = Sc(S^n) = n(n-1)$ and rewrite the above as

$$\frac{f'}{1+f^2} = (\arctan f)' = \frac{n}{2}$$

and

$$f = f(t) = \tan\frac{n}{2}t$$

which is a function defined on the $\frac{2\pi}{n}$-interval $(-\frac{\pi}{n}, +\frac{\pi}{n})$.

This settles the matter for flat manifolds Y and the general case follows by rescaling general metrics in Y with a large constants. $\square$

3 Toric Bands in Spheres and Lower Bounds on Lipschitz Constants of Map $X \to S^n$ in terms of $Sc(X)$

Suppose, there is a *toric band of width* d in the unit n-sphere S^n that is a domain $\underline{V} \subset S^n$ which is homeomorphic to $\mathbb{T}^{n-1} \times [-1,1]$ and such that the distance between the two boundary components $\partial_\pm(\underline{V})$ of $\underline{V}$ is equal to d and let f be a continuous map of non-zero degree from an oriented Riemannian n-manifold X to S^n.

Recall that saying "degree" presupposes that f is locally constant at infinity, i.e. constant on each boundary component of X and, if X is non-compact, on every component of the complement to some (large) compact subset in X, and let us additionally assume that the (finite) f-image of the so defined infinity *does not intersect* V. (This is relevant only if ∂X is disconnected and/or if X is disconnected at infinity.)

Then the pullback $V = f^{-1}(\underline{V}) \subset X$ is a Riemannian over-torical band, such that the distance between the two parts $\partial_\pm(V)$ of its boundary is $\geq \lambda^{-1}d$, and the inequality

$$d = width(V) \leq 2\pi\sqrt{\frac{n-1}{\sigma n}}$$

(this is $[dist_\pm \leq \frac{2\pi d}{n}]$ formulated for $\sigma = n(n-1)$ in the previous section) shows that

$$\bigcirc^n_{Lip} \qquad [Sc(X) \geq \sigma] \Rightarrow \left[Lip(f) \geq \frac{d}{2\pi}\sqrt{\frac{\sigma n}{n-1}}\right],$$

where, recall,

$$Lip(f) = \sup_{x_1 \neq x_2} \frac{dist_{S^n}(f(x_1), f(x_2))}{dist_X(x_1, x_2)}.$$

Notice that the $\frac{2\pi}{n}$-inequality $[dist_\pm \leq \frac{2\pi}{n}]$, that is (essentially) $\bigcirc^n_{Lip}$ applied to the identity map, shows that torical bands in S^n have widths $d \leq \frac{2\pi}{n}$.

Conjecturally, the maximal widths d for large $n \to \infty$ must be asymptotic to $\frac{1}{n^{1+\alpha}}$ for some $\alpha > 0$.

Round Tori and in S^n and in $\mathbb{R}^n$. Let us show that this α must be $\leq \frac{1}{2}$ by exhibiting embedded tori $\mathbb{T}^{n-1}$ with bands of width $\approx \frac{1}{n^{\frac{3}{2}}}$ around them, where we use the following terminology.

Over-Torical Width $width_{\hat{\mathcal{T}}}(X)$**.** This is defined for Riemannian manifolds X as the supremum of numbers d, such that X admits an equidimensional locally isometric (not necessarily globally one-to-one) immersion from an overtorical Riemannian band of width d.

For instance, it is obvious that

$$width_{\hat{\mathcal{T}}}(S^2) = \pi.$$

More significantly, since the Clifford torus in S^3 has $rad^{\odot} = \pi/4$, (see section 1)

$$width_{\hat{\mathcal{T}}}(S^3) \geq \pi/2$$

and consequently,

all continuous maps f from a Riemannian (possibly incomplete) 3-manifold X with $Sc(X) \geq 6 = Sc(S^3)$ to S^3 which are constant at infinity and have $deg(f) \neq 0$, satisfy

$$Lip(f) \geq \frac{3}{4}.$$

This improves the inequality $Lip(f) \geq \frac{3}{8\pi}$ from [GL83] but falls short of the conjectural bound $Lip(f) \geq 1$.

Another natural conjecture is the *equality*

$$width_{\hat{\mathcal{T}}}(S^3) = \pi/2.$$

Moreover, one expects that

all (possibly incomplete) 3-manifolds X with sectional curvatures ≥ 1 satisfy

$$width_{\hat{\mathcal{T}}}(X) \leq \pi/2.$$

Starting from $n = 4$, codimension one tori in S^n can't be rotationally invariant any more; we construct certain "roundish" ones with relatively large focal coradii

$r = rad^{\odot}$, i.e. with the normal exponential maps of these tori is one-to-one within distance $\leq r$ from them.

We construct these tori in the unit Euclidean n-balls (rather than in the unit spheres) by induction as follows.

Given codimension one tori $Y_1 \subset B^{n_1} \subset \mathbb{R}^{n_1}$, and $Y_2 \subset B^{n_2} \subset \mathbb{R}^{n_2}$ with focal coradii r_1 and r_2, take $c_1, c_2 > 0$, such that

$$c_1^2 + c_2^2 = 1 \text{ and } c_1 r_1 = c_2 r_2,$$

observe that the product of the c_i-scaled Y_i in $\mathbb{R}^{n_i}$ is contained in the unit ball

$$Y_\times = c_1 Y_1 \times c_2 Y_2 \subset B^{n_1+n_2} \subset \mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.$$

and

$$rad^{\odot}(Y_\times) = r_\times = c_1 r_1 = c_2 r_2.$$

Then let

$$Y_{\times_+} = \frac{1}{1+\delta}(Y_\times)_{+\delta} \subset B^{n_1+n_2}$$

be the $\frac{1}{1+\delta}$-scaled boundary of the δ-neighbourhood of $Y_\times$ in $\mathbb{R}^{n_1+n_2}$ with $\delta = \frac{1}{2} r_\times$ and observe that

$$rad^{\odot}(Y_{\times_+}) = rad^{\odot}(Y_\times) \cdot \frac{1}{2} \cdot \left(\frac{1}{1 + \frac{1}{2} r_\times} \right) = \frac{r_\times}{2 + r_\times}.$$

In particular, if a torus $Y = Y_1 = Y_2 = Y(n) \subset B^n$ has normal focal radius $r = r(n)$, the resulting $Y(2n) = (Y \times Y)_+ \subset B^{2n}$ satisfies

$$r(2n) = rad^{\odot}(Y \times Y)_+ = \frac{r(n)}{2\sqrt{2} + r(n)}$$

and the normal focal radius of

$$Y(2n+1) = ((c_1 Y(n)) \times (c_2 Y(n+1)))_+ \subset B^{2n+1}.$$

satisfies a similar relation.

Then, starting from $Y(2) = S^1 \subset \mathbb{R}^2$ with $r(2) = 1$ one obtains $Y(4)$, $Y(4)$,..., such that

$$\begin{aligned} r(4) &\geq \frac{1}{2^{\frac{3}{2}} + 1} > \frac{1}{4} = \frac{2}{4^{\frac{3}{2}}}, \\ r(8) &> \frac{1}{8\sqrt{2} + 1} = \frac{1}{\frac{1}{2} 8^{\frac{3}{2}} + 1} > \frac{1}{13}, \end{aligned}$$

and, in general,

$$r(n) \geq c n^{-\frac{3}{2}},$$

where a (very) rough estimate is $c > 1$ for $n = 2^i$ and $c > \frac{1}{3}$ for all n.

Eventually, since the normal bands around these tori $Y(n)$, can be transported from B^n to S^n by the obvious expanding map $B^n \to S^n$, we conclude that

$$width_{\hat{\mathcal{T}}}(S^n) \geq 2c \cdot n^{-\frac{3}{2}}$$

which combined with $\bigcirc^n_{Lip}$ implies the following.

Spherical Lipschitz Bound Theorem. *If the scalar curvature of a (possibly incomplete) Riemannian n-manifold is bounded from below by $n(n-1) = Sc(S^n)$, then all continuous maps f from X to the sphere S^n (and also to the hemisphere to S^n_+) of non-zero degrees*[5] *satisfy*

$$Lip(f) > \frac{c}{\pi\sqrt{n}} \text{ for the above } c > \frac{1}{3}.$$

(This c is not optimal; but since this inequality is unlikely to be qualitatively sharp anyway there is no point in fiddling with constants.)

Remarks. (a) If X is a *complete spin*[6] manifold, then the *sharp* spherical Lipschitz bound $Lip(f) \geq 1$ is known to hold for these maps $f : X \to S^n$ by the work of Llarull [Lla98]. This is accomplished by carefully analysing the *algebraic Schroedinger–Lichnerowicz–Weitzenboeck formula* for *the Dirac operator on X twisted with the spin bundle $\mathbb{S}^+(S^n)$ pulled back to X* and applying the index theorem.

In fact, this Dirac operator proof rules out smooth proper maps $f : X \to U \subset S^n$ of non-zero degrees, which *strictly decrease areas of surfaces $S \subset X$* (such f may have $Lip(f) >> 1$) and where the complements to the (open) subsets $U \subset S^n$ are *zero dimensional*, or, more generally, where *all connected subsets $A \subset S^n \setminus U$ are trees and/or closed curves with trivial (i.e. identity) Levi-Civita monodromy transformations around them* (see section 10).

(b) It remains unknown:
- *if the spin condition is essential for ruling out maps f for which area $(f(S)) < area(S)$,*
- *if the completeness condition is essential for $Lip(f) \geq 1$,*
- *if one may allow closed curves in $S^n \setminus U$ with nontrivial Levi-Civita monodromies even if X complete and spin.* (See section 10 for further questions of this kind.)

(c) The above inequality $Lip(f) \geq \frac{c}{\pi\sqrt{n}}$ (which applies to incomplete non-spin manifolds) improves upon $Lip(f) \geq \frac{n}{2^n\pi}$ in [GL83].[7]

[5] Here such a map $X \to S^n$ is supposed to be constant at infinity, including ∂X and to be proper from the interior of X to that of S^n_+.

[6] In fact, it suffices to have the universal covering of X spin—we return to this later on'; here we recall that an orientable smooth manifold X is *spin* if the restrictions of the tangent bundle $T(X)$ to all surfaces $S \subset X$ are trivial bundles.

[7] The inequality $Lip(f) \geq \frac{c}{\pi\sqrt{n}}$ with $c = 1/3$ gains over $Lip(f) \geq \frac{n}{2^n\pi}$ only for $n \geq 6$ but a more precise evaluation of $rad^{\odot}(Y(n))$ shows this gain for all n.

This gains in significance as $n \to \infty$, where the proof for $n \geq 9$ depends on the controlled singularity results by Lohkamp and Schoen–Yau, which the present author has not studied in detail.

(d) The above estimates of torical width of S^n and of focal radii of tori in S^n raise a multitude of questions concerning $width_{\hat{\mathcal{T}}}(X)$, $rad^{\odot}(Y \subset X)$ and their generalisations for various X and Y. These will be briefly discussed in section 7.

4 $\frac{4\pi}{n}$-Bound on Width and Related Inequalities for Iso-Enlargeable Bands

Hypersphericity and Iso-Enlargeability. An oriented Riemannian manifold X is called *hyperspherical* if it admits continuous maps f to S^n, $n = dim(X)$ with arbitrarily small $Lip(f) > 0$, which are *constant at infinity* which have *non-zero degrees*.

A Riemannian manifold X is called *iso-enlargeable* if there exists a sequence of Riemannian manifolds $\tilde{X}_i$ of dimension $n = dim(X)$ and of locally isometric maps $\tilde{X}_i \to X$, such that $\tilde{X}_i$ admit continuous maps constant at infinity

$$f_i : \tilde{X}_i \to S^n,$$

such that

$$deg(f_i) \neq 0 \text{ and } Lip(f_i) \to 0 \text{ for } i \to \infty.$$

Examples. (a) The archetypical hyperspherical manifolds are the Euclidean spaces $\mathbb{R}^n$.

(b) Complete simply connected manifolds X with *non-positive sectional curvatures* κ are also hyperspherical.
This follows from (a), since the the inverse exponential maps $\exp^{-1} : X \to \mathbb{R}^n = T_{x_0}(X)$ satisfy $Lip(\exp^{-1} \leq 1$ for $\kappa(X) \leq 0$.

(c) If a compact manifold X is fibered over an $\underline{X}$, where $\kappa(\underline{X}) \leq 0$ and where the fibers also admit metrics with $\kappa \leq 0$ then the universal covering of X is hyperspherical by an easy argument.

(d) Compact locally symmetric spaces Y that have no (local) factors isometric to real and/or complex hyperbolic spaces are enlargeable but *not overtorical*, since the homology groups $H_1(Y)$ are *finite* for these Y.

Instances of such Y are compact quotients $H^n_{\mathbb{H}}/\Gamma$ of *quaternion hyperbolic spaces* (here the sectional curvature $\kappa(Y) < 0$) and compact quotients
$SO(n)\backslash SL(n)/\Gamma$, $n \geq 3$ (here $\kappa(Y) \leq 0$).

Remark/Question. If the, locally isometric maps $\tilde{X}_i \to X$ in the definition of iso-enlargeability are required to be *covering maps*, which is equivalent to *completeness* of $\tilde{X}_i$ in the case where X itself is complete (e.g. compact), then X is called *enlargeable*, see [GL83, Dra00, DFW03, HS06, BH09, Han11].

It is obvious that

$$enlargeable \Rightarrow iso\text{-}enlargeable,$$

Also one may expect that the reverse implication holds for compact manifolds, since sequences $\tilde{X}_i$ (sub)converge in a natural way to some $\tilde{X}$, where the maps $\tilde{X}_i \to X$ (sub)converge to a covering map $\tilde{X} \to X$ and where properly scaled maps $\tilde{X}_i \to S^n$ (sub)converge to Lipschitz maps $\tilde{f}_i : X \to S^n$.

But, in general, these $\tilde{f}_i$ are neither constant at infinity nor do they have non-zero degree, at least not in the ordinary sense (even if $\tilde{f}_i : \tilde{X}_i \to X$ were covering maps to start with). Thus

enlargeability of compact iso-enlargeable manifolds remains problematic even for compact *aspherical*[8] manifolds X.

(Examples of enlargeable manifolds with non-hyperspherical *universal* coverings exhibited in [BH09] tilts one toward accepting a possibility of iso-enlargeable but non-enlargeable compact manifolds X.)

On the other hand, there is the following relation between iso-enlargeability and the overtorical width $width_{\hat{\mathcal{T}}}(X)$ which was defined in the previous section.

If X is compact, then

$$[width_{\hat{\mathcal{T}}}(X) = \infty] \Leftrightarrow [X \text{is iso-enlargeable}].$$

In fact, the (quantitative form of the obvious) implication "⇐" has been already established the previous section.

Now, to prove "⇒", we observe that the maps $f : V \to \mathbb{T}^{n-1} \times [-1, 1]$ used in the definition of "over-torical" can be assumed Lipschitz, where, moreover, the corresponding maps (coordinate projections) $V \to [-1, 1]$ can be arranged to have their Lipschitz constants equal to

$$\frac{2}{width(V)}.$$

These f, by passing to the $\mathbb{Z}^{n-1}$-coverings $\tilde{V} \to V$, become Lipschitz maps $\tilde{f} : \tilde{V} \to \mathbb{R}^{n-1} \times [-1, 1]$. which, by scaling $\varepsilon : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, turn to maps

$$\tilde{f}_\varepsilon : \tilde{V} \to \mathbb{R}^{n-1} \times [-1, 1]$$

with Lipschitz constants arbitrarily close to $\frac{2}{width(V)}$ and which remain proper with degrees $\neq 0$.

Finally, we compose these $\tilde{f}_\varepsilon$ with the obvious map $\mathbb{R}^{n-1} \times [-1, 1] \to S^n$ of degree one and $Lip = \pi$ and obtain maps

$$\tilde{F}_\varepsilon : \tilde{V} \to S^n, \text{ where } deg(\tilde{F}) \neq 0 \text{ and } Lip(\tilde{F}) \leq \frac{2\pi}{width(V)} + \varepsilon'$$

[8] A manifold is called *aspherical* if its universal covering is contractible.

with arbitrarily small ε', and the implication

$$[width_{\hat{\mathcal{T}}}(X) = \infty] \Rightarrow [X \text{is iso-enlargeable}].$$

is thus established.

$\mathcal{V}$-Width and $\mathcal{IE}$-Width. Given a class $\mathcal{V}$ of Riemannian bands V define $width_{\mathcal{V}}(X)$ of a Riemannian manifold X as we did it for $width_{\hat{\mathcal{T}}}$, namely, as

the supremum of numbers d, such that X admits an equidimensional locally isometric (not necessarily globally one-to-one) immersion from a band $V \in \mathcal{V}$ with $width(V) = d$.

Here, we are concerned with the class of *iso-enlargeable orientable bands* V which admits proper maps (i.e. boundary to boundary) $f : V \to Y \times [-1, 1]$, where Y must be compact orientable iso-enlargeable manifolds without boundaries[9] and where $deg(f) \neq 0$.

Iso-enlargeable $\frac{4\pi}{n}$-Inequality. *The iso-enlargeable widths of n-dimensional Riemannian manifolds X are bounded by the over-torical widths as follows.*

$$width_{\hat{\mathcal{T}}}(X) \leq width_{\mathcal{IE}}(X) \leq 2width_{\hat{\mathcal{T}}}(X).$$

Consequently, if $Sc(X) \geq \sigma > 0$, then

$$width_{\mathcal{IE}}(X) \leq 4\pi\sqrt{\frac{n-1}{\sigma n}}.$$

Proof. The inequality $width_{\hat{\mathcal{T}}} \leq width_{\mathcal{IE}}$ is obvious.

To prove $width_{\mathcal{IE}} \leq 2width_{\hat{\mathcal{T}}}$ let us show that iso-enlargeable bands V with width d contain over-torical ones with width $d/2$.

In fact, since the above Y is iso-enlargeable, there exist locally isometric immersions of $(n-1)$-dimensional over-torical bands Y_D to Y with $width(Y_D) \geq D$ for all $D > 0$. Then the pullbacks[10] of $Y_D \times [-1, 1]$ under the maps $f : V \to Y \times [-1, 1]$ come with natural maps

$$f^{-1}(Y_D \times [-1, 1]) \to [0, D] \text{ and } f^{-1}(Y_D \times [-1, 1]) \to [0, d],$$

both with $Lip \leq 1$.

Then the pull back of the circle of radius $d/2$ in $[0, D] \times [0, d]$ under the pair of these maps (which may be assumed smooth and transversal to this circle) serves as the required overtorical band of width $\geq d/2$.

Finally, we recall the $\frac{2\pi}{n}$-inequality for over-torical bands in section 2 and obtain our $\frac{4\pi}{n}$-inequality for iso-enlargeable bands. □

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[9] A more general definition of iso-enlargeability for bands with no reference to closed manifolds is given in section 11.7.

[10] Even if $g : A \to B$ is a non-injective map, we speak of the f-pullback of A for a map $f : C \to B$, where $f^{-1}(A)$ is understood as the set of pairs $\{a, c\}_{g(a)=f(c)} \in A \times C$ which comes with the map $f^{-1}(A) \to C$, $(a, c) \mapsto c$ which has the same kind of (non)-injectivity as $g : A \to B$.

IMPROVEMENT. *The* $\frac{4\pi}{n}$*-inequality for compact iso-enlargeable bands* V *can be upgraded to* $width_{\mathcal{IE}}(V) \le 2\pi\sqrt{\frac{n-1}{\sigma n}}$ as it is explained at the end of section 11.6.[11] Accordingly, the curvature decay estimate below can be improved by the factor of 2.

Iso-enlargeable $[\asymp \frac{8\pi^2}{R^2}]$ -Decay Theorem. Let a manifold X admit a proper map of non-zero degree to the total space $\underline{X}$ of a two dimensional vector bundle $\underline{X} \to Y$ where Y is a compact iso-enlargeable (e.g. admitting a metric with non-positive curvature) manifold.

If the bundle $\underline{X} \to Y$ *is trivial then the scalar curvatures of all complete Riemannian metrics* g *in* X *restricted to concentric balls* $B(R) = B_{x_0}(R) \subset X$ *satisfy*

$$\left[\asymp \frac{8\pi^2}{R^2}\right] \qquad \min_{B(R)} Sc(X) \le \frac{8\pi^2}{(R-R_0)^2} \text{ for some } R_0 = R_0(X, g, x_0) \text{ and all } R \ge R_0.$$

Proof. This follows word for word the argument for the quadratic decay theorem in section 1 and its generalisation in section 2 with "iso-enlargeable " for "over-torical". □

What happens to nontrivial bundles $\underline{X} \to Y$? The above argument applies to non-trivial bundles, where the (total spaces of the) corresponding *circle bundles are iso-enlargeable*, which is so, for instance by the above (c) for Y which admit metrics with non-positive sectional curvatures.

In general, the examples in [BH09] indicate a possibility of non-enlargeable circle bundles over enlargeable Y; yet, it seems hard(er) to find such examples, where the corresponding $\underline{X}$ would admit complete metrics with $Sc \ge \sigma > 0$.

Remarks and Questions. (□) Let V be an n-dimensional manifold with sectional curvatures $\kappa(V) \ge 1$ which admits a proper map $\Phi : V \to [0, d]^n$ given by n functions $\phi_i(v)$ with $Lip(\phi_i) \le 1$ and such that $deg(\Phi) \ne 0$.

Is the maximal d for these V achieved by the regular cube □ in the hemisphere S^n_+ with the boundary $\partial\square \subset S^{n-1} = \partial S^n_+$?

(△) The same question for spherical simplices $\triangle \subset S^n_+$ with $\partial\triangle \subset S^{n-1}$:

do these simplices have maximal distances between opposite faces among all simplices with $\kappa \ge 1$*?*

From $\mathcal{V}$-manifolds to $\mathcal{V}$-Enlargeable ones. $[\mathcal{V} \rightsquigarrow \mathcal{VE}]$: Given a "natural" class $\mathcal{V}$ of manifolds one defines an, a priori larger, class $\mathcal{VE}$ of $\mathcal{V}$-enlargeable manifolds X by the condition

$$width_{\mathcal{V}} = \infty.$$

Thus, for instance the class $\hat{\mathcal{T}}$ of over-torical manifolds leads to the class $\hat{\mathcal{T}}\mathcal{E} \supsetneqq \hat{\mathcal{T}}$ of $\hat{\mathcal{T}}$-enlargeable manifolds, which, as we know, is equal to the class $\mathcal{IE}$ of iso-enlargeable manifolds,

11 The proof in the original version of this paper was incorrect, as it was pointed out to me by the referee.

On the other hand, if we depart from the class $\mathcal{IE} = \hat{\mathcal{T}}\mathcal{E}$, then the new class $\mathcal{IEE}$ defined by $width_{\mathcal{IE}} = \infty$ will coincide with $\mathcal{IE}$.

In the following section, following Schoen–Yau and Schick, we define class $\mathcal{SYS} \supsetneqq \hat{\mathcal{T}}$, where the corresponding class $\mathcal{SYSE}$ of $\mathcal{SYS}$-enlargeable manifolds is *strictly greater than the class of iso-enlargeable ones.*

5 Schoen–Yau–Schick Manifolds and $\mathcal{SYS}$-Bands

Schoen–Yau Definition. [SY79b], [SY17]. A compact orientable n-manifold X is *SYS*, if there exist $n-2$ integer homology classes $h_1, h_2, \dots, h_{n-2} \in H_1(X)$, such that their intersection consecutive

$$h_1 \frown h_2 \frown \cdots \frown h_{n-2} \in H_2(X)$$

is *non-spherical*, i.e. it is *not contained* in the image of the *Hurewicz homomorphism* $\pi_2(X) \to H_2(X)$, or, equivalently, it doesn't lift to the universal covering of X.

Schick Definition. [Sch98] A homology class $h \in H_n(K)$, where $K = K(\Pi, 1)$ is the Eilenberg–MacLane space for an Abelian group Π, is called SYS, if its consecutive *cap-producs* with some cohomology classes $h_1, h_2, \dots, h_{n-2} \in H^1(K, \mathbb{Z})$ are non-zero,

$$(\dots((h \cap h_1) \cap h_2) \cap \dots \cap h_{n-2}) = h \cap (h_1 \smile, \dots, \smile h_{n-2}) \neq 0 \in H_2(K).$$

(Geometrically speaking, generic 2-dimensional intersections of the n-cycles $C \subset K$ representing h with $(n-2)$-codimensional pullbacks of generic points of, some, say piecewise linear, maps $K \to \mathbb{T}^{n-2}$ are non-homologous to zero.)

Then a manifold X is SYS if the Abel classifying map $X \to K(\Pi, 1)$ for $\Pi = H_1(X)$ sends the fundamental class $[X] \in H_n(X)$ to a SYS class in this $K(\Pi, 1)$.

(Recall that, by definition, the spaces $K(\Pi, 1)$ have *contractible* universal coverings and fundamental groups *isomorphic* to Π. The standard finite dimensional approximations to these K are products of tori and *lens spaces* $L_i = S^N/\mathbb{Z}_{l_i}$, where the latter, observe, carry natural metrics with $Sc > 0$.

Abel's $X \to K$ maps, which are unique up-to homotopy, are characterised by inducing isomorphisms on the 1-dimensional homology groups.)

Historical Remark. In 1979 Schoen and Yau proved that SYS manifolds (defined slightly differently in [SY79b] with incorporation of some spin manifolds) of dimensions $n \leq 7$ *carry no metrics with* $Sc > 0$. Then, in the recent paper [SY17], they published the proof for all n.

Meanwhile, Schick [Sch98] has shown that *no available Dirac operator methods can rule out* $Sc > 0$ *on these manifolds.*

Examples.

- $\bullet_1$ Overtorical manifolds are SYS.
- $\bullet_2$ Let X be obtained by a surgery applied on a closed curve C in the n-torus as in [Sch98].
  If $n \geq 4$, then X is SYS if and only if C represents a *divisible* homology class in $H_1(\mathbb{T}^n)$.
  (Such an X is over-torical if and only if C is *homologous to zero*.)
- $\bullet_3$ If a compact orientable manifold X admits a map f of degree one to a SYS manifold that X is SYS.
  But if $deg(f) > 1$ then X is not necessarily SYS, unlike the case of the over-torical and iso-enlargeable manifolds. For instance if the curve C in $\bullet_2$ is m-divisible, than the some m-sheeted covering of X is non-SYS.
  Probably, these non-SYS coverings carry metrics with $Sc > 0$.
- $\bullet_4$ Products of SYS manifolds by overtorical ones are SYS.
  But products SYS $\times$ SYS and SYS $\times$ [iso-enlargeable] are, in general, not SYS.

  SYS-Bands. A band V is called SYS if it admits a band map $(\partial_\pm \to \partial_\pm)$ of degree ± 1 to $Y \times [-1, 1]$ where Y is a compact SYS manifold.[12]

Accordingly, define the $\mathcal{SYS}$-width of $width_{\mathcal{SYS}}(X)$ of Riemannian manifolds X based on the class $\mathcal{SYS}$ as we did it for $\mathcal{IE}$ in the previous section.

$\frac{4\pi}{n}$-Inequality for SYS-Bands. *All Riemannian manifolds X with $Sc(X) \geq \sigma > 0$ satisfy*

$$width_{\mathcal{SYS}}(X) \leq 4\pi\sqrt{\frac{n-1}{\sigma n}}.$$

Consequently, compact manifolds without boundaries, which have

$$width_{\mathcal{SYS}}(X) = \infty$$

admit no metrics with positive scalar curvatures.

Proof. By symmetrising a SYS-band $V \to Y \times [-1, 1]$ as in the proof of the overtorical $\frac{2\pi}{n}$-Inequality in section 2 (now Y plays the role of the torus $\mathbb{T}^{n-1}$ in section 2) we arrive at $V_{\circ^{n-3}}$ with $\mathbb{T}^{n-3}$-invariant metric with $Sc \geq \sigma$, such that

the quotient space $\underline{V}^3 = V_{\circ^{n-3}}/\mathbb{T}^{n-3}$ is an orientable 3-manifold with the boundary decomposed into two (possibly disconnected) disjoint parts say

$$\partial\underline{V}^3 = S_- \cup S_+,$$

where

$$dist_{\underline{V}^3}(S_-, S_+) \geq d$$

[12] A more general definition of SYS for bands with no reference to closed manifolds is given in section 11.7.

for d equal to the distance between the two boundary components in V,
and where the Schoen–Yau–Schick property of Y implies that

if a closed surface $S \subset \underline{V}^3$ separates S_- from S_+, then the homomorphism

$$\pi_1(S) \to \pi_1(\underline{V}^3)$$

has infinite image.

Therefore the $d/2$-equidistance surface to S_- (or to S_+) contains a circle C which has infinite order in $\pi_1(\underline{V}^3)$ and, by the Poincaré duality, the covering $\tilde{\underline{V}}^3$ of $\underline{V}^3$ with the cyclic $\pi_1(\tilde{\underline{V}}^3)$ generated by the (homotopy class of) C contains a relative 2-cycle $\tilde{C}^{\perp}$[13] with non-zero intersection index with the lift $\tilde{C}$ of C to $\tilde{\underline{V}}^3$.

Take the pull back of the cycle $\tilde{C}^{\perp}$ to the corresponding covering $\tilde{V}_{\circ^{n-3}}$ of

$$V_{\circ^{n-3}} = (V_{\circ^{n-3}}/\mathbb{T}^{n-3}) \times \mathbb{T}^{n-3},$$

write this pullback cycle as

$$\tilde{C}^{\perp} \times \mathbb{T}^{n-3} \subset \tilde{V}_{\circ^{n-3}},$$

and symmetrize the minimal cycle in the $(n-1)$-homology class of $\tilde{C}^{\perp} \times \mathbb{T}^{n-3}$.

Since $dist(\tilde{C}, \partial\tilde{\underline{V}}^3) = dist(C, \partial\underline{V}^3) \geq d/2$, the quotient surface of the resulting $\tilde{V}_{\circ^{n-2}}$ contains a point within distance $\geq d/2$ from its boundary, which implies (compare p. 310 in [GL83]) that

$$d/2 \leq 2\pi\sqrt{(n-1)/\sigma n}. \qquad \square$$

Question. Can one replace the above $4\pi\sqrt{\frac{n-1}{\sigma n}}$ by $2\pi\sqrt{\frac{n-1}{\sigma n}}$?

6 $\mathcal{SYS}$-Enlargeable Manifolds and Codimension Two Depth Inequalities

A Riemannian manifold X is called *$\mathcal{SYS}$-Enlargeable* if it has *infinite* $\mathcal{SYS}$-width.

For instance, SYS manifolds and iso-enlargeable manifolds are *$\mathcal{SYS}$-Enlargeable.*

What is more interesting is that

if an n-manifold X admits a proper Lipschitz map ϕ (Lipschitz means $Lip(\phi) < \infty$) *to an iso-enlargeable manifold of dimension $n-2$, say $\phi: X \to \underline{X}$, such that the homological pullback $\phi^![\underline{x}] \in H_2(X)$, $[\underline{x}] = 1 \in H_0(\underline{X}) = \mathbb{Z}$, is non-spherical (as in the first definition of SYS in the previous section), then X is $\mathcal{SYS}$-enlargeable.*

Therefore, by the above $\frac{4\pi}{n}$-inequality,

If such an X is compact, then it admits no metric with $Sc > 0$.

[13] Relative means relative to $\partial\tilde{\underline{V}}^3 + \partial_\infty\tilde{\underline{V}}^3$ where ∂_∞ stands for the complement of a large ball in $\tilde{\underline{V}}^3$.

Thus, for example,

products X of SYS manifolds by compact iso-enlargeable ones (e.g. those which admit metrics with $\kappa(X_2) \le 0$) admit no metrics with positive scalar curvatures.

(These X, in general, are neither iso-enlargeable nor SYS.)

$\frac{8\pi}{n}$-Inequality for SYSE-Bands. Denote by $\mathcal{SYSE}$ the class of SYS-enlargeable manifolds, say that a compact band V is SYSE if it admits a map of degree ±1 to $Y \times [-1,1]$, where Y is SYSE and accordingly define $width_{SYSE}(X)$ for Riemannian manifold X (see $[\mathcal{V} \rightsquigarrow \mathcal{VE}]$ in section 4).

Then by arguing as in the proof of the iso-enlargeable $\frac{4\pi}{n}$-inequality in section 4 we conclude that

$$width_{\mathcal{SYS}}(X) \le width_{\mathcal{SYSE}}(X) \le 2width_{\mathcal{SYS}}(X)$$

for all Riemannian manifolds X.

Consequently,

if $Sc(X) \ge \sigma > 0$ then

$$width_{\mathcal{SYSE}}(X) \le 8\pi\sqrt{\frac{n-1}{\sigma n}}.$$

Question. Can one improve 8π to 2π or, at least, to 4π?

Depth Inequalities. Define *the depth of a homology class h* in a Riemannian manifold X with boundary as the supremum of $d \ge 0$ such that h can be represented by a cycle positioned within distance $\ge d$ from the boundary of X. (If X is incomplete, we include the points obtained by completion of X in the boundary of X.)

Let Y be a closed $(n-2)$-dimensional manifold and $p : \underline{X} \to Y$ be a disc bundle, e.g. the trivial one $\underline{X} = Y \times B^2$.

Let X be a compact n-manifold with boundary and $f : X \to \underline{X}$ be a proper continuous map where *proper*, means *boundary* → *boundary*. Let $h = f^!([Y] \in H_{n-2}(X)$ be the homology pull-back of the homology class of the zero section $Y = Y_{\mathbf{0}} \subset \underline{X}$.

Let $\underline{X}_{-\varepsilon} \subset \underline{X}$ be the complement of the open ε-neighbourhood of $Y_{\mathbf{0}}$ in $\underline{X}$ and observe that the boundary of $\underline{X}_{-\varepsilon}$ consists of two components, call them $\underline{\partial}_\pm$ which are canonically homeomorphic to the total space of the circle bundle associated to $\underline{X} \to Y$, denoted $p_\circ : Y_\circ \to Y$.

Let $\partial_\pm = \partial_\pm(X) \subset \partial X$ be the two parts of the boundary of X which are sent by the map $f : X \to \underline{X}$ to $\underline{\partial}_+$ and to $\underline{\partial}_-$ correspondingly.

Observe that

- $\underline{X} = Y_\circ \times [\varepsilon, 1]$;
- If Y is iso-enlargeable then $Y_\circ$ is also iso-enlargeable.
- if the fibration $p : \underline{X} = Y$ is trivial, $\underline{X} = Y \times B^2$, and

if Y is over-toric then also $Y_\circ$ is over-toric,
if Y is SYS then $Y_\circ$ is also SYS,

if Y is SYSE then $Y_\circ$ is also SYSE.
Now
let the fibration $p:\underline{X}\to Y$ *be trivial* and let

$$Sc(X)\geq n(n-1)=Sc(S^n).$$

Observe that the band-width $\frac{k\pi}{n}$-inequalities, (for $k=2,4,8$ see sections 2, 4, 5) imply the following bounds on the depths of $h\in H_{n-2}(X)$ by the argument that we have already used several times, e.g. in the proofs of the quadratic decay inequality in section 1.

$[\hat{\mathcal{T}}]_\circ$ *If* Y *is over-torical, i.e. if it admits a map to the torus* $\mathbb{T}^{n-2}$ *with degree* $\neq 0$, *then*

$$depth(h)\leq\frac{2\pi}{n}.$$

This is the only case where our inequality is (known to be) sharp,

$[\mathcal{IE}]_\circ$ *If* Y *is iso-enlargeable, e.g. if it admits a metric with non-positive sectional curvature, then*

$$depth(h)\leq\frac{4\pi}{n}.\text{[14]}$$

(Here the fibration p need not be trivial.)

$[\mathcal{SYS}]_\circ$ *If* Y *is SYS and if the map* $f:X\to\underline{X}$ *has* $deg(f)=\pm1$, *then*

$$depth(h)\leq\frac{4\pi}{n}.$$

(The simplest example of a non-overtoric SYS manifold Y for $n-2\geq 4$ is obtained from the $(n-2)$-torus by attaching a 2-handle based on a k-multiple of closed curve in this torus where $k\neq\pm1$. In this case one only need $deg(f)$ to be non-divisible by k.)

$[\mathcal{SYSE}]_\circ$ *If* Y *is SYSE and if the map* $f:X\to\underline{X}$ *has* $deg(f)=\pm1$, *then*

$$depth(h)\leq\frac{8\pi}{n}.$$

(Recall, this was stated earlier, here as everywhere in this paper the above inequalities are established unconditionally for $n\leq 8$, while the case $n\geq 9$ relies on the recent partial regularity results by Lohkamp and by Schoen and Yau which the present author has not studied in detail.)

On nontrivial bundles $p:\underline{X}\to Y$. Here, similarly to where we addressed this issue in section 4, one may drop the triviality of p assumption, if, for instance, Y admits a metric with $\kappa\leq 0$.

No reasonable assumption of this kind, however, seems in view for SYS and SYSE manifolds.

In fact, circle bundles over many SYS manifolds, say on those obtained by surgery on closed curves in $\mathbb{T}^n$ (see $\bullet_2$ in section 5) are very likely to carry metrics with $Sc>0$

and so the above inequality can't hold with any constant for non-trivial fibrations $p : \underline{X} \to Y$.

On Complete Manifolds and Dirac Operators. The inequality $depth(h) < \infty$ implies that the interiors of the manifolds X in $[\mathcal{IE}]_\circ$ and $[\mathcal{SYSE}]_\circ$

admit no complete metrics g with $Sc(g) \geq \sigma > 0$.

(The inequality $depth(h) < \infty$ in the remaining cases follow from these two.)

Strangely enough, even if X is spin, this was proven by the Dirac operator methods for *enlargeable and related* manifolds Y [GL83, HPS15] *only under additional geometric assumptions* on X in spirit of "bounded geometry".

(To be honest, I am not 100% certain this is the case for [HPS15]. The main result is stated in this paper for closed manifolds and I have not followed the proofs in sufficient details to understand what is actually proven there for complete non-compact manifolds.[15])

Question. Do all products manifolds $Y \times \mathbb{R}^2$, and, more generally, the total spaces of all $\mathbb{R}^2$-bundles admit complete metrics g with $Sc(g) \geq 0$?

Do, for example, such metrics g exist for compact manifolds Y which admit metrics with strictly negative sectional curvatures?

If there are no such g among rotationally symmetric warped product metrics,[16] then, probably, no complete metric g on $Y \times \mathbb{R}^2$ has $Sc(g) \geq 0$, where the best candidates of this kind of manifolds with no complete metrics on them with $Sc \geq 0$ are non-trivial $\mathbb{R}^2$-bundles over surfaces of genera ≥ 2.

7 External Curvature, Focal Radius and Depth in Codimension> 2

Observe that by Gauss theorema egregium the scalar curvature of hypersurfaces $Y \subset S^n$, $n \geq 2$, with principal curvatures $c_i = c_i(y)$, $y \in Y$, $i = 1, \dots, n-1$, satisfies

$$Sc(Y) = Sc(S^{n-1}) + \left(\sum_i c_i\right)^2 - \sum_i c_i^2 \geq (n-1)(n-2) - \sum_i c_i^2.$$

It follows that if an $(n-1)$-dimensional manifold manifold Y *admits no metric with $Sc > 0$*, that the suprema of the principal curvatures of all smooth immersions from Y to the unit sphere S^n satisfy

$$\sup_{i,y} |c_i(y)| \geq \sqrt{n-2}$$

This is significantly weaker then the $\frac{\pi}{n}$-inequality for the normal radius of $\mathbb{T}^{n-1} \subset S^n$, which implies that $\sup_{i,y} c_i(y) \geq \frac{(1+\varepsilon_n)n}{\pi}$. But it applies to such manifolds, for instance, as certain exotic spheres Y of dimensions $8m+1$ and $8m+2$ which carry

[15] Bernhard Hanke told me that the results from [HPS2015] apply to complete manifolds by the C^*-arguments in Roe's partitioned index theorem and in [HS 2007].

[16] Figuring this out does not seem hard, but I have not tried doing this.

no metrics with $Sc > 0$ by a theorem of Hitchin [Hit74], yet are immersible (but not embeddable!)[17] to S^n by Smale-Hirsch theorem.

Besides, this $\sup c_i$ inequality obviously generalises to Y in S^n of all codimensions k where it reads

$$\sup_{i,j,y}|c_{i,j}(y)| \geq \frac{\sqrt{n-k-1}}{k},\ i = 1,\dots,n-1,\ j = 1,\dots,k,\ y \in Y,$$

for all Y which admit no metric with $Sc > 0$.

Then, obviously, the same holds true for Riemannian manifolds $X \supset Y$ with *sectional curvatures* $\kappa \geq 1$.

More interestingly, a similar inequality holds for immersions to unit Euclidean balls $B(1) \subset \mathbb{R}^n$ Namely,

if an $(n-k)$ dimensional Y admits no metric with $Sc > 0$, then the principal curvatures of all smooth immersions $Y \to B(1) \subset \mathbb{R}^n$ are bounded from below by

$$\sup_{i,j,y}|c_{ij}(y)| \geq \frac{1}{const}\frac{\sqrt{n-k-1}}{k}$$

for some universal positive constant $const \leq 100$.

Proof. The Euclidean case reduces to the spherical one, since the standard projective map $\mathbb{R}^n \supset B(1) \to S^n$ distorts curvatures of the curves in $B(1)$ by a bounded amount. (Compare Lemma (C′) in 3.2.3 in [Gro86]). □

Remark. This $\sup c_{i,j}$-inequality also holds in the balls in the hyperbolic spaces with sectional curvature $\kappa = -1$.

Also, the following weaker form of this inequality holds for the unit balls in all n-dimensional Riemannian manifolds X with $-1 \leq \kappa(X) \leq 1$.

$$\sup_{i,j,y}|c_{ij}(y)| \geq \frac{1}{const}\frac{\sqrt{n-k-1}}{k} - const'.$$

In fact—this is obvious by today's standards—the exponential maps $exp : T_x(X) \subset B(1) \to X$ in these X can be approximated by maps with controlled distortion of curvatures of the curves in $B(1)$.

Discussion. There is a huge gap between the above lower bounds on the curvatures of submanifolds in S^n (and/or in $B(1) \subset \mathbb{R}^n$) and the observed curvatures in the available examples $Y \subset S^n$.

Probably, certain homogeneous submanifolds $Y \subset S^n$, such as

[17] According to textbooks' terminology, a smooth map $A \to B$ is an *immersion* if it is *locally* one-to-one and the inverse map is smooth, while *embeddings* are immersions which are globally one-to-one and, if Y is non-compact, are additionally required to be homeomorphisms from Y to their (possibly, non-closed in B) images.

- real and complex projective spaces *Veronese* represented by symmetric/Hermitian forms of rank one,
- Grassmannians *Plücker* embedded to exterior powers of linear spaces,
- the same Grassmannians represented by projectors in spaces of operators,

give a fair idea of embeddings with economical c_{ij}.

For instance, the curvature of the obvious embedding of the product of spheres

$$Y = S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k} \subset S^{n_1+n_2+\ldots+n_j+j-1} = \partial B(1) \subset \mathbb{R}^{n_1+n_2+\ldots+n_j+j}$$

has $\max c_{ij} = \sqrt{k}$ and it is plausible (?) that

no embedding/immersion of this Y to S^n may have a (significantly) smaller curvatures c_{ij}.

Notice that above local bound $\max c_{ij} \gtrsim \sqrt{n}/k$ is non-vacuous only if all spheres are one dimensional, while the only known improvement of this bound is the inequality $\max c_{ij} \gtrsim n$ which was established in the previous section only for codimensions 1 and 2 and only for $\mathcal{SYSE}$-manifolds Y (e.g. for Y which admits metrics with $\kappa \le 0$.)

This, for instance, leaves the following questions open.

(a) *Does the torus*

$$\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n-3}$$

embed to S^n with the principal curvature $c_{ij} \le 100/n$?

(b) *Does the product*

$$\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n-3} \times S^2$$

embed to S^n with the principal curvature $c_i \le 10$?

In fact, we are more interested in *depth of homology and cohomology classes* in Riemannian manifolds V rather than in their curvatures, where,

by definition, *depth*$(h) \ge d$ for an $h \in H^*(V)$ if *the restriction of h to the subset* $V_{-d} \subset V$ of the points within distance $\ge d$ from the boundary of V (including the infinity for non-compact V, as in the previous section) *does not vanish.*

Problem. *Bound "complexity" of an h in terms of d = depth(h).*

For instance, let the sectional curvature of V be bounded from below by $\kappa(V) \ge 1$ and let h be induced by a continuous map from the fundamental cohomology class of a product of spheres,

$$h = f^*[Y] \text{ for } f : V \to Y = S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}.$$

Does the depth of h necessarily tend to zero for $k \to \infty$?

8 Symmetrization of Riemannian Manifolds with Point-wise Control of the Scalar Curvature

Step 1 in Symmetrization by Reflections. Let X be a compact Riemannian manifold, with a (possibly empty) boundary, and let $(Y_0, \partial Y_0) \subset (X, \partial X)$ be a cooriented hypersurface which is *strictly locally volume minimising*, i.e. all sufficiently close to Y hypersurfaces $(Y, \partial Y) \subset (X, \partial X)$ different from Y_0 satisfy

$$vol_{n-1}(Y) > vol_{n-1}(Y_0).$$

Let $U \subset X$ be a (small) neighbourhood of Y_0 in X which is divided by Y_0 into two "halves", denoted $U_\pm \subset U$, and let $Y_{\pm\varepsilon} \subset U_\pm$ be hypersurfaces homologous to Y_0 in $U_\pm$ which minimise the functionals

$$Y \mapsto vol_{n-1}(Y) - \varepsilon \cdot vol_n(U_{\pm\varepsilon})$$

where

$$U_{\pm\varepsilon} = U(Y_{\pm\varepsilon}) \subset U_\pm$$

denote the regions bounded by $Y_{\pm\varepsilon}$ and Y_0.

By the basic regularity theorems of Simons-Federer-Almgren-Allard these $Y_{\pm\varepsilon}$ do exist for small $\varepsilon \geq 0$ and they are smooth away from closed subsets of Hausdorff codimension ≥ 7.

Moreover, if $n = dim(X) = 8$, then, according to [Sma93],[18] these $Y_\pm(\varepsilon)$ everywhere smooth for an open dense set of $\varepsilon > 0$ [Sma93].[19]

The mean curvatures of all these $Y_{\pm\varepsilon}$ at the regular points satisfies

$$mean.curv(Y_{\pm\varepsilon}) = \varepsilon$$

and the dihedral angles between the tangent spaces to $Y_{\pm\varepsilon}$ and those to ∂X at all regular points of $Y_{\pm\varepsilon}$ on the boundary $\partial Y_{\pm\varepsilon}$ are $\leq \frac{\pi}{2}$. (If the boundary $\partial X \subset X$ is totally geodesic then $Y_{\pm\varepsilon}$ is normal to ∂X.)

On non-strictly minimal Y_0. If Y_0 is *non-strictly* volume minimising, then there are hypersurfaces in X with the same volume as Y_0, which lie ε-close to Y_0 for all small ε. These do not intersect Y_0 and each of them lies on one side of Y_0, where it plays the role of $Y_{-\varepsilon}$ or of $Y_{+\varepsilon}$.

Alternatively, one may slightly perturb the metric in X, such that $Y_0 = Y_{0,\epsilon}$ becomes strictly minimising. Then this ϵ, which goes through the following stages of symmetrisation, is sent to 0 at the end of the symmetrization process.

Let

$$U_{[-\varepsilon,\varepsilon]} = U_{-\varepsilon} \cup U_{+\varepsilon}$$

[18] What we need is not formulated in [Sma93] but the argument from [Sma93] does apply.

[19] I am not certain this is formulated in [Sma93].

and let $\tilde{U}_{[-\varepsilon,\varepsilon]}$ be obtained by reflecting $U_{[-\varepsilon,\varepsilon]}$ in the two parts $Y_{\pm\varepsilon}$ of the (relative) boundary of $U_{[-\varepsilon,\varepsilon]}$ in X (i.e. with the exclusion of $U_{[-\varepsilon,\varepsilon]} \cap \partial X$), denoted

$$\partial_{\pm} U_{[-\varepsilon,\varepsilon]} = Y_{-\varepsilon} \cup Y_{+\varepsilon}.$$

In other words, $\tilde{U}_{[-\varepsilon,\varepsilon]}$ is a space, which is acted upon by the semidirect product group $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}_2$, such that

- $$\tilde{U}_{[-\varepsilon,\varepsilon]}/\Gamma = U_{[-\varepsilon,\varepsilon]},$$
- there is an embedding $E : U_{[-\varepsilon,\varepsilon]} \hookrightarrow \tilde{U}_{[-\varepsilon,\varepsilon]}$ which is inverse to the quotient map $Q : \tilde{U}_{[-\varepsilon,\varepsilon]} \to U_{[-\varepsilon,\varepsilon]}$,
$$Q \circ E = Id : U_{[-\varepsilon,\varepsilon]} \to U_{[-\varepsilon,\varepsilon]},$$
- the group Γ is generated by two involutions (reflections) of $\tilde{U}_{[-\varepsilon,\varepsilon]}$, one of them fixing $E(Y_{-\varepsilon}) \subset E(\partial_{\pm} U_{[-\varepsilon,\varepsilon]})$ and the other one $E(Y_{+\varepsilon}) \subset E(\partial_{\pm} U_{[-\varepsilon,\varepsilon]})$.

Thus, the action of our $\Gamma = \Gamma_{\varepsilon}$ on $\tilde{U}_{[-\varepsilon,\varepsilon]}$ mimics the action of the same group on the line (∞, ∞), which is generated by the transformations

$$t \mapsto \pm\varepsilon - t$$

and where $\tilde{U}_{[-\varepsilon,\varepsilon]}$ admits a Γ-equivariant map to (∞, ∞), such that the pullback of $[-\varepsilon, \varepsilon] \subset (-\infty, \infty)$ is equal to $E(U_{[-\varepsilon,\varepsilon]}) \subset \tilde{U}_{[-\varepsilon,\varepsilon]}$.

In particular, the action of the group $\mathbb{Z} = 2\varepsilon\mathbb{Z} \subset \Gamma$ on $\tilde{U}_{[-\varepsilon,\varepsilon]}$ is free and the quotient space is equal to the double of $U_{[-\varepsilon,\varepsilon]}$,

$$\tilde{U}_{[-\varepsilon,\varepsilon]}/2\varepsilon\mathbb{Z} = U_{[-\varepsilon,\varepsilon]} \bigcup_{\partial_{\pm} U_{[-\varepsilon,\varepsilon]}} U_{[-\varepsilon,\varepsilon]},$$

where the boundary of this double is the double of the region $U_{[-\varepsilon,\varepsilon]} \cap \partial X$ across the boundary of this region in ∂X,

$$\partial \left(U_{[-\varepsilon,\varepsilon]} \bigcup_{\partial_{\pm} U_{[-\varepsilon,\varepsilon]}} U_{[-\varepsilon,\varepsilon]} \right) = \left(U_{[-\varepsilon,\varepsilon]} \cap \partial X \right) \bigcup_{\partial'} \left(U_{[-\varepsilon,\varepsilon]} \cap \partial X \right)$$

for

$$\partial' = \partial \left(U_{[-\varepsilon,\varepsilon]} \cap \partial X \right) = \partial_{\pm} U_{[-\varepsilon,\varepsilon]} \cap \partial X.$$

The Riemannian metric on $U_{[-\varepsilon,\varepsilon]} \subset X$, that is the restriction of the metric of X to $U_{[-\varepsilon,\varepsilon]} \subset X$, naturally induces a Γ_{ε}-invariant path metric in $\tilde{U}_{[-\varepsilon,\varepsilon]}$, call it $\tilde{g}_{\varepsilon}$; if the hypersurfaces $Y_{\pm\varepsilon} \subset X$ are non-singular, e.g. if $n = dim(X) \leq 7$, then this metric is C^0-Riemannian.

And if, moreover, the minimal hypersurface $X_0 \subset X$ is *non-singular*, e.g. for $n \le 7$, then the spaces $(\tilde{U}_{[-\varepsilon,\varepsilon]}, \tilde{g}_\varepsilon)$ Hausdorff converge to a smooth Riemannian manifold, which we denote

$$(\underset{\rightarrow}{\tilde{U}}_0, \underset{\rightarrow}{\tilde{g}}_0) = \lim_{\varepsilon\to 0}(\tilde{U}_{[-\varepsilon,\varepsilon]}, \tilde{g}_\varepsilon),$$

which is isometrically acted upon by the group $\mathbb{R} \rtimes \mathbb{Z}_2 = \lim_{\varepsilon\to 0}\Gamma_\varepsilon$ for the above $\Gamma_\varepsilon \subset \mathbb{R} \rtimes \mathbb{Z}_2$, where the action of $\mathbb{R} \subset \mathbb{R} \rtimes \mathbb{Z}_2$ is free.

In fact, $\underset{\rightarrow}{\tilde{U}}_0 = Y_0 \times \mathbb{R}$ and

$$\underset{\rightarrow}{\tilde{g}}_0 = dy^2 + \tilde{\phi}(y)^2 dt^2$$

where $\tilde{\phi}$ is a smooth function on Y_0, which, probably,[20] is equal to ϕ from section 2, which was defined there via the second differential (variation) of the function $Y \mapsto vol_{n-1}(Y)$.

In general, the spaces $(\tilde{U}_{[-\varepsilon,\varepsilon]}, \tilde{g}_\varepsilon)$ Hausdorff converge away from the Γ_ε-orbits of the δ-neighbourhoods of the singularity of $E(Y_0) \subset \tilde{U}_{[-\varepsilon,\varepsilon]}$, where $\delta = \delta_\varepsilon \to 0$ for $\varepsilon \to 0$.

Then $(\underset{\rightarrow}{\tilde{U}}_0, \underset{\rightarrow}{\tilde{g}}_0)$ stands for the metric completion of the resulting (smooth Riemannian) limit space for $\varepsilon, \delta \to 0$.

Question. Does $\mathbb{R}$ act *freely* on $\underset{\rightarrow}{\tilde{U}}_0$ in the case where the minimal hypersurface $Y_0 \subset X$ has *singularities*?

Consecutive Torical Symmetrization of $(n-m)$-Overtorical Manifolds. A compact oriented n-dimensional Riemannian manifold X with (possibly empty) boundary is called $(n-m)$-*overtorical*, if it comes along with a continuous map $f : X \to \mathbb{T}^{n-m}$, such that the "homological pullback" of a point $\theta_0 \in \mathbb{T}^{n-m}$, denoted $f^*[\theta_0] \in H_m(X, \partial X)$, doesn't vanish.

(To clarify, recall that if f_o is a smooth map homotopic to f, then the pullbacks $f_o^{-1}(\theta) \subset X$ of *generic* $\theta \in \mathbb{T}^{n-m}$ are smooth oriented submanifolds of dimensions m which are homologous to $f^*[\theta_0]$, i.e. $[f_o^{-1}(\theta)] = f^*[\theta_0]$.

Alternatively, $f^*[\theta_0]$ can be defined as the Poincare dual of the image of the fundamental cohomology class $[\mathbb{T}^{n-m}]^* \in H^{n-m}(\mathbb{T}^{n-m};\mathbb{Z})$ under the cohomology homomorphism $f^* : H^{n-m}(\mathbb{T}^{n-m};\mathbb{Z}) \to H^{n-m}(X;\mathbb{Z})$.

For instance, $\mathbb{T}^{n-m} \times B$ is $(n-m)$-overtorical for all compact m-dimensional manifolds B, possibly with a boundary.)

If X is $(n-m)$-overtorical and if $Y_0 \subset X$ represents the homology class in $H_{n-1}(X, \partial X)$ corresponding to the (homological) f-pullback of a codimension 1 torus in $\mathbb{T}^{n-m}$, then the manifold $X(\varepsilon) = \tilde{U}_{[-\varepsilon,\varepsilon]}/2\varepsilon\mathbb{Z}$ is also $(n-m)$-*overtorical* for all sufficiently small $\varepsilon > 0$.

[20] This is, of course, easy to check but we do not need it here. In any case, we prefer the Haussdorf limit definition of $\underset{\rightarrow}{\tilde{g}}_0$, since it is less demanding on the regularities of X and Y_0.

In fact, this has nothing to do with minimality of Y_0. It is, obviously, true for all hypersurfaces $Y_0 \subset X$ in suitable homology classes in $H_{n-1}(X, \partial X)$ and their small neighbourhoods $U_{[\pm\varepsilon]} \subset X$ say with smooth boundaries or even with singular boundaries, provided the singular loci have codimensions ≥ 2.

Now let us iterate

$$X \mapsto X(\varepsilon_1) \mapsto X(\varepsilon_1, \varepsilon_2) = X(\varepsilon_1)(\varepsilon_2) \mapsto X(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_i) \mapsto \dots,$$

where each step

$$X(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_i) \mapsto X(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_i, \varepsilon_{i+1})$$

depends on the choice of a codimension 1 subtorus in the corresponding torus $\mathbb{T}^{n-m}$, where each minimal

$$Y_{0_i} \subset X(\varepsilon_1, \dots, \varepsilon_i) \text{ and the corresponding } Y_{\pm\varepsilon_i} \subset X(\varepsilon_1, \dots, \varepsilon_i)$$

are taken in the non-singular loci of $X(\varepsilon_1, \dots, \varepsilon_i)$ modulo their complements (which have positive codimensions.

It should be noted that the $(n-m)$-tori serving $X(\varepsilon_1, \dots, \varepsilon_i)$ and $X(\varepsilon_1, \dots, \varepsilon_j)$ with $j \neq i$ are *not canonically isomorphic.* However, it make sense of taking a generic infinite sequence of these subtori, and then to send all $\varepsilon_i \to 0$.

Let us incorporate the relevant properties of the resulting limit space, call it $\tilde{X}_\infty$, in the following.

Definition of $\mathbb{R}^{n-m} \rtimes O(n-m)$-Symmetrization. Given an $(n-m)$-overtorical manifold X, call an oriented manifold $\tilde{X}_{sm}$ with (possibly empty) boundary an $\mathbb{R}^{n-m} \rtimes O(n-m)$-*symmetrization of* X if it satisfies the following nine conditions.

- $\bullet_1$ $\tilde{X}_{sm}$ is *isometrically acted upon by* $\mathbb{R}^{n-m} \rtimes O(m)$, that is the isometry group of $\mathbb{R}^{n-m}$, such that *the orbits of this action are equal to the* $\mathbb{R}^{n-m}$-*orbits.*
- $\bullet_2$ The action of $\mathbb{R}^{n-m}$ on $\tilde{X}_{sm}$ is *free.*
- $\bullet_3$ The quotient map $\tilde{X}_{sm} \to \tilde{X}_{sm}/\mathbb{R}^{n-m}$ admits an *inverse*, say $E_\infty : \tilde{X}_{sm}/\mathbb{R}^{n-m} \to \tilde{X}_{sm}$, where the image

$$E_\infty(\tilde{X}_{sm}/\mathbb{R}^{n-m}) \subset \tilde{X}_{sm}$$

  is *normal* to the $\mathbb{R}^{n-m}$-orbits in $\tilde{X}_{sm}$.
- $\bullet_4$ there exists *a continuous map*

$$\tilde{\varphi}_\infty : \tilde{X}_{sm} \to X,$$

  with the following properties.
- $\bullet_5$ The map $\tilde{\varphi}_\infty$ is $\mathbb{R}^{n-m} \rtimes O(n-m)$-*invariant.*
- $\bullet_6$ The corresponding map φ_∞ from the quotient manifold $\tilde{X}_{sm}/\mathbb{R}^{n-m}$ to X,

$$\varphi_\infty = \tilde{\varphi}_\infty/\mathbb{R}^{n-m} : \tilde{X}_{sm}/\mathbb{R}^{n-m} \to X,$$

  sends the fundamental homology class

$$[\tilde{X}_{sm}/\mathbb{R}^{n-m}] \in H^m(\tilde{X}_{sm}/\mathbb{R}^{n-m}, \partial\tilde{X}_{sm}/\mathbb{R}^{n-m}) \textit{ to } f^*[\theta_0] \in H^m(X, \partial X).$$

$\bullet_7$ The map $\tilde{\varphi}_\infty$ is *1-Lipschitz.*
$\bullet_8$ The map $\tilde{\varphi}_\infty$ is *scalar curvature non-increasing*,

$$Sc(X)(\tilde{\varphi}_\infty(\tilde{x})) \leq Sc(\tilde{X}_{sm})(\tilde{x}),\ \tilde{x} \in \tilde{X}_{sm},$$

$\bullet_9$ The map $\tilde{\varphi}_\infty$ sends $\partial \tilde{X}_{sm} \to \partial X$ and it is *mean curvature non-increasing*,

$$mn.curv(\partial X)(\tilde{\varphi}_\infty(\tilde{x})) \leq mn.curv(\tilde{\partial} X_{sm})(\tilde{x}),\ \tilde{x} \in \tilde{\partial} X_\infty.$$

In short, $\bullet_6$ -$\bullet_9$ say that symmetrization $X \mapsto \tilde{X}_\infty$ must be

the topology and distance in X non-increasing

and, at the same time,

the scalar and mean curvatures non-decreasing.

Symmetrization Theorem. *Every $(n-m)$-overtorical manifold X of dimension $n \leq 7$ admits an $\mathbb{R}^{n-m} \rtimes O(n-m)$-symmetrization.*

Proof. If $n \leq 7$, then there is no serious problem with singularities, and the above limit space $\tilde{X}_\infty$ satisfies the above conditions $\bullet_1$ - $\bullet_9$.

In fact, even though the natural metrics on $X(\varepsilon_1, \ldots, \varepsilon_i)$ are only piecewise smooth, the limit space $\tilde{X}_{sm}$ of the $\mathbb{Z}^{n-m}$-covers of $X(\varepsilon_1, \ldots, \varepsilon_i)$ for $\varepsilon_i \to 0$, $i = 1, 2, \ldots$, is a *smooth* Riemannian manifold with boundary.

Then everything becomes fairly obvious, except for $\bullet_8$ and $\bullet_9$.

To prove these what we use is that, at every symmetrization step $X \mapsto X(\varepsilon)$, the Riemannian metric on $X(\varepsilon)$, which is, a priori, only continuous and which may have a corner at the boundary, can be

C^∞-smoothed with an arbitrarily small decrease of the scalar curvature of X as well as of the mean curvature of ∂X.

This is explained in section 11.2.

Then the proof follows by *semicontinuity* of the scalar curvature and mean curvatures (the latter is essentially obvious) under C^0-limits of Riemannian metrics, see [Gro14a] and [Bam16]. □

About $n > 7$. It is not impossible that the Symmetrization Theorem remains valid for all n but $\tilde{X}_\infty$ can be used for this purpose only for $m \leq 6$.

For instance, suppose X is homeomorphic to $\mathbb{T}^1 \times S^7$. If our minimizing hypersurface $Y \subset X$ in the homology class of $S^7 \subset X$ has a singularity, then, clearly, the space $\tilde{X}_\infty$ is also singular because $\tilde{X}_\infty / \mathbb{R}^1 = Y$.

On the other hand if X is homeomorphic to, say, $\mathbb{T}^{n-m} \times S^m$ for $m \leq 6$, then $\tilde{X}_\infty$ and $\tilde{X}_\infty / \mathbb{R}^{n-m}$ can be, a priori, non-singular.

This makes the Symmetrization Theorem plausible for all n and $m \leq 6$.

And the techniques (results?) from [SY17], probably, yield the symmetrization theorem for all n and $m \leq 2$.

Almost Symmetrization. This means that the conditions $\bullet_8$ and $\bullet_9$ need to be satisfied up to an arbitrarily small error $\epsilon > 0$:

$$Sc(X)(\tilde{\varphi}_\infty(\tilde{x})) \leq Sc(\tilde{X}_{sm})(\tilde{x}) + \epsilon$$

and

$$mn.curv(\partial X)(\tilde{\varphi}_\infty(\tilde{x})) \le mn.curv(\tilde{\partial} X_\infty)(\tilde{x}) + \epsilon$$

Now, since by [Sma93] minimal hypersurface in generic 8-manifolds are non-singular, the argument used for $n \le 7$, implies the following.

Almost Symmetrization Theorem for $n = 8$. *All $(n-m)$-overtorical manifods of dimension $n \le 8$ admit almost $\mathbb{R}^{n-m} \rtimes O(n-m)$-symmetrizations.*

9 Application of Symmetrization to Manifolds with Positive and with Negative Scalar Curvatures

Let V be a Riemannian band and let $Z_0 \subset V$ be a closed hypersurface which separates $\partial_- V$ from $\partial_+ V$ and, thus, divides V into two halves $V_\pm \supset \partial_\pm(V)$. Let

- the mean curvatures of $\partial_\pm V$ are bounded from below by some constants $M_\pm$;
- the scalar curvature of V is bounded by a given function $\sigma = \sigma(d)$ of the signed distance $d = d(v)$ from v to Z_0, that is

  $$Sc(V)(v) \ge \sigma(dist_\pm(v, Z_0)),$$

  where

  $$dist_\pm(v, Z_0) = dist(v, Z_0) \text{ for } v \in V_+ \text{ and } dist_\pm(v, Z_0) = -dist(v, Z_0) \text{ for } v \in V_-.$$

Since $\mathbb{R}^{n-1} \rtimes O(n-1)$-symmetric metrics on $\mathbb{R}^{n-1} \times [-l, l]$ cane be written as

$$\hat{g} = \hat{\varphi}^2 g_{Eu} + dt^2$$

where g_{Eu} is the flat Euclidean metric on $\mathbb{R}^{n-1}$ and where the scalar curvature of $\hat{g}$, which depends only on $t \in [-l, l]$, satisfies

$$◑ \qquad Sc(\hat{g}) = -2(n-1)\frac{\hat{\varphi}''}{\hat{\varphi}} - (n-1)(n-2)\frac{(\hat{\varphi}')^2}{\hat{\varphi}^2},$$

the net effect of $\mathbb{R}^{n-1} \rtimes O(n-1)$-symmetrization of V can be stated in concrete terms as follows.

Symmetrization of Bands. *If the above band V is $\mathbb{R}^{n-1} \rtimes O(n-1)$-symmetrisable, then there exists a smooth function $\hat{\varphi}(t) = \hat{\varphi}_\sigma(t)$, on the segment $[-l, +l]$ such that*

$$\pm l \ge dist(Z, \partial_\pm V),$$

$$\frac{\hat{\varphi}'(-l)}{\hat{\varphi}(l)} \le \frac{-M_-}{n-1}, \ \frac{\hat{\varphi}'(l)}{\hat{\varphi}(l)} \ge \frac{M_+}{n-1}$$

and

$$-2(n-1)\frac{\hat{\varphi}''(t)}{\hat{\varphi}} - (n-1)(n-2)\frac{\hat{\varphi}'(t)^2}{\hat{\varphi}(t)^2} \ge \sigma(t),$$

that is

$$\hat{◑}_l \qquad -2f'(t) - nf(t)^2 \geq \frac{\sigma(t)}{n-1} \text{ for } f(t) = \frac{\hat{\varphi}'(t)}{\hat{\varphi}(t)} \text{ and all } t \in [-l, l].$$

Symmetrization Corollary for $Sc \geq 0$. Let V be an isoenlargeable band and $Z_0 \subset V$ be a hypersurface which separates $\partial_-(V)$ from $\partial_+(V)$. Let $Sc(V) \geq 0$ and let

$$Sc(V) \geq \sigma_0 > 0 \textit{ on the } \delta_0\textit{-neighbourhood of } Z_0.$$

Then

the distance from Z *to the boundary* ∂V *is bounded by a constant which depends only on the dimension of* V*, on* $\sigma_0 > 0$ *and on* $\delta_0 > 0$*,*

$$dist(Z, \partial V) \leq C = C_n(\sigma_0, \delta_0).$$

Moreover, this remains true

if the inequality $Sc(V) \geq 0$ *is replaced by* $Sc(V) \geq -\varepsilon$ *for a small positive* $\varepsilon \leq \varepsilon_n(\sigma_0, \delta_0) > 0$.

Proof. If $\sigma(t) \geq \sigma_0$ for $t \in [-\delta_0, \delta_0] \subset [-l, l]$ and $\sigma(t) \geq -\varepsilon$ for all $t \in [-l, l]$, where $\varepsilon << \delta_0, \sigma_0$, then the inequality $\hat{◑}_l$ implies that

$$l \leq C = C_n(\sigma_0, \delta_0)$$

and the proof follows. □

Notice that no condition $mean.curv(\partial_\pm(V) \geq M_\pm$ has been used at this point.

Sub-corollary for Complete Manifolds with $Sc \geq 0$**.** *Open isoenlargeable bands carry no complete metrics with scalar curvatures* $Sc > 0$*.*

Moreover,

complete metrics with $Sc \geq 0$ *on such bands are Riemannian flat.*

In fact, a deformation theorem by Kazdan and Warner together with the Cheeger-Gromoll splitting theorem imply that if such a band admits no complete metric with $Sc > 0$ then every complete metric with $Sc \geq 0$ is flat.

Representative Example. *If a compact manifold* Z *admits a metric with negative sectional curvature, then there is no complete metrics with* $Sc \geq 0$ *on the connected sums*

$$X = (Z \times \mathbb{R}) \#_i P_i$$

for compact manifolds P_i*.*

(A similar result is proven in 6.12 and 6.13 of [GL83] for spin manifolds X.)

Symmetrization Corollary for $Sc \geq \sigma < 0$. To get a perspective look at the following

Model Example. Let $V_{[-l,l]}$ be the band of width $2l$ between concentric horospheres in the hyperbolic space H^n of constant curvature -1, which is the product

$$V_{[-l,l]} = \mathbb{R}^{n-1} \times [-l,l] \subset \mathbb{R}^{n-1} \times (-\infty + \infty) = H^n,$$

where the hyperbolic metric in these coordinates is $g_{hyp} = e^{2t} g_{Eu} + dt^2$.

The scalar curvature of g_{hyp} in these coordinates, in agreement with ◑, is $-n(n-1)$, while the mean curvatures of the boundaries $\partial_\pm V_{[-l,l]} = \mathbb{R}^{n-1} \times \{\pm l\}$ are

$$mean.curv(\partial_\pm V_{[-l,l]}) = \pm(n-1).$$

Such a band becomes compact if divided by the action of $\mathbb{Z}^{n-1} \subset \mathbb{R}^{n-1}$ for

$$\mathbb{R}^{n-1} \times [a,b]/\mathbb{Z}^{n-1} = \mathbb{T}^{n-1} \times [a,b].$$

Now,

let V be a compact Riemannian overtorical band, where the scalar curvature and the mean curvatures of the boundaries satisfy

$$\begin{aligned} Sc(V) &\geq -n(n-1), \\ mean.curv(\partial_-(V)) &\geq -(n-1), \\ mean.curv(\partial_+(V)) &\geq (n-1). \end{aligned}$$

then, in fact,

$$\begin{aligned} Sc(V) &= -n(n-1), \\ mean,curv(\partial_-(V)) &= -(n-1), \\ mean.curv(\partial_+(V)) &= (n-1). \end{aligned}$$

Proof. If either of the above three inequalities is *strict* (i.e. ">") at some point, then, by slightly conformally perturbing the metric of V (see section 11.2), one can make *all three strict at all points.*

This, by symmetrization, would result in a function $\hat{f} = \frac{\varphi'}{\varphi}$ on some segment $[-l,l]$, $l > 0$, such that

$$f(-l) < 1, \ f(l) > 1$$

and

$$2f' < -n(f^2 - 1)$$

which is, obviously, impossible. QED. □

Sub-corollary: Weak Rigidity of $H^n/\mathbb{Z}^{n-1}$. Let $X = H^n/\mathbb{Z}^{n-1}$, where H^n is the hyperbolic space with the sectional curvature $\kappa(g_{hyp}) = -1$, and the group $\mathbb{Z}^{n-1}$ discretely and isometrically acts on H^n by parabolic transformations, i.e. preserving a horosphere in H^n.[21]

If a Riemannian metric on X, which coincides with the hyperbolic one (descended from H^n to X) outside a compact subset in X, satisfies

$$Sc(g) \geq Sc(H^n) = -n(n-1)$$

then

$$Sc(g) = -n(n-1)$$

everywhere on X.

Soap Bubbles and Rigidity of Bands. A sharper version of the above sub-corollary, namely the implication

$$[\circledcirc_{-1}] \qquad Sc(g) \geq -n(n-1) \Rightarrow \kappa(g) = -1$$

follows from the existence of *stable minimal bubbles* in X, which are closed hypersurfaces Y which separate the two ends in X and which minimise the functional

$$Y \mapsto vol_{n-1}(Y) - (n-1)vol(X_{\prec Y}),$$

where $X_{\prec Y} \subset X$ is the part of X which is bounded by Y and which has $vol < \infty$.

Proof. The relation $[\circledcirc_{-1}]$ for $n \leq 8$ follows from $\circledcirc[M]$ below by taking $M = n-1$.

$\circledcirc[M]$. Let a compact Riemannian band V admit a function $M = M(v)$ such that

$$M_{|\partial_- V} \leq -mean.curv(\partial_- V) \text{ and } M_{|\partial_+ V} \geq mean.curv(\partial_+ V)$$

and

$$\frac{n}{n-1}M^2 - 2\|dM\| + Sc(V) \geq 0.$$

□

If $n = dim(V) \leq 8$, then either

there is a closed hypersurface $Y_\circ \subset V$, which separates $\partial_- V$ from $\partial_+ V$ and which admits a metric with $Sc > 0$,

or

V decomposes into the (warped) product, $V = Y \times [-l, l]$ with the metric

$$\varphi(t)^2 g_Y + dt^2,$$

where the Riemannian metric g_Y on Y has zero Ricci curvature.

---

[21] If $n \geq 3$ then all discrete isometric actions of $\mathbb{Z}^{n-1}$ on H^n are parabolic.

This is shown for $n \leq 7$ in section $5\frac{5}{6}$ in [Gro96]) and the case $n = 8$ can be taken care of with a help of ideas from [Sma93].

But it seems that the regularisation techniques of [Loh16] and/or of [SY17] do not apply, at least not directly, to this case and the validity of the above statement for $n \geq 9$ remains quite problematic.

On the other hand [$\circledcirc_{-1}$], where the implied $Y_\circ$ is the $n-1$ torus, must follow from these techniques which are, in principle, applicable whenever torical symmetrization works.

On Min-Oo Rigidity Theorem. By adapting an idea of Witten to a "hyperbolically modified" Dirac operator, [Min89] proved a version of the positive mass theorem for H^n. In particular he has shown the following.

[MRT] *If a complete spin manifold X is isometric to H^n outside a compact subset and if $Sc(X) \geq -n(n-1)$ then X is isometric to H^n.*

Since compact perturbations of H^n can be periodically extended by discrete actions of isometry groups Γ on H^n, e.g. for the above parabolic $\mathbb{Z}^n$,

[MRT] *follows from* [$\circledcirc_{-1}$].

Thus, **[MRT]** *remains valid without assuming X is spin* (but with some reservations for $n \geq 9$).

Moreover, this is shown in [ACG07], that [MRT] combined with an argument from [Loh99] implies the positive mass theorem for H^n and the spin condition can be disposed of in the context of the full Min-Oo(-Wang Chruściel–Herzlich) theorem (unconditionally for $n \leq 8$).

Proving Rigidity by Symmetrization. The rigidity of bands V in the symmetrization context says that the universal coverings of

the extremal bands, where our $\frac{2\pi}{n}$-inequality becomes an equality, must be $\mathbb{R}^{n-1} \rtimes O(n-1)$-invariant.

We shall indicate below the proof of this for $n \leq 7$, where the dimension $n = 8$ needs a little effort, and where the regularisation as developed in [Loh16] and in [SY17] for $n \geq 9$ may need an additional refinement to yield rigidity.

Now, assuming minimal varieties are non-singular, we observe that

the symmetrization process *strictly enlarges* the scalar curvature of V, unless the minimal hypersurfaces $Y \subset V$ used for this process are *totally geodesic*.

In fact, by the second variation formula in the form given to it in [SY79a], the corresponding operator L from section 2 is *strictly positive*, which implies increase of the scalar curvature under symmetrization. And this also work for symmetrization by reflection in section 2 if one replaces the smoothing of edges argument by an appeal to the corresponding operator L.

Thus, one represent all our homology classes in $H_{n-1}(V, \partial V)$ by totally geodesic submanifolds. This strongly restricts the geometry of V but does not, at least not obviously, imply the required $\mathbb{R}^{n-1} \rtimes O(n-1)$-symmetry of V.

However, by applying the same argument to the soap bubbles $Y_{\pm\varepsilon} \subset X$ which lie close to minimal Y and minimise the functional $Y \to vol_{n-1}(Y) - \varepsilon \cdot vol_n U_{\pm\varepsilon}$ as in section 8 one sees that no minimal Y can be locally strictly minimising in either of the two halves it divides V into.

This shows, that minimal Y in all homology classes, besides being totally geodesic, are "freely movable" in V, namely, they serve as fibers of a fibrations of V over the circle.

Then the required $\mathbb{R}^{n-1} \rtimes O(n-1)$-symmetry of V easily follows.[22]

10 Comparison with Results Obtained with Twisted Dirac Operators

Besides the method of of minimal hypersurfaces, a non-trivial information on geometry (and topology) of Riemannian manifolds X with $Sc(X) \geq \sigma$, $\sigma \in (-\infty, \infty)$, can be obtained by confronting

I: Atiyah-Singer type *index theorems for Dirac operators* which yield *non-zero harmonic spinors* on X
with

II: the *twisted Schroedinger–Lichnerowicz–Weitzenboeck formula* for manifolds with *lower bounds on their scalar curvatures* which rules out, or significantly restricts, such spinors.

Comparison of (partly overlapping) results obtainable with minimal hypersurfaces and with Dirac operators exposes *limitations of both methods* and exhibits wide gaps in our understanding of scalar curvatures; this begs for a new approach.

Let us briefly demonstrate this on a few simple examples.[23].

(1) Spin, Spinor Bundles and Dirac Operators. Since the fundamental group of the special (i.e. orientation preserving) orthogonal group $SO(n)$ for $n \geq 3$ is $\mathbb{Z}/2\mathbb{Z}$, there are exactly *two* different orientable bundles of rank $n \geq 3$ over closed connected surfaces. The trivial bundle is, by definition, *spin* and the non-trivial one is *non-spin*.

An orientable manifold X is called *spin* if the restrictions of the tangent bundle $T(X)$ to all surfaces $S \subset X$ are spin (i.e trivial).[24]

For instance, all orientable hypersurfaces $X^n \subset \mathbb{R}^{n+1}$ are spin, all 3-manifolds are spin and

simply connected n-manifolds with trivial second homotopy groups are spin.

The simplest non-spin manifolds are the complex projective spaces $\mathbb{C}P^n$ of even complex dimensions n and connected sums of other manifolds with these $\mathbb{C}P^n$.

The *spinor bundle* of a Riemannian spin manifold X of dimension n, denoted $\mathbb{S}(X)$, is a unitary vector bundle of vector bundle of rank 2^n with a unitary

[22] Since I have not written this down in detail, I might have missed some hidden difficulty in this apparently quite innocuous argument.

[23] See [Ros07, BER17, Gro17] for more elaborated techniques and examples

[24] "Spin" makes sense also for non-orientable bundles and manifolds but we do not need them at this point.

connection associated to the Levi-Civita connection in $T(X)$. If n is even, the bundle $\mathbb{S}(X)$ splits, $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$.

The Dirac operator is a canonically defined first order differential operator D represented as a certain "natural" linear combination of covariant derivatives which act on $\mathbb{S}(X)$. (See [BHMM15] for definitions, basic results and geometric applications of the Dirac operator.)

When n is even the Dirac operator splits:

$D = D^+ \oplus D^-$, where the operators D^+ and D^- are mutually adjoint for $D^+ : C^\infty(\mathbb{S}^+) \to C^\infty(\mathbb{S}^-)$ and $D^- : C^\infty(\mathbb{S}^-) \to C^\infty(\mathbb{S}^+)$ and where $ind(D) = dim(ker D^+) - dim(ker D^-)$. The solutions of $D(s) = 0$ are called harmonic spinors on X.

Twisted Dirac operator. Given a complex vector bundle $L = (L, \nabla)$ with a linear connection, one naturally defines

$$D^{\pm}_{\otimes L} : C^\infty(\mathbb{S}^\pm \otimes L) \to C^\infty(\mathbb{S}^\mp \otimes L),$$

where the sections of $S^\pm \otimes L$ in the kernel of $D_{\otimes L} = D^-_{\otimes L} \oplus D^+_{\otimes L}$ are called *L-twisted harmonic spinors.*

(2) Chern character and Todd Genus. The Chern character of a complex vector bundle L over X is a certain polynomial in the Chern classes $c_i \in H^i(X;\mathbb{Z})$ of L in the rational cohomology $H^*(X;\mathbb{Q})$ starting from $c_0 = rank(L)$,

$$ch(L) = c_0 + c_1 + \frac{1}{2}(c_1^2 + c_2) + \frac{1}{6}c_1^3 + c_1c_2 + 3c_3 + \dots + \frac{1}{i!}(c_1^i + \dots + k_ic_i) + \dots$$

where, observe, all $k_i \neq 0$. The basic properties of ch (which essentially define it) are additivity and multiplicativity:

$$ch(L_1 \oplus L_2) = ch(L_1) + ch(L_2) \text{ and } ch(L_1 \otimes L_2) = ch(L_1) \smile ch(L_2).$$

The $\hat{A}$-genus is another polynomial, now in the Pontryagin classes $p_i = p_i(T(X)) \in H^*(X;\mathbb{Q})$,

$$\hat{A}(X) = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(-4p_2 + 7p_1^2) + \frac{1}{967680}(-16p_3 + 44p_2p_1 - 31p_1^3) + \cdots$$

where again the coefficients at $p_i \in H^{4i}(X;\mathbb{Z})$ are non-zero.

(3) Topological Index I. Let $\hat{X}$ be an oriented Riemannian spin Γ-manifold, which means $\hat{X}$ is acted upon by a group Γ and let let $\hat{L}_1 = (\hat{L}_1, \nabla_1)$ and $\hat{L}_2 = (\hat{L}_2, \nabla_2)$ be complex vector bundles with unitary connections such that Γ acts on $\hat{L}_1$ and on $\hat{L}_2$ by fiber-wise unitary (linear isometric) connection preserving transformations compatible with the action of Γ on $\hat{X}$, such that the following conditions are satisfied.

(i) The action of Γ on $\hat{X}$ is *proper, isometric and orientation preserving*, where "proper" means that there are at most finitely many $\gamma \in \Gamma$, such that for all compact subsets $K \subset \hat{X}$ the intersections $K \cap \gamma(K)$ are empty for all but finitely many $\gamma \in \Gamma$.
preserves the connections in these bundles.

(ii) There exists a unitary connection preserving Γ-equivariant isomorphism between the bundles $\hat{L}_1$ and $\hat{L}_2$ at infinity, that is on the complement to a Γ-invariant subset $V \subset \hat{X}$ such that V/Γ is compact.

Let

$$I = I(\hat{X}, \hat{L}_1 \ominus \hat{L}_2) = (\underline{\hat{A} \smile (ch(\hat{L}_1) - ch(\hat{L}_2))}[\hat{X}/\Gamma]$$

be defined by representing $\hat{A}$, $ch(\hat{L}_1)$ and $ch(\hat{L}_2)$ by the Chern-Weil differential forms on $\hat{X}$, call them $\alpha, \lambda_1, \lambda_2 \in \bigwedge^*(\hat{X})$ which, clearly, are Γ-invariant and where $\lambda_1 - \lambda_2$ vanishes outside V.

Since the form $\iota = \alpha \wedge (\lambda_1 - \lambda_2)$ vanishes outside V, since it is Γ-invariant and since the action of Γ on $\hat{X}$ is proper, ι descends to a form $\underline{\iota}$ on the quotient space $\hat{X}/\Gamma$, which vanishes outside a compact subset[25] and defines the cohomology class $(\underline{\hat{A} \smile (ch(\hat{L}_1) - ch(\hat{L}_2))}$ of $\hat{X}/\Gamma$ with compact supports,

$$[\underline{\iota}] = (\underline{\hat{A} \smile (ch(\hat{L}_1) - ch(\hat{L}_2))} \in H^n_{comp}(\hat{X}/\Gamma; \mathbb{R}).$$

Then the index $I = \underline{\iota}[\hat{X}/\Gamma]$ can be defined as the integral

$$\int_{\hat{X}/\Gamma} \underline{\iota} = \int_\Delta \iota$$

for a fundamental domain $\Delta \subset \hat{X}$.

(4) Atiyah's L_2-Index Theorem. Let the following conditions be satisfied.

(a) The manifold $\hat{X}$ is *complete*.

(b) The connections in $\hat{L}_1$ and in $\hat{L}_2$ are *unitarizable*. This means these bundles admit unitary structures, i.e. fiberwise Hermitian scalar products $\langle \dots \rangle$, preserved by the parallel transport in these connections.

(c) The above unitary structures, (which are unique up to scaling) are Γ-*invariant*.

(d) The operators $D^2_{\otimes \hat{L}_1}$ and $D^2_{\otimes \hat{L}_2}$ are *uniformly positive at infinity*/Γ, where a differential operator $\mathcal{D}$ on sections $s = s(\hat{x})$ of a unitary bundle on a manifold $\hat{X}$ with a Γ action is called uniformly positive at infinity/Γ, if

$$\int_{\hat{X}} \langle \mathcal{D}(s(\hat{x})), s(\hat{x}) \rangle_{\hat{x}} d\hat{x} \ge c \cdot \int_{\hat{X}} ||s(\hat{x})||^2$$

for a constant $c > 0$ and all sections s with compact supports outside a certain subset $V \subset \hat{X}$ such that V/Γ is compact.

If the topological index $I = I = I(\hat{X}, \hat{L}_1 \ominus \hat{L}_2)$ does not vanish, then there exists either an $\hat{L}_1$- or $\hat{L}_2$-twisted harmonic square integrable spinor on $\hat{X}$.

In fact,

[25] We do not assume the action of Γ on $\hat{X}$ to be free and the space $\hat{X}/\Gamma$ may be singular but our forms are defined on it anyway.

the von-Neumann dimensions of the kernels $\hat{K}^{\pm}_{1,2}$ *of the operators* $D^{\pm}_{\otimes \hat{L}_{1,2}}$ *satisfy*

$$dim_\Gamma(\hat{K}^+_1) - dim_\Gamma(\hat{K}^-_1) - dim_\Gamma(\hat{K}^+_2) + dim_\Gamma(\hat{K}^-_2) = I(\hat{X}, \hat{L}_1 \ominus \hat{L}_2).$$

About the Proof. The equality

$$dim_\Gamma(ker D^+_{\otimes \hat{L}}) - dim_\Gamma(ker D^-_{\otimes \hat{L}_2}) = I(\hat{X}, \hat{L})$$

in the case of *compact* $\hat{X}/\Gamma$ is proven in [Ati76].

(If $\hat{X}/\Gamma$ is compact only a single bundle $\hat{L} = \hat{L}_1$ is needed, since one may take the trivial bundle of rank zero for $\hat{L}_2$; then the conditions (a) and (d) are irrelevant.)

The case of *non-compact manifolds* with no Γ-actions is treated in [GL83].

The compatibility of the two arguments was pointed out in [Gro86], where one finds further references.

Suggestion. It would be interesting to remove or to relax some of the conditions in the formulation of the index theorem.

$\widetilde{spin}$-Example. Let $\hat{X}$ be the universal covering $\tilde{X}$ of a manifold X. If X is spin then the spin bundle $\mathbb{S}(X)$ and the Dirac operator in it are defined and lift Γ-equivariantly to $\hat{X} = \tilde{X}$ for the Galois action of $\Gamma = \pi_1(X)$ on $\tilde{X}$.

But if X is non-spin, yet $\tilde{X}$ is spin, then the group which acts on $\mathbb{S}(\tilde{X})$ is the semidirect product $\mathbb{Z}_2 \rtimes \Gamma$ where $\mathbb{Z}_2$ acts by the ± 1-involution on spinors which corresponds to the Galois involutive transformation on the double covering of the principal bundle associated to the tangent bundle $T(\tilde{X})$.

Thus,

Atiyah's L_2-index theorem applies to the Galois coverings $\hat{X}$ of non-spin manifolds X whenever these $\hat{X}$ are spin.

(5) Twisted Schroedinger–Lichnerowicz–Weitzenboeck Formula. This formula relates the squares of L-twisted Dirac operators with the *rough Laplacians* $\nabla^*\nabla$ in the bundles $L = (L, \nabla)$ on X with unitary connections, where, recall, the operators $\nabla^*\nabla$ acts on sections of L; they are (non-strictly) positive

$$\int \nabla^*\nabla = \int ||\nabla||^2;$$

thus their kernels consist of ∇-parallel sections of L and $rank(ker(\nabla^*\nabla) \leq rank(L)$.

Here is the formula.

$$D^2_{\otimes l} = \nabla^*\nabla + \frac{1}{4} Sc(x) \cdot Id + \mathcal{R},$$

where $\mathcal{R}$ is a linear self adjoint endomorphism (zero order operator) of $\mathbb{S} \times L$ defined by the *operator valued curvature form R of L* coupled by the *Clifford multiplication* in $\mathbb{S}$ as follows.

$$\mathcal{R}(s \otimes l) = \frac{1}{2} \sum_{1 \leq i < j \leq n} e_i e_j s \otimes R(e_i \wedge e_j)(l),$$

where $Id : \mathbb{S} \times L$ is the identity operator, where $e_i \in T_x(X) \subset T(X)$ is an orthonormal frame at the point $x \in X$, where the above formula applies and where $s \in \mathbb{S}_x$ and $l \in L_x$.

Since the Clifford multiplication operators $e_i : s \mapsto e_i s$ are unitary,

$$||\mathcal{R}(s \otimes l)l| \leq \frac{n(n-1)}{4}||R|| \cdot ||s|| \cdot ||l||$$

where $||R||$ is the supremum of the norms of the curvature operator over all unit bivectors in the tangent spaces $T_x(X)$.

It follows then the norm of the operator $\mathcal{R}$ is bounded by

$$||\mathcal{R}|| \leq const_n ||R||$$

for

$$const_n = \frac{n(n-1)}{4}\sqrt{rank(\mathbb{S})} = n(n-1)2^{n-2}.$$

(6) Let $\hat{X}$ be a Γ-manifold with Γ-invariant bundles $\hat{L}_{1,2}$, such that the assumptions (a), (b) and (c) in the above Atiyah's L_2-index theorem are satisfied.

Let, moreover, the norms of the curvature operators R_1 and R_2 of the (unitary) connections in $\hat{L}_1$ and $\hat{L}_2$ be bounded by

$$Sc(\hat{X})(\hat{x}) \geq \varepsilon + 4const_n \cdot max(||R_1||_{\hat{x}}, ||R_2||_{\hat{x}})$$

for the above $const_n = n(n-1)2^{n-2}$, some $\varepsilon > 0$ and all $\hat{x} \in \hat{X} \smallsetminus V$ for a subset $V \subset \hat{X}$ with compact quotient V/Γ.

Then the above **(4)** and **(5)** yield the following.

Theorem. *If the topological index*

$$I = I(\hat{X}, \hat{L}_1 \ominus \hat{L}_2) = \underline{\hat{A} \smile (ch(\hat{L}_1) - ch(\hat{L}_2))}[X/\Gamma]$$

doesn't vanish, then there exists a point $\hat{x} \in \hat{X}$, where

$$Sc(\hat{X})(\hat{x}) \leq 4const_n \cdot max(||R_1||_{\hat{x}}, ||R_2||_{\hat{x}}).$$

(7) Area Enlargeable Manifolds. Recall that an n-dimensional Riemannian manifold X is called *area enlargeable* if it admits a sequence of orientable coverings $\tilde{X}_i \to X$ and of smooth maps $f_i : \tilde{X}_i \to S^n$ which are

- $\bullet_1$ constant at infinity,
- $\bullet_2$ have non-zero degree,
- $\bullet_3$ contract the areas of the surfaces $\Sigma \subset \tilde{X}_i$ by

$$area(f_i(\Sigma)) \leq \alpha_i area(\Sigma) \text{ for } \alpha_i \underset{i\to\infty}{\to} 0$$

Observe that area enlargeability is a weaker condition than enlargeability, where instead of $\bullet_3$ one requires $Lip(f_i) \to 0$ (see section 4), and that area enlargeability, similarly to enlargeability, is a *homotopy invariant* of *compact* manifolds X.

Let us show of that area enlargeability is incompatible with $Sc > 0$.

[□] *Complete area enlargeable manifolds X the universal coverings of which are spin can't have $Sc(X) \geq \varepsilon > 0$.*

Proof. Let's first assume that $n = 2m$ and let L be a complex vector bundle of some rank N over S^n with *non-zero* Chern class $c_m \in H^n(S^n)$.

Let X be the universal covering $\tilde{X}$ acted upon by $\Gamma = \pi_1(X)$, let L_1 be the trivial bundle $X \times \mathbb{C}^N$ and let L_i be induced from L by the composed map

$$X = \tilde{X} \to \tilde{X}_i \underset{f_i}{\to} S^n.$$

It is easy to see that non-vanishing of c_m implies non-vanishing of the topological index I and that the curvature of L_i tends to zero for $i \to \infty$

Therefore, the above **(6)** applies to (X, L_1, L_i) for a sufficiently large i and yields the proof for even n, while the case of $n = 2m-1$ reduces to $n = 2m$ by taking $X \times S^1$. [26] □

(8) Llarull's Rigidity Theorem. The above, as it is shown in [Lla98] can be rendered sharp by taking the positive (or negative) spin bundle $\mathbb{S}^+(S^n)$ for $\underline{L}$.

The Chern character of $\mathbb{S}^+(S^n)$ for $n = 2m$ is equal to the fundamental cohomology class $[S^n] \in H^n(S^n)$ and the norm of the Levi-Civita connection in this bundle equals $\frac{1}{2}$—all this is more or less obvious.

What is less obvious (see [Lla98], [Min02]) is that the lowest eigenvalue of the operator $\mathcal{R}$ on $\mathbb{S} \otimes \mathbb{S}^+$ on S^n is equal $-\frac{n(n-1)}{4}$, which, by **(6)** (and a trifle of linear algebra) implies the following

◯ Let X be a Riemannian manifold, such that
- X is complete,
- $Sc(X)(x) \geq \varepsilon > 0$ for all $\underline{x}$ outside a compact subset in X.
- the universal covering $\tilde{X}$ of X is spin.

Let a continuous map $f : X \to S^n$ satisfy the following conditions.

($*_{/\infty}$) f is constant at infinity (i.e. constant outside a compact subset in X);
($*_{deg}$) f has non-zero degree;
($*_{C^1}$) f is C^1-smooth;
($*_{ar}$) The map f (non-strictly) decreases integrals of the scalar curvature of X over all smooth surfaces $\Sigma \subset X$. (Since S^n has constant scalar curvature $n(n-1)$ this amounts to the inequality

$$\int_S Sc(X)(\sigma) d\sigma \geq n(n-1) area(f(\Sigma)).)$$

[26] "Area enlargeable" appears as "Λ^2-enlargeable" in [GL83], where the coverings $\tilde{V}_i$ are assumed spin.

Then

The map f is a homothety: there exists a constant $\lambda > 0$, such that

$$dist_{S^n}(f(x_1), f(x_1)) = \lambda \cdot dist_X(x_1, x_1) \text{ for all } x_1, x_2 \in X.$$

About the Proof. Here, the Dirac operator on X is twisted with the bundles $L = L_1$, which is induced by $f : X \to S^n$ from $\mathbb{S}^+(S^n)$, and where one takes the trivial bundle of the same rank as L for L_2.

In this case, the formula for $\mathcal{R} : \mathbb{S} \otimes L \to \mathbb{S} \otimes L$ from the above **(5)**, that is

$$\mathcal{R}(s \otimes l) = \frac{1}{2} \sum_{1 \le i < j \le n} e_i e_j s \otimes R(e_i \wedge e_j)(l),$$

written in the frames of vectors $e_i \in T_{\underline{x}}$, which simultaneously diagonalize the Riemannian metric of X and the metric induced by f from S^n effectively describes the action of $\mathcal{R}$ on the corresponding (Clifford) basis in $\mathbb{S}(X) \otimes f^!(\mathbb{S}^+(S^n)$, which is $\{\underline{e}_{i_1}\underline{e}_{i_2}\dots\underline{e}_{i_n} \otimes \underline{e}_{j_1}e_{j_2}\dots\underline{e}_{j_n}\}$. Then a straightforward computation in [Lla98] (and/or a more conceptual argument in [Min02]) shows that the spectrum of $\mathcal{R}$ is bounded from below by $-\frac{n(n-1)}{4}$ and the above **(6)** applies.

The above settles the case of even n.

If n is odd one uses area contracting maps $X \times S^1(R) \to S^{n+1}$ for large R where the corresponding $\mathcal{R}$ is still bounded by by $\frac{1}{4}(n(n+1) = \frac{1}{4}Sc(S^n)$ because the natural splitting of metric in $X \times S^1 \to S^{n+1}$ (see [Lla98]).

Alternatively, one can construct (non-split) metrics g_ε for on $X \times S^1 \to S^{n+1}$, for all $\varepsilon > 0$, with $Sc(g_\varepsilon) \ge (n+1)(n+2) - \varepsilon = Sc(S^{n+1}) - \varepsilon$, such that area non-increasing maps $X \to S^n$ suspend to area non-increasing maps $(X \times S^1, g_\varepsilon) \to S^{n+1}$.

Generalisation. It is shown in [GS02] that the above remain valid for S^n if the standard metric g on S^n is replaced by g' with *positive curvature operator*. This, shows, in particular, that Llarull's theorem is *stable under small perturbations* of the spherical metric g_0.

(9) Discussion. There are two drawbacks of the above results compared to what can be done with minimal hypersurfaces.

I. Spin. In the original paper [Lla98] the manifold X was assumed spin, which we have relaxed to requiring the universal covering of X to be spin. Yet, we still can't prove, $\bigcirc$ or even $\square$ for all complete manifolds.

II. Completeness. Neither $\bigcirc$ or $\square$ hold true as they stand for incomplete manifolds and it is unclear what their correct reformulations should be.

And even if the area decreasing condition for maps $f : X \to S^n$ is strengthened to to $Lip(f) \le 1$, one can't get any bound on $Sc(X)$ with Dirac operator methods for incomplete X, while minimal hypersurface do allow such bounds (see section 3).

On the other hand, the Dirac operator results also have two advantages over those achieved with minimal hypersurfaces.

[i] Area Versus Length. Application of minimal hypersurfaces depends on distance rather than area estimates of metrics involved.

[ii] Non-Abelian Symmetries. Dirac operator effectively accommodates symmetries of underlying (model) manifolds.

For instance, one can not prove with minimal hypersurfaces that *no metric* $g \geq g_0$ on S^n, where g_0 is the standard metric with the sectional curvature 1, *can have* $Sc(g) \geq n(n-1) = Sc(g_0)$.[27]

Specific Problem. Let $Z \subset S^n$ be a closed subset of codimension $k \geq 2$, let X be an orientable n-dimensional Riemannian manifold and let

$$f : X \to S^n \setminus Z$$

be a smooth proper map of non-zero degree which is distance decreasing or, more generally, area decreasing.

When and how can one bound the scalar curvature of $\boldsymbol{X}$?

Example. If Z is a piecewise smooth one-dimensional subset (graph) with trivial Levi-Civita holonomies along all it cycles, e.g. a disjoint union of trees, and if X complete, then—compare with remark (a) in section 3,

$$\inf_{x \in X} Sc(X)(x) < n(n-1) = Sc(S^n).$$

Proof. Let $\epsilon : S^n \to S^n$ be an arbitrarily small perturbation of the identity map which sends a small neighbourhood of Z to Z. Then the bundle L on X which is induced from $\mathbb{S}^+(S^n)$ by the composed map $\epsilon \circ f : \underline{X} \to S^n$ is trivial at infinity and the above proof of ◯ applies.

More generally, the same argument applies to closed subsets $Z \subset S^n$ admit sequences of maps

$$\epsilon_i : S^n \to S^n$$

such that

- the maps ϵ_i send small neighbourhoods of Z in S^n to subsets $Z_i \subset S^n$ as above, namely i.e. piecewise smooth with trivial holonomies over all cycles in Z_i;
- the maps ϵ_i converge, for $i \to \infty$, to the identity map in the C^1-topology. □

Questions.

(a) Can one more effectively describe these Z e.g. those of the topological dimension zero?

(b) Does the above inequality $\inf_x Sc(X)(x) < n(n-1)$ hold true for smooth closed curves $Z \subset S^n$, $n \geq 3$, with *non-trivial* holonomy?.

[27] Such a proof may be possible for $n = 3$ with suitable boundary conditions for minimizing surfaces.

(c) Does S^n minus a point admit a *incomplete* metric $g \geq g_0$ with $Sc(g) > n(n-1) = Sc(g_0)$ (where g_0 is the spherical metric)?

Let us generalise the class of overtorical manifolds X, where non-zero multiples of the fundamental cohomology classes, denoted $[X]^\circ \in H^n(X;\mathbb{Z})$, decompose into products of one dimensional classes,

$$k[X]^\circ = h_1 \smile \ldots \smile h_n, \ \ h_i \in H^1(X;\mathbb{Z}),$$

as follows.

(10) Oversymplectic Manifolds. A compact orientable n-dimensional manifold X is oversymplectic if a multiple of the fundamental cohomology class of X, decomposes into product of one and two dimensional classes,

$$k \cdot [X]^\circ = h_1 \smile \ldots \smile h_m,$$

and such an X is called $[\tilde{\uparrow}0]$-oversymplectic, if

the classes h_i vanish in the cohomology of the universal covering $\tilde{X}$ under the natural homomorphism $H^*(X) \to H^*(\tilde{X})$.

Notice that $[\tilde{\uparrow}0]$ is automatic for 1-dimensional classes.

Also note that if $n = 2m$, then, by grouping 1-dimensional h_i into pairs, one can make all $h_i \in H^2(X;\mathbb{Z})$, $i = 1,\ldots,m$, and if $n = 2m+1$ all but one among h_i can be brought to $H^2(X;\mathbb{Z})$.

Moreover, the a priori different 2-dimensional classes h_i, can be replaced by a single one, namely by a generic linear combination h of h_i, since $\smile_i h_i = k' \cdot h^{\smile m}$.

It follows that X of dimension $n = 2m$ is oversymplectic if and only if it admits a map of *non-zero degree* to the complex projective space $\mathbb{C}P^m$, where the condition $[\tilde{\uparrow}0]$ says in effect that *the pull back of the symplectic (Kähler) 2-form on $\mathbb{C}P^m$ to the universal covering $\tilde{X}$ of X is exact.*

And if $n = 2m+1$ is odd, there is such a map $X \times S^1 \to \mathbb{C}P^{m+1}$.

Observe that $[\tilde{\uparrow}0]$-oversymplecticity, similarly to overtoricity and to iso-enlrageability of manifolds X is inherited by X' which admit maps $X' \to X$ of non-zero degrees and also by the products $X' = X \times \mathbb{T}^k$.

Still, $[\tilde{\uparrow}0]$-oversymplicity seems significantly different from iso-enlargeability, and, probably, there are many examples of $[\tilde{\uparrow}0]$- $[\tilde{\uparrow}0]$-oversymplectic manifolds, even among projective algebraic ones, which are not (iso)enlargeable.

The reason we brought forth this oversymplecticity is the following proposition.

$(\star\tilde{\uparrow}0)$ *If X is $[\tilde{\uparrow}0]$-oversymplectic, then it admits no metric with $Sc > 0$, provided the universal covering $\tilde{X}$ is spin.*[28]

(This, as it was mentioned earlier, implies that the only possibility for $Sc(X) \geq 0$ is X being flat.

Also recall that vanishing of the second homotopy group $\pi_2(X)$ implies that $\tilde{X}$ is spin and observe that $\pi_2(X) = 0$ also implies $[\tilde{\uparrow}0]$.)

[28] $\tilde{X}$ is spin if and only if the restrictions of the tangent bundle $T(X)$ to all 2-spheres in X are trivial; if $n \geq 5$ this is the same as triviality of the normal bundles of embedded 2-spheres in X.

Proof of $(\star)$. Let $n = 2m$ and $\tilde{l}$ be the lift of the canonical line bundle of $\mathbb{C}P^m$ to $\tilde{X}$. Because of $[\tilde{\uparrow}0]$, this bundle is trivial there are the p-th order roots $\sqrt[p]{\tilde{l}}$ for all $p = 1, 2, \ldots$, which are represented by the p-sheeted coverings of the total space of the circle bundles associated to $\tilde{l}$.

And albeit the Galois' actions of the fundamental group $\Gamma = \pi_1(X)$ on $\tilde{X}$ and on $\tilde{l}$ does not extend to $\sqrt[p]{\tilde{l}}$, the semidirect product $\mathbb{Z}_p \rtimes \Gamma$ does act on $\sqrt[p]{\tilde{l}}$.

Since $\tilde{X}$ is spin, the twisted Dirac operator $D_{\otimes \sqrt[p]{\tilde{l}}}$, i.e. D with coefficients in $\sqrt[p]{\tilde{l}}$, is defined and the corresponding space of harmonic L_2-spinors is acted upon by the group $\mathbb{Z}_p \times \mathbb{Z}_2 \rtimes \Gamma$.

Then an elementary computation shows that the topological index $D_{\otimes \sqrt[p]{\tilde{l}}}$ does not vanish for infinitely many p and then, by the Atiyah L_2-index theorem, $D_{\otimes \sqrt[p]{\tilde{l}}}$-harmonic L_2-spinors exist for arbitrarily large p.

But since the curvatures of the bundles $\sqrt[p]{\tilde{l}}$ tend to 0 for $p \to \infty$, uniform positivity of the scalar curvature of $\tilde{X}$ would not allow such spinors for large p according to the twisted Schroedinger- Lichnerowicz-Weitzenboeck vanishing theorem. QED.

11 Continuation of Discussion. On the surface of things, $(\star\tilde{\uparrow}0)$ generalizes Schoen–Yau theorem on non-existence of metrics with $Sc > 0$ on overtorical manifolds, but...

(1) Here again there is an annoying spin condition in the statement of $(\star\tilde{\uparrow}0)$, which, for all we know must be unnecessary.
(2) More seriously, we can say preciously little about incomplete manifolds.

For instance,

one can't bound with the present day techniques the width of product bands $(Y \times [-1, 1], g)$ with metrics g which have $Sc(g) \geq \sigma > 0$ for $[\tilde{\uparrow}0]$-oversymplectic manifolds Y.

(The same can be said about all other non-$\mathcal{SYSE}$-manifolds Y which are known not to to admit metrics with $Sc > 0$).

Because of this,

one is unable to rule out complete metrics with $Sc > 0$ on $Y \times \mathbb{R}$ and complete metrics with $Sc \geq \sigma > 0$ on $X \times \mathbb{R}^2$ for $[\tilde{\uparrow}0]$-oversymplectic manifolds Y.

What is not hard to show, however, is the following

$(\star_{\times\mathbb{R}})$ *Products* $X = Y \times \mathbb{R}$ *carry no complete metrics* g *with* $Sc(g) \geq \sigma > 0$ *for all* $[\tilde{\uparrow}0]$*-oversymplectic manifolds* Y *the universal coverings of which are spin.*

Sketch of the Proof. Since $X = (X, g)$ is complete and two-ended, it admits a proper 1-Lipschitz function onto $\mathbb{R}$, which we call it $\phi : X \to \mathbb{R}$.

Let $X' = X \times \mathbb{R}$ and let

$$\Phi_\varepsilon = (\varepsilon \cdot \phi, \varepsilon \cdot \phi') : X' \to \mathbb{R}^2$$

where $\phi' : X' = X \times \mathbb{R} \to \mathbb{R}$ is the coordinate projection.

Let l'_ε be the Φ_ε-pullback of l_0 to X' and let l°_ε be the formal difference between l'_ε and the trivial complex line bundle with the trivial connection.

Let l_0 be a complex line bundle over $\mathbb{R}^2$ with a unitary connection, which is isomorphic to the trivial bundle outside a compact subset in $\mathbb{R}^2$ and such that the curvature ω_0 of l_0 is $\omega_0 = p_0(t_1,t_2)dt_1 \wedge dt_2$ for a non vanishing function $p_0 \geq 0$.

Let $dim(Y) = 2m$, let l be the line bundle over X' induced by the composed map $X' \to Y \to \mathbb{C}P^m$ from the canonical line bundle and let

$$l^\circ_\varepsilon = l \otimes l^\circ_\varepsilon.$$

Pass to the universal covering $\tilde{X}$ and observe as earlier, that Atiyah's L_2-index theorem, applied to the Dirac operator twisted with $\sqrt[p]{\tilde{l}^\circ_\varepsilon}$ and combined with Schroedinger- Lichnerowicz-Weitzenboeck vanishing theorem for small $\varepsilon << \sigma$ and for $p \to \infty$, rules out $Sc \geq \sigma > 0$ for complete metrics on X. QED

Generalisation to Non-Compact X. The above $(\star)$ generalises to complete $[\tilde{\uparrow}0]$-oversymplectic manifolds X, where the fundamental class of X in the cohomology with *compact supports*, denoted $H^n(X,[\infty])$,[29] decomposes into 1- and 2-classes also with compact supports and where these classes must vanish in the cohomology $H^{1,2}(\tilde{X},\widetilde{[\infty]})$.

For instance, if X of dimension $2m$ admits a proper map of non-zero degree to a complement of a subset $Z \subset \mathbb{C}P^m$, this condition is satisfied if $H_1(Z) = 0$ and the symplectic form of $\mathbb{C}P^m$ vanishes on Z.

12 Min-Oo - Goette - Semmelmann Rigidity Theorem. A (very) special case of this theorem (2.10 in [GS01]) reads as follows.

Let X b a compact orientable Riemannian manifold of dimension $2m$ and let $f: X \to \mathbb{C}P^m$ be a C^1-smooth area non-increasing *spin map* of non-zero degree where f is called *spin* if the restriction of the tangent bundle $T(X)$ to $\Sigma \subset X$ is trivial if an only the restriction $T(\mathbb{C}P^n)_{|f(\Sigma)}$ is trivial for all surfaces $\Sigma \subset X$.

For instance, if m is odd and X is spin then all maps $X \to \mathbb{C}P^n$ are spin.

⊛ *If*

$$Sc(X)(x) \geq Sc(\mathbb{C}P^m)(f(x)) \textit{ for all } x \in X,$$

then f is an isometry.

About the Proof. The $\mathcal{R}$-term in the Schroedinger–Lichnerowicz–Weitzenboeck formula **(5)**) for D twisted with *line bundles* l shows (see [Hit74]) that that if the curvature form ω of an l (where the cohomology class of $2\pi\omega$ equals the Chern class $c_1(l)$) on a Riemannian manifold of dimension $2m$ diagonalises as

$$\sum_{i=1,\ldots,m} \lambda_i e_{2i-1} \wedge e_{2i}$$

for an orthonormal frame $e_1, e_2, \ldots, e_{2m}$, then

$$||\mathcal{R}|| \leq 4 \sum_i \lambda_i.$$

[29] This $[\infty]$ stands for the complement to a (large) non-specified compact subset in X.

If l equals the (anti)canonical bundle $l_0 = \bigwedge^m T(\mathbb{C}P^m)$ then, its curvature form for the Levi- Civita connection of the Fubini Studi metric g_0 has

$$\lambda_1 = \lambda_2 = \ldots = \lambda_m = m+1$$

and $g_0 = m(m+1)$.

On the other hand, an easy homological computation shows that the topological index $I(D, l_0)$ does not vanish on $\mathbb{C}P^m$ and since $deg(f) \neq 0$ it doesn't vanish on X either. This shows that f can't be *strictly* area decreasing, while the equality case needs an additional argument (see [GS01]).

The above applies, strictly speaking, to odd m, where $\mathbb{C}P^m$ is spin, and if m is even, one twists D with the *virtual square root of* l_0 (see [Hit74,Min88,GS01]).

(13) (Interpolating between $(\star\tilde{\uparrow}0)$ with $(\circledast)$. Unlike the (obvious) implication $\bigcirc \Rightarrow \square$ the (sharp) theorem $(\circledast)$ by no means implies (rough) $(\star\tilde{\uparrow}0)$.

But an obvious combination of the proofs of these theorems brings the two together as follows.

Let X be a complete oriented Riemannian $2m$-manifold and $f : X \to \mathbb{C}P^m \smallsetminus Z$ be a proper C^1-smooth *area non-increasing* map of *non-zero* degree, where $Z \subset \mathbb{C}P^m$ a smooth submanifold on which the *symplectic form of* $\mathbb{C}P^m$ *vanishes* and which has $H_1(Z) = 0$.

Let the composed map $\tilde{X} \to X \to \mathbb{C}P^m \smallsetminus Z \subset \mathbb{C}P^m$ from the universal covering of X to $\mathbb{C}P^m$, call it $\tilde{f} : \tilde{X} \to \mathbb{C}P^m$, be spin.

$\left(\circledast\tilde{\uparrow}\frac{1}{p}\right)$ *If the* $\tilde{f}$*-pullback of the generator* $c \in H^2(\mathbb{C}P^m);\mathbb{Z})$, *that is* $\tilde{f}^*(c) \in H^2(\tilde{X};\mathbb{Z})$, *is divisible by a positive integer* p, *then*

$$\inf_x Sc(X)(x) < \frac{1}{p} Sc(\mathbb{C}P^m),$$

unless X *is compact,* Z *is empty,* $p = 1$ *and* f *is an isometry.*

One can only wonder if there is anything of this kind that may come from minimal hypersurfaces.

Acknowledgments

I want to express my gratitude to the anonymous referee for the time and effort he/she has spent reading this article. Besides pointing out inadequacy of the first version of our treatment of smoothing multiple corners and of minimal hypersurfaces in non-compact manifolds, the referee has indicated many other inconsistencies and errors in the paper—more than 60 in the first draft of the paper and more than 100 in the present version. Also he/she has made several insightful suggestions which necessitated the appearance of the appendix (section 11 below) in this article.

11 Appendix

In this section, following the suggestions by the referee, we explain in a greater detail the following.

(a) *Smoothing hypersurfaces with no decrease of their mean curvatures.*
(b) *Smoothing Riemannian metrics with with no decrease of their scalar curvatures.*

In the case (a) the principal step of smoothing consists of "rounding corners" of piecewise smooth hypersurfaces which makes our hypersurfaces C^1-smooth.
In the case (b) one C^1-smoothes continuous piecewise smooth Riemannian metrics by similarly "rounding them by bending" along their singular loci.
Both rounding constructions, however simple, depend on *specific* geometric properties of the mean curvature and the scalar curvature correspondingly.
Then, in both cases, the final step of smoothing $C^1 \rightsquigarrow C^\infty$ follows by *homotopy extension* construction for solutions of *general* differential inequalities which is explained in section 11.1 below.
Next, we

(c) *elucidate some properties of on minimal hypersurfaces in open manifolds needed for the width inequalities.*
Finally, as it was suggested by the referee, we
(d) *summarise topological obstruction for $Sc > 0$ on closed manifolds which follow from our inequalities*

and
(e) *highlight several conjectures mentioned in the main body of the article.*

11.1 Universal Constructions of Smoothing and Bending. A. Linear Smoothing Operators. The most common kind of smoothing in linear analysis is achieved by applying convolution-like operators to objects you want to smooth.
For instance, let X be a Riemannian manifold and $Y \subset X$ be a compact C^1-smooth cooriented hypersurface. Let $U \supset Y$ be a small C^∞-split neighbourhood of Y, say

$$U = \underline{Y} \times (-\delta, \delta),$$

where Y is represented by a graph of C^1-smooth function $f(\underline{y})$ and let Y_ε, $\varepsilon > 0$ be the graphs of the the ε-smoothed functions $f_\varepsilon(\underline{y})$, for

$$f(\underline{y}) \mapsto f_\varepsilon(\underline{y}) = \int_{\underline{Y}} K_\varepsilon(\underline{y}, \underline{y}') f(\underline{y}') d\underline{y}'.$$

Then, for the usual K_ε,

$$Y_\varepsilon \underset{C^1}{\rightarrow} Y \text{ for } \varepsilon \to 0$$

and, since the mean curvature of Y is *linearly* expressible in terms of the second derivatives of f, the mean curvatures of Y_ε satisfy almost the same bounds as those of Y, whenever the latter are defined.

For instance, if Y is piecewise C^2, or more generally if the above (C^1-smooth) function f is $C^{1,1}$, i.e. df is Lipschitz, and if

$$mn.curv(Y)(y) \geq \phi(y)$$

for a continuous function ϕ on $U \supset Y$ and almost all $y \in Y$, then

$$mn.curv(Y_\varepsilon)(y_\varepsilon) \geq \phi(y_\varepsilon) + o(1),\ \varepsilon \to 0,$$

for all $y_\varepsilon \in Y_\varepsilon$. (A slightly different format of this smoothing is suggested on pp. 939 and 949 in section 3.4 and 5.7 in [Gro14b].)
Similarly, $C^{1,1}$-smooth Riemannian metrics g on a manifold X can be C^1-approximated by C^∞-smooth g_ε such that if $Sc(g)(x) \geq \phi(x)$, then

$$Sc(g_\varepsilon) \geq \phi(x) + o(1),$$

because the operator $g \mapsto Sc(g)$ is linear in the second derivatives of g.

B. Smoothing by Local Bending. Smoothing a function f on V, where for instance, the second derivatives jump across a double sided hypersurface $\Sigma \subset V$, can be achieved by deforming, we call it *bending*, f on one side of V, say to the "left" of Σ, such that the derivatives on the left side become equal to the derivatives on the right other side of σ at all points $v \subset \Sigma$.
Linearity of the differential inequality we want to keep preserved by such bending, e.g. of $mn.curv > \phi$ or $Sc > \phi$, is not indispensable. What is essential for $C^1 \rightsquigarrow C^\infty$ smoothing is the *connectivity* rather than convexity of the subsets in the sets of values of derivatives of functions and/or of metrics defined by the required inequalities.
Let us formulate the relevant general bending property of solutions of such inequalities in terms of "cut-offs of deformations", where "connectivity", is hidden the concept of "deformation/homotopy".
Let $Z \to V$ be a smooth fibration and let $Z^{[r]}$ be that space of the r-jets of germs of C^r-smooth sections $f : V \to Z$.
Let $J^r_f : V \to Z^{[r]}$ denote the r-th jet of f and recall that by the definition of jets, $J^r_{f_1}(v) = J^r_{f_2}(v)$ *if and only the values of the sections* f_1 *and* f_2 *as well as of all their partial derivative of orders 1,2,... r, in some local coordinates, are equal at* v.
In fact, this property *defines* jets as as well as the spaces $Z^{[r]}$.
Let $\Sigma \subset V$ be a piecewise smooth subset,[30] let $U \supset \Sigma$ be a neighbourhood of Σ in V and let $\mathcal{R} \subset Z^{[r]}$ be an open subset.

★ **Cut-off Homotopy Lemma.**[31] Let $f_t : V \to Z$, $t \in [0,1]$, be a C^r-continuous family of smooth sections, such that

(i) $J^r_{f_t}(V) \subset \mathcal{R}$ for all $t \in [0,1]$
and
(ii) $J^{r-1}_{f_t}(v)$ is constant in t for all $v \in \Sigma$.

[30] Probably, what follows holds true for all closed subsets $\Sigma \subset V$.
[31] This is a reformulation of *the weak flexibility lemma* given as an exercise on p. 111 in [Gro86].

Then there exists a smaller neighbourhood $\bar{U} \subset U$ of Σ and another C^r-continuous family of smooth sections $\bar{f}_t : V \to Z$, such that, similarly to f_t,

$$J^r_{\bar{f}_t}(V) \subset \mathcal{R} \quad \text{for all } t \in [0,1]$$

and

$$\bar{f}_t \text{ is equal to } f_t \text{ on } \bar{U}$$

while at the same time

$$\bar{f}_t \text{ is constant in } t \text{ outside } U,$$

i.e. $\bar{f}_t(v) = f_0(v)$ for all t and all $v \in V \setminus U$.

Moreover, if f_t was constant in the neighbourhood of a closed subset $V_0 \subset V$ then $\bar{f}_t$ can be taken constant in t on V_0.

REMARK. The general case of the lemma easily reduces to that where V is *compact* and Σ is a *smooth submanifold* of codimension *one*.

Warning and Perturbative Generalisation. The conclusion of ★ by no means holds true in general without assumption (ii). However,

"constant" in (ii) can be replaced by "almost constant" as follows.

★′ *Let $f_{t,\theta}$ $t, \theta \in [0,1]$ be a two parameter C^r-continuous family of sections $V \to Z$ where $f_{t,0}$ satisfies (ii). Then there exists an $\varepsilon_0 > 0$. such that the conclusion of ★ holds for f_{t,θ_0} for all $\theta_0 \leq \varepsilon_0$.*

This follows from ★ and from what is called *microflexibility* of differential inequalities defined by *open* subsets in the jet spaces, which is, of course, fully trivial.

(More interestingly, there are classes of *flexible* maps, e.g. *smooth immersions* $V^n \to \mathbb{R}^{n+1}$, which are defined with certain $\mathcal{R}$, where extension of homotopies f_t (called *regular homotopies* for immersions) is possible for *all* f_t. But the proofs of this in interesting cases don't, unlike ★ and ★′, reduce to generalities but depend on specific constructions adapted to specific properties of particular $\mathcal{R}$. See [Gro86] and references therein.)

1-D -*Example.* Let $V = [0,\infty)$ and $Z = [0,\infty) \times \mathbb{R} \to [0,\infty)$ be the trivial fibration. Then sections of Z correspond to real functions $f(v)$, $v \geq 0$,

$$Z^{[r]} = [0,\infty) \times \mathbb{R}^{r+1}$$

and the r-jets are maps

$$J^{r-1}_f = \left(f, \frac{df}{dv}, \dots, \frac{d^r f}{dv^r}\right) : [0,\infty) \to \mathbb{R}^{r+1}.$$

⋆ *Given continuous functions $a_0(v), \dots, a_r(v)$, $b_0(v) > a_0(v), \dots, b_r(v) > a_r(v)$, a number $\delta > 0$ and a real function $f(v)$, $v \geq 0$, such that*

$$a_i(v) < \frac{d^i f(v)}{dv^i} < b_i(v) \text{ for } i = 0, \dots, r \text{ and all } v \geq 0,$$

there exists a function $\bar{f}(x)$, which also satisfies these inequalities,

$$a_i(v) < \frac{d^i \bar{f}(v)}{dv^i} < b_i(v) \text{ for } i = 0, \dots, r \text{ and all } v \geq 0,$$

and such that

$$\frac{d^r \bar{f}(0)}{dv^r} = c_r \text{ and } \bar{f}(v) = f(v) \text{ for } v \geq \delta,$$

where c_r is a given number in the interval $a_r(0) < c_r < b_r(0)$.

The essential and essentially obvious case of this example is where $r = 1$, $a_0 = a_1 = 0$ and $b_0 = b_1 = \infty$. When this is understood, all of ★ becomes obvious as well.[32]

⦶*Mean Curvature Example.* Let $Y \subset X$ be a piecewise smooth C^1-hypersurface, where the singular locus is a smooth hypersurface $\Sigma \subset Y$ (i.e. $dim(\Sigma) = dim(Y) - 1$), where the two smooth parts, say V_1 and V_2 of Y meet. This Y can be obviously C^∞-smoothed *along* Σ by deforming V_1 and V_2 near Σ such that they would C^∞-match at $\Sigma = V_1 \cap V_2$ and with almost no decrease of their mean curvatures.[33]

Then ★—here $r = 2$—allows an extension of these deformations/homotopies to *all of* $Y = V_1 \cup V_2$, such that the resulting smoothed hypersurface, say $\bar{Y} = \bar{Y}_\varepsilon$, for given positive continuous function $\varepsilon = \varepsilon(x) > 0$ on X, satisfies.

- The C^1-distance between $\bar{Y}_\varepsilon$ and Y is $\leq \varepsilon$.
- $\bar{Y}_\varepsilon$ coincides with V outside the ε-neighbourhood of Σ.
- The mean curvatures of $\bar{Y}_\varepsilon$ satisfy the same, up to ε, inequalities as the mean curvatures of Y.

In particular, if $mn.curv(Y)(y) \geq \phi(y)$ for a continous function ϕ on $X \supset Y$, then $mn.curv(\bar{Y}_\varepsilon)(\bar{y}) \geq \phi(\bar{y}) + \varepsilon(\bar{y})$.

Similarly one can smooth more general *piecewise smooth* C^1-hypersurfaces and also *piecewise smooth C^1-Riemannian metrics* with *negligible decrease* of their scalar curvatures.

11.2 ε-Redistribution of Curvature. In smoothing and bending constructions it is easier deal with with strict inequalities, such as $Sc > 0$, rather than with non strict ones, such as $Sc \geq 0$

Below are two simple (and, probably, known) propositions which allows one to relax "partially strict" inequalities.

Redistribution of the Mean Curvature. Let X be a C^∞-smooth Riemannian n-manifold and $V \subset X$ a domain with *cosimplicial corners*, i.e. each point at the boundary of V admits a neighbourhood in V which is diffeomorphic to a neighbourhood in the positive "octant" $\mathbb{R}^n_+ \subset \mathbb{R}^n$.

Let $\partial_i \subset \partial = \partial V$ denote the $(n-1)$-faces of V and ∂_{ij} be the $(n-2)$-faces which we call *edges*. Let ϕ_i and ψ_{ij} be smooth functions on X, such that

[32] This "obvious" presupposes familiarity with the basic geometry of the jet spaces.

[33] This deformation at a point $\sigma \in \Sigma = V_1 \cap V_2$ does not C^2-significantly move a neighbourhood $U_1 \subset V_1$ of $\sigma \in V_1$, unless $mn.curv(V_1)(\sigma) < mn.curv(V_2)(\sigma)$ and the same applies to V_2.

(i) the mean curvatures of the $(n-1)$-faces of V satisfy

$$mn.curv(\partial_i)(x) \geq \phi_i(x),$$

for all ∂_i and all $x \in \partial_i$;

(ii) the dihedral angles between the $(n-1)$-faces $\partial_i, \partial_j \subset \partial V$, satisfy

$$\angle_v(\partial_i, \partial_j) \leq \psi_{ij}(x)$$

for all (non-empty) edges $\partial_{ij} = \partial_i \cap \partial_j$ and all $x \in \partial_{ij}$.

($\star_>$) If V is *compact* and the boundary ∂V of V is *connected*, then *either*

$$mn.curv(\partial_i)(x) = \phi_i(x) \text{ and } \angle_x(\partial_i, \partial_j) = \psi_{ij}(x)$$

for all points $x \in \partial V$ (where these equalities make sense),
or there exists an arbitrarily small C^∞*-perturbation* V' *of* V *(by a* C^∞*-diffeomorphism close to the identity), such that*

$$mn.curv(\partial_i')(x) > \phi_i(x) \text{ and } \angle_x(\partial_i', \partial_j') < \psi_{ij}(x)$$

for all $\partial_i' \subset \partial V'$ *and* $\partial_{ij}' = \partial_i' \cap \partial_j'$ *and all* $x \in \partial_i'$ *and* $x \in \partial_{ij}'$ *correspondingly.*

Sketches of Three Different Proofs. **(1)** *Fredholm+Unique Continuation.* To get the idea, let $Y = \partial V$ be smooth and let $\phi_0(x)$ be a smooth function on X which extends the function $y \mapsto mn.curv(\partial V)(y)$ from $Y = \partial V$ to X. Let $U_0 \subset X$ be a neighbourhood of a point $y_0 \in Y$.
If a smooth function ϕ_0' is sufficiently C^∞-close to ϕ_0, then there exists
($\star_=$) *a* C^∞*-perturbation* Y' *of the hypersurface* $Y = \partial V$*, such that*

$$mn.curv(Y')(x) = \phi_0'(x)$$

for all $x \in Y' \setminus U_0$.
This follows by the implicit function theorem for the operator $Y \overset{\mathcal{M}}{\mapsto} mn.curv(Y)$, since

- $\bullet_1$ the linearisation $L = L_{\mathcal{M},Y}$ of $\mathcal{M}$ at Y, being Fredholm, has *finite codimensional* image;
- $\bullet_2$ the adjoint operator of L (which happens to be equal to L) has the *unique continuation* property for *connected*[34] hypersurfaces Y.

This ($\star_=$) also holds for certain domains where the boundary ∂V is non-smooth. For instance if V has no corners, i.e. if there is no triple intersections of faces, $\partial_{ijk} = \partial_i \cap \partial_j \cap \partial_k$, then a version of ($\star_=$), which is significantly stronger than ($\star_>$), follows by by perturbing the faces ∂_i one by one.

[34] This property (obviously) fails to be true if ∂V is disconnected and ($\star$) doesn't have to hold anymore.

But if the linearized boundary value problem loses regularity at the corners (this, probably, doesn't happen if all dihedral angles $\angle(\partial_i, \partial_j)$ are 90°), then it becomes unclear if $(\star_=)$ remains true.
However, $(\star_>)$ is taken care of by the following.

Local Mnc-Lemma.[35] Let Y be a smooth hypersurface in a Riemannian manifold X of dimension n.
Then there exists a continuous function $\varepsilon = \varepsilon_{X,Y}(y) > 0$, on Y, which is also continuous with respect to the C^∞-topology in the space of hypersurfaces $Y \subset X$ with the following property.
Let $S_y(\epsilon) = S_y^{n-2}(\epsilon) \subset Y$ be the ϵ-sphere around a point $y \in Y$ for some positive $\epsilon \leq \varepsilon$ and let $\delta > 0$ be a positive number.
Then there exists a diffeomorphism $\phi : Y \to X$, such that
$[\star_\delta]$ ϕ is δ-close to the identity diffeomorphism $id : Y \to Y \subset X$ in the C^2-topology;
$[\star_\epsilon]$ the diffeomorphism ϕ is equal to the identity outside a narrow band around the sphere $S_y(\epsilon)$,

$$\phi(y) = y,$$

unless

$$v \in B_y(\epsilon) \smallsetminus (S_y(\epsilon) \cup B_y(\epsilon')), \text{ where } \epsilon' = \epsilon\left(1 - \tfrac{1}{10^{10n}}\right),$$

and where $B_y(\epsilon)$ is the ball bounded by $S_y(\epsilon)$;
$[\star_>]$ the diffeomorphism ϕ strictly increases the mean curvature of Y in the interior of $B_y(\epsilon)$ close to $S_y(\epsilon)$, namely at all points

$$v \in B_y(\epsilon) \smallsetminus (S_y(\epsilon) \cup B_y(\epsilon'')) \text{ for } \epsilon'' = \epsilon\left(1 - \tfrac{1}{20^{10n}}\right).$$

The proof of the lemma is accomplished by applying the initial stage of *bending the hypersurface* $V \smallsetminus B_y(\epsilon''))$ near its boundary, which is described in the next section and where the existence of the *initial* bending we need here is fully obvious.

(2) *Spread of Positivity.* The above lemma allows an extension of the strict inequality $mn.curv(Y)(y) > \psi(y)$ from (controllably small) balls of radii ϵ' in hypersurfaces $Y \subset X$ to larger balls of radii

$$\epsilon = \epsilon' \left(1 - \frac{1}{10^{10n}}\right)^{-1}$$

and, consequently, from arbitrary open subsets $U' \subset Y$ to larger $U \subset Y$, without moving the complements $Y \smallsetminus U$.

Localization Corollary. The local lemma allows perturbations Y' required by $(\star_>)$ to be localised in a given domain $U \subset Y$, which, in turn, yields $(\star_>)$ for *non-compact* Y.

[35] Albeit 100% elementary, this lemma heavily relies on the specifically Riemannian/Pythagorean nature of our problem.

(3) ☾ *Variational Proof of* ($\star_>$). One can also construct a perturbation of $\partial = \partial V$ with a required control over $mn.curv(\partial)$, say in the smooth (no edges) case, by minimizing the functional

$$V \mapsto vol_{n-1}(\partial V) - \int_V \varphi'(x)dx$$

for a suitably chosen function $\varphi'(x)$.
To see what such a $\varphi'(x)$ should be, let $\mu(x)$ be a smooth extension of the mean curvature function from ∂ to X and let $\delta(x)$ be the signed distance function to ∂, i.e. $\delta(x) = dist(x, \partial)$ for $x \in V$ and $\delta(x) = -dist(x, \partial)$ for $x \in X \smallsetminus V$.
Observe that the original V locally strictly and stably minimizes the functional

$$V \mapsto vol_{n-1}(\partial V) - \int_V \varphi(x)dx \text{ for } \varphi(x) = \lambda\delta(x) - \mu(x)$$

where $\lambda = \lambda(X, \partial)$ is a sufficiently large constant, namely $\lambda >> \sup_\partial(curv^2(\partial)\| + \sup_X |Ricci(X)|$, see section 10.2 in [Gro12].
Thus, the functional $V \mapsto vol_{n-1}(\partial V) - \int_V \phi'(x)$ has a unique local minimum V' whenever φ' is sufficiently close to the above φ, and where, observe,

$$mn.curv(\partial V')(x) = \varphi'(x) \text{ for all } x \in \partial V'.$$

Then it is not hard to arrange such a φ' that would make the mean curvature of V increase outside U_0 (compare section 11.9).

Redistribution of the Scalar Curvature. Let, besides the above functions ϕ_i and ψ_{ij} on X, we are given a continuous function $\sigma(x)$ such that the scalar curvature of the Riemannian metric g in X satisfies

$$Sc(g)(x) \geq \sigma(x) \text{ for all } x \in X.$$

If V is connected (non-compact is allowed) then
either

$$Sc(g)(x) = \sigma(x) \text{ for all } x \in V,\ mn.curv(\partial_i)(x) = \phi_i(x), \text{ and } \angle_x(\partial_i, \partial_j) = \psi_{ij}(x),$$

where the latter two equalities hold for all points $x \in \partial V$, where they make sense,
Moreover,
one can keep $g' = g$ on a given closed subset in X on which $Sc(g)(x) > \sigma(x)$.
Sketches of two Proofs. [**1**] If V is compact with smooth boundary, the (linearization of the) operator $f \mapsto Sc(f^2g)$, $f > 0$ is Fredholm with the unique continuation property and the above argument via linearization applies.
[**2**] The initial stage of what is called "intrinsic bending" in section 11.5 yields the following simple proposition that fully accomplishes our purpose. (As in the mean curvature case, the existence of *initial* intrinsic bending is obvious.)
Local Sc-Lemma. Let $X = (X, g)$ be a smooth n-dimensional Riemannian manifold. Then there exists a continuous function $\varepsilon = \varepsilon_g(x) > 0$ on X, which is also

continuous with respect to the C^∞-topology in the space of Riemannian metrics g on X, with the following property.
Let $S_x(\epsilon) = S_x^{n-1}(\epsilon) \subset X$ be the ϵ-sphere around a point $x \in X$ for some positive $\epsilon \leq \varepsilon$ and let $\delta > 0$ be a positive number.
Then there exists a smooth Riemannian metric, g' on X, such that

[$\star_\delta$] *the metric g' is δ-close to g in the C^3-topology;*
[$\star_\epsilon$] *the metric g' is equal to g in a narrow band around the sphere $S_x(\epsilon)$,*

$$g'(x) = g(x),$$

unless

$$x \in B_x(\epsilon) \smallsetminus (S_x(\epsilon) \cup B_x(\epsilon')), \text{ where } \epsilon' = \epsilon\left(1 - \frac{1}{10^{10n}}\right),$$

and where $B_x(\epsilon)$ is the ball bounded by $S_x(\epsilon)$;
[$\star_>$] *the scalar curvature of g' is strictly greater then that of g in the interior of $B_v(\epsilon)$ close to $S_x(\epsilon)$, namely at all points*

$$v \in B_x(\epsilon) \smallsetminus (S_x(\epsilon) \cup B_x(\epsilon'')) \textit{ for } \epsilon'' = \epsilon\left(1 - \frac{1}{20^{10n}}\right).$$

11.3 Rounding and Smoothing Hypersurfaces with no Decrease of their Mean Curvatures. Let $X = X^n$ be a C^∞-smooth Riemannian manifold of dimension n e.g. $X = \mathbb{R}^n$, and let $Y = Y^{n-1} \subset X$ be a *cooriented* hypersurface which is
Y is locally C^∞-diffeomorphic to a convex polyhedral hypersurface in $\mathbb{R}^n$.
In other words, Y is *piecewise C^∞-smooth* with all dihedral angles between the tangent spaces of the smooth regions $Y_i \subset Y$ at their meeting points in the singular locus of Y being *strictly less than π.*

$$\angle_{ij} = \angle(T_y(Y_i), T_y(Y_j)) < \pi.$$

For instance, the boundaries Y of finite intersections of domains $U_i \subset X$ with smooth boundaries which all intersect *transversally* are of this kind.
Let us agree that the mean curvatures of cooriented hypersurfaces are evaluated with the *outward looking* normal vectors, where this sign convention makes mean curvatures of boundaries of *convex* domain *positive.*

[${}^\varepsilon\leftrightarrows_\varepsilon$]-Rounding.

Move Y inward equidistantly by ε and then by the same ε outward. If Y is a *compact closed* hypersurface and $\varepsilon > 0$ is sufficiently small, then the resulting hypersurface, call it $Y \pm \varepsilon]$, is

C^1-smooth and piecewise C^∞-smooth.

To see this clearly, let $Y_{-\varepsilon]}$ be the inward ε-equidistant hypersurface to Y, which, in the case $Y = \partial U$, is equal to the boundary of the ε-neighbourhood of the complement $X \smallsetminus U$.

Observe that if ε is small, than

$Y_{-\varepsilon]}$ *has the same corner pattern as* Y.

Now the piecewise structure of $Y\pm\varepsilon]$, that is the outward ε-equidistant to $Y_{-\varepsilon]}$, can be seen with *the normal (nearest point) projection* $Y_{\pm\varepsilon]}\to Y_{-\varepsilon]}$, call it $p_{-\varepsilon]}: Y\pm\varepsilon]\to Y_{-\varepsilon]}$, where each $y\in Y_{\pm\varepsilon]}$ is sent by $p_{-\varepsilon}$ to the unique (ε is small!) nearest point in $Y_{-\varepsilon]}$: *the smooth pieces of* $Y_{\pm\varepsilon]}$ *are the* $p_{-\varepsilon]}$*-pullbacks of the* ($(k-1)$*-codimensional) "faces"* $Y^{n-k}_{-\varepsilon]}\subset Y_{-\varepsilon]}$, $k=1,2,\dots,n$.

For instance, if Y is the boundary of the intersection of smooth domains with *transversally meeting* boundaries, then

these faces correspond to meeting points of k*-tuples of such boundaries.*

Observe that the subset $Y^{n-1}_{\pm\varepsilon]}=p^{-1}_{-\varepsilon]}\left(Y^{n-1}_{-\varepsilon]}\right)\subset Y_{\pm\varepsilon]}$ satisfies

$$Y^{n-1}_{-\varepsilon]}=Y\cap Y_{\pm\varepsilon]}$$

and that it is also equal to the set of points $y\in Y$, such that $dist(y,sing(Y_{-\varepsilon]})>\varepsilon$, where $sing(Y_{-\varepsilon]})$ is the set of points where Y is non-smooth, i.e. union of what we call *the edges* or $(n-2)$*-faces* of $Y_{-\varepsilon]}$.

Also observe that that the mean curvatures of the C^2-smooth pieces $Y^{n-k}_{\pm\varepsilon]}=p^{-1}_{-\varepsilon]}\left(Y^{n-k}_{-\varepsilon]}\right)$ of $Y^{n-k}_{\pm\varepsilon]}$ for $k\ge 2$ are *large positive,*

$$mn.curv\left(Y^{n-k}_{\pm\varepsilon]}\right)=\frac{k-1}{\varepsilon}+O(1) \text{ for } k\ge 2 \text{ and } \varepsilon\to 0.$$

Thus, the normal (nearest point) projection $p_{\pm\varepsilon]}: Y\to Y_{\pm\varepsilon]}$, which is defined for small $\varepsilon\ge 0$, is *mean curvature non-decreasing,*

$$mn.curv\left(Y_{\pm\varepsilon]}\right)(p_{\pm\varepsilon]}(y))\ge mn.curv(Y)(y)$$

for all $y\in Y$ and small $\varepsilon>0$.

From C^1 to C^∞. According to ★ from section 11.1, the C^1-hypersurfaces $Y_{\pm\varepsilon]}$ can be C^1-approximated by C^∞-smooth ones, call them $Y'_{\pm\varepsilon]}$, such that

* $Y'_{\pm\varepsilon]}$ coincide with $Y_{\pm\varepsilon]}$ outside the ε'-neighbourhood of the singular locus of $Y_{\pm\varepsilon]}$ (where C^∞-pieces of $Y_{\pm\varepsilon]}$ meet) where $0<\varepsilon'<<\varepsilon$ can be taken arbitrarily small.
* The normal (nearest point) projection $p'_{\pm\varepsilon]}: Y\to Y'_{\pm\varepsilon]}$ is defined for small ε and it moves Y *inward.*
* The mean curvature is almost non decreasing under this projection,

$$mn.curv\left(Y'_{\pm\varepsilon]}\right)(p'_{\pm\varepsilon]}(y))\ge mn.curv(Y)(y)+O(\varepsilon').$$

Finally, one can, if one wishes, C^∞-approximate $Y'_{\pm\varepsilon]}$ by Y'', where mean curvatures $\ge$ than those at (suitably) corresponding points of the original Y and where, moreover, $(\star_>)$ from the previous section allows one to achieve a strict inequality

$mn.curv(Y'') > mn.curv(Y)$, unless (a connected component of) Y was smooth to start with.
For instance, if the mean curvature of Y were ≥ 0 at the regular points of Y then one can obtain a smooth approximation Y'' of Y also with $mn.curv \geq 0$.
In fact, with a little care, (arguing as in compare 11.6) one can arrange our C^∞-smooth $Y'_{\pm\varepsilon]}$ itself, such that

$$mn.curv\left(Y'_{\pm\varepsilon]}\right)(p'_{\pm\varepsilon]}(y)) \geq mn.curv(Y)(y),$$

but this is not essential for the present paper.

Smoothing non-compact Y. If Y is a non-compact hypersurface, then instead of small constant ε one takes a small and fast decaying function $\varepsilon = \varepsilon(y) > 0$.
A direct construction of satisfactory $Y_{\pm\varepsilon]}$ with variable ε, however trivial, is cumbersome. A better approach is via a local version of $[^\varepsilon\leftrightarrows_\varepsilon]$-*rounding by bending* procedure described below.

Bending Hypersurfaces Near their Boundaries and Localisation of Smoothing. Smoothing the edge, where two smooth hypersurfaces Y_1 and Y_2 in X meet, can be achieved by *inward bending* of one of them say of Y_1 near its boundary $\partial Y_1 = Y_1 \cap Y_2$, such that

- the bending doesn't decrease the mean curvature of Y_1,
- the bending increases the dihedral angle from the original $\angle(Y_1, Y_2) < \pi$ to $\angle(Y_{1\varepsilon}, Y_2) = \pi$,
  where $\varepsilon > 0$, which can be chosen arbitrarily small, signifies that
- the bent hypersurface $Y_{1\varepsilon}$ coincides with Y_1 within distance $\geq \varepsilon$ from ∂Y_1 and
- $Y_{1\varepsilon}$ is everywhere ε-close to Y_1.

The technical advantage of such a bending is that it is easily *localisable*: you need to bend Y_1 only at the points you want to.
Namely, let $Y_1, Y_2 \subset X$ be smooth cooriented hypersurfaces meeting at their common boundaries, $\partial Y_1 = \partial Y_2 = Y_1 \cap Y_2$, denote this intersection Y_{12}, and let $\alpha : Y_{12} \to [0, 2\pi]$ be a smooth function, bounded from below by the dihedral angles between Y_1 and Y_2 at all points in Y_{12},

$$\alpha(y) \geq \angle_y(Y_1, Y_2),\ y \in Y_{12}.$$

Then, for all $\varepsilon > 0$, there exists a smooth embedding $\phi = \phi_\varepsilon : Y_1 \to X$ with the following properties.

$\bullet_1$ *$mn.curv(\phi(Y_1), y) \geq mn.curv(Y_1, y)$ for all $y \in Y_1$;*
$\bullet_2$ *$\phi(y) = y$ for all $y \in Y_{12}$;*
$\bullet_3$ *$\angle_y(\phi(Y_1), Y_2) = \alpha(y)$ for all $y \in Y_{12}$;*
$\bullet_4$ *the map ϕ is ε-close to the original embedding $Y_1 \hookrightarrow X$ (in the C^0-topology);*

- $\bullet_5$ *the map ϕ coincides with the embedding $Y_1 \hookrightarrow X$ at the points $y \in Y_1$ within distance $\geq \varepsilon$ from the the subset $Y'_{12} \subset Y_{12} \subset Y_1$, where $\alpha(y') > \angle_{y'}(Y_1, Y_2)$.*
- $\bullet_6$ *The intersection of the image $\phi(Y_1) \subset X$ with Y_2 is equal to Y_{12}.*

About the Proof. This easily accomplished by an isotopy of Y_1 which, moreover, moves the equidistant hypersurfaces $Y_{12,\varepsilon} \subset Y_1$ (of dimensions $n-2$) by at most ε in the C^∞ topology.

REMARKS. (a) The condition $\bullet_5$ can be sharpened as follows.

$\bullet_5^\star$ *the (open) subset $Y_{\phi\neq} \subset Y_1$ of the points $y \in Y_1$, where $\phi(y) \neq y$, lies ε-close to Y_{12} and the closure of $Y_{\phi,\neq}$ intersects the (closed) subset $Y_{=\angle} \subset Y_{12}$, where $\alpha(y) = \angle_y(Y_1, Y_2)$, only at the boundary of $Y_{=\angle}$ in Y_{12}.*

Achieving this, which is unneeded for our applications anyway, requires a little bit of extra effort.

(b) If one doesn't insist on $\bullet_6$, then one can bend/rotate Y_1 by an arbitrary "angle" in the interval $[\angle(Y_1, Y_2), \infty)$ in the spirit of the *Frizzing Lemma* in [LM84].

11.4 Rounding Edges of Riemannian Doubles with no Decrease of their Scalar Curvatures. Let $V = (V, g)$ be a smooth Riemannian n-manifold with boundary and let $W = V \bigcup_{\partial V} V$ be the double of V. This W carries a natural continuous Riemannian metric, call it $\tilde{g} = g\&g$, which equals g on both V-halves of W.

Let the boundary ∂V has *positive mean curvature* and let us explain following [GL80a] how

$\tilde{g}$ can be C^0-approximated by smooth metrics with their scalar curvatures bounded from below by $Sc(g)$.

$[{}^\varepsilon\!\leftrightarrows_\varepsilon]^{\mathbb{D}}$-Rounding. *Let $V_{-\varepsilon} \subset V$ be the complement of the ε-neighbourhood of $\partial V \subset V$*

and

let $W_\varepsilon \subset V \times \mathbb{R}$ be the boundary of the ε-neighbourhood of

$$V_{-\varepsilon} = V_{-\varepsilon} \times \{0\} \subset V \times \mathbb{R}.$$

This W_ε consists of two ε-equidistant copies of V_ε and of a semicircular part $W_{\mathbb{D}} = \partial V_\varepsilon \times S^1_+(\varepsilon)$, that is a half of the boundary of the ε-neighbourhood of $\partial V_\varepsilon \subset V \times \mathbb{R}$, as depicted in figure 8 on p. 227 in [GL80a]).

If V is compact and $\varepsilon \to 0$, then the principal curvatures $\lambda_1, \dots, \lambda_n$ of $W_{\mathbb{D}}$ are evaluated in terms of the principal curvatures $\mu_1, \dots, \mu_{n-1}$ of the boundary $\partial V \subset V$ as follows.[36]

$$\lambda_i = (\mu_i + O(\varepsilon)) \cdot \cos\theta + o(\varepsilon) \text{ for } i = 1, \dots, n-1 \text{ and } \lambda_n = \varepsilon^{-1} + O(1),$$

where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ denotes the angular parameter of the (right) semicircle $S^1_+(\varepsilon)$.[37]

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[36] We correct here a minor error from [GL80a].

[37] In fact, $\lambda_n = \varepsilon^{-1} + o(1)$, but this is unneeded for our present purpose.

Then the scalar curvature of $W_{\mathbb{D}}$, which is expressed by the Gauss theorema egregium satisfies

$$Sc(W_{\mathbb{D}})(v,\theta) = Sc(V)(v) + (2\varepsilon^{-1}mn(v) + O(1)) \cdot \cos\theta + o(1),$$

where $mn(v) = mn.curv(\partial V)(v)$, where $v \in \partial V$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
Now we bring into play the inequality

$$mn.curv(\partial(V) > 0$$

and see that the scalar curvature of $W_{\mathbb{D}}$ is bounded from below by that of V up to an error $\to 0$. And since the part of the hypersurface W_ε, which is parallel to V has the same scalar curvature as V, the scalar curvature of W_ε is everywhere bounded from below, up to an error $\to 0$, by that of V.

From C^1 to C^∞. The submanifold $W_\varepsilon \subset V \times \mathbb{R}$ has the same type of regularity as $Y_{\pm\varepsilon]}$ from the previous section, namely, it is C^1 *and piecewise* C^∞, and, similarly to $Y_{\pm\varepsilon]}$, it can be C^∞-smoothed by applying ★ from section 11.1 with a negligible decrease of the scalar curvature, where, moreover, this smoothing can be performed equivariantly for the obvious involution of $W = V \bigcup_{\partial V} V$. (And as in the case of $Y_{\pm\varepsilon]}$, one can, with a little care, do all this with no decrease of the scalar curvature at all.[38]

Non-Compact Manifolds, C^0-Approximation and C^∞-Smoothing.
Similarly to how that was done in the previous section, one extends the above to *non-compact manifolds* V with a use of small positive *functions* $\varepsilon(v)$, $v \in \partial V$ instead of constant ε.
Besides controlling scalar curvature of W_ε, we want the metric on W_ε to be C^0-close to $\tilde{g}$ on W for the former brought to the latter by a suitable diffeomorphism $W_\varepsilon \to W$.
Firstly, such a diffeomorphism is constructed from $W_{\mathbb{D}}$ to the $\frac{\varepsilon\pi}{2}$-neighbourhood $U \subset W$of $\partial V \subset W$ with a use of normal decompositions of $W_{\mathbb{D}}$ and U as $\partial V \times [-\frac{\varepsilon\pi}{2}, \frac{\varepsilon\pi}{2}]$ and then it is extended to all of W_ε.

11.5 "Intrinsic Bending" of Riemannian Manifolds along the Boundaries with no Decrease of their Scalar Curvatures. The rounding construction from [GL80a], which we presented in the previous section, was described in the normal coordinates in [Alm85] and, a more general form of it appears in [Mia02], in [BMN10] and in [MS12].[39]

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[38] In fact, the required smoothing can be easily done directly in this case, as it is indicated in [GL80a], where, in truth—this was pointed out by the referee—it is only claimed that $\inf Sc(W) \geq \frac{1}{2} \inf Sc(V)$ and removing "$\frac{1}{2}$" from this inequality *without an appeal to* 11.1★ requires a bit of attention.

[39] It was pointed out by the referee of the present paper that the smoothing proposition 3.1 in [Mia02], due to an extra term in there, doesn't imply the corresponding edge smoothing results from [GL80a] and in [Alm85], and, in the report on the corrected version of this paper, the referee pointed out that what we call "bending" is proven in [BMN10].

Let us present a construction of "bending" (Theorem 5 from [BMN 2010]) following [Gro14a].

$\circlearrowleft_\varepsilon$-Family. Let h be a smooth Riemannian metric on a manifold Y, let A_{old}, A_{new} be smooth quadratic differential forms (i.e. symmetric 2-tensors) on Y and let

$$h_\varepsilon(t) = h + tA_{new} + \frac{t^2}{2\varepsilon}(A_{old} - A_{new}),\ 0 \le t \le \varepsilon. \tag{++}$$

Then clearly,

$$h_\varepsilon(0) = h \tag{$*_0$}$$

while

$$\frac{dh_\varepsilon(0)}{dt} = A_{new} \text{ and } \frac{dh_\varepsilon(\varepsilon)}{dt} = A_{old}. \tag{$*_1$}$$

Assume at this point that Y is compact. Then
$*_2$ the metrics $h_\varepsilon(t)$ C^∞-converge to h for $\varepsilon \to 0$, where this convergence is uniform in $t \in [0, \varepsilon]$. In fact,

$$h_\varepsilon(t) = h + o(\varepsilon),$$

which means that the C^r-distances from $h_\varepsilon(t)$ to h satisfy

$$||h_\varepsilon(t) - h||_{C^r} = o(\varepsilon) \text{ for all } r = 0, 1, 2, \dots\ .$$

In particular, the deformation $h \rightsquigarrow h_\varepsilon(t)$ is almost circular in t for small ε,

$$h_\varepsilon(\varepsilon) \to h \quad \text{ for } \varepsilon \to 0.$$

Also,

$$\frac{d^2h_\varepsilon(t)}{dt^2} = \frac{1}{\varepsilon}(A_{old} - A_{new}) \text{ for all } (y,t) \in Y \times [0,\varepsilon]. \tag{$*_3$}$$

Now, let us incorporate the family $h_\varepsilon(t)$ into family of Riemannian metrics

$$g_\varepsilon = h_\varepsilon(t) + dt^2$$

on the manifold $Y \times [0, \varepsilon]$.
Then, clearly,
[i] *the second fundamental forms of the two boundary parts $Y \times \{0\}$ and $Y \times \{\varepsilon\}$ in the manifold $Y \times [0, \varepsilon]$ with the metric g_ε are equal to A_{new} and A_{old} correspondingly, where these forms are evaluated on the (same unit) vector field $\frac{d}{dt}$.*

Furthermore, the scalar curvature of this metric satisfies,

$$Sc(g_\varepsilon)(y,t) = \frac{1}{\varepsilon}trace(A_{old} - A_{new}) + O(1). \tag{[ii]}$$

Indeed, the Ricci curvature $Ricci_\varepsilon$ of g_ε at the field $\frac{d}{dt}$ is expressed in terms of $A_\varepsilon(t) = \frac{dh_\varepsilon(t)}{dt}$ by Hermann Weyl's formula

$$Ricci_\varepsilon\left(\frac{d}{dt}, \frac{d}{dt}\right) = trace\left(\frac{d}{dt}A_\varepsilon(t)^*\right) + (A_\varepsilon(t)^*)^2$$

where $A_\varepsilon^*(t)$ denote the (selfadjoint shape) operators associated to the quadratic forms $A_\varepsilon(t)$ via the metrics $h_\varepsilon(t)$ on Y.
Therefore, in view of $*_2$ and $*_3$,

$$Ricci_\varepsilon\left(\frac{d}{dt}, \frac{d}{dt}\right) = trace\frac{1}{\varepsilon}trace(A_{old} - A_{new}) + O(1)$$

and **[ii]** follows by the Gauss theorema egregium for the hypersurfaces $Y \times \{t\}$ in the manifold $Y \times [0, \varepsilon]$ with the metric g_ε.

Bending Lemma. Let $X = (X, g)$ be a smooth (possibly non-compact) Riemannian manifold with boundary $Y = \partial X$ and let A_{old} denote the second fundamental form of Y with respect to the *inward* normal field. (Notice that the boundaries of *convex* domains have *positive definite* second forms for such fields.) Let A_{new} be another smooth quadratic form on Y.
If

$$trace(A_{new}) < trace(A_{old})$$

then there exits a family of C^∞-smooth metrics $g_\epsilon = g_{new,\epsilon}$, $\epsilon > 0$, on X, such that

[I] *The restrictions of the Riemannian metrics g_ϵ to $Y = \partial X \subset X$ are equal to such restrictions of g for all $\epsilon > 0$.*
[II] *The second fundamental forms of $Y \in X$ with respect to $g_{new,\epsilon} = g_\epsilon$, are equal to A_{new} for all $\epsilon > 0$.*
[III] *The scalar curvatures of g_ϵ are bounded from below by those of g,*

$$Sc(g_\epsilon) \geq Sc(g).$$

[IV] *The metrics g_ϵ C^0-converge to g for $\epsilon \to 0$.*
[V] *The metrics g_ϵ are equal to g within distance $\geq \epsilon$ from Y.*

Proof. Let $\varepsilon << \epsilon$ be a (very small) positive number, such that the the ε-neighbourhood U_ε of Y *normally splits*,

$$U_\varepsilon = Y \times [0, \varepsilon],$$

where, by the definition of "normally", the hypersurfaces $Y \times \{t\}$ are t-equidistant to $Y = Y \times \{0\} \in X$.
Let us replace the metric g in U_ε by the above g_ε.
Observe that the Riemannian form g_ε on the hypersurface $Y \times \{\varepsilon\}$ as well as second fundamental form of $Y \times \{\varepsilon\}$ with respect to g_ε are ε-close to the g-related forms. Therefore, according to $\bigstar'$ from section 11.1 applied to sections of the bundle of

quadratic forms on the tangents to submanifolds $Y \times \{t\} \in X$, the form g_ε extends to all of X with the required properties except [III]; this can be guaranteed by $\bigstar'$ only up to an arbitrarily small error. Yet, this error can be taken care of by redistribution of the scalar curvature from section 11.2.
(If our "redistribution" is obtained with a use of the Fredholm + unique continuation properties of the operator $f \mapsto Sc(f^2 g)$, then V needs to be compact and also the condition [V] suffer. But none of this happens if we rely on the Local Sc-Lemma.[40]) □

Half Way $[^\varepsilon \leftrightarrows_\varepsilon]^{\mathbb{D}}$-Rounding and Making Doubles with $Sc \geq \sigma$. If $X = \partial X$ has strictly positive mean curvature, then the above applies to $A_{new} = 0$ and allows a "bending" (of the Riemannian metric in) X near the boundary $Y = \partial X$, which makes Y *totally geodesic* and such that
the scalar curvature of $X_{bent} = (X, g_{new})$ *is bounded from below by the scalar curvature of the original metric* g.
Then the metric $g_{new} \& g_{new}$ on the double $(X, g_{new}) \cup_Y (X, g_{new})$ is C^1-smooth and, by $\bigstar$, it can be C^∞-smoothed keeping its scalar curvature bounded from below by $Sc(g_{new})$ which itself is bounded from below by $S(g)$ according to the above Bending Lemma.
VI$_{loc}$ *Localisation of Bending.* Let the form A_{new} be equal to A_{old} on a (now possibly non-empty) compact subset $Y_0 \subset Y = \partial X$ and the inequality $trace(A_{new}) < trace(A_{old})$ holds in the complement $Y \setminus Y_0$.
If X is connected and the complement $Y \setminus Y_0$ is non-empty, then there exists a family $g^\star_\varepsilon$, which satisfies the above [I]-[IV] and where [V] is strengthened to the following
$[V_\star]$ *The metrics g_ε are equal to g within distance $\geq \varepsilon$ from $Y \setminus Y_0$.*

About the Proof. Construction of $g^\star_\varepsilon$, which satisfies this condition (parallel to the condition $\bullet_5$ in 11.3 for bending hypersurfaces with controlled mean curvatures) is achieved by an obvious smooth cut off of the above g_ε, where the arising error term in $Sc(g^\star_\varepsilon)$ can be compensated due to the $\frac{1}{\varepsilon}$-contribution in the positivity of the scalar curvature.

11.6 Minimal Hypersurfaces in Non-compact Manifolds. Let us return to M-bubbles from section 9: these are closed cooriented hypersurfaces in Riemannian manifolds, say $Y_{min} \subset X$, which locally minimise the functional

$$Y \mapsto vol_{n-1}(Y) - \int_{U_-} M(x)dx,$$

where M is a function defined in a neighbourhood of $U_* \supset Y_{min}$ and where $U_- \subset U$ is the interior part of U_*.
Unconditionally, *the existence* of such a Y_{min}, possibly a singular one if $n = dim(X) \geq 8$, is ensured only for *closed* manifolds X.

[40] There is no circularity here, since Local Sc-Lemma, albeit being a special case of the Bending Lemma, admits an independent (and obvious) proof.

And if X is a compact manifold with boundary, then the existence of Y_{min} follows from a suitable bound on the mean curvature of the boundary as in section 9,

$$M_{|\partial_- X} \leq -mean.curv(\partial_- X) \text{ and } M_{|\partial_+ X} \geq mean.curv(\partial_+ X)$$

where $\partial_\pm$ are the parts of the boundary of X positioned in the interior/exterior region of X with respect to Y_{min}. (This makes sense since the decomposition $\partial X = \partial_- X \cup \partial_- X$ depends only on the homology class of $Y_{\min}$.)
Now we look for this kind of condition for non-compact manifolds, where our motivations are twofold:

(A) Finding geometric conditions on X, such that a given hypersurface $Y \subset X$ would admit a metric with positive scalar curvature.
(B) Finding least demanding constraints on a perturbation g_ε of a Riemannian metric g on X, such that a hypersurface $Y \subset X$ would admit an ε-close to it hypersurface with the mean curvature with respect to g_ε being close to $mn.curv_g(Y)$.

An obvious (over-optimistic?) conjecture in the direction of (A) is as follows.
(A **?**) Let X be a complete Riemannian manifold of dimension ≥ 6 and let $M(x)$ be a continuous function on X, such that

$$\frac{n}{n-1} M(x)^2 - 2||dM(x)|| + Sc(X)(x) \geq 0.$$

Then every closed cooriented hypersurface $Y_0 \subset X$, for which the inclusion homomorphism between the fundamental groups

$$\pi_1(Y_0) \to \pi_1(X)$$

is injective, admits a a metric with $Sc > 0$.
(This generalises Conjecture 1.24 in [Ros07] for $X = Y_0 \times S^1$ and $M = 0$ and a similar conjecture—I recall seeing it in a paper by Rosenberg and/or Stolz—for complete X homeomorphic to $Y_0 \times \mathbb{R}$.)

Motivating Example. Let let X be a *complete two-ended* Riemannian manifold and let Y_0 be a smooth closed hypersurface, such that the following four conditions are satisfied.
[○|○] Y_0 is *connected* and it *separates the two ends of* X. Thus, both components of the complement $X \smallsetminus Y_0$ are infinite and both are one-ended. For instance, X is homeomorphic to $Y_0 \times \mathbb{R}$.
(We agree that the only "infinities of" $X \smallsetminus Y_0$ which count are parts of the "infinity of" X.)
$[\circ{=}\circ]_{H_1}$ The inclusion homology homomorphism $H_1(Y_0) \to H_1(X)$ is an *isomorphism.* (If the inclusion homomorphism between the fundamental groups $\pi_1(Y_0) \to \pi_1(X)$ is injective then the covering of X with the fundamental group equal to the image $\pi_1(Y_0) \subset \pi_1(X)$ has this property.)
$[\circ \hookrightarrow \circ]_{\pi_1}$ The inclusion homomorphism between the fundamental groups $\pi_1(Y_0) \to \pi_1(X)$ is injective.

(This implies $[\circ\!\!=\!\!\circ]_{H_1}$ for the covering of X with fundamental group equal the image $\pi_1(Y_0) \subset \pi_1(X)$, but the roles played by H_1 and π_1 in the arguments below are different.)
$[vol = \infty]$ Both connected components of the complement $X \setminus Y_0$, call them $X_\pm \subset X$, have *infinite* volumes.
[spin] Manifold X is *spin*.
$[\odot_{\lambda,\rho}]$ There exist constants $\lambda, \rho > 0$, such that all balls in X of radii $r \leq \rho$ are λ-*Lipschitz contractible in* X (but not necessarily within themselves), i.e. there exist λ-Lipschitz maps

$$\phi = \phi_{x,r} : B_x(r) \times [0,r] \to X,\ x \in X,\ r \in [0,\rho],$$

where the maps $\phi(\dots,0)$ are the original imbeddings $B_x(r) \subset X$ and such that the maps $\phi(\dots,r) : B_x(r) \to X$ *are constant.*
(Coverings of closed manifolds, obviously, satisfy this condition.)
$[\bullet]$ If the scalar curvature of X is everywhere bounded from below by $\sigma_- \in (-\infty,+\infty)$ and $Sc(X)(x) \geq \sigma_+ > 0$ in the ε-neighbourhood of $Y_0 \subset X$, such that

$$[\sigma_+ >> |\sigma_-|] \qquad \sigma_+ \geq \frac{2}{\varepsilon}\sqrt{\frac{|\sigma_-|(n-1)}{n}}.$$

Then Y_0 admits a metric with $Sc > 0$, provided $6 \leq dim(X) \leq 8$.
Prior to explaining the proof, a few remarks are in order. $(*)$ The spin condition can be significantly relaxed and, possibly, fully removed. (We shall explain this in the course of the proof of $[\bullet]$.)

- $(*)$ The condition $[\odot_{\lambda,\rho}]$ implies $[vol = \infty]$ as we shall see in the course of the proof of $[\bullet]$; however, the two play opposite roles in the proof of $[\bullet]$. But in any case, we would rather get rid of $[\odot_{\lambda,\rho}]$ altogether.
- $(*)$ The main function of the inequality $dim(X) \geq 6$ is to rule out 4-manifolds Y_0, where there are topological obstructions to the existence of metrics with $Sc > 0$ which have no counterparts for other dimensions, see [Ros07] and references therein.
- $(*)$ The inequality $dim(X) \leq 8$ is, most likely, unnecessary—it is due to our inability to handle singularities of minimal hypersurfaces in manifolds of dimensions ≥ 9.

Proof of $[\bullet]$. Step 1. The inequality $[\sigma_+ >> |\sigma_-|]$ implies that there exists a function $M(x)$, which vanishes on Y_0 which is positive on one of the components of $X \setminus Y_0$, say on X_+ and negative on X_-, which is constant 2ε-far from Y_0 and such that

$$\frac{n}{n-1} M(x)^2 - 2\|dM(x)\| + Sc(X)(x) > 0$$

at all points $x \in X$.

Step 2. Since $vol(X_\pm) = \infty$, the integrals of M over both components $X_\pm$ of $X \smallsetminus Y_0$ are $\pm$infinite. Therefore, there exists a *compact connected* subset $V_0 \subset X$, which contains Y_0 and such that

$$\int_{V_0 \cap X_\pm} |M(x)| dx > vol_{n-1}(Y_0).$$

Consequently, every closed hypersurface Y' in the complement $X \smallsetminus V_0$, which is homologous to Y_0 satisfies

$$vol_{n-1}(Y') - \int_{X'_+ \smallsetminus \infty_+} M(x)dx \geq vol_{n-1}(Y_0) - \int_{X_+ \smallsetminus \infty_+} M(x)dx + \mu_0,$$

where

- ∞_+ denotes a unspecified subset in X_+, which has a sufficiently large compact complement; in particular, ∞_+ doesn't intersect V_0,
- $X'_+ \subset X$ is the (infinite) connected component of $X \smallsetminus Y'$ which contains ∞_+.
- $\mu_0 = \mu_0(V_0)$ is a positive constant.

Since the difference

$$\int_{X'_+ \smallsetminus \infty_+} M(x)dx - \int_{X_+ \smallsetminus \infty_+} M(x)dx$$

does not depend on the choice of $\infty_+ \subset X_+$, the above inequality is unambiguous.

It follows that

Y_{min} CAN'T ESCAPE FROM V_0:

all closed hypersurfaces $Y' \subset X$, which are homologous to Y_0 and which almost (up to μ_0) minimize the functional

$$Y' \mapsto vol_{n-1}(Y') - \int_{X'_+} M(x)dx =_{def} vol_{n-1}(Y') - \int_{X'_+ \smallsetminus \infty_+} M(x)dx$$

intersect V_0.

And since these Y', because of the above [o|o] Y_0 and $[\circ\!=\!\circ]_{H_1}$, are connected, the minimization process for this functional converges (in the sense of the geometric measure theory) to a minimum, call it $Y_{min} \subset X$: this is a possibly infinite hypersurface in X of *finite volume*, where this Y_{min} *doesn't fully escape* V_0. Namely

- Y_{min} is *proper*, i.e. it is a closed as a subset in X, but it *may be non-compact* (hence, unbounded for complete X);
- $$vol_{n-1}(Y_{min}) \leq vol(Y_0) < \infty$$
- The hypersurface Y_{min} has *non-empty* intersection with V_0.
  (Notice that completeness of X is non-essential at this point.)

Step 3. Let us bring forth the above local Lipschitz contractibility condition $[\odot_{\lambda,\rho}]$, invoke the *cone/filling inequality* 3.4.C from [Gro83]. This shows that $[\odot_{\lambda,\rho}]$ implies the following *filling inequalities* $[\odot^{n-1}_{n-2}]$ *and* $[\odot^{n}_{n-1}]$.
$[\odot^{n-1}_{n-2}]$ *All* $(n-2)$*-cycles* β *in all* ρ*-balls,* $\beta = \beta_{n-2} \subset B_x(\rho) \subset X$, $n = dim(X)$, *bound* $(n-1)$*-chains* $\alpha \subset X$, *such that*

$$vol_{n-1}(\alpha) \leq c \cdot vol_{n-2}(\beta)^{\frac{n-1}{n-2}},$$

for some constant $c = c(n, \lambda, \rho)$.
$[\odot^{n}_{n-1}]$ *All subdomains* $U \subset B_x(\rho)$ *satisfy*

$$vol_n(U) \leq c' \cdot vol_{n-1}(\partial U)^{\frac{n}{n-1}},$$

for some $c' = c'(n, \lambda, \rho)$.
Then these inequalities easily yield the following lower bound on the volume of the balls in the minimizer $Y_{min} \subset X$,

$$vol_{n-1}(Y_{min} \cap B_x(\rho)) \geq \nu = \nu(n, c, c') = \nu(n, \lambda, \rho) > 0$$

for all $x \in Y_{min}$.[41]
Since Y_{min} is connected, the above volume bound $vol_{n-1}(Y_{min}) \leq vol(Y_0)$ implies a bound on its diameter,

$$diam(Y_{min}) \leq R = \frac{2\rho \cdot vol_{n-1}(Y_0)}{\nu} < \infty.$$

Then, by combining this inequality with the above intersection property for our minimizer, $Y_{min} \cap V_0 \neq \emptyset$, we conclude that

$Y_{min} \subset X$ is trapped[42] in the R-neighbourhood of V_0.

And since X is complete, Y_{min} is *compact.*

Step 4. By the classical regularity theorem(s) of Simons-Federer-Almgren-Allard this Y_{min} is a smooth hypersurface for $dim(X) \leq 7$; if $n = 8$, then the regularity is achieved by a small perturbation of the Riemannian metric in X.[43]
Then, the inequality $\frac{n}{n-1}M(x)^2 - 2\|dM(x)\| + Sc(X)(x) > 0$, implies that the induced metric in $\tilde{Y}_{min}$ is conformal to a metric with $Sc > 0$ by the M-version of the Schoen–Yau argument (see §$5\frac{5}{6}$ in [Gro96]).

Step 5. If Y_{min} is smooth and $dim(Y_{min}) \geq 5$ then the kernel of the inclusion homomorphism

$$\pi_1(Y_{min}) \to \pi_1(X) = \pi_1(Y)$$

[41] This is explained in [Gro83], where the corresponding inequalities are formulated for the functional $Y \mapsto vol_{n-1}(Y)$ and where $X \supset Y$ is often required to be compact. But all (relevant) arguments from [Gro83] apply in the present case.
[42] This means "contained".
[43] The argument from [Sma93] easily generalises to our Y_{min}.

can be "killed" by 2-dimensional surgery that results in another hypersurface, say $Y \subset X$, which is also homologous to Y_0 and which admits a metric with $Sc \geq 0$.
If, moreover, the inclusion homomorphism $\pi_1(Y_0) \to \pi_1(X)$ is *an isomorphism,* and X is spin, then Y is spin-bordant to Y_0 in the classifying space of $\pi_1(Y_0)$ and the existence of metric with $Sc > 0$ on Y_0, follows from the theorem 1.5 in [Sto01]. (Possibly—I am not certain—this theorem also covers the non-spin case.)
Finally, since the homomorphism $\pi_1(Y_0) \to \pi_1(X)$ is injective according to $[\circ \hookrightarrow \circ]_{\pi_1}$, all of the above applies to the covering space $\tilde{X} \to X$ with the fundamental group $\pi_1(\tilde{X}) - \pi)Y_0)$. QED.

Discussion. The above argument can be generalised, refined and made effective, which would result in specific *inequalities for the "relative size"* of pairs $(X, M(x))$, where $Sc(X)(x)$ is suitably bounded from below in terms of $M(x)$ and where certain hypersurfaces $Y_0 \subset X$ admit *no metrics* with $Sc > 0$.
However, it remains unclear if the condition $[\odot_{\lambda,\rho}]$, or anything of this kind, is truly necessary.
In fact, we do know that *no such condition is needed* for the bounds on the width of overtorical and similar band-shaped manifolds to which our symmetrization with point-wise control of the scalar curvature applies (see sections 7 and 8).
Also, even the present form of $[\bullet]$ remains problematic for manifolds X with *dim* $(X) \geq 9$.

11.7 Bounds on Widths of Non-Compact Riemannian Bands. The band inequalities in sections 2, 4, 5 generalise to certain *non-compact complete* bands.

Case I. Let V be a *proper* (see section 2) band with *compact boundaries*

$$\partial V = \partial_- \cup \partial_+$$

The concepts of "overtorical", "isoenlargeable", and "SYS" make sense for these V, where the instances of such non-compact bands are, topologically, obtained from compact ones by removing isolated points from their interiors.
proper bands V with compact boundaries which have $Sc(V) \geq \sigma > 0$ satisfy the same width bound as their compact counterparts in sections 2, 4, 5.

Proof. Let $V' \subset V$ be a compact manifold obtained by cutting of the infinity of V far away from ∂V. Thus V' has an extra set of components, call their union ∂', where one can make make the distance $dist(\partial(V), \partial')$ as large as one wants.
Thus we arrange such a V', where the the ρ-neighbourhood $U'_\rho \subset V'$ of $\partial' \subset V'$ for a large ρ does not disturb the essential topology of V:

$[\bigstar_\rho]$ the subset $U'_\rho \subset V'$ *does not intersect $\partial = \partial V$ and, moreover, $V' \setminus U'_\rho$ is in the same topological largeness class as V, namely overtorical, "isoenlargeable or SYS correspondingly.*
Clearly, this property is inherited by minimal hypersurfaces constructed in the suitable homology classes as in sections 2, 4, 5, where the final stage of the symmetrization goes through if ρ is sufficiently large.

For instance, $\rho \geq \frac{4\pi}{\sqrt{\sigma}}$ is sufficient for this purpose but this bound doesn't seem sharp. QED □

Case II. Let V be a proper complete orientable n-dimensional band, such that V admits a proper 1-Lipschitz map $\psi : V \to \mathbb{R}^{n-1}$, such that the restriction of ψ to ∂_- (*and hence, on* ∂_+) has degree $d \neq 0$.
If the scalar curvature of V *is bounded from below by* $Sc(V) \geq \sigma > 0$, *then the width of* V *is bounded as follows.*

$$width(V) = dist(\partial_-, \partial_+) \leq 2\pi\sqrt{\frac{n-1}{\sigma n}}.$$

Proof. Let $S^{n-3}(R) \subset \mathbb{R}^{n-1}$ be a very large codimension 2 sphere say of radius $\geq 100^{n-1}$ and let

$$\Sigma = \partial V \cap \psi^{-1}(S^{n-3}(R)) \subset V,$$

where one may assume if one wishes—this is not truly necessary—that the map ψ is transversal to the sphere $S^{n-3}(R)$.
This Σ, which has codimension 2 in V, is meant to serve as the boundary condition for a minimizing hypersurface, Y, namely, the boundary of Y must be contained in Σ and the relative homology class of Y in $H_{n-1}(V, \Sigma)$ must be equal to the homology pullback of the $(n-1)$-ball bounded by this sphere.
To avoid unnecessary non-compactness problems,[44] we cut off V and thus Y, by taking a large $V' \subset V$ which contains $\psi^{-1}(S^{n-3}(R))$, say the ψ-pullback of the R'-neighbourhood $U = U(R, R') \subset \mathbb{R}^{n-1}$ of the sphere $S^{n-3}(R) \subset \mathbb{R}^{n-1}$ for $R' \geq 100R$.
This Y serves as the first step of the inductive symmetrization, except that unlike the original V it is compact and has extra part of the boundary, namely, the ψ-pullback of the boundary of U.
Thus, in order to have a proper inductive scheme , one should drop the completeness assumption on V, and require instead that
V admits a proper 1-Lipschitz map to the R_n-ball in $\mathbb{R}^{n-1}$ for a sufficiently large R_n, say for $R_n \geq 1000^n$, such that the restriction of ψ to ∂_- has degree $\neq 0$.
At this point, we invite the reader to fill in the details in the above argument. □

Corollary. ***Sharp Bound on Widths of Iso-Enlargeable Bands***.
C*ompact Iso-enlargeable bands V (e.g., those homeomorphic to $V_0 \times [-1, 1]$ where V_0 admits a metric with non-positive curvature) with their scalar curvatures $\geq \sigma$ satisfy the $\frac{2\pi}{n}$-Inequality.*

$$width(V) \leq 2\pi\sqrt{\frac{n-1}{\sigma n}}.$$

[44] There is no such problem if V has *locally bounded geometry*, e.g. if it is a covering space of a compact band, then the minimizing Y is compact since locally bounded geometry implies uniform local Lipschitz contractibility ($[\odot_{\lambda,\rho}]$ from the previous section.

Proof. It follows from the definition of iso-enlargeability of V that there exist Riemannian bands V_R for all $R > 0$, such that

- $\bullet_1$ there exist locally isometric band maps $V_R \to V$,
- $\bullet_2$ there exist proper 1-Lipschitz maps from V_R to the R-balls in $\mathbb{R}^{n-1}$, such that the restrictions of ψ to $\partial_\pm$ have (equal) degrees $\neq 0$.

  Thus, the above applies and the proof follows. □

Remark. Minimal hypersurfaces with mixed boundary conditions in our argument are similar to those in §12 in [GL83], where, in fact, a non-sharp version of the above inequality is proven.

On the Definition of Iso-enlargeability. These $\bullet_1$ and $\bullet_2$ can be taken for the definition of iso-enlargeability of bands as it was mentioned in footnote 9 in section 4.

11.8 Stable Enlargeability and Stable Bounds on the Scalar Curvature. Let X be an orientable Riemannian manifold of dimension n and P an orientable pseudomanifold of dimension N.

Let us consider three different cases indexed by $i = 0, 1, 2$, each associated with a map

$$f = f_i : X \times P \to B \subset \mathbb{R}^{n+N-i},\ i = 0, 1, 2,$$

where B is an open unit ball, and where f is a smooth proper, such that *the restrictions of f to the the submanifolds $X_p = X \times \{p\} \subset P$, $p \in P$, are λ-Lipschitz for some constant $\lambda > 0$* and

- $\bullet_0$ if $i = 0$, then the map f has *non-zero degree*;
- $\bullet_1$ if $i = 1$, then the cap product of the pullback of the fundamental cohomology class of B with compact support with the fundamental class of $X \times P$,

$$f^*[B]^{n+N-1} \cap [X \times P]_{N+n} \in H_1(X \times P),$$

  does not vanish when taken with $\mathbb{Q}$-coefficients, i.e. under the homomorphism $H_1(X \times P) \to H_1(X \times P)$;
- $\bullet_2$ if $i = 2$ then the cap product class[45]

$$f^*[B]^{n+N-2} \cap [X \times P]_{N+n} \in H_2(X \times P)$$

  is *aspherical*, i.e. it is not contained in the image of the the Hurewicz homomorphism, or, equivalently, it does not lift to the universal covering of $X \times P$.

Then the infimum of the scalar curvature of X is bounded by

$$\underline{\kappa}_X = \inf_{x \in X} Sc(X)(x) \le \varepsilon \lambda^{-2}$$

where $\varepsilon > 0$ depends only on n and N.

[45] Geometrically, this class is represented by (possibly singular) surfaces in $X \times P$, which are the f-pullbacks of generic points $b \in B$.

Proof. First, let P be a manifold and let us endow it with a very (arbitrarily) large metric which has scalar curvature $\geq -\delta$ for a given arbitrarily small $\delta > 0$. This makes $Sc(X \times P) \geq \kappa - \delta$ and, at the same time, the Lipschitz constant of f as close to λ as you wish.

Then, the proof follows either by arguing as in the proof of the spherical Lipschitz bound theorem in section 3 with a torical band in B and a use of width inequalities for compact bands or doing it more directly with open bands as in the previous section. (This directly applies to $i = 0, 2$ and the case $i = 1$ trivially reduces to $i = 0$.)

In general, if P is a pseudomanifold, we take a similarly large metric in P, where $Sc \geq -\delta$ on all N-faces Q of P. Then, clearly, there exist □

(*) a face Q,
(**) a (large) open subset $X' \subset X$,
(***) a (small) open subball $B' = B'_{Q,X'} \subset B$ around some point $b \in B$,

such that the above applies to the restriction f' of the map f to some the subproduct $X' \times Q' \subset f^{-1}(B') \cap (X' \times Q)$ for some open subset $Q' \subset Q$

$$f' : X' \times Q' \to B'.$$

Complaint. One can't help but to be annoyed by the *the dimension being brought up* by incorporating *purely topological* parameters P into the *geometry* of $X \times P$, only to be immediately *brought down* by constructing a decreasing chain of minimal hypersurfaces.[46]

11.9 Mean Curvature Stability of Polyhedral Domains. Let Y be a closed smooth cooriented hypersurface in a Riemannian manifold $X = (X, g)$ and let g_ε be a family of smooth Riemannian metrics on X which C^0-converge to g for $\varepsilon \to 0$. It is shown in section 10.2 of [Gro12] that such small perturbations g_ε can be accompanied by small perturbations of Y which only slightly change the mean curvature of Y. Namely we have the following perturbation stability property

$[\bigcirc_\varepsilon]$. *there exists a family of diffeomorphisms $\psi_\varepsilon : X \to X$, $\varepsilon > 0$, such that*

(1) *the diffeomorphisms ψ_ε, $\varepsilon \to 0$, converge to the identity map $id : X \to X$ in the C^0-topology;*
(2) *the $(n-1)$-volumes of the hypersurfaces $Y_\varepsilon = \psi_\varepsilon(Y) \subset X$ with respect to g_ε converge to $vol_{n-1}(Y)$ for g;*
(3) *the g_ε-mean curvatures of Y_ε converge to $mn.curv_g(Y)$. For instance, if $mn.curv\,(Y) > c$ for some number $c \in (-\infty, +\infty)$, then the hypersurfaces Y_ε satisfy the same inequality for all sufficiently small $\varepsilon > 0$.*

[46] If X is *complete* and the universal covering $\tilde{X}$ of X is *spin*, then the bound $\inf_{x \in X} Sc(X)(x) \leq \varepsilon\lambda^{-2}$ for $i = 0, 1$ (but not for $i = 2$) follows by applying a suitable index theorem for the P-family of Dirac operators on $\tilde{X}$, keeping the P-parameters at their proper place.

The essential steps in the proof of this can be seen as ε-miniaturised versions of the first three steps in the proof of [•] in section 11.6, which now take place in a small tubular neighbourhood $U \supset Y$.

Step$_\varepsilon$ 1+2. Take a suitable function $M_\varepsilon(x)$ in U for which Y strictly minimizes the functional

$$Y \mapsto vol_{n-1}(Y) - \int_{U_+} M_\varepsilon(x)dx,$$

where $U_+ \subset U$ is the part of U positioned *inward of* Y, such that the corresponding minimizer Y_ε of this functional for g_ε. (Such an M on Y must be equal to $mn.curv(Y)$)

CAN'T FULLY ESCAPE FROM THE ε-NEIGHBOURHOOD $U_\varepsilon \subset U$ OF Y.

Step$_\varepsilon$ 3. Observe that the filling inequalities $[\odot^{n-1}_{n-2}]$ and $[\odot^{n}_{n-1}]$ are stable under small perturbations g_ε of g and conclude that

Y_ε IS TRAPPED IN U_ε.

(This ε may be slightly larger than the above one.)

Then the smoothness of Y_ε ($n \geq 8$ included) follows from Almgren's optimal isoperimetric inequality (see [Alm86]), which also allows a construction of diffeomorphisms $\psi_\varepsilon : X \to X$ which send $Y \to Y_\varepsilon$ (see [Gro12] for details).

Warning. The hypersurfaces Y_ε do not, in general, C^1-converge to Y and, conceivably, there are examples (I have not scrutinised the literature), where there are *no diffeomorphisms* ψ_ε having the norms of their differentials (and/or of their inverses) *bounded* by $1+\varepsilon$.

On the other hand, one can control some *Hölder norm* of ψ_ε according to *Reifenberg's topological disk theorem.*

Also *Reifenberg's flatness condition* implies relative versions of the filling inequalities $[\odot^{n-1}_{n-2}]$ and $[\odot^{n}_{n-1}]$ from the previous section that is useful for smoothing "intrinsic edges and corners" in manifolds with $Sc \geq \sigma$. (see the next section and [Gro14a]).

Localization of $[\bigcirc_\varepsilon]$. $[\bigcirc^{loc}_\varepsilon]$. *If the metrics g_ε are equal to g on a neighbourhood $U_0 \subset X$ of compact subset $X_0 \subset X$ then the above diffeomorphisms ψ_ε can be taken equal to the identity map on another (smaller) neighbourhood $U_\varepsilon \supset X_0$.*

About the Proof. This is achieved with suitable functions M_ε defined on the complement of $X \smallsetminus U_\varepsilon$. And here, as on other localization occasions, we leave the actual proof to the reader, since the corresponding localised properties are only marginally used in the present paper.

Stability Relative to ∂X***.*** In this paper, we need a relative version of $[\bigcirc_\varepsilon]$, and also of $[\bigcirc^{loc}_\varepsilon]$, where X is a manifold *with boundary* and our hypersurface $Y \subset X$ has $\partial Y \subset \partial X$.

The above argument applies (almost) word for word to such Y, where, additionally, one has to keep track of the dihedral angles between (the tangent spaces of) Y and ∂X along $\partial Y \subset \partial X$.

For instance,

if these angles on the inward side of Y *satisfy* $\angle_{in}(Y, \partial X) < \frac{\pi}{2}$, *then also* $\angle_{in}(Y_\varepsilon, \partial X) < \frac{\pi}{2}$.

This together with Bending Lemma (section 11.5) combined with ε-redistribution of curvature (section 11.2) yield the following.

$\mathbb{O}_\varepsilon$-FLATTENING COROLLARY. Let X be a Riemannian manifold with smooth boundary and let $Y \subset X$ me a compact smooth cooriented hypersurface, where $\partial Y \subset \partial X$ and such that $mn.curv(Y) > 0$ and the inward (with respect to the coorientation of Y) dihedral angles between Y and ∂X are everywhere $\leq \frac{\pi}{2}$.

Then there exists a family of smooth Riemannian metrics g_ε, $\varepsilon > 0$, *on* X, *such that*

- $\bullet_1$ *the hypersurface* Y *is totally geodesic in* X *with respect to* g_ε *for all* $\varepsilon > 0$;
- $\bullet_2$ $mn.curv_{g_\varepsilon}(\partial X) \geq mn.curv_g(\partial X)$ *and the inward dihedral angles with respect to* g_ε *between* Y *and* ∂X *are everywhere* $\leq \frac{\pi}{2}$ *for all* $\varepsilon > 0$;
- $\bullet_3$ *the scalar curvatures of* g_ε *are bounded from below at all points* $x \in X$ *by* $Sc(g)(x)$;
- $\bullet_4$ *the metrics* g_ε *are equal to* g *outside the* ε*-neighbourhood (with respect to* g*) of the union* $Y \cup \partial X$;
- $\bullet_5$ *the quadratic forms* $g_\varepsilon - g$ *are positive semidefinite for all* $\varepsilon > 0$.

$\bullet_?$ *On Convergence* $g_\varepsilon \to g$. Ideally, one would like to have C^0-convergence $g_\varepsilon \to g$ for $\varepsilon \to 0$ but all we are able to show is that only the *negative part* of the difference $g_\varepsilon - g$ tends to zero, which, by small perturbations of g_ε, allows one to make $g_\varepsilon - g$ positive semidefinite.

To see where the problem resides, let us return to a closed smooth cooriented hypersurface Y in a Riemannian manifold $X = (X, g)$ at the beginning of this section, and a family of smooth Riemannian metrics let g_ε on X which C^0-converge to g for $\varepsilon \to 0$.

Assume, for simplicity's sake, that $mn.curv(Y) = 0$ and let Y_ε be the perturbations of Y which have $mn.curv(Y_\varepsilon) \to 0$ and which themselves uniformly converge to Y.

We do know that these Y_ε are diffeomorphic to Y but we do not expect their Riemannin metrics $h_\varepsilon = g_{|Y_\varepsilon}$ induced from g_ε to converge to $h = g_{|Y}$. All we can say is that the g-normal projections $Y_\varepsilon \to Y$, besides being homotopic to diffeomorphisms, are $(1 + o(1))$-Lipschitz. This eventually transforms to the above $\bullet_5$.

And albeit our construction of g_ε can't deliver the C^0-convergence $g_\varepsilon \to g$, probably, one can show that the distance functions $dist_{g_\varepsilon}$ on $X \times X$ uniformly converge to $dist_g$ for $\varepsilon \to 0$.

ON STABILITY OF PIECEWISE SMOOTH HYPERSURFACES.

Here, as everywhere in this paper, "piecewise smooth" hypersurfaces Y in X are, by definition, locally diffeomorphic to polyhedral hypersurfaces in $\mathbb{R}^n$, $n = dim(X)$.

If X comes with a Riemannian metric g, these Y, if they are cooriented, are characterised by the mean curvatures of their $n-1$-faces and the dihedral angles between these faces.

Apparently, as indicated (without proof) in section 4.8 in [Gro14a] these Y satisfy a piecewise-smooth version of the above [$\bigcirc_\varepsilon$].

This means that,
given an ε-family of smooth Riemannian metrics on X, which C^0-converge to a metric g as in $[\bigcirc_\varepsilon]$,

$$g_\varepsilon \underset{\varepsilon\to 0}{\to} g,$$

there exists a family of piecewise smooth homeomorphisms $\psi_\varepsilon : X \to X$, $\varepsilon > 0$, such that

- *ψ_ε are smooth on the $(n-1)$-faces of Y away from the $(n-3)$-faces,*
- *ψ_ε converge to the identity map $id : X \to X$ in the C^0-topology;*
- *the $(n-1)$-volumes of the hypersurfaces $Y_\varepsilon = \psi_\varepsilon(Y) \subset X$ with respect to g_ε converge to $vol_{n-1}(Y)$ for g;*
- *the g_ε-mean curvatures of the faces of Y_ε converge to $mn.curv_g$ of the corresponding faces of Y;*
- *the g_ε-dihedral angles between the faces of Y_ε converge to the g-dihedral angles between the corresponding faces of Y_ε.*

The relative version of this in manifolds X with corners (i.e. with piecewise smooth boundaries), which also seems to follow by the available techniques[47], would automatically yield smoothing of metrics on manifolds obtained by reflections of manifolds with corners with no decrease of their scalar curvatures (see section 4.8 in [Gro14a]).
On the other hand, one can prove the existence of such smoothings in the essential cases by some roundabout argument as we shall explain in the next section.[48]

11.10 Flattening of Faces and Regularisation of Reflections.

□-Flattening Lemma. Let $X = (X, g)$ be a Riemannian manifold with corners, where all faces of the boundary ∂X have positive mean curvatures and where the dihedral angles between the pairs of $(n-1)$-faces in ∂X, wherever they meet, are $\leq \frac{\pi}{2}$.
Then there exists a family of smooth metrics g_δ, $\delta > 0$, such that

(1) *the faces of ∂X are totally geodesic with respect to g_δ for all $\delta > 0$;*
(2) *all dihedral angles in ∂X with respect to g_δ are $\frac{\pi}{2}$ for all $\delta > 0$;*
(3) *the scalar curvatures of g_δ are bounded from below at all $x \in X$ by $Sc(g)(x)$;*
(4) *the metrics g_δ coincide with g outside the δ-neighbourhood of the boundary ∂X;*
(5) *the differences $g_\delta - g$ are positive semidefinite for all $\delta > 0$.*

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[47] The local version of this must be also true

[48] An unpleasant technical difficulty in the proof of the mean curvature stability in the piecewise smooth case, which one has to (?) go around, is the absence (?) of C^1 regularity theorem for minimal $Y \subset X$ at the singular boundary points of X.

This is a special case of the Approximation/Reflection Lemma in section 4.9 in [Gro14a] where this is proven not only for g itself, but for metrics g_ε which C^0-converge to g.[49]
Granted that, one can reflect X around the faces (see **2** in section 11.1) smoothly the metric in the resulting manifold $\tilde{X}$ by using Cut-off Homotopy Lemma (★ in 11.1) and applying a corresponding inequality for manifolds (bands) without corners.
Thus, for instance, one shows in [Gro14a] that
Riemannian manifolds X with $Sc(X) > 0$ can't contain mean curvature convex (e.g. convex) cubical domains Q, where all dihedral angles, are non-obtuse.
Below is another EXAMPLE.
Sub-Rectangular $\frac{2\pi}{n}$-Inequality. Let X be a Riemannian n-manifold, let $Q \subset X$ be a domain diffeomorphic to the n-cube $[-1,+1]^n$ and let $Q_i^\pm \subset \partial Q \subset Q$, $i = 1,\dots,n$, denote the pairs of opposite codimension 1 faces in Q which correspond to such pairs in the cube.
Let

(i) the faces $Q_i^\pm$ for $i = 1,\dots,n-1$, are mean curvature convex, i.e.

$$mn.curv(Q_i^\pm) \geq 0,$$

(ii) the dihedral angles $\angle_{\pm i,\pm j} = \angle(Q_i^\pm, Q_j^\pm)$ between these faces are non-obtuse at all points in the $(n-2)$-"edges" where these faces meet,

$$\angle_{\pm i,\pm j} \leq \pi/2, \text{ for all } i,j = 1,\dots,n-1, i \neq j,$$

(iii) the scalar curvature of X satisfies $Sc(X) > n(n-1)$.

Then the distance between the two remaining opposite faces satisfy

$$\left[\square_\pm < \tfrac{2\pi}{n}\right] \qquad dist_\pm = dist_X(Q_n^+, Q_n^-) < \frac{2\pi}{n}.$$

Proof. Reflect Q, but now only only in the faces $Q_i^\pm$ with $i < n$. Thus we construct a torical band with $Sc > \sigma$ and apply $\left[\circledcirc_{\frac{2\pi}{\sqrt{\sigma}}}\right]$ to this band. □

FLATTENING AND GLUING WITH $Sc > 0$. The proof of the Approximation/Reflection Lemma in [Gro14a] proceeds by induction on the number of faces with application of Reifenberg's flatness property of minimal varieties at each step of induction.

But if the boundary of X consists of *only two* (possibly disconnected) $(n-1)$-faces then the □-Flattening Lemma follows from the $\mathbb{(}_\varepsilon$-Flattening Corollary from the previous section.

[49] This lemma is formulated in [Gro14a] only for $Sc(g) > 0$ and without formulating the above (4) and (5). However the proof of this lemma indicated in [Gro14a] automatically deliver these properties.

Indeed if $Y \subset$X is a *totally geodesic* hypersurface *normal* to ∂X, then the intrinsic bending delivered by the proof of the Bending Lemma from section 11.5 does not disturb Y and we have both Y and ∂X totally geodesic as well as mutually normal. This also applies if the set of $(n-1)$-faces one can be divided into *two* subsets of mutually disjoint ones, i.e. if the incidence graph $\mathcal{I}$ between the faces is *bipartite*.
And assuming no three $(n-1)$-faces meet, $\mathcal{I}$ can be artificially made bipartite by subdividing the faces.
This is done by creating "new narrow" $(n-1)$-faces positioned close to the $(n-2)$-faces on one side of them. (The $(n-2)$ faces are assumed *two-sided* , i.e. coorientable in ∂X.)
Observe that the the dihedral angles between the "new" $(n-1)$-faces and the "old" ones are equal to π, which, however, leads to no problem due to the possible localization of bending (section 11.5) and of $[\bigcirc_\varepsilon]$ (section 11.7).
Now let us explain how one can get rid of "higher order corners" in X by paying the price of a change of their topologies. In fact this "price" limits the application of such gluing to $dim(X) = 3$
Given an n-dimensional cosimplicial manifold X with corners, e.g. diffeomorphic to the n-cube $[0,1]^n$, one may *double* it over the set of its (preliminarily cut off) *highest order corners*, where the *maximal numbers*, say m, of the $(n-1)$-faces in ∂X meet. For instance, if $X = [0,1]^n$ then $m = n$ where these highest corners are the ordinary vertices in $[0,1]^n$.
The resulting double, call it $X^{[1]}$ has a natural corner structure where the highest corners have order $m^{[1]} = m-1$. Thus we can continue unless we arrive at $X^{[m]}$ with smooth boundary (see section 1.1 in [Gro14a]).
Geometrically, if X is the ordinary cube $[0,1]^n \subset \mathbb{R}^n$, (almost but not quite exactly) this can be achieved by cutting the faces of this cube of dimensions $\leq n-2$ by hyperplanes and then by reflecting the resulting polyhedron around its (new as well as old) $(n-1)$-faces.
Now we want to stop at $X^{[m]}$, for $m \leq n-2$, such that the only singularities are $(n-2)$-dimensional faces/"edges", where pairs of $(n-1)$-faces meet.
Gluing/Surgery Lemma. Let g be a Riemannian metric on X, for which all faces have $mn.curv_g > 0$ and all dihedral angles are $\leq \alpha \leq \pi$.
Then $X^{[m\leq n-2]}$ *admits a smooth metric* $g^{[m]}$*, where the faces also have* $mn.curv_{g^{[m]}} > 0$ *and and all dihedral angles are* $\leq \alpha$*, and such that the scalar curvature of* $g^{[m]}$ *is bounded from below (in a natural sense) by that of* g*.*
For instance, if $Sc(g) > 0$*, then also* $Sc(g^{[m]}) > 0$*.*

Sketch of the Proof. Let first X be diffeomorphic to the 3-cube $[0,1]^3$. By consecutively applying the Cut-off Homotopy Lemma at a vertex $p \in [0,1]^3$ to the three 2-faces at p, one can "infinitesimally straighten" these faces at p, i.e. make them geodesic at p, while keeping $mn.curv \geq 0$.

If there are two 3-manifolds, X and X' with such vertices, one can arrange the connected sum

$$(X,p)\#(X,p')$$

by "gluing" them with an arbitrarily small decrease of their scalar curvatures by the same construction as it is done for the ordinary connected sum in [GL80b], where one can preliminarily enlarge the scalar curvature of the two at the points p and p' by means of the Cut-off Homotopy Lemma (as it is done on p. 111 in [Gro86] in a similar context.)

Then the required double of our cubical X is achieved by "infinitesimally straightening" the faces at all eight vertices in X and then taking the double with a "glueing metric" at all vertices.

The above applies to 0-faces (vertices) p of all X for all $n = dim(X)$. In general, if P is an m-face for $0 < m \leq n-2$, one "infinitesimally straighten" the $(n-1)$-faces *normally to* P, where a face $P^+ \supset P$ is regarded infinitesimally straight normally to P if the second fundamental form of $P \subset X$ *vanishes* on the tangent subspaces $T_x \subset T_x(P^+)$ *normal to* $P \subset P^+$ at all points $x \in P$, where, observe, $rank(T_x) = dim(P^+) - dim(P) \geq 2$.

Clarifying Topological Remark. To see what makes the difference between dimensions $n = 3$ and $n \geq 4$, let V be an n-dimensional manifold X minus an open tubular neighbourhood of a closed submanifold Y.

If X carries a metric with $Sc > 0$ and $codim(Y) \geq 3$, then the double $W = V \cup_{\partial V} V$ also carries such metric.

Now, if X is overtorical and the submanifold Y can be homotoped to a single point in X, e.g. Y is a finite union of points in X, then W is also overtorical.

But if, for instance, $X = \mathbb{T}^n$ and Y is the union of n disjoint circles which generate $H_1(X)$, then W is *not* overtorical. In fact, the corresponding W in this case does carry a metric with positive scalar curvature.

11.11 Non-existence Results and Conjectures. Following a suggestion by the referee, we briefly overview in this section a few constrains, some of which proved in the main body of this article and some conjectural, on the topology of manifolds which carry metrics with $Sc > 0$.[50]

Say that a closed oriented manifold X is SYS (Schoen–Yau–Schick) *over a cohomology class* h *in a topological space* K, where

$$h \in H^{n-2}(K;\mathbb{Z}),\ n = dim(X),$$

if there is a continuous map $A : X \to K$, such that the 2-dimensional homology class in X which is the Poincaré dual of the cohomology pullback $A^*(h) \in H^{n-2}(X;\mathbb{Z})$ of h,

$$A^{\perp}(h) = PD(A^*(h)) \in H_2(X),$$

[50] Unsolved problems on $Sc > 0$ are collected in [Gro17].

is *non-spherical*, i.e. it is *not contained* in the image of the *Hurewicz homomorphism* $\pi_2(X) \to H_2(X)$.
For example, "SYS over the fundamental cohomology class of the torus $\mathbb{T}^{n-2}$" is the same as "SYS" defined in section 5.
Let us generalise the Schoen–Yau theorem ([SY79b] for $n \le 7$ and [SY17] for all n) on *non-existence of metrics with positive scalar curvatures on SYS-manifolds* and (some case of) theorem 13.8 in [GL83] as follows.

$\mathcal{SYSE}$-Non-existence Theorem. Let K be a manifold which admits *a complete metric with non-positive sectional curvature* and let X be a closed manifold X which is *SYS over an integer cohomology class $h \in H^{n-2}(K)$ which does not vanish in $h \in H^{n-2}(K;\mathbb{Q})$.*
Then X admits no metric with positive scalar curvature.

Proof. Start with the case where K is compact oriented of dimension $n-2 = dim(X)-2$ and $h \in H^{n-2}(K;\mathbb{Z})$ is the fundamental class of K. Let $\tilde{K}$ be the universal covering of K and $\tilde{X}$ be the covering of X induced by the above map $A : X \to K$.
Since $\tilde{K}$ has non-positive curvature, it admits proper λ-Lipschitz maps F_λ to the unit ball $B = B^{n-2}(1) \subset \mathbb{R}^{n-2}$ of degree 1 for all $\lambda > 0$ and then, when $\lambda \to 0$, the proof follows from the stable SYSE-bound on the scalar curvature from section 6 applied to the composed maps

$$\tilde{X} \overset{\tilde{A}}{\to} \tilde{K} \to B$$

in the role of $f = f_\lambda$ and $X = X \times \{p\}$, where $\{p\}$ a single point space for P.
Now let us turn to the case of a complete orientable K of dimension $m \ge n-2$ and let $h \in H^{n-2}(K;\mathbb{Z})$. Assume K is parallelizable, otherwise, pass to the total space of the normal (for an embedding $K \to \mathbb{R}^M$) vector bundle $T^\perp(K) \to K$, which (by an easy argument) also carries a metric with non-positive sectional curvatures.
In this case, there exists continuous proper maps $F_\lambda : \tilde{K} \times K \to B = B^m(1)$, $\varepsilon > 0$, which are λ-Lipschitz (diffeomorphisms) on all "slices " $\tilde{K} \times k$, $k \in K$: these maps are constructed as above with the use of inverse exponential maps in $\tilde{K}$ at the points over $k \in K$, where the tangent spaces $T_k(K)$ are identified with a single $\mathbb{R}^m$ with a use of a frame in $T(K)$ as it is done in section 13 in [GL83]).
Let $P \subset K$ be a subpseudomanifold of codimension $n-2$, which represents the Poincare dual of h (which is a homology class with, a priori, infinite support) and let $f_\varepsilon : \tilde{X} \times P \to B$ be the maps obtained by composing $\tilde{A} : \tilde{X} \to \tilde{K}$ and the restriction of F_λ to $\tilde{X} \times P$,

$$f = f_\lambda(\tilde{x}, p) = F_\lambda(\tilde{A}(\tilde{x}), p).$$

Clearly, this map $f : X \times P \to B$, satisfies the assumptions on f in section 11.6, and stable SYSE-bound on the scalar curvature from 11.6 applies. QED. □

CONJECTURE **A**. If a closed manifold X is SYS over a non-torsion cohomology class h in an aspherical space K, then X admits no metrics with $Sc > 0$.

This conjecture may be unrealistically strong but the special case of this, where *K admits a complete (possibly singular) metric with non-positive curvature* seems within reach.
Recall (see section 10) that a closed oriented manifold X is called $[\tilde{\uparrow}0]$-oversymplectic if
• a multiple of the fundamental cohomology class of X decomposes into product of one and two dimensional classes,

$$k \cdot [X]^\circ = h_1 \smile \ldots \smile h_m,$$

and
• X the classes h_i vanish in the cohomology of the universal covering $\tilde{X}$.
Also recall (see section 10) that $[\tilde{\uparrow}0]$-*oversymplectic manifolds, the universal covers of which are spin, carry no metrics with* $Sc > 0$.
Probably, the spin condition is redundant; moreover, one may merge this with the above **A** as follows.
Conjecture **B**. Products of manifolds X as in the above **A** by $[\tilde{\uparrow}0]$-oversymplectic ones admit no metrics with $Sc > 0$.

11.12 A Few Geometric Problems and Conjectures. Conjecture **C**. If a closed manifold V_0 of dimension $n-1 \geq 5$, admits no metric with $Sc > 0$ then Riemannian bands V diffeomorphic to $V_0 \times [-1,1]$ which have $Sc(V) \geq \sigma > 0$, satisfy the sharp width inequality,

$$width(V) \leq 2\pi\sqrt{\frac{n-1}{\sigma n}}.$$

Conjecture **D**. Let g_0 stands for the standard Riemannian metric on the unit sphere S^n with the sectional curvature 1.
If a RIemannian metric g on S^n minus a point satisfies

$$g \geq g_0 \text{ and } Sc(g) \geq Sc(g_0) = n(n-1),$$

then $g_0 = g$.
(If g is complete, this follows, by the relative index theorem for the Dirac operator, see [Lla98] and (8) in section 10)
Conjecture **D'**. Let X be closed n-manifold, such that X minus a point admits *no complete metric with* $Sc > 0$.
Let V be obtained by removing a small open n-ball from X, i.e. $V = X \smallsetminus B_{x_0}(\varepsilon)$, and let g be a metric on V with $Sc(g) \geq \sigma > 0$. If the ρ-neighbourhood with respect to g of the boundary sphere $S^{n-1} = \partial V = \partial B_{x_0}(\varepsilon)$ is homeomorphic to $S^{n-1} \times [0,1]$, then

$$\rho \leq \frac{20}{\sqrt{\sigma}}.$$

(If X is a SYS-manifold, then metrics g with $Sc(g) \geq \sigma$ on V do satisfy this inequality as it follows by Schoen–Yau's kind of argument adapted to manifolds with boundaries as in section 11.6.
On the other hand, the conjecture must be vacuous for *simply connected* manifolds X, since $X \setminus \{x_0\}$ for such an X contracts to an $(n-2)$-subpolyhedron in $X \setminus \{x_0\}$, which, most probably, implies that $X \setminus \{x_0\}$ admits a complete metric with $Sc > 0$.)
Conjecture **E**. Let V be a Riemannian manifold homeomorphic to $\mathbb{T}^2 \times [-1.1]$ with sectional curvature everywhere ≥ 1. Then (this was already mentioned in section 3)

$$width(V) = dist(\partial_-(V), \partial_+(V)) \leq \pi/2,$$

(The simplest unsettled case is where V is a domain in S^3 or, more generally if it admits an isometric embedding or immersion into S^3.
Below is a far reaching generalisation of the spherical case of **E**.
Conjecture $\mathbf{E}_{+\infty}$. Let $Y \subset S^N$, $N = n, n+1, \ldots, \infty$, be a submanifold homeomorphic to the product of n closed manifolds of dimensions ≥ 1, e.g. homeomorphic to the N-torus and let $U \supset Y$ be a neighbourhood of Y in S^N which admits a retraction to Y. Then

$$dist(Y, \partial U) \leq \arcsin \frac{1}{\sqrt{n}}.$$

But, in reality, one has no estimate for this distance even for high codimensional tori in spheres, which suggests the following conjecture opposite to $\mathbf{E}_\infty$.
Conjecture $-$**E**. Every compact smooth manifold Y^n of dimension n admits a smooth embedding to the sphere S^{2n} such that all principal curvatures of the image satisfy

$$curv(Y^n \subset S^{2n}) \leq const < \infty,$$

say for $const = 1000$.
It is hard to believe in the validity of either $\mathbf{E}_{+\infty}$ or **-E**, but something in between may be true, e.g. the following.
Conjecture $\mathbf{E}_{\mathbb{T}^n}$. The minimal constant β, such that the n-torus admits an smooth immersion to S^{n+1} with principal curvatures $\leq \beta$ is asymptotic, for $n \to \infty$, to

$$const \cdot n^\beta \text{ for some } \beta > 1 .$$

Problem **F**. Let $U \subset \mathbb{R}^N$ and $V \subset \mathbb{R}^n$, $n \leq N$, be open subsets, e.g. balls $B^N(r)$ and $B^n(R)$, $R \geq r$. Evaluate the minimal $\beta = \beta(U, V) > 0$, such that V admits a smooth *locally expanding immersion/embedding*[51] to U with the principal curvatures $\leq \beta$.

In Conclusion. The above **A** - **F** are only tips of the iceberg of what we don't know about the scalar curvature and nearabouts.
(An outline of this "iceberg" is given in [Gro17].)

[51] A a smooth map $f : V \to U$ is *locally expanding* if the differential $Df : T(V) \to T(U)$ doesn't decrease the norms of the tangent vectors, $||Df(\tau)|| \geq ||\tau||$ for all $\tau \in T(V)$.

References

[Alm85] Sebastiao Almeida. Minimal Hypersurfaces of a Positive Scalar Curvature. *Math. Z.* 190 (1985), 73–82.

[Alm86] F.J. Almgren Jr. Optimal isoperimetric inequalities, Indiana Univ. *Math. J.* 35 (1986), 451–547

[ACG07] Lars Andersson, Mingliang Cai and Gregory J. Galloway. Rigidity and positivity of mass for asymptotically hyperbolic manifolds. *Ann. Henri Poincaré* (1)9 (2008), 1–33.

[Ati76] M. F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. *Astérisque* 32–3 (1976), 43–72.

[Bam16] R. Bamler. A Ricci flow proof of a result by Gromov on lower bounds for scalar curvature. Mathematical Research Letters Volume 23 (2016). Number 2, Pages 325 - 337.

[BER17] Boris Botvinnik, Johannes Ebert, Oscar Randal-Williams. *Infinite loop spaces and positive scalar curvature, Inventiones mathematicae* (3)209 (2017), 749–835.

[BH09] M. Brunnbauer, B. Hanke. Large and small group homology. *J. Topology* 3 (2010), 463–486.

[BHMM15] Jean-Pierre Bourguignon, Oussama Hijazi, Jean-Louis Milhorat, Andrei Moroianu and Sergiu Moroianu. A Spinorial Approach to Riemannian and Conformal Geometry,EMS Monographs in Mathematics (2015).

[BMN10] S Brendle, F.C. Marques, A. Neves. Deformations of the hemisphere that increase scalar curvature, arXiv:1004.3088[math.DG]

[DFW03] A. N. Dranishnikov, Steven C. Ferry and Shmuel Weinberger. Large Riemannian manifolds which are flexible. Annals of Mathematics 157 (2003), 919–938

[Dra00] A. Dranishnikov. Asymptotic topology. *Russian Math. Surveys* (6)55 (2000), 71–116

[Dra06] A. N. Dranishnikov. On hypereuclidean manifolds Geom. *Dedicata* 117 (2006), 215–231

[FS80] D. Fischer-Colbrie, R. Schoen. The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature. *Comm. Pure Appl. Math.* 33 (1980), 199–211

[GL80a] M. Gromov, B. Lawson. Spin and Scalar Curvature in the Presence of a Fundamental Group I Annals of Mathematics, 111 (1980), 209-230.

[GL80b] M. Gromov and H. B. Lawson. The classification. of simply connected. manifolds of positive scalar curvature, *Annals of Mathematics*, 11, pp 423–434 (1980).

[GL83] M. Gromov and H. B. Lawson. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Etudes Sci. Publ. Math.* 58 (1983), 83–196.

[Gro83] M. Gromov. Filling Riemannian manifolds. *J. Differential Geom.* 18, no. 1, 1–147. (1983).

[Gro86] M. Gromov. *Partial differential relations*. Springer (1986)

[Gro96] M. Gromov. Positive curvature, macroscopic dimension, spectral gaps and higher signatures. In: *Functional analysis on the eve of the 21st century*, Vol. II (New Brunswick, NJ, 1993) ,volume 132 of Progr. Math., Birkhäuser, (1996), pp. 1–213.

[Gro12] M. Gromov. Hilbert volume in metric spaces. *Part 1. Cent. Eur. J. Math.* (2)10(2012), 371–400.

[Gro14a] M. Gromov, Dirac and Plateau billiards in domains with corners. *Central European Journal of Mathematics* (8)12 (2014), 1109–1156

[Gro14b] M. Gromov. M. Plateau-Stein manifolds, Cent. Eur. J. Math., 12(7), 923–951 (2014).

[Gro17] M. Gromov. 101 Questions, Problems and Conjectures around Scalar Curvature. http://www.ihes.fr/~gromov/PDF/101-problemsOct1-2017.pdf (2017).

[GS01] S. Goette, U. Semmelmann. Spin^c Structures and Scalar Curvature Estimates. *Annals of Global Analysis and Geometry.* (4)20 (2001) pp 301–324

[GS02] S. Goette and U. Semmelmann. Scalar curvature estimates for compact symmetric spaces. *Differential Geom. Appl.* (1)16 (2002), 65–78

[Han11] B. Hanke. Positive scalar curvature, K-area and essentialness, Global Differential Geometry pp 275–302, (2011).

[HS06] B. Hanke, T. Schick. Enlargeability and index theory. *J. Differential Geom.* (2)74 (2006), 293–320.

[HPS15] Bernhard Hanke, Daniel Pape and Thomas Schick. Codimension two index obstructions to positive scalar curvature. *Annales de l'institut Fourier* (6)65 (2015), 2681–2710.

[Hit74] N. Hitchin. Harmonic spinors, Advances in Math. 14 (1974), 1–55.

[Lla98] M. Llarull. Sharp estimates and the Dirac operator. *Mathematische Annalen-January* (1)310 (1998), 55–71.

[LM84] H.B., Jr Lawson and M.-L. Michelsohn. Approximation by positive mean curvature immersions: frizzing. *Inventiones mathematicae* 77 (1984), 421–426

[Loh99] J. Lohkamp. Scalar curvature and hammocks. *Math. Ann.* 313 (1999), 385–407

[Loh16] J. Lohkamp. The Higher Dimensional Positive Mass Theorem II, arXiv:1612.07505 (2016).

[MS12] Donovan McFeron, Gábor Székelyhidi. On the positive mass theorem for manifolds with corners, Communications in Mathematical Physics (2)313 (2012).

[Mia02] P. Miao. Positive mass theorem on manifolds admitting corners along a hypersurface. Adv. Theor. Math. Phys., (6)6 (2002), 1163–1182

[Min88] M. Min-Oo. Scalar curvature rigidity of certain symmetric spaces. In Geometry, topology, and dynamics (Montreal, PQ, 1995), volume 15 of CRM Proc. Lecture Notes , pages 127–136

[Min89] M. Min-Oo, Scalar curvature rigidity of asymptotically hyperbolic spin manifolds. *Math. Ann.* 285 (1989), 527–539

[Min02] M. Min-Oo. K-Area, mass and asymptotic geometry, http://ms.mcmaster.ca/minoo/mypapers/crm_es.pdf (2002)

[Sma93] N. Smale. Generic regularity of homologically area minimizing hyper surfaces in eight-dimensional mani- folds. *Comm. Anal. Geom.* (2)1 (1993), 217–228.

[Ros07] J. Rosenberg. Manifolds of positive scalar curvature: a progress report. In: *Surveys on Differential Geometry, vol. XI: Metric and Comparison Geometry*, International Press (2007).

[Sch98] Thomas Schick. A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture. *Topology* (6)37 (1998).

[SY79a] R. Schoen and S. T. Yau. Existence of incompressible minimal surfaces and the topology of three dimensional manifolds of non-negative scalar curvature.

Ann. of Math. 110 (1979), 127–142.

[SY79b] R. Schoen and S. T. Yau. On the structure of manifolds with positive scalar curvature. *Manuscripta Math.* 28 (1979), 159–183

[SY17] Richard Schoen, Shing-Tung Yau. Positive Scalar Curvature and Minimal Hypersurface Singularities, arXiv:1704.05490 (2017)

[Sto01] S. Stolz. Manifolds of a positive scalar curvature, in: T. Farrell etal. (eds.), Topology of high dimensional manifolds, ICTP Lect. Notes, vol. 9, 665–706. 1.1, Trieste (2001).

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Received: August 11, 2017
Revised: February 13, 2018
Accepted: April 5, 2018