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NEARLY PARALLEL VORTEX FILAMENTS IN THE 3D GINZBURG–LANDAU EQUATIONS

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Abstract. We introduce a framework to study the occurrence of vortex filament concentration in $3D$ Ginzburg–Landau theory. We derive a functional that describes the free-energy of a collection of nearly-parallel quantized vortex filaments in a cylindrical 3-dimensional domain, in certain scaling limits; it is shown to arise as the Γ-limit of a sequence of scaled Ginzburg–Landau functionals. Our main result establishes for the first time a long believed connection between the Ginzburg–Landau functional and the energy of nearly parallel filaments that applies to many mathematically and physically relevant situations where clustering of filaments is expected. In this setting it also constitutes a higher-order asymptotic expansion of the Ginzburg–Landau energy, a refinement over the arclength functional approximation. Our description of the vorticity region significantly improves on previous studies and enables us to rigorously distinguish a collection of multiplicity one vortex filaments from an ensemble of fewer higher multiplicity ones. As an application, we prove the existence of solutions of the Ginzburg–Landau equation that exhibit clusters of vortex filaments whose small-scale structure is governed by the limiting free-energy functional.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^3$ and $\varepsilon > 0$ small. For $u \in H^1(\Omega;\mathbb{C})$, the Ginzburg–Landau energy is given by

$$F_\varepsilon(u;\Omega) = F_\varepsilon(u) := \int_\Omega e_\varepsilon(u) \ , \qquad e_\varepsilon(u) := \frac{1}{2}\,|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2. \tag{1.1}$$

We want to derive an effective interaction energy for $n \geq 2$ vortex filaments in the context of Ginzburg–Landau theory; this will allow us to prove the existence of solutions of the Ginzburg–Landau equations—that is, critical points of $F_\varepsilon(\cdot)$—with a particular geometric structure that we detail below.

A model setting for studying nearly parallel vortex filaments is one found in fluid dynamics [KMD95] which we adopt. Thus, we will always consider Ω of the form

$$\Omega := \omega \times (0,L), \qquad \omega \subset \mathbb{R}^2 \text{ bounded, open, simply connected, } \partial\omega \text{ smooth.} \tag{1.2}$$

Throughout this paper, we write points in Ω in the form (x,z) with $x \in \omega$ and $z \in (0,L)$. We always assume that $0 \in \omega$, and the configurations of interest to us are those with n vortex lines close to the vertical $\{0\} \times (0,L)$.

Given $u = u^1 + iu^2 \in H^1(\Omega;\mathbb{C})$, we define the *momentum* and *vorticity* vector fields,[1] denoted $j(u)$ and $\mathcal{J}u$ respectively, by

$$j(u) := Im(\bar{u}\,\nabla u), \qquad \mathcal{J}u := \frac{1}{2}\nabla \times j(u) = \nabla u^1 \times \nabla u^2. \tag{1.3}$$

It is known (see [MSZ04, ABO05]) that for every $n \in \mathbb{N}$, there exist solutions (u_ε) of the Ginzburg–Landau equations for which the energy and vorticity concentrate around $\{0\} \times (0,L)$ in the sense that

$$\int_\Omega \phi \frac{e_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} dx\,dz \to n\pi \int_0^L \phi(0,z)\; dz \qquad \text{for all } \phi \in C(\Omega) \tag{1.4}$$

and

$$\int_\Omega \varphi \cdot \mathcal{J}u_\varepsilon \, dx\,dz \;\to n\pi \int_0^L \varphi(0,z)\cdot e_z dz \qquad \text{for all } \varphi \in C^1_c(\Omega;\mathbb{R}^3), \tag{1.5}$$

where e_z is the standard unit vector in the z direction. These are interpreted as stating that the solutions (u_ε) exhibit 1 or more vortex filaments, carrying a total of n quanta of vorticity, clustering near the segment $\{0\} \times (0,L)$.

Our results give a precise description of the way in which this clustering occurs. In particular for $0 < \varepsilon \ll 1$ we find solutions with the following properties:

- each solution possesses n distinct filaments, identified as curves along which the vorticity concentrates, each of multiplicity 1 (rather than a smaller number of filaments of higher multiplicity);
- these filaments are separated by distances of order $|\log \varepsilon|^{-1/2}$;
- after dilating horizontal distances by a factor of $|\log \varepsilon|^{1/2}$, the limiting geometry of the vortex filaments is governed by a particular free-energy functional, see (1.6) below.

For the solutions we find, the limiting vorticity (after rescaling in the horizontal variables) is concentrated on n curves of the form

$$z \in (0,L) \quad \longmapsto \quad (f_i(z), z)$$

where the function $f = (f_1, \dots, f_n)$ minimizes

$$G_0(f) := \pi \int_0^L \left(\frac{1}{2}\sum_{i=1}^n |f_i'|^2 - \sum_{i\neq j} \log|f_i - f_j| \right) dz \tag{1.6}$$

[1] The vorticity can also be defined as a 2-form, and indeed this is the perspective we will adopt throughout most of this paper.

in $H^1\left((0,L);(\mathbb{R}^2)^n\right)$, subject to certain boundary conditions. The length scale of vortex separation

$$h_\varepsilon = |\log \varepsilon|^{-1/2} \tag{1.7}$$

is critical in the sense that it gives rise to a limiting functional in which the $\frac{1}{2}|f'|^2$ term and the logarithmic repulsion terms roughly balance.

The solutions we find with the above properties will be obtained as minimizers and local minimizers of the Ginzburg–Landau energy with suitable boundary conditions. The description of the fine structure of the vorticity in these solutions will be deduced from a detailed asymptotic description of the energy and vorticity of sequences $(u_\varepsilon) \subset H^1(\Omega;\mathbb{C})$ of functions with n vortex filaments clustering on a scale h_ε around the segment $\{0\} \times (0,L)$. Very roughly speaking, we will prove that in certain regimes, if $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is a sequence with limiting rescaled vorticity described by $f \in H^1((0,L);(\mathbb{R}^2)^n)$, then

$$\text{“}F_\varepsilon(u_\varepsilon) \approx \text{ logarithmically divergent term } + G_0(f)\text{''} \quad \text{as } \varepsilon \to 0. \tag{1.8}$$

See Theorem 3 below for a precise statement. Thus, the functional G_0 may be seen as an asymptotic energy associated to the family $(F_\varepsilon)_{\varepsilon\in(0,1]}$, after renormalizing by subtraction of the divergent term. In fact, $G_0(\cdot)$ has already been identified as a candidate for the asymptotic renormalized energy of a family of nearly parallel vortex filaments in [PK08]. A related effective energy funtional is found via formal arguments, and in a somewhat different setting, in [AR01].

The divergent term in (1.8) is related to the arclength (with multiplicity) of the limiting vorticity. This reflects the well-known connection between the Ginzburg–Landau energy and the arclength of limiting vortex filaments. Numerous specific results of this sort are known, including for example [Riv96, San01, LR01, BBO01, JS02, BBM04, ABO05]. In this context, the term $\frac{\pi}{2}\int_0^L \sum_{i=1}^n |f_i'|^2 dz$ in $G_0(\cdot)$ may be seen as the linearization of arclength, the leading-order asymptotic energy.

In $2D$, an asymptotic expansion of the energy, in the spirit of (1.8), was first carried out in the seminal work of Bethuel, Brezis and Hélein [BBH94] and later extended to several other two-dimensional contexts (see for example [SS07]). The term $-\pi\int_0^L \sum_{i\neq j} \log|f_i - f_j| dz$ in $G_0(\cdot)$ in essence arises from a fundamental object, the $2D$ “renormalized energy”, introduced in [BBH94].

Until now, higher dimensional counterparts of the results of Bethuel *et al* [BBH94] with a comparable degree of precision have been very elusive. The corresponding 3D results—the rigorous version of (1.8), stated in Theorem 3—are the main contribution of this paper. However, since they require considerable notation to state, we first describe our existence theorems in more detail.

1.1 Solutions of the Ginzburg–Landau equations with vortex clustering. We will study minimizers and local minimizers of the Ginzburg–Landau energy $F_\varepsilon(\cdot\,;\Omega)$, for $\Omega = \omega \times (0,L)$ as described in (1.2), with Dirichlet data on $\omega \times \{0,L\}$ and natural boundary conditions on $\partial\omega \times (0,L)$; see (1.13) below for the precise formulation. The Dirichlet conditions may be understood to require that n vortex filaments enter and leave Ω at certain points within a distance at most $O(h_\varepsilon)$ from the ends of the segment $\{0\} \times (0, L)$.

More precisely, we will consider boundary data w_ε^z, for $z \in \{0, L\}$, of the form

$$w_\varepsilon^z(x) = \prod_{j=1}^{n} \Big[\exp\left(i\beta(x, p_{\varepsilon,j}(z))\right) \zeta_\varepsilon(x - p_{\varepsilon,j}(z)) \Big] \tag{1.9}$$

where

- $\beta(\cdot,\cdot)$ is defined so that $\frac{w_\varepsilon^z}{|w_\varepsilon^z|}$ is *exactly* the canonical harmonic map of Bethuel *et al* (see [BBH94], section I.3) with singularities $(p_{\varepsilon,j}(z))_{j=1}^n$ and natural boundary conditions on $\partial\omega$; see (5.10) for the details;
- ζ_ε has the form $\zeta_\varepsilon(x) = \rho_\varepsilon(|x|)\frac{x_1+ix_2}{|x|}$, and $\rho_\varepsilon : [0,\infty) \to [0,1]$ satisfies

$$\rho_\varepsilon(0) = 0, \qquad 0 \le \rho_\varepsilon' \le C/\varepsilon, \qquad \rho_\varepsilon(s) \ge (1 - C\varepsilon/s)_+ ; \tag{1.10}$$

- $(p_{\varepsilon,j}(z))$ are sequences in ω such that

$$q_{\varepsilon,j}(z) := h_\varepsilon^{-1} p_{\varepsilon,j}(z) \longrightarrow q_j^0(z) \qquad \text{as } \varepsilon \to 0, \text{ for some } q_j^0(z),\ j = 1, \dots, n. \tag{1.11}$$

Note that we do *not* assume that the points $(p_{\varepsilon,j}(z))$ are distinct. For example, $p_{\varepsilon,j}(z) = 0$ for all j and for $z = 0, L$ is allowed by our assumptions.

In considering minimizers, we will assume that $\Omega = \omega \times (0, L)$ satisfies

$$L < 2\,\mathrm{dist}(0, \partial\omega). \tag{1.12}$$

This condition is close to necessary; see Remark 1 below.

Throughout this paper we write $B(r, x)$ or $B_r(x)$ to denote the *open* ball in $\mathbb{R}^2$ of radius r and center x, and $B(r) := B(r, 0)$.

We present our first result relating minimizers of F_ε to those of the reduced functional G_0. The $W^{-1,1,}$ norm, which appears in the statement, is defined in (1.22) below.

Theorem 1. *Assume that $\Omega = \omega \times (0, L)$ satisfies (1.12), and that (u_ε) minimizes $F_\varepsilon(\cdot; \Omega)$ in the space*

$$\mathcal{A}_\varepsilon := \{u \in H^1(\Omega; \mathbb{C}) : \ u(x, 0) = w_\varepsilon^0(x), \ \ u(x, L) = w_\varepsilon^L(x)\} \tag{1.13}$$

for a sequence of boundary data $\{w_\varepsilon^0, w_\varepsilon^L\}_{\varepsilon\in(0,1]} \subset H^1(\omega; \mathbb{C})$ as described in (1.9)–(1.11).

Then setting $v_\varepsilon(x, z) = u_\varepsilon(h_\varepsilon x, z)$, the vorticities $\{\mathcal{J} v_\varepsilon\}_{\varepsilon\in(0,1]}$ are precompact in $W^{-1,1}(B(R)\times(0, L))$ for every $R > 0$, and any limit J^* of a convergent subsequence, as $\varepsilon \to 0$, is a vector-valued measure of the form

$$\int \varphi \cdot dJ^* \ = \ \pi \sum_{i=1}^{n} \int_0^L \varphi(\gamma_i^*(z)) \cdot \gamma_i^{*\prime}(z)\, dz \qquad \text{for } \varphi \in C_c(\mathbb{R}^2 \times (0, L); \mathbb{R}^3). \tag{1.14}$$

Here $\gamma_i^*(z) = (f_i^*(z), z)$, *and* $f^* = (f_1^*, \ldots, f_n^*) \in H^1((0,L);\mathbb{R}^2)$ *minimizes* $G_0(\cdot)$ *in*

$$\mathcal{A}_0 := \left\{ f \in H^1((0,L);(\mathbb{R}^2)^n) \ : \sum_i \delta_{f_i(z)} = \sum_i \delta_{q_i^0(z)} \text{ for } z \in \{0, L\} \right\} \tag{1.15}$$

for $q_i^0(z)$ *appearing in* (1.11).

After we present Theorem 2 about local minimizers, we discuss both results in the context of $3D$ Ginzburg–Landau theory and on the study of concentration phenomena in elliptic PDE's at large.

Remark 1. Given any $\bar{x} \in \partial\Omega$ and $\delta < L/2$, one can construct a sequence of functions $(u_\varepsilon) \subset H^1(\Omega;\mathbb{C})$ satisfying the boundary conditions of Theorem 1 above, with energy and vorticity concentrating around the straight line segments connecting $(0,0)$ to $(\bar{x}, \delta)$ and $(\bar{x}, L-\delta)$ to $(0, L)$, and such that

$$\lim_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|} \int_\Omega e_\varepsilon(u_\varepsilon) = n\pi 2(|\bar{x}|^2 + \delta^2)^{1/2}.$$

This follows from results in [ABO05]. In particular, if $\operatorname{dist}(0, \partial\omega) < \frac{1}{2}L$, one can choose $\bar{x}$ and δ so that the right-hand side is strictly less that $n\pi L$. Then Theorem 3 below implies that for any minimizing sequence, vorticity cannot concentrate around $\{0\} \times (0, L)$.

Next we state a result about local minimizers. We will say that $f^* \in \mathcal{A}_0$ is a strict local minimizer of G_0 if there exists $\delta > 0$ such that

$$G_0(f^*) < G_0(f) \quad \text{ for all } f \in \mathcal{A}_0 \text{ such that } 0 < \|f - f^*\|_{H^1((0,L);(\mathbb{R}^2)^n)} < \delta\ .$$

The point is that we understand "local" with respect to the natural topology which here is $H^1((0,L);(\mathbb{R}^2)^n)$. Similarly, u_ε is a strict local minimizer of F_ε in $\mathcal{A}_\varepsilon$ if there exists $\delta > 0$ such that

$$F_\varepsilon(u_\varepsilon) < F_\varepsilon(u) \quad \text{ for all } u \in \mathcal{A}_\varepsilon \text{ such that } 0 < \|u - u_\varepsilon\|_{H^1(\Omega;\mathbb{C})} < \delta.$$

In particular, a local minimizer u_ε of F_ε is a solution of the Ginzburg–Landau equations,

$$\boxed{-\Delta u_\varepsilon = \frac{1}{\varepsilon^2}(1 - |u_\varepsilon|^2)u_\varepsilon,} \tag{1.16}$$

and a local minimizer f^* of $G_0(\cdot)$ satisfies

$$\boxed{-f_i^{*\prime\prime} - \sum_{j\neq i} \frac{f_i^* - f_j^*}{|f_i^* - f_j^*|^2} = 0 \qquad \text{for } i = 1, \ldots, n.} \tag{1.17}$$

The system (1.17) appears in various contexts; its solutions are equilibria of reduced systems in fluid mechanics [KMD95, KPV03, LM2000] and supplemented with suitable periodic conditions they correspond to trajectories in the planar n-body problem with logarithmic potential (examples of periodic orbits with or without collisions and for different potentials may be found in [CM99, FT04, Che08, BFT08]).

Theorem 2. *Let* $(w_\varepsilon^{0,L})_{\varepsilon\in(0,1]} \subset H^1(\omega;\mathbb{C})$ *be sequences satisfying* (1.9)–(1.11)*, and assume that* f^* *is a strict local minimizer of* G_0 *in* $\mathcal{A}_0$*, and that* $f_i^*(z) \neq f_j^*(z)$ *for all* $i \neq j$ *and* $z \in (0,L)$*.*

Then there exists a sequence of local minimizers of $F_\varepsilon(\cdot;\Omega)$ *in* $\mathcal{A}_\varepsilon$ *such that for* $v_\varepsilon(x,z) := u_\varepsilon(h_\varepsilon x, z)$*, the vorticities* $\mathcal{J}v_\varepsilon$ *converge in* $W^{-1,1}(B(R)\times(0,L))$ *for all* $R>0$ *to the vector-valued measure of the form* (1.14)*, where* $\gamma_i^*(z) = (f_i^*(z), z)$ *for* $i = 1,\dots,n$*.*

Theorems 1 and 2 are the first results that show existence of solutions to the Ginzburg–Landau equation (1.16) whose vorticity asymptotically concentrates along the graphs of solutions of the system (1.17). A connection between (1.16) and (1.17) has long been suspected and supporting evidence can be found in [MSZ04, PK08, JS09] and the references therein. It can be seen from the proofs that our results do not only describe the asymptotic geometry of the vorticity, but as a by-product also give a very precise asymptotic expansion of the energy of the maps u_ε in terms of the functional G_0 from (1.6), which may be seen as a $3D$ renormalized energy of the vortex filaments, in the spirit of the $2D$ renormalized energy introduced in [BBH94]. One of our achievements here is an improved compactness for the vorticity of configurations satisfying (1.23)–(1.25) below.

In the scalar setting of the Allen–Cahn equation, recent results that describe interface clustering have been obtained, see for example [PKW08, PKPW10, PKWY10]. In the scalar case interfaces are of codimension 1; higher condimension defects introduce new difficulties we have to deal with when proving our results, which may be seen as first analogs of the clustering phenomenon for the vector-valued Ginzburg–Landau equation.

We note that Theorem 2 does not require condition (1.12), and also has the additional advantage of not requiring that $f_i^*(z) \neq f_j^*(z)$ for $i \neq j$, when $z \in \{0, L\}$. We remark however that if one wants to allow collisions between filaments (that is, values of $z \in (0,L)$ for which $f_i^*(z) = f_j^*(z)$ for some $i \neq j$), then the right definition of "local minimizer" for our purposes would become more complicated, since there are then multiple essentially different $f \in H^1((0,L);(\mathbb{R}^2)^n)$ that represent the same vortex paths. As a result, one would need a notion of local minimizers in a suitable quotient space of $H^1((0,L);(\mathbb{R}^2)^n)$. We prefer to avoid these technicalities here, since we do not know any examples of local minimizers, in this sense, for which the filaments collide. However, related considerations appear in the proofs of Lemmas 12 and 13 for example.

1.2 Some definitions and notation.

As remarked above, the vorticity can be realized as either a vector field or a 2-form. For $u = u^1 + iu^2$, we will write

$$Ju = du^1 \wedge du^2 = \frac{1}{2} d(Im(\bar{u}\, du))$$

for the vorticity 2-form, compare (1.3).

It is convenient to state some of our results in the language of geometric measure theory.

If U is an open subset of some Euclidean space, then a k-current U is a bounded linear functional on the space $\mathcal{D}^k(U)$ of smooth k-forms with compact support in U.

We will often encounter 1-currents (as well as 0-currents, which can be identified with distributions). We will be especially interested in some particular classes of 1-currents. First, to any Lipschitz curve $\gamma : (a, b) \to U$, there is a corresponding 1-current T_γ, defined by

$$T_\gamma(\varphi) := \int_\gamma \varphi \ = \ \int_a^b \langle \varphi(\gamma(t)), \gamma'(t) \rangle dt.$$

Here and below, $\langle \cdot, \cdot \rangle$ denotes the dual pairing between covectors and vectors, so that if $\varphi = \phi_1 dx^1 + \phi_2 dx^2 + \phi_3 dz$ and $v = (v^1, v^2, v^3)$, then

$$\langle \varphi, v \rangle = \ \phi_i v^i.$$

(Throughout this work we implicitly sum over repeated indices.) Thus, T_γ acts on a 1-form φ via integration of φ over the curve parametrized by γ.

Given a Lipschitz function $f : (0, L) \to \mathbb{R}^2$, we will write

$$\Gamma_f := T_\gamma \quad \text{ for } \gamma : (0, L) \to \mathbb{R}^2 \times (0, L) \text{ defined by } \gamma(z) = (f(z), z). \tag{1.18}$$

Thus, Γ_f is the 1-current in $\mathbb{R}^2 \times (0, L)$ corresponding to the graph of f over the segment $(0, L)$ of the z-axis. More generally, if $f \in H^1((0, L); \mathbb{R}^2)$, then we define $\Gamma_f = T_\gamma$, where γ is a Lipschitz reparametrization of the curve $z \mapsto (f(z), z)$. Such a reparametrization exists, since the condition $f \in H^1$ guarantees that the curve has finite arclength.

As a special case of the above, we will write Γ_0 to denote the current corresponding to the vertical segment $z \in (0, L) \mapsto (0, z) \in \omega \times (0, L)$. With this notation, the limit in (1.5) is written as $n\pi\Gamma_0(\varphi)$ (if we view it as acting on 1-forms rather than vector fields).

The other class of 1-currents that often arises in this paper is the following: given $u = u^1 + iu^2 \in H^1(\Omega; \mathbb{C})$, we will write $\star Ju$ to denote the 1-current defined by

$$\star\, Ju(\varphi) := \int_\Omega \langle \varphi, \mathcal{J}u \rangle dx \ = \ \int_\Omega \varphi \wedge Ju, \qquad\qquad \varphi \in \mathcal{D}^1(\Omega). \tag{1.19}$$

In what follows, it will often be the case that most of the information encoded in the vorticity Ju (or its distributional realization $\star Ju$) is already contained in its z component, that is, the part of the vorticity that describes rotation in the $x_1 x_2$ plane, orthogonal to the z-axis. This will be denoted by

$$J_x u := \partial_1 u^1 \, \partial_2 u^2 - \partial_1 u^2 \, \partial_2 u^1.$$

This is the Jacobian determinant with respect to the x variables. Observe that if $\varphi \in \mathcal{D}^1(\Omega)$ has the form $\varphi = \phi\, dz$, for ϕ a smooth compactly supported function, then

$$\star\, Ju(\phi\, dz) = \int_\Omega \phi(x,z) J_x u(x,z) dx\, dz \tag{1.20}$$

as follows directly from the definitions (1.3), (1.19).

If S is a k-current on an open subset U of a Euclidean space, we use the notation

$$\|S\|_{F(U)} := \sup\{S(\varphi) : \varphi \in D^k(U), \max(\|\varphi\|_\infty, \|d\varphi\|_\infty) \le 1\} \tag{1.21}$$

for the *flat norm* of S. This quantity will be finite for every current S that arises in this paper.

We note that a 0-current S on a set $U \subset \mathbb{R}^m$ is just a distribution—that is, a bounded linear functional on the space $\mathcal{D}^0(U)$ of smooth, compactly supported 0-forms, or functions. If φ is a 0-form, then $\|d\varphi\|_\infty = \|\nabla\varphi\|_\infty$, and it follows that

$$\begin{aligned}\|S\|_{F(U)} &= \sup\{S(\varphi) : \varphi \in D^0(\Omega), \max(\|\varphi\|_\infty, \|\nabla\varphi\|_\infty) \le 1\} \\ &=: \|S\|_{W^{-1,1}(U)},\end{aligned} \tag{1.22}$$

so we will sometimes use these two notations interchangeably for 0-currents. For a 1-current, the flat norm is somewhat stronger than the $W^{-1,1}$ norm.

The *mass* of a k-current S on an open set U is defined by

$$M(S) = \sup\{S(\varphi) : \varphi \in D^k(U), \|\varphi\|_\infty \le 1\}.$$

If γ is an injective Lipschitz curve then it is easy to check that

$$M(T_\gamma) = \text{length}(\gamma).$$

1.3 Configurations with nearly parallel vortex lines. As suggested above, our main PDE results will be obtained from a careful study of the energy $F_\varepsilon(u_\varepsilon)$ for certain sequences of functions $(u_\varepsilon) \subset H^1(\Omega;\mathbb{C})$ with properties that will be easily verified in PDE applications.

First, we are interested in sequences that exhibit $n \ge 2$ vortex lines clustering around the segment $\{0\} \times (0, L)$. More precisely, we will assume that

$$\int_0^L \|J_x u_\varepsilon(\cdot, z) - \pi n \delta_0\|_{F(\omega)}\, dz \to 0 \qquad \text{as } \varepsilon \to 0. \tag{1.23}$$

Remark 2. We show below that $\int_0^L \|J_x u_\varepsilon(\cdot,z) - \pi n\delta_0\|_{F(\omega)}\, dz \le \|\star J u_\varepsilon - n\pi\Gamma_0\|_{F(\Omega)}$, see Sect. 2.1. As a result, (1.23) follows from the assumption

$$\|\star\, Ju_\varepsilon - n\pi\Gamma_0\|_{F(\Omega)} \to 0 \qquad \text{as } \varepsilon \to 0,$$

which may appear more natural than (1.23).

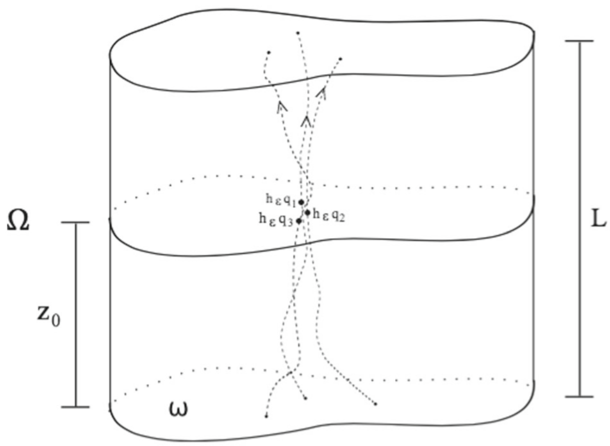

Figure 1: Example of a vortex configuration satisfying (1.23)–(1.25)

We will also assume that there is at least one height $z_0 \in [0, L]$ and points $q_1^0, \dots, q_n^0 \in \mathbb{R}^2$, not necessarily distinct, such that

$$\|J_x u_\varepsilon(\cdot, z_0) - \pi \sum_{i=1}^{n} \delta_{h_\varepsilon q_i^0}\|_{F(\omega)} = o(h_\varepsilon) \qquad \text{as } \varepsilon \to 0, \quad \text{and} \tag{1.24}$$

$$\int_\omega e_\varepsilon^{2d}(u_\varepsilon)(x, z_0)\, dx \le M|\log \varepsilon| \qquad \text{for some } M > 0, \tag{1.25}$$

where

$$e_\varepsilon^{2d}(u) := \frac{1}{2}\,|\nabla_x u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2, \qquad \nabla_x := (\partial_{x_1}, \partial_{x_2}).$$

Condition (1.24) implies that at the height z_0, there are exactly n vortices, all within distance $O(h_\varepsilon)$—the critical length scale—of the vertical axis. The energy bound (1.25) acts to ensure that the behaviour of u_ε at height z_0 carries meaningful information about its behaviour at nearby heights.

It turns out—this is a consequence of Theorem 3 below—that under the above assumptions,

$$F_\varepsilon(u_\varepsilon) \ge n\pi L\,|\log \varepsilon| + \pi n(n-1)L\,|\log h_\varepsilon| - O(1).$$

We therefore introduce

$$G_\varepsilon(u) := F_\varepsilon(u) - \left[n\pi L\,|\log \varepsilon| + \pi n(n-1)L\,|\log h_\varepsilon| + \kappa_n(\Omega)\right], \tag{1.26}$$

where the constant $\kappa_n(\Omega)$ is defined in (2.3) below; it is connected to the *renormalized energy* introduced in [BBH94]. Finally, we will assume[2] that there exists some

[2] We will number certain specific constants $c_1, c_2, \dots$ that appear repeatedly in our arguments, whereas other generic constants will be denoted just by C. Throughout, all constants are independent of ε, but may depend for example on Ω and on parameters such as c_1.

constant c_1 such that

$$G_\varepsilon(u_\varepsilon) \le c_1. \tag{1.27}$$

This is a stringent energy bound that will be shown to require that the vortices are nearly parallel. Our arguments will show that once there is some height z_0 at which there are n vortices at distance $O(h_\varepsilon)$ from the z axis, as in (1.24), (1.25), the energy bound (1.27) together with (1.23) essentially forces the filaments to remain $O(h_\varepsilon)$ from the z-axis throughout their entire length.

We remark that (1.27) implies the much less precise bound

$$F_\varepsilon(u_\varepsilon) \le C|\log\varepsilon| \tag{1.28}$$

which is used numerous times throughout this paper.

Assumptions (1.23)–(1.25) above are adapted to the study of problems for which boundary data is prescribed on $\omega \times \{0, L\}$. Natural assumptions on the boundary data, such as those described in (1.9)–(1.11), then guarantee that (1.24), (1.25) are satisfied. Most of our results will remain valid if one does not assume (1.24), (1.25), but assumption (1.23) is replaced by the stronger condition

$$\int_0^L \|J_x u(\cdot, z) - \pi n\delta_0\|_{F(\omega)}\, dz \le Ch_\varepsilon. \tag{1.29}$$

This is often adequate for the construction of *local* minimizers of F_ε, and under this assumption some of our arguments can be simplified. However, (1.29) is hard to verify directly for sequences of energy-minimizers, as considered in Theorem 1. As in Remark 2, assumption (1.29) follows directly if one instead assumes the (arguably more natural) condition $\| \star Ju_\varepsilon - n\pi\Gamma_0\|_{F(\Omega)} \le Ch_\varepsilon$.

1.4 Asymptotic behavior of energy and vorticity. Our main result is the Γ-convergence of G_ε to G_0.

Theorem 3. (a) *Assume that $(u_\varepsilon) \subset H^1(\Omega;\mathbb{C})$ is a sequence satisfying (1.27), together with either (1.23)–(1.25) or (1.29).*

Then, setting $v_\varepsilon(x,z) = u_\varepsilon(h_\varepsilon x, z)$ for $x \in \omega_\varepsilon = h_\varepsilon^{-1}\omega$ and $z \in (0,L)$, there exists some $f = (f_1,\dots,f_n) \in H^1((0,L),(\mathbb{R}^2)^n)$ such that after passing to a subsequence if necessary:

$$J_x v_\varepsilon \to \pi\delta_{[f(\cdot)]} \quad \text{in } W^{-1,1}(B(R)\times(0,L)) \text{ for every } R>0 \tag{1.30}$$

where $\delta_{[f(\cdot)]}$ denotes the measure on $\mathbb{R}^2 \times (0,L)$ defined by

$$\int \phi\delta_{[f(\cdot)]} := \sum_{i=1}^n \int_0^L \phi(f_i(z),z)\, dz \;=\; \sum_{i=1}^n \Gamma_{f_i}(\phi\, dz). \tag{1.31}$$

Moreover, there exists some $\sigma \in S^n$ (the symmetric group of degree n) such that $f_i(z_0) = q^0_{\sigma(i)}$ for $i = 1,\dots,n$, where $z_0, (q^0_i)$ appear in (1.24). And finally,

$$G_0(f) \le \liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon). \tag{1.32}$$

(b) *Conversely, given* $f \in H^1((0,L),(\mathbb{R}^2)^n)$*, there exists* $(u_\varepsilon) \subset H^1(\Omega,\mathbb{C})$ *such that* (1.30) *holds, together with* (1.23)–(1.25)*, and in addition* [3]

$$G_0(f) \geq \limsup_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon). \tag{1.33}$$

(c) *In addition, whenever* (u_ε) *satisfies* (1.30), (1.33)*, we have the improved compactness*

$$\| \star Jv_\varepsilon - \pi \sum_{i=1}^{n} \Gamma_{f_i} \|_{F(B(R)\times(0,L))} \to 0 \quad \textit{as } \varepsilon \to 0, \qquad \textit{for every } R > 0. \tag{1.34}$$

In its full strength, Theorem 3 does not only lend itself to applications such as Theorems 1 and 2 but it also provides a framework to analyze clustering of filaments in more general $3D$ Ginzburg–Landau problems. Instances where this occurrence is expected are considered in the works [MSZ04, PK08]. Moreover, the lengthscale h_ε is rather natural and it can be imposed by the geometry [MSZ04] or by physical effects as in [Con11] where distinct filaments are shown to concentrate at this lengthscale due to an external magnetic field in a much simpler setting. It is also a starting point towards establishing a correspondence between solutions, not necessarily stable, of (1.17) and (1.16) (see [JS09] for an appropriate strategy).

Some of the most salient features of the Γ-convergence of G_ε to G_0 that may be applied to other situations are the accurate characterization of the vorticity and the expansion of the energy $F_\varepsilon(u_\varepsilon)$ up to $o(1)$ for the solutions predicted in Theorems 1 and 2. Following [JS02, ABO05] we would be only able to conclude that

$$\frac{1}{\pi} \star Ju_\varepsilon \to n\Gamma_0 \text{ in } W^{-1,1}(\Omega),$$

and that

$$F_\varepsilon(u_\varepsilon) = n\pi L|\log\varepsilon| + o(|\log\varepsilon|).$$

In comparison (1.34) allows us to identify the vorticity in a very precise manner that in particular lets us distinguish n distinct filaments, while (1.32) and (1.33) capture a renormalized energy of the filaments that stores fine information about the geometry of the filaments and their mutual interaction. This realization of minimizers of F_ε is a $3D$ analog, in the present setting, of the asymptotic description in [BBH94].

Theorem 3 is also robust in the sense that rather small adaptations of the proof could be used to establish variants, such as a corresponding Γ-convergence result with Dirichlet boundary conditions on $\partial\omega \times (0,L)$. For example, given $g \in C^\infty(\partial\omega; S^1)$ of degree n, the conclusions of Theorem 3 still hold for a sequence (u_ε) in

$$\{u \in H^1(\Omega;\mathbb{C}) : u(x,z) = g(x) \text{ for } x \in \partial\omega, z \in (0,L)\} \tag{1.35}$$

[3] In the special case when ω is a ball, results similar to those of part (*b*) above are proved in [PK08].

satisfying the hypotheses of Theorem 3, as long as the constant $\kappa_n(\Omega)$ appearing in the definition (1.26) of G_ε is changed to a new constant $\kappa_n(\Omega; g)$, which we display in (2.6). Briefly, the point is that $\kappa_n(\Omega)$ is constructed from the Bethuel-Brezis-Hélein renormalized energy on ω with natural boundary conditions, whereas $\kappa_n(\Omega; g)$ uses the renormalized energy on ω with Dirichlet data $u = g$ on $\partial\omega$. The only places where this requires any changes in the proof of Theorem 3 are the following:

(1) the proof of Lemma 2. Here (3.8) would need to be modified by changing the natural (Neumann) renormalized energy to the Dirichlet renormalized energy for g, which is then carried through the rest of the proof.
We remark that the verification of (3.8) relies on Theorem 2 in [JS08]. This result remains true for Dirichlet boundary conditions, with only cosmetic changes in the proof, although we do not know any reference that presents all the details.

(2) the construction of the recovery sequence in Sect. 5.2. Here one would need to build the recovery sequence out of the canonical harmonic map with Dirichlet, rather than Neumann, boundary conditions. This would simply entail a change in the boundary conditions for the phase factor β, defined in (5.10).

More generally, if one fixes $A \subset \partial\Omega$ and considers a sequence of functions of the form

$$(u_\varepsilon) \subset \{u \in H^1(\Omega; \mathbb{C}) \ : \ u = g_\varepsilon \text{ on } A\} \tag{1.36}$$

and satisfying the hypotheses of Theorem 3, then one expects parallel results to hold, if κ_n in the definition of G_ε is modified to a suitable $\kappa_n(\Omega; A; g_\varepsilon) \geq \kappa_n(\Omega)$.

In particular, we consider boundary data of the form (1.36) in Theorems 1 and 2, with $A = \omega \times \{0, L\}$. In order to avoid technicalities related to estimates of the impact of general boundary data on the energy, we focus on carefully chosen data, for which in fact $\kappa_n(\Omega; A; g_\varepsilon) = \kappa_n(\Omega)$.

1.5 Outline of the paper. In the following lines, we explain the main ideas in the proofs. We begin by presenting the key steps in establishing Theorem 3. In what follows, all the statements about the asymptotic behavior of objects depending on the family (u_ε) are understood to hold up to passing to a subsequence.

Proof of Theorem 3. The general strategy consists in splitting the contributions of the energy into two parts, one coming from $e_\varepsilon^{2d}(u)$, the other from $|\partial_z u|^2$, and to obtain sharp lower bounds for each piece.

Roughly speaking, if one knew the vorticity of a sequence of maps (u_ε) obeyed (1.30) and (1.31) for some $f \in H^1((0, L), (\mathbb{R}^2)^n)$, then one would expect that on a typical height z_0, the graph of f should cross $\mathbb{R}^2 \times \{z_0\}$ transversally and, accordingly, for small enough ε, the restricted maps $(v_\varepsilon(\cdot, z_0))$ should have n well defined vortices close to $f_1(z_0), \ldots, f_n(z_0)$. Then, arguments in Ginzburg–Landau theory should yield the lower bound for u_ε,

$$\int_\omega e_\varepsilon^{2d}(u_\varepsilon(x,z_0))dx \geq \pi n|\log\varepsilon| - \pi n(n-1)\log\|J_x(u_\varepsilon(\cdot,z_0)) - n\pi\delta_0\|_{F(\omega)} - C$$

on such a height. From this one should be able to deduce after some work

$$\int_\Omega e_\varepsilon^{2d}(u_\varepsilon) \geq (\pi n|\log\varepsilon| + \pi n(n-1)|\log h_\varepsilon|)\,|S| - \text{logarithmic interaction terms} - C,$$

where S is the set of these typical heights, which one expects to be close to having full measure in $(0,L)$. On the other hand, for maps independent of the z-variable, this lower bound holds with $|S| = L$ and corresponds to the total energy, up to a constant. This suggests that the remainder should account for variations in the energy from that of perfectly parallel filaments, that is,

$$\frac{1}{2}\int_\Omega |\partial_z u_\varepsilon|^2 \geq \frac{\pi}{2}\int_0^L \sum_i |f_i'|^2 dz - o(1) \qquad \text{as } \varepsilon \to 0. \tag{1.37}$$

In light of this, the first step to a rigorous argument is to establish some preliminary lower bounds under assumptions (1.23), (1.27) and a corresponding initial compactness for the vorticity, somewhat weaker than (1.30) and (1.31). This will later allow for improving both the lower bounds and compactness property, upon subsequent analysis using (1.24) and (1.25).

First estimates. In Sect. 3 we define a notion of 'good height' (see (3.11) below). Heuristically speaking, a height z_ε is considered good for our purposes if $J_x u_\varepsilon(\cdot, z_\varepsilon)$ is close to $n\pi\delta_0$ on many balls centered at the origin. This choice makes possible to deduce that for any such height either

$$\int_\omega e_\varepsilon^{2d}(u_\varepsilon(x,z_\varepsilon))dx \geq \pi(n+\theta)|\log\varepsilon| \qquad \text{for some } \theta > 0, \text{ see Lemma 2 below,}$$

in which case we have a considerable energy excess of order $\mathcal{O}(|\log\varepsilon|)$, or else a very detailed description of the vortex structure is available which allows for very precise lower bounds in terms of the renormalized energy (2.2) (see Lemma 2). Right from the outset, elementary inequalities using this fact tell us that (1.23) implies that the set $\mathcal{G}^\varepsilon$ of good heights has measure $|\mathcal{G}^\varepsilon| = L - o(1)$, which is not enough for obtaining a lower bound that accounts for all the divergent part of the energy, but at least let us conclude in Lemma 5 below that for any interval $(a,b) \subseteq (0,L)$,

$$\int_a^b \int_\omega e_\varepsilon^{2d}(u_\varepsilon)dxdz = n\pi(b-a)|\log\varepsilon| + o(|\log\varepsilon|). \tag{1.38}$$

This fact will be used to prove the key Proposition 1 and a first compactness property of the vorticity of suitably rescaled maps.

A key estimate. We define $\mathcal{B}^\varepsilon := (0,L) \setminus \mathcal{G}^\varepsilon$. The next task is to exploit the definition of good height and an auxiliary result, Lemma 7, to show that if $z_\varepsilon \in \mathcal{B}^\varepsilon$, somewhat surprisingly, we have the much stronger lower bound on the $2d$ energy,

$$\int_\omega e_\varepsilon^{2d}(u_\varepsilon)(x,z_\varepsilon)dx \geq \varepsilon^{-\alpha}, \tag{1.39}$$

for some absolute positive constant $\alpha > 0$. Thus $\mathcal{B}^\varepsilon$ can be understood either as the set of "bad heights", on which we do not have detailed information about vortex structure, or as the set of "better heights", which enjoy a very strong lower energy bound. Note that (1.39) immediately implies, thanks to (1.27), that $\mathcal{B}^\varepsilon$ is smaller in measure than a power of ε.

The idea here is that with the aid of Lemma 7, we can show that there is a constant c such that for any "bad" height b, there are many cylinders $\omega \times (a, b)$, such that at least one of the following holds

$$\int_{\omega\times\{b,a\}} e_\varepsilon^{2d}(u_\varepsilon)\ dx \geq 2\varepsilon^{-\alpha}, \text{ or } \int_{\omega\times(a,b)} e_\varepsilon^{2d}(u_\varepsilon)\ dxdz \geq c|\log\varepsilon|.$$

However, choosing a close enough to b, which we show can be done, we can rule out the latter thanks to (1.38). We also show that we can choose a so that $\int_{\omega\times\{a\}} e_\varepsilon^{2d}(u_\varepsilon)\ dx < \varepsilon^{-\alpha}$ and thus we deduce the desired bound (1.39) from the former possibility.

Characterizing the vorticity. Section 4 deals with the compactness of (u_ε), in particular we show the vorticity concentrates along n vortex filaments which we identify as H^1-curves over $(0, L)$, and we prove the lower bound (1.37). We remark that results similar to (1.37) are known in somewhat different contexts, in which one has for example strong bounds on $\int_\omega e_\varepsilon^{2d}(u_\varepsilon(x, z))dx$ that are *uniform* with respect to the z variable, as well as limiting vortex curves that are known not to intersect; see [Jer99b, Li99, SS04]. Our proof here borrows some ideas from these earlier works. We choose to follow [Jer99b], but one could also give a different proof that relies more on ideas introduced in [SS04].

We start by noticing that thanks to (1.38) we have

$$\int_\Omega |\partial_z u_\varepsilon|^2 = o(|\log\varepsilon|),$$

and therefore we may find a natural scale ℓ_ε such that $h_\varepsilon \leq \ell_\varepsilon = o(1)$, which we show at the end of the proof to be equal to h_ε, such that the rescaled maps $v_\varepsilon(x, z) := u_\varepsilon(\ell_\varepsilon x, z)$ satisfy that their normalized energies

$$\frac{1}{|\log\varepsilon'|}\int_{\Omega_\varepsilon} e_{\varepsilon'}(v_\varepsilon)$$

are uniformly bounded, where $\varepsilon' := \varepsilon/\ell_\varepsilon$ and $\Omega_\varepsilon := \omega_\varepsilon \times (0, L)$; $\omega_\varepsilon := \ell_\varepsilon^{-1}\omega$. This puts us in a position to apply an abstract compactness result from [JS02](see also [ABO05]), which we will do at several stages in the proof of part (c) of Theorem 3. A key point is to show that there is a dense subset H of $(0, L)$, such that for any heights $z_1 < z_2 \in H$, we have that

$$\frac{\pi}{2}\min_{\sigma\in S^n}\sum_{i=1}^n \frac{\left|p_i(z_1) - p_{\sigma(i)}(z_2)\right|^2}{z_2 - z_1} \leq \liminf_{\varepsilon\to 0}\int_{\omega_\varepsilon\times(z_1,z_2)} \frac{|\partial_z v_\varepsilon(x, z)|^2}{2|\log\varepsilon|}dxdz, \tag{1.40}$$

where for $j = 1, 2$, it holds that

$$J_x v_\varepsilon(\cdot, z_j) \to \pi \sum_{i=1}^{n} \delta_{p_i(z_j)} \qquad \text{in } W^{-1,1}(B(R)), \quad \text{for all } R > 0.$$

In fact something stronger is proved in Lemmas 10–13. This type of estimates let us relate the information of the vorticity of nearby heights, whose variations can be controlled by $|\partial_z v_\varepsilon|^2/|\log \varepsilon|$ at any scale. In particular, we make use of (1.40) to identify the limiting vortex filaments and to gain the anticipated control on the modulus of continuity of the corresponding f.

Refined lower bounds and compactness. We complete the proof of the compactness and lower bounds, that is part (*a*) of Theorem 3, in Sect. 4.4 by combining the previous steps and appealing to (1.24), (1.25). We obtain a very precise lower bound for $e_\varepsilon^{2d}(u_\varepsilon)$ in terms of the intermediate scale ℓ_ε in Lemma 14; it follows from a fact we establish a bit earlier: given $z \in H$, it holds that

$$\int_\omega e_\varepsilon^{2d}(u_\varepsilon(x,z))\,dx - \big[n(\pi|\log\varepsilon| + \gamma) + n(n-1)\pi|\log \ell_\varepsilon| + n^2 H_\omega(0,0)\big] \geq -\pi \sum_{i \neq j} \log|q_i(z) - q_j(z)|, \quad (1.41)$$

where $J_x v_\varepsilon(\cdot, z) \to \pi \sum_{i=1}^n \delta_{q_i(z)}$ in $W^{-1,1}(B(R))$ for all $R > 0$. For more details see Lemmas 11 and 12.

An application of Fatou's lemma to (1.41) and all the previous analysis yield the lower bound

$$G_\varepsilon(u_\varepsilon) \geq \pi n(n-1)\log(h_\varepsilon/\ell_\varepsilon) + G_0(f) + o(1).$$

Finally, we proceed to conclude the proof of part (*a*) by using (1.24), (1.25) to show that the intermediate scale ℓ_ε actually coincides with h_ε in this case.

Stronger compactness and the construction of a recovery sequence. We turn to the proof of part (*c*) of Theorem 3 in Sect. 5.1. So far we know that $J_x v_\varepsilon \to \pi\delta_{[f(\cdot)]}$ in $W^{-1,1}(B(R) \times (0,L))$ for every $R > 0$; here we show that if there is no loss in (1.32) then in fact f captures all of the limiting vorticity, that is

$$\frac{1}{\pi} \star Jv_\varepsilon \text{ converges to } \sum_{i=1}^{n} \Gamma_{f_i} \text{ in the flat norm } F(B(R) \times (0,L)), \text{ for every } R > 0.$$

The proof hinges upon the measure-theoretic Lemma 15, which tells us that the limiting vorticity J (whose existence is guaranteed thanks to Theorem 4) is given by $\sum_{i=1}^n \Gamma_{f_i}$ plus some indecomposable pieces supported on a union of horizontal planes. However, the tightness in the energy precludes the presence of these horizontal components.

To conclude the proof of Theorem 3, it remains to show the existence of a recovery sequence for the effective energy $G_0(f)$ for any admissible f. We construct such families (u_ε) based on canonical harmonic maps and estimates from [JS07,JS08]. This is done in Sect. 5.2.

Theorems 1 and 2. These results rest on the Γ-convergence result. The extra technical difficulty we need to address is the refinement of the construction of a recovery sequence that is now additionally required to satisfy prescribed boundary data; this is a delicate matter because for us boundary conditions with multiple degree vortices are perfectly acceptable. This is taken care of in Sect. 6 where we present the proof of Theorem 1. To prove Theorem 2 we need to relate local minimizers with respect to two different topologies; after this is done the existence of local minimizers with the desired properties follows from standard arguments (see [MSZ04]). This is achieved in Sect. 7.

The article concludes with an appendix where the proofs of some technical results from Sect. 3 are included.

2 Preliminaries

A key object in our study, and more generally in the asymptotic analysis of the Ginzburg–Landau functional, is the *renormalized energy*, introduced by Bethuel, Brezis, and Hélein [BBH94], which in our context may be defined as follows.

For $y \in \omega$, we write $H_\omega(\cdot, y)$ to denote the solution to

$$\Delta_x H_\omega(\cdot, y) = 0 \text{ in } \omega, \qquad H_\omega(x,y) = -\log|x-y| \text{ for } x \in \partial\omega. \tag{2.1}$$

For distinct points $p_1, \dots, p_n \in \omega$ we define

$$W_\omega(p_1,\dots,p_n) = -\pi\left(\sum_{i\neq j}\log|p_i-p_j| + \sum_{i,j} H_\omega(p_i,p_j)\right). \tag{2.2}$$

The constant $\kappa_n(\Omega)$ appearing in (1.26) is given by

$$\kappa_n(\Omega) := -\pi n^2 L H_\omega(0,0) + nL\gamma, \tag{2.3}$$

Here γ is the constant defined in [BBH94] as

$$\gamma := \lim_{\varepsilon\to 0}(I(1,\varepsilon) + \pi\log\varepsilon), \tag{2.4}$$

where

$$I(R,\varepsilon) := \min\left\{\frac{1}{2}\int_{B(0,R)} e_\varepsilon^{2d}(u)dx\ ;\ u \in H^1(B(R),\mathbb{C}), u = \frac{x}{|x|} \text{ on } \partial B(R)\right\}. \tag{2.5}$$

Remark 3. We remark that natural boundary conditions on $\partial\omega$ are built into the definition of H_ω and hence also W_ω and $\kappa_n(\Omega)$. In particular, if we wish to consider Dirichlet data on $\partial\omega \times (0,L)$ of the form contemplated in (1.35), then the correct substitute for κ_n is

$$\kappa_n(\Omega, g) = -\pi n^2 L H(0,0;g) + nL\gamma \tag{2.6}$$

where $H(\cdot, y; g)$ is harmonic in ω for every y, and

$$\nu(x) \cdot \nabla_x H(x, y; g) = j_\tau g(x) + \nu \cdot \frac{(x-y)}{|x-y|^2} \quad \text{for } x \in \partial\omega. \tag{2.7}$$

Here $j_\tau g(x) = Im(\bar{g}, \nabla_\tau g)(x)$. Similarly, the correct renormalized energy for (1.35) is defined as in (2.2), but with $H_\omega(p_i, p_j; g)$ in place of $H_\omega(p_i, p_j)$.

2.1 More about currents. In Sect. 1.2, we defined k-currents, and we introduced several particular classes of currents of interest, including for example the 1-current T_γ associated to a Lipschitz curve γ.

In general, if T is a k-current, then ∂T is the $(k-1)$-current defined by $\partial T(\phi) := T(d\phi)$. For example, if U is an open set in $\mathbb{R}^n$ and $\gamma : [a, b] \to \bar{U}$ is a Lipschitz curve such that $\gamma(s) \in U$ for $s \in (a, b)$, then

$$\partial T_\gamma(\phi) = \phi(\gamma(b)) - \phi(\gamma(a)) \qquad \text{for } \phi \in \mathcal{D}^0(U).$$

In particular, $\partial T_\gamma = 0$ in U if $\gamma(a)$ and $\gamma(b)$ belong to ∂U, or if $\gamma(a) = \gamma(b)$.

We will frequently encounter *integer multiplicity rectifiable* 1-currents in various 3-dimensional open sets U. Such a current T admits a moderately complicated description in general, but if $M(T) < \infty$ and $\partial T = 0$ in U, which will always be the case for us, then it can always be represented in the form

$$\begin{aligned} T &= \sum_{i \in I} T_{\gamma_i}, \qquad\qquad \partial T_{\gamma_i} = 0 \text{ in } U \text{ for all } i \in I, \\ M(T) &= \sum_{i \in I} M(T_{\gamma_i}) \ = \sum_{i \in I} \mathcal{H}^1(\gamma_i) < \infty, \end{aligned} \tag{2.8}$$

for some family of Lipschitz maps $\{\gamma_i\}_{i \in I}$, with I at most countable. (Here and below, $\mathcal{H}^k$ and $\mathcal{L}^k$ denote k-dimensional Hausdorff and Lebesgue measure, respectively.)

The remainder of this section is used only rarely and can be skipped until needed.

It will be useful to consider slices of various currents, including $\star Ju$ and Γ_0, by the function $\zeta(x, z) = z$. In general a slice[4] of a 1-current S in Ω by $\zeta^{-1}\{z\}$ is a 0-current supported in $\zeta^{-1}\{z\}$, which is denoted $\langle S, \zeta, z\rangle$. For the currents we are interested in, we have explicit formulas:

$$\begin{aligned} \langle \star Ju, \zeta, z\rangle(g) &= \int_\omega J_x u(x, z) g(x, z)\, dx \quad \text{a.e. } z, \qquad \text{for } u \in H^1(\Omega; \mathbb{C}), \\ \langle \Gamma_f, \zeta, z\rangle &= \delta_{(f(z), z)} \quad \text{a.e. } z, \qquad \text{for } f \in H^1((0, L); \mathbb{R}^2), \end{aligned}$$

using notation introduced in (1.18). In particular,

$$\langle \Gamma_0, \zeta, z\rangle = \delta_{(0,z)}.$$

[4] In order for the slices of S to be well-defined, S must satisfy some mild regularity hypotheses, which will always hold for us.

Note that $\langle \star Ju, \zeta, z\rangle$ can (for *a.e.* z) be identified with the Jacobian of $u(\cdot, z) \in H^1(\omega; \mathbb{C})$, and as a result

$$\|\langle \star Ju - n\pi\Gamma_0, \zeta, z\rangle\|_{F(\Omega)} = \|J_x u(\cdot, z) - n\pi\delta_0\|_{F(\omega)}.$$

In Remark 2 above, we have asserted that

$$\int_0^L \|J_x u(\cdot, z) - \pi n\delta_0\|_{F(\omega)}\, dz \ \leq \| \star Ju - n\pi\Gamma_0\|_{F(\Omega)}.$$

This estimate, which is not really used in this paper, is a direct consequence of the above considerations and the general fact that

$$\int_0^L \|\langle S, \zeta, z\rangle\|_{F(\omega)} dz \ \leq \ \|S\|_{F(\Omega)} \qquad \text{for any current } S \text{ in } \Omega, \tag{2.9}$$

proved in Federer [Fed69] 4.2.1.

3 Preliminary lower bounds for the 2d energy

In this section we prove lower bounds for the $2d$ energy under assumption (1.23).

3.1 A criterion for vorticity. We first introduce a criterion that will allow us to detect, roughly speaking, when a function $w \in H^1(\omega; \mathbb{C})$ has n vortices rather near the origin. There are many ways of doing this; the one we choose is designed to facilitate the proof of a key estimate that we establish in Proposition 1 below.

We henceforth write

$$r^* := 1 \wedge \mathrm{dist}(0, \partial\omega) = \min\{1, \mathrm{dist}(0, \partial\omega)\}.$$

Given $w \in H^1(\omega; \mathbb{C})$, we will use the notation

$$\mathcal{S}_n(w) := \left\{ s \in \left(\frac{r^*}{2}, r^*\right) : \ \left| \int_{B(s)} J_x w(x)\ dx - n\pi \right| \leq 1 \right\}. \tag{3.1}$$

We will later show that w has various good properties if $\mathcal{S}_n(w)$ is large in the sense that

$$|\mathcal{S}_n(w)| \geq \frac{r^*}{4}. \tag{3.2}$$

This says that on a majority of balls $B(s)$, $\frac{r^*}{2} < s < r^*$, the vorticity contained in $B(s)$ is not too far from $n\pi$.

We first establish some estimates that will later allow us to show that if $(u_\varepsilon) \subset H^1(\Omega, \mathbb{C})$ satisfies assumption (1.23) and ε is small, then $w(\cdot) = u_\varepsilon(\cdot, z)$ satisfies (3.2) for most values of z.

Lemma 1. *If $w \in H^1(\omega;\mathbb{C})$, then*

$$|\mathcal{S}_n(w)| \geq \frac{r^*}{2} - \|J_x w - n\pi\delta_0\|_{F(\omega)}. \tag{3.3}$$

Also, if $|\mathcal{S}_n(w)| \geq \frac{r^*}{4}$, then there exists $\phi \in W_0^{1,\infty}(\omega)$ such that $0 \leq \phi \leq 1$, $\|\nabla\phi\|_\infty \leq 4/r^*$, supp$(\phi) \subset B(r^*)$, $\phi = 1$ in $B(r^*/2)$, and

$$\left| \int_\omega \phi(x)\, J_x w(x)\, dx - n\pi \right| \leq 1. \tag{3.4}$$

Proof. Consider a measurable subset $A \subset \left(\frac{r^*}{2}, r^*\right)$ and let $g : (0,\infty) \to \mathbb{R}$ be the (unique) compactly supported Lipschitz continuous function such that

$$g'(s) = -\mathbf{1}_{s\in A}\, \mathrm{sign}\left(\int_{B(s)} J_x w\, dx - n\pi \right) \quad \text{a.e. } s.$$

Since $g'(s) = 0$ for $s \geq r^*$, it is clear that $g(s) = 0$ for $s \geq r^*$, and hence

$$g(s) = -\int_s^{r^*} g'(t)\ dt \qquad \text{for } 0 \leq s \leq r^*.$$

Thus

$$\begin{aligned} \int_{s\in A} \left| \int_{B(s)} (J_x w - n\pi\delta_0) \right| ds &= -\int_0^{r^*} g'(s) \left(\int_{B(s)} (J_x w - n\pi\delta_0) \right) ds \\ &= -\int_{B(r^*)} \int_{|x|}^{r^*} g'(s)(J_x w - n\pi\delta_0)\ ds \\ &= \int_\omega g(|x|)(J_x w - n\pi\delta_0). \end{aligned}$$

If we take $A := \left(\frac{r^*}{2}, r^*\right) \setminus \mathcal{S}_n(w)$, then it follows from the definition of $\mathcal{S}_n(w)$ that

$$\frac{r^*}{2} - |\mathcal{S}_n(w)| \ = \ |A| \ \leq \ \int_{s\in A} \left| \int_{B(s)} (J_x w - n\pi\delta_0) \right| ds.$$

Also, $\max(\|g\|_\infty, \|g'\|_\infty) \leq 1$ so

$$\int_\omega g(|x|)(J_x w - n\pi\delta_0) \leq \|J_x w - n\pi\delta_0\|_{F(\omega)}.$$

The first conclusion of the lemma follows by combining these inequalities.

To prove the final conclusion of the lemma, assume that $|\mathcal{S}_n(w)| \geq \frac{r^*}{4}$, and let $g(s)$ be the (unique) compactly supported Lipschitz continuous function such that

$$g'(s) = -|\mathcal{S}_n(w)|^{-1} \mathbf{1}_{\mathcal{S}_n(w)} \qquad a.e.\ s.$$

Then $0 \le g \le 1$, and $\|g'\|_\infty = |\mathcal{S}_n(w)|^{-1} \le 4/r^*$. Moreover, arguing as above one computes that

$$\int_\omega g(|x|)(J_x w - n\pi\delta_0) = -\int_0^{r^*} g'(s)\left(\int_{B(s)} J_x w - n\pi\delta_0\right) ds$$
$$= \frac{1}{|\mathcal{S}_n(w)|}\int_{\mathcal{S}_n(w)}\left(\int_{B(s)} J_x w - n\pi\delta_0\right) ds.$$

Then the final conclusion (3.4), with $\phi(x) = g(|x|)$, now follows directly from the definition of $\mathcal{S}_n(w)$. □

We now identify the above-mentioned good properties enjoyed by a function w satisfying (3.2). We show that such a function either has a very well-defined vortex structure and associated sharp lower energy bounds, or else it has excess energy, in the sense that condition (3.5) below fails. More precisely, we have

LEMMA 2. *There exist positive numbers θ, a, b, depending on n and r^*, such that $b < a$, and the following holds:*

Assume that $w \in H^1(\omega;\mathbb{C})$ and that $|\mathcal{S}_n(w)| \ge \frac{r^*}{4}$ and

$$\int_\omega e_\varepsilon^{2d}(w)(x)\,dx \le \pi(n+\theta)|\log\varepsilon|. \tag{3.5}$$

If $0 < \varepsilon < \varepsilon_0 = \varepsilon_0(\theta, a, b, n)$, then there exist $p_1^\varepsilon, \dots, p_n^\varepsilon \in \omega$, such that

$$\|J_x w - \pi\sum \delta_{p_i^\varepsilon}\|_{F(\omega)} \le \varepsilon^a\ , \tag{3.6}$$

$$\mathrm{dist}(p_i^\varepsilon, \partial\omega) \ge \frac{r^*}{8} \text{ for all } i, \qquad |p_i^\varepsilon - p_j^\varepsilon| \ge \varepsilon^b \text{ for } i \ne j, \qquad \text{and} \tag{3.7}$$

$$\int_\omega e_\varepsilon^{2d}(w)dx \ \ge\ n(\pi|\log\varepsilon| + \gamma) + W_\omega(p_1^\varepsilon, \dots, p_n^\varepsilon) - C(n,\theta)\varepsilon^{(a-b)/2} \tag{3.8}$$

where W_ω is the renormalized energy defined in (2.2) and γ is defined in (2.4).

The proof of Lemma 2 is postponed to "Appendix A". For now we only remark that the verification of (3.6), (3.7) involves a vortex ball argument, and that we will deduce (3.8) by appealing to results in [JS08], for which (3.6) and (3.7) supply the hypotheses. Incidentally, the reason we have assumed that ω is simply connected is that this condition is imposed in [JS08].

Our next result is a corollary of Lemma 2.

LEMMA 3. *Assume that $u_\varepsilon(\cdot, z) \in H^1(\omega;\mathbb{C})$, that $|\mathcal{S}_n(u_\varepsilon(\cdot, z))| \ge \frac{r^*}{4}$, and that (3.5) holds.*

Then for $0 < \varepsilon < \varepsilon_0$,

$$\int_\omega e_\varepsilon^{2d}(u_\varepsilon(x,z))dx \ge n\pi\,|\log\varepsilon| - C - \pi n(n-1)\log\|J_x u_\varepsilon(\cdot, z) - n\pi\delta_0\|_{F(\omega)} \tag{3.9}$$

and

$$\int_\omega e_\varepsilon^{2d}(u_\varepsilon(x,z))dx \geq n\pi\,|\log\varepsilon| - C. \tag{3.10}$$

Proof. We wish to deduce the conclusions of the lemma from (3.8). To do this, we must estimate

$$W_\omega(p_1^\varepsilon,\dots,p_n^\varepsilon) = -\pi\left(\sum_{i\neq j}\log\left|p_i^\varepsilon - p_j^\varepsilon\right| + \sum_{i,j} H_\omega(p_i^\varepsilon,p_j^\varepsilon)\right)$$

where H_ω is defined in (2.1).

It is clear that H_ω is smooth in the interior of $\omega\times\omega$, so (3.7) implies that $|H_\omega(p_i^\varepsilon,p_j^\varepsilon)|\leq C$ for all i,j.

To estimate the other terms, we write $s_\varepsilon := \|J_x u_\varepsilon(\cdot,z) - n\pi\delta_0\|_{F(\omega)}$ for convenience. Then it follows from (3.6) and the triangle inequality that

$$\|\pi\sum_{i=1}^n(\delta_{p_i^\varepsilon}-\delta_0)\|_{W^{-1,1}} \leq s_\varepsilon + \varepsilon^a.$$

On the other hand, setting $\phi(x) = (r^* - |x|)^+$,

$$\|\pi\sum_{i=1}^n(\delta_{p_i^\varepsilon}-\delta_0)\|_{W^{-1,1}} \geq \int_\omega \phi(x)\pi\sum_{i=1}^n(\delta_0 - \delta_{p_i^\varepsilon}) = \pi\sum|p_i^\varepsilon|.$$

Thus $|p_i^\varepsilon - p_j^\varepsilon| \leq s_\varepsilon + \varepsilon^a \leq 2s_\varepsilon$ for all $i\neq j$. (The last inequality follows from (3.7), which with the above shows that $s_\varepsilon \geq \varepsilon^b - \varepsilon^a \geq \frac12\varepsilon^b$.) These imply that

$$W_\omega(p_1^\varepsilon,\dots,p_n^\varepsilon) \geq -\pi n(n-1)\log s_\varepsilon - C.$$

Also, since $|p_i^\varepsilon| \leq \operatorname{diam}(\omega)$ for all i, it is clear that $W_\omega(p_1^\varepsilon,\dots,p_n^\varepsilon)\geq -C$. □

3.2 Good heights. We now fix $u_\varepsilon \in H^1(\Omega;\mathbb{C})$ satisfying (1.23).

We will say that a height z is *good* if $u_\varepsilon(\cdot,z)$ satisfies (3.2), and we define $\mathcal{G}_1^\varepsilon$ to be the set of good heights:

$$\mathcal{G}_1^\varepsilon := \left\{z\in(0,L) : |\mathcal{S}_n(u_\varepsilon(\cdot,z))| \geq \frac{r^*}{4}\right\}. \tag{3.11}$$

We also define

$$\mathcal{B}_1^\varepsilon := (0,L)\setminus\mathcal{G}_1^\varepsilon.$$

The results of the previous sections immediately imply that the good set is big:

Lemma 4. *For $u_\varepsilon\in H^1(\Omega;\mathbb{C})$ satisfying (1.23), we have the estimates*

$$L - |\mathcal{G}_1^\varepsilon| = |\mathcal{B}_1^\varepsilon| \;\leq \frac{4}{r^*}\int_0^L \|J_x u_\varepsilon(\cdot,z) - n\pi\delta_0\|_{F(\omega)}dz.$$

Proof. By (3.3), we know that if $z \in \mathcal{B}_1^\varepsilon$, then $\|J_x u_\varepsilon(\cdot, z) - n\pi\delta_0\|_{F(\omega)} > \frac{r^*}{4}$. In addition, Chebyshev's inequality implies that for any $s > 0$,

$$\mathcal{L}^1\left(\left\{z \in (0,L) : \|J_x u(\cdot,z) - \pi n\delta_0\|_{F(\omega)} \geq s\right\}\right) \leq \frac{1}{s}\int_0^L \|J_x u(\cdot,z) - \pi n \delta_0\|_{F(\omega)}\, dz. \tag{3.12}$$

The lemma follows by combining these facts. □

As shown in Lemma 3, if $z \in \mathcal{G}_1^\varepsilon$, then $u_\varepsilon(\cdot, z)$ satisfies good lower energy bounds. Perhaps surprisingly, under hypotheses that are satisfied by minimizers and local minimizers whose vorticity clusters around the vertical segment, $u_\varepsilon(\cdot, z)$ satisfies *much stronger* lower energy bounds for $z \in \mathcal{B}_1^\varepsilon$. Indeed, in Sect. 3.4 we will prove

Proposition 1. *Assume that u_ε satisfies (1.23) and (1.27). Then for every $\alpha \in \left(0, \frac{2}{3}\right)$, there exists ε_0 such that if $0 < \varepsilon \leq \varepsilon_0$, then*

$$\int_\omega e_\varepsilon(u_\varepsilon(x,z))\, dx \;\geq\; \varepsilon^{-\alpha} \qquad \textit{for every } z \in \mathcal{B}_1^\varepsilon. \tag{3.13}$$

As a result, using the upper bound (1.28) on F_ε, if $\varepsilon < \varepsilon_0$ then $|\mathcal{B}_1^\varepsilon| \leq C\varepsilon^\alpha |\log\varepsilon|$.

This proposition will play an important role in the proof of Theorem 3, although there in fact a much weaker estimate than (3.13) would suffice.

We continue with some easier estimates that will be used several times, including in the proof of Proposition 1.

Lemma 5. *Assume that (u_ε) satisfies (1.23), (1.27). If A is any measurable subset of $(0, L)$, then*

$$\lim_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\int_A\int_\omega e_\varepsilon(u_\varepsilon)dx\ dz = \lim_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\int_A\int_\omega e_\varepsilon^{2d}(u_\varepsilon)dx\ dz = n\pi|A| \tag{3.14}$$

and as a consequence

$$\int_\Omega |\partial_z u_\varepsilon|^2\, dx\, dz \;=\; o(|\log\varepsilon|). \tag{3.15}$$

Proof. If $z \in \mathcal{G}_1^\varepsilon$, then

$$\int_\omega e_\varepsilon(u_\varepsilon(x,z))dx \geq \int_\omega e_\varepsilon^{2d}(u_\varepsilon(x,z))dx \geq n\pi|\log\varepsilon| - C.$$

This follows from Lemma 3 if (3.5) holds, and if not it is immediate. Thus

$$\begin{aligned}\frac{1}{|\log\varepsilon|}\int_A\int_\omega e_\varepsilon(u_\varepsilon)dx\ dz &\geq \frac{1}{|\log\varepsilon|}\int_{\mathcal{G}_1^\varepsilon\cap A}\int_\omega e_\varepsilon^{2d}(u_\varepsilon)\, dx\, dz\\ &\geq |A\cap\mathcal{G}_1^\varepsilon|(n\pi - o(1)) \quad \text{ as } \varepsilon\to 0.\end{aligned}$$

Also, it is clear from (1.23) and Lemma 4 that $|A \cap \mathcal{G}_1^\varepsilon| \geq |A| - |\mathcal{B}_1^\varepsilon| \to |A|$ as $\varepsilon \to 0$. Thus

$$\liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_A \int_\omega e_\varepsilon(u_\varepsilon) dx\ dz \geq \liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_A \int_\omega e_\varepsilon^{2d}(u_\varepsilon) dx\ dz \geq n\pi|A|. \tag{3.16}$$

Then using (1.27) and applying (3.16) to $\tilde{A} = (0, L) \setminus A$,

$$\begin{aligned}\limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_A \int_\omega e_\varepsilon^{2d}(u_\varepsilon) dx\ dz &\leq \limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_A \int_\omega e_\varepsilon(u_\varepsilon) dx\ dz \\ &\leq n\pi L - n\pi|\tilde{A}| = n\pi|A|.\end{aligned}$$

This is (3.14). Since $\frac{1}{2}|\partial_z u|^2 = e_\varepsilon(u) - e_\varepsilon^{2d}(u)$, we obtain (3.15) as a direct consequence of (3.14) and (1.27). □

We next define another "good set", consisting of the set of points z such that $u_\varepsilon(\cdot, z)$ satisfies the hypotheses of Lemmas 2 and 3. This will appear often in the proof of the Γ-limit lower bound and compactness assertions.

Lemma 6. *Assume that (u_ε) satisfies (1.23) and (1.27), and define*

$$\mathcal{G}_2^\varepsilon := \left\{ z \in \mathcal{G}_1^\varepsilon \ : \int_\omega e_\varepsilon^{2d}(u_\varepsilon(x, z)) dx \leq \pi(n + \theta)|\log \varepsilon| \right\}, \qquad \mathcal{B}_2^\varepsilon := (0, L) \setminus \mathcal{G}_2^\varepsilon, \tag{3.17}$$

where θ is the constant from Lemma 2. Then

$$|\mathcal{B}_2^\varepsilon| = o(1) \qquad |\mathcal{G}_2^\varepsilon| = L - o(1) \qquad \text{as } \varepsilon \to 0.$$

Proof. We will write

$$\widetilde{\mathcal{B}}_2^\varepsilon := \left\{ z \in \mathcal{G}_1^\varepsilon \ : \int_\omega e_\varepsilon^{2d}(u_\varepsilon(x, z)) dx > \pi(n + \theta)|\log \varepsilon| \right\}.$$

Note that $\mathcal{B}_2^\varepsilon = \mathcal{B}_1^\varepsilon \cup \widetilde{\mathcal{B}}_2^\varepsilon$. Lemma 4 and (1.23) imply that $L - |\mathcal{G}_1^\varepsilon| = |\mathcal{B}_1^\varepsilon| \to 0$, so we only need to prove that $|\widetilde{\mathcal{B}}_2^\varepsilon| \to 0$ as $\varepsilon \to 0$. Toward this end we use Lemma 3 and the definition of $\widetilde{\mathcal{B}}_2^\varepsilon$ to compute

$$\begin{aligned}|\log \varepsilon|(Ln\pi + o(1)) &\overset{(1.27)}{\geq} \int_\Omega e_\varepsilon^{2d}(u_\varepsilon)\, dx\, dz \\ &\geq \int_{\mathcal{G}_1^\varepsilon \setminus \widetilde{\mathcal{B}}_2^\varepsilon} \int_\omega e_\varepsilon^{2d}(u_\varepsilon)\, dx\, dz + \int_{\widetilde{\mathcal{B}}_2^\varepsilon} \int_\omega e_\varepsilon^{2d}(u_\varepsilon)\, dx\, dz \\ &\geq (|\mathcal{G}_1^\varepsilon| - |\widetilde{\mathcal{B}}_2^\varepsilon|)(n\pi|\log \varepsilon| - C) + |\widetilde{\mathcal{B}}_2^\varepsilon|(\pi(n + \theta)|\log \varepsilon| \\ &= |\mathcal{G}_1^\varepsilon| n\pi|\log \varepsilon| + \theta\pi|\log \varepsilon||\widetilde{\mathcal{B}}_2^\varepsilon| - C.\end{aligned}$$

As already noted, $|\mathcal{G}_1^\varepsilon| \to L$ as $\varepsilon \to 0$, so the conclusions follow. □

3.3 A Jacobian estimate with weak boundary control. The following result plays an important role in the proof of Proposition 1.

LEMMA 7. *Let $D := (\mathbb{R}/\ell\mathbb{Z}) \times I$ for some interval I and some $\ell > 0$, and assume that $w \in H^1(D;\mathbb{C})$. Then for every $\alpha \in (0, \frac{2}{3})$, there exists $c_2 = c_2(\alpha)$ such that for all sufficiently small $\varepsilon > 0$ (where "sufficiently small" depends on α and D), at least one of the following holds:*

$$\int_{\partial D} e_\varepsilon(w) d\mathcal{H}^1 \ \geq \varepsilon^{-\alpha} \tag{3.18}$$

or

$$\int_D e_\varepsilon(w) \ \geq c_2 \left|\log \varepsilon\right| \left|\int_D Jw\right| - c_2 \varepsilon^{1-\alpha} \left|\log \varepsilon\right|. \tag{3.19}$$

In fact we will show that one may take $c_2 = \frac{\pi}{8}(1 - \frac{3\alpha}{2})$.

Conclusion (3.19) is an example of a Jacobian estimate, relating the Ginzburg–Landau energy and the Jacobian of a complex-valued function w on a domain D. Well-known examples show that an like (3.19) cannot hold without some information about the behaviour of w on ∂D. The lemma shows that the very weak bounds $\int_{\partial D} e_\varepsilon(w) d\mathcal{H}^1 < \varepsilon^{-\alpha}$ —that is, the failure of (3.18)—is sufficient information for this purpose.

REMARK 4. A very similar argument shows that the same result is true if D is a bounded, open subset of $\mathbb{R}^2$, with smooth boundary. In this case the geometry of sets such as $\partial B \cap D$ and $B \cap \partial D$ (if B is a ball) becomes more complicated. However, on small scales, which are all that matter for this argument, these complications basically vanish.

In the rest of the subsection we will make use of the symbol ∂ to represent either the topological boundary of a set, or the boundary in the sense of Stoke's theorem, the meaning being clear from the context.

Proof. **1**. (*Preliminaries*). By a density argument, we may assume that w is smooth in $\bar{D}$. We assume that

$$e_{\partial D} := \int_{\partial D} e_\varepsilon(w)\, d\mathcal{H}^1 < \varepsilon^{-\alpha}, \tag{3.20}$$

and we will show that then (3.19) holds for all sufficiently small ε.

We first recall that if Σ is a Lipschitz arc (that is, the image of an injective Lipschitz curve) in $\bar{D}$ and $\mathcal{H}^1(\Sigma) \geq \varepsilon$, then

$$\int_\Sigma e_\varepsilon(|w|) d\mathcal{H}^1 \ \geq \ \frac{c}{\varepsilon} \|1 - |w|\ \|^2_{L^\infty(\Sigma)}. \tag{3.21}$$

See for example [JS02, Lemma 2.3] for a proof. In particular this applies to $\Sigma := \partial D$, so (3.20) implies that

$$\|1 - |w|\,\|_{L^\infty(\partial D)} \ \leq \ c\varepsilon^{(1-\alpha)/2} \leq \frac{1}{3} \qquad \text{for sufficiently small } \varepsilon. \tag{3.22}$$

We record a couple of consequences of this fact. First, basic properties of the degree imply that for $v := w/|w|$ (well-defined on ∂D)

$$2\pi d^* := 2\pi \deg(w;\partial D) = \int_{\partial D} j(v)\cdot\tau \;=\; \int_{\partial D} \frac{1}{|w|^2} j(w)\cdot\tau.$$

Also, an integration by parts shows that

$$\int_D Jw \;=\; \frac{1}{2}\int_{\partial D} j(w)\cdot\tau.$$

Thus, since that $|j(w)| \le |w|\,|\nabla w|$ we deduce from (3.22) that

$$\begin{aligned}\left|\pi d^* - \int_D Jw\right| &= \left|\frac{1}{2}\int_{\partial D}\left(\frac{1}{|w|^2}-1\right) j(w)\cdot\tau\right| \\ &\le 2\int_{\partial D}\left|1-|w|^2\right|\,|\nabla w| \le 4\varepsilon e_{\partial D} \overset{(3.20)}{\le} C\varepsilon^{1-\alpha}. \end{aligned}\tag{3.23}$$

We may thus assume that

$$d^* \neq 0 \tag{3.24}$$

since otherwise (3.19) follows immediately from (3.23). Next, we again use (3.22) to find that

$$\left|\int_D Jw\right| = \left|\frac{1}{2}\int_{\partial D} j(w)\cdot\tau\right| \le \int_{\partial D}|\nabla w| \;\le\; C\sqrt{e_{\partial D}} \le C\varepsilon^{-\alpha/2}. \tag{3.25}$$

What follows is a modification, in which we exploit (3.20) to control certain boundary terms, of the classical procedure of obtaining lower bounds for Ginzburg–Landau functionals in terms of the Jacobian by means of a ball construction [Jer99a, San98].

2. (*Basic estimates*). For $v = \frac{w}{|w|}$ as above, recall that

$$e_\varepsilon(w) = \frac{1}{2}|w|^2|\nabla v|^2 + e_\varepsilon(|w|), \qquad e_\varepsilon(|w|) := \frac{1}{2}|\nabla|w||^2 + \frac{1}{4\varepsilon^2}(|w|^2-1)^2. \tag{3.26}$$

In particular this implies that $|\nabla v| \le \frac{1}{|w|}|\nabla w|$.

Next, let B be a ball[5] of radius r. Fix $r_0 > 0$, depending on D, such that if $r < r_0$, then $\partial D\cap B$ consists of at most one line segment, necessarily of length at most $2r$ and $\partial B\cap D$ is an arc of a circle (It suffices to take $r_0 < \frac{1}{2}\min(\ell,|I|)$). Then (3.22) and elementary inequalities imply that

$$\int_{\partial D\cap B}|\partial_\tau v|d\mathcal{H}^1 \;\le\; 2\int_{\partial D\cap B}|\partial_\tau w|d\mathcal{H}^1 \;\le 4\sqrt{r}\sqrt{e_{\partial D}},$$

where $\partial_\tau w$ denotes the tangential derivative. Thus

$$\int_{\partial D\cap B}|\partial_\tau v|d\mathcal{H}^1 \;\le \pi, \qquad\qquad \text{if } r\le r_1 := \min\Big(\frac{\pi^2}{16}e_{\partial D}^{-1}, r_0\Big). \tag{3.27}$$

[5] We have in mind a ball with respect to the natural notion of distance in $(\mathbb{R}/\ell\mathbb{Z})\times\mathbb{R}$.

On the other hand, if $d := \deg(w; \partial(B\cap D))$ is well-defined and nonzero, then

$$d = \frac{1}{2\pi}\int_{\partial(B\cap D)} j(v)\cdot\tau.$$

Since $|j(v)\cdot\tau| \le |v|\ |\partial_\tau v| \ = \ |\partial_\tau v|$, we combine this with (3.27) to find that

$$\int_{\partial B\cap D} |\partial_\tau v| \ \ge \ (2\pi|d| - \pi) \ge \pi|d| \qquad\qquad \text{if } r \le r_1.$$

In particular, if $m := \min_{\partial B\cap D}|w|$, it follows from this, (3.26) and (3.22) that

$$\begin{aligned}\int_{\partial B\cap D} e_\varepsilon(w) &\ge \frac{m^2}{2}\int_{\partial B\cap D}|\partial_\tau v|^2 + \int_{\partial B\cap D} e_\varepsilon(|w|)\\ &\ge \frac{m^2}{2\mathcal{H}^1(\partial B\cap D)}\left(\int_{\partial B\cap D}|\partial_\tau v|\right)^2 + \frac{c}{\varepsilon}(1-m)^2\\ &\ge \frac{m^2\pi d^2}{4r} + \frac{c}{\varepsilon}(1-m)^2 \qquad\qquad \text{if } r\le r_1.\end{aligned}$$

If we define

$$\lambda_\varepsilon(r,d) := \min_{m\in[0,1]} \frac{m^2\pi d^2}{4r} + \frac{c}{\varepsilon}(1-m)^2\ , \tag{3.28}$$

then it follows that for any ball B,

$$\int_{\partial B\cap D} e_\varepsilon(w) \ge \lambda_\varepsilon(r,d)$$

$$\text{if } d := \deg(w;\partial(B\cap D)) \text{ and } r = \text{radius}(B) \le r_1 := \min\Big(\frac{\pi^2}{16}e_{\partial D}^{-1}, r_0\Big). \tag{3.29}$$

3. (*Lower bounds via a ball construction*). With estimate (3.29) in hand, we can carry out a vortex ball construction, as described in "Appendix A", as long as all balls have radius at most r_1. We sketch the main steps, following the presentation in [Jer99a, JS02]. To get started, we invoke Proposition 3.3 in [Jer99a], which shows that there exists a finite collection $\{B_i^0\}$ of closed, pairwise disjoint balls such that

$$S_E \subset \cup B_i^0 \qquad \text{for } S_E \text{ defined in (A.6)}, \tag{3.30}$$

$$r_i^0 \ge \varepsilon \text{ for all } i, \tag{3.31}$$

$$\int_{B_i^0\cap S_E} e_\varepsilon(w) \ge \frac{c_0}{\varepsilon} r_i^0 \ge \Lambda_\varepsilon(r_i^0) := \int_0^{r_i^0} \lambda_\varepsilon(r,1)\wedge\frac{c_0}{\varepsilon}\, dr. \tag{3.32}$$

Here r_i^0 denotes the radius of B_i^0 and c_0 a constant, independent of w and ε.

If $\sum_i r_i^0 \ge r_1$, then it follows from (3.32), the choice of r_1 [see (3.29)] and (3.20) that

$$\int_D e_\varepsilon(w) \ \ge \frac{c_0}{\varepsilon} r_1 \ge \ c\varepsilon^{\alpha-1}.$$

Since $\alpha < 2/3$, this together with (3.25) implies that (3.19) holds for all small ε. We may therefore assume that $\sum r_i^0 < r_1$. Next, from the additivity properties of the degree, (3.30), and the definition (A.6) of S_E, we see that[6]

$$0 \overset{(3.24)}{\neq} d^* = \deg(w;\partial D) = \sum_i \deg(w;\partial(B_i^0 \cap D)) =: \sum_i d_i^0.$$

Thus at least one ball has nonzero degree, and hence

$$\sigma^0 := \min_i \frac{r_i^0}{|d_i^0|} < r_1.$$

We may now follow a standard vortex ball argument construction as summarized in Lemma 22, but using (3.28), (3.29) in place of the usual estimates (A.9), (A.10). For a range of $\sigma > \sigma^0$, this yields a collection $\mathcal{B}(\sigma) = \{B_k^\sigma\}_{k=1}^{k(\sigma)}$ of balls such that

$$S_E \subset \cup_k B_k^\sigma, \tag{3.33}$$

$$\int_{B_k^\sigma \cap \omega} e_\varepsilon^{2d}(w)\,dx \geq \frac{r_k^\sigma}{\sigma}\Lambda_\varepsilon(\sigma), \qquad \text{for } r_k^\sigma := \text{radius}(B_k^\sigma), \tag{3.34}$$

$$r_k^\sigma \geq \sigma|d_k^\sigma|, \qquad \text{for } d_k^\sigma := \deg(w;\partial(B_k^\sigma \cap D)). \tag{3.35}$$

Moreover, $\sigma \mapsto \sum_k r_k^\sigma$ is a continuous, nondecreasing function. This process may be continued as long as all balls in the collection have radius at most r_1.

The estimates we obtain in this way are both worse and better than the classical ones, summarized in Lemma 22 for example. They are worse in that we have a somewhat weaker lower bound (compare (3.28) and (A.9)) and we can only continue as long as every ball has radius at most r_1; but better in that all the estimates we obtain apply to all balls, even those that intersect ∂D. This is not true in the classical case, compare for example (3.35).

We stop the ball construction when the sum of the radii is exactly r_1. Then (3.34), (3.35), and the fact[7] that $s \mapsto \frac{1}{s}\Lambda_\varepsilon(s)$ is nonincreasing, imply the lower bound

$$\int_D e_\varepsilon(w) \;\geq\; |d^*|\Lambda_\varepsilon(\frac{r_1}{|d^*|}), \qquad \text{where } d^* = \deg(w;\partial D) = \sum d_k^\sigma. \tag{3.36}$$

The last equality follows from (3.33) and the additivity of the degree.

4. (*Estimating the right-hand side of* (3.36)).

In view of (3.23), in order to prove (3.19), it suffices to show that

$$\Lambda_\varepsilon(\frac{r_1}{|d^*|}) \geq c|\log\varepsilon| \qquad \text{for all sufficiently small } \varepsilon > 0.$$

[6] Strictly speaking, if V is a set such that $\partial V \cap S_E \neq \emptyset$, then here and below, $\deg(w;\partial V)$ should be replaced by $\text{dg}(w;\partial V_i)$, see (A.8) for the definition. This does not change the argument in any essential way.

[7] This is easily checked, and a proof can be for example [Jer99a], Proposition 3.1.

It is straightforward to check (see [Jer99a] or [JS02]) that

$$\Lambda_\varepsilon(\sigma) \geq \frac{\pi}{4}\log\frac{\sigma}{\varepsilon} - C.$$

Also, it follows from (3.23) and (3.25) that $|d^*| \leq C\sqrt{e_{\partial D}} \leq C\varepsilon^{-\alpha/2}$. As a result, if $r_1 = r_0$, then

$$\Lambda(\frac{r_1}{|d^*|}) \geq \frac{\pi}{2}\log(\frac{r_0}{C\varepsilon^{1-\alpha/2}}) - C = \frac{\pi}{2}(1-\frac{\alpha}{2})|\log\varepsilon| - C \geq \frac{\pi}{4}(1-\frac{\alpha}{2})|\log\varepsilon|$$

if ε is small enough. On the other hand, if $r_1 = \frac{\pi^2}{16}e_{\partial D}^{-1}$ then

$$\begin{aligned}\Lambda_\varepsilon(\frac{r_1}{|d^*|}) &\geq \Lambda_\varepsilon(\frac{e_{\partial D}^{-1}}{C\sqrt{e_{\partial D}}}) \geq \frac{\pi}{2}\log(\frac{e_{\partial D}^{-3/2}}{\varepsilon}) - C \\ &\overset{(3.20)}{\geq} \frac{\pi}{2}(1-\frac{3\alpha}{2})|\log\varepsilon| - C \geq \frac{\pi}{4}(1-\frac{3\alpha}{2})|\log\varepsilon|\end{aligned}$$

if ε is small enough. ◻

Remark 5. The cylinder $\partial B(t)\times I \subset \mathbb{R}^3$ may be parametrized by the map

$$i : (\mathbb{R}/2\pi t)\times I \to \mathbb{R}^3, \qquad i(s,z) = (t\cos(\frac{s}{t}), t\sin(\frac{s}{t}), z).$$

and then

$$\begin{aligned}\int_{\partial B(s)\times I} Ju &= \int_{(\mathbb{R}/2\pi s)\times I} i^* Ju = \int_{(\mathbb{R}/2\pi s)\times I} i^*u^*(d\,\text{area}) \\ &= \int_{(\mathbb{R}/2\pi s)\times I}(u\circ i)^*(d\,\text{area}) = \int_{(\mathbb{R}/2\pi s)\times I} J(u\circ i).\end{aligned}$$

Here $u^*(d\,\text{area})$ denotes the pullback by $u : \Omega \to \mathbb{C} \cong \mathbb{R}^2$ of the area form $dx_1\wedge dx_2$ on $\mathbb{R}^2$. Thus, writing $u = (u_1, u_2)$, we have $u^*(dx_1\wedge dx_2) = du_1\wedge du_2 = Ju$. Similarly i^*Ju denotes the pullback by i of Ju.

Also, the area formula implies that

$$\int_{\partial B(s)\times I} e_\varepsilon(u_\varepsilon)d\mathcal{H}^2 \;\geq \int_{(\mathbb{R}/2\pi s)\times I} e_\varepsilon^{2d}(u_\varepsilon\circ i)d\mathcal{H}^2.$$

Thus Lemma 7 immediately implies the same result for cylinders in $\mathbb{R}^3$.

3.4 Proof of Proposition 1. Here we combine the definition of a good height and the implicit global information about the vorticity provided by Lemma 7.

Proof of Proposition 1. **Step 1**. Fix $\alpha \in (0, \frac{2}{3})$, and fix $z_b \in \mathcal{B}_1^\varepsilon$.

We define a new "good set" :

$$\mathcal{G}_3^\varepsilon := \left\{ z \in (0,L) : \|J_x u_\varepsilon(\cdot, z) - \pi n \delta_0\|_{F(\omega)} \le \frac{r^*}{16}, \int_\omega e_\varepsilon(u_\varepsilon(x,z))dx \le \frac{r^*}{32}\varepsilon^{-\alpha} \right\}.$$

It follows from (3.12), and (1.27) and Chebyshev's inequality (for the condition involving the energy) that

$$|(0,L) \setminus \mathcal{G}_3^\varepsilon| \le C(\int_0^L \|J_x u_\varepsilon(\cdot, z) - n\pi\delta_0\|_{F(\omega)} + \varepsilon^{\alpha/2}).$$

Thus if ε is small enough, in view of (1.23), we may fix $z_g \in \mathcal{G}_3^\varepsilon$ such that

$$\frac{1}{2}\delta \le |z_b - z_g| \le \delta \tag{3.37}$$

for $\delta = \delta(\alpha)$ to be fixed below.

If we define

$$\widetilde{\mathcal{S}}^\varepsilon(z) := \left\{ s \in \left(\frac{r^*}{2}, r^*\right) : \left| \int_{B(s)} J_x u_\varepsilon(x,z)\ dx - n\pi \right| \le \frac{1}{2} \right\},$$

then the proof of Lemma 1 shows that for every z,

$$\left|\widetilde{\mathcal{S}}^\varepsilon(z)\right| \ge \frac{r^*}{2} - 2\|J_x u(\cdot, z) - n\pi\delta_0\|_{F(\omega)},$$

and thus

$$\text{if } z \in \mathcal{G}_3^\varepsilon, \text{ then } \left|\widetilde{\mathcal{S}}^\varepsilon(z)\right| \ge \frac{3r^*}{8}.$$

In particular, this holds for $z = z_g$.

On the other hand, the definition of $\mathcal{B}_1^\varepsilon$ implies that the set

$$\left\{ s \in \left(\frac{r^*}{2}, r^*\right) : \left| \int_{B(s)} J_x u_\varepsilon(x,z_b)\ dx - n\pi \right| \ge 1 \right\}$$

has measure at least $r^*/4$. Thus the intersection of this set with $\widetilde{\mathcal{S}}^\varepsilon(z_g)$ has measure at least $r^*/8$. As a result, the set

$$\mathcal{T} := \left\{ s \in \left(\frac{r^*}{2}, r^*\right) : \left| \int_{B(s)} J_x u_\varepsilon(x,z_b)\ dx - \int_{B(s)} J_x u_\varepsilon(x,z_g) dx \right| \ge \frac{1}{2} \right\} \tag{3.38}$$

satisfies

$$|\mathcal{T}| \ge \frac{r^*}{8}. \tag{3.39}$$

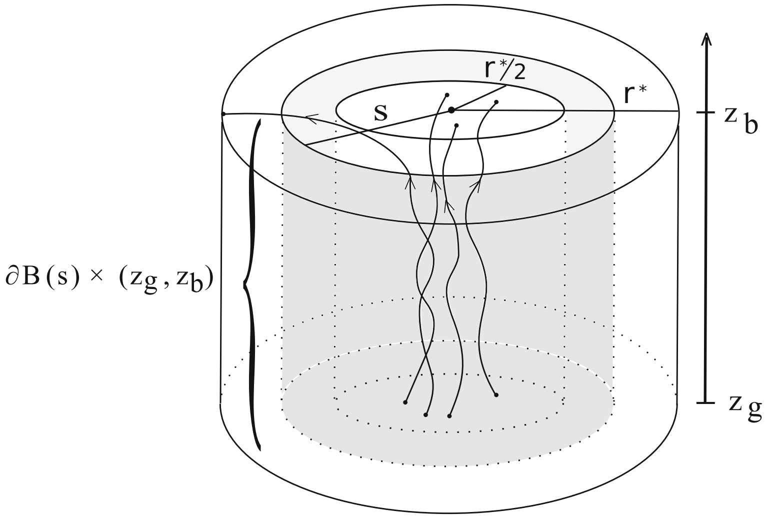

Figure 2: Configuration with good height z_g and bad height z_b

Step 2. In what follows, if M is an oriented 2d submanifold of Ω, we write $\int_M Ju_\varepsilon$ to denote the integral of Ju_ε over M in the standard sense of differential geometry (recalling that Ju_ε is a 2-form). Then in particular[8]

$$\int_{B(s)} J_x u_\varepsilon(x,z)\ dx = \int_{B(s)\times\{z\}} Ju_\varepsilon. \tag{3.40}$$

Now let $I = (z_0, z_1)$, where $z_0 = \min(z_b, z_g)$ and $z_1 = \max(z_g, z_b)$. Since $d\,Ju_\varepsilon = 0$, we find from Stokes Theorem that

$$0 \ = \ \int_{B(s)\times I} d\,Ju_\varepsilon \ = \ \int_{\partial(B(s)\times I)} Ju_\varepsilon,$$

and upon breaking $\partial(B(s)\times I)$ into pieces in the natural way, we find that

$$\int_{\partial B(s)\times I} Ju_\varepsilon = \int_{B(s)\times\{z_0\}} Ju_\varepsilon - \int_{B(s)\times\{z_1\}} Ju_\varepsilon.$$

It thus follows from (3.40) and the definition (3.38) of $\mathcal{T}$ that

$$\left|\int_{\partial B(s)\times I} Ju_\varepsilon\right| \geq \frac{1}{2} \qquad \text{if } s\in\mathcal{T}. \tag{3.41}$$

Step 3. Now define

$$\mathcal{T}_* := \left\{ s\in\mathcal{T}\ :\ \int_{\partial B(s)\times\{z_g,z_b\}} e_\varepsilon(u_\varepsilon) d\mathcal{H}^1 < \varepsilon^{-\alpha} \right\}, \qquad \mathcal{T}^* := \mathcal{T}\setminus\mathcal{T}_*.$$

[8] This identity specifies our convention for orienting $B(s)\times\{z\}$, which is the standard one.

We will show that $|\mathcal{T}^*| \geq \frac{r^*}{16}$. To do this, in view of (3.39), it suffices to show that $|\mathcal{T}_*| < \frac{r^*}{16}$ for all small ε. To do this, note that the coarea formula, Lemma 7 (see also Remark 5) and (3.41) imply that

$$\int_{\omega\times I} e_\varepsilon(u_\varepsilon)\ dx\ dz \geq \int_{s\in\mathcal{T}_*}\left(\int_{\partial B(s)\times I} e_\varepsilon(u_\varepsilon)\ d\mathcal{H}^2\right)$$
$$\geq \frac{1}{2}c_2|\log\varepsilon||\mathcal{T}_*| - c\varepsilon^{1-\alpha}|\log\varepsilon|.$$

On the other hand, we know from Lemma 5 and (3.37) that

$$\int_{\omega\times I} e_\varepsilon(u_\varepsilon)\ dx\ dz \leq n\pi|\log\varepsilon|(|I| + o(1))\ \ \leq n\pi|\log\varepsilon|(\delta + o(1)).$$

By a suitable choice of δ, we can therefore guarantee that $|\mathcal{T}_*| < \frac{r^*}{16}$ for all sufficiently small $\varepsilon > 0$.

Step 4. Since $|\mathcal{T}^*| \geq \frac{r^*}{16}$, we easily see that

$$\int_{\omega\times\{z_g,z_b\}} e_\varepsilon(u_\varepsilon)d\mathcal{H}^2 \geq \int_{s\in\mathcal{T}^*}\int_{\partial B(s)\times\{z_g,z_b\}} e_\varepsilon(u_\varepsilon) \geq \frac{r^*}{16}\varepsilon^{-\alpha}.$$

Since

$$\int_{\omega\times\{z_g\}} e_\varepsilon(u_\varepsilon)d\mathcal{H}^2 \leq \frac{r^*}{32}\varepsilon^{-\alpha}$$

by definition of $\mathcal{G}_3^\varepsilon$, we conclude that

$$\int_{\omega\times\{z_b\}} e_\varepsilon(u_\varepsilon)d\mathcal{H}^2 \geq \frac{r^*}{32}\varepsilon^{-\alpha}$$

for all sufficiently small ε. This is the conclusion (3.13) of the Proposition, up to the factor $\frac{r^*}{32}$, which can be absorbed by taking a larger choice of α and a correspondingly smaller ε_0. □

Remark 6. The proof shows that $e_\varepsilon(u_\varepsilon)$ may be replaced by $e_\varepsilon^{2d}(u_\varepsilon)$ in the conclusion (3.13), since in fact only tangential components of the boundary energy appear in the $\int_{\partial D} e_\varepsilon d\mathcal{H}^1 \geq \varepsilon^{-\alpha}$ part of the possibilities contemplated in Lemma 7.

3.5 Alternate hypothesis (1.29). Recall that in Theorem 3, we have assumed energy bounds (1.27), together with either (1.23)–(1.25) or (1.29). In this section we demonstrate a couple of ways in which the latter assumption is stronger than the former. The first shows that the (1.29) case of Theorem 3 implies the (1.23)–(1.25) case, and the second indicates some ways in which our arguments can be simplified if we assume (1.29).

Following this section we will focus on hypotheses (1.23)–(1.25) which, in addition to being more subtle, are also the hypotheses we need for our applications in Theorem 1.

Lemma 8. *If* $(u_\varepsilon) \subset H^1(\Omega;\mathbb{C})$ *is a sequence satisfying* (1.27) *and* (1.29), *then there is a subsequence along which it also satisfies* (1.23)–(1.25).

Proof. Since (1.23) follows immediately from (1.29), we only need to find a point $z \in (0, L)$ and a subsequence along which assumptions (1.24)–(1.25) hold.

To do this, we define yet another "good set",

$$\mathcal{G}_4^\varepsilon := \{z \in \mathcal{G}_2^\varepsilon : \|J_x u_\varepsilon(\cdot, z) - \pi n \delta_0\|_{F(\omega)} \le 3Ch_\varepsilon\},$$

where C is the same constant appearing in (1.29). It follows from (1.29), (3.12) and Lemma 6 that $|\mathcal{G}_4^\varepsilon| \ge \frac{L}{2}$. By the Borel–Cantelli Lemma, we can therefore find some $z_0 \in (0, L)$ and a subsequence ε_k such that $z_0 \in \mathcal{G}_4^{\varepsilon_k}$ for all k.

The fact that $z_0 \in \mathcal{G}_2^{\varepsilon_k}$ for all k implies that (1.25) holds. To prove (1.24), Lemma 2 implies that for all sufficiently large k there exist points $\{p_i^\varepsilon\}_{i=1}^n$ such that (3.6) holds, *i.e.*

$$\|J_x u_\varepsilon(\cdot, z_0) - \pi \sum_{i=1}^n \delta_{p_i^\varepsilon}\|_{F(w)} \le \varepsilon^a.$$

(Here and below, we often write ε instead of ε_k, to reduce clutter.) Since $z_0 \in \mathcal{G}_4^{\varepsilon_k}$,

$$\|n\pi\delta_0 - \pi \sum_{i=1}^n \delta_{p_i^\varepsilon}\|_{W^{-1,1}(\omega)} \le \varepsilon^a + 3Ch_\varepsilon \le Ch_\varepsilon.$$

It follows from the definition (1.22) of the flat norm (for example by testing with $\varphi = (1 - |x|)^+$ or a regularization thereof) that $\sum |p_i^\varepsilon| \le Ch_\varepsilon$. If we let $q_i^\varepsilon = p_i^\varepsilon / h_\varepsilon$, then $\sum |q_i^\varepsilon| \le C$, and we may pass to a further subsequence such that $q_i^\varepsilon \to q_i^0$ for $i = 1, \dots, n$. Then for this subsequence, by the triangle inequality

$$\|J_x u_\varepsilon(\cdot, z_0) - \pi \sum_{i=1}^n \delta_{h_\varepsilon q_i^0}\|_{F(w)} \le \varepsilon^a + \pi \sum_{i=1}^n \|\delta_{p_i^\varepsilon} - \delta_{h_\varepsilon q_i^0}\|_{F(\omega)}.$$

It follows again from the definition (1.22) of the flat norm that the right-hand side is bounded by $\varepsilon^a + \sum |p_i^\varepsilon - h_\varepsilon q_i^0| = \varepsilon^a + h_\varepsilon |q_i^\varepsilon - q_i^0|$, and this is clearly $o(h_\varepsilon)$ as $\varepsilon \to 0$. □

Our next result is not used anywhere in this paper. Its significance is that it shows that, if we allow ourselves the stronger assumption (1.29), then certain difficulties in the proof of Theorem 3 can be avoided.

Lemma 9. *If* $(u_\varepsilon) \subseteq H^1(\Omega;\mathbb{C})$ *is a family satisfying* (1.27) *and* (1.29), *then*

$$\int_\Omega e_\varepsilon^{2d}(u_\varepsilon) \ge n\pi L\,|\log \varepsilon| + \pi n(n-1)L\,|\log h_\varepsilon| - C \tag{3.42}$$

and

$$\int_\Omega |\partial_z u_\varepsilon|^2 \,dx\,dz \le C. \tag{3.43}$$

In the next section, when proving Theorem 3 under the weaker hypotheses (1.23)–(1.25), we only know at the outset that $\int_\Omega |\partial_z u_\varepsilon|^2 dx\, dz = o(|\log\varepsilon|)$, see (3.15). This is considerably weaker than (3.43), and this weakness introduces some complications into our arguments.

Proof. According to Lemma 3, if z belongs to the set $\mathcal{G}_2^\varepsilon$, defined in (3.17), then the lower energy bound (3.9) holds. In addition, it follows from Proposition 1 and the definition of $\mathcal{G}_2^\varepsilon$ that

$$\int_\omega e_\varepsilon^{2d}(u_\varepsilon(x,z))dx \;\geq \pi(n+\theta)|\log\varepsilon| \qquad \text{for } z\notin \mathcal{G}_2^\varepsilon.$$

From this and (3.9),

$$\begin{aligned}\int_\Omega e_\varepsilon^{2d}(u_\varepsilon)dx\,dz \;\geq\;& (L-|\mathcal{G}_2^\varepsilon|)\pi(n+\theta)|\log\varepsilon| \\ &+ \int_{z\in\mathcal{G}_2^\varepsilon}\big(n\pi|\log\varepsilon| - \pi n(n-1)\log\|J_x u_\varepsilon(\cdot,z)-n\pi\delta_0\|_{F(\omega)} - C\big)\,dz \\ \geq\;& n\pi L|\log\varepsilon| - C - \pi n(n-1)\int_{z\in\mathcal{G}_2^\varepsilon}\log\|J_x u_\varepsilon(\cdot,z)-n\pi\delta_0\|_{F(\omega)}\,dz.\end{aligned}$$

To estimate the integral, note that by Jensen's inequality and (1.29),

$$\begin{aligned}&-\int_{\mathcal{G}_2^\varepsilon}\log\|J_x u_\varepsilon(\cdot,z)-n\pi\delta_0\|_{F(\omega)}dz \\ &\quad\geq -|\mathcal{G}_2^\varepsilon|\log\left(\frac{1}{|\mathcal{G}_2^\varepsilon|}\int_{\mathcal{G}_2^\varepsilon}\|J_x u_\varepsilon(\cdot,z)-n\pi\delta_0\|_{F(\omega)}dz\right)\geq -L\log h_\varepsilon - C.\end{aligned}$$

This proves (3.42). Finally, (3.43) is immediate from (3.42) and (1.27), since $e_\varepsilon(u) = e_\varepsilon^{2d}(u) + \frac{1}{2}|\frac{\partial u}{\partial z}|^2$. □

4 Compactness and lower bound

In this section we prove part (*a*) of Theorem 3, consisting of the compactness and lower bound assertions. In view of Lemma 8, it suffices to consider assumptions (1.23)–(1.25), together with (1.27).

Our strategy will be to rescale on a scale ℓ_ε chosen to facilitate the proof of the compactness assertions. We will also obtain energy lower bounds that depend on ℓ_ε in a way that will allow us to conclude, only in the final step of the proof, that in fact $\ell_\varepsilon = h_\varepsilon$. This will require us to be rather careful about how some of our estimates depend on certain constants, such as c_3, defined below.

Thus the rescaling will turn out to be the same as that in the statement of the theorem, and the lower energy bounds we prove will reduce to (1.32).

Thus, we fix a sequence (u_ε) satisfying (1.23) and (1.27), and we define

$$\ell_\varepsilon := \max\left\{ h_\varepsilon\,,\ \left(\frac{1}{c_3|\log\varepsilon|}\int_\Omega |\partial_z u_\varepsilon|^2\,dx\,dz\right)^{1/2}\right\} \tag{4.1}$$

for a constant c_3 that will be specified later. (We will invoke assumptions (1.24) and (1.25) only when they are needed). It follows from the definition of ℓ_ε and from (3.15) that

$$h_\varepsilon \le \ell_\varepsilon = o(1) \qquad \text{as } \varepsilon \to 0. \tag{4.2}$$

Throughout the rest of this section, we will use the notation

$$\omega_\varepsilon = \ell_\varepsilon^{-1}\omega, \qquad v_\varepsilon(x,z) = u_\varepsilon(\ell_\varepsilon x, z) \quad \text{ for } (x,z)\in\Omega_\varepsilon := \omega_\varepsilon\times(0,L). \tag{4.3}$$

A change of variables shows that

$$\frac{1}{|\log\varepsilon|}\int_{\Omega_\varepsilon} |\partial_z v_\varepsilon|^2\,dx\,dz = \frac{1}{\ell_\varepsilon^2|\log\varepsilon|}\int_\Omega |\partial_z u_\varepsilon|^2\,dx\,dz \le c_3 \tag{4.4}$$

and that, for $\varepsilon' = \varepsilon/\ell_\varepsilon$,

$$\int_{\Omega_\varepsilon} e^{2d}_{\varepsilon'}(v_\varepsilon)\,dx\,dz = \int_\Omega e^{2d}_\varepsilon(u_\varepsilon)\,dx\,dz \le C|\log\varepsilon|.$$

From (4.2) and (1.7),

$$\frac{|\log\varepsilon'|}{|\log\varepsilon|} \to 1 \text{ as } \varepsilon\to 0, \tag{4.5}$$

and recalling that $e_{\varepsilon'}(v_\varepsilon) = e^{2d}_{\varepsilon'}(v_\varepsilon) + \frac12|\partial_z v_\varepsilon|^2$, it follows that

$$\frac{1}{|\log\varepsilon'|}\int_{\Omega_\varepsilon} e_{\varepsilon'}(v_\varepsilon)\,dx\,dz \le C. \tag{4.6}$$

At different stages in the proof of (1.30) we will need to invoke a general compactness result for Ginzburg–Landau functionals of [JS02, ABO05].

Theorem 4. *Let U be a bounded, open subset of $\mathbb{R}^3$. Assume $(u_\varepsilon)\subseteq W^{1,2}(U;\mathbb{C})$ is such that*

$$\limsup_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\int_U e_\varepsilon(u_\varepsilon)dx < \infty.$$

Then, there exists a subsequence $\varepsilon_k\to 0$ and an integer multiplicity rectifiable 1-current J such that $\frac{1}{\pi}Ju_{\varepsilon_k}$ converges to J in $W^{-1,1}(U)$. In addition, one has the following uniform lower semicontinuity:

$$\liminf_{k\to\infty}\frac{1}{|\log\varepsilon_k|}\int_U e_{\varepsilon_k}(u_{\varepsilon_k})dx \ge \pi M_U(J).$$

This was first proved in [JS02], which in fact established compactness in the $(C_c^{0,\alpha}(U))^*$ norm for all $0 < \alpha \le 1$. A different proof, which established compactness in the flat norm $F(U)$, was subsequently given in [ABO05], which also proved a corresponding upper bound.

We immediately conclude from (4.6) and Theorem 4 (the statement considers a family of maps indexed by ε' but it can be easily seen to also apply to (v_ε)) that there exists an integer multiplicity 1-current J in $\mathbb{R}^2 \times (0, L)$ such that

$$\frac{1}{\pi} \star Jv_\varepsilon \to J \text{ in } W^{-1,1}(B(R) \times (0, L)) \quad \text{for all } R > 0. \tag{4.7}$$

Our goal is to show that, roughly speaking, J consists of n graphs of H^1 functions over the vertical segment $(0, L)$, possibly together with other pieces that the vertical component $J_x v_\varepsilon$ of the vorticity fails to record.

In view of (4.4), we may also assume, after passing to a further subsequence, that there exists a measure μ on $[0, L]$ such that

$$\mu_\varepsilon \rightharpoonup \mu \quad \text{weakly as measures, where} \quad \mu_\varepsilon(A) := \int_{\omega_\varepsilon \times A} \frac{|\partial_z v_\varepsilon(x, z)|^2}{|\log \varepsilon|} dx\, dz. \tag{4.8}$$

For this subsequence, general properties of weak convergence of measures imply that

$$\mu((0, L)) \le \liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \frac{|\partial_z v_\varepsilon(x, z)|^2}{|\log \varepsilon|} dx\, dz = \liminf_{\varepsilon \to 0} \int_{\Omega} \frac{|\partial_z u_\varepsilon(x, z)|^2}{\ell_\varepsilon^2 |\log \varepsilon|} dx\, dz.$$

The objective now is to identify the filaments. To that end we will find a countable dense subset of heights such that, among other things, the slices of the Jacobians at these heights converge to π times the sum of n Dirac masses; these are the candidates for the values of f at these heights. We then establish the existence of a unique $H^1((0, L))$ extension of f. This will require control on the modulus of continuity of f; obtaining this control is our first task.

4.1 Modulus of continuity. We first establish the basic estimate, see (4.11) below, that lets us control $\|f'\|_{L^2}^2$ by $\int_\Omega |\partial_z u_\varepsilon|^2$. At this stage we do not yet need all the hypotheses of Theorem 3.

LEMMA 10. *Assume that (u_ε) satisfies (1.23), (1.27), define (v_ε) by rescaling as in (4.1), (4.3), and define μ by (4.8).*

Assume that $\{z_1^\varepsilon\}$ and $\{z_2^\varepsilon\}$ are sequences in $[0, L]$ such that $z_j^\varepsilon \to z_j$ for $j = 1, 2$, with $0 \le z_1 < z_2 \le L$, and that the following conditions hold for $j = 1, 2$ (perhaps after passing to a subsequence):

$$J_x v_\varepsilon(\cdot, z_j^\varepsilon) \to \pi \sum_{i=1}^{n(z_j)} \delta_{p_i(z_j)} \qquad \text{in } W^{-1,1}(B(R)), \quad \text{for all } R > 0, \tag{4.9}$$

(for certain points $\{p_i(z_j)\}_{i=1}^{n(z_j)}$*, not necessarily distinct) and*

$$\limsup_{\varepsilon\to 0} |\log\varepsilon|^{-1}\int_\omega e_\varepsilon^{2d}(u_\varepsilon(x,z_j^\varepsilon))dx \le M \tag{4.10}$$

for some $M>0$*. Finally, assume that* $n(z_i)=n$ *for either* $i=1$ *or* 2*. Then* $n(z_1)=n(z_2)=n$*, and*

$$\frac{\pi}{2}\min_{\sigma\in S^n}\sum_{i=1}^n \frac{\left|p_i(z_1)-p_{\sigma(i)}(z_2)\right|^2}{z_2-z_1} \ \le\ \frac{1}{2}\mu([z_1,z_2]). \tag{4.11}$$

Here we follow the idea in the proof of Proposition 3 in [Jer99b], where however one has somewhat more information, such as bounds on $\int_{\omega_\varepsilon} e_\varepsilon^{2d}(v_\varepsilon)$ that are uniform for $z\in(z_1,z_2)$, as well as distinct limiting vortex curves that are known not to intersect.

Proof. We first present the proof in the basic case when $z_j^\varepsilon = z_j$ for all ε, for $j=1,2$. Given a small number $0<\tau$, we define $\psi_\varepsilon^\tau:\omega_\varepsilon\times\mathbb{R}\to\mathbb{R}^2$ as follows:

$$\psi_\varepsilon^\tau(x,z) = \begin{cases} v_\varepsilon(x,z_1) & \text{if } z\le z_1/\tau\\ v_\varepsilon(x,\tau z) & \text{if } z_1/\tau < z < z_2/\tau\\ v_\varepsilon(x,z_2) & \text{if } z_2/\tau \le z.\end{cases} \tag{4.12}$$

Step 1. Let us write $n_i = n(z_i)$ for $i=1,2$, and (as above) $\varepsilon' = \varepsilon/\ell_\varepsilon$. We first prove that $n_1=n_2=n$, and that

$$\begin{aligned}\liminf_{\varepsilon\to 0}\frac{1}{|\log\varepsilon'|}&\int_{z_1/\tau}^{z_2/\tau}\int_{\omega_\varepsilon} e_{\varepsilon'}(\psi_\varepsilon^\tau)dx\,dz\\ &\ge \pi\min_{\sigma\in s^n}\sum_{i=1}^n\left(\tau^{-2}(z_1-z_2)^2+|p_i(z_1)-p_{\sigma(i)}(z_2)|^2\right)^{\frac12}.\end{aligned} \tag{4.13}$$

For $\delta\ge 0$ we will use the notation

$$I_\delta = (\frac{z_1}{\tau}-\delta,\frac{z_2}{\tau}+\delta),\qquad U_\delta = \mathbb{R}^2\times I_\delta.$$

First, some changes of variable show that

$$\begin{aligned}\int_{I_\delta}\int_{\omega_\varepsilon} e_{\varepsilon'}(\psi_\varepsilon^\tau)dx\,dz &= \delta\sum_{i=1,2}\int_\omega e_\varepsilon^{2d}(u_\varepsilon(x,z_i))dx\\ &\quad+\frac{1}{\tau}\int_{z_1}^{z_2}\int_{\omega_\varepsilon} e_{\varepsilon'}^{2d}(v_\varepsilon)dx\,dz+\frac{\tau}{2}\int_{z_1}^{z_2}\int_{\omega_\varepsilon}|\partial_z v_\varepsilon|^2dx\,dz.\end{aligned} \tag{4.14}$$

In light of (4.4), (4.5), (4.6), and (4.10), there is thus a constant $C=C(\tau,\delta)$ such that

$$\frac{1}{|\log\varepsilon'|}\int_{I_\delta}\int_{\omega_\varepsilon} e_{\varepsilon'}(\psi_\varepsilon^\tau)dx\,dz\le C\qquad \text{for all } \varepsilon\in(0,1].$$

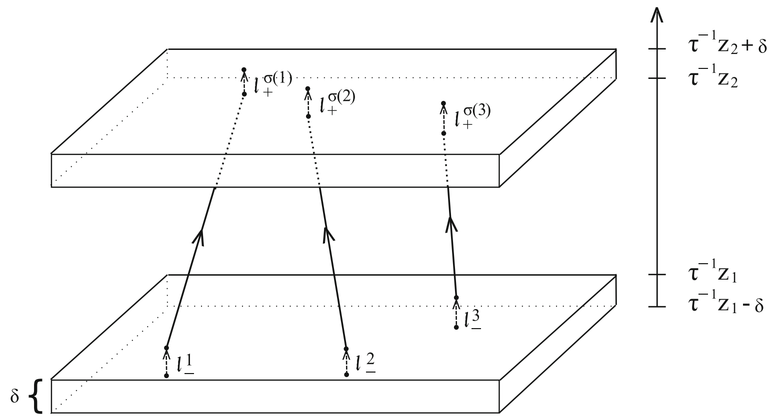

Figure 3: Depiction of a $\hat{J}$ whose associated g_i's are of minimal length

It therefore follows from Theorem 4 that there exists a 1-current $\hat{J}$ in U_δ such that, after passing to a subsequence if necessary,

$$\frac{1}{\pi} J\psi_\varepsilon^\tau \to \hat{J} \qquad \text{in } W^{-1,1}(B(R)\times I_\delta), \quad \text{for all } R>0.$$

Moreover, $\hat{J}$ is integer multiplicity rectifiable, with $\partial\hat{J}=0$ in U_δ, and

$$\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon'|}\int_{I_\delta}\int_{\omega_\varepsilon} e_{\varepsilon'}(\psi_\varepsilon^\tau)dx\,dz \ge \pi M_{U_\delta}(\hat{J}). \tag{4.15}$$

We want to estimate $M_{U_\delta}(\hat{J})$. Note first from (4.9) and definition of ψ_ε^τ that, using the notation (1.18), we have

$$\hat{J} = \pi\sum_{i=1}^{n_1} T_{l_-^i} \text{ in } \mathbb{R}^2\times(\tau^{-1}z_1-\delta,\tau^{-1}z_1), \quad \text{for } l_-^i(z)=(p_i(z_1),z), \tag{4.16}$$

$$\hat{J} = \pi\sum_{i=1}^{n_2} T_{l_+^i} \text{ in } \mathbb{R}^2\times(\tau^{-1}z_2,\tau^{-1}z_2+\delta), \quad \text{for } l_+^i(z)=(p_i(z_2),z). \tag{4.17}$$

On the other hand, recalling (2.8), there exist Lipschitz curves $\{g_i\}_{i\in I}$ such that

$$\hat{J}=\sum_{i\in I} T_{g_i}, \qquad \partial T_{g_i}=0 \text{ in } U_\delta, \quad \text{for all } i,$$

and $M(\hat{J})=\sum_{i\in I}\text{length}(g_i)$. In particular, certain of these curves must coincide (after reparametrization) with $l_-^i(z)$ for $z\in(\tau^{-1}z_1-\delta,\tau^{-1}z_1)$. We may thus choose to label and parametrize these $\{g_i\}$ so that $g_i(z)=l_-^i(z)$ for $z\in(\tau^{-1}z_1-\delta,\tau^{-1}z_1)$, for $i=1,\dots,n_1$ (see figure 3 below).

Furthermore, because $\partial T_{g_i}=0$ for all i, and all $T_{l_\pm^i}$ are oriented in the same way, we conclude each g_i, for $i=1,\dots,n_1$, must connect to one of the curves l_+^j, $j=1,\dots,n_2$, and each l_+^j must connect to one g_i. It follows that $n_1=n_2=n$ (since

$n_i = n$ for one of $i = 1, 2$, by assumption), and that there is some $\sigma \in S_m$ such that g_i connects to $l_+^{\sigma(i)}$. Then elementary geometry implies that

$$\text{length}(g_i) \geq 2\delta + |l_-^i(\frac{z_1}{\tau}) - l_+^{\sigma(i)}(\frac{z_2}{\tau})| = 2\delta + \left(\frac{(z_2 - z_1)^2}{\tau^2} + |p_i(z_1) - p_{\sigma(i)}(z_2)|^2 \right)^{\frac{1}{2}}.$$

Adding over $i = 1, \ldots, n$, we deduce that

$$\liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon'|} \int_{I_\delta} \int_{\omega_\varepsilon} e_{\varepsilon'}(\psi_\varepsilon^\tau) dx\, dz$$
$$\geq \pi \min_{\sigma \in S^n} \sum_{i=1}^n \left(\tau^{-2}(z_1 - z_2)^2 + |p_i(z_1) - p_{\sigma(i)}(z_2)|^2 \right)^{\frac{1}{2}} + 2\pi n\delta.$$

Then (4.13) follows by sending $\delta \searrow 0$. This involves an interchange of limits on the left-hand side, which is justified since

$$\frac{1}{|\log \varepsilon'|} \int_{I_\delta \setminus I_0} \int_{\omega_\varepsilon} e_{\varepsilon'}(\psi_\varepsilon^\tau) dx\, dz \overset{(4.10)}{\leq} \frac{|\log \varepsilon|}{|\log \varepsilon'|} M\pi\delta \overset{(4.5)}{\leq} 2M\pi\delta$$

for all positive δ and all sufficiently small $\varepsilon > 0$.

Step 2. We next show that

$$\limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon'|} \int_{z_1/\tau}^{z_2/\tau} \int_{\omega_\varepsilon} e_{\varepsilon'}^{2d}(\psi_\varepsilon^\tau) dx\, dz = \frac{n\pi}{\tau}(z_2 - z_1). \tag{4.18}$$

Indeed, changing variables as in (4.14) we have

$$\int_{z_1/\tau}^{z_2/\tau} \int_{\omega_\varepsilon} e_{\varepsilon'}^{2d}(\psi_\varepsilon^\tau) dx\, dz = \frac{1}{\tau} \int_{z_1}^{z_2} \int_{\omega_\varepsilon} e_{\varepsilon'}^{2d}(v_\varepsilon) dx\, dz = \frac{1}{\tau} \int_{z_1}^{z_2} \int_{\omega} e_{\varepsilon}^{2d}(u_\varepsilon) dx\, dz.$$

Then the claim follows by dividing by $|\log \varepsilon'|$, recalling (4.5), using (3.14), with $S = (z_1, z_2)$, and sending $\varepsilon \to 0$.

Step 3. Standard properties of weak convergence imply that

$$\mu([z_1, z_2]) \geq \liminf_{\varepsilon \to 0} \mu_\varepsilon([z_1, z_2]) = \liminf_{\varepsilon \to 0} \frac{1}{2} \int_{z_1}^{z_2} \int_{\omega_\varepsilon} \frac{|\partial_z v_\varepsilon|^2}{|\log \varepsilon|} dx\, dz.$$

And from the previous steps we deduce that

$$\liminf_{\varepsilon \to 0} \frac{1}{2} \int_{z_1}^{z_2} \int_{\omega_\varepsilon} \frac{|\partial_z v_\varepsilon|^2}{|\log \varepsilon|} dx\, dz = \liminf_{\varepsilon \to 0} \frac{1}{\tau} \int_{\tau^{-1} z_1}^{\tau^{-1} z_2} \int_{\omega_\varepsilon} \frac{|\partial_z \psi_\varepsilon^\tau|^2}{2\, |\log \varepsilon|} dx\, dz$$
$$= \liminf_{\varepsilon \to 0} \frac{1}{\tau} \int_{\tau^{-1} z_1}^{\tau^{-1} z_2} \int_{\omega_\varepsilon} \frac{e_\varepsilon(\psi_\varepsilon^\tau) - e_\varepsilon^{2d}(\psi_\varepsilon^\tau)}{|\log \varepsilon|} dx\, dz$$
$$\geq \frac{\pi}{\tau} \min_{\sigma \in S^m} \sum_{i=1}^n \left(\left(\frac{(z_1 - z_2)^2}{\tau^2} + |p_i(z_1) - p_{\sigma(i)}(z_2)|^2 \right)^{\frac{1}{2}} - \frac{(z_2 - z_1)}{\tau} \right).$$

Since the left-hand side is independent of τ, we can take the $\tau \to 0$ limit of the right-hand side to deduce (4.11).

Step 4. Now assume that $z_1^\varepsilon, z_2^\varepsilon$ depend nontrivially on ε. For each ε, define

$$V_\varepsilon(x,z) = \begin{cases} v_\varepsilon(x, z_1^\varepsilon) & \text{if } z \le z_1^\varepsilon \\ v_\varepsilon(x,z) & \text{if } z_1^\varepsilon < z < z_2^\varepsilon \\ v_\varepsilon(x, z_2^\varepsilon) & \text{if } z_2^\varepsilon \le z. \end{cases}$$

Then for any $Z_1 < z_1 < z_2 < Z_2$, we may apply the previous case on the (fixed) interval (Z_1, Z_2), since $v_\varepsilon(x, z_j^\varepsilon) = V_\varepsilon(x, Z_j)$ for $j = 1, 2$ and all sufficiently small ε. Then (4.11) implies that

$$\frac{\pi}{2} \min_{\sigma \in S^m} \sum_{i=1}^n \frac{\left|p_i(z_1) - p_{\sigma(i)}(z_2)\right|^2}{Z_2 - Z_1} \;\le\; \frac{1}{2}\tilde{\mu}([Z_1, Z_2]) \;\le \frac{1}{2}\mu([Z_1, Z_2]),$$

where $\tilde{\mu}$ is the measure generated as in (4.8), but by $|\partial_z V_\varepsilon|^2$ rather than $|\partial_z v_\varepsilon|^2$. We conclude the proof by letting $Z_1 \nearrow z_1$ and $Z_2 \searrow z_2$. □

4.2 Compactness and lower bounds at *a.e* height.

We will use the notation

$$\xi_\varepsilon(z) := \int_\omega e_\varepsilon^{2d}(u_\varepsilon(x,z)\,dx - \big[n(\pi|\log\varepsilon| + \gamma) + n(n-1)\pi|\log \ell_\varepsilon| - n^2\pi H_\omega(0,0)\big]. \quad (4.19)$$

LEMMA 11. *Assume that $(u_\varepsilon) \subset H^1(\Omega; \mathbb{C})$ satisfies (1.23)–(1.25) and (1.27).*

(a) *There exist $\varepsilon_0 > 0$ and $C > 0$*

$$\xi_\varepsilon(z) \ge -C \qquad \text{for all } z \in \mathcal{G}_2^\varepsilon \text{ and } 0 < \varepsilon < \varepsilon_0, \quad (4.20)$$

where $\mathcal{G}_2^\varepsilon$ was defined in (3.17).

(b) *In addition, suppose that for some z and some sequence $\varepsilon_k \searrow 0$,*

$$z \in \mathcal{G}_2^{\varepsilon_k} \qquad \text{for all sufficiently large } k. \quad (4.21)$$

Then, after possibly passing to a further subsequence, there exist points $q_i(z), i = 1, \dots, n$ such that

$$J_x v_\varepsilon(\cdot, z) \to \pi \sum_{i=1}^n \delta_{q_i(z)} \qquad \text{in } W^{-1,1}(B(R)), \text{ for all } R > 0, \quad (4.22)$$

$$\liminf_{k\to\infty} \xi_{\varepsilon_k}(z) \ge -\pi \sum_{i \ne j} \log|q_i(z) - q_j(z)|, \quad (4.23)$$

and, setting $c_4 := \max_i |q_i^0|$, where $q_1^0, \dots, q_n^0$ appear in (1.24),

$$|q_i(z)| \le c_4 + \Big(\frac{c_3\, L}{\pi}\Big)^{1/2} \qquad \text{for all } i. \quad (4.24)$$

In our notation, as with $q_i(z)$ above, we will consistently fail to indicate the dependence of various limiting quantities on the subsequence that generates them.

The uniform lower bound (4.20), needed for our Γ-limit lower bound, is proved using a compactness argument, and the proof of the lemma begins by assembling the necessary compactness assertions.

Proof. **Step 1.** Recall that if $z \in \mathcal{G}_2^\varepsilon$, then $u_\varepsilon(\cdot, z)$ satisfies the hypotheses of Lemma 2. There thus exist points $\{p_i^\varepsilon(z)\}_{i=1}^n$ that satisfy (3.6), (3.7), (3.8). To express these conclusions in terms of v_ε, note that by rescaling, one has

$$\|J_x v_\varepsilon(\cdot, z) - \pi \sum_{i=1}^n \delta_{\ell_\varepsilon^{-1} p_i^\varepsilon(z)}\|_{F(\omega_\varepsilon)} \le \ell_\varepsilon^{-1} \|J_x u_\varepsilon(\cdot, z) - \pi \sum_{i=1}^n \delta_{p_i^\varepsilon(z)}\|_{F(\omega)}.$$

This is straightforward to check from the definition (1.22) of the flat norm. Thus (3.6), (3.7) imply that for $q_i^\varepsilon(z) := p_i^\varepsilon(z)/\ell_\varepsilon$,

$$\|J_x v_\varepsilon(\cdot, z) - \pi \sum_{i=1}^n \delta_{q_i^\varepsilon(z)}\|_{F(\omega_\varepsilon)} \le \ell_\varepsilon^{-1} \varepsilon^a, \tag{4.25}$$

$$|q_i^\varepsilon(z) - q_j^\varepsilon(z)| \ge \ell_\varepsilon^{-1} \varepsilon^b \qquad \text{for } i \ne j. \tag{4.26}$$

Now consider any sequence (ε_k) tending to 0 and a sequence of points $(z^{\varepsilon_k}) \subset \mathcal{G}_2^{\varepsilon_k}$ and $z^{\varepsilon_k} \to z$, for some $z \in [0, L]$. We may pass to a further subsequence and relabel if necessary to find some integer $n(z) \le n$, and points $q_i(z) \in \mathbb{R}^2$ for $i = 1, \ldots, n(z)$, such that

$$q_i^{\varepsilon_k}(z^{\varepsilon_k}) \to q_i(z) \text{ for } 1 \le i \le n(z), \qquad |q_i^{\varepsilon_k}| \to \infty \text{ for } n(z) < i \le n. \tag{4.27}$$

Then it follows easily from (4.25) and standard properties of the $W^{-1,1}$ norm (and is easily checked from the definition (1.22)) that

$$J_x v_{\varepsilon_k}(\cdot, z^{\varepsilon_k}) \to \pi \sum_{i=1}^{n(z)} \delta_{q_i(z)} \text{ in } W^{-1,1}(B(R)), \quad \text{for every } R > 0. \tag{4.28}$$

Step 2. We claim that under these conditions, $n(z) = n$, and (4.24) holds.

We first assume that $z \ne z_0$, where z_0 is the height appearing in (1.24), (1.25). For concreteness we assume that $z < z_0$; the other case is identical. We want to apply Lemma 10 along the subsequence fixed above, with $z_1^{\varepsilon_k} = z^{\varepsilon_k}$ and $z_2^{\varepsilon_k} = z_0$. To verify the hypotheses of the lemma, we first rescale (1.24) to find that

$$\|J_x v_\varepsilon(\cdot, z_0) - \pi \sum_{i=1}^n \delta_{h_\varepsilon q_i^0/\ell_\varepsilon}\|_{W^{-1,1}(\omega_\varepsilon)} = o(\frac{h_\varepsilon}{\ell_\varepsilon}) = o(1)$$

as $\varepsilon \to 0$. Since $\ell_\varepsilon \ge h_\varepsilon$ by construction, we may assume after passing to a subsequence that $h_\varepsilon/\ell_\varepsilon \to \alpha$ as $\varepsilon \to 0$, for some $\alpha \in [0, 1]$. Then

$$J_x v_\varepsilon(\cdot, z_0) \to \pi \sum_{i=1}^n \delta_{\alpha q_i^0} \quad \text{in } W^{-1,1}(B(R)) \text{ for all } R > 0. \tag{4.29}$$

This is one of the hypotheses on $(z_2^{\varepsilon_k})$ in Lemma 10. The other hypothesis follows directly from (1.25). The same hypotheses are satisfied by $(z_1^{\varepsilon_k})$, by (4.28) and the fact that $z_1^{\varepsilon_k} \in \mathcal{G}_2^{\varepsilon_k}$. We may therefore apply this lemma to find that $n(z) = n$ if $z \neq z_0$. In addition, since $\mu([0,L]) \leq c_3$ due to (4.4) and (4.8), we deduce from (4.11) that for some $\sigma \in S^n$,

$$\sum_{i=1}^{n} |q_i(z) - \alpha q^0_{\sigma(i)}|^2 \leq \frac{c_3}{\pi}|z - z_0| \leq c_3\frac{L}{\pi}$$

which implies that $\max |q_i(z)| \leq c_4 + \sqrt{c_3 L/\pi}$ if $z \neq z_0$.

If $z = z_0$, we apply Lemma 10 with $z^\varepsilon = z^1_\varepsilon$, and z^2_ε any sequence in $\mathcal{G}_k^\varepsilon$ such that $z_2^\varepsilon \to z_2 \neq z_0$, along some subsequence such that $J_x v_\varepsilon(\cdot, z_z^\varepsilon)$ converges to a limit as in (4.9). Since we may take z_2 as close as we like to z_0, we may repeat the arguments from above to easily conclude that (4.24) holds in this case as well.

Step 3. We claim that there exists $\varepsilon_0 > 0$ such that

$$\max_i |q_i^\varepsilon(z)| < c_4 + 2\sqrt{c_3 L/\pi} \qquad \text{for all } 0 < \varepsilon < \varepsilon_0 \text{ and } z \in \mathcal{G}_2^\varepsilon. \tag{4.30}$$

Assume toward a contradiction that (4.30) fails. Then we may find a sequence (ε_k) tending to 0 and a sequence of points $(z^{\varepsilon_k}) \subset \mathcal{G}_2^{\varepsilon_k}$ such that

$$\max_i |q_i^{\varepsilon_k}(z^{\varepsilon_k})| \geq c_4 + 2\sqrt{c_3 L/\pi} \qquad \text{for all } k. \tag{4.31}$$

We may pass to a subsequence such that z_{ε_k} converges to a limit z, and in addition (4.27) and (4.28) hold, with $n(z) = n$. Clearly, (4.31) implies that $\max_i |q_i(z)| \geq c_4 + 2\sqrt{c_3 L/\pi}$, which is impossible in view of (4.24). This contradiction completes the proof.

Step 4. For $z \in \mathcal{G}_2^\varepsilon$, estimate (3.8) states that

$$\int_\omega e_\varepsilon^{2d}(u_\varepsilon(x,z))dx \geq n(\pi|\log\varepsilon| + \gamma) + W_\omega(p_1^\varepsilon(z), \ldots, p_n^\varepsilon(z)) - C\varepsilon^{(a-b)/2},$$

where according to (2.2),

$$W_\omega(p_1^\varepsilon(z), \ldots, p_n^\varepsilon(z)) = -\pi\left(\sum_{i\neq j} \log|p_i^\varepsilon(z) - p_j^\varepsilon(z)| + \sum_{i,j} H_\omega(p_i^\varepsilon(z), p_j^\varepsilon(z))\right).$$

We also know from (4.30) that $|p_i^\varepsilon(z)| \leq C\ell_\varepsilon$ for $0 < \varepsilon < \varepsilon_0$, with a constant independent of z. Since H_ω is smooth in the interior of $\omega \times \omega$, it follows that $H_\omega(p_i^\varepsilon(z), p_j^\varepsilon(z)) = H_\omega(0,0) + O(\ell_\varepsilon)$ for every i,j. Writing $p_i^\varepsilon = \ell_\varepsilon q_i^\varepsilon$ as in Step 1, we deduce that

$$\xi_\varepsilon(z) \geq -\pi \sum_{i\neq j} \log|q_i^\varepsilon(z) - q_j^\varepsilon(z)| - C\ell_\varepsilon - C\varepsilon^{(a-b)/2}. \tag{4.32}$$

Conclusion (4.20) follows immediately from this estimate together with (4.30).

Now assume that z satisfies (4.21). From Steps 1 and 2 we know that we may find a subsequence such that (4.28) holds, with $n(z) = n$. This is (4.22). In addition, it follows immediately from (4.32) that (4.23) is satisfied as well. We have already verified in Step 2 above that (4.24) holds, so this completes the proof. □

4.3 Identifying the filaments. We next use the basic estimate (4.11) to choose a subsequence for which the vorticities converge at *a.e.* height. To this end, it is convenient to introduce some notation. Let X denote the quotient space $(\mathbb{R}^2)^n/S^n$. Thus, points in X consist of equivalence classes in $(\mathbb{R}^2)^n$, where $p \sim p'$ if there exists some permutation σ such that $p_i = p'_{\sigma(i)}$ for all $i = 1, \dots, n$. The equivalence class containing p will be denoted $[p]$. The natural notion of distance in X is

$$d_X([p],[p'])^2 = \min_{\sigma \in S^n} \sum_{i=1}^{n} |p_i - p'_{\sigma(i)}|^2, \qquad \text{for } p, p' \in (\mathbb{R}^2)^n. \tag{4.33}$$

We will write

$$\delta_{[p]} := \sum_{i=1}^{n} \delta_{p_i}. \tag{4.34}$$

Note that this is well-defined in the sense that the measure on the right-hand side depends only on the equivalence class. Similarly, the function

$$[q] \in X \mapsto -\pi \sum_{i \neq j} \log |q_i(z) - q_j(z)|$$

is also well-defined, since the right-hand side is invariant under permutations of the indices. We remark that we adopt the convention that $-\log(0) = +\infty$.

In the following lemma, we identify the limiting vorticity by a map $(0, L) \to X$, which in effect means that at this stage we do not worry about labelling the points.

LEMMA 12. *Assume that $(u_\varepsilon) \subset H^1(\Omega; \mathbb{C})$ satisfies (1.23)–(1.25) and (1.27).*

Then there exists a set $H_G \subset (0, L)$ of full measure, a subsequence (ε_k), and a function $[q] : (0, L) \to X$, such that for every $z \in H_G$ (as well as for $z = z_0$, from (1.24)),

$$J_x v_{\varepsilon_k}(\cdot, z) \to \pi \delta_{[q](z)} \quad \text{in } W^{-1,1}(B(R)) \quad \text{for every } R > 0 \tag{4.35}$$

and

$$\liminf_{\varepsilon \to 0} \ \xi_\varepsilon(z) \geq -\pi \sum_{i \neq j} \log |q_i(z) - q_j(z)|. \tag{4.36}$$

Moreover, for every $z < z'$,

$$\pi \frac{d_X([q](z), [q](z'))^2}{|z - z'|} \leq \mu((z, z')). \tag{4.37}$$

Proof. Recall from Lemma 6 that under our hypotheses, $|\mathcal{B}_2^\varepsilon| = o(1)$ as $\varepsilon \to 0$. We may therefore choose a subsequence (ε_k) such that $\sum |\mathcal{B}_2^{\varepsilon_k}| < \infty$. (Note, we will shortly pass to further subsequences.) Then by the Borel–Cantelli Lemma, the set

$$H_G := \bigcup_{\ell=1}^{\infty} \bigcap_{k=\ell}^{\infty} \mathcal{G}_2^{\varepsilon_k}$$

has full measure in $(0, L)$.

In view of Lemma 11, we may now choose subsequences and invoke a diagonal argument to find a set $H_G^0 = \{z_i\}_{i=0}^{\infty} \subset H_G$, dense in $(0, L)$, such that (4.22) and (4.23) hold for every z_i. Applying Lemma 10 along this subsequence with $z_1^\varepsilon = z_i, z_2^\varepsilon = z_j$ for pairs of points z_i, z_j in H_G^0, we find that

$$\pi \frac{d_X([q(z_i)], [q(z_j)])^2}{|z_j - z_i|} \leq \mu([z_i, z_j]) \leq \mu([0, L]) \qquad \text{whenever } z_i < z_j. \tag{4.38}$$

In particular, this states that the map

$$z_i \in H_G^0 \mapsto [q(z_i)] \in X$$

is Hölder continuous. It thus has a unique extension to a continuous map, say $[q] : (0, L) \to X$, such that $[q](z_i) = [q(z_i)]$ for all i. For any $z < z'$, we may fix sequences $(z_j), (z_j') \in H_G$ such that $z_j \searrow z, z_j' \nearrow z'$. Then

$$\pi\, d_X([q](z), [q](z'))^2 = \pi \lim_{j\to\infty} d_X([q](z_j), [q](z_j'))^2 \leq |z - z'|\mu((z, z')).$$

This proves (4.37).

We finally claim that *without passing to any further subsequences*, (4.35) holds for *every* $z \in H_G$. If not, then we could find some $z \in H_G$ and a further subsequence of the chosen subsequence (ε_k) such that (4.22) holds but (4.35) fails. Then we see from (4.11) that

$$\frac{\pi}{2} d_X([q(z_i)], [q(z)])^2 \;\leq\; |z_i - z| \;\mu([0, L])$$

for every $z_i \in H_G^0$. Since H_G^0 is dense and $[q(z_i)] = [q](z_i) \to q[z]$ as $z_i \to z$, it follows that $[q(z)] = [q](z)$, a contradiction. This proves the claim. □

We next show that $[q] : (0, L) \to X = (\mathbb{R}^2)^n / S^n$ admits a suitable lifting to a map $(0, L) \to (\mathbb{R}^2)^n$. This will complete the identification of the limiting vortex filaments.

LEMMA 13. *Let* $[q] : (0, L) \to X$ *satisfy* (4.37) *for some measure* μ *on* $(0, L)$.

Then there exists a function $f \in H^1((0,L),(\mathbb{R}^2)^n)$ *such that* $[f(z)] = [q](z)$ *for all* $z \in (0,L)$*. Moreover, whenever* $0 \le z < z' \le L$*,*

$$\pi \sum_{j=1}^{n} \frac{|f_j(z) - f_j(z')|^2}{|z-z'|} \le \mu((z,z'))\,, \qquad \textit{and} \tag{4.39}$$

$$\pi \int_z^{z'} \sum_i |f_i'|^2\, dz \le \mu((z,z'))\,. \tag{4.40}$$

Proof. We will write

$$\mathcal{D}^i := \{kL/2^i \ : k = 1,\dots,2^i-1\}, \qquad \mathcal{D} := \cup_i \mathcal{D}^i.$$

We refer to elements of $\mathcal{D}$ as dyadic heights. For every i, we can choose $f^i : \mathcal{D}^i \to (\mathbb{R}^2)^n$ such that

$$[f^i(z)] = [q](z) \quad \text{for all } z \in \mathcal{D}^i, \tag{4.41}$$

and

$$\sum_{j=1}^{n} |f_j^i(z) - f_j^i(z')|^2 = d_X([q](z),[q](z'))^2, \qquad \text{if } z,z' \in \mathcal{D}^i,\ |z-z'| = 2^{-i}L. \tag{4.42}$$

Now fix i_0, and for $l > i_0$, and let $f^{i_0,l}$ denote the restriction of f^l to $\mathcal{D}^{i_0}$. For every i, it is clear that $f^{i_0,l}$ satisfies (4.41). For every $[q] \in X$, there are at most $n!$ points $p \in (\mathbb{R}^2)^n$ such that $[p] = [q]$, and hence there are only finitely many maps $\mathcal{D}^i \to (\mathbb{R}^2)^n$ that satisfy (4.41). By the pigeonhole principle, we can thus find a subsequence $l_m \to \infty$ along which $f^{i_0,l_m}(z)$ is independent of l_m, for all large enough m, for every $z \in \mathcal{D}^{i_0}$.

We then define $f(z)$ for $z \in \mathcal{D}^{i_0}$ by requiring that

$$f(z) = f^{i_0,l_m}(z) \quad \text{for all sufficiently large } m. \tag{4.43}$$

Now fix $z,z' \in \mathcal{D}^{i_0}$ and some l from the subsequence (l_m) such that (4.43) holds. We assume that $z < z'$, and we let $z_s := z + 2^{-l}s$. Then Jensen's inequality implies that

$$\begin{aligned}
\pi \sum_{j=1}^{n} \frac{|f_j(z) - f_j(z')|^2}{z'-z} &= \pi \sum_{j=1}^{n} \frac{1}{z'-z} \left| \sum_{s=1}^{2^l(z'-z)} f_j^l(z_s) - f_j^l(z_{s-1}) \right|^2 \\
&\le \pi \sum_{j=1}^{n} \sum_{s=1}^{2^l(z'-z)} 2^l \left| f_j^l(z_s) - f_j^l(z_{s-1}) \right|^2 \\
&\overset{(4.42)}{=} \pi \sum_{s=1}^{2^l(z'-z)} 2^l\, d_X(\,[q](z_s),[q](z_{s-1})\,)^2 \\
&\overset{(4.37)}{\le} \sum_{s=1}^{2^l(z'-z)} \mu((z_{s-1},z_s)\,) \le \mu(\,(z,z')\,).
\end{aligned} \tag{4.44}$$

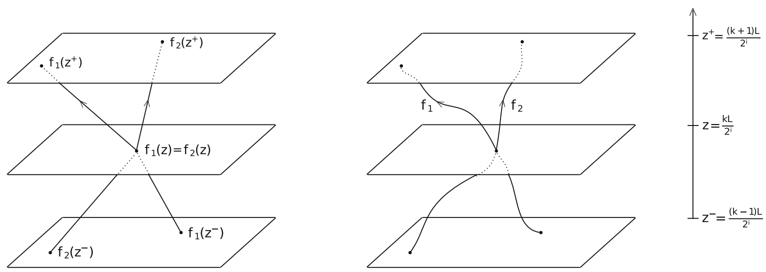

Figure 4: 2^{-i}-scale approx. of f with labels and its Lipschitz continuous limit (right). This example of f admits two consistent labellings

When invoking (4.42), we have used the fact that every z_s belongs to D^l.

Now let $i_1 > i_0$ be some element of the chosen subsequence, for example l_1, and let f^{i_1,l_m} be the restriction of f^{l_m} to $\mathcal{D}^{i_1}$. Arguing as above, we find a further subsequence, still denoted (l_m), along which f^{i_1,l_m} is eventually independent of m, and we define $f(z)$ for $z \in \mathcal{D}^{i_1}$ as in (4.43). Note that this is consistent with the earlier definition of points in $\mathcal{D}^{i_0} \subset \mathcal{D}^{i_1}$, and that (4.44) holds exactly as before.

Continuing in this way, we define $f(z)$ for every $z \in \mathcal{D}$, such that $[f(z)] = [q](z)$, and (4.39) holds for every pair of points in $\mathcal{D}$. In particular, it follows that f is continuous, as a map $\mathcal{D} \to (\mathbb{R}^2)^n$, and hence has a unique continuous extension to a function, still denoted f, mapping $(0, L) \to (\mathbb{R}^2)^n$. It then easily follows that (4.39) holds and that $[f(z)] = [q](z)$ for all $z \in (0, L)$.

Finally, we may approximate f by a sequence of maps $f^\ell : (0, L) \to (\mathbb{R}^2)^n$ that agree with f at a finite number of points and interpolate linearly between these. It follows from (4.39) that such maps belong to $H^1((0, L); (\mathbb{R}^2)^n)$, with $\pi \sum \|(f_i^\ell)'\|_{L^2}^2 \le \mu((0, L))$. Then standard arguments imply that

$$(f_i^\ell)' \rightharpoonup f_i' \text{ weakly in } L^2((0, L)), \text{ for every } i,$$

and as a result, further standard arguments imply that

$$\pi \sum_{i=1}^n \int_0^L |f_i'(z)|^2 \, dz \le \liminf_\ell \pi \sum_{i=1}^n \int_0^L |f_i^{\ell\prime}(z)|^2 \, dz \le \mu((0, L)).$$

This proves that $f \in H^1((0, L); (\mathbb{R}^2)^n)$. The same argument may be carried out on any given subinterval $(z, z') \subset (0, L)$, which proves that (4.40) holds. □

4.4 Conclusion of the proof of compactness and lower bound.

LEMMA 14. *Along the subsequence (ε_k) found in Lemma 12,*

$$\liminf \int_0^L \xi_\varepsilon(z) \, dz \ge -\pi \int_0^L \sum_{i \neq j} \log |f_i(z) - f_j(z)| \, dz. \tag{4.45}$$

Proof. We have already proved in (4.36) that

$$\liminf_{\varepsilon\to 0} \xi_\varepsilon(z) \geq -\pi \sum_{i\neq j} \log|f_i(z) - f_j(z)|$$

for $a.e.\ z$. Moreover, it follows from Proposition 1 that there exists some $\varepsilon_0 > 0$ such that $\xi_\varepsilon(z) \geq 0$ for all $z \in \mathcal{B}_1^\varepsilon$, when $0 < \varepsilon < \varepsilon_0$. And we have shown in Lemma 11 that, taking ε_0 smaller if necessary, $\xi_\varepsilon(z) \geq -C$ for all $z \in \mathcal{G}_1^\varepsilon$, for $\varepsilon < \varepsilon_0$. Thus the conclusion follows from Fatou's Lemma. □

We can now present the

Conclusion of the proof of part (a) of Theorem 3. **Step 1**. We first claim that if the parameter c_3 in the definition of ℓ_ε is taken to be large enough, then

$$\ell_\varepsilon = h_\varepsilon \qquad \text{for all small } \varepsilon. \tag{4.46}$$

If this fails, then we see from the definition (4.1) of ℓ_ε that

$$\int_\Omega |\partial_z u_\varepsilon|^2 dx\, dz = c_3 (\frac{\ell_\varepsilon}{h_\varepsilon})^2.$$

Then the definitions of G_ε and ξ_ε (see (1.26), (4.19)) imply that

$$G_\varepsilon(u_\varepsilon) = \int_0^L \xi_\varepsilon(z)\, dz - n(n-1)\pi L \log(\frac{\ell_\varepsilon}{h_\varepsilon}) + \frac{c_3}{2}(\frac{\ell_\varepsilon}{h_\varepsilon})^2. \tag{4.47}$$

Recalling from (1.27) that $G_\varepsilon(u_\varepsilon) \leq c_1 < \infty$ for all ε, we deduce from (4.45) that

$$\limsup_{\varepsilon\to 0} \left[-n(n-1)\pi L \log(\frac{\ell_\varepsilon}{h_\varepsilon}) + \frac{c_3}{2}(\frac{\ell_\varepsilon}{h_\varepsilon})^2 \right] \leq c_1 + \pi \int_0^L \sum_{i\neq j} \log|f_i(z) - f_j(z)|\, dz.$$

If $c_3 \geq n(n-1)\pi L$, then the function $s \mapsto -n(n-1)\pi L \log s + \frac{c_3}{2} s^2$ is increasing on $[1,\infty)$. Since $\frac{\ell_\varepsilon}{h_\varepsilon} \geq 1$, it follows that the left-hand side of the above inequality is greater than or equal to $\frac{c_3}{2}$. On the other hand, it follows from (4.24) that $|f_i(z)| \leq c_4 + \sqrt{Lc_3/\pi}$ for all i and z. Putting these together, we obtain

$$\frac{1}{2} c_3 \leq c_1 + n(n-1)L\pi \log(2c_4 + 2\sqrt{Lc_3/\pi}).$$

We now fix c_3 large enough that this yields a contradiction; then (4.46) follows.

Step 2. Since $\ell_\varepsilon = h_\varepsilon$, and recalling (4.4) and (4.8), we can rewrite

$$G_\varepsilon = \int_0^L \xi_\varepsilon(z)\, dz + \frac{1}{2}\mu_\varepsilon([0,L]).$$

Then the Γ-limit lower bound (1.32) follows immediately from (4.45) and (4.40).

Step 3. The claim that $[f(z_0)] = [q^0]$ is a consequence of the proof of Lemma 12 (which in particular shows that (4.35) holds for $z = z_0$), assumption (1.24), and Lemma 13.

It therefore only remains to improve the convergence already established in Lemmas 12, 13 above, by showing that $J_x v_\varepsilon \to \delta_{[f(\cdot)]}$ in $W^{-1,1}(B(R) \times (0,L))$ for every $R > 0$. First, recall from (4.7) that the family of 1-currents $(\star J v_\varepsilon)$ is precompact in $\cup_R W^{-1,1}(B(R) \times (0,L))$. Then the relationship (1.20) between $\star J v_\varepsilon$ and $J_x v_\varepsilon$ implies that $(J_x v_\varepsilon)$ is also precompact in the same topology. So we only need to identify the limit. To do this, fix $\phi \in W_c^{1,\infty}(\mathbb{R}^2 \times (0,L))$, and let $\phi^\varepsilon(x,z) := \phi(\frac{x}{h_\varepsilon}, z)$ and

$$\Phi_\varepsilon(z) := \int_{\omega_\varepsilon} \phi(x,z) J_x v_\varepsilon(x,z)\, dx = \int_\omega \phi^\varepsilon(x,z) J_x u_\varepsilon(x,z)\, dx.$$

It follows from Lemma 12 that $\Phi_\varepsilon(z) \to \Phi(z) = \pi \int \phi(x,z)\delta_{[f(z)]}$ for *a.e.* z, along the chosen subsequence. In addition, Theorem 2.1 [JS02] implies that for every $z \in (0,L)$,

$$|\Phi_\varepsilon(z)| \leq C(\omega)(\|\phi^\varepsilon\|_{L^\infty} + \varepsilon^\alpha \|\phi^\varepsilon\|_{W^{1,\infty}}) \left(1 + \int_\omega \frac{e_\varepsilon^{2d}(u_\varepsilon(\cdot,z))}{|\log\varepsilon|} dx\right)$$

for some $\alpha > 0$. We then see from Lemma 5 that for any measurable $S \subset (0,L)$,

$$\int_S |\Phi_\varepsilon(z)|\, dz \leq C\|\phi\|_{W^{1,\infty}}(|S| + o(1)) \qquad \text{as } \varepsilon \to 0,$$

where the $o(1)$ term is independent of S. Thus the sequence (Φ_ε) is asymptotically uniformly integrable, so the Vitali Convergence Theorem implies that in fact $\Phi_\varepsilon \to \Phi$ in L^1. This implies that

$$\int_{\Omega_\varepsilon} \phi\ J_x v_\varepsilon\ dx\, dz = \int_0^L \Phi_\varepsilon(z)\, dz \to \int_0^L \Phi(z)\, dz = \int \phi\delta_{[f(\cdot)]}. \qquad \square$$

5 Upper bound and improved compactness

In this section we prove the remaining results of Theorem 3.

5.1 Improved compactness for "tight" sequences We first prove that sequences which attain the Γ-limit lower bound are compact in a stronger sense than previously established in (1.30). The results of the previous section are all available, as we continue to assume hypotheses (1.27), together with either (1.23)–(1.25) or (1.29), although some of these are by now redundant.

The proof requires a measure-theoretic lemma that is proved at the end of this subsection.

Proof of part (*c*) *of Theorem 3*. Let (u_ε) be a sequence satisfying (1.30) and (1.33). Recall (see (4.7), (4.8)) that, up to subsequence, there exists an integer multiplicity rectifiable 1-current J such that $\frac{1}{\pi}\star Jv_\varepsilon \to J$ in $W^{-1,1}(B(R)\times(0,L))$ for every $R>0$.

We have also shown (see (4.19), (4.45), and recall that in fact $\ell_\varepsilon = h_\varepsilon$) that

$$\int_0^L\int_\omega e_\varepsilon^{2d}(u_\varepsilon)dx\,dz \geq n\pi L\left|\log\varepsilon\right| + \pi n(n-1)L\left|\log h_\varepsilon\right| \\ -\pi\sum_{i\neq j}\int_0^L \log\left|f^i(z)-f^j(z)\right|dz + \kappa_n(\Omega) + o_\varepsilon(1), \tag{5.1}$$

where $\kappa_n(\Omega)$ was defined in (2.3). Finally, recall that there exists a measure μ on $[0,L]$ such that $\mu_\varepsilon \rightharpoonup \mu$ weakly as measures, where

$$\mu_\varepsilon(A) := \int_{\omega_\varepsilon\times A}\frac{\left|\partial_z v_\varepsilon(x,z)\right|^2}{\left|\log\varepsilon\right|}dx\,dz,$$

and that (see (4.40)) for all measurable $A\subseteq(0,L)$

$$\mu(A) \geq \pi\int_A\left|f'(z)\right|^2dz, \qquad \text{where } |f'|^2 := \sum_j |f_j'|^2. \tag{5.2}$$

But then (1.33) implies that

$$\begin{array}{c}\int_0^L\int_\omega e^{2d}(u_\varepsilon)dx\,dz - n\pi L\left|\log\varepsilon\right| \\ -\pi n(n-1)L\left|\log h_\varepsilon\right|\end{array} \longrightarrow -\pi\sum_{i\neq j}\int_0^L\log\left|f^i(z)-f^j(z)\right|dz + \kappa_n(\Omega),$$

as $\varepsilon\to 0$; and

$$\mu((0,L)) = \pi\int_0^L\left|f'\right|^2dz. \tag{5.3}$$

Since $|f'|^2$ is nonnegative, we deduce from (5.2) and (5.3) that μ is absolutely continuous with respect to Lebesgue measure with density

$$d\mu = \pi\left|f'\right|^2dz.$$

In particular μ has no atoms.

Our goal is to show that

$$S := J - \sum_{i=1}^n \Gamma_{f_i} = 0.$$

We know that for every $\phi\in C_c^\infty(\mathbb{R}^2\times(0,L))$,

$$\frac{1}{\pi}\star Jv_\varepsilon(\phi\,dz) = \int\phi J_x v_\varepsilon\,dx\,dz \to \int\phi\delta_{[f(\cdot)]} = \sum_{i=1}^n\Gamma_{f_i}(\phi\,dz),$$

and it follows that $S(\phi\, dz) = 0$ for all such ϕ. Then it follows from Lemma 15 below that S may be written as a linear combination of currents

$$S = \sum_{j\in I} T_{\gamma_j}$$

where each T_{γ_j} is supported in a horizontal plane $\{z = z_j\}$. Assume toward a contradiction that T_{γ_j} is nonzero for $j = 1$, say, and let $\varepsilon_0, \delta_0 > 0$ be two small numbers to be chosen later. We know from (3.14), (4.5) and a change of variables that

$$\lim_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|} \int_{z_1-\delta_0}^{z_1+\delta_0} \int_{\omega_\varepsilon} e^{2d}_{\varepsilon'}(v_\varepsilon) dx\, dz = 2n\pi\delta_0$$

for $\varepsilon' = \varepsilon/h_\varepsilon$. At the same time we know that

$$\frac{1}{2}\mu_\varepsilon((z_1 - \delta_0, z_1 + \delta_0)) \to \frac{\pi}{2} \int_{z_1-\delta_0}^{z_1+\delta_0} |f'|^2\, dz.$$

We can choose δ_0 small enough so that

$$\max\left\{2n\pi\delta_0, \frac{\pi}{2}\int_{z_1-\delta_0}^{z_1+\delta_0} |f'|^2\, dz\right\} < \frac{\varepsilon_0}{2}.$$

In light of this we have

$$\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|} \int_{z_1-\delta_0}^{z_1+\delta_0} \int_{\omega_\varepsilon} e_{\varepsilon'}(v_\varepsilon) dx\, dz < \varepsilon_0. \tag{5.4}$$

On the other hand, for every $\delta_0 > 0$, the mass of the limiting current J in $B(R) \times (z_1 - \delta_0, z_1 + \delta_0)$ is at least $M(T_{\gamma_1})$, for R large enough, depending on the support of T_{γ_1}. Thus, by applying Theorem 4 in $B(R) \times (z_1 - \delta_0, z_1 + \delta_0)$ for this choice of R, we find that

$$\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|} \int_{z_1-\delta_0}^{z_1+\delta_0} \int_{\omega_\varepsilon} e_{\varepsilon'}(v_\varepsilon) dx\, dz \geq \pi M(T_{\gamma_1}). \tag{5.5}$$

Choosing $\varepsilon_0 < \pi M(T_{\gamma_1})$ makes (5.4) and (5.5) incompatible. Hence $S = 0$, which completes the proof of (1.34). □

LEMMA 15. *Let S be an integer multiplicity rectifiable 1-current in $\mathbb{R}^2 \times (0, L)$ such that $M(S) < \infty$ and $\partial S = 0$ in $\mathbb{R}^2 \times (0, L)$. Assume in addition that $S(\phi\, dz) = 0$ for all $\phi \in C_c^\infty((\mathbb{R}^2 \times (0, L))$.*

Then S may be written as a sum of 1-currents

$$S = \sum_{i\in I} T_{\gamma_i}$$

where each T_{γ_i} is supported in a hyperplane $\{z = a_i\}$ for some $a_i \in (0, L)$.

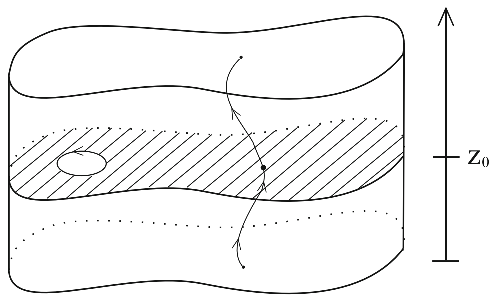

Figure 5: A current with an indecomposable component supported at $\{z = z_0\}$

Proof. We first claim that $\partial S = 0$ in $\mathbb{R}^3$. To see this, fix any $\phi \in C_c^\infty(\mathbb{R}^3)$. For any $\chi \in C_c^\infty((0,L))$ (which we identify in the natural way with a function on $\mathbb{R}^2 \times (0,L)$ depending only on the z variable) we have

$$S(\chi d\phi) = S(d(\chi\phi)) - S(\phi d\chi) = \partial S(\chi\phi) - S(\phi\chi'\, dz) = 0.$$

We can thus take a sequence of functions χ_k such that $\chi_k \nearrow \mathbf{1}_{(0,L)}$ to conclude that $S(d\phi) = 0$. Since ϕ was arbitrary, it follows that $\partial S = 0$ in $\mathbb{R}^3$ as claimed.

The remainder of the proof is classical; we recall the arguments for the convenience of the reader. Using the decomposition theorem for 1-dimensional integer multiplicity currents, see [Fed69] 4.2.25 (to which we refer for the definition of indecomposable) we may write S as a sum of indecomposable 1-currents, say $S = \sum_{i\in I} T_{\gamma_i}$, with $\sum_I M(T_{\gamma_i}) = M(S)$ and $\sum_I M(\partial T_{\gamma_i}) = M(\partial S) = 0$. It follows that $\partial T_{\gamma_i} = 0$ and that $T_{\gamma_i}(\phi\, dz) = 0$ for every i and all $\phi \in C_c^\infty$. By basic properties of slicing of currents (see for example [Fed69] 4.3.2) this implies that $\langle T_{\gamma_i}, \zeta, z\rangle = 0$ for *a.e.* $z \in (0,L)$, where $\zeta : \mathbb{R}^2 \times (0,L) \to (0,L)$ is the projection onto the vertical axis, $\zeta(x,z) = z$. Then Solomon's Separation Lemma [Sol84] implies that every T_{γ_i} is supported in a level set of ζ.

We remark that Solomon's Lemma applies to indecomposable currents T in $\mathbb{R}^n$ such that $M(T)+M(\partial T) < \infty$; it is for this reason that we verified that $\partial S = 0 < \infty$ in $\mathbb{R}^3$. □

5.2 Constructing a recovery sequence. In this part, given $f \in H^1((0,L);(\mathbb{R}^2)^n)$ we build sequences of maps $(u_\varepsilon) \subset H^1(\Omega;\mathbb{C})$ whose Ginzburg–Landau energy recovers the nearly parallel vortex filaments energy $G_0(f)$. Part (b) of Theorem 3 follows from

PROPOSITION 2. *(Recovery Sequence). Let* $f \in H^1\left((0,L);(\mathbb{R}^2)^n\right)$. *Then there exists a sequence* $(u_\varepsilon) \subset H^1(\Omega;\mathbb{C})$ *such that*

(a) $\|\star J v_\varepsilon - \pi\sum_{i=1}^n \Gamma_{f_i}\|_{F(B(R)\times(0,L))} \to 0$ *for every* $R > 0$, *as* $\varepsilon \to 0$,
(b) $\lim_{\varepsilon\to 0} G_\varepsilon(u_e) \le G_0(f)$.

As it is usual in this kind of construction, we define a sequence of trial maps (u_ε) based on canonical harmonic maps with prescribed singularities in ω, introduced in [BBH94], section I.3. This provides us with the right estimates for e_ε^{2d}. Then, we show that the L^2-norm of f' is well approximated by $\pi \int_\Omega |\partial_z u_\varepsilon|^2 \, dx \, dz$. Finally, the task is to show that a sequence defined in this way satisfies (a).

First note that to prove Proposition 2, it suffices to prove it for smooth f satisfying

$$\inf_{\substack{z \in (0,L) \\ i \neq j \in \{1,\dots,n\}}} \left| f^i(z) - f^j(z) \right| > 0. \tag{5.6}$$

In fact, once we prove Proposition 2 for such configurations, a density (and diagonal) argument yields the result.

We will write $f_\varepsilon = (f_{\varepsilon,1}, \dots, f_{\varepsilon,n}) = h_\varepsilon f$. We will take our trial functions to have the form

$$u_\varepsilon(x,z) := \prod_{j=1}^n \left[e^{i\beta(x, f_{\varepsilon,j}(z))} U_\varepsilon(x - f_{\varepsilon,j}(z)) \right] \tag{5.7}$$

for certain functions β, U_ε that we now describe. First, if we write $x \in \mathbb{R}^2$ in polar coordinates $(r\cos\theta, r\sin\theta)$, then we define

$$U_\varepsilon(x) = e^{i\theta} \qquad \text{if } |x| \geq \sqrt{\varepsilon}, \tag{5.8}$$

and in $B(\sqrt{\varepsilon})$, we choose U_ε to minimize $\int_{B(\sqrt{\varepsilon})} e_\varepsilon^{2d}(u)$ among all functions $u : B(\sqrt{\varepsilon}) \to \mathbb{C}$ such that $u = e^{i\theta}$ on $\partial B(\sqrt{\varepsilon})$. Thus, using notation introduced in (2.5),

$$\int_{B(\sqrt{\varepsilon})} e_\varepsilon^{2d}(U_\varepsilon) \, dx \; = I(\sqrt{\varepsilon}, \varepsilon). \tag{5.9}$$

The intermediate length scale $\sqrt{\varepsilon}$ is chosen rather arbitrarily—we just need some radius r_ε such that $\varepsilon \ll r_\varepsilon \ll h_\varepsilon$. Theorem 11.2 in [PR2000] implies that U_ε has the form

$$U_\varepsilon = \rho_\varepsilon(r) e^{i\theta} \qquad \text{with } 0 \leq \rho_\varepsilon \leq 1 \text{ for all } r > 0$$

for all sufficiently small $\varepsilon > 0$. Next, $\beta(x,y)$ is chosen so that for every $y \in \omega$,

$$\begin{cases} \Delta_x \beta(x,y) = 0 & \text{for } x \text{ in } \omega, \\ \nu(x) \cdot \nabla_x \left(e^{i\beta(x,y)} U_\varepsilon(x-y) \right) = 0 & \text{for } x \in \partial\omega, \end{cases} \tag{5.10}$$

for all ε small enough such that $|U_\varepsilon(x-y)| = 1$ for $x \in \partial\omega$. This defines β up to a constant, which we fix by requiring that $\int_\omega \beta(x,y) dx = 0$ for all y.

We will only need a couple of facts about β. First,

$$|\nabla_y \beta(x,y)| \leq C(r) \text{ for all } (x,y) \in \omega \times B(r), \qquad \text{if } B(r) \subset \omega. \tag{5.11}$$

This is a straightforward consequence of standard elliptic regularity results. Second, we have the estimates

$$\int_\omega e^{2d}(u_\varepsilon(x,z)\,)\,dx \le n(\pi\,|\log\varepsilon| + \gamma) + W_\omega(f_{\varepsilon,1}(z),\dots,f_{\varepsilon,n}(z)) + C\varepsilon \tag{5.12}$$

and

$$\left\| J_x u_\varepsilon(\cdot,z) - \pi\delta_{[f_\varepsilon(z)]}\right\|_{W^{-1,1}(\omega)} \le C\varepsilon \tag{5.13}$$

with constants independent of z. These are proved[9] in Lemma 14 in [JS08].

Rewriting the right-hand side of (5.12) using (2.2) and (2.3), and integrating in z, we have

$$\int_\Omega e^{2d}_\varepsilon(u_\varepsilon(x,z)\,)dx\,dz \le \pi Ln\,|\log\varepsilon| + \pi Ln(n-1)|\log h_\varepsilon|$$
$$-\pi\sum_{i\neq j}\int_0^L \log\left|f^i(z) - f^j(z)\right| + \kappa_n(\Omega) + \mathcal{O}\left(h_\varepsilon\right). \tag{5.14}$$

The dominant contribution to the $\mathcal{O}(h_\varepsilon)$ errors term comes from the integral of $\sum_{i,j}(H(f_{\varepsilon,i}(z), f_{\varepsilon,j}(z)) - H(0,0))$.

Also, by setting $v_\varepsilon(x,z) = u_\varepsilon(h_\varepsilon x, z)$ and rescaling, we deduce from (5.13) that

$$\sup_{z\in(0,L)}\left\| J_x v_\varepsilon(\cdot,z) - \pi\delta_{[f(z)]}\right\|_{W^{-1,1}(\omega_\varepsilon)} \le C\varepsilon h_\varepsilon^{-1} \to 0, \quad \text{as } \varepsilon\to 0\,. \tag{5.15}$$

In light of this, to prove part (b) of Proposition 2, it remains to show

Lemma 16.

$$\limsup_{\varepsilon\to 0}\frac{1}{2}\int_\Omega |\partial_z u_\varepsilon|^2\,dx\,dz \le \frac{\pi}{2}\int_0^L \left|f'\right|^2 dz. \tag{5.16}$$

Remark 7. We note that, in light of (5.15), once (5.16) is established, we can appeal to Theorem 3 (c) to conclude $\star Jv_\varepsilon \to \pi\sum\Gamma_{f_i}$, thereby proving part (a) of Proposition 2.

Proof of Lemma 16. Let us write $\beta_\varepsilon(x,z) := \sum_{i=1}^n \beta(x; f_{\varepsilon,i}(z))$, so that

$$u_\varepsilon(x,z) = e^{i\beta_\varepsilon(x,z)}\prod_{j=1}^n U_\varepsilon(x - f_{\varepsilon,j}(z)).$$

[9] Although the descriptions are a little different, our function $u_\varepsilon(\cdot,z)$ is *exactly* the same as $u_\star^{r,\varepsilon}(\cdot,a,d)$ from [JS08], once we set $r=\sqrt{\varepsilon}$, $a=(f_{\varepsilon,1}(z),\dots,f_{\varepsilon,n}(z))$ and $d=(1,\dots,1)$. If one wants to check this, it is helpful to note that $\nabla_x\beta(x,y) = \nabla_x^\perp H(x,y)$, for H solving (2.1).
Experts will recognize that the definition of β is such that $\frac{u_\varepsilon(\cdot,z)}{|u_\varepsilon(\cdot,z)|}$ is exactly the canonical harmonic map with singularities at $(f_{\varepsilon,1}(z),\dots,f_{\varepsilon,n}(z))$ and natural boundary conditions, as introduced in [BBH94] section I.3, and (5.12) expresses this fact.

Then since $\|U_\varepsilon\|_{L^\infty} = 1$,

$$|\partial_z u_\varepsilon(x,z)|^2 \le |\partial_z \beta_\varepsilon(x,z)|^2 + \sum_{i-1}^{n} |\partial_z (U_\varepsilon(x - f_{\varepsilon,i}(z)))|^2 .$$

To estimate the first term, note that

$$\partial_z \beta_\varepsilon(x,z) = \sum_i \nabla_y \beta(x; f_{\varepsilon,i}(z)) \cdot f'_{\varepsilon,i}(z).$$

Since $f'_{\varepsilon,i}(z) = h_\varepsilon f'_i(z)$, it follows from (5.11) that

$$|\partial_z \beta_\varepsilon(x,z)|^2 \le C h_\varepsilon^2 \sum_{i=1}^{n} |f'_i(z)|^2 = C h_\varepsilon^2 |f'(z)|^2.$$

Next, for every $i = 1, \dots, n$,

$$|\partial_z (U_\varepsilon(x - f_{\varepsilon,i}(z)))|^2 = h_\varepsilon^2 \left|\nabla_x U_\varepsilon(x - f_{\varepsilon,i}(z)) \cdot f'_i(z)\right|^2 .$$

Thus, fixing $R > \mathrm{diam}(\omega)$,

$$\begin{aligned}\int_\omega |\partial_z (U_\varepsilon(x - f_{\varepsilon,i}(z)))|^2 \, dx &\le \int_{B(R, f_{\varepsilon,i}(z))} h_\varepsilon^2 \left| f'_i(z) \cdot \nabla_x U_\varepsilon(x - f_{\varepsilon,i}(z))\right|^2 \\ &= \int_{B(R,0)} h_\varepsilon^2 \left| f'_i(z) \cdot \nabla U_\varepsilon(x)\right|^2 .\end{aligned}$$

But the rotational symmetry of U_ε implies that for any vector $v \in \mathbb{R}^2$,

$$\int_{B(R,0)} h_\varepsilon^2 \, |v \cdot \nabla U_\varepsilon(x)|^2 = \int_{B(R,0)} h_\varepsilon^2 \left| v^\perp \cdot \nabla U_\varepsilon(x)\right|^2 .$$

And since

$$|v \cdot \nabla U_\varepsilon(x)|^2 + \left| v^\perp \cdot \nabla U_\varepsilon(x)\right|^2 = |v|^2 \, |\nabla U_\varepsilon(x)|^2$$

we conclude that

$$\int_\omega |\partial_z (U_\varepsilon(x - f_{\varepsilon,i}(z)))|^2 \, dx \ \le \frac{1}{2} |f'_i(z)|^2 \int_{B(R,0)} \frac{|\nabla U_\varepsilon(x)|^2}{|\log \varepsilon|} \, dx.$$

To estimate the right-hand side, we first remark that the constant $I(r,\varepsilon)$ appearing on the right-hand side of (5.9) is known to satisfy

$$I(r,\varepsilon) = \pi \log\left(\frac{r}{\varepsilon}\right) + \gamma + \mathcal{O}\left(\left(\frac{\varepsilon}{r}\right)^2\right),$$

see [JS07], Lemma 6.8. It follows from this and (5.8) that

$$\frac{1}{2}\int_{B(R,0)} \frac{|\nabla U_\varepsilon(x)|^2}{|\log \varepsilon|} \, dx \le \frac{1}{|\log \varepsilon|}\left(\pi \log \frac{R}{\varepsilon} + \gamma + \mathcal{O}(\varepsilon)\right) = \pi + \mathcal{O}\left(\frac{1}{|\log \varepsilon|}\right).$$

The above estimates combine to show that

$$\int_\omega |\partial_z u_\varepsilon(x,z)|^2\,dx \le \pi \sum_{i=1}^n |f_i'(z)|^2 + \mathcal{O}\left(\frac{1+|f_i'(z)|^2}{|\log\varepsilon|}\right)$$

where the error terms are uniform with respect to z. The proof is concluded by integrating in z. □

6 The proof of Theorem 1

To deduce Theorem 1 from Theorem 3, we must first verify that it is possible to construct recovery sequences—that is, sequences verifying the conclusions of part (b) of Theorem 3—that in addition satisfy the boundary conditions (1.13). Our first lemma addresses difficulties that arise when the boundary vortex locations $(q_1^0(z),\dots,q_n^0(z))$, for $z\in\{0,L\}$, are not distinct.

Lemma 17. *Let $f=(f_1,\dots,f_n)\in H^1([0,L];(\mathbb{R}^2)^n)$. Then for all sufficiently small $\delta>0$, where the smallness condition may depend on f, there exists $f^\delta\in H^1([0,L];(\mathbb{R}^2)^n)$ of the form*

$$f_i^\delta = \begin{cases} f_i & \text{in } \{0,L\}\\ f_i + a_i^\delta & \text{in } [\delta^{1/2}, L-\delta^{1/2}]\\ \text{affine} & \text{in } [0,\delta^{1/2}] \text{ and in } [L-\delta^{1/2},L]\end{cases} \tag{6.1}$$

for certain $a_i^\delta\in\mathbb{R}^2$ such that $|a_i^\delta|\le\delta^{1/3}$ and

$$\max_{z\in[0,L]} |f_i^\delta(z)-f_i(z)| \le o(\delta^{1/4}) \qquad \text{for all } i=1,\dots n, \tag{6.2}$$

$$|f_i^\delta(z)-f_j^\delta(z)| \ge \delta^{1/2}\min\{z, L-z, \delta^{1/2}\}, \qquad \text{for } i\ne j, \tag{6.3}$$

$$\int_0^{\sqrt\delta} |(f^\delta)'|^2 + \int_{L-\sqrt\delta}^L |(f^\delta)'|^2 = o(1) \quad \text{as } \delta\searrow 0, \tag{6.4}$$

and

$$\int_{\sqrt\delta}^{L-\sqrt\delta}\left(\frac{\pi}{2}|(f^\delta)'|^2 - \pi\sum_{i\ne j}\log|f_i^\delta - f_j^\delta|\right)dz \to G_0(f) \text{ as } \delta\to 0. \tag{6.5}$$

Proof. **Step 1**. We only need to explain how to choose the points a_i^δ such that the stated properties hold. For this, we introduce some notation: let

$$\begin{aligned} f_{ij}(z) &:= f_i(z)-f_j(z),\\ N_{ij}^\delta &:= \{x\in\mathbb{R}^2 : \operatorname{dist}(x, Image(f_{ij}))<\delta\},\\ \mathcal{N}_{ij}^\delta &:= \{a\in(\mathbb{R}^2)^n : a_i-a_j\in N_{ij}^\delta\}.\end{aligned}$$

We will show below that we can find

$$a^\delta \in \{a \in (\mathbb{R}^2)^n : |a_i| \le \delta^{1/3} \text{ for all } i, \;\; a \notin \cup_{i,j}\mathcal{N}_{ij}^\delta\}. \tag{6.6}$$

First, we assume that we have found such a point a^δ, and we check that f^δ then has the desired properties. For $z \in [\delta^{1/2}, L-\delta^{1/2}]$, we deduce from (6.1) and (6.6) that $|f_i^\delta(z) - f_i(z)| \le \delta^{1/3}$, which implies (6.2) for these z. In particular, since

$$|f^i(z) - f^i(0)| \le \sqrt{z}(\int_0^z |f'|\,dz)^{1/2} = o(\sqrt{z}) \quad \text{ as } z \searrow 0,$$

it follows that $|f_i^\delta(\delta^{1/2}) - f_i(0)| \le o(\delta^{1/4})$. Then for $z \in [0, \delta^{1/2}]$, it follows that

$$|f_i^\delta(z) - f_i(z)| \le |f_i^\delta(z) - f_i(0)| + |f_i(0) - f_i(z)| = o(\delta^{1/4}) \quad \text{ as } \delta \to 0.$$

A similar argument of course holds near $z = L$, completing the proof of (6.2). To verify (6.3), first suppose that $\delta^{1/2} < z < L - \delta^{1/2}$. Since $a^\delta \notin \mathcal{N}_{ij}^\delta$, the definitions imply that for every $z \in (0, L)$

$$\delta \le |f_{ij}(z) - (a_i^\delta - a_j^\delta)| = |f_i^\delta(z) - f_j^\delta z)|.$$

This is (6.3) for $z \in [\delta^{1/2}, L-\delta^{1/2}]$. For other values of z it follows from the fact that f^δ is affine in $[0, \delta^{1/2}]$ and in $[L-\delta^{1/2}, L]$.

One may deduce (6.4) rather directly from the above estimates, which imply that $|(f^\delta)'(z)| = o(\delta^{-1/4})$ for $z \in [0, \delta^{1/2}] \cup [L-\delta^{1/2}, L]$.

Next we observe that for $z \in [\delta^{1/2}, L-\delta^{1/2}]$, if $i \ne j$ and δ is small enough, then

$$|f_i^\delta(z) - f_j^\delta(z)| \ge \begin{cases} |f_i(z) - f_j(z)|^4 & \text{if } |f_i(z) - f_j(z)| \le \delta^{1/4} \\ \frac{1}{2}|f_i(z) - f_j(z)| & \text{otherwise.} \end{cases}$$

This follows from (6.3) in the first case, and in the second case from the fact that $|a_i^\delta| \le \delta^{1/3}$. Since f is bounded, it follows that $|f_i^\delta - f_j^\delta| \ge C^{-1}|f_i - f_j|^4$ for $C = C(f)$, and hence that

$$-\pi \sum_{i\ne j} \log|f_i^\delta(z) - f_j^\delta(z)| \le -4\pi \sum_{i \ne j} \log|f_i(z) - f_j(z)| + C$$

if $z \in [\delta^{1/2}, L-\delta^{1/2}]$. Thus we can apply the dominated convergence theorem to the logarithmic interaction terms in (6.5). Since $(f^\delta)' = f'$ in $[\delta^{1/2}, L-\delta^{1/2}]$, the other terms converge trivially, and (6.5) follows.

Step 2. To find a^δ satisfying (6.6), it suffices to show that

$$\mathcal{L}^{2n}\left(\cup_{i\ne j}\mathcal{B}_{ij}^\delta\right) \ll \delta^{2n/3}, \qquad \text{for } \mathcal{B}_{ij}^\delta := \{a \in \mathcal{N}_{ij}^\delta : |a_i| < \delta^{1/3} \;\forall\; i\;\}, \tag{6.7}$$

since then the set in (6.6), which is the complement of $\cup\mathcal{B}_{ij}^\delta$ in $\{a : \max|a_i| \le \delta^{1/3}\}$, must have positive measure. To prove (6.7), first note that for every i, j

$$\int_0^L |f_{ij}'(z)|\,dz \le \sqrt{L}\left(\int_0^L |f_i'(z) - f_j'(z)|^2\right)^{1/2} \le C\|f'\|_{L^2((0,L)} = C.$$

The image of f_{ij} is therefore a connected curve of length at most C. It follows that $\mathcal{L}^2(N^\delta_{ij}) \le C\delta$. Next, if we define $P_{ij} : (\mathbb{R}^2)^n \to \mathbb{R}^2$ by $P_{ij}(a_1, \dots, a_n) = a_i - a_j$, then $\mathcal{N}^\delta_{ij} = P^{-1}_{ij}(N^\delta_{ij})$, and one easily computes that the Jacobian of P_{ij} (in the sense of the coarea formula) is $JP_{ij} = 1$. Then the coarea formula implies that

$$\begin{aligned}\mathcal{L}^{2n}(\mathcal{B}^\delta_{ij}) &= \int_{\{\max_i |a_i| \le \delta^{1/3}\}} JP_{ij}\, \mathbf{1}_{P_{ij}(a)\in N^\delta_{ij}}\, d\mathcal{L}^{2n}(a)\\ &= \int_{N^\delta_{ij}} \mathcal{H}^{2n-2}\left(\{a : \max_i |a_i| \le \delta^{1/3}, P_{ij}(a) = x\}\right) dx \le C\delta^{(2n-2)/3}\mathcal{L}^2(N^\delta_{ij}).\end{aligned}$$

Thus $\mathcal{L}^{2n}(\mathcal{B}^\delta_{ij}) \le C\delta^{(2n+1)/3}$ for all $i \ne j$. This implies (6.7) for all sufficiently small $\delta > 0$. □

LEMMA 18. *Assume that* $(w^z_\varepsilon) \subset H^1(\omega; \mathbb{C})$ *are sequences satisfying* (1.9)–(1.11) *for* $z \in \{0, L\}$, *and that* $f \in H^1([0, L]; (\mathbb{R}^2)^n)$ *satisfies*

$$f_i(0) = q^0_i(0), \qquad f_i(L) = q^0_{\sigma(i)}(L) \quad \text{for } i = 1, \dots, n$$

for some $\sigma \in S^n$. *Then there is a sequence* $(u_\varepsilon) \subset H^1(\Omega; \mathbb{C})$ *such that (defining* v_ε *as usual by rescaling)*

$$\star J v_\varepsilon \to \pi \sum_{i=1}^n \Gamma_{f_i} \text{ in } \cup_{R>0} F(B(R) \times (0, L)), \qquad \liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon) \le G_0(f)$$

and $u_\varepsilon(x, z) = w^z_\varepsilon(x)$ *for* $x \in \omega$ *and* $z \in \{0, L\}$.

Proof. Let $f_\varepsilon = h_\varepsilon f^{\delta_\varepsilon}$, where f^δ is the regularization of f provided by Lemma 17 and $\delta_\varepsilon = \varepsilon^{1/3}$. We then define

$$\tilde{u}_\varepsilon(x, z) := \prod_{j=1}^n \left[e^{i\beta(x, f_{\varepsilon,j}(z))}\tilde{U}_\varepsilon(x - f_{\varepsilon,j}(z), z)\right]$$

where β is defined in (5.10), and [for $\hat{U}_\varepsilon$ from (5.8), (5.9)] we set

$$\tilde{U}_\varepsilon(x, z) := \begin{cases} \zeta_\varepsilon(x) + \varepsilon^{-1/6} z(\hat{U}_\varepsilon(x) - \zeta_\varepsilon(x)) & \text{if } 0 \le z \le \varepsilon^{1/6}\\ \hat{U}_\varepsilon(x) & \text{if } \varepsilon^{1/6} \le z \le L - \varepsilon^{1/6}\\ \zeta_\varepsilon(x) + \varepsilon^{-1/6}(L - z)(\hat{U}_\varepsilon(x) - \zeta_\varepsilon(x)) & \text{if } L - \varepsilon^{1/6} \le z \le L.\end{cases}$$

(Recall that ζ_ε is defined in Sect. 1.1.) The flat-norm convergence of $\star J v_\varepsilon$ follows very much as in the proof of Proposition 2, so we omit the details.

Step 1. For $\delta^{1/2} = \varepsilon^{1/6} \le z \le L - \varepsilon^{1/6}$, Lemma 17 implies that

$$|f_{\varepsilon,i} - f_{\varepsilon,j}| \ge 2\sqrt{\varepsilon} \qquad \text{for sufficiently small } \varepsilon. \tag{6.8}$$

The significance of this stems from the fact that $\sqrt{\varepsilon}$ is the intermediate length scale that we chose (rather arbitrarily) in the construction of $\hat{U}_\varepsilon$. Once vortices are separated by larger distances (that is, once (6.8) holds), all the estimates in the proofs of Proposition 2 and Lemma 16 are uniform, and it follows that

$$\int_{\varepsilon^{1/6}}^{L-\varepsilon^{1/6}} \int_\omega e_\varepsilon(\tilde{u}_\varepsilon)\,dx\,dz - n\pi L(|\log\varepsilon| + \gamma) - n(n-1)\pi L|\log h_\varepsilon| - \kappa_n(\Omega)$$
$$\leq \int_{\varepsilon^{1/6}}^{L-\varepsilon^{1/6}} \frac{\pi}{2}|(f^{\delta_\varepsilon})'|^2 - \pi\sum_{i\neq j}\log|f_i^{\delta_\varepsilon} - f_j^{\delta_\varepsilon}|\,dz + o(1)$$

as $\varepsilon \to 0$. In view of (6.5), we therefore only need to show that

$$\int_0^{\varepsilon^{1/6}} \int_\omega e_\varepsilon(\tilde{u}_\varepsilon) + \int_{L-\varepsilon^{1/6}}^{L} \int_\omega e_\varepsilon(\tilde{u}_\varepsilon) \to 0. \tag{6.9}$$

We will only consider the first integral, since the estimate of the second is identical.

Step 2. Both $\hat{U}_\varepsilon$ and ζ_ε have the form $\rho_\varepsilon(r)e^{i\theta}$ for ρ_ε satisfying

$$\rho_\varepsilon(0) = 0, \qquad 0 \leq \rho_\varepsilon' \leq \frac{C}{\varepsilon}, \qquad 1 \geq \rho_\varepsilon(s) \geq 1 - C\frac{\varepsilon}{s}. \tag{6.10}$$

Thus $\tilde{U}_\varepsilon(\cdot, z)$ also has this form for every z. It follows that $\partial_z\tilde{u}_\varepsilon$ is a sum of terms of the form already estimated in Lemma 16, together with some new terms of the form

$$\varepsilon^{-1/6}(U_\varepsilon(\cdot - a) - \zeta_\varepsilon(\cdot - a)) * (\text{smooth functions})$$

where the smooth functions have modulus less than 1. It also follows from (6.10) that $|(\hat{U}_\varepsilon - \zeta_\varepsilon)(x)| \leq \min\{1, C\varepsilon/|x|\}$ and hence that for every a,

$$\|\hat{U}_\varepsilon(\cdot - a) - \zeta_\varepsilon(\cdot - a)\|_{L^2(\omega)}^2 \leq \int_0^{\mathrm{diam}(\omega)} \min(1, C\varepsilon/s)^2 s\,ds \leq C\varepsilon^2|\log\varepsilon|$$

as long as $0 < \varepsilon < 1/2$ say, where $C = C(\omega)$. Using these facts and arguing as in the proof of Lemma 16, one checks that

$$\int_0^{\varepsilon^{1/6}} \int_\omega |\partial_z\tilde{u}_\varepsilon|^2 dx\,dz \leq C\int_0^{\varepsilon^{1/6}} |(f^{\delta_\varepsilon})'|^2 + o(1) \overset{(6.4)}{=} o(1)$$

as $\varepsilon \to 0$.

Step 3. We now write

$$\tilde{u}_\varepsilon^j := e^{i\beta(x, f_{\varepsilon,j}(z))}\tilde{U}_\varepsilon(x - f_{\varepsilon,j}(z), z),$$

so that $\tilde{u}_\varepsilon = \prod_{i=1}^n \tilde{u}_\varepsilon^j$. Since $\|\tilde{u}_\varepsilon^j\|_{L^\infty} \leq 1$ for every j, it is clear that

$$|\nabla_x\tilde{u}_\varepsilon|^2 \leq (\sum_j |\nabla_x\tilde{u}_\varepsilon^j|)^2 \leq n\sum_j |\nabla_x\tilde{u}_\varepsilon^j|^2.$$

Also, for $a, b \in [0,1]$, by rearranging the inequality $0 \le (1-a^2)(1-b^2)$, we find that $(1-a^2b^2) \le (1-a^2) + (1-b^2)$. By induction, we deduce that

$$(1 - |\prod \tilde{u}_\varepsilon^j|^2)^2 \le \Big(\sum (1 - |\tilde{u}_\varepsilon^j|^2)\Big)^2 \le n \sum (1 - |\tilde{u}_\varepsilon^j|^2)^2.$$

Therefore

$$e_\varepsilon^{2d}(\tilde{u}_\varepsilon) \le n \sum_{j=1}^n e^{2d}(\tilde{u}_\varepsilon^j).$$

We can then appeal to rather standard estimates of $e^{2d}(\tilde{u}_\varepsilon^j)$ to conclude that

$$\int_\omega e^{2d}(\tilde{u}_\varepsilon^j(x,z))dz \le \pi |\log \varepsilon| + C$$

for every z, so it follows that

$$\int_0^{\varepsilon^{1/6}} \int_\omega e^{2d}(\tilde{u}_\varepsilon) dx\, dz \le \varepsilon^{1/6} n^2 \sup_{j,z} \int_\omega e^{2d}(\tilde{u}_\varepsilon^j(x,z)) dx \le \varepsilon^{1/6} n^2 (\pi |\log \varepsilon| + C)\ ,$$

completing the proof of (6.9). □

Proof of Theorem 1. Let u_ε minimize F_ε in

$$\mathcal{A}_\varepsilon := \{u \in H^1(\Omega; \mathbb{C}) : \ u(x,0) = w_\varepsilon^0(x), \quad u(x,L) = w_\varepsilon^L(x)\}$$

for boundary data as described in (1.9). We want to verify that the sequence (u_ε) generated in this way satisfies the hypotheses of part (a) of Theorem 3. It is straightforward to check from (1.9)–(1.11) that assumptions (1.24) and (1.25) are satisfied, and (1.27) follows from Lemma 18. The only remaining hypothesis to check is that

$$\| \star Ju_\varepsilon - n\pi\Gamma_0 \|_{W^{-1,1}(\Omega)} \to 0 \qquad \text{as } \varepsilon \to 0. \tag{6.11}$$

To verify this we argue as in Lemma 10. Thus, for $\delta > 0$ to be specified below, we let $\Omega^\delta := \omega \times (-\delta, L+\delta)$, and we extend each u_ε to a function on Ω^δ, still denoted u_ε, such that $u_\varepsilon(x,z) = w_\varepsilon^0(x)$ for $z < 0$, and $u_\varepsilon(x,z) = w_\varepsilon^L(x)$ for $z > L$. From (1.27) and Theorem 4, we may pass to a subsequence and find an integer multiplicity 1-current J in Ω^δ such that $\partial J = 0$ in Ω^δ,

$$\frac{1}{\pi} \star Ju_\varepsilon \to J \text{ in } W^{-1,1}(\Omega^\delta), \qquad\qquad M_\Omega(J) < M_{\Omega^\delta}(J) \le nL + 2\delta M.$$

We must show that the restriction of J to Ω is $n\Gamma_0$. It is for this that we will need our assumption that

$$L < 2R \qquad \text{where } R = \text{dist}(0, \partial\omega).$$

Due to our assumptions on $w_\varepsilon^{0,L}$, it is straightforward to check that the restriction of J to $\omega \times (-\delta, 0)$, for example, consists of n copies of the segment $\{0\} \times (-\delta, 0)$

(with the correct, "upward" orientation, henceforth not mentioned), with a similar statement in $\omega \times (L, L+\delta)$.

We may write $J = \sum_{i\in I} T_{\gamma_i}$, where each γ_i is a Lipschitz curve without boundary in Ω^δ. Then there must be subsets I^0, I^1 of I, both with cardinality n, such that

$$\text{for } i \in I^0,\ T_{\gamma_i} = \{0\} \times (-\delta, 0) \qquad \text{in } \omega \times (-\delta, 0),$$
$$\text{for } i \in I^1,\ T_{\gamma_i} = \{0\} \times (L, L+\delta) \quad \text{in } \omega \times (L, L+\delta).$$

Clearly, if $i \in I^0 \cap I^1$, then γ_i connects a copy of $\{0\} \times (-\delta, 0)$ to a copy of $\{0\} \times (L, L+\delta)$, and must have length at least $L + 2\delta$.

On the other hand, if $i \in I^0 \setminus I^1$, then γ_i must connect a copy of $\{0\} \times (-\delta, 0)$ to $\partial\omega \times [0, L]$, and must have length at least $\delta + R$. The same applies to $i \in I^1 \setminus I^0$. Thus, if $n_0 := \#(I^0 \cap I^1)$, then

$$\begin{aligned} nL + 2M\delta \ge M_{\Omega^\delta}(J) &\ge \sum_{i\in(I^0\cup I^1)} M_{\Omega^\delta}(T_{\gamma_i}) \\ &\ge n(L+2\delta) + (n-n_0)(2R-L). \end{aligned}$$

Since δ may be chosen arbitrarily small, we may deduce that $n = n_0$, and hence that $I^0 = I^1$. Similar considerations show that $I = I^0$—that is, there are no indecomposable pieces other than the n curves that connect $\{0\} \times (-\delta, 0)$ to $\{0\} \times (L, L+\delta)$. Finally, the same argument shows that none of these n curves can have length greater than $L + 2\delta$. Thus every curve coincides with $\{0\} \times (0, L) = \Gamma_0$ in Ω. This says that the restriction of J to Ω is exactly $n\Gamma_0$, which completes the proof of (6.11).

Having verified (6.11), we may apply part (a) of Theorem 3 to conclude that $\star\frac{1}{\pi} Jv_\varepsilon$ is precompact in $W^{-1,1}(B(R) \times (0, L))$ for every $R > 0$, and that every limit of a convergent subsequence has the form $\pi \sum_{i=1}^n \Gamma_{f_i^*}$, where

$$f^* \in \mathcal{A}_0 := \{f \in H^1([0, L]; (\mathbb{R}^2)^n)\ : \exists \sigma \in S^n,\ f_i(0) = q_i^0(0),\ f_i(L) = q^0_{\sigma(i)}(L)\}$$

for $(q_i^0(z))$, $z = 0, L$ appearing in (1.9), (1.11); and that

$$G_0(f^*) \le \liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon)$$

along the subsequence. Since u_ε minimizes F_ε in $\mathcal{A}_\varepsilon$, it follows from Lemma 18 that

$$\liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon) \le \inf_{f\in\mathcal{A}_0} G_0(f).$$

Therefore f^* minimizes $G_0(\cdot)$ in $\mathcal{A}_0$, as was to be shown. □

7 The proof of Theorem 2

We first need a Lemma that relates local minimizers with respect to two different topologies.

Lemma 19. *Assume that $f^* \in H^1((0,L);(\mathbb{R}^2)^n)$ is a strict local minimizer of G_0 in $\mathcal{A}_0$ and that $f_i^*(z) \neq f_j^*(z)$ for all $z \in (0,L)$ and $i \neq j$.*

Then there exists $\delta^*, R_1 > 0$ such that

$$\text{if } f \in \mathcal{A}_0 \quad \text{and} \quad 0 < \|\pi \sum_{i=1}^n (\Gamma_{f_i} - \Gamma_{f_i^*})\|_{F(B(R)\times(0,L))} < \delta^* \quad \text{for some } R \geq R_1,$$

then $G_0(f) > G_0(f^*)$.

Proof. Consider any sequence $(f^k) \in \mathcal{A}_0$ such that

$$G_0(f^k) \leq G_0(f^*), \qquad \|\sum_{i=1}^n (\Gamma_{f_i^k} - \Gamma_{f_i^*})\|_{F(B(R)\times(0,L))} \to 0 \tag{7.1}$$

for some $R \geq R_1$, where R_1 will be fixed below. We will show that $f^k \to f^*$ in $H^1((0,L);(\mathbb{R}^2)^n)$ as $k \to \infty$, after a possible relabelling. Since f^* is a local minimizer, this will prove that $f^k = f^*$ for all sufficiently large k. Because the sequence (f^k) is arbitrary, this will establish the lemma.

First, by a 1-dimensional Sobolev embedding theorem,

$$\|f^k\|_{L^\infty}^2 \leq \|f^k(0)\|^2 + L\|(f^k)'\|_{L^2}^2 = C + C\|(f^k)'\|_{L^2}^2,$$

where the constants depend on the boundary data for $\mathcal{A}_0$ and on L. Also, it is clear that

$$-\pi \int_0^L \sum_{i\neq j} \log|f_i^k - f_j^k| \geq -\pi n(n-1)L\log(2\|f^k\|_{L^\infty}).$$

It follows that

$$\|f^k\|_{L^\infty}^2 - C\log(\|f^k\|_{L^\infty}) - C \leq CG_0(f^k) \leq CG_0(f^*).$$

Since $x^2 - C\log x \to +\infty$ as $x \to +\infty$, there exists some constant C such that

$$\int_0^L \sum_{i\neq j} \log|f_i^k - f_j^k| \leq C, \qquad \|f^k\|_\infty + \|(f^k)'\|_{L^2} \leq C \qquad \text{for all } k. \tag{7.2}$$

We now fix $R_1 := \sup_k \|f^k\|_\infty + 1$.

Now for $\delta \in (0, L/2)$, define $r = r(\delta)$ by

$$r(\delta) := \min\left(\{1\} \cup \left\{ \frac{1}{4}|f_i^*(z) - f_j^*(z)| : i \neq j, \delta \leq z \leq L - \delta \right\} \right).$$

We claim that for every δ as above, if k is large enough, then for every i,

$$\min_j |f_i^*(z) - f_j^k(z)| < r(\delta) \text{ for } \delta < z < L - \delta. \tag{7.3}$$

Suppose toward a contradiction that (7.3) fails for some i and z. Since both f^* and f^k are $C^{0,\frac{1}{2}}$, with modulus of continuity depending on f^* but independent of k, see (7.2), it follows that there is an interval I^k such that

$$\min_j |f_i^*(z) - f_j^k(z)| \geq \frac{1}{2} r(\delta) \text{ for all } z \in I^k, \qquad |I^k| \geq c = c(f^*).$$

For such z, it follows that

$$\begin{aligned} &\| \sum_{j=1}^n (\delta_{f_j^*(z)} - \delta_{f_j^k(z)}) \|_{F(B(R))} \\ &\qquad \geq \int_{B(R)} (\frac{1}{2} r(\delta) - |f_i^*(z) - x|)^+ \Big(\sum_{j=1}^n (\delta_{f_j^*(z)} - \delta_{f_j^k(z)}|) \Big)(dx) = \frac{1}{2} r(\delta). \end{aligned}$$

Hence for all k,

$$\int_0^L \| \sum_{j=1}^n (\delta_{f_j^*(z)} - \delta_{f_j^k(z)}) \|_{F(B(R))} dz \geq \frac{1}{2} \, |I^k| \, r(\delta) > c > 0.$$

On the other hand, we may use (2.9) to estimate

$$\begin{aligned} \int_0^L \| \sum_{i=1}^n (\delta_{f_i^*(z)} - \delta_{f_i^k(z)}) \|_{F(B(R))} \, dz &= \int_0^L \| \langle \sum_{i=1}^n (\Gamma_{f_i^*} - \Gamma_{f_i^k}), \zeta, z \rangle \|_{F(B(R))} \, dz \\ &\leq \| \sum_{i=1}^n (\Gamma_{f_i^*} - \Gamma_{f_i^k}) \|_{F(B(R) \times (0,L))} \to 0, \end{aligned}$$

a contradiction. Hence (7.3) holds.

Since the balls $B(f_i^*(z), r(\delta))$ are disjoint for $\delta < z < L - \delta$, by definition of $r(\delta)$, it follows from (7.3) that for every z in this range, each of these balls contains exactly one point $f_j^k(z)$. We may relabel the (f_j^k) such that at height $z = L/2$ for example, $|f_i^*(L/2) - f_i^k(L/2)| < r(\delta)$ for all i. Then (7.3) and the continuity of f^*, f^k imply that $|f_i^*(z) - f_i^k(z)| < r(\delta)$ for all i and all $z \in (\delta, L - \delta)$.

Now we let $k \to \infty$ and, using (7.2), extract a subsequence such that f^k converges weakly in H^1, and thus uniformly, to a limit $f^\infty \in \mathcal{A}_0$. It follows from the above that $|f_i^*(z) - f_i^\infty(z)| \leq r(\delta)$ for all i and all $z \in (\delta, L - \delta)$. Since δ is arbitrary, we conclude that $f^\infty = f^*$. Thus in fact $f^k \rightharpoonup f^*$ in H^1, without passing to a subsequence. Then the choice of (f^k) and standard lower semicontinuity arguments imply that

$$G_0(f^*) \geq \liminf G_0(f^k) \geq G_0(f^*).$$

It follows in particular that $\int |(f^k)'|^2 dz \to \int |(f^*)'|^2 dz$. This allows us to improve weak H^1 convergence to strong H^1 convergence, completing the proof. □

The proof of Theorem 2 now follows standard arguments.

Proof of Theorem 2. Fix $0 < \delta^1 < \delta^*$, for δ^* from Lemma 19, and let u_ε minimize F_ε in

$$\bar{\mathcal{A}}_{\varepsilon,\delta^1} := \{u \in \mathcal{A}_\varepsilon \ : \ \| \star Ju - \pi \sum_{i=1}^n \Gamma_{h_\varepsilon f_i^*} \|_{F(\Omega)} \leq h_\varepsilon \delta^1 \}.$$

Existence of a minimizer is rather standard; see for example [MSZ04], Theorem 4.2 for a very similar argument. We claim that if ε is small enough

$$\| \star Ju_\varepsilon - \pi \sum_{i=1}^n \Gamma_{h_\varepsilon f_i^*} \|_{F(\Omega)} < h_\varepsilon \delta^1 \qquad \text{or equivalently,} \quad u_\varepsilon \in \mathcal{A}_{\varepsilon,\delta^1}. \tag{7.4}$$

Toward this end, consider a hypothetical sequence in $\bar{\mathcal{A}}_{\varepsilon,\delta^1}$ for which (7.4) fails, so that the constraint holds with equality. Then Lemma 18 and the definition of $\mathcal{A}_\varepsilon$ imply that (u_ε) satisfies the hypotheses of Theorem 3. Thus, defining as usual $v_\varepsilon(x,z) = u_\varepsilon(h_\varepsilon x, z)$, we conclude that

$$\star J v_\varepsilon \to \pi \sum_{i=1}^n \Gamma_{f_i} \qquad \text{in } F(B(R) \times (0,L)) \text{ for every } R > 0.$$

for some $f \in H^1((0,L);(\mathbb{R}^2)^n)$. Moreover, again appealing to Lemma 18 (the recovery sequence with correct boundary conditions) and Theorem 3, we find that

$$G_0(f) \leq G_0(f^*).$$

Also, by rescaling our assumption that $\| \star Ju_\varepsilon - \pi \sum_{i=1}^n \Gamma_{h_\varepsilon f_i^*} \|_{F(\Omega)} = h_\varepsilon \delta^1$ and taking limits, we conclude that $\|\pi \sum_{i=1}^n (\Gamma_{f_i} - \Gamma_{f_i^*})\|_{F(B(R)\times(0,L))} = \delta^1$ for every $R > 0$.

This is impossible, in view of Lemma 19, so (7.4) must be true.

Then it is well-known (see for example [MSZ04]) and not hard to check that $\mathcal{A}_{\varepsilon,\delta^1}$ is an open set in H^1, and so (7.4) implies that u_ε is a local minimizer. By arguing in this way for a sequence $\delta^k \searrow 0$, and employing a diagonal argument, we may generate a sequence (u_ε) such that the rescaled Jacobians (Jv_ε) converge as required to $\pi \sum \Gamma_{f_i^*}$. □

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A

We finally present the proof of Lemma 2, which establishes certain properties of $u_\varepsilon(\cdot, z)$ for $z \in \mathcal{G}_2^\varepsilon$. The main point of the proof is contained in the following

LEMMA 20. *There exist positive numbers θ, a, b, depending on n and ω, such that $b < a$, and the following holds:*
Assume that $w \in H^1(\omega; \mathbb{C})$ satisfies

$$\int_\omega e^{2d}(w)(x)\,dx \le \pi(n+\theta)|\log\varepsilon| \tag{A.1}$$

and

$$\int_\omega \phi(x) J_x w(x)\,dx \ge n\pi - 1 \quad \text{for some } \phi \in W_0^{1,\infty}(B(r^*)) \text{ with } Lip(\phi) \le 4/r^*, \tag{A.2}$$

where $r^* := \min\{1, \operatorname{dist}(0, \partial\omega)\}$.
Then if ε is sufficiently small, there exists $p = (p_1, \ldots, p_n)$ such that

$$\|J_x w - \pi \sum \delta_{p_i}\|_{W^{-1,1}(\omega)} \le \varepsilon^a \tag{A.3}$$

and

$$\operatorname{dist}(p_i, \partial\omega) \ge \frac{r^*}{8} \text{ for all } i, \qquad |p_i - p_j| \ge \varepsilon^b \quad \text{for all } i \ne j. \tag{A.4}$$

To obtain Lemma 2 from the above, note that if z satisfies the hypotheses of Lemma 2, then $w = u_\varepsilon(\cdot, z)$ satisfies (A.1), (A.2). Indeed, (A.1) is exactly (3.5), and (A.2) is a consequence of Lemma 1, see (3.4). Thus (3.6) (3.7) follow directly from (A.3), (A.4). The final conclusion (3.8) is then an immediate consequence[10] of Theorem 2 in [JS08], whose hypotheses are implied by (3.6), (3.7).
Throughout the proof of Lemma 20, we will assume that w is smooth. The general case follows from this by a standard mollification argument.
Our proof relies on a vortex ball construction, as introduced by [Jer99a] and [San98]. We recall some ingredients that we will need below. Our presentation mostly follows that of [Jer99a] and [JS02], which can also be used as sources, adapted to our needs, for background on topics such as the degree $\deg(w; \partial O)$.
We will use the notation

$$S := \{x \in \omega : |w(x)| \le \frac{1}{2}\}, \tag{A.5}$$

$$S_E := \cup\{\text{components } S_i \text{ of } S \ : \ \deg(w; \partial S_i) \ne 0\}, \tag{A.6}$$

$$S_E^\varepsilon := \cup\{\text{components } S_i \text{ of } S \ : \ \deg(w; \partial S_i) \ne 0,\ \operatorname{dist}(S_i, \partial\omega) \ge \varepsilon\}. \tag{A.7}$$

If O is an open subset of ω such that $\partial O \cap S_E = \emptyset$, then

$$\operatorname{dg}(w; \partial O) := \sum \big\{ \deg(u; \partial S_i) : \text{components } S_i \text{ of } S_E \text{ such that } S_i \subset O \big\}. \tag{A.8}$$

[10] If one wants to check this, note from (3.6), (3.7) that $\rho_\alpha \ge \frac{1}{2}\varepsilon^b$, and that one may take $s_\varepsilon = \varepsilon^a$, where the notation ρ_α and s_ε appears in [JS08]. From this one easily checks that hypotheses in [JS08] relating s_ε and ρ_α are satisfied here. Note also that $\Sigma_\Omega^\varepsilon(u; \alpha, d)$ appearing in [JS08] is defined to be $\int_\Omega e_\varepsilon^{2d}(u))dx \ - \ n(\pi|\log\varepsilon| + \gamma) + W_\omega(\alpha_1, \ldots, \alpha_n)$ when $d = (1, \ldots, 1)$, which is the relevant case here. So conclusion (3.8) is exactly a lower bound for $\Sigma_\omega^\varepsilon(u(\cdot, z), p^\varepsilon, d)$ with d as above.

We also define

$$\lambda_\varepsilon(r,d) := \min_{m\in[0,1]} \left(\frac{m^2 d^2 \pi}{r} + \frac{1}{c\varepsilon}(1-m)^2 \right), \tag{A.9}$$

$$\Lambda_\varepsilon(s) := \int_0^s (\lambda_\varepsilon(r;1) \wedge \frac{1}{C\varepsilon})\, dr.$$

LEMMA 21. *Assume that $B(r,x) \subset \omega$ for some $r \geq \varepsilon$, and that $|\deg(w;\partial B(r,x))| = d > 0$. Then*

$$\int_{\partial B(r,x)} e_\varepsilon^{2d}(w)\, dx \ \geq\ \lambda_\varepsilon(r,d) \ \ \geq\ \lambda_\varepsilon(\frac{r}{d},1). \tag{A.10}$$

For a proof, see for example [Jer99a] Theorem 2.1, or consult Lemma 7 above for a very similar argument.
By integrating (A.10), one obtains lower bounds for the energy $e_\varepsilon^{2d}(w)$ on an annulus on which one has some information about the degree. These bounds are naturally expressed in terms of Λ_ε. The main point of the vortex ball construction is to assemble these estimates in a clever way. These arguments lead for example to the following:

LEMMA 22. *For all $\sigma \geq r_0 := C\varepsilon \int_\omega e_\varepsilon^{2d}(w)dx$, there exists a collection $\mathcal{B}(\sigma) = \{B_k^\sigma\}_{k=1}^{k(\sigma)}$ of disjoint balls such that*

$$S_E^\varepsilon \subset \cup_k B_k^\sigma \tag{A.11}$$

$$\int_{B_k^\sigma \cap \omega} e_\varepsilon^{2d}(w)\, dx \geq \frac{r_k^\sigma}{\sigma}\Lambda_\varepsilon(\sigma), \qquad \textit{for } r_k^\sigma := \text{radius}(B_k^\sigma) \tag{A.12}$$

$$r_k^\sigma \geq \sigma |d_k^\sigma| \qquad \textit{if } B_k^\sigma \subset \omega, \qquad \textit{for } d_k^\sigma := \text{dg}(w, \partial B_k^\sigma). \tag{A.13}$$

Moreover, if x_k^σ denotes the center of B_k^σ, then

$$\|Jw - \pi \sum d_k \delta_{x_k^\sigma}\|_{W^{-1,1}(\omega)} \leq C(r_0 + \sum_k r_k^\sigma) \int_\omega e_\varepsilon^{2d}(w)\, dx. \tag{A.14}$$

Finally, $\Lambda_\varepsilon(\sigma) \geq \pi \log \frac{\sigma}{\varepsilon} - C$ for all σ.

Proof. In Proposition 6.4 in [JS02] it is proved that a collection of balls satisfying (A.11)–(A.13) exists for every σ larger than some r_0. The fact that one may take $r_0 = C\varepsilon \int_\omega e_\varepsilon^{2d}(w)dx$ follows from the proofs of Propositions 6.2 and 6.4, [JS02].
To prove (A.14), one modifies $\{B_k^\sigma\}$ to obtain a collection of balls that covers all of S, and whose radii sum to at most $r_0 + \sum r_k^\sigma$. This relies on a lemma due to [San98], which shows that S may be covered by a collection of disjoint balls whose radii sum to at most r_0. Then, (A.14) follows from standard arguments, see for example [SS07, Theorem 6.1]. □

With the above result as our starting point, we can now present the

proof of Lemma 20. We fix positive numbers θ, a, b such that

$$\frac{n+\theta}{1-2a} < n+1, \qquad \theta \leq b < a.$$

For example, if $2b = a = \frac{1}{3(n+1)}$ then both inequalities may be satisfied by a sufficiently small positive θ.

Step 1. First, consider the collection of balls $\mathcal{B}(\sigma)$ from Lemma 22, with $\sigma = \varepsilon^{2a}$. We will write $\{x_i^\sigma\}$ for the center of B_i^σ. Then (A.12) implies that

$$\sum_k \int_{B_k^\sigma \cap \omega} e_\varepsilon^{2d}(w)dx \geq \sum_k \frac{r_k^\sigma}{\sigma}\Lambda_\varepsilon(\sigma) \geq \sum \frac{r_k^\sigma}{\sigma}(1-2a)\left(\pi|\log\varepsilon| - C\right). \tag{A.15}$$

Then from (A.1) and the choice of a, we find that for small enough ε,

$$\sum_k \frac{r_k^\sigma}{\sigma} \leq \frac{(n+\theta)}{(1-2a)}\frac{|\log\varepsilon|}{|\log\varepsilon| - C} < n+1 \qquad \text{and thus} \qquad \sum_k |d_k^\sigma| \leq n. \tag{A.16}$$

On the other hand, from (A.14), (A.16) and (A.1) we have

$$\|J_x w - \pi \sum d_i^\sigma \delta_{x_i^\sigma}\|_{W^{-1,1}(\omega)} \leq C\sigma(n+1)\int_\omega e_\varepsilon^{2d}(w) \leq \varepsilon^a \tag{A.17}$$

if ε is small enough. This and (A.2) imply that

$$n\pi - 1 \leq \pi \sum_k d_k^\sigma \phi(x_k^\sigma) + 4\varepsilon^a/r^*.$$

If ε is small enough, then by comparing this with (A.16) and recalling that $0 \leq \phi \leq 1$, we see that

$$d_k^\sigma \geq 0 \text{ for all } k, \qquad \sum_k d_k^\sigma = n \tag{A.18}$$

and

$$\phi(x_k^\sigma) \geq 1 - \tfrac{1}{\pi} > \tfrac{1}{2} \text{ for all } k \text{ such that } d_k^\sigma > 0.$$

Since $\mathrm{Lip}(\phi) \leq 4/r^*$ and $\mathrm{supp}(\phi) \subset B(1)$, it follows that

$$|x_i^\sigma| \leq \frac{7}{8}r^* \qquad \text{for } k \text{ such that } d_k^\sigma > 0. \tag{A.19}$$

We also remark that (A.13), (A.16), and (A.18) imply that

$$n\sigma \leq \sum r_k^\sigma < (n+1)\sigma \quad \text{for small } \varepsilon$$

and thus

$$d_k^\sigma \sigma \leq r_k^\sigma < (d_k^\sigma + 1)\sigma \quad \text{for all } k, \qquad \sum_{d_k^\sigma = 0} r_k^\sigma < \sigma. \tag{A.20}$$

In particular, $\max_k r_k^\sigma \leq (n+1)\sigma$.
To complete the proof of the lemma, it suffices to show that

$$d_k^\sigma \leq 1 \text{ for all } k, \qquad \text{and} \qquad |x_k^\sigma - x_\ell^\sigma| \geq \varepsilon^b \text{ if } d_k^\sigma, d_\ell^\sigma \neq 0 \text{ and } k \neq \ell. \tag{A.21}$$

Indeed, once we know (A.21), then it follows from (A.18) that there are exactly n points x_k^σ for which d_k^σ is nonzero, and that $d_k^\sigma = 1$ for all of these. We take $\{p_1, \ldots, p_n\}$ to be these points. Then (A.19), (A.21) imply that (A.4) holds, and (A.17) reduces to (A.3).
Step 2. To start the proof of (A.21), let

$$\widetilde{\mathcal{B}}^0 := \{B_k^\sigma \in \mathcal{B}(\sigma) \ : d_k^\sigma \neq 0\}.$$

We claim that there is a collection $\widetilde{\mathcal{B}}^1 = \{\widetilde{B}^1_k\}$ of (at most n) closed pairwise disjoint balls such that

$$\bigcup_{\widetilde{\mathcal{B}}^0} B^\sigma_k \subset \bigcup_{\widetilde{\mathcal{B}}^1} \widetilde{B}^1_k \tag{A.22}$$

and every ball in $\widetilde{\mathcal{B}}^1$ has the same radius $\widetilde{r}^1$, with

$$\widetilde{r}^1 \le C(n)\sigma. \tag{A.23}$$

Such a collection can be found as follows:

- first replace every $B^\sigma_k \in \widetilde{\mathcal{B}}^0$ by a concentric ball of radius $\max_{\widetilde{\mathcal{B}}^0} r^\sigma_k \le (n+1)\sigma$;
- enclose intersecting balls in larger balls, without increasing the sum of the radii, to obtain a new pairwise disjoint collection, with fewer balls;
- repeat: increase the radii of the remaining balls, as necessary, until they are the same size, then combine balls that intersect. After finitely many steps (at most $n-1$ mergings) this produces a collection satisfying (A.22), (A.23).

Let

$$\widetilde{R}^1 := \sup\Big\{R \in (\widetilde{r}^1, \frac{1}{8}) \ : \ \{B(R, \widetilde{x}^1_i)\} \text{ are pairwise disjoint}\Big\}$$

where $\{\widetilde{x}^1_i\}$ denotes the centers of the balls in $\mathcal{B}^1$. We now proceed inductively, using the same procedure to find collections $\widetilde{B}^j = \{\widetilde{B}^j_i\}$ of (at most $n+1-j$) balls such that for $j \ge 2$,

$$\bigcup_{\widetilde{\mathcal{B}}^{j-1}} B(\widetilde{R}^{j-1}, \widetilde{x}^{j-1}_k) \subset \bigcup_{\widetilde{\mathcal{B}}^j} \widetilde{B}^j_k \tag{A.24}$$

and all balls are closed and pairwise disjoint, with the same radius

$$\widetilde{r}^j \le C(n)\widetilde{R}^{j-1} \tag{A.25}$$

and where

$$\widetilde{R}^j := \sup\Big\{R \in (\widetilde{r}^j, \frac{1}{8}) \ : \ \{B(R, \widetilde{x}^j_i)\} \text{ are pairwise disjoint}\Big\}.$$

Let J denote the first j for which either $\widetilde{R}^j = \frac{r^*}{8}$ or $\widetilde{r}^{j+1} \ge \frac{r^*}{8}$. With each step the number of balls decreases, and if there is only one ball left, it can expand unimpeded, so it is clear that $J \le n$. It follows from (A.19) that the interiors of all the balls are contained in ω. Note also that

$$\widetilde{R}^J \ge \frac{1}{8C(n)}. \tag{A.26}$$

This is clear if $\widetilde{R}^J = \frac{r^*}{8}$, and if $\widetilde{r}^{J+1} \ge \frac{r^*}{8}$ then it follows from (A.25).
To prove (A.21), it now suffices to show that

$$\widetilde{\mathcal{B}}^1 \text{ consists of } n \text{ balls, all of degree 1,} \qquad \text{and} \quad \widetilde{R}^1 > \varepsilon^b/2. \tag{A.27}$$

Step 3. We now write $\widetilde{A}^j_k := B(\widetilde{R}^j, \widetilde{x}^j_k) \setminus B(\widetilde{r}_j, \widetilde{x}^j_k)$, and we estimate the energy contained in these annuli.
Let us write $\widetilde{d}^j_k := \mathrm{dg}(u; \partial\widetilde{B}^j_k)$, and note that

$$\widetilde{d}^j_k > 0 \text{ for all } j,k, \qquad \textstyle\sum_k \widetilde{d}^j_k = n \text{ for every } j, \qquad \max_j \widetilde{d}^j_k \ge 2 \ \text{ for } \ j \ge 2. \tag{A.28}$$

For every j and k we deduce from (A.10) that

$$\int_{\widetilde{A}_k^j} e_\varepsilon^{2d}(w)dx = \int_{\widetilde{r}^j}^{\widetilde{R}^j} \int_{\partial B(r,\widetilde{x}_k^j)} e_\varepsilon^{2d}(w)d\mathcal{H}^1 \; dr$$
$$\geq \int_{\widetilde{r}^j}^{\widetilde{R}^j} \int_{\partial B(r,\widetilde{x}_k^j)} \lambda_\varepsilon(r,\widetilde{d}_k^j)\, \mathbf{1}_{r\notin A_k^j} \; dr \;, \tag{A.29}$$

for

$$A_k^j := \{r \in (\widetilde{r}^j, \widetilde{R}^j) : \mathrm{dg}(u, \partial B(r,\widetilde{x}_k^j)) \neq \widetilde{d}_k^j\}.$$

In general, since $r \mapsto \lambda_\varepsilon(r,d)$ is a nonincreasing function for every d, if A is a measurable subset of an interval (a,b), then

$$\int_a^b \lambda_\varepsilon(r,d)\mathbf{1}_{r\notin A}\, dr \geq \int_{a+|A|}^b \lambda_\varepsilon(r,d)\, dr.$$

Thus we would like to estimate the measure of A_k^j. Toward this end, let

$$Z := \cup_{\{B_k^\sigma \in \mathcal{B}(\sigma)\, :\, d_k^\sigma = 0\}} B_k^\sigma$$

and note that if $\partial B(r,\widetilde{x}_k^j) \cap Z = \emptyset$, then $\mathrm{dg}(u, \partial B(r,\widetilde{x}_k^j))$ is well-defined and equals $\widetilde{d}_k^j$, as a consequence of the definition of dg together with (A.11), (A.22), (A.24), and the definition of Z. So

$$A_k^j \subset \{r \in (\widetilde{r}^j, \widetilde{R}^j) : \partial B(r,\widetilde{x}_k^j) \cap Z \neq \emptyset\}.$$

Since Z is a union of balls whose radii sum to at most σ, see (A.20), we conclude that $|A_k^j| \leq 2\sigma$. Thus for every j,k,

$$\int_{\widetilde{A}_k^j} e_\varepsilon^{2d}(w)dx \geq \int_{\widetilde{r}_*^j}^{\widetilde{R}^j} \lambda_\varepsilon(r,\tilde{d}_k^j)\, dr, \qquad \widetilde{r}_*^j := \min\{\widetilde{r}^j + 2\sigma, \widetilde{R}^j\} \overset{(A.25)}{\leq} C(n)\widetilde{R}^{j-1}. \tag{A.30}$$

Also, it is straightforward to check from the definition of λ_ε that if $r \geq \varepsilon^a$ and $d \leq n$, then

$$\lambda_\varepsilon(r,d) \geq \frac{\pi d^2}{r}(1 - C(n)\varepsilon^{1-a}).$$

Thus

$$\sum_{j=1}^J \sum_k \int_{\widetilde{A}_k^j} e_\varepsilon^{2d}(w)\, dx \geq (1 - C\varepsilon^{1-a}) \sum_j \pi \log(\frac{\widetilde{R}^j}{\widetilde{r}_*^j}) \left(\sum_k (d_k^j)^2\right).$$

Step 4. We wish to show that neither of the conditions appearing in (A.21) can be violated. We thus consider two cases.

Case 1: $d_k^1 > 1$ **for some** k**.** Then it follows from (A.28) that $\sum_k (d_k^j)^2 > n+2$ for all j, and hence that

$$\sum_{j=1}^J \sum_k \int_{\widetilde{A}_k^j} e_\varepsilon^{2d}(w)\, dx \geq \pi(n+2)(1 - C\varepsilon^{1-a}) \sum_{j=1}^J \log(\frac{\widetilde{R}^j}{\widetilde{r}_*^j})$$
$$= \pi(n+2)(1 - C\varepsilon^{1-a}) \left[\log(\frac{\widetilde{R}^J}{\widetilde{r}_*^1}) + \sum_{j=2}^J \log(\frac{\widetilde{R}^{j-1}}{\widetilde{r}_*^j})\right].$$

But it follows from (A.30), (A.26) and (A.23)

$$\frac{\widetilde{R}^{j-1}}{\widetilde{r}_*^j} \geq \frac{1}{C}, \qquad \frac{\widetilde{R}^J}{\widetilde{r}_*^1} \geq \frac{1}{C\sigma} = \frac{1}{C\varepsilon^{2a}}$$

for constants depending on n. It follows that

$$\sum_{j=1}^{J}\sum_k \int_{\widetilde{A}_k^j} e_\varepsilon^{2d}(w)\,dx \geq \pi(n+2)2a|\log\varepsilon| - C.$$

Combining this with (A.15) we find that

$$\int_\omega e_\varepsilon^{2d}(w) \geq \pi(n+4a)|\log\varepsilon| - C,$$

contradicting (A.1) when ε is small enough.

Case 2: $d_k^1 = 1$ for all k, but $\widetilde{R}^1 \leq \varepsilon^b/2$. This implies that $\widetilde{r}_*^2 \leq Ce^b$. Then essentially the same arguments as above show that

$$\sum_{j=1}^{J}\sum_k \int_{\widetilde{A}_k^j} e_\varepsilon^{2d}(w)\,dx \geq (1 - C\varepsilon^{1-a})\left[n\pi\log(\frac{\widetilde{R}^1}{\widetilde{r}_*^1}) + (n+2)\pi\sum_{j=2}^{J}\log(\frac{\widetilde{R}^j}{\widetilde{r}_*^j})\right].$$

Continuing to follow the previous case, from this one deduces that

$$\int_\omega e_\varepsilon^{2d}(w) \geq \pi(n+2b)|\log\varepsilon| - C,$$

again contradicting (A.1). This verifies (A.27) and completes the proof of the lemma. □

References

[AR01] A. Aftalion and T. Rivière, "Vortex energy and vortex bending for a rotating Bose-Einstein condensate, *Phys Rev A.*, 64 (2001) 043611.

[ABO05] G. Alberti, S. Baldo and G. Orlandi, "Variational convergence for functionals of Ginzburg–Landau type", *Indiana Univ. Math. J.*, (5)54 (2005), 1411–1472

[BFT08] V. Barutello, D. Ferrario and S. Terracini, "On the singularities of generalized solutions to n-body-type problems". *Int. Math. Res. Not.* IMRN (2008), Art. ID rnn 069, 78 pp.

[BBH94] F. Bethuel, H. Brezis and F. Hélein, "Ginzburg–Landau Vortices", *Progress in Nonlinear Differential Equations and their Applications 13,* Birkhäuser Boston, Boston, MA, (1994)

[BBO01] F. Bethuel, H. Brezis, and G. Orlandi, " Asymptotics for the Ginzburg–Landau equation in arbitrary dimensions", *J. Funct. Anal.* (2)186 (2001), 432–520

[BBM04] J. Bourgain, H. Brezis, and P. Mironescu, "$H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg–Landau equation", *Publ. Math. Inst. Hautes tudes Sci.* (99) (2004), 1–115

[Che08] K. Chen, "Existence and minimizing properties of retrograde orbits to the three-body problem with various choices of masses", *Ann. of Math. (2)* (2)167 (2008), 325–348

[CM99] A. Chenciner and R. Montgomery, "A remarkable periodic solution of the three body problem in the case of equal masses", *Ann. of Math.* 152 (1999), 881–901

[Con11] A. Contreras, "On the First critical field in Ginzburg–Landau theory for thin shells and manifolds", *Archive for Rational Mechanics and Analysis*, (2)200, 563–611. (2011)

[PK08] M. del Pino and M. Kowalczyk, "Renormalized energy of interacting Ginzburg–Landau vortex filaments", *J. Lond. Math. Soc. (2)* (3)77 (2008), 647–665

[PKPW10] M. del Pino, M. Kowalczyk, F. Pacard, and J Wei, "The Toda system and multiple-end solutions of autonomous planar elliptic problems", *Adv. Math.* (4)224 (2010), 1462–1516

[PKW08] M. del Pino, M. Kowalczyk, and J Wei, "The Toda system and clustering interfaces in the Allen–Cahn equation", *Arch. Ration. Mech. Anal.* (1)190 (2008), 141–187

[PKWY10] M. del Pino, M. Kowalczyk, J. Wei, and J. Yang, "Interface foliation near minimal submanifolds in Riemannian manifolds with positive Ricci curvature." *Geom. Funct. Anal.* (4)20 (2010), 918–957

[Fed69] H. Federer, "Geometric Measure Theory" *Die Grundlehren der mathematischen Wissenschaften, Band 153*, Springer-Verlag, New York (1969)

[FT04] D. Ferrario and S. Terracini, "On the existence of collisionless equivariant minimizers for the classical n-boby problem", *Invent. Math.* 155 (2004) 305–362

[Jer99a] R. Jerrard, "Lower bounds for generalized Ginzburg–Landau functionals", SIAM Math. Anal. (4)30 (1999a) 721–746

[Jer99b] R. Jerrard, "Vortex dynamics for the Ginzburg–Landau wave equation", *Calc. Var. and PDE,* 9 (1999b), 1–30

[JS02] R. Jerrard and H. Soner, "The Jacobian and the Ginzburg–Landau energy", *Calc. Var and PDE,* 14 (2002), 151-191

[JS07] R. Jerrard and D. Spirn, "Refined Jacobian estimates for Ginzburg–Landau functionals", *Indiana Univ. Math. Jour.,* 56 (2007), 135–186

[JS08] R. Jerrard and D. Spirn, "Refined Jacobian estimates and Gross–Pitaevsky vortex dynamics", *Arch. Rat. Mech. Anal.,* 190 (2008), 425–475

[JS09] R. Jerrard and P. Sternberg, "Critical points via Gamma-convergence: general theory and applications", *Jour. Eur. Math. Soc.,* (4)11 (2009), 705–753

[KPV03] C. Kenig, G. Ponce and L. Vega, "On the interaction of nearly parallel vortex filaments", *Comm. Math. Phys.* 243 (2003) 471–483

[KMD95] R. Klein, A. Majda, and K. Damodaran, "Simplified equations for the interaction of nearly parallel vortex filaments", *J. Fluid Mech.* 228 (1995), 201–248

[Li99] F. H. Lin, "Vortex dynamics for the nonlinear wave equation", *Comm. Pure Appl. Math.* (6)52 (1999), 737–761

[LR01] F.-H. Lin and T. Rivière, "A quantization property for static Ginzburg–Landau vortices", *Comm. Pure Appl. Math.* (2)54 (2001), 206–228

[LM2000] P. L. Lions and A. Majda. "Equilibrium Statistical Theory for Nearly Parallel Vortex Filaments " , *Communications on Pure and Applied Mathematics*, LIII (2000), 0076–0142

[MSZ04] J. A. Montero, P. Sternberg, and W.P. Ziemer, "Local minimizers with vortices in the Ginzburg–Landau system in three dimensions". *Comm. Pure Appl. Math.* (1)57 (2004), 99–125

[PR2000] F. Pacard and T. Rivière, "Linear and nonlinear aspects of vortice. The Ginzburg–Landau model", *Progress in Nonlinear Differential Equations and their Applications 39,* Birkhäuser Boston, Boston, MA, (2000)

[Riv96] T. Rivière, ' Line vortices in the $U(1)$-Higgs model", *ESAIM Contrôle Optim. Calc. Var.* 1 (1995/96), 77–167

[San98] E. Sandier. "Lower bounds for the energy of unit vector fields and applications", *J. Funct. Anal.* 152 (1998), 379–403

[San01] E. Sandier, " Ginzburg–Landau minimizers from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ and minimal connections", *Indiana Univ. Math. J.* (4)50 (2001), 1807–1844

[SS04] E. Sandier and S. Serfaty, " A product-estimate for Ginzburg–Landau and corollaries", *J. Funct. Anal.* (1)211 (2004), 219–244

[SS07] E. Sandier and S. Serfaty. "Vortices in the magnetic Ginzburg–Landau model". *Progress in Nonlinear Differential Equations and their Applications*, 70. Birkhäuser Boston, Inc., Boston, MA, (2007)

[Sol84] B. Solomon, "A new proof of the closure theorem for integral currents", *Indiana Univ. Math. J.,* (3)33 (1984), 393–418

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