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KAM FOR THE NONLINEAR BEAM EQUATION

L. Hakan Eliasson, Benoît Grébert and Sergei B. Kuksin

Abstract. In this paper we prove a KAM theorem for small-amplitude solutions of the non linear beam equation on the d-dimensional torus

$$u_{tt} + \Delta^2 u + mu + \partial_u G(x,u) = 0, \quad t \in \mathbb{R},\ x \in \mathbb{T}^d, \quad (*)$$

where $G(x,u) = u^4 + O(u^5)$. Namely, we show that, for generic m, many of the small amplitude invariant finite dimensional tori of the linear equation $(*)_{G=0}$, written as the system

$$u_t = -v, \quad v_t = \Delta^2 u + mu,$$

persist as invariant tori of the nonlinear equation $(*)$, re-written similarly. The persisted tori are filled in with time-quasiperiodic solutions of $(*)$. If $d \geq 2$, then not all the persisted tori are linearly stable, and we construct explicit examples of partially hyperbolic invariant tori. The unstable invariant tori, situated in the vicinity of the origin, create around them some local instabilities, in agreement with the popular belief in the nonlinear physics that small-amplitude solutions of space-multidimensional Hamiltonian PDEs behave in a chaotic way.

Contents

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1 Introduction

1.1 The beam equation and KAM for PDE's. The paper deals with small-amplitude solutions of the multi-dimensional nonlinear beam equation on the torus:

$$u_{tt} + \Delta^2 u + mu = -g(x,u), \quad u = u(t,x),\ t \in \mathbb{R}, \quad x \in \mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d, \tag{1.1}$$

where g is a real analytic function of $x \in \mathbb{T}^d$ and of u in the vicinity of the origin in $\mathbb{R}$. We shall consider functions g of the form

$$g = \partial_u G, \quad G(x,u) = u^4 + O(u^5). \tag{1.2}$$

The polynomial u^4 is the *main* part of G and $O(u^5)$ is the *higher order* part. m is the mass parameter and we assume that $m \in [1,2]$.

This equation is interesting by itself. Besides, it is a good model for the Klein–Gordon equation

$$u_{tt} - \Delta u + mu = -\partial_u G(x,u), \quad x \in \mathbb{T}^d, \tag{1.3}$$

which is among the most important equations of mathematical physics. We feel confident that the ideas and methods of our work apply—with additional technical efforts—to (1.3). Our approach is adequate to a number of other space-multidimensional Hamiltonian PDEs with a mass parameter. We have chosen the beam equation (1.1) for an object of study in this work simply since this is the easiest space-multidimensional equation of mathematical physics to which the approach applies. Our results, presented below, hold for solutions of eq. (1.1) with a *typical positive* value of the mass parameter. The situation with the nonlinear wave Eq. $(1.3)_{m=0}$, as well as with the zero-mass beam equation, may be quite different.

Our goal is to develop a general KAM-theory for small-amplitude solutions of (1.1). To do this we compare them with time-quasi-periodic solution of the linearised at zero equation

$$u_{tt} + \Delta^2 u + mu = 0. \tag{1.4}$$

Decomposing real functions $u(x)$ on $\mathbb{T}^d$ to Fourier series

$$u(x) = \sum_{a \in \mathbb{Z}^d} u_a e^{\mathbf{i}\langle a,x\rangle} + \text{c.c.}$$

(here c.c. stands for "complex conjugated"), we write time-quasiperiodic solutions for (1.4), corresponding to a finite set of excited wave-vectors $\mathcal{A} \subset \mathbb{Z}^d$, as

$$u(t,x) = \sum_{a \in \mathcal{A}} \left(\xi_a e^{\mathbf{i}\lambda_a t} + \eta_a e^{-\mathbf{i}\lambda_a t}\right) e^{\mathbf{i}\langle a,x\rangle} + \text{c.c.}, \tag{1.5}$$

where $\lambda_a = \sqrt{|a|^4 + m}$. We examine these solutions and their perturbations in Eq. (1.1) under the assumption that the action-vector $I = \{\frac{1}{2}(|\xi_a|^2 + |\eta_a|^2),\ a \in \mathcal{A}\}$ is small. In our work this goal is achieved provided that

- the finite set $\mathcal{A}$ is typical in a probabilistic sense;

- the mass parameter m does not belong to a certain set of zero measure.

The linear stability of the obtained solutions for (1.1) is under control. If $d \geq 2$, and $|\mathcal{A}| \geq 2$, then some of them are linearly unstable.

Before giving exact statement of the result, we discuss the state of affairs in the KAM for PDE theory. The theory started in late 1980's and originally applied to 1d Hamiltonian PDEs, see in [Kuk87, Kuk93, Cra00]. The first works on this theory treated

(a) perturbations of linear Hamiltonian PDE, depending on a vector-parameter of the dimension equal to the number of frequencies of the unperturbed quasiperiodic solution of the linear system (for solutions (1.5) this is $|\mathcal{A}|$). Usually the parameter enters the equation through the potential term $V(x)u(t,x)$, where the potential V depends on the parameter. Next the theory was applied to

(b) perturbations of integrable Hamiltonian PDE, e.g. of the KdV or Sine-Gordon equations, see [Kuk00]. In paper [BK95]

(c) small-amplitude solutions of the 1d Klein–Gordon equation (1.3) with $G(x,u) = -u^4 + O(u^4)$ were treated as perturbed solutions of the Sine-Gordon equation, and a singular version of the KAM-theory (b) was developed to study them (Notice that for suitable a and b we have $mu - u^3 + O(u^4) = a \sin bu + O(u^4)$. So the 1d Eq. (1.3) is the Sine-Gordon equation, perturbed by a small term $O(u^4)$).

It was proved in [BK95] that for a.a. values of m and for any finite set $\mathcal{A}$ most of the small-amplitude solutions (1.5) for the linear Klein-Gordon equation (with $\lambda_a = \sqrt{|a|^2 + m}$) persist as linearly stable time-quasiperiodic solutions for (1.3). In [KP96] it was realised that it can be fruitful in 1d equations like (1.3), just as it is in finite-dimensional Hamiltonian systems (see for example [Eli88]), to study small solutions not as perturbations of solutions for an integrable PDE, but rather as perturbations of solutions for a Birkhoff–integrable system, after the equation is normalised by a Birkhoff transformation. The paper [KP96] deals not with 1d Klein–Gordon equation (1.3), but with 1d NLS equation, which is similar to (1.3) for the problem under discussion; in [Pös96] the method of [KP96] was applied to the 1d Eq. (1.3). The approach of [KP96] turned out to be very efficient and later was used for many other 1d Hamiltonian PDEs. In [GY06] it was applied to the d-dimensional beam equation (1.1) with an x-independent nonlinearity g and allowed to treat perturbations of some special solutions (1.5).

Space-multidimensional KAM for PDE theory started 10 years later with the paper [Bou98] and, next, publications [Bou04, EK10]. The just mentioned works deal with perturbations of parameter-depending linear equations (cf. (a)), and in difference with the 1d case the parameter enters the equations through physically unnatural convolutive term $V(x) * u(t,x)$.[1] The approach of [EK10] is different from that

[1] Since for $d \geq 2$ the spectral theory of differential operators with mutiplicative potentials is too complicated.

of [Bou98, Bou04] and allows to analyse the linear stability of the obtained KAM-solutions. Also see [BB12, BB13, GP16]. Since integrable space-multidimensional PDE (practically) do not exist, then no multi-dimensional analogy of the 1d theory (b) is available.

Efforts to create space-multidimensional analogies of the KAM-theory (c) were made in [Wan, PP12, PP13], using the KAM-techniques of [Bou98, Bou04, EK10], respectively. All the three works deal with the NLS equation. Their main disadvantage (compare to the 1d theory (c)) is severe restrictions on the finite set $\mathcal{A}$ (i.e. on the class of unperturbed solutions which the methods allow to perturb). The result of [Wan] gives examples of some sets $\mathcal{A}$ for which the KAM-persistence of the corresponding small-amplitude solutions (1.5) holds, while the result of [PP12, PP13] applies to solutions (1.5), where the set $\mathcal{A}$ is nondegenerate in certain very non-explicit way.

Some KAM-theorems for small-amplitude solutions of multidimensional beam equations (1.1) with typical m were obtained in [GY06, GY06]. Both works treat equations with a constant-coefficient nonlinearity $g(x,u) = g(u)$, which is significantly easier than the general case (cf. the linear theory, where constant-coefficient equations may be integrated by the Fourier method). Similar to [Wan, PP12, PP13], the theorems of [GY06, GY06] only allow to perturb solutions (1.5) with very special sets $\mathcal{A}$ (see also "Appendix B"). Solutions of (1.1), constructed in these works, all are linearly stable.

1.2 Beam equation in real and complex variables. Introducing $v = u_t \equiv \dot{u}$ we rewrite (1.1) as

$$\begin{cases} \dot{u} = -v, \\ \dot{v} = \Lambda^2 u + g(x,u), \end{cases} \tag{1.6}$$

where $\Lambda = (\Delta^2 + m)^{1/2}$. Defining $\psi(t,x) = \frac{1}{\sqrt{2}}(\Lambda^{1/2}u + \mathbf{i}\Lambda^{-1/2}v)$ we get for the complex function $\psi(t,x)$ the equation

$$\frac{1}{\mathbf{i}}\dot{\psi} = \Lambda\psi + \frac{1}{\sqrt{2}}\Lambda^{-1/2} g\left(x, \Lambda^{-1/2}\left(\frac{\psi+\bar{\psi}}{\sqrt{2}}\right)\right).$$

Thus, if we endow the space $L^2(\mathbb{T}^d, \mathbb{C})$ with the standard real symplectic structure, given by the two-form $-\mathbf{i}d\psi \wedge d\bar{\psi}$, then Eq. (1.1) becomes a Hamiltonian system

$$\dot{\psi} = \mathbf{i}\,\partial h/\partial\bar{\psi}$$

with the Hamiltonian function

$$h(\psi,\bar{\psi}) = \int_{\mathbb{T}^d} (\Lambda\psi)\bar{\psi}\mathrm{d}x + \int_{\mathbb{T}^d} G\left(x, \Lambda^{-1/2}\left(\frac{\psi+\bar{\psi}}{\sqrt{2}}\right)\right)\mathrm{d}x.$$

The linear operator Λ is diagonal in the complex Fourier basis

$$\{e_a(x) = (2\pi)^{-d/2} e^{\mathbf{i}\langle a,x\rangle},\ a \in \mathbb{Z}^d\}.$$

Namely,

$$\Lambda e_a = \lambda_a e_a, \quad \lambda_a = \sqrt{|a|^4 + m}, \quad \forall\, a \in \mathbb{Z}^d.$$

Let us decompose ψ and $\bar\psi$ in the basis $\{e_a\}$:

$$\psi = \sum_{a\in\mathbb{Z}^d} \xi_a e_a, \quad \bar\psi = \sum_{a\in\mathbb{Z}^d} \eta_a e_{-a}\,.$$

Let

$$\begin{cases} p_a = \frac{1}{\sqrt{2}}(\xi_a + \eta_a), \\ q_a = \frac{\mathbf{i}}{\sqrt{2}}(\xi_a - \eta_a), \end{cases} \tag{1.7}$$

and denote by ζ_a the pair (p_a, q_a).[2]

We fix any $m_* > d/2$ and define the Hilbert space

$$Y = \left\{ \zeta = (p,q) \in \ell^2(\mathbb{Z}^d,\mathbb{C}) \times \ell^2(\mathbb{Z}^d,\mathbb{C}) \mid \|\zeta\|^2 = \sum_a \langle a\rangle^{2m_*} |\zeta_a|^2 < \infty \right\}, \tag{1.8}$$

where $\langle a\rangle = \max(1,|a|)$, corresponding to Fourier coefficients of complex functions $(\psi(x), \bar\psi(x))$ from the Sobolev space $H^{m_*}(\mathbb{T}^d, \mathbb{C}^2)$. A vector $\zeta \in Y$ is *real* if, and only if, all its components are real.

Let us endow Y with the symplectic structure

$$\big(dp \wedge dq\big)(\zeta,\zeta') = \sum_a \langle J\zeta_a, \zeta'_a\rangle, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.9}$$

and consider there the Hamiltonian system

$$\dot\zeta_a = J\frac{\partial h}{\partial \zeta_a}, \quad a \in \mathbb{Z}^d, \tag{1.10}$$

where the Hamiltonian function h equals the quadratic part

$$h_2 = \frac{1}{2}\sum_{a\in\mathbb{Z}^d} \lambda_a (p_a^2 + q_a^2) \tag{1.11}$$

plus the higher order term

$$h_{\geq 4} = \int_{\mathbb{T}^d} G\left(x, \sum_{a\in\mathbb{Z}^d} \frac{(p_a - \mathbf{i}q_a)e_a + (p_{-a} + \mathbf{i}q_{-a})e_a}{2\sqrt{\lambda_a}}\right) \mathrm{d}x. \tag{1.12}$$

The beam Eq. (1.6), considered in the Sobolev space $\{(u,v) \mid (\psi,\bar\psi) \in H^{m_*}\}$, is equivalent to the Hamiltonian system (1.10).

We will write the Hamiltonian h as

$$h = h_2 + h_{\geq 4} = h_2 + h_4 + h_{\geq 5}, \tag{1.13}$$

[2] ζ_a will be considered as a line-vector or a colon-vector according to the context.

where

$$h_4 = \int_{\mathbb{T}^d} u^4 \mathrm{d}x = \int_{\mathbb{T}^d} \left(\sum_{a\in\mathbb{Z}^d} \frac{(p_a - \mathbf{i}q_a)e_a + (p_{-a} + \mathbf{i}q_{-a})e_a}{2\sqrt{\lambda_a}} \right)^4 \mathrm{d}x, \tag{1.14}$$

$h_{\geq 5} = O(u^5)$ comprises the remaining higher order terms and $h_{\geq 4} = h_4 + h_{\geq 5}$. Note that h_4 satisfies the *zero momentum condition*, i.e.

$$h_4 = \sum_{a,b,c,d\in\mathbb{Z}^d} C(a,b,c,d)(\xi_a + \eta_{-a})(\xi_b + \eta_{-b})(\xi_c + \eta_{-c})(\xi_d + \eta_{-d}),$$

where $C(a,b,c,d) \neq 0$ only if $a+b+c+d = 0$. This condition turns out to be useful to restrict the set of small divisors that have to be controlled. If the function G does not depend on x, then h satisfies a similar property at any order.

1.3 Invariant tori and admissible sets. The quadratic Hamiltonian h_2 (which is h when $G = 0$ in (1.1)) is integrable and its phase-space is foliated into (Lagrangian or isotropic) invariant tori. Indeed, take a finite subset $\mathcal{A} \subset \mathbb{Z}^d$ and let

$$\mathcal{L} = \mathbb{Z}^d \backslash \mathcal{A}.$$

For any subset X of $\mathbb{Z}^d$, consider the projection

$$\pi_X : (\mathbb{C}^2)^{\mathbb{Z}^d} \to (\mathbb{C}^2)^X = \left\{ \zeta \in (\mathbb{C}^2)^{\mathbb{Z}^d} : \zeta_a = 0 \ \forall a \notin X \right\}.$$

We can thus write $(\mathbb{C}^2)^{\mathbb{Z}^d} = (\mathbb{C}^2)^X \oplus (\mathbb{C}^2)^{\mathbb{Z}^d\backslash X}$, $\zeta = (\zeta_X, \zeta_{\mathcal{Z}\backslash X})$, and when X is finite this gives an injection

$$(\mathbb{C}^2)^{\#X} \hookrightarrow (\mathbb{C}^2)^{\mathbb{Z}^d}$$

whose image is $(\mathbb{C}^2)^X$.[3]

For any real vector with positive components $I_{\mathcal{A}} = (I_a)_{a\in\mathcal{A}}$, the $|\mathcal{A}|$-dimensional torus

$$T_{I_{\mathcal{A}}} = \begin{cases} p_a^2 + q_a^2 = 2I_a, & p_a, q_a \in \mathbb{R}, \ a \in \mathcal{A}, \\ p_a = q_a = 0, & a \in \mathcal{L}, \end{cases} \tag{1.15}$$

is invariant under the flow of h_2. $T_{I_{\mathcal{A}}}$ is the image of the torus

$$\mathbb{T}^{\mathcal{A}} = \{r_{\mathcal{A}} = 0\} \times \{\theta_a \in \mathbb{T} : a \in \mathcal{A}\} \times \{\zeta_{\mathcal{L}} = 0\} \tag{1.16}$$

under the embedding

$$U_{I_{\mathcal{A}}} : \theta_{\mathcal{A}} \mapsto \begin{cases} p_a - \mathbf{i}q_a = \sqrt{2I_a}\, e^{\mathbf{i}\theta_a}, & a \in \mathcal{A}, \\ p_a = q_a = 0, & a \in \mathcal{L}, \end{cases} \tag{1.17}$$

the pull-back, by $U_{I_{\mathcal{A}}}$, of the induced flow is simply the translation

$$\theta_{\mathcal{A}} \mapsto \theta_{\mathcal{A}} + t\omega_{\mathcal{A}}, \tag{1.18}$$

[3] We shall frequently, without saying, identify $(\mathbb{C}^2)^X$ and $(\mathbb{C}^2)^{\#X}$.

where we have denoted the translation vector (the tangential frequencies) by $\omega_{\mathcal{A}}$, i.e. $\lambda_a = \omega_a$ for $a \in \mathcal{A}$. The parametrised curve

$$t \mapsto U_{I_{\mathcal{A}}}(\theta + t\omega)$$

is thus a quasi-periodic solution of the beam Eq. (1.10) when $G = 0$.

When $G \neq 0$ the higher order terms in h give rise to a perturbation of h_2—a perturbation that gets smaller, the smaller is I. Our goal is to prove the persistency of the invariant torus $T_I^{\mathcal{A}}$, or, more precisely, of the invariant embedding $U_I^{\mathcal{A}}$, for most values of I when the higher order terms are taken into account. The problem doing this for this model is twofold. First the integrable Hamiltonian h_2 is completely degenerate in the sense of KAM-theory: the frequencies $\omega_{\mathcal{A}}$ do not depend on I. One can try to improve this by adding to h_2 an integrable part of the Birkhoff normal form. This will, in "generic" situations, correct this default. However, and that's the second problem, our model is far from "generic" since the eigenvalues $\{\lambda_a : a \in \mathbb{Z}^d\}$ are very resonant. This has the effect that the Birkhoff normal form is not integrable, and therefore is difficult to use.

An important part of our analysis will be to show that this program can be carried out if we exclude a zero-measure set of masses m and restrict the choice of $\mathcal{A}$ to *admissible* or *strongly admissible* sets.

Let $|\cdot|$ denote the euclidean norm in $\mathbb{R}^d$. For vectors $a, b \in \mathbb{Z}^d$ we define

$$a \angle b \quad \text{iff} \quad \#\left\{x \in \mathbb{Z}^d \mid |x| = |a| \text{ and } |x-b| = |a-b|\right\} \leq 2. \tag{1.19}$$

Relation $a \angle b$ means that the integer sphere of radius $|b-a|$ with the centre at b intersects the integer sphere $\{x \in \mathbb{Z}^d \mid |x| = |a|\}$ in at most two points.

Definition 1.1. A finite set $\mathcal{A} \in \mathbb{Z}^d$ is called admissible iff

$$a, b \in \mathcal{A}, \quad a \neq b \Rightarrow |a| \neq |b|\,.$$

An admissible set $\mathcal{A}$ is called strongly admissible iff

$$a, b \in \mathcal{A}, \quad a \neq b \Rightarrow a \angle a + b\,.$$

Certainly if $|\mathcal{A}| \leq 1$, then $\mathcal{A}$ is admissible, but for $|\mathcal{A}| > 1$ this is not true. For $d \leq 2$ every admissible set is strongly admissible, but in higher dimension this is no longer true: see for example the set (B.2) in "Appendix B".

However, strongly admissible, and hence admissible sets are typical: see "Appendix E" for a precise formulation and proof of this statement.

We shall define a subset of $\mathcal{L}$, important for our construction:

$$\mathcal{L}_f = \{a \in \mathcal{L} \mid \exists\, b \in \mathcal{A} \quad \text{such that} \quad |a| = |b|\}. \tag{1.20}$$

Clearly $\mathcal{L}_f$ is a finite subset of $\mathcal{L}$. For example, if $d = 1$ and $\mathcal{A}$ is admissible, then $\mathcal{A} \cap -\mathcal{A} \subset \{0\}$ and $\mathcal{L}_f = -(\mathcal{A} \backslash \{0\})$.

1.4 The Birkhoff normal form. In a neighbourhood of an invariant torus $T_{I_\mathcal{A}}$ we introduce (partial) action-angle variables $(r_\mathcal{A}, \theta_\mathcal{A}, \xi_\mathcal{L}, \eta_\mathcal{L})$ by the relation

$$\frac{1}{\sqrt{2}}(p_a - \mathbf{i}q_a) = \sqrt{I_a + r_a}\, e^{\mathbf{i}\theta_a}, \quad a \in \mathcal{A}. \tag{1.21}$$

These variables define a diffeomorphism from a neighbourhood of $\mathbb{T}^\mathcal{A}$ in (the Hilbert manifold)

$$\mathbb{C}^\mathcal{A} \times (\mathbb{C}/2\pi\mathbb{Z})^\mathcal{A} \times \pi_\mathcal{L} Y \tag{1.22}$$

to a neighbourhood of $T_{I_\mathcal{A}}$ in Y. It is *real* in the sense that it gives real values to real arguments.

The symplectic structure on Y is pull-backed to

$$dr_\mathcal{A} \wedge d\theta_\mathcal{A} + d\xi_\mathcal{L} \wedge d\eta_\mathcal{L}, \tag{1.23}$$

which endows the space (1.22) with a symplectic structure.

In these variables h will depend on I, but its integrable part h_2 becomes, up to an additive constant,

$$\sum_{a\in\mathcal{A}} \omega_a r_a + \frac{1}{2}\sum_{a\in\mathcal{L}} \lambda_a (p_a^2 + q_a^2)$$

which does not depend in I.[4] The Birkhoff normal form will provide us with an integrable part that does depend on I. We shall prove

Theorem 1.2. *There exists a zero-measure Borel set $\mathcal{C} \subset [1,2]$ such that for any $m \notin \mathcal{C}$, any admissible set $\mathcal{A}$, any $c_* \in (0, 1/2]$ and any analytic nonlinearity of the form (1.2), there exist $\nu_0 > 0$ and $\beta_0 > 0$ such that for any $0 < \nu \le \nu_0$, $0 < \beta_\# \le \beta_0$ there exists an open set $Q \subset [\nu c_*, \nu]^\mathcal{A}$,*

$$\operatorname{meas}([\nu c_*, \nu]^\mathcal{A} \backslash Q) \le C\nu^{\#\mathcal{A}+\beta_\#},$$

and for every $I = I_\mathcal{A} \in Q$ there exists a real symplectic holomorphic diffeomorphism Φ_I, defined in a neighbourhood (that depends on c_* and ν) of $\mathbb{T}^\mathcal{A}$ such that the transformed Hamiltonian

$$\begin{aligned} &h \circ \Phi_I(r_\mathcal{A}, \theta_\mathcal{A}, p_\mathcal{L}, q_\mathcal{L}) = \langle \Omega(I), r_\mathcal{A}\rangle + \tfrac{1}{2}\textstyle\sum_{a\in\mathcal{L}\backslash\mathcal{L}_f} \Lambda_a(I)(p_a^2 + q_a^2) \\ &+\tfrac{1}{2}\textstyle\sum_{b\in\mathcal{L}_f\backslash\mathcal{F}} \Lambda_b(I)(p_b^2 + q_b^2) + \langle K(I)\zeta_\mathcal{F}, \zeta_\mathcal{F}\rangle + f_I(r_\mathcal{A}, \theta_\mathcal{A}, p_\mathcal{L}, q_\mathcal{L}) \end{aligned} \tag{1.24}$$

(where $\mathcal{F} = \mathcal{F}_I$ is a subset of $\mathcal{L}_f$, possible empty) has the following properties:

(i) *$\Omega(I) = \omega_\mathcal{A} + MI$ and the matrix M is invertible;*
(ii) *each $\Lambda_a(I)$, $a \in \mathcal{L}\backslash\mathcal{L}_f$, is real and close to λ_a,*

$$|\Lambda_a(I) - \lambda_a| \le C\,|I|\,\langle a\rangle^{-2};$$

[4] Both h_2 and the higher order terms of h depend on the mass m.

(iii) *each* $\Lambda_b(I)$, $b \in \mathcal{L}_f \backslash \mathcal{F}$, *is real and non-zero,*

$$C^{-1}\,|I|^{1+c\beta_\#} \le |\Lambda_b(I)| \le C\,|I|^{1-c\beta_\#}\,;$$

(iv) *the operator* $K(I)$ *is real symmetric and satisfies* $\|K(I)\| \le C\,|I|^{1-c\beta_\#}$. *The Hamiltonian operator* $JK(I)$ *is hyperbolic (unless* $\mathcal{F}_I$ *is empty), and the moduli of the real parts of its eigenvalues are bigger than* $C^{-1}\,|I|^{1+\beta_\#}$;

(v) *The function* f_I *is much smaller than the quadratic part.*

Moreover, all objects depend C^∞ *on* I. *The set* $\mathcal{F}_I$ *does not depend on* I *when* I *stays in the same connected component of* Q, *but may be different for different components.*

This result is proven in Part II. For a more precise formulation, giving in particular the domain of definition of Φ_I, the smallness in f_I and estimates of the derivatives with respect to I, see Theorem 5.1. The matrix M is explicitly defined in (4.44), and the functions Λ_a are explicitly defined in (4.45). An interesting information is that the mapping Φ_I and the domain Q only depend on $h_2 + h_4$, and that the set $\mathcal{F}_I$ is empty on some connected components of Q.

1.5 The KAM theorem. The Hamiltonian $h_I \circ \Phi_I$ (1.24) is much better than h_I since its integrable part depends on I in a non-degenerate way because M is invertible. Does the invariant torus (1.16) persist under the perturbation f_I? ... and, if so, is the persisted torus reducible?

In finite dimension the answer is yes under very general conditions—for the first proof in the purely elliptic case see [Eli88], and for a more general case see [You99]. These statements say that, under general conditions, the invariant torus persists and remains reducible under sufficiently small perturbations for a subset of parameters of large Lebesgue measure.

In infinite dimension the situation is more delicate, and results can only be proven under quite severe restrictions on the normal frequencies Λ_a; see the discussion above in Sect. 1.1. A result for the beam equation (which is a simpler model than the Schrödinger and wave equations) was first obtained in [GY06, GY06]. Here we prove a KAM-theorem which improves on these results in at least two respects:

- We have imposed no "conservation of momentum" on the perturbation, which allows us to treat equations (1.1) with x-dependent nonlinearities g. This has the effect that our normal form is not diagonal in the purely elliptic directions. In this respect it resembles the normal form obtained in [EK10] for the nonlinear Schrödinger equation, and where the block diagonal form is the same.
- We have a finite-dimensional, possibly hyperbolic, component, whose treatment requires higher smoothness in the parameters.

The proof has the structure of a classical KAM-theorem carried out in a complex infinite-dimensional situation. The main part is, as usual, the solution of the homological equation with reasonable estimates. The fact that the block structure is not

diagonal complicates a lot: see for example, [EK10] where this difficulty was also encountered. The iteration combines a finite linear iteration with a "super-quadratic" infinite iteration. This has become quite common in KAM and was also used in [EK10].

A technical difference, with respect to [EK10], is that here we use a different matrix norm which has much better multiplicative properties. This simplifies a lot the functional analysis which is described in Part I.

A special difficulty in our setting is that we are facing a *singular perturbation problem*. The perturbation f_I becomes small only by taking I small, but when I gets smaller the integrable part becomes more degenerate. This is seen for example in the lower bounds for $\Lambda_b(I)$ and for the real parts of the eigenvalues of $JK(I)$. So there is a competition between the smallness condition on the perturbation and the degeneracies of the integrable part which requires quite careful estimates.

A KAM-theorem which is adapted to our beam equation is proven in Part III and formulated in Theorem 6.7 and its Corollary 6.9.

1.6 Small amplitude solutions for the beam equation. Applying to the normal form of Part II, the KAM theorem of Part III, we obtain in Part IV the main results of this work. To state them we recall that a Borel subset $\mathfrak{J} \subset \mathbb{R}_+^{\mathcal{A}}$ is said to have a *positive density* at the origin if

$$\liminf_{\nu \to 0} \frac{\operatorname{meas}(\mathfrak{J} \cap \{x \in \mathbb{R}_+^{\mathcal{A}} \, |x| < \nu\})}{\operatorname{meas}\{x \in \mathbb{R}_+^{\mathcal{A}} \, |x| < \nu\}} > 0. \tag{1.25}$$

The set $\mathfrak{J}$ has the *density one* at the origin if the lim inf above equals one (so the ratio of the measures of the two sets converges to one as $\nu \to 0$).

Theorem 1.3. *There exists a zero-measure Borel set $\mathcal{C} \subset [1,2]$ such that for any strongly admissible set $\mathcal{A} \subset \mathbb{Z}^d$, any $m \notin \mathcal{C}$ and any analytic nonlinearity (1.2), there exist constants $\aleph_1 \in (0, 1/16], \aleph_2 > 0$, only depending on $\mathcal{A}$ and m, and a set $\mathfrak{J} = \mathfrak{J}_{\mathcal{A}} \subset]0,1]^{\mathcal{A}}$, having density one at the origin, with the following property: There exist a constant $C > 0$, a real continuous mapping $U' = U'_{\mathcal{A}} : \mathbb{T}^{\mathcal{A}} \times \mathfrak{J} \to Y$, analytic in the first argument, satisfying*

$$\big|\big|U'(\theta, I) - U_I(\theta)\big|\big| \le C|I|^{1-\aleph_1} \tag{1.26}$$

(see (1.17)) for all $(\theta, I) \in \mathbb{T}^{\mathcal{A}} \times \mathfrak{J}$, and a continuous mapping $\Omega' = \Omega'_{\mathcal{A}} : \mathfrak{J} \to \mathbb{R}^{\mathcal{A}}$,

$$|\Omega'(I) - \omega_{\mathcal{A}} - MI| \le C|I|^{1+\aleph_2}, \tag{1.27}$$

where the matrix M is the same as in (1.24), such that:

(i) *for any $I \in \mathfrak{J}$ and $\theta \in \mathbb{T}^{\mathcal{A}}$ the parametrised curve*

$$t \mapsto U'(\theta + t\Omega'(I), I) \tag{1.28}$$

is a solution of the beam Eqs. (1.10)–(1.12), and, accordingly, the analytic torus $U'(\mathbb{T}^{\mathcal{A}}, I)$ is invariant for this equation;

(ii) *the set $\mathfrak{J}$ may be written as a countable disjoint union of compact sets $\mathfrak{J}_j$, such that the restrictions of the mappings U' and Ω' to the sets $\mathbb{T}^{\mathcal{A}} \times \mathfrak{J}_j$ are C^1 Whitney -smooth;*

(iii) *the solution (1.28) is linearly stable if and only if in (1.24) the operator $K(I)$ is trivial (i.e. the set $\mathcal{F} = \mathcal{F}_I$ is empty). The set of $I \in \mathfrak{J}$ such that $K(I)$ is trivial is always of positive measure, and it equals $\mathfrak{J}$ if $d = 1$ or $|\mathcal{A}| = 1$, but for $d \geq 2$ and for some choices of the set $\mathcal{A}$ its complement has positive measure.*

If the set $\mathcal{A}$ is admissible but not strongly admissible, then a weaker version of the theorem above is true.

Theorem 1.4. *There exists a zero-measure Borel set $\mathcal{C} \subset [1, 2]$ such that for any admissible set $\mathcal{A} \subset \mathbb{Z}^d$, any $m \notin \mathcal{C}$ and any analytic nonlinearity (1.2), there exist constants $\aleph_1 \in (0, 1/16], \aleph_2 > 0$, only depending on $\mathcal{A}$ and m, and a set $\mathfrak{J} = \mathfrak{J}_{\mathcal{A}} \subset]0, 1]^{\mathcal{A}}$, having positive density at the origin, such that all assertions of Theorem 1.3 are true.*

Remark 1.5.

1. The torus $U_I(\mathbb{T}^{\mathcal{A}}, I)$ (see (1.15)), invariant for the linear beam Eq. $(1.10)_{G=0}$, is of size $\sim \sqrt{I}$. The constructed invariant torus $U'_{\mathcal{A}}(\mathbb{T}^{\mathcal{A}}, I)$ of the nonlinear beam equation is a small perturbation of $U_I(\mathbb{T}^{\mathcal{A}}, I)$ since by (1.26) the Hausdorff distance between $U'_{\mathcal{A}}(\mathbb{T}^{\mathcal{A}}, I)$ and $U_I(\mathbb{T}^{\mathcal{A}})$ is smaller than $C|I|^{1-2\aleph_1} \leq C|I|^{7/8}$.
2. Denote by $\mathcal{T}_{\mathcal{A}}$ the image of the mapping $U'_{\mathcal{A}}$. This set is invariant for the beam equation and is filled in with its time-quasiperiodic solutions. By the item ii) of Theorem 1.3 its Hausdorff dimension equals $2|\mathcal{A}|$. Now consider $\mathcal{T} = \cup \mathcal{T}_{\mathcal{A}}$, where the union is taken over all strongly admissible sets $\mathcal{A} \subset \mathbb{Z}^d$. This invariant set has infinite Hausdorff dimension. Some time-quasiperiodic solutions of (1.1), lying on $\mathcal{T}$, are linearly stable, while, if $d \geq 2$, some others are unstable.
3. Our result applies to Eq. (1.1) with any d. Notice that for d sufficiently large the global in time well-posedness of this equation is unknown.
4. The construction of solutions (1.28) crucially depends on certain equivalence relation in $\mathbb{Z}^d$, defined in terms of the set $\mathcal{A}$ (see (5.15)). This equivalence is trivial if $d = 1$ or $|\mathcal{A}| = 1$ and is non-trivial otherwise.
5. We discuss in "Appendix B" examples of sets $\mathcal{A}$ for which the operator $K(I)$ is non-trivial for certain values of I. Note that examples of partially hyperbolic KAM-solutions for the 2d cubic NLS equation were constructed earlier in the work [GXY11].
6. The solutions (1.28) of Eq. (1.10), written in terms of the $u(x)$-variable as solutions $u(t, x)$ of Eq. (1.1), are H^{m_*+1}-smooth as functions of x and analytic as functions of t. Here m_* is a parameter of the construction for which we can take any real number $> d/2$ (see (1.8)). The set $\mathfrak{J}$ depends on m_*, so the assertion of the theorem does not imply immediately that the solutions $u(t, x)$ are C^∞–smooth in x. Still, since

$$-(\Delta^2 + m)u = u_{tt} + \partial_u G(x,u),$$

where G is an analytic function, then the theorems imply by induction that the solutions $u(t,x)$ define analytic curves $\mathbb{R} \to H^m(\mathbb{T}^d)$, for any m. In particular, they are smooth functions.

Structure of Text. The paper consists of Introduction and four parts. Part I comprises general techniques needed to read the paper. The main Parts II–III are independent of each other, and the final Part IV, containing the proofs of Theorems 1.3, 1.4, uses only the main theorems of Parts II-III, and the intermediate results are not needed to understand it.

Some Notation and Agreements. We denote a cardinality of a set X as $|X|$ or as $\#X$. For $a \in \mathbb{Z}^N$ we denote $\langle a \rangle = \max(1, |a|)$.

In any finite-dimensional space X we denote by $|\cdot|$ the Euclidean norm. For subsets X and Y of a Euclidean space we denote

$$\underline{\operatorname{dist}}\,(X,Y) = \inf_{x\in X, y\in Y} |x-y|, \quad \operatorname{diam}(X) = \sup_{x,y\in X} |x-y|.$$

The distance on a torus induced by the Euclidean distance (on the tangent space) will be denoted $|\cdot -_* \cdot|$.

For any matrix A, finite or infinite, we denote by ${}^tA$ the transposed matrix. I stands for the identity matrix of any dimension.

The space of bounded linear operators between Banach spaces X and Y is denoted $\mathcal{B}(X,Y)$. Its operator norm will be usually denoted $\|\cdot\|$ without specification the spaces. If A is a finite matrix, then $\|A\|$ stands for its operator-norm.

We call analytic mappings between domains in complex Banach spaces *holomorphic* to reserve the name *analytic* for mappings between domains in real Banach spaces. This definition extends from Banach spaces to Banach manifolds.

Pairings in l^2-spaces. The scalar product on any complex Hilbert space is, by convention, complex anti-linear in the first variable and complex linear in the second variable. For any l^2-space X of finite or infinite dimension, the natural complex-bilinear pairing is denoted

$$\langle \zeta, \zeta' \rangle = \langle \bar{\zeta}, \zeta' \rangle_{l^2}, \quad \zeta, \zeta' \in X. \tag{1.29}$$

This is a symmetric complex-bilinear mapping.

Constants. The numbers d (the space-dimension) and $\#\mathcal{A}$, as well as s_*, m_* and $\#\mathcal{P}, \#\mathcal{F}$ (that will occur in Part II) will be fixed in this paper. Constants depending only on these numbers and on the choice of finite-dimensional norms are regarded as absolute constants. An absolute constant only depending on x is thus a constant that, besides these factors, only depends on x. Arbitrary constants will often be denoted by Ct., ct. and, when they occur as an exponent, by exp. Their values may change from line to line. For example we allow ourselves to write $2Ct. \le Ct.$

PART I. SOME FUNCTIONAL ANALYSIS

2 Matrix Algebras and Function Spaces

2.1 The phase space. Let $\mathcal{A}$ and $\mathcal{F}$ be two finite sets in $\mathbb{Z}^d$ and let $\mathcal{L}_\infty$ be an infinite subset of $\mathbb{Z}^d$. Let $\mathcal{L}$ be the disjoint union $\mathcal{F} \sqcup \mathcal{L}_\infty$.[5] Let $\mathcal{Z}$ be the disjoint union $\mathcal{A} \sqcup \mathcal{F} \sqcup \mathcal{L}_\infty$ and consider $(\mathbb{C}^2)^{\mathcal{Z}}$.

For any subset X of $\mathcal{Z}$, consider the projection

$$\pi_X : (\mathbb{C}^2)^{\mathcal{Z}} \to (\mathbb{C}^2)^X = \left\{\zeta \in (\mathbb{C}^2)^{\mathcal{Z}} : \zeta_a = 0 \ \forall a \notin X\right\}.$$

We can thus write $(\mathbb{C}^2)^{\mathcal{Z}} = (\mathbb{C}^2)^X \times (\mathbb{C}^2)^{\mathcal{Z}\setminus X}$, $\zeta = (\zeta_X, \zeta_{\mathbb{Z}\setminus X})$, and when X is finite this gives an injection

$$(\mathbb{C}^2)^{\#X} \hookrightarrow (\mathbb{C}^2)^{\mathcal{Z}}$$

whose image is $(\mathbb{C}^2)^X$.

In $\mathbb{R}^2$ we consider the partial ordering $(\gamma_1', \gamma_2') \le (\gamma_1, \gamma_2)$ if, and only if $\gamma_1' \le \gamma_1$ and $\gamma_2' \le \gamma_2$.

Let $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$ and let Y_γ be the Hilbert space of sequences $\zeta \in (\mathbb{C}^2)^{\mathcal{Z}}$ such that

$$||\zeta||_\gamma = \sqrt{\sum_{a\in\mathcal{Z}} |\zeta_a|^2 e^{2\gamma_1|a|}\langle a\rangle^{2\gamma_2}} < \infty \tag{2.1}$$

provided with the scalar product[6]

$$\langle \zeta, \zeta'\rangle_\gamma = \sum_{a\in\mathcal{Z}} \langle \zeta_a, \zeta_a'\rangle_{\mathbb{C}^2} e^{2\gamma_1|a|}\langle a\rangle^{2\gamma_2}.$$

If $\gamma_1 \ge 0$ and $\gamma_2 > d/2$, then this space is an algebra with respect to the convolution. If $\gamma_1 = 0$, this is a classical property of Sobolev spaces. For the case $\gamma_1 > 0$ see [EK08], Lemma 1.1. (There the space $Y_{(0,m_*)}$ coincides with the space Y, defined in (1.8), while $Y_{(0,0)}$ is the l^2-space of complex sequences $(\mathbb{C}^2)^{\mathcal{Z}}$.)

Example 2.1. Let $\mathcal{A} = \mathcal{F} = \emptyset$, $\mathcal{L}_\infty = \mathbb{Z}^d$ and $\varrho > 0$. Then any vector $\hat f = (\hat f_a, a \in \mathbb{Z}^d) \in Y_\varrho$ defines a holomorphic vector-function $f(y) = \sum \hat f_a e^{\mathbf{i}\langle a,y\rangle}$ on the ϱ-vicinity $\mathbb{T}^n_\varrho$ of the torus $\mathbb{T}^n$, $\mathbb{T}^n_\varrho = \{y \in \mathbb{C}^n/2\pi\mathbb{Z}^n \mid |\Im y| < \sigma\}$, where its norm is bounded by $C_d\|\hat f\|_\varrho$. Conversely, if $f : \mathbb{T}^n_\varrho \to \mathbb{C}^2$ is a bounded holomorphic function, then its Fourier coefficients satisfy $|\hat f_a| \le \mathrm{Const}\, e^{-|a|\varrho}$, so $\hat f \in Y_{\varrho'}$ for any $\varrho' < \varrho$.

[5] This is a more general setting than in the introduction, where $\mathcal{L}$ and $\mathcal{A}$ were two disjoint subsets of $\mathbb{Z}^d$.

[6] Complex linear in the second variable and complex anti-linear in the first.

Write $\zeta_a = (p_a, q_a)$ and let

$$\Omega = \sum_{a \in \mathcal{Z}} dp_a \wedge dq_a.$$

Ω is an anti-symmetric bi-linear form which is continuous on

$$Y_\gamma \times Y_{-\gamma} \cup Y_{-\gamma} \times Y_\gamma \to \mathbb{C}$$

with norm $\|\Omega\| = 1$. The subspaces $(\mathbb{C}^2)^{\{a\}}$ are symplectic subspaces of two (complex) dimensions carrying the canonical symplectic structure.

Ω defines (by contraction on the second factor) a bounded bijective operator

$$Y_\gamma \ni \zeta \mapsto \Omega(\cdot, \zeta) \in Y_{-\gamma}^*$$

where $Y_{-\gamma}^*$ denotes the Banach space dual of $Y_{-\gamma}$. (Notice that $\zeta' \mapsto \Omega(\zeta', \zeta)$ is a well-defined bounded linear form on $Y_{-\gamma}$.) We shall denote its inverse by

$$J_\Omega : Y_{-\gamma}^* \to Y_\gamma.$$

We shall also let J_Ω act on operators

$$J_\Omega : \mathcal{B}(X, Y_{-\gamma}^*) \to \mathcal{B}(X, Y_\gamma)$$

through $(J_\Omega H)(x) = J_\Omega(H(x))$ for any bounded operator $H : X \to Y_{-\gamma}^*$.

Remark 2.2. The complex-bilinear pairing (1.29) on the l^2-space $Y_{(0,0)}$ extends to a continuous mapping $Y_\gamma \times Y_{-\gamma} \to \mathbb{C}$ which allows to identify $Y_{-\gamma}$ with the dual space Y_γ^*. Then

$$\Omega(\zeta, \zeta') = \langle J\zeta, \zeta' \rangle, \tag{2.2}$$

where J here stands for the linear operator $\zeta \mapsto J\zeta$ defined by

$$(J\zeta)_a = J\zeta_a \quad \forall a \in \mathcal{Z},$$

where the 2×2-matrix J (in the right hand side) is defined in (1.9).[7] Then we have

$$J_\Omega \zeta = J\zeta \quad \forall\, \zeta \in Y_{-\gamma}^*, \tag{2.3}$$

where ζ in the r.h.s. is regarded as a vector in Y_γ, and we shall frequently denote the operator J_Ω by J. (It will be clear from the context which of the two operators J denotes.)

[7] Sorry for the abuse of notation.

A bijective bounded operator $A : Y_\gamma \to Y_\gamma$, $\gamma \geq (0,0)$, is *symplectic* if, and only if,

$$\Omega(A\zeta, A\zeta') = \Omega(\zeta, \zeta') \quad \forall\ \zeta, \zeta' \in Y_\gamma\,.$$

Writing Ω in the form (2.2) we see that A is symplectic if and only if ${}^tAJA = J$. Here ${}^tA$ stands for the operator, symmetric to A with respect to the pairing $\langle\cdot,\cdot\rangle$ (its matrix is transposed to that of A).

Let

$$\mathbb{A}^{\mathcal{A}} = \mathbb{C}^{\mathcal{A}} \times (\mathbb{C}/2\pi\mathbb{Z})^{\mathcal{A}}$$

and consider the Hilbert manifold $\mathbb{A}^{\mathcal{A}} \times_{\pi_{\mathcal{L}}} Y_\gamma$ whose elements are denoted $x = (r, \theta = [z], w)$. We provide this manifold with the metric[8]

$$\big\|x - x'\big\|_\gamma = \inf_{p\in\mathbb{Z}^d} ||(r, z + 2\pi p, w) - (r', z', w')||_\gamma.$$

We provide $\mathbb{A}^{\mathcal{A}} \times \pi_{\mathcal{L}} Y_\gamma$ with the symplectic structure Ω. To any C^1-function $f(r,\theta,w)$ on (some open set in) $\mathbb{A}^{\mathcal{A}} \times \pi_{\mathcal{L}} Y_\gamma$ it associates a vector field $X_f = J(df)$—the Hamiltonian vector field of f—which in the coordinates (r,θ,w) takes the form[9]

$$\begin{pmatrix} \dot r_a \\ \dot\theta_a \end{pmatrix} = J \begin{pmatrix} \frac{\partial}{\partial r_a} f(r,\theta,w) \\ \frac{\partial}{\partial \theta_a} f(r,\theta,w) \end{pmatrix} \quad \begin{pmatrix} \dot p_a \\ \dot q_a \end{pmatrix} = J \begin{pmatrix} \frac{\partial}{\partial p_a} f(r,\theta,w) \\ \frac{\partial}{\partial q_a} f(r,\theta,w) \end{pmatrix}.$$

2.2 A matrix algebra. The mapping

$$(a,b) \mapsto [a-b] = \min(|a-b|, |a+b|) \tag{2.4}$$

is a pseudo-metric on $\mathbb{Z}^d$, i.e. it verifies all the relations of a metric with the only exception that $[a-b]$ is $= 0$ for some $a \neq b$. This is most easily seen by observing that $[a-b] = \mathrm{d}_{\mathrm{Hausdorff}}(\{\pm a\}, \{\pm b\})$. We have $[a-0] = |a|$.

Define, for any $\gamma = (\gamma_1, \gamma_2) \geq (0,0)$ and $\varkappa \geq 0$,

$$e_{\gamma,\varkappa}(a,b) = Ce^{\gamma_1[a-b]} \max([a-b],1)^{\gamma_2} \min(\langle a\rangle, \langle b\rangle)^{\varkappa}. \tag{2.5}$$

LEMMA 2.3. (i) *If* $\gamma_1, \gamma_2 - \varkappa \geq 0$, *then*

$$e_{\gamma,\varkappa}(a,b) \leq e_{\gamma,0}(a,c) e_{\gamma,\varkappa}(c,b), \quad \forall a,b,c,$$

if C *is sufficiently large (bounded with* $\gamma_2, \varkappa$*).*

(ii) *If* $-\gamma \leq \tilde\gamma \leq \gamma$, *then*

$$e_{\tilde\gamma,\varkappa}(a,0) \leq e_{\gamma,\varkappa}(a,b) e_{\tilde\gamma,\varkappa}(b,0), \quad \forall a,b$$

if C *is sufficiently large (bounded with* $\gamma_2, \varkappa$*).*

[8] Using this notation for the metric on the manifold will not confuse it with the norm on the tangent space, which is also denoted $||\cdot||_\gamma$, we hope.

[9] There is no agreement as to the sign of the Hamiltonian vectorfield —we've used the choice of Arnold [Arn06].

Proof. (i) Since $[a-b] \le [a-c]+[c-b]$ it is sufficient to prove this for $\gamma_1 = 0$. If $\gamma_2 = 0$ then the statement holds for any $C \ge 1$, so it is sufficient to consider $\gamma_2 > 0$. This reduces easily to $\gamma_2 = 1$ and, hence, $\varkappa \le 1$. Then we want to prove

$$\begin{aligned}&\max([a-b],1)\min(\langle a\rangle,\langle b\rangle)^{\varkappa}\\ &\le C\max([a-c],1)\max([c-b],1)\min(\langle c\rangle,\langle b\rangle)^{\varkappa}.\end{aligned}$$

This gives the estimate.

(ii) Again it suffices to prove this for $\gamma_1 = 0$ and $\gamma_2 = 1$. Then we want to prove

$$\max(|a|,1)^{\tilde\gamma_2} \le C\max([a-b],1)\min(\langle a\rangle,\langle b\rangle)^{\varkappa}\max(|b|,1)^{\tilde\gamma_2}.$$

The inequality is fulfilled with $C \ge 1$ if a or b equal 0. Hence we need to prove

$$|a|^{\tilde\gamma_2} \le C\max([a-b],1)\min(\langle a\rangle,\langle b\rangle)^{\varkappa}|b|^{\tilde\gamma_2}.$$

Suppose $\tilde\gamma_2 \ge 0$. If $|a| \le 2|b|$ then this holds for any $C \ge 2$. If $|a| \ge 2|b|$ then $[a-b] \ge \frac{1}{2}|a|$ and the statement holds again for any $C \ge 2$.

If instead $\tilde\gamma_2 < 0$, then we get the same result with a and b interchanged. □

2.2.1 The space $\mathcal{M}_{\gamma,\varkappa}$. We shall consider matrices $A : \mathcal{Z}\times\mathcal{Z} \to gl(2,\mathbb{C})$, formed by 2×2-blocs, (each A_a^b is a complex 2×2-matrix). Define

$$|A|_{\gamma,\varkappa} = \max\begin{cases}\sup_a \sum_b \left\|A_a^b\right\| e_{\gamma,\varkappa}(a,b),\\ \sup_b \sum_a \left\|A_a^b\right\| e_{\gamma,\varkappa}(a,b),\end{cases} \tag{2.6}$$

where the norm on A_a^b is the matrix operator norm.

Let $\mathcal{M}_{\gamma,\varkappa}$ denote the space of all matrices A such that $|A|_{\gamma,\varkappa} < \infty$. Clearly $|\cdot|_{\gamma,\varkappa}$ is a norm on $\mathcal{M}_{\gamma,\varkappa}$—this is indeed true for all $(\gamma_1,\gamma_2,\varkappa) \in \mathbb{R}^3$. It follows by well-known results that $\mathcal{M}_{\gamma,\varkappa}$, provided with this norm, is a Banach space.

Transposition—$({}^tA)_a^b = {}^tA_b^a$—and $\mathbb{C}$-conjugation—$(\overline{A})_a^b = \overline{A_a^b}$)— do not change this norm. The identity matrix is in $\mathcal{M}_{\gamma,\varkappa}$ if, and only if, $\varkappa = 0$, and then $|I|_{\gamma,0} = C$.

Remark. The "l^1-norm" used here is a bit more complicated than the "sup-norm" used in [EK10], but it gives rise, as we shall see, to much better multiplicative properties.

2.2.2 Matrix multiplication. We define (formally) the *matrix product*

$$(AB)_a^b = \sum_c A_a^c B_c^b.$$

Notice that complex conjugation, transposition and taking the adjoint behave in the usual way under this formal matrix product.

PROPOSITION 2.4. *Let $\gamma_2 \geq \varkappa$. If $A \in \mathcal{M}_{\gamma,0}$ and $B \in \mathcal{M}_{\gamma,\varkappa}$, then AB and $BA \in \mathcal{M}_{\gamma,\varkappa}$ and*

$$|AB|_{\gamma,\varkappa} \quad \text{and} \quad |BA|_{\gamma,\varkappa} \leq |A|_{\gamma,0} |B|_{\gamma,\varkappa}.$$

Proof. (i) We have, by Lemma 2.3(i),

$$\begin{aligned}\sum_b \left\|(AB)_a^b\right\| e_{\gamma,\varkappa}(a,b) &\leq \sum_{b,c} \|A_a^c\| \left\|B_c^b\right\| e_{\gamma,\varkappa}(a,b) \\ &\leq \sum_{b,c} \|A_a^c\| \left\|B_c^b\right\| e_{\gamma,0}(a,c) e_{\gamma,\varkappa}(c,b)\end{aligned}$$

which is $\leq \|A\|_{\gamma,0} \|B\|_{\gamma,\varkappa}$. This implies in particular the existence of $(AB)_a^b$.

The sum over a is shown to be $\leq |A|_{\gamma,0} |B|_{\gamma,\varkappa}$ in a similar way. The estimate of BA is the same. □

Hence $\mathcal{M}_{\gamma,0}$ is a Banach algebra, and $\mathcal{M}_{\gamma,\varkappa}$ is an ideal in $\mathcal{M}_{\gamma,0}$ when $\varkappa \leq \gamma_2$.

2.2.3 The space $\mathcal{M}_{\gamma,\varkappa}^b$. We define (formally) on Y_γ

$$(A\zeta)_a = \sum_b A_a^b \zeta_b.$$

PROPOSITION 2.5. *Let $-\gamma \leq \tilde{\gamma} \leq \gamma$. If $A \in \mathcal{M}_{\gamma,\varkappa}$ and $\zeta \in Y_{\tilde{\gamma}}$, then $A\zeta \in Y_{\tilde{\gamma}}$ and*

$$\|A\zeta\|_{\tilde{\gamma}} \leq |A|_{\gamma,\varkappa} \|\zeta\|_{\tilde{\gamma}}.$$

Proof. Let $\zeta' = A\zeta$. We have

$$\sum_a |\zeta_a'|^2 e_{\tilde{\gamma},0}(a,0)^2 \leq \sum_a \left(\sum_b \left\|A_a^b\right\| |\zeta_b| e_{\tilde{\gamma},0}(a,0) \right)^2.$$

Write

$$\left\|A_a^b\right\| |\zeta_b| e_{\tilde{\gamma},0}(a,0) = I \times (I |\zeta_b| e_{\tilde{\gamma},0}(b,0)) \times J,$$

where

$$I = I_{a,b} = \sqrt{\|A_a^b\| e_{\gamma,\varkappa}(a,b)}$$

and

$$J = J_{a,b} = \frac{e_{\tilde{\gamma},0}(a,0)}{e_{\gamma,\varkappa}(a,b) e_{\tilde{\gamma},0}(b,0)}.$$

Since, by Lemma 2.3(ii), $J \le 1$ we get, by Hölder,

$$\sum_a \left|\zeta'_a\right|^2 e_{\tilde\gamma,0}(a,0)^2 \le \sum_a \left(\sum_b I_{a,b}^2\right)\left(\sum_b I_{a,b}^2 \left|\zeta_b\right|^2 e_{\tilde\gamma,0}(b,0)^2\right)$$
$$\le |A|_{\gamma,\varkappa} \sum_{a,b} I_{a,b}^2 \left|\zeta_b\right|^2 e_{\tilde\gamma,0}(b,0)^2 \le |A|_{\gamma,\varkappa} \sum_b \left|\zeta_b\right|^2 e_{\tilde\gamma,0}(b,0)^2 \sum_a I_{a,b}^2$$
$$\le |A|_{\gamma,\varkappa}^2 \|\zeta\|_{\tilde\gamma}^2 .$$

This shows that y_a exists for all a, and it also proves the estimate. □

We have thus, for any $-\gamma \le \tilde\gamma \le \gamma$, a continuous embedding of $\mathcal{M}_{\gamma,\varkappa}$,

$$\mathcal{M}_{\gamma,\varkappa} \hookrightarrow \mathcal{M}_{\gamma,0} \to \mathcal{B}(Y_{\tilde\gamma}, Y_{\tilde\gamma}),$$

into the space of bounded linear operators on $Y_{\tilde\gamma}$. Matrix multiplication in $\mathcal{M}_{\gamma,\varkappa}$ corresponds to composition of operators.

For our applications (see Lemma 2.7) we shall consider a somewhat larger sub algebra of $\mathcal{B}(Y_\gamma, Y_\gamma)$ with somewhat weaker decay properties. Let

$$\mathcal{M}^b_{\gamma,\varkappa} = \mathcal{B}(Y_\gamma, Y_\gamma) \cap \mathcal{M}_{(\gamma_1,\gamma_2-m_*),\varkappa} \tag{2.7}$$

which we provide with the norm

$$\|A\|_{\gamma,\varkappa} = \|A\|_{\mathcal{B}(Y_\gamma,Y_\gamma)} + |A|_{(\gamma_1,\gamma_2-m_*),\varkappa}. \tag{2.8}$$

When $\gamma = (\gamma_1,\gamma_2) \ge \gamma_* = (0, m_* + \varkappa)$, Proposition 2.4 shows that this norm makes $\mathcal{M}^b_{\gamma,0}$ into a Banach sub-algebra of $\mathcal{B}(Y_\gamma, Y_\gamma)$ and $\mathcal{M}^b_{\gamma,\varkappa}$ becomes an ideal in $\mathcal{M}^b_{\gamma,0}$.

2.3 Functions. For $\sigma, \mu \in (0,1]$ let

$$\mathcal{O}_\gamma(\sigma,\mu) = \left\{x = (r_\mathcal{A}, \theta_\mathcal{A}, w) \in \mathbb{A}^\mathcal{A} \times \pi_\mathcal{L} Y_\gamma : |r_\mathcal{A}| < \mu,\ |\,|\Im\theta_\mathcal{A}| < \sigma,\ |\,\|w\|_\gamma < \mu\right\}. \tag{2.9}$$

It is often useful to scale the action variables r by μ^2 and not by μ, but in our case μ will be ≈ 1, and then there is no difference (on the contrary, in Sect. 4.2 we scale $r_\mathcal{A}$ as μ^2 to simplify the calculations we perform there). The advantage with our scaling is that the Cauchy estimates becomes simpler.

Let

$$\gamma = (\gamma_1,\gamma_2) \ge \gamma_* = (0, m_* + \varkappa), \tag{2.10}$$

We shall consider perturbations

$$f : \mathcal{O}_{\gamma_*}(\sigma,\mu) \to \mathbb{C}$$

that are *real holomorphic and continuous up to the boundary* (rhcb). This means that it gives real values to real arguments and extends continuously to the closure of $\mathcal{O}_{\gamma_*}(\sigma,\mu)$. f is clearly also rhcb on $\mathcal{O}_\gamma(\sigma,\mu)$ for any $\gamma \ge \gamma_*$, and

$$df : \mathcal{O}_\gamma(\sigma,\mu) \to Y_\gamma^*$$

and

$$J_\Omega d^2 f : \mathcal{O}_\gamma(\sigma,\mu) \to \mathcal{B}(Y_\gamma, Y_{-\gamma})$$

are rhcb.

REMARK 2.6. Identifying Y_γ^* with $Y_{-\gamma}$ via the paring $\langle\cdot,\cdot\rangle$ we will interpret the differential $df(\zeta)$ as a gradient $\nabla f(\zeta) \in Y_{-\gamma}$,

$$df(\zeta)(\zeta') = \langle \nabla f(\zeta), \zeta' \rangle \quad \forall\, \zeta' \in Y_\gamma\,.$$

As classically, $\nabla f(\zeta)$ is the vector $\nabla f(\zeta) = (\nabla_a f(\zeta), a \in \mathcal{Z})$, where for $\zeta = \big(\zeta_a = (p_a, q_a), a \in \mathcal{Z}\big)$, $\nabla_a f$ is the 2-vector $(\partial f/\partial p_a, \partial f/\partial q_a)$.

Similar we will interpret $d^2 f$ as the Hessian $\nabla^2 f$, which is an operator with the matrix $((\nabla^2 f)^b_a, a, b \in \mathcal{Z})$, formed by the 2×2-blocks $(\nabla^2 f)^b_a = \nabla_a \nabla_b f$. The Hessian defines bounded linear operators $\nabla^2 f(\zeta) : Y_\gamma \to Y_{-\gamma}$, and

$$d^2 f(\zeta)(\zeta^1, \zeta^2) = \langle \nabla^2 f(\zeta)\zeta^1, \zeta^2 \rangle \quad \forall\, \zeta^1, \zeta^2 \in Y_\gamma\,.$$

We shall require that the mappings df and $d^2 f$ posses some extra smoothness:

R1 –*first differential* There exists a $\gamma \geq \gamma_*$ such that

$$J_\Omega df = J\nabla f : \mathcal{O}_{\gamma'}(\sigma,\mu) \to Y_{\gamma'}$$

is rhcb for any $\gamma_* \leq \gamma' \leq \gamma$.

This is a natural smoothness condition on the space of holomorphic functions on $\mathcal{O}_{\gamma_*}(\sigma,\mu)$. Via the Cauchy estimate it implies some bounds on the Hessian of f. But we need stronger restriction on the Hessian and assume

R2 –*second differential*

$$J_\Omega d^2 f = J\nabla^2 f : \mathcal{O}_{\gamma'}(\sigma,\mu) \to \mathcal{M}^b_{\gamma',\varkappa}$$

is rhcb for any $\gamma_* \leq \gamma' \leq \gamma$.

Such decay conditions do not seem to be naturally related to any smoothness condition of f, but they are instrumental in the KAM-theory for multidimensional PDE's: see for example [EK10] where such conditions were used to build a KAM-theory for some multidimensional non-linear Schrödinger equations.

2.3.1 The function space $\mathcal{T}_{\gamma,\varkappa}$. Consider the space of functions $f : \mathcal{O}_{\gamma_*}(\sigma,\mu) \to \mathbb{C}$ which are real holomorphic and continuous up to the boundary (rhcb) of $\mathcal{O}_{\gamma_*}(\sigma,\mu)$. We define $\mathcal{T}_{\gamma,\varkappa}(\sigma,\mu)$ to be the space of all such functions which verify $R1$ and $R2$.

We provide $\mathcal{T}_{\gamma,\varkappa}(\sigma,\mu)$ with the norm

$$|f|^{\sigma,\mu}_{\gamma,\varkappa} = \max \begin{cases} \sup_{x\in\mathcal{O}_{\gamma_*}(\sigma,\mu)} |f(x)|, \\ \sup_{\gamma_*\le\gamma'\le\gamma} \sup_{x\in\mathcal{O}_{\gamma'}(\sigma,\mu)} ||J_\Omega df(x)||_{\gamma'} = \|\nabla f(x)\|_{\gamma'}, \\ \sup_{\gamma_*\le\gamma'\le\gamma} \sup_{x\in\mathcal{O}_{\gamma'}(\sigma,\mu)} ||J_\Omega d^2 f(x)||_{\gamma',\varkappa} = ||\nabla^2 f(x)||_{\gamma',\varkappa} \end{cases} \tag{2.11}$$

making it into a Banach space. (It is even a Banach algebra with the constant function $f = 1$ as unit, but we shall be concerned with Poisson products rather than with products.)

This space is relevant for our application because

LEMMA 2.7. *Let $\mathcal{Z} = \mathcal{L}_\infty = \mathbb{Z}^d$ and $\varkappa = 2$. Then the Hamiltonian function $h_{\ge 4}$, defined in (1.12), belongs to $\mathcal{T}_{\gamma_g,2}(1,\mu_g)$ for suitable $\mu_g \in (0,1]$ and $\gamma_g > \gamma_*$.*

The lemma is proven in "Appendix A". Notice that we would not have been able to prove this if we had used the matrix norm (2.6) instead of (2.8).

The higher differentials $d^{k+2}f$ can be estimated by Cauchy estimates on some smaller domain in terms of this norm.

REMARK. The higher order differential $d^{k+2}f(x)$, $x \in \mathcal{O}_\gamma(\sigma,\mu)$, is canonically identified with three bounded symmetric multi-linear maps

$$\begin{aligned} (Y_\gamma)^{k+2} &\longrightarrow \mathbb{C}, \\ (Y_\gamma)^{k+1} &\longrightarrow Y^*_\gamma, \\ (Y_\gamma)^{k} &\longrightarrow \mathcal{B}(Y_\gamma, Y^*_\gamma). \end{aligned}$$

Due to the smoothing condition R1 the second one takes its values in the subspace $Y^*_{-\gamma}$. Due to the smoothing condition R2 $Jd^{k+2}f(x)$ is a bounded symmetric multilinear map

$$(Y_\gamma)^k \longrightarrow \mathcal{M}^b_{\gamma,\varkappa}. \tag{2.12}$$

Alternatively, identifying $d^2 f$ with the hessian $\nabla^2 f$, we may identify $d^{k+2}f$ with a continuous symmetric multilinear mapping of the form (2.12).

2.3.2 The function space $\mathcal{T}_{\gamma,\varkappa,\mathcal{D}}$. Let $\mathcal{D}$ be an open set in $\mathbb{R}^{\mathcal{P}}$. We shall consider functions

$$f : \mathcal{O}_{\gamma^*}(\sigma,\mu) \times \mathcal{D} \to \mathbb{C}$$

which are of class $\mathcal{C}^{s_*}$ for some integer $s_* \ge 0$. We say that $f \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$ if, and only if,

$$\frac{\partial^j f}{\partial \rho^j}(\cdot,\rho) \in \mathcal{T}_{\gamma,\varkappa}(\sigma,\mu)$$

for any $\rho \in \mathcal{D}$ and any $|j| \leq s_*$. We provide this space by the norm

$$|f|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}} = \max_{|j|\leq s_*} \sup_{\rho\in\mathcal{D}} |\frac{\partial^j f}{\partial \rho^j}(\cdot,\rho)|^{\sigma,\mu}_{\gamma,\varkappa}. \tag{2.13}$$

This norm makes $\mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$ a Banach space.

2.3.3 Jets of functions. For any function $f \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$ we shall consider the following Taylor polynomial of f at $r = 0$ and $w = 0$:

$$f^T(x) = f(0,\theta,0) + d_r f(0,\theta,0)[r] + d_w f(0,\theta,0)[w] + \frac{1}{2} d^2_w f(0,\theta,0)[w,w]. \tag{2.14}$$

Functions of the form f^T will be called *jet-functions.*

PROPOSITION 2.8. *Let $f \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$. Then $f^T \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$ and*

$$|f^T|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}} \leq C|f|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}.$$

(C is an absolute constant.)

Proof. The proof follows the general argument. Look for example on

$$g(x) = d^2_w f \circ p(x)[w,w], \quad x = (r,\theta,w),$$

where $p(x)$ is the projection onto $(0,\theta,0)$. This function g is rhcb on $\mathcal{O}_{\gamma_*}(\sigma,\mu)$, being a composition of such functions. A bound for its sup-norm is obtained by a Cauchy estimate of f:

$$\left\|d^2_w f(p(x))\right\|_{\mathcal{B}(Y_{\gamma_*}, Y^*_{-\gamma_*})} \|w\|^2_{\gamma'} \leq \text{Ct.}\frac{1}{\mu^2} \sup_{\mathcal{O}_{\gamma_*}(\sigma,\mu)} |f(y)| \, \|w\|^2_{\gamma_*} \leq \text{Ct.} \sup_{y\in\mathcal{O}_{\gamma_*}(\sigma,\mu)} |f(y)|\,.$$

Since $Jdg(x)[\cdot]$ equals

$$\big(Jdd^2_w f \circ p(x)[w,w]\big)[dp[\cdot]] + 2\big(Jd^2_w f \circ p(x)[w]\big)[\cdot],$$

and

$$Jd^2_w f : \mathcal{O}_{\gamma'}(\sigma,\mu) \to \mathcal{B}(Y_{\gamma'}, Y_{\gamma'})$$

and

$$Jdd^2_w f = Jd^2_w df : \mathcal{O}_{\gamma'}(\sigma,\mu) \to \mathcal{B}(Y_{\gamma'}, \mathcal{B}(Y_{\gamma'}, Y_{\gamma'}))$$

are rhcb, it follows that dg verifies R1 and is rhcb. The norm $\|Jdg(x)\|_{\gamma'}$ is less than

$$\left\|Jd^2_w df(p(x))\right\|_{\mathcal{B}(Y_{\gamma'}, \mathcal{B}(Y_{\gamma'}, Y_{\gamma'}))} \|w\|^2_{\gamma'} + 2\left\|Jd^2_w f(p(x))\right\|_{\mathcal{B}(Y_{\gamma'}, Y_{\gamma'})} \|w\|_{\gamma'}\,,$$

which is $\leq \text{Ct.} \sup_{y\in\mathcal{O}_{\gamma'}(\sigma,\mu)} \|Jdf(y)\|_{\gamma'}$—this follows by Cauchy estimates of derivatives of Jdf.

Since $Jd^2 g(x)[\cdot,\cdot]$ equals

$$\big(Jd^2 d^2_w f \circ p(x)[w,w]\big)[dp[\cdot], dp[\cdot]] + 2J\big(dd^2_w f \circ p(x)[w]\big)[\cdot, dp[\cdot] + 2Jd^2_w f \circ p(x)[\cdot,\cdot],$$

and

$$Jd^2_w f : \mathcal{O}_{\gamma'}(\sigma,\mu) \to \mathcal{M}^b_{\gamma',\varkappa},$$

$$Jdd^2_w f = Jd^2_w df : \mathcal{O}_{\gamma'}(\sigma,\mu) \to \mathcal{B}(Y_{\gamma'}, \mathcal{M}^b_{\gamma',\varkappa}),$$

and

$$Jd^2 d^2_w f = Jd^2_w d^2 f : \mathcal{O}_{\gamma'}(\sigma,\mu) \to \mathcal{B}(Y_{\gamma'}, \mathcal{B}(Y_{\gamma'}, \mathcal{M}^b_{\gamma',\varkappa}))$$

are rhcb, it follows that Jdg^2 verifies R2 and is rhcb. The norm $\left\|Jd^2 g\right\|_{\gamma',\varkappa}$ is less than

$$\left\|Jd^2_w d^2 f(p(x))\right\|_{\mathcal{B}(Y_{\gamma'},\mathcal{B}(Y_{\gamma'},\mathcal{M}^b_{\gamma',\varkappa}))} \|w\|^2_{\gamma'} + 2\left\|Jd_w d^2 f(p(x))\right\|_{\mathcal{B}(Y_{\gamma'},\mathcal{M}^b_{\gamma',\varkappa})} \|w\|_{\gamma'} + 2\left\|Jd^2 f(x)\right\|_{\gamma',\varkappa},$$

which is $\leq$ Ct. $\sup_{y\in\mathcal{O}_{\gamma'}(\sigma,\mu)} \left\|Jd^2 f(y)\right\|_{\gamma',\varkappa}$—this follows by a Cauchy estimate of $Jd^2 f$.

The derivatives with respect to ρ are treated alike. □

2.4 Flows

2.4.1 Poisson brackets. The Poisson bracket $\{f,g\}$ of two $\mathcal{C}^1$-functions f and g is (formally) defined by

$$\{f,g\} = \Omega(Jdf, Jdg) = \langle J\nabla f, \nabla g\rangle = -df[Jdg] = dg[Jdf].$$

If one of the two functions verify condition R1, this product is well-defined. Moreover, if both f and g are jet-functions, then $\{f,g\}$ is also a jet-function.

PROPOSITION 2.9. *Let $f,g \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$, and let $\sigma' < \sigma$ and $\mu' < \mu \leq 1$. Then*

(i) $\{g,f\} \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma',\mu')$ *and*

$$|\{g,f\}|^{\sigma',\mu'}_{\gamma,\varkappa,\mathcal{D}} \leq C^{\mu-\mu'}_{\sigma-\sigma'} |g|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}} |f|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}$$

for

$$C^{\mu-\mu'}_{\sigma-\sigma'} = C\big(\frac{1}{(\sigma-\sigma')} + \frac{1}{(\mu-\mu')}\big).$$

(ii) *the n-fold Poisson bracket $P^n_g f \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$ and*

$$\left|P^n_g f\right|^{\sigma',\mu'}_{\gamma,\varkappa,\mathcal{D}} \leq \big(C^{\mu-\mu'}_{\sigma-\sigma'} |g|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}\big)^n |f|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}$$

where $P_g f = \{g,f\}$.

(C is an absolute constant.)

Proof. (i) We must first consider the function $h = \Omega(Jdg, Jdf)$ on $\mathcal{O}_{\gamma_*}(\sigma,\mu)$ Since $Jdg,\ Jdf : \mathcal{O}_{\gamma_*}(\sigma,\mu) \to Y_{\gamma_*}$ are rhcb, it follows that $h : \mathcal{O}_{\gamma_*}(\sigma,\mu) \to \mathbb{C}$ is rhcb, and

$$|h(x)| \le \|Jdg(x)\|_{\gamma_*} \|Jdf(x)\|_{\gamma_*} .$$

The vector $Jdh(x)$ is a sum of

$$J\Omega(Jd^2g(x), Jdf(x)) = Jd^2g(x)[Jdf(x)]$$

and another term with g and f interchanged. Since $Jd^2g : \mathcal{O}_{\gamma'}(\sigma,\mu) \to \mathcal{B}(Y_{\gamma'}, Y_{\gamma'})$ and $Jdg,\ Jdf : \mathcal{O}_{\gamma'}(\sigma,\mu) \to Y_{\gamma'}$ are rhcb, it follows that Jdh verifies R1 and is rhcb. Moreover

$$\big\|Jd^2g(x)[Jdf(x),\cdot]\big\|_{\gamma'} \le \big\|Jd^2g(x)\big\|_{\mathcal{B}(Y_{\gamma'},Y_{\gamma'})} \|Jdf(x)\|_{\gamma'}$$

and, by definition of $\mathcal{M}^b_{\gamma,\varkappa}$,

$$\big\|Jd^2g(x)\big\|_{\mathcal{B}(Y_{\gamma'},Y_{\gamma'})} \le \big\|Jd^2g(x)\big\|_{\gamma',0} .$$

The operator $Jd^2h(x) = d(Jdh)(x)$ is a sum of

$$Jd^3g(x)[Jdf(x)]$$

and

$$Jd^2g(x)[Jd^2f(x)]$$

and two other terms with g and f interchanged.

Since $Jd^3g : \mathcal{O}_{\gamma'}(\sigma,\mu) \to \mathcal{B}(Y_{\gamma'}, \mathcal{M}^b_{\gamma',\varkappa})$ and $Jdf : \mathcal{O}_{\gamma'}(\sigma,\mu) \to Y_{\gamma'}$ are holomorphic functions, it follows that the first function $\mathcal{O}_{\gamma'}(\sigma,\mu) \to \mathcal{B}(Y_{\gamma'}, \mathcal{M}^b_{\gamma',\varkappa})$ also is holomorphic. It can be estimated on a smaller domain using a Cauchy estimate for $Jd^3g(x)$.

The second term is treated differently. Since

$$Jd^2f,\ Jd^2g : \mathcal{O}_{\gamma'}(\sigma,\mu) \to \mathcal{M}^b_{\gamma,\varkappa}$$

are rhcb, and since, by Proposition 2.4, taking products is a bounded bi-linear map with norm ≤ 1, it follows that the second function $\mathcal{O}_{\gamma'}(\sigma,\mu) \to \mathcal{M}^b_{\gamma',\varkappa}$ is rhcb and

$$\big\|Jd^2g(x)[Jd^2f(x)]\big\|_{\gamma',\varkappa} \le \big\|Jd^2g(x)\big\|_{\gamma',\varkappa} \big\|Jd^2f(x)\big\|_{\gamma',\varkappa} .$$

The derivatives with respect to ρ are treated alike.

(ii) That $g_n = P_g^n f \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma',\mu')$ follows from (ii), but the estimate does not follow from the estimate in (ii). The estimate follows instead from Cauchy estimates of n-fold product $P_g^n f$ and from the following statement:

for any $n \geq 1$ and any $k \geq 0$, $\left|d^k g_n(x)\right|$, $x \in \mathcal{O}_{\gamma'}(\sigma,\mu)$, is bounded by a sum of terms of the form[10]

$$|d^{m_1} g(x)| \dots |d^{m_n} g(x)|\, |d^{m_{n+1}} f(x)|$$

with $\sum m_j = n+1+k$ and each $m_j \geq 1$. The number of terms in the sum is $\leq 2^{nk}$ (This is proven above for $n = 1$ and $k \leq 2$. It follows for $k \geq 3$ by the product formula for derivatives. It follows then for all $n \geq 2$ and any $k \geq 0$ by an easy induction).

Let now $m'_j = 2$ if $m_j \geq 3$ and $= m_j$ if $m_j \leq 2$. Then the term above can be estimated by Cauchy estimates:

$$\leq (C^{\mu-\mu'}_{\sigma-\sigma'})^{\sum(m_j-m'_j)} \left|d^{m'_1} g(x)\right| \dots \left|d^{m'_n} g(x)\right| \left|d^{m'_{n_1}} f(x)\right|$$

$$\leq (C^{\mu-\mu'}_{\sigma-\sigma'})^{\sum(m_j-m'_j)} (|g|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}})^n\, |f|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}$$

The result now follows by observing that $\sum(m_j - m'_j) \leq \max(n+k-2, 0)$ and taking $k = 2$. [Indeed, if $\sum(m_j - m'_j)$ were $\geq n+k-1$, then $\sum m'_j \leq \sum m_j - (n+k-1) = 2$. Since $m'_j \geq 1$ this forces n to be $= 1$ and all m'_j to be $= 1$. Hence $m_j = m'_j$ and $\sum(m_j - m'_j) = 0$.] □

Remark 2.10. The proof shows that the assumptions can be relaxed when g is a jet function: it suffices then to assume that $g \in \mathcal{T}_{\gamma,0,\mathcal{D}}(\sigma,\mu)$ and $g - \hat{g}(\cdot,0,\cdot) \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$.[11]

Then $\{g,f\}$ will still be in $\mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$ but with the bound

$$|\{g,f\}|^{\sigma',\mu'}_{\gamma,\varkappa,\mathcal{D}} \leq C^{\mu-\mu'}_{\sigma-\sigma'} \left(|g|^{\sigma,\mu}_{\gamma,0,\mathcal{D}} + |g - \hat{g}(\cdot,0,\cdot)|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}} \right) |f|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}\,.$$

To see this it is enough to consider a jet-function g which does not depend on θ. The only difference with respect to case (i) is for the second differential. The second term is fine since, by Proposition 2.4, $\mathcal{M}^b_{\gamma',\varkappa}$ is a two-sided ideal in $\mathcal{M}^b_{\gamma',0}$ and

$$\left\|Jd^2 g(x)[Jd^2 f(x)]\right\|_{\gamma',\varkappa} \leq \left\|Jd^2 g(x)\right\|_{\gamma',0} \left\|Jd^2 f(x)\right\|_{\gamma',\varkappa}.$$

For the first term we must consider $Jd^3 g(x)[Jdf(x)]$ which, a priori, takes its values in $\mathcal{M}^b_{\gamma',0}$ and not in $\mathcal{M}^b_{\gamma',\varkappa}$. But since g is a jet-function independent of θ this term is $= 0$.

[10] In the norms of the appropriate Banach spaces.

[11] $\hat{g}(\cdot,0,\cdot)$ is the 0:th Fourier coefficient of the function $\theta \mapsto g(\cdot,\theta,\cdot)$.

2.4.2 Hamiltonian flows. The Hamiltonian vector field of a $\mathcal{C}^1$-function g on (some open set in) Y_γ is Jdg. Without further assumptions it is an element in $Y_{-\gamma}$, but if $g \in \mathcal{T}_{\gamma,\varkappa}$, then it is an element in Y_γ and has a well-defined local flow $\{\Phi_g^t\}$. Clearly $(d/dt)f(\Phi_g^t) = \{f,g\} \circ \Phi_g^t$ for a C^1-smooth function f.

PROPOSITION 2.11. *Let $g \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$, and let $\sigma' < \sigma$ and $\mu' < \mu \le 1$. If*

$$|g|_{\sigma,\mu\ \gamma,\varkappa,\mathcal{D}} \le \frac{1}{C}\min(\sigma-\sigma',\mu-\mu'),$$

then

(i) *the Hamiltonian flow map $\Phi^t = \Phi_g^t$ is, for all $|t| \le 1$ and all $\gamma_* \le \gamma' \le \gamma$, a $\mathcal{C}^{s_*}$-map*

$$\mathcal{O}_{\gamma'}(\sigma',\mu') \times \mathcal{D} \to \mathcal{O}_{\gamma'}(\sigma,\mu)$$

which is real holomorphic and symplectic for any fixed $\rho \in \mathcal{D}$. Moreover,

$$\left\|\partial_\rho^j(\Phi^t(x,\rho)-x)\right\|_{\gamma'} \le C\,|g|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}},$$

and

$$\left\|\partial_\rho^j(d\Phi^t(x)-I)\right\|_{\gamma',\varkappa} \le C\,|g|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}},$$

for any $x \in \mathcal{O}_{\gamma'}(\sigma',\mu')$, $\gamma_* \le \gamma' \le \gamma$, and $0 \le |j| \le s_*$.

(ii) *$f \circ \Phi_g^t \in \mathcal{T}_{\gamma,\varkappa}(\sigma',\mu',\mathcal{D})$ for $|t| \le 1$ and*

$$\left|f \circ \Phi_g^t\right|_{\substack{\sigma',\mu'\\ \gamma,\varkappa,\mathcal{D}}} \le C\,|f|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}$$

for any $f \in \mathcal{T}_{\gamma,\varkappa}(\sigma,\mu,\mathcal{D})$.

(C is an absolute constant.)

Proof. It follows by general arguments that the local flow $\Phi = \Phi_g : U \to \mathcal{O}_\gamma(\sigma,\mu)$ is real holomorphic in (t,ζ) in some $U \subset \mathbb{C} \times \mathcal{O}_\gamma(\sigma,\mu)$, and that it depends smoothly on any smooth parameter in the vector field. Clearly, for $|t| \le 1$ and $x \in \mathcal{O}_\gamma(\sigma',\mu')$

$$\left\|\Phi^t(x,\rho)-x\right\|_\gamma \le \sup_{x\in\mathcal{O}_\gamma(\sigma,\mu)} \|Jdg(x)\|_\gamma \le |g|^{\sigma,\mu}_{\gamma,0,\mathcal{D}}$$

as long as $\Phi^t(x)$ stays in the domain $\mathcal{O}_\gamma(\sigma,\mu)$. It follows by classical arguments that this is the case if

$$|g|^{\sigma,\mu}_{\gamma,0,\mathcal{D}} \le \text{ct.}\min(\sigma-\sigma',\mu-\mu').$$

The differential. We have

$$\frac{d}{dt}d\Phi^t(x) = -Jd^2g(\Phi^t(x))d\Phi^t(x) = B(t)d\Phi^t(x),$$

where $B(t) \in \mathcal{M}^b_{\gamma,\varkappa}$. By re-writing this equation in the integral form $d\Phi^t(x) = \mathrm{Id} + \int_0^t B(s) d\Phi^s(x) \mathrm{d}s$ and iterating this relation, we get that $d\Phi^t(x) - \mathrm{Id} = B^\infty(t)$ with

$$B^\infty(t) = \sum_{k\geq 1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \prod_{j=1}^k B(t_j) \mathrm{d}t_k \cdots \mathrm{d}t_2 \, \mathrm{d}t_1.$$

We get, by Proposition 2.4, that $d\Phi^t(x) - \mathrm{Id} \in \mathcal{M}^b_{\gamma,\varkappa}$ and, for $|t| \leq 1$,

$$\left\| d\Phi^t(x) - \mathrm{Id} \right\|_{\gamma,\varkappa} \leq \sum_{k\geq 1} \left\| Jd^2 g(\Phi^t(x)) \right\|^k_{\gamma,\varkappa} \frac{t^k}{k!} \leq \left\| Jd^2 g(\Phi^t(x)) \right\|_{\gamma,\varkappa}.$$

In particular, $A = d\Phi^t(x)$ is a bounded bijective operator on Y_γ. Since $Jd^2 g$ is a Hamiltonian vector field we clearly have that

$$\Omega(A\zeta, A\zeta') = \Omega(\zeta, \zeta'), \quad \forall \zeta, \zeta' \in Y_\gamma,$$

so A is symplectic.

Parameter Dependence. For $|j| = 1$, we have

$$\frac{d}{dt} Z(t) = \frac{d}{dt} \frac{\partial^j \Phi^t(x,\rho)}{\partial \rho^j} = B(t,\rho) Z(t) - \frac{\partial^j Jdg(\Phi^t(x,\rho),\rho)}{\partial \rho^j} = B(t) Z(t) + A(t).$$

Since

$$\|A(t)\|_\gamma + \|B(t)\|_{\gamma,\varkappa} \leq \mathrm{Ct.} \, |g|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}},$$

it follows by classical arguments, using Gronwall, that

$$\|Z(t)\|_{\gamma,0} \leq \mathrm{Ct.} \, |g|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}} \, |t| \, .$$

The higher order derivatives (with respect to ρ) of $\Phi^t(x,\rho)$, and the derivatives of $d\Phi^t(x,\rho)$ are treated in the same way.

The same argument applies to any $\gamma_* \leq \gamma' \leq \gamma$.

Since

$$f \circ \Phi^t_g = \sum_{n\geq 0} \frac{1}{n!} t^n P^n_{-g} f,$$

(ii) is a consequence of Proposition 2.9(ii). □

REMARK 2.12. If the set $\mathcal{Z}$ is such that $\mathcal{A} = \mathcal{F} = \emptyset$ and $\mathcal{L}_\infty = \mathbb{Z}^d$ (so $\mathcal{Z} = \mathbb{Z}^d$), then the domains $\mathcal{O}_\gamma(\sigma,\mu)$ and the functional spaces on these domains which we introduced do not depend on σ. In this case in our notation we will chose the dumb parameter σ to be 1. The assertions of the Propositions 2.9 and 2.11 remain true if we there take $\sigma = \sigma' = 1$ and drop the assumptions, related to σ and σ' (in particular, replace there $\min(\sigma - \sigma', \mu - \mu')$ by $\mu - \mu'$, and replace $1/(\sigma - \sigma')$ by 0).

PART II. A BIRKHOFF NORMAL FORM

3 Small Divisors

3.1 Non resonance of basic frequencies. In this subsection we assume that the set $\mathcal{A} \subset \mathbb{Z}^d$ is admissible, i.e. it only contains integer vectors with different norms (see Definition 1.1).

We consider the vector of basic frequencies

$$\omega \equiv \omega(m) = (\omega_a(m))_{a\in\mathcal{A}}, \quad m \in [1,2], \tag{3.1}$$

where $\omega_a(m) = \lambda = \sqrt{|a|^4 + m}$. The goal of this section is to prove the following result:

PROPOSITION 3.1. *Assume that $\mathcal{A}$ is an admissible subset of $\mathbb{Z}^d$ of cardinality n included in $\{a \in \mathbb{Z}^d \mid |a| \le N\}$. Then for any $k \in \mathbb{Z}^{\mathcal{A}}\backslash\{0\}$, any $\kappa > 0$ and any $c \in \mathbb{R}$ we have*

$$\operatorname{meas}\left\{ m \in [1,2] \;\middle|\; \left| \sum_{a\in\mathcal{A}} k_a \omega_a(m) + c \right| \le \kappa \right\} \le C_n \frac{N^{4n^2} \kappa^{1/n}}{|k|^{1/n}},$$

where $|k| := \sum_{a\in\mathcal{A}} |k_a|$ and $C_n > 0$ is a constant, depending only on n.

The proof follows closely that of Theorem 6.5 in [Bam03] (also see [BG06]); a weaker form of the result was obtained earlier in [Bou95]. Non of the constants C_j etc. in this section depend on the set $\mathcal{A}$.

LEMMA 3.2. *Assume that $\mathcal{A} \subset \{a \in \mathbb{Z}^d \mid |a| \le N\}$. For any $p \le n = |\mathcal{A}|$, consider p points $a_1, \dots, a_p$ in $\mathcal{A}$. Then the modulus of the following determinant*

$$D := \begin{vmatrix} \frac{d\omega_{a_1}}{dm} & \frac{d\omega_{a_2}}{dm} & \dots & \frac{d\omega_{a_p}}{dm} \\ \frac{d^2\omega_{a_1}}{dm^2} & \frac{d^2\omega_{a_2}}{dm^2} & \dots & \frac{d^2\omega_{a_p}}{dm^2} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \frac{d^p\omega_{a_1}}{dm^p} & \frac{d^p\omega_{a_2}}{dm^p} & \dots & \frac{d^p\omega_{a_p}}{dm^p} \end{vmatrix}$$

is bounded from below:

$$|D| \ge C N^{-3p^2+p},$$

where $C = C(p) > 0$ is a constant depending only on p.

Proof. First note that, by explicit computation,

$$\frac{d^j\omega_i}{dm^j} = (-1)^j \Upsilon_j \left(|i|^4 + m\right)^{\frac{1}{2}-j}, \quad \Upsilon_j = \prod_{l=0}^{j-1} \frac{2l-1}{2}. \tag{3.2}$$

Inserting this expression in D, we deduce by factoring from each $l-th$ column the term $(|a_\ell|^4+m)^{-1/2}=\omega_\ell^{-1}$, and from each $j-th$ row the term Υ_j that the determinant, up to a sign, equals

$$\left[\prod_{l=1}^{p}\omega_{a_\ell}^{-1}\right]\left[\prod_{j=1}^{p}\Upsilon_j\right]\times\begin{vmatrix} 1 & 1 & 1 & \dots & 1\\ x_{a_1} & x_{a_2} & x_{a_3} & \dots & x_{a_p}\\ x_{a_1}^2 & x_{a_2}^2 & x_{a_3}^2 & \dots & x_{a_p}^2\\ . & . & . & \dots & .\\ . & . & . & \dots & .\\ . & . & . & \dots & .\\ x_{a_1}^p & x_{a_2}^p & x_{a_3}^p & \dots & x_{a_p}^p \end{vmatrix},$$

where we denoted $x_a:=(|a|^4+m)^{-1}=\omega_a^{-2}$. Since $|\omega_{a_k}|\le 2|a_k|^2\le 2N^2$ for every k, the first factor is bigger than $(2N^2)^{-p}$. The second is a constant, while the third is the Vandermond determinant, equal to

$$\prod_{1\le l<k\le p}(x_{a_\ell}-x_{a_k})=\prod_{1\le l<k\le p}\frac{|a_k|^4-|a_\ell|^4}{\omega_{a_\ell}^2\omega_{a_k}^2}=:V.$$

Since $\mathcal{A}$ is admissible, then

$$|V|\ge\prod_{1\le l<k\le p}\frac{|a_k|^2+|a_\ell|^2}{\omega_{a_\ell}^2\omega_{a_k}^2}\ge\left(\frac{1}{4}\right)^{p(p-1)}N^{-3p(p-1)},$$

where we used that each factor is bigger than $\frac{1}{16}N^{-6}$ using again that $|\omega_{a_k}|\le 2|a_k|^2\le 2N^2$ for every k. This yields the assertion. □

LEMMA 3.3. *Let $u^{(1)},\dots,u^{(p)}$ be p independent vectors in $\mathbb{R}^p$ of norm at most one, and let $w\in\mathbb{R}^p$ be any non-zero vector. Then there exists $i\in[1,\dots,p]$ such that*

$$\left|\langle u^{(i)},w\rangle\right|\ge C_p|w|\left|\det\left(u^{(1)},\dots,u^{(p)}\right)\right|.$$

Proof. Without lost of generality we may assume that $|w|=1$.

Let $|\langle u^{(i)},w\rangle|\le a$ for all i. Consider the p-dimensional parallelogram Π, generated by the vector $u^{(1)},\dots,u^{(p)}$ in $\mathbb{R}^p$ (i.e., the set of all linear combinations $\sum x_j u^{(j)}$, where $0\le x_j\le 1$ for all j). It lies in the strip of width $2pa$, perpendicular to the vector w, and its projection to the $p-1$-dimensional space, perpendicular to w, lies in the ball around zero of radius p. Therefore the volume of Π is bounded by $C_p p^{p-1}(2pa)=C_p' a$. Since this volume equals $|\det(u^{(1)},\dots,u^{(p)})|$, then $a\ge C_p|\det(u^{(1)},\dots,u^{(p)})|$. This implies the assertion. □

Consider vectors $\frac{d^i\omega}{dm^i}(m)$, $1\le i\le n$, denote $K_i=|\frac{d^i\omega}{dm^i}(m)|$ and set

$$u^{(i)}=K_i^{-1}\frac{d^i\omega}{dm^i}(m),\quad 1\le i\le n\,.$$

From (3.2) we see that[12] $K_i \le C_n$ for all $1 \le i \le n$ (as before, the constant does not depend on the set $\mathcal{A}$). Combining Lemmas 3.2 and 3.3, we find that for any vector w and any $m \in [1,2]$ there exists $r = r(m) \le n$ such that

$$\begin{aligned}\Big|\langle \frac{d^r\omega}{dm^r}(m), w\rangle\Big| &= K_r\big|\langle u^{(r)}, w\rangle\big| \ge K_r C_n |w| (K_1 \dots K_n)^{-1}|D| \\ &\ge C_n |w| N^{-3n^2+n}\,. \end{aligned} \tag{3.3}$$

Now we need the following result (see Lemma B.1 in [Eli98]):

LEMMA 3.4. *Let $g(x)$ be a C^{n+1}-smooth function on the segment [1,2] such that $|g'|_{C^n} = \beta$ and $\max_{1\le k\le n} \min_x |\partial^k g(x)| = \sigma$. Then*

$$\operatorname{meas}\{x \,||g(x)| \le \rho\} \le C_n(\frac{\beta}{\sigma} + 1)(\frac{\rho}{\sigma})^{1/n}.$$

Consider the function $g(m) = |k|^{-1}\sum_{a\in\mathcal{A}} k_a\omega_a(m) + |k|^{-1}c$. Then $|g'|_{C^n} \le C_n'$, and $\max_{1\le k\le n}\min_m |\partial^k g(m)| \ge C_n N^{-3n^2+n}$ in view of (3.3). Therefore, by Lemma 3.4,

$$\operatorname{meas}\{||g(m)| \le \tfrac{\kappa}{|k|}\} \le C_n N^{3n^2-n}(\tfrac{\kappa}{|k|}N^{3n^2-n})^{1/n} = C_n N^{3n^2+2n-1}(\tfrac{\kappa}{|k|}\,)^{1/n}.$$

This implies the assertion of the proposition.

3.2 Small divisors estimates. We recall the notation (1.20), (3.1), and note the elementary estimates

$$\langle a\rangle^2 < \lambda_a(m) < \langle a\rangle^2 + \frac{m}{2\langle a\rangle^2} \quad \forall\, a \in \mathbb{Z}^d, \qquad m \in [1,2], \tag{3.4}$$

where $\langle a\rangle = \max(1, |a|^2)$. In this section we study four type of linear combinations of the frequencies $\lambda_a(m)$:

$$\begin{aligned} D_0 &= \langle\omega, k\rangle, \quad k \in \mathbb{Z}^{\mathcal{A}}\backslash\{0\}, \\ D_1 &= \langle\omega, k\rangle + \lambda_a, \quad k \in \mathbb{Z}^{\mathcal{A}},\ a \in \mathcal{L}, \\ D_2^{\pm} &= \langle\omega, k\rangle + \lambda_a \pm \lambda_b, \quad k \in \mathbb{Z}^{\mathcal{A}},\ a, b \in \mathcal{L}\,. \end{aligned}$$

In subsequent sections they will become divisors for our constructions, so we call these linear combinations "divisors".

DEFINITION 3.5. Consider independent formal variables $x_0, x_1, x_2, \dots$. Now take any divisor of the form D_0, D_1 or $D_2^{\pm}$, write there each $\omega_a, a \in \mathcal{A}$, as λ_a, and then replace every $\lambda_a, a \in \mathbb{Z}^d$, by $x_{|a|^2}$. Then the divisor is called resonant if the obtained algebraical sum of the variables $x_j, j \ge 0$, is zero. Resonant divisors are also called trivial resonances.

[12] In this section C_n denotes any positive constant depending only on n.

Note that a D_0-divisor cannot be resonant since $k \neq 0$ and the set $\mathcal{A}$ is admissible; a D_1-divisor $\langle \omega, k\rangle + \lambda_a$ is resonant only if $a \in \mathcal{L}_f$, $|k| = 1$ and $\langle \omega, k\rangle = -\omega_b$, where $|a| = |b|$. Finally, a D_2^+-divisor or a D_2^- divisor with $k \neq 0$ may be resonant only when $(a, b) \in \mathcal{L}_f \times \mathcal{L}_f$, while the divisors D_2^- of the form $\lambda_a - \lambda_b$ are resonant if and only if $|a| = |b|$. So there are finitely many trivial resonances of the form D_0, D_1, D_2^+ and of the form D_2^- with $k \neq 0$, but infinitely many of them of the form D_2^- with $k = 0$.

Our first aim is to remove from the segment $[1, 2] = \{m\}$ a small subset to guarantee that for the remaining m's the moduli of all non-resonant divisors admit positive lower bounds. Below in this section

$$\begin{aligned}&\text{constants } C, C_1 \text{ etc. depend on the admissible set } \mathcal{A},\\ &\text{while the exponents } c_1, c_2 \text{ etc depend only on } |\mathcal{A}|. \text{ Borel}\\ &\text{sets } \mathcal{C}_\kappa \text{ etc. depend on the indicated arguments and } \mathcal{A}.\end{aligned} \tag{3.5}$$

We begin with the easier divisors D_0, D_1 and D_2^+.

PROPOSITION 3.6. *Let $1 \geq \kappa > 0$. There exists a Borel set $\mathcal{C}_\kappa \subset [1, 2]$ and positive constants C (cf. (3.5)), satisfying* meas $\mathcal{C}_\kappa \leq C\kappa^{1/(n+2)}$, *such that for all $m \notin \mathcal{C}_\kappa$, all k and all $a, b \in \mathcal{L}$ we have*

$$|\langle \omega, k\rangle| \geq \kappa \langle k\rangle^{-n^2}, \quad \text{except if } k = 0, \tag{3.6}$$

$$|\langle \omega, k\rangle + \lambda_a| \geq \kappa \langle k\rangle^{-3(n+1)^3}, \quad \text{except if the divisor is a trivial resonance}, \tag{3.7}$$

$$|\langle \omega, k\rangle + \lambda_a + \lambda_b| \geq \kappa \langle k\rangle^{-3(n+2)^3}, \text{ except if the divisor is a trivial resonance}. \tag{3.8}$$

Here $\langle k\rangle = \max(|k|, 1)$.

Besides, for each $k \neq 0$ there exists a set $\mathfrak{A}_\kappa^k$ whose measure is $\leq C\kappa^{1/n}$ such that for $m \notin \mathfrak{A}_\kappa^k$ we have

$$|\langle \omega, k\rangle + j| \geq \kappa \langle k\rangle^{-(n+1)n} \text{for all } j \in \mathbb{Z}\ . \tag{3.9}$$

Proof. We begin with the divisors (3.6). By Proposition 3.1 for any non-zero k we have

$$\text{meas}\{m \in [1, 2] \mid |\langle \omega, k\rangle| \leq \kappa |k|^{-n^2}\} < C\kappa^{1/n} |k|^{-n-1/n}\,.$$

Therefore the relation (3.6) holds for all non-zero k if $m \notin \mathfrak{A}_0$, where meas $\mathfrak{A}_0 \leq C\kappa^{1/n} \sum_{k\neq 0} |k|^{-n-1/n} = C\kappa^{1/n}$.

Let us consider the divisors (3.7). For $k = 0$ the required estimate holds trivially. If $k \neq 0$, then the relation, opposite to (3.7) implies that $|\lambda_a| \leq C|k|$. So we may assume that $|a| \leq C|k|^{1/2}$. If $|a| \notin \{|s| \mid s \in \mathcal{A}\}$, then Proposition 3.1 with $n := n+1$, $\mathcal{A} := \mathcal{A} \cup \{a\}$ and $N = C|k|^{1/2}$ implies that

$$\begin{aligned}&\text{meas}\{m \in [1, 2] \mid |\langle \omega, k\rangle + \lambda_a| \leq \kappa |k|^{-3(n+1)^3}\}\\ &\leq C\kappa^{1/(n+1)} |k|^{2(n+1)^2 - 3(n+1)^2 - \frac{1}{n+1}} \leq C\kappa^{1/(n+1)} |k|^{-(n+1)^2}.\end{aligned}$$

This relation with $n+1$ replaced by n also holds if $|a| = |s|$ for some $s \in \mathcal{A}$, but $\langle \omega, k\rangle + \lambda_a$ is not a trivial resonant. Since for fixed k the set$\{\lambda_a \mid |a|^2 \le C|k|\}$ has cardinality less than $2C|k|$, then the relation $|\langle \omega, k\rangle + \lambda_a| \le \kappa |k|^{-3(n+1)^3}$ holds for a fixed k and all a if we remove from [1,2] a set of measure $\le C\kappa^{1/(n+1)}|k|^{-(n+1)^2+1} \le C\kappa^{1/(n+1)}|k|^{-n-1}$. So we achieve that the relation (3.7) holds for all k if we remove from $[1,2]$ a set $\mathfrak{A}_1$ whose measure is bounded by $C\kappa^{1/(n+1)} \sum_{k\neq 0} |k|^{-n-1} = C\kappa^{1/(n+1)}$.

For a similar reason there exist a Borel set $\mathfrak{A}_2$ whose measure is bounded by $C\kappa^{1/(n+2)}$ and such that (3.8) holds for $m \notin \mathfrak{A}_2$. Taking $\mathcal{C}_\kappa = \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \mathfrak{A}_2$ we get (3.6)-(3.8). Proof of (3.9) is similar. □

Now we control divisors $D_2^- = \langle \omega, k\rangle + \lambda_a - \lambda_b$.

PROPOSITION 3.7. *There exist positive constants C, c, c_- and for $0 < \kappa$ there is a Borel set $\mathcal{C}'_\kappa \subset [1,2]$ (cf. (3.5)), satisfying*

$$\text{meas } \mathcal{C}'_\kappa \le C\kappa^c, \tag{3.10}$$

such that for all $m \in [1,2]\backslash\mathcal{C}'_\kappa$, all $k \neq 0$ and all $a, b \in \mathcal{L}$ we have

$$R(k;a,b) := |\langle \omega, k\rangle + \lambda_a - \lambda_b| \ge \kappa |k|^{-c_-}, \tag{3.11}$$

except if the divisor is a trivial resonance.

Proof. We may assume that $|b| \ge |a|$. We get from (3.4) that

$$|\lambda_a - \lambda_b - (|a|^2 - |b|^2)| \le m|a|^{-2} \le 2|a|^{-2}.$$

Take any $\kappa_0 \in (0,1]$ and construct the set $\mathfrak{A}^k_{\kappa_0}$ as in Proposition 3.6. Then $\text{meas}\,\mathfrak{A}^k_{\kappa_0} \le C\kappa_0^{1/n}$ and for any $m \notin \mathfrak{A}^k_{\kappa_0}$ we have

$$R := R(k;a,b) \ge \left|\langle \omega, k\rangle + |a|^2 - |b|^2\right| - 2|a|^{-2} \ge \kappa_0 |k|^{-(n+1)n} - 2|a|^{-2}.$$

So $R \ge \frac{1}{2}\kappa_0 |k|^{-(n+1)n}$ and (3.11) holds if

$$|b|^2 \ge |a|^2 \ge 4\kappa_0^{-1}|k|^{(n+1)n} =: Y_1.$$

If $|a|^2 \le Y_1$, then

$$R \ge \lambda_b - \lambda_a - C|k| \ge |b|^2 - Y_1 - C|k| - 1.$$

Therefore (3.11) also holds if $|b|^2 \ge Y_1 + C|k| + 2$, and it remains to consider the case when $|a|^2 \le Y_1$ and $|b|^2 \le Y_1 + C|k| + 2$. That is (for any fixed non-zero k), consider the pairs (λ_a, λ_b), satisfying

$$|a|^2 \le Y_1, \quad |b|^2 \le Y_1 + 2 + C|k| =: Y_2\,. \tag{3.12}$$

There are at most $CY_1^{d/2}Y_2^{d/2}$ pairs like that. Since the divisor $\langle\omega,k\rangle+\lambda_a-\lambda_b$ is not resonant, then in view of Proposition 3.1 with $N=Y_2^{1/2}$ and $|\mathcal{A}|\le n+2$, for any $\tilde\kappa>0$ there exists a set $\mathfrak{B}^k_{\tilde\kappa}\subset[1,2]$, whose measure is bounded by

$$C\tilde\kappa^{1/(n+2)}\kappa_0^{-c_1}|k|^{c_2},\quad c_j=c_j(n)>0,$$

such that $R\ge\tilde\kappa$ if $m\notin\mathfrak{B}^k_{\tilde\kappa}$ for all pairs (a,b) as in (3.12) (and k fixed).

Let us choose $\tilde\kappa=\kappa_0^{2c_1(n+2)}$. Then $\operatorname{meas}\mathfrak{B}^k_{\tilde\kappa}\le C\kappa_0^{c_1}|k|^{c_2}$ and $R\ge\kappa_0^{2c_1(n+2)}$ for a,b as in (3.12). Denote $\mathfrak{C}^k_{\kappa_0}=\mathfrak{A}^k_{\kappa_0}\cup\mathfrak{B}^k_{\tilde\kappa}$. Then $\operatorname{meas}\mathfrak{C}^k_{\kappa_0}\le C\big(\kappa_0^{1/n}+\kappa_0^{c_1}|k|^{c_2}\big)$, and for m outside this set and all a,b (with k fixed) we have $R\ge\min\big(\frac12\kappa_0|k|^{-(n+1)n},\kappa_0^{2c_1(n+2)}\big)$. We see that if $\kappa_0=\kappa_0(k)=2\kappa^{c_3}|k|^{-c_4}$ with suitable $c_3,c_4>0$, then

$$\operatorname{meas}\big(\mathcal{C}'_\kappa=\cup_{k\ne0}\mathfrak{C}^k_{\kappa_0}\big)\le C\kappa^{c_3},$$

and, if m is outside $\mathcal{C}'_\kappa$, then $R(k;a,b)\ge\kappa|k|^{-c_-}$ with a suitable $c_->0$. □

It remains to consider the divisors D_2^- with $k=0$, i.e. $D_2^-=\lambda_a-\lambda_b$. Such a divisor is resonant if $|a|=|b|$.

LEMMA 3.8. *Let $m\in[1,2]$ and the divisor $D_2^-=\lambda_a-\lambda_b$ is non-resonant, i.e. $|a|\ne|b|$. Then $|\lambda_a-\lambda_b|\ge\frac14$.*

Proof. We have

$$|\lambda_a-\lambda_b|=\frac{\big||a|^4-|b|^4\big|}{\sqrt{|a|^4+m}+\sqrt{|b|^4+m}}\ge\frac{|a|^2+|b|^2}{\sqrt{|a|^4+m}+\sqrt{|b|^4+m}}\ge\frac14.$$ □

By construction the sets $\mathcal{C}_\kappa$ and $\mathcal{C}'_\kappa$ decrease with κ. Let us denote

$$\mathcal{C}=\bigcap_{\kappa>0}(\mathcal{C}_\kappa\cup\mathcal{C}'_\kappa)\,.\tag{3.13}$$

From Propositions 3.6, 3.7 and Lemma 3.8 we get:

PROPOSITION 3.9. *The set $\mathcal{C}$ is a Borel subset of $[1,2]$ of zero measure. For any $m\notin\mathcal{C}$ there exists $\kappa_0=\kappa_0(m)>0$ such that the relations (3.6), (3.7), (3.8) and (3.11) hold with $\kappa=\kappa_0$.*

In particular, if $m\notin\mathcal{C}$ then any of the divisors

$$\langle\omega,s\rangle,\quad\langle\omega,s\rangle\pm\lambda_a,\quad\langle\omega,s\rangle\pm\lambda_a\pm\lambda_b,\quad s\in\mathbb{Z}^d,\quad a,b\in\mathcal{L},$$

vanishes only if this is a trivial resonance. If it is not, then its modulus admits a qualified estimate from below.

The zero-measure Borel set $\mathcal{C}$ serves a fixed admissible set $\mathcal{A}$, $\mathcal{C}=\mathcal{C}_\mathcal{A}$. But since the set of all admissible sets is countable, then replacing $\mathcal{C}$ by $\cup_\mathcal{A}\mathcal{C}_\mathcal{A}$ we obtain a zero-measure Borel set which suits all admissible sets $\mathcal{C}$. For further purposes we modify $\mathcal{C}$ as follows:

$$\mathcal{C}=:\mathcal{C}\cup\{\tfrac43,\tfrac53\}.\tag{3.14}$$

4 The Birkhoff Normal Form I

In Sects. 4 and 5 we construct a symplectic change of variable that puts the Hamiltonian (1.12) to a normal form. In these sections constants in the estimates may depend on

$$d,\ G,\ \mathcal{A} \text{ and constants with lower index } * \text{ (including } c_*) \tag{4.1}$$

without saying. Their dependence on other parameters will be indicated. This does not contradicts Agreements (see the end of Introduction) since in these sections the set $\mathcal{F}$ is defined in terms of $\mathcal{A}$ and $\mathcal{P}$ does not occur.

4.1 Statement of the Result. The goal of this section is to get a normal form for the Hamiltonian $h = h_2 + h_4 + h_{\geq 5}$ of the beam equation, written in the form (1.10), in toroidal domains in the space which are complex neighbourhoods of the n-dimensional real tori $T_{I_\mathcal{A}}$ (see (1.15)). We scale the parameters $I_\mathcal{A}$ as $\nu\rho$ where $\nu > 0$ is small and $\rho = (\rho_a, a \in \mathcal{A})$ belongs to the domain

$$\mathcal{D} = [c_*, 1]^\mathcal{A}. \tag{4.2}$$

In this section $c_* \in (0, \frac{1}{2}]$ is regarded as a fixed parameter.

Consider the complex vicinity of the torus $T_{\nu\rho\,\mathcal{A}}$ (see (1.15))

$$\mathbf{T}_\rho(\nu,\sigma,\mu,\gamma) = \{(p_\mathcal{A}, q_\mathcal{A}, p_\mathcal{L}, \zeta_\mathcal{L}) : \begin{cases} |\frac{1}{2}(p_a^2+q_a^2) - \nu\rho_a| < \nu c_*^2\mu^2, & a \in \mathcal{A}, \\ |\Im\theta_a| < \sigma, & a \in \mathcal{A}, \\ \|(p_\mathcal{L}, q_\mathcal{L})\|_\gamma < \nu^{1/2} c_*\mu, \end{cases} \tag{4.3}$$

where θ_a is related to p_a, q_a through $\frac{p_a - \mathbf{i}q_a}{\sqrt{p_a^2+q_a^2}} = e^{\mathbf{i}\theta_a}$—this is well-defined when $\mu \leq 1$ because then $p_a^2 + q_a^2 \neq 0$ for all $a \in \mathcal{A}$ whenever the point belongs to this vicinity.

In this section we use the complex coordinates $(\xi_a, \eta_a), a \in \mathbb{Z}^d$, defined in (1.7), denoting $(\xi_a, \eta_a) = \zeta_a$. So we will write points of $\mathbf{T}_\rho(\nu,\sigma,\mu,\gamma)$ as $\zeta = (\zeta_\mathcal{A}, \zeta_\mathcal{L})$. We recall (see (1.20)) that we have split the set $\mathcal{L} = \mathbb{Z}^d \backslash \mathcal{A}$ into the union $\mathcal{L} = \mathcal{L}_f \cup \mathcal{L}_\infty$. We will write $\zeta_\mathcal{L} = (\zeta_f, \zeta_\infty)$ and will use the notation of Sect. 2.1 with $\mathcal{Z} = \mathbb{Z}^d$, $\mathcal{Z} = \mathcal{A} \cup \mathcal{L}_f \cup \mathcal{L}_\infty$ (i.e. with $\mathcal{F} = \mathcal{L}_f$).

Proposition 4.1. *There exists a zero-measure Borel set $\mathcal{C} \subset [1,2]$ such that for any admissible set $\mathcal{A}$, any $c_* \in (0, 1/2]$ and $m \notin \mathcal{C}$ we can find real numbers $\gamma_g > \gamma_* = (0, m_* + 2)$ and $\nu_0 > 0$, where ν_0 depends on m, with the following property.*

For any $0 < \nu \leq \nu_0$ and $\rho \in [c_*, 1]^\mathcal{A}$ there exists a holomorphic diffeomorphism (onto its image)

$$\Phi_\rho : \mathcal{O}_{\gamma_*}\left(\frac{1}{2}, \mu_*^2\right) \to \mathbf{T}_\rho(\nu, 1, 1, \gamma_*), \quad \mu_* = \tfrac{c_*}{2\sqrt{2}}, \tag{4.4}$$

which defines holomorphic transformations

$$\Phi_\rho : \mathcal{O}_\gamma\left(\tfrac{1}{2}, \mu_*^2\right) \to \mathbf{T}_\rho(\nu, 1, 1, \gamma), \quad \gamma_* \leq \gamma \leq \gamma_g,$$

such that

$$\Phi_\rho^*\big(-\mathbf{i}dp\wedge dq\big)=\nu dr_{\mathcal{A}}\wedge d\theta_{\mathcal{A}}\ -\mathbf{i}\ \nu d\xi_{\mathcal{L}}\wedge d\eta_{\mathcal{L}},$$

and such that Φ *and the transformed Hamiltonian*

$$\begin{aligned}\tfrac{1}{\nu}h\circ\Phi_\rho(r,\theta,\xi_{\mathcal{L}},\eta_{\mathcal{L}})=\langle\Omega(\rho),r\rangle+\textstyle\sum_{a\in\mathcal{L}_\infty}\Lambda_a(\rho)\xi_a\eta_a\\ +\tfrac{\nu}{2}\left\langle K(\rho)\zeta_f,\zeta_f\right\rangle+f(r,\theta,\zeta_{\mathcal{L}};\rho)\end{aligned}\tag{4.5}$$

(where h *is the Hamiltonian* (1.11)+(1.12)*) satisfy:*

(i) Φ_ρ *depends smoothly (even analytically) on* ρ*, and*

$$\begin{aligned}||\ \Phi_\rho(r,\theta,\xi_{\mathcal{L}},\eta_{\mathcal{L}})-(\sqrt{\nu\rho}\cos(\theta),\sqrt{\nu\rho}\sin(\theta),0,0)\ ||_\gamma\\ \leq C(\sqrt{\nu}\,|r|+\sqrt{\nu}\,\|(\xi_{\mathcal{L}},\eta_{\mathcal{L}})\|_\gamma+\nu^{\frac{3}{2}})\end{aligned}\tag{4.6}$$

for all $(r,\theta,\xi_{\mathcal{L}},\eta_{\mathcal{L}})\in\mathcal{O}_\gamma(\frac{1}{2},\mu_*^2)\cap\{\theta\ \text{real}\}$ *and all* $\gamma_*\leq\gamma\leq\gamma_g$;

(ii) *the vector* Ω *and the scalars* $\Lambda_a, a\in\mathcal{L}_\infty$ *are affine functions of* ρ*, explicitly defined by* (4.44) *and* (4.45)*;*

(iii) K *is a symmetric real matrix. It is a quadratic polynomial of* $\sqrt{\rho}=(\sqrt{\rho}_1,\dots,\sqrt{\rho}_n)$*, explicitly defined by relation* (4.47)*;*

(iv) *the remaining term* f *belongs to* $\mathcal{T}_{\gamma_g,\varkappa=2,\mathcal{D}}(\frac{1}{2},\mu_*^2)$ *and satisfies*

$$|f|_{\substack{1/2,\mu_*^2\\ \gamma_g,2,\mathcal{D}}}\leq C\nu,\quad |f^T|_{\substack{1/2,\mu_*^2\\ \gamma_g,2,\mathcal{D}}}\leq C\nu^{3/2}\,.\tag{4.7}$$

Finally, Φ_ρ *is not a real diffeomorphism, but verifies the "conjugate-reality" condition:*

$$\Phi_\rho(r,\theta,\xi_{\mathcal{L}},\eta_{\mathcal{L}})\quad\textit{is real if, and only if,}\quad\eta_{\mathcal{L}}=\overline{\xi}_{\mathcal{L}}.$$

The constant C *depends on* m *(we recall* (4.1)*) but not on* ν*.*

REMARK 4.2. 1. Φ_ρ is close to the scaling by the factor $\nu^{1/2}$ on the $\mathcal{L}_\infty$-modes but not on the $(\mathcal{A}\cup\mathcal{L}_f)$-modes, where it is close to a certain affine transformation, depending on θ.

2. All the objects, involved in this proposition, except the remaining term f in (4.5), depend only on the main part u^4 of G, and not on the higher order correction.

The rest of this section is devoted to the proof of Proposition 4.1. From now on we arbitrarily enumerate the set $\mathcal{A}$ of excited modes, i.e. we write $\mathcal{A}$ as

$$\mathcal{A}=\{a_1,\dots,a_n\},\tag{4.8}$$

so that the cardinality of $\mathcal{A}$ is n, and accordingly identify $\mathbb{R}^{\mathcal{A}}$ with $\mathbb{R}^n$ and identify various $\mathcal{A}$-valued maps with maps, valued in the set $\{1,\dots,n\}$.

4.2 Resonances and the Birkhoff Procedure. Instead of the domains $\mathcal{O}_\gamma(\sigma,\mu)$, in this section we will use domains

$$\mathcal{O}_\gamma(\sigma,\mu^2,\mu)=\{(r,\theta,w):|r|<\mu^2,|\Im\theta|<\sigma,\|w\|_\gamma<\mu\},\tag{4.9}$$

more convenient for the normal form calculation. The space of functions on $\mathcal{O}_\gamma(\sigma,\mu^2,\mu)$, defined similar to the space $\mathcal{T}_{\gamma,\varkappa}(\sigma,\mu)$, will be denoted $\mathcal{T}_{\gamma,\varkappa}(\sigma,\mu^2,\mu)$. The norm $|f|_{\underset{\gamma,\varkappa}{\sigma,\mu,\mu^2}}$ in this space is defined by the relation (2.11), where the first line is given the weight $\mu^0=1$, the second line—the weight μ^1, and the third line—μ^2. Note that

$$\mathcal{O}_\gamma(\sigma,\mu^2,\mu)\subset\mathcal{O}_\gamma(\sigma,\mu)\subset\mathcal{O}_\gamma(\sigma,\mu,\sqrt{\mu}),\tag{4.10}$$

and that $|\cdot|_{\underset{\gamma,\varkappa}{\sigma,\mu,\mu^2}}$ and $|\cdot|_{\underset{\gamma,\varkappa}{\sigma,\mu}}$ are equivalent if $\mu\sim 1$.

In the situation of Remark 2.12, when $\mathcal{Z}=\mathbb{Z}^d$ and $\mathcal{A}=\mathcal{F}=\emptyset$, we have $\mathcal{T}_{\gamma,\varkappa}(1,\mu^2,\mu)=\mathcal{T}_{\gamma,\varkappa}(1,\mu)$, and

$$|f|_{\underset{\gamma,\varkappa}{1,\mu,\mu^2}}\le|f|_{\underset{\gamma,\varkappa}{1,\mu}}\le\mu^{-2}|f|_{\underset{\gamma,\varkappa}{1,\mu,\mu^2}}\tag{4.11}$$

for any $0<\mu\le 1$.

Example 4.3 (homogeneous functionals). Let $\mathcal{Z}=\mathbb{Z}^d$ and $\mathcal{A}=\mathcal{F}=\emptyset$ and let $f(w)\in\mathcal{T}_{\gamma,\varkappa}(1,1,1)=\mathcal{T}_{\gamma,\varkappa}(1,1)$ be an r-homogeneous function, $r\le 2$ integer. Then df and d^2f are, accordingly, $(r-1)$—and $(r-2)$—homogeneous. So for any $0<\mu\le 1$ we have

$$|f|_{\underset{\gamma,\varkappa}{1,\mu,\mu^2}}=\mu^r|f|_{\underset{\gamma,\varkappa}{1,1,1}}.\tag{4.12}$$

If for $j=1,2$ $f_j(w)\in\mathcal{T}_{\gamma,\varkappa}(1,1,1)$ is an r_j—homogeneous functional, $r_j\ge 2$, then the functional $\{f_1,f_2\}$ is r_1+r_2-2—homogeneous. So the relation above and Proposition 2.9 imply that

$$|\{f_1,f_2\}|_{\underset{\gamma,\varkappa}{1,1,1}}\le C|f_1|_{\underset{\gamma,\varkappa}{1,1,1}}\cdot|f_2|_{\underset{\gamma,\varkappa}{1,1,1}}.\tag{4.13}$$

Let us consider the quartic part h_2+h_4 of the Hamiltonian h,

$$h_2=\sum_{a\in\mathbb{Z}^d}\lambda_a\xi_a\eta_a,\quad h_4=(2\pi)^{-d}\sum_{(i,j,k,\ell)\in\mathcal{J}}\frac{(\xi_i+\eta_{-i})(\xi_j+\eta_{-j})(\xi_k+\eta_{-k})(\xi_\ell+\eta_{-\ell})}{4\sqrt{\lambda_i\lambda_j\lambda_k\lambda_\ell}}$$

(the variables ξ,η are defined in (1.7)), where $\mathcal{J}$ denotes the zero momentum set:

$$\mathcal{J}:=\{(i,j,k,\ell)\subset\mathbb{Z}^d\mid i+j+k+\ell=0\}.$$

We decompose $h_4 = h_{4,0} + h_{4,1} + h_{4,2}$ according to

$$\begin{aligned} h_{4,0} =& \frac{1}{4}(2\pi)^{-d} \sum_{(i,j,k,\ell)\in\mathcal{J}} \frac{\xi_i\xi_j\xi_k\xi_\ell + \eta_i\eta_j\eta_k\eta_\ell}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_\ell}}, \\ h_{4,1} =& (2\pi)^{-d} \sum_{(i,j,k,-\ell)\in\mathcal{J}} \frac{\xi_i\xi_j\xi_k\eta_\ell + \eta_i\eta_j\eta_k\xi_\ell}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_\ell}}, \\ h_{4,2} =& \frac{3}{2}(2\pi)^{-d} \sum_{(i,j,-k,-\ell)\in\mathcal{J}} \frac{\xi_i\xi_j\eta_k\eta_\ell}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_\ell}}, \end{aligned}$$

and define

$$\mathcal{J}_2 = \{(i,j,k,\ell) \subset \mathbb{Z}^d \mid (i,j,-k,-\ell) \in \mathcal{J},\ \sharp\{i,j,k,\ell\} \cap \mathcal{A} \geq 2\}\,.$$

By Proposition 3.9 we have

LEMMA 4.4. *If $m \notin \mathcal{C}$, then there exists $\kappa(m) > 0$ such that for all $(i,j,k,\ell) \in \mathcal{J}_2$*

$$\begin{aligned} &|\lambda_i + \lambda_j + \lambda_k - \lambda_\ell| \geq \kappa(m)\,; \\ &|\lambda_i + \lambda_j - \lambda_k - \lambda_\ell| \geq \kappa(m), \quad \textit{except if } \{|i|,|j|\} = \{|k|,|\ell|\}\,. \end{aligned}$$

For $\gamma = (\gamma_1, \gamma_2)$, where $0 \leq \gamma_1 \leq 1$, $\gamma_2 \geq m_*$, and for $\mathcal{Z} = \mathbb{Z}^d$ as above consider the space Y_γ as in Sect. 2.1, written in terms of the complex coordinates $\zeta_a = (\xi_a, \eta_a), a \in \mathbb{Z}^d$. In these variables the symplectic form Ω reads $\Omega = -i\sum d\xi_a \wedge d\eta_a$. For $0 < \mu \leq 1$ consider the ball $\mathcal{O}_\gamma(1,\mu^2,\mu) = \mathcal{O}_\gamma(1,\mu) = \{|\zeta|_\gamma < \mu\}$.

For any vector $\zeta = (\zeta_a = (\xi_a, \eta_a), a \in \mathbb{Z}^d)$, we will write $\zeta_a^+ = \xi_a$ and $\zeta_a^- = \eta_a$. For an integer $r \geq 2$ we abbreviate $a = (a_1, \dots, a_r) \in (\mathbb{Z}^d)^r$, $\varsigma = (\varsigma_1, \dots, \varsigma_r) \in \{+,-\}^r$, and consider a homogeneous polynomial

$$P^r(\zeta) = M \sum_{a\in(\mathbb{Z}^d)^r} \sum_{\varsigma\in\{+,-\}^r} A_a^\varsigma\, \zeta_{a_1}^{\varsigma_1} \dots \zeta_{a_r}^{\varsigma_r}\,.$$

Here M is a positive constant, the moduli of all coefficients A_a^ς are bounded by 1, and

$$A_a^\varsigma = 0 \quad \text{unless} \quad a_1\varsigma_1^0 + \dots a_r\varsigma_r^0 = 0$$

for some fixed boolean vector $\varsigma^0 \in \{+,-\}^r$. Denote by D^- the block-diagonal operator

$$D^- = \operatorname{diag}\{|\lambda_a|^{-1/2} I, a \in \mathbb{Z}^d\}, \quad I \in M(2\times 2), \tag{4.14}$$

and set $Q^r(\zeta) = P^r(D^-\zeta)$.

LEMMA 4.5. *For any γ as above, $Q^r \in \mathcal{T}_{\gamma,2}(1,1,1)$ and*

$$|Q^r|^{1,1,1}_{\gamma,2} \leq CM, \quad C = C(r). \tag{4.15}$$

The lemma is proved in "Appendix A".

Note that by this lemma, (4.11) and (4.12), $|Q^r|_{\substack{1,\mu\\ \gamma,2}} \le CM\mu^{r-2}$. So by Lemma 2.11 if the function Q^r is real, then the hamiltonian flow-maps $\Phi^t = \Phi^t_{Q^r}$, $|t| \le 1$, define real-holomorphic symplectic mappings

$$\begin{aligned}&\Phi^t : \mathcal{O}_\gamma(1,\mu^2,\mu) \to \mathcal{O}_\gamma(1,4\mu^2,2\mu) \quad \text{if } r \ge 4 \text{ and } \mu \le \mu_1,\ \mu_1 = \mu_1(M) > 0,\\ &\text{or if } r = 3 \text{ and } M > 0 \text{ is sufficiently small}\end{aligned} \tag{4.16}$$

(we recall that now $\mathcal{O}_\gamma(1,\mu^2,\mu) = \mathcal{O}_\gamma(1,\mu)$).

Proposition 4.6. *For $m \notin \mathcal{C}$ and $\mu_g > 0$, $\gamma_g > \gamma_*$ as in Lemma 2.7 there exists $\mu \in (0,\mu_g]$ and a real holomorphic symplectomorphism*

$$\tau : \mathcal{O}_{\gamma_*}(1,\mu) = \{|\zeta|_{\gamma_*} < \mu\} \to \mathcal{O}_{\gamma_*}(1,2\mu)$$

which is a diffeomorphism on its image and which for $\gamma_* \le \gamma \le \gamma_g$ defines analytic mappings $\tau : \mathcal{O}_\gamma(1,\mu) \to \mathcal{O}_\gamma(1,2\mu)$, such that

$$\|\tau^{\pm 1}(\zeta) - \zeta\|_\gamma \le C\|\zeta\|^3_\gamma \quad \forall\, \zeta \in \mathcal{O}_\gamma(1,\mu). \tag{4.17}$$

It transforms the Hamiltonian $h = h_2 + h_4 + h_{\ge 5}$ as follows:

$$h \circ \tau = h_2 + z_4 + q_4^3 + r_6^0 + h_{\ge 5} \circ \tau, \tag{4.18}$$

where

$$z_4 = \frac{3}{2}(2\pi)^{-d} \sum_{\substack{(i,j,k,\ell)\in\mathcal{J}_2\\ \{|i|,|j|\}=\{|k|,|\ell|\}}} \frac{\xi_i\xi_j\eta_k\eta_\ell}{\lambda_i\lambda_j},$$

and $q_4^3 = q_{4,1} + q_{4,2}$ with[13]

$$\begin{aligned} q_{4,1} &= (2\pi)^{-d} \sum_{(i,j,-k,\ell)\notin\mathcal{J}_2} \frac{\xi_i\xi_j\xi_k\eta_\ell + \eta_i\eta_j\eta_k\xi_\ell}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_\ell}},\\ q_{4,2} &= \frac{3}{2}(2\pi)^{-d} \sum_{(i,j,k,\ell)\notin\mathcal{J}_2} \frac{\xi_i\xi_j\eta_k\eta_\ell}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_\ell}}. \end{aligned}$$

The functions $z_4, q_4^3, r_6^0, h_{\ge 5}\circ\tau$ are real holomorphic on $\mathcal{O}_\gamma(1,\mu)$ for each $\gamma_* \le \gamma \le \gamma_g$. Besides r_6^0 and $h_{\ge 5}\circ\tau$ are, respectively, functions of order 6 and 5 at the origin. For any $0 < \mu' \le \mu$ the functions

z_4, q_4^3, r_6^0 and $h_{\ge 5}\circ\tau$ belong to $\mathcal{T}_{\gamma_g,2}(1,(\mu')^2,\mu')$, and

$$|z_4|_{\substack{1,\mu',(\mu')^2\\ \gamma_g,2}} + |q_4^3|_{\substack{1,\mu',(\mu')^2\\ \gamma_g,2}} \le C(\mu')^4, \tag{4.19}$$

$$|r_6^0|_{\substack{1,\mu',(\mu')^2\\ \gamma_g,2}} \le C(\mu')^6, \tag{4.20}$$

$$|h_{\ge 5}\circ\tau|_{\substack{1,\mu',(\mu')^2\\ \gamma_g,2}} \le C(\mu')^5. \tag{4.21}$$

The constants C and μ depend on m (we recall (4.1)).

[13] The upper index 3 signifies that q_4^3 is at least cubic in the transversal directions $\{\zeta_a, a \in \mathcal{L}\}$.

Proof. We use the classical Birkhoff normal form procedure. We construct the transformation τ as the time one flow $\Phi^1_{\chi_4}$ of a Hamiltonian χ_4, given by

$$\begin{aligned}\chi_4 = &-\tfrac{\mathbf{i}}{4}(2\pi)^{-d}\textstyle\sum_{(i,j,k,\ell)\in\mathcal{J}} \frac{\xi_i\xi_j\xi_k\xi_\ell-\eta_i\eta_j\eta_k\eta_\ell}{(\lambda_i+\lambda_j+\lambda_k+\lambda_\ell)\sqrt{\lambda_i\lambda_j\lambda_k\lambda_\ell}}\\ &-\mathbf{i}(2\pi)^{-d}\textstyle\sum_{(i,j,-k,\ell)\in\mathcal{J}_2} \frac{\xi_i\xi_j\xi_k\eta_\ell-\eta_i\eta_j\eta_k\xi_\ell}{(\lambda_i+\lambda_j+\lambda_k-\lambda_\ell)\sqrt{\lambda_i\lambda_j\lambda_k\lambda_\ell}}\\ &-\tfrac{3\mathbf{i}}{2}(2\pi)^{-d}\textstyle\sum_{\substack{(i,j,k,\ell)\in\mathcal{J}_2\\ \{|i|,|j|\}\neq\{|k|,|\ell|\}}} \frac{\xi_i\xi_j\eta_k\eta_\ell}{(\lambda_i+\lambda_j-\lambda_k-\lambda_\ell)\sqrt{\lambda_i\lambda_j\lambda_k\lambda_\ell}}.\end{aligned} \tag{4.22}$$

The Hamiltonian χ_4 is 4-homogeneous and real (it takes real values if $\xi_a = \bar\eta_a$ for each a). If $m\notin\mathcal{C}$, then by Lemma 4.5 $\chi_4\in\mathcal{T}_{\gamma,2}(1,1,1)$, and by Lemma 2.11 and (4.16) the time-one flow-map of this Hamiltonian, $\tau=\Phi^1_{\chi_4}$ is a real holomorphic and symplectic change of coordinates, defined in the μ—neighbourhood of the origin in Y_γ for any $\gamma_*\le\gamma\le\gamma_g$ and a suitable positive $\mu=\mu(m)$. The relation (4.12) implies that on $\mathcal{O}_\gamma(1,2\mu)$ the norm of the hamiltonian vector field is bounded by $C\mu^3$. This implies (4.17).

Since the Poisson bracket, corresponding to the symplectic form $-\mathbf{i}d\xi\wedge d\eta$ is $\{F,G\}=\mathbf{i}\langle\nabla_\eta F,\nabla_\xi G\rangle-\mathbf{i}\langle\nabla_\xi F,\nabla_\eta G\rangle$, and since $\nabla_{\eta_s}h_2=\lambda_s\xi_s$, $\nabla_{\xi_s}h_2=\lambda_s\eta_s$, then we calculate

$$\begin{aligned}\{\chi_4,h_2\} = &\frac{1}{4}(2\pi)^{-d}\sum_{(i,j,k,\ell)\in\mathcal{J}} \frac{\xi_i\xi_j\xi_k\xi_\ell+\eta_i\eta_j\eta_k\eta_\ell}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_\ell}}\\ &+(2\pi)^{-d}\sum_{(i,j,-k,\ell)\in\mathcal{J}_2} \frac{\xi_i\xi_j\xi_k\eta_\ell+\eta_i\eta_j\eta_k\xi_\ell}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_\ell}}\\ &+\frac{3}{2}(2\pi)^{-d}\sum_{\substack{(i,j,k,\ell)\in\mathcal{J}_2\\ \{|i|,|j|\}\neq\{|k|,|\ell|\}}} \frac{\xi_i\xi_j\eta_k\eta_\ell}{\sqrt{\lambda_i\lambda_j\lambda_k\lambda_\ell}}.\end{aligned} \tag{4.23}$$

Therefore the transformed quartic part of the Hamiltonian h, i.e. $(h_2+h_4)\circ\tau$, equals

$$\begin{aligned}h_2+(h_4+\{\chi_4,h_2\})+(\{\chi_4,h_4\}+\int_0^1(1-t)\{\chi_4,\{\chi_4,h_2+h_4\}\}\circ\Phi^t_{\chi_4}\mathrm{d}t)\\ =h_2+(z_4+q_4^3)+r_6^0\end{aligned}$$

with z_4 and q_4^3 as in the statement of the proposition and

$$r_6^0=\{\chi_4,h_4\}+\int_0^1(1-t)\{\chi_4,\{\chi_4,h_2+h_4\}\}\circ\Phi^t_{\chi_4}\mathrm{d}t.$$

The reality of the functions z_4 and q_4^3 follow from the explicit formulas for them, while the inclusion of these functions to $\mathcal{T}_{\gamma_g,2}(1,1,1)$ and the estimate (4.19) for any $0<\mu'\le\mu$ hold by Lemma 4.5 and (4.12).

To verify (4.20) we first note that $\{\chi_4, h_4\}$ is a 6-homogeneous function, belonging to $\mathcal{T}_{\gamma_g,2}(1,1,1) =: \mathcal{T}$ by (4.13). It satisfies the estimate in (4.20) by (4.12). Next, $\{\chi_4, h_2\}$ is a 4-homogeneous function, given by (4.23). By Lemma 4.5 it belongs to $\mathcal{T}$. The function $\{\chi_4, h_4\}$ is 6-homogeneous and belongs to $\mathcal{T}$ by (4.13). So $\{\chi_4, \{\chi_4, h_2+h_4\}\}$ is a sum of a 6- and 8-homogeneous functions, belonging to $\mathcal{T}$ by (4.13). Now the estimate (4.20) for the second component of r_6^0 follows from (4.16), Lemma 2.11 and (4.12).

Finally, the estimate (4.21) follows by applying the argument above to homogeneous components of $h_{\geq 5}$ and noting that the obtained sum converges, if μ is sufficiently small. We skip the details. □

Clearly $\mathbf{T}_\rho(\nu,1,1,\gamma) \subset \mathcal{O}_\gamma(1,\mu)$ if $\nu \leq C^{-1}\mu^2$ (see (4.3)). Due to (4.17), if $\zeta \in \mathbf{T}_\rho(\nu,1/2,1/2,\gamma)$ and $\gamma_* \leq \gamma \leq \gamma_g$, then $\|\tau^{\pm 1}(\zeta) - \zeta\|_\gamma \leq C'(m)\nu^{\frac{3}{2}}$. Therefore

$$\tau^{\pm 1}(\mathbf{T}_\rho(\nu,1/2,1/2,\gamma)) \subset \mathbf{T}_\rho(\nu,1,1,\gamma) \subset \mathcal{O}_\gamma(1,\mu), \tag{4.24}$$

provided that $\nu \leq C^{-1}\mu^2$, $\gamma_* \leq \gamma \leq \gamma_g$ and $\rho \in [c_*,1]^{\mathcal{A}}$.

4.3 Normal form, corresponding to admissible sets $\mathcal{A}$. Everywhere below in Sects. 4, 5 the set $\mathcal{A}$ is assumed to be admissible in the sense of Definition 1.1.

In the domains $\mathbf{T}_\rho = \mathbf{T}_\rho(\nu,\sigma,\mu,\gamma)$ we pass from the complex variables $(\zeta_a, a \in \mathcal{A})$, to the corresponding complex action-angles (I_a, θ_a), using the relations

$$\xi_a = \sqrt{I_a}e^{\mathbf{i}\theta_a}, \quad \eta_a = \sqrt{I_a}e^{-\mathbf{i}\theta_a}, \quad a \in \mathcal{A}. \tag{4.25}$$

By $\mathbf{T}_\rho^{I,\theta} = \mathbf{T}_\rho^{I,\theta}(\nu,\sigma,\mu,\gamma)$ we will denote a domain $\mathbf{T}_\rho(\nu,\sigma,\mu,\gamma)$, written in the variables $(I,\theta,\xi_{\mathcal{L}},\eta_{\mathcal{L}})$, and will denote by ι the corresponding change of variables,

$$\iota : \mathbf{T}_\rho^{I,\theta} \to \mathbf{T}_\rho, \quad (I,\theta,\xi_{\mathcal{L}},\eta_{\mathcal{L}}) \mapsto \zeta. \tag{4.26}$$

Thus, $\iota^{-1}T_{\nu\rho\mathcal{A}} = \{(I,\theta,0,0) : I = \nu\rho, \theta \in \mathbb{T}^n\}$.

The Hamiltonian z_4 contains the integrable part, formed by monomials of the form $\xi_i\xi_j\eta_i\eta_j = I_iI_j$ that only depend on the actions $I_n = \xi_n\eta_n$, $n \in \mathbb{Z}^d$. Denote it z_4^+ and denote the rest z_4^-. It is not hard to see that

$$z_4^+ \circ \iota = \frac{3}{2}(2\pi)^{-d} \sum_{\ell\in\mathcal{A},\ k\in\mathbb{Z}^d} (4 - 3\delta_{\ell,k})\frac{I_\ell I_k}{\lambda_\ell\lambda_k}. \tag{4.27}$$

To calculate z_4^-, we decompose it according to the number of indices in $\mathcal{A}$: a monomial $\xi_i\xi_j\eta_k\eta_\ell$ is in z_4^{-r} $(r = 0,1,2,3,4)$ if $(i,j,-k,-\ell) \in \mathcal{J}$ and $\sharp\{i,j,k,\ell\}\cap\mathcal{A} = r$. We note that, by construction, $z_4^{-0} = z_4^{-1} = \emptyset$.

Since $\mathcal{A}$ is admissible, then in view of Lemma 4.4 for $m \notin \mathcal{C}$ the set z_4^{-4} is empty. The set z_4^{-3} is empty as well:

LEMMA 4.7. *If $m \notin \mathcal{C}$, then $z_4^{-3} = \emptyset$.*

Proof. Consider any term $\xi_i\xi_j\eta_k\eta_\ell \in z_4^{-3}$, i.e. $\{i,j,k,\ell\} \cap \mathcal{A} = 3$. Without lost of generality we can assume that $i,j,k \in \mathcal{A}$ and $\ell \in \mathcal{L}$. Furthermore we know that $i+j-k-\ell = 0$ and $\{|i|,|j|\} = \{|k|,|\ell|\}$. In particular we must have $|i| = |k|$ or $|j| = |k|$ and thus, since $\mathcal{A}$ is admissible, $i = k$ or $j = k$. Let for example, $i = k$. Then $|j| = |\ell|$. Since $i + j = k + \ell$ we conclude that $\ell = j$ which contradicts our hypotheses. □

Recall that the finite set $\mathcal{L}_f \subset \mathcal{L}$ was defined in (1.20). The mapping

$$\ell : \mathcal{L}_f \to \mathcal{A}, \quad a \mapsto \ell(a) \in \mathcal{A} \text{ if } |a| = |\ell(a)|, \tag{4.28}$$

is well defined since the set $\mathcal{A}$ is admissible. Now we define two subsets of $\mathcal{L}_f \times \mathcal{L}_f$:

$$(\mathcal{L}_f \times \mathcal{L}_f)_+ = \{(a,b) \in \mathcal{L}_f \times \mathcal{L}_f \mid \ell(a) + \ell(b) = a + b\}, \tag{4.29}$$

$$(\mathcal{L}_f \times \mathcal{L}_f)_- = \{(a,b) \in \mathcal{L}_f \times \mathcal{L}_f \mid a \neq b \text{ and } \ell(a) - \ell(b) = a - b\}. \tag{4.30}$$

Example 4.8. If $d = 1$, then $\ell(a) = -a$ and the sets $(\mathcal{L}_f \times \mathcal{L}_f)_\pm$ are empty. If d is any, but $\mathcal{A}$ is a one-point set $\mathcal{A} = \{b\}$, then $\mathcal{L}_f$ is the punched discrete sphere $\{a \in \mathbb{Z}^d \mid |a| = |b|, a \neq b\}$, $\ell(a) = b$ for each a, and the sets $(\mathcal{L}_f \times \mathcal{L}_f)_\pm$ again are empty. If $d \geq 2$ and $|\mathcal{A}| \geq 2$, then in general the sets $(\mathcal{L}_f \times \mathcal{L}_f)_\pm$ are non-trivial. See in "Appendix B".

Obviously

$$(\mathcal{L}_f \times \mathcal{L}_f)_+ \cap (\mathcal{L}_f \times \mathcal{L}_f)_- = \emptyset. \tag{4.31}$$

For further reference we note that

Lemma 4.9. *If* $(a,b) \in (\mathcal{L}_f \times \mathcal{L}_f)_+ \cup (\mathcal{L}_f \times \mathcal{L}_f)_-$ *then* $|a| \neq |b|$.

Proof. If $(a,b) \in (\mathcal{L}_f \times \mathcal{L}_f)_+$ and $|a| = |b|$ then $\ell(a) = \ell(b)$ and we have

$$|a + b| = |2\ell(a)| = 2|a| = |a| + |b|$$

which is impossible since b is not proportional to a. If $(a,b) \in (\mathcal{L}_f \times \mathcal{L}_f)_-$ and $|a| = |b|$ then $\ell(a) = \ell(b)$ and we get $a - b = 0$ which is impossible in $(\mathcal{L}_f \times \mathcal{L}_f)_-$. □

Our notation now agrees with that of Sect. 2.1, where $\mathcal{Z} = \mathbb{Z}^d$ is the disjoint union $\mathbb{Z}^d = \mathcal{A} \cup \mathcal{L}_f \cup \mathcal{L}_\infty$. Accordingly, the space $Y_\gamma = Y_{\gamma\mathbb{Z}^d}$ decomposes as

$$Y_\gamma = Y_\mathcal{A} \oplus Y_{\mathcal{L}_f} \oplus Y_{\gamma\mathcal{L}_\infty}, \quad Y_\gamma = \{\zeta = (\zeta_\mathcal{A}, \zeta_f, \zeta_\infty)\}, \tag{4.32}$$

where $Y_\mathcal{A} = \operatorname{span}\{\zeta_s, s \in \mathcal{A}\}$, etc. Below in this Section and in Sect. 5, the domains $\mathcal{O}_\gamma(\sigma, \mu^2, \mu)$ and $\mathcal{O}_\gamma(\sigma, \mu)$, as well as the corresponding function spaces, refer to $\mathcal{Z}$ as above.

LEMMA 4.10. *For $m \notin \mathcal{C}$ the part z_4^{-2} of the Hamiltonian z_4 equals*

$$3(2\pi)^{-d}\Big(\sum_{(a,b)\in(\mathcal{L}_f\times\mathcal{L}_f)_+}\frac{\xi_{\ell(a)}\xi_{\ell(b)}\eta_a\eta_b+\eta_{\ell(a)}\eta_{\ell(b)}\xi_a\xi_b}{\lambda_a\lambda_b} + 2\sum_{(a,b)\in(\mathcal{L}_f\times\mathcal{L}_f)_-}\frac{\xi_a\xi_{\ell(b)}\eta_{\ell(a)}\eta_b}{\lambda_a\lambda_b}\Big). \tag{4.33}$$

Proof. Let $\xi_i\xi_j\eta_k\eta_\ell$ be a monomial in z_4^{-2}. We know that $(i,j,-k,-\ell)\in\mathcal{J}$ and $\{|i|,|j|\}=\{|k|,|\ell|\}$. If $i,j\in\mathcal{A}$ or $k,\ell\in\mathcal{A}$ then we obtain the finitely many monomials as in the first sum in (4.33). Now we assume that $i,\ell\in\mathcal{A}$ and $j,k\in\mathcal{L}$. Then we have that, either $|i|=|k|$ and $|j|=|\ell|$ which leads to finitely many monomials as in the second sum in (4.33). Or $i=\ell$ and $|j|=|k|$. In this last case, the zero momentum condition implies that $j=k$ which is not possible in z_4^-. □

4.4 Eliminating the non integrable terms. For $\ell\in\mathcal{A}$ we introduce the variables $(I_a,\theta_a,\zeta_{\mathcal{L}})$ as in (4.25), (4.26). Now the symplectic structure $-\mathbf{i}d\xi\wedge d\eta$ reads

$$-\sum_{a\in\mathcal{A}} dI_a\wedge d\theta_a - \mathbf{i}d\xi_{\mathcal{L}}\wedge d\eta_{\mathcal{L}}. \tag{4.34}$$

In view of (4.27), (4.18) and Lemma 4.10, for $m\notin\mathcal{C}$ the Hamiltonian h, transformed by $\tau\circ\iota$, may be written as

$$\begin{aligned} h\circ\tau\circ\iota = \langle\omega,I\rangle + \sum_{s\in\mathcal{L}}\lambda_s\xi_s\eta_s + \frac{3}{2}(2\pi)^{-d}\sum_{\ell\in\mathcal{A},\ k\in\mathbb{Z}^d}(4-3\delta_{\ell,k})\frac{I_\ell\xi_k\eta_k}{\lambda_\ell\lambda_k} \\ + 3(2\pi)^{-d}\Big(\sum_{(a,b)\in(\mathcal{L}_f\times\mathcal{L}_f)_+}\frac{\xi_{\ell(a)}\xi_{\ell(b)}\eta_a\eta_b+\eta_{\ell(a)}\eta_{\ell(b)}\xi_a\xi_b}{\lambda_a\lambda_b} \\ + 2\sum_{(a,b)\in(\mathcal{L}_f\times\mathcal{L}_f)_-}\frac{\xi_a\xi_{\ell(b)}\eta_{\ell(a)}\eta_b}{\lambda_a\lambda_b}\Big) + q_4^3\circ\iota + r_5^0, \end{aligned}$$

where $r_5^0 = h_{\geq 5}\circ\tau\circ\iota + r_6^0\circ\iota$ (recall that $\omega=(\lambda_a, a\in\mathcal{A})$). The first line contains the integrable terms. The second and third lines contain the lower-order non integrable terms, depending on the angles θ; there are finitely many of them. The third line also contains the remaining high order terms, where q_4^3 is of total order (at least) 4 and of order 3 in the normal directions ζ, while r_5^0 is of total order at least 5. The latter is the sum of $r_6^0\circ\iota$ which comes from the Birkhoff normal form procedure (and is of order 6) and $h_{\geq 5}\circ\tau\circ\iota$ which comes from the term of order ≥ 5 in the nonlinearity (1.2). Here I is regarded as a variable of order 2, while θ has zero order. The terms $q_4^3\circ\iota$ and r_5^0 should be regarded as a perturbation.

To deal with the non integrable terms in the second and third lines, following the works on the finite-dimensional reducibility (see [Eli01]), we introduce a change of variables

$$\Psi : (\tilde I,\tilde\theta,\tilde\xi,\tilde\eta)\mapsto(I,\theta,\xi,\eta),$$

symplectic with respect to (4.34), but such that its differential at the origin is not close to the identity. It is defined by the following relations:

$$I_\ell = \tilde I_\ell - \textstyle\sum_{|a|=|\ell|,\ a\neq\ell} \tilde\xi_a\tilde\eta_a, \quad \theta_\ell = \tilde\theta_\ell \quad \ell\in\mathcal{A};$$
$$\xi_a = \tilde\xi_a e^{\mathbf{i}\tilde\theta_{\ell(a)}}, \quad \eta_a = \tilde\eta_a e^{-\mathbf{i}\tilde\theta_{\ell(a)}} \quad a\in\mathcal{L}_f; \quad \xi_a=\tilde\xi_a, \quad \eta_a=\tilde\eta_a \quad a\in\mathcal{L}_\infty.$$

For any $(\tilde I,\tilde\theta,\tilde\zeta)\in \mathbf{T}^{I,\theta}_\rho(\nu,\sigma,\mu,\gamma)$ denote by $y=\{y_l, l\in\mathcal{A}\}$ the vector, whose l-th component equals $y_l=\sum_{|a|=|l|,a\neq l}\tilde\xi_a\tilde\eta_a$. Then

$$|I-\nu\rho| \le |\tilde I-\nu\rho| + |y| \le c_*^2\nu\mu^2 + \sum_{a\in\mathcal{L}_f}|\tilde\xi_a\tilde\eta_a| \le 2c_*^2\nu\mu^2. \tag{4.35}$$

This implies that

$$\Psi^{\pm1}\left(\mathbf{T}^{I,\theta}_\rho\left(\nu,\tfrac12,\frac{1}{2\sqrt2},\gamma\right)\right)\subset \mathbf{T}^{I,\theta}_\rho\left(\nu,\tfrac12,\tfrac12,\gamma\right) =: \mathbf{T}^{I,\theta}_\rho. \tag{4.36}$$

The transformation Ψ is identity on each torus $\{(I,\theta,\zeta_\mathcal{L}) : I=\text{const}, \theta\in\mathbb{T}^n, \zeta_\mathcal{L}=0\}$. Writing it as $(I,\theta,\zeta_\mathcal{L})\mapsto(\tilde I,\tilde\theta,\tilde\zeta_\mathcal{L})$ we see that

$$|\tilde I_a - I_a| \le \|\zeta_\mathcal{L}\|^2_\gamma,\ a\in\mathcal{A},\ \tilde\theta=\theta \quad \text{and} \quad \|\tilde\zeta_\mathcal{L}\|_\gamma = \|\zeta_\mathcal{L}\|_\gamma, \tag{4.37}$$

and that $(\xi,\eta)=\iota(\tilde I,\tilde\theta,\tilde\zeta_\mathcal{L})$ satisfies

$$\xi_l = \sqrt{I_l}\, e^{\mathbf{i}\theta_l} = \sqrt{\tilde I_l}\, e^{\mathbf{i}\tilde\theta_l} + O(\nu^{-1/2})\,O(|\zeta_\mathcal{L}|^2), \quad l\in\mathcal{A}. \tag{4.38}$$

Accordingly, dropping the tildes, we write the restriction to $\mathbf{T}^{I,\theta}_\rho$ of the transformed Hamiltonian $h^1=h\circ\tau\circ\iota\circ\Psi$ as

$$\begin{aligned} h^1 =& \langle\omega,I\rangle + \sum_{a\in\mathcal{L}_\infty}\lambda_a\xi_a\eta_a + 6(2\pi)^{-d}\sum_{\ell\in\mathcal{A},\ k\in\mathcal{L}}\frac{1}{\lambda_\ell\lambda_k}(I_\ell - \sum_{\substack{|a|=|\ell|\\ a\in\mathcal{L}_f}}\xi_a\eta_a)\xi_k\eta_k \\ &+ \frac32(2\pi)^{-d}\sum_{\ell,k\in\mathcal{A}}\frac{4-3\delta_{\ell,k}}{\lambda_\ell\lambda_k}(I_\ell - \sum_{\substack{|a|=|\ell|\\ a\in\mathcal{L}_f}}\xi_a\eta_a)(I_k - \sum_{\substack{|a|=|k|\\ a\in\mathcal{L}_f}}\xi_a\eta_a) \\ &+ 3(2\pi)^{-d}\sum_{(a,b)\in(\mathcal{L}_f\times\mathcal{L}_f)_+}\frac{\sqrt{I_{\ell(a)}I_{\ell(b)}}}{\lambda_a\lambda_b}(\eta_a\eta_b+\xi_a\xi_b) \\ &+ 6(2\pi)^{-d}\sum_{(a,b)\in(\mathcal{L}_f\times\mathcal{L}_f)_-}\frac{\sqrt{I_{\ell(a)}I_{\ell(b)}}}{\lambda_a\lambda_b}\xi_a\eta_b + q_4^{3'} + r_5^{0'} + \nu^{-1/2}r_5^{4'}. \end{aligned}$$

Here $q_4^{3'}$ and $r_5^{0'}$ are the function q_4^3 and r_5^0, transformed by Ψ, so the former satisfy the same estimates as the latter, while $r_5^{4'}$ is a function of forth order in the

normal variables. The latter comes from re-writing terms like $\xi_{\ell(a)}\xi_{\ell(b)}\eta_a\eta_b$, using (4.38) and expressing η_a, η_b via the tilde-variables. Or, after a simplification:

$$\begin{aligned} h^1 = {} & \langle \omega, I\rangle + \textstyle\sum_{a\in\mathcal{L}_\infty} \lambda_a\xi_a\eta_a + \frac{3}{2}(2\pi)^{-d}\sum_{\ell,k\in\mathcal{A}} \frac{4-3\delta_{\ell,k}}{\lambda_\ell\lambda_k} I_\ell I_k \\ & +3(2\pi)^{-d}\Big(2\sum_{\ell\in\mathcal{A},\ a\in\mathcal{L}_\infty} \frac{1}{\lambda_\ell\lambda_a} I_\ell\xi_a\eta_a - \sum_{\ell\in\mathcal{A},\ a\in\mathcal{L}_f} \frac{(2-3\delta_{\ell,|a|})}{\lambda_\ell\lambda_a} I_\ell\xi_a\eta_a\Big) \\ & +3(2\pi)^{-d}\sum_{(a,b)\in(\mathcal{L}_f\times\mathcal{L}_f)_+} \frac{\sqrt{I_{\ell(a)}I_{\ell(b)}}}{\lambda_a\lambda_b}(\eta_a\eta_b + \xi_a\xi_b) \\ & +6(2\pi)^{-d}\sum_{(a,b)\in(\mathcal{L}_f\times\mathcal{L}_f)_-} \frac{\sqrt{I_{\ell(a)}I_{\ell(b)}}}{\lambda_a\lambda_b}\xi_a\eta_b + q_4^{3'} + r_5^{0'} + \nu^{-1/2}r_5^{4'}. \end{aligned} \tag{4.39}$$

We see that the transformation Ψ removed from $h\circ\tau\circ\iota$ the non-integrable lower-order terms on the price of introducing "half-integrable" terms which do not depend on the angles θ, but depend on the actions I and quadratically depend on the finitely many variables ξ_a, η_a with $a\in\mathcal{L}_f$.

The Hamiltonian $h\circ\tau\circ\iota\circ\Psi$ should be regarded as a function of the variables $(I,\theta,\zeta_{\mathcal{L}})$. Abusing notation, below we often drop the lower-index $\mathcal{L}$ and write $\zeta_{\mathcal{L}} = (\xi_{\mathcal{L}}, \eta_{\mathcal{L}})$ as $\zeta = (\xi,\eta)$.

4.5 Rescaling the variables and defining the transformation Φ. Our aim is to study the transformed Hamiltonian h^1 on the domains $\mathbf{T}^{I,\theta}_\rho = \mathbf{T}^{I,\theta}_\rho(\nu, \frac{1}{2}, \frac{1}{2\sqrt{2}}, \gamma)$, $0\le\gamma\le\gamma_g$ (see (4.36)). To do this we re-parametrise points of $\mathbf{T}^{I,\theta}_\rho$ by mean of the scaling

$$\chi_\rho : (\tilde r, \tilde\theta, \tilde\xi, \tilde\eta) \mapsto (I,\theta,\xi,\eta), \tag{4.40}$$

where $I = \nu\rho + \nu\tilde r$, $\theta = \tilde\theta$, $\xi = \sqrt{\nu}\,\tilde\xi$, $\eta = \sqrt{\nu}\,\tilde\eta$. Clearly,

$$\chi_\rho : \mathcal{O}_\gamma(\tfrac{1}{2}, \mu_*^2, \mu_*) \to \mathbf{T}^{I,\theta}_\rho$$

for $0\le\gamma\le\gamma_g$, where μ_* is defined in (4.4), and in the new variables the symplectic structure reads

$$-\nu\sum_{\ell\in\mathcal{A}} d\tilde r_\ell\wedge d\tilde\theta_\ell \ - \mathbf{i}\,\nu\sum_{a\in\mathcal{L}} d\tilde\xi_a\wedge d\tilde\eta_a.$$

Denoting

$$\Phi = \Phi_\rho = \tau\circ\iota\circ\Psi\circ\chi_\rho,$$

we see that this transformation is analytic in $\rho\in\mathcal{D}$. In view of (4.37), $\zeta = (\xi,\eta) = \Phi(\tilde r,\tilde\theta,\tilde\zeta)$ satisfies

$$\|\zeta - \zeta'\|_\gamma \le C(\sqrt{\nu}\,(|\tilde r| + \|\zeta\|_\gamma)), \quad \zeta' = \big(\sqrt{\nu\rho}\,e^{\mathbf{i}\tilde\theta}, \sqrt{\nu\rho}\,e^{\mathbf{i}\tilde\theta}, 0\big).$$

This relation and (4.17) imply (4.6), so the assertion (i) of the proposition holds.

Dropping the tildes and forgetting the irrelevant constant $\nu\langle\omega,\rho\rangle$, we have

$$\begin{aligned} h\circ\Phi(r,\theta,\zeta) = \nu\Big[&\langle\omega,r\rangle + \textstyle\sum_{a\in\mathcal{L}_\infty}\lambda_a\xi_a\eta_a + (2\pi)^{-d}\nu\Big(\tfrac{3}{2}\sum_{\ell,k\in\mathcal{A}}\tfrac{4-3\delta_{\ell,k}}{\lambda_\ell\lambda_k}\rho_\ell r_k \\ &+6\sum_{\ell\in\mathcal{A},\ a\in\mathcal{L}_\infty}\tfrac{1}{\lambda_\ell\lambda_a}\rho_\ell\xi_a\eta_a - 3\sum_{\ell\in\mathcal{A},\ a\in\mathcal{L}_f}\tfrac{(2-3\delta_{\ell,|a|})}{\lambda_\ell\lambda_a}\rho_\ell\xi_a\eta_a \\ &+3\sum_{(a,b)\in(\mathcal{L}_f\times\mathcal{L}_f)_+}\tfrac{\sqrt{\rho_{\ell(a)}}\sqrt{\rho_{\ell(b)}}}{\lambda_a\lambda_b}(\eta_a\eta_b+\xi_a\xi_b) \\ &+6\sum_{(a,b)\in(\mathcal{L}_f\times\mathcal{L}_f)_-}\tfrac{\sqrt{\rho_{\ell(a)}}\sqrt{\rho_{\ell(b)}}}{\lambda_a\lambda_b}\xi_a\eta_b\Big)\Big] \\ &+\left(\left(q_4^{3'}+r_5^{0'}+\nu^{-1/2}r_5^{4'}\right)(I,\theta,\sqrt{\nu}\zeta)\right)|_{I=\nu\rho+\nu r}, \end{aligned} \tag{4.41}$$

where $\zeta=\zeta_{\mathcal{L}}=(\zeta_a)_{a\in\mathcal{L}},\ \zeta_a=(\xi_a,\eta_a)$, and $\zeta_f=(\zeta_a)_{a\in\mathcal{L}_f}$. So,

$$\nu^{-1}h\circ\Phi=\tilde{h}_2+f, \tag{4.42}$$

where f is the perturbation, given by the last line in (4.41),

$$f=\nu^{-1}\left(\left(q_4^{3'}+r_5^{0'}+\nu^{-1/2}r_5^{4'}\right)(I,\theta,\nu^{1/2}\zeta)\right)|_{I=\nu\rho+\nu r}, \tag{4.43}$$

and $\tilde{h}_2=\tilde{h}_2(I,\xi,\eta;\rho,\nu)$ is the quadratic part of the Hamiltonian, which is independent from the angles θ:

$$\tilde{h}_2=\langle\Omega,r\rangle+\sum_{a\in\mathcal{L}_\infty}\Lambda_a\xi_a\eta_a+\nu\langle K(\rho)\zeta_f,\zeta_f\rangle.$$

Here $\Omega=(\Omega_k)_{k\in\mathcal{A}}$ with

$$\Omega_k=\Omega_k(\rho,\nu)=\omega_k+\nu\sum_{\ell\in\mathcal{A}}M_k^\ell\rho_l,\quad M_k^\ell=\frac{3(4-3\delta_{\ell,k})}{(2\pi)^d\lambda_k\lambda_\ell}, \tag{4.44}$$

$$\Lambda_a=\Lambda_a(\rho,\nu)=\lambda_a+6\nu(2\pi)^{-d}\sum_{\ell\in\mathcal{A}}\frac{\rho_\ell}{\lambda_\ell\lambda_a}, \tag{4.45}$$

and $K(\rho)$ is a symmetric complex matrix, acting in the space

$$Y_{\mathcal{L}_f}=\{\zeta_f\}\simeq\mathbb{C}^{2|\mathcal{L}_f|}, \tag{4.46}$$

such that the corresponding quadratic form is

$$\begin{aligned} \langle K(\rho)\zeta_f,\zeta_f\rangle = 3(2\pi)^{-d}\Big(&\sum_{\ell\in\mathcal{A},\ a\in\mathcal{L}_f}\frac{(3\delta_{\ell,|a|}-2)}{\lambda_\ell\lambda_a}\rho_\ell\xi_a\eta_a \\ &+\sum_{(a,b)\in(\mathcal{L}_f\times\mathcal{L}_f)_+}\frac{\sqrt{\rho_{\ell(a)}}\sqrt{\rho_{\ell(b)}}}{\lambda_a\lambda_b}(\eta_a\eta_b+\xi_a\xi_b) \\ &+2\sum_{(a,b)\in(\mathcal{L}_f\times\mathcal{L}_f)_-}\frac{\sqrt{\rho_{\ell(a)}}\sqrt{\rho_{\ell(b)}}}{\lambda_a\lambda_b}\xi_a\eta_b\Big). \end{aligned} \tag{4.47}$$

Note that the matrix M in (4.44) is invertible since

$$\det M=3^n(2\pi)^{-dn}\left(\Pi_{k\in\mathcal{A}}\lambda_k\right)^{-2}\det\left(4-3\delta_{\ell,k}\right)_{\ell,k\in\mathcal{A}}\neq 0.$$

The explicit formulas (4.44)–(4.47) imply the assertions (ii) and (iii).

The transformations $\Psi \circ \chi_\rho$ and $\tau \circ \iota$ both are real if we use in the spaces Y_γ and $Y_{\gamma\mathcal{L}}$ the real coordinates (p_a, q_a), see (1.7). This implies the stated "conjugate-reality" of Φ_ρ.

It remains to verify (iv). By Proposition 4.6 the function f belongs to the class $\mathcal{T}_{\gamma_g,2}(\frac{1}{2}, \mu_*^2, \mu_*)$. Since the remainder f has the form (4.43) then in view of (4.19)-(4.21) for $(r, \theta, \zeta) \in \mathcal{O}_\gamma(\frac{1}{2}, \mu_*^2, \mu_*)$ it satisfies the estimates

$$|f| \le C\nu, \quad \|\nabla_\zeta f\|_\gamma \le C\nu, \quad \|\nabla_\zeta^2 f\|_{\gamma,2}^b \le C\nu.$$

Now consider the f^T-component of f. Only the second term in (4.43) contributes to it and we have that

$$|f^T| + \|\nabla_\zeta f^T\|_\gamma + \|\nabla_\zeta^2 f^T\|_{\gamma,2}^b \le C\nu^{3/2}.$$

This implies the assertion (iv) of the proposition in view of (4.10) and (4.11).

We will provide the domains $\mathcal{O}_\gamma\big(\frac{1}{2}, \mu_*^2\big) \subset \mathcal{O}_\gamma\big(\frac{1}{2}, \mu_*^2, \mu_*\big) = \{(r, \theta, \xi, \eta)\}$ with the symplectic structure $-\sum_{\ell\in\mathcal{A}} dr_\ell \wedge d\theta_\ell \ -\mathbf{i}\sum_{a\in\mathcal{L}} d\xi_a \wedge d\eta_a$. Then the transformed Hamiltonian system, constructed in Proposition 4.1 has the Hamiltonian, given by the r.h.s. of (4.5).

5 The Birkhoff Normal Form II

In this section we shall refine the normal form (4.5) further. We shall construct a ρ-dependent transformation which diagonalises the Hamiltonian operator (modulo the term f) and shall examine its smoothness in ρ. So here we are concerned with analysis of the finite-dimensional linear hamiltonian operator $\mathbf{i}JK(\rho)$ defined by the Hamiltonian (4.47). To do this we will have to restrict ρ to some (large) subset $Q \subset \mathcal{D} = [c_*, 1]^{\mathcal{A}}$. In this section and below c_* is regarded as a parameter of the construction, belonging to an interval $(0, \frac{1}{2}c_0]$, where $c_0 > 0$ depends on m and on the constants in (4.1). This c_0 is introduced in Lemma 5.4 and is fixed after it. The parameter c_* will be fixed till Sect. 10.2 (the last in our work), where we will vary it.

In this section we shall also shift from the conjugate-reality to the ordinary reality, thus restoring the original real character of the system.

Theorem 5.1. *There exists a zero-measure Borel set $\mathcal{C} \subset [1, 2]$ such that for any admissible set $\mathcal{A}$ and any $m \notin \mathcal{C}$ there exist real numbers $\gamma_g > \gamma_* = (0, m_* + 2)$ and $\beta_0, \nu_0, c_0 > 0$, where c_0, β_0, ν_0 depend on m, such that, for any $0 < c_* \le c_0$, $0 < \nu \le \nu_0$ and $0 < \beta_\# \le \beta_0$ there exists an open set $Q = Q(c_*, \beta_\#, \nu) \subset [c_*, 1]^{\mathcal{A}}$, increasing as $\nu \to 0$ and satisfying*

$$\mathrm{meas}([c_*, 1]^{\mathcal{A}} \backslash Q) \le C\nu^{\beta_\#}, \tag{5.1}$$

with the following property.

For any $\rho \in Q$ there exists a real holomorphic diffeomorphism (onto its image)

$$\Phi_\rho : \mathcal{O}_{\gamma_*}\big(\tfrac{1}{2}, \mu_*^2\big) \to \mathbf{T}_\rho(\nu, 1, 1, \gamma_*), \quad \mu_* = \tfrac{c_*}{2\sqrt{2}}, \tag{5.2}$$

which defines analytic diffeomorphisms $\Phi_\rho : \mathcal{O}_\gamma\big(\frac{1}{2}, \mu_*^2\big) \to \mathbf{T}_\rho(\nu, 1, 1, \gamma)$, $\gamma_* \le \gamma \le \gamma_g$, such that

$$\Phi_\rho^*\big(d\xi \wedge d\eta\big) = \nu dr_{\mathcal{A}} \wedge d\theta_{\mathcal{A}} \ + \ \nu dp_{\mathcal{L}} \wedge dq_{\mathcal{L}}, \tag{5.3}$$

and, up to a constant,

$$\begin{aligned} \tfrac{1}{\nu}\, h \circ \Phi_\rho(r, \theta, p_{\mathcal{L}}, q_{\mathcal{L}}) &= \langle \Omega(\rho), r\rangle + \tfrac{1}{2}\textstyle\sum_{a\in\mathcal{L}_\infty} \Lambda_a(\rho)(p_a^2 + q_a^2) \\ &\quad + \tfrac{1}{2}\textstyle\sum_{b\in\mathcal{L}_f\setminus\mathcal{F}} \Lambda_b(\rho)(p_b^2 + q_b^2) + \nu\langle K(\rho)\zeta_{\mathcal{F}}, \zeta_{\mathcal{F}}\rangle + f(r, \theta, \zeta_{\mathcal{L}}; \rho), \end{aligned} \tag{5.4}$$

where $\mathcal{F} = \mathcal{F}_\rho \subset \mathcal{L}_f$ (only depending on the connected component of Q containing ρ), and h is the Hamiltonian (1.11) + (1.12). Φ_ρ satisfies:

(i) *Φ_ρ depends smoothly on ρ and*

$$\begin{aligned} \| \Phi_\rho\,(r, \theta, \xi_{\mathcal{L}}, \eta_{\mathcal{L}}) - \big(\sqrt{\nu\rho}\cos(\theta), \sqrt{\nu\rho}\sin(\theta), \sqrt{\nu\rho}\xi_{\mathcal{L}}, \sqrt{\nu\rho}\eta_{\mathcal{L}}\big)\, \|_\gamma \\ \le C\left(\sqrt{\nu}\,|r| + \sqrt{\nu}\,\|(\xi_{\mathcal{L}}, \eta_{\mathcal{L}})\|_\gamma + \nu^{\frac{3}{2}}\right)\nu^{-\hat{c}\beta_\#} \end{aligned} \tag{5.5}$$

for all $(r, \theta, \xi_{\mathcal{L}}, \eta_{\mathcal{L}}) \in \mathcal{O}_\gamma(\frac{1}{2}, \mu_*^2) \cap \{\theta \text{ real}\}$ and all $\gamma_* \le \gamma \le \gamma_g$;

(ii) *the vector Ω and the scalars $\Lambda_a, a \in \mathcal{L}_\infty$, are affine functions of ρ, explicitly defined (4.44), (4.45);*

(iii) *the functions $\Lambda_b(\rho)$, $b \in \mathcal{L}_f\setminus\mathcal{F}$, are smooth in Q,*

$$\|\Lambda_b\|_{C^j(Q)} \le C_j \nu^{-\beta_\# \beta(j)}\nu, \quad \forall j \ge 0, \tag{5.6}$$

where $0 < \beta(1) \le \beta(2) \le \dots$, and satisfy (5.38). In some open subset of $[c_*, 1]^{\mathcal{A}}$ they also satisfy (5.29);

(iv) *K is a symmetric real matrix that depends smoothly on $\rho \in Q$, and*

$$\sup_{\rho\in Q} \|\partial_\rho^j K(\rho)\| \le C_j \nu^{-\beta_\#\beta(j)}, \quad \forall j \ge 0; \tag{5.7}$$

the set $\mathcal{F} = \mathcal{F}_\rho$ is void for some ρ (in which case the operator $K(\rho)$ is trivial);

(v) *the eigenvalues $\{\pm \mathbf{i}\Lambda_a, a \in \mathcal{F}\}$ of JK are smooth in Q, satisfy (5.6) and*

$$\inf_{\rho\in Q} |\Im\Lambda_a(\rho)| \ge C^{-1}\nu^{\bar{c}\beta_\#}, \quad \forall a \in \mathcal{F}; \tag{5.8}$$

(vi) *There exists a complex symplectic operator $U(\rho)$ such that*

$$U(\rho)^{-1} JK(\rho) U(\rho) = \mathbf{i}\,\mathrm{diag}\{\pm\Lambda_a(\rho), a \in \mathcal{F}\};$$

the operator $U(\rho)$ smoothly depends on ρ and satisfies

$$\sup_{\rho\in Q} \big(\|\partial_\rho^j U(\rho)\| + \|\partial_\rho^j U(\rho)^{-1}\|\big) \le C_j \nu^{-\beta_\#\beta(j)}, \quad \forall j \ge 0. \tag{5.9}$$

(vii) *f belongs to $\mathcal{T}_{\gamma,\varkappa=2,Q}(\frac{1}{2},\mu_*^2)$ and satisfies*

$$|f|^{1/2,\mu_*^2}_{\gamma_g,2,Q} \le C\nu^{-\hat{c}\beta_\#}\nu, \quad |f^T|^{1/2,\mu_*^2}_{\gamma_g,2,Q} \le C\nu^{-\hat{c}\beta_\#}\nu^{3/2}. \tag{5.10}$$

The set Q and the matrix $K(\rho)$ do not depend on the function G (having the form (1.2)). The constants C, C_j are as in (4.1), while the exponents $\bar{c}, \hat{c}$ and $\beta(j)$ depend on m (we recall (4.1)).

Remark. 1. By (5.3) the transformation Φ_ρ transforms the beam equation, written in the form (1.10), to a system, which has the Hamiltonian (5.4) with respect to the symplectic structure $dr_{\mathcal{A}} \wedge d\theta_{\mathcal{A}} + \nu dp_{\mathcal{L}} \wedge dq_{\mathcal{L}}$.

2. We also have $\Phi_\rho\big(\mathcal{O}_\gamma(\frac{1}{2},\mu_*^2)\big) \subset \mathbf{T}_\rho(\nu,1,1,\gamma)$ for $\gamma_* \le \gamma \le \gamma_g$.

The remaining part of this section is devoted to the proof of this result.

5.1 Matrix $K(\rho)$. Recalling (4.8) and (4.2), we write the symmetric matrix $K(\rho)$, defined by relation (4.47), as a block-matrix, polynomial in $\sqrt{\rho} = (\sqrt{\rho}_1, \dots, \sqrt{\rho}_n)$. We write it as $K(\rho) = K^d(\rho) + K^{n/d}(\rho)$. Here K^d is the block-diagonal matrix

$$\begin{aligned} K^d(\rho) &= \operatorname{diag}\left(\begin{pmatrix} 0 & \mu(a,\rho) \\ \mu(a,\rho) & 0 \end{pmatrix},\ a \in \mathcal{L}_f\right), \\ \mu(a,\rho) &= C_*\left(\tfrac{3}{2}\,\rho_{\ell(a)}\lambda_a^{-2} - \lambda_a^{-1}\textstyle\sum_{l\in\mathcal{A}}\rho_l\lambda_l^{-1}\right), \quad C_* = 3(2\pi)^{-d}. \end{aligned} \tag{5.11}$$

Note that[14]

$$\mu(a,\rho) \quad \text{is a function of } |a| \text{ and } \rho. \tag{5.12}$$

The non-diagonal matrix $K^{n/d}$ has zero diagonal blocks, while for $a \ne b$ its block $K^{n/d}(\rho)^b_a$ equals

$$C_*\frac{\sqrt{\rho_{l(a)}\rho_{l(b)}}}{\lambda_a\lambda_b}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\chi^+(a,b) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\chi^-(a,b)\right),$$

where

$$\chi^+(a,b) = \begin{cases} 1, & (a,b) \in (\mathcal{L}_f \times \mathcal{L}_f)_+, \\ 0, & \text{otherwise}, \end{cases}$$

and χ^- is defined similar in terms of the set $(\mathcal{L}_f \times \mathcal{L}_f)_-$. In view of (4.31),

$$\chi^+(a,b)\cdot\chi^-(a,b) \equiv 0.$$

Accordingly, the Hamiltonian matrix $\mathcal{H}(\rho) = \boldsymbol{i}JK(\rho)$ equals $\big(\mathcal{H}^d(\rho) + \mathcal{H}^{n/d}(\rho)\big)$, where

$$\begin{aligned} \mathcal{H}^d(\rho) &= \mathbf{i}\operatorname{diag}\left(\begin{pmatrix} \mu(a,\rho) & 0 \\ 0 & -\mu(a,\rho) \end{pmatrix},\ a \in \mathcal{L}_f\right), \\ \mathcal{H}^{n/d}(\rho)^b_a &= \mathbf{i}C_*\tfrac{\sqrt{\rho_{\ell(a)}}\sqrt{\rho_{\ell(b)}}}{\lambda_a\lambda_b}\left[J\chi^+(a,b) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\chi^-(a,b)\right]. \end{aligned} \tag{5.13}$$

[14] Here and in similar situations below we do not mention the obvious dependence on the parameter $m \in [1,2]$.

Note that all elements of the matrix $\mathcal{H}(\rho)$ are pure imaginary, and

$$\text{if } (\mathcal{L}_f \times \mathcal{L}_f)_+ = \emptyset, \text{ then } -\mathbf{i}\mathcal{H}(\rho) \text{ is real symmetric,} \tag{5.14}$$

in which case all eigenvalues of $\mathcal{H}(\rho)$ are pure imaginary. In "Appendix B" we show that if $d \geq 2$, then, in general, the set $(\mathcal{L}_f \times \mathcal{L}_f)_+$ is not empty and the matrix $\mathcal{H}(\rho)$ may have hyperbolic eigenvalues.

EXAMPLE 5.2. In view of Example 4.8, if $d = 1$ then the operator $\mathcal{H}^{n/d}$ vanishes. We see immediately that in this case $\mathcal{H}^d$ is a diagonal operator with simple spectrum.

Let us introduce in $\mathcal{L}_f$ the relation $\sim$, where

$$a \sim b \text{ if and only if } \quad a = b \text{ or } (a,b) \in (\mathcal{L}_f \times \mathcal{L}_f)_+ \cup (\mathcal{L}_f \times \mathcal{L}_f)_-. \tag{5.15}$$

It is easy to see that this is an equivalence relation. By Lemma 4.9

$$a \sim b,\ a \neq b \ \Rightarrow\ |a| \neq |b|\,. \tag{5.16}$$

The equivalence $\sim$, as well as the sets $(\mathcal{L}_f \times \mathcal{L}_f)_\pm$, depends only on the lattice $\mathbb{Z}^d$ and the set $\mathcal{A}$, not on the eigenvalues λ_a and the vector ρ. It is trivial if $d = 1$ or $|\mathcal{A}| = 1$ (see Example 4.8)) and, in general, is non-trivial otherwise. If $d \geq 2$ and $|\mathcal{A}| \geq 2$ it is rather complicated.

The equivalence relation divides $\mathcal{L}_f$ into equivalence classes, $\mathcal{L}_f = \mathcal{L}_f^1 \cup \cdots \cup \mathcal{L}_f^M$. The set $\mathcal{L}_f$ is a union of the punched spheres $\Sigma_a = \{b \in \mathbb{Z}^d \mid |b| = |a|, b \neq a\}$, $a \in \mathcal{A}$, and by (5.16) each equivalence class $\mathcal{L}_f^j$ intersects every punched sphere Σ_a at at most one point.

Let us order the sets $\mathcal{L}_f^j$ in such a way that for a suitable $0 \leq M_0 \leq M$ we have

– $\mathcal{L}_f^j = \{b_j\}$ (for a suitable point $b_j \in \mathbb{Z}^d$) if $j \leq M_0$;

– $|\mathcal{L}_f^j| = n_j \geq 2$ if $j > M_0$.

Accordingly the complex space $Y_{\mathcal{L}_f}$ (see (4.32)) decomposes as

$$Y_{\mathcal{L}_f} = Y^{f1} \oplus \cdots \oplus Y^{fM}, \quad Y^{fj} = \operatorname{span}\{\zeta_s, s \in \mathcal{L}_f^j\}. \tag{5.17}$$

Since each $\zeta_s, s \in \mathcal{L}_f$, is a 2-vector, then

$$\dim Y^{fj} = 2|\mathcal{L}_j^j| = 2n_j, \quad \dim Y_{\mathcal{L}_f} = 2|\mathcal{L}_f| = 2\sum_{j=1}^{M} n_j := 2\mathbf{N}.$$

So $\dim Y^{fj} = 2$ for $j \leq M_0$ and $\dim Y^{fj} \geq 4$ for $j > M_0$. In view of (5.16),

$$|\mathcal{L}_f^j| = n_j \leq |\mathcal{A}| \quad \forall\, j. \tag{5.18}$$

We readily see from the formula for the matrix $\mathcal{H}(\rho) = \mathbf{i}JK(\rho)$ that the spaces Y^{fj} are invariant for the operator $\mathcal{H}(\rho)$. So

$$\mathcal{H}(\rho) = \mathcal{H}^1(\rho) \oplus \cdots \oplus \mathcal{H}^M(\rho), \quad \mathcal{H}^j = \mathcal{H}^{j\,d} + \mathcal{H}^{j\,n/d}, \tag{5.19}$$

where $\mathcal{H}^j$ operates in the space Y^{fj}, so this is a block of the matrix $\mathcal{H}(\rho)$. The operators $\mathcal{H}^{j\,d}$ and $\mathcal{H}^{j\,n/d}$ are given by the formulas (5.13) with $a, b \in \mathcal{L}_f^j$. The Hamiltonian operator $\mathcal{H}^j(\rho)$ polynomially depends on $\sqrt{\rho}$, so its eigenvalues form an algebraic function of ρ. Since the spectrum of $\mathcal{H}^j(\rho)$ is an even set, then we can write branches of this algebraic function as $\{\pm\mathbf{i}\Lambda_1^j(\rho), \dots, \pm\mathbf{i}\Lambda_{n_j}^j(\rho)\}$ (the factor $\mathbf{i}$ is convenient for further purposes). The eigenvalues of $\mathcal{H}(\rho)$ are given by another algebraic function and we write its branches as $\{\pm\mathbf{i}\Lambda_m(\rho), 1 \le m \le \mathbf{N} = |\mathcal{L}_f|\}$. Accordingly,

$$\{\pm\Lambda_1(\rho), \dots, \pm\Lambda_{\mathbf{N}}(\rho)\} = \cup_{j\le M}\{\pm\Lambda_k^j(\rho), k \le n_j\}, \tag{5.20}$$

and $\Lambda_j = \Lambda_1^j$ for $j \le M_0$.

The functions Λ_k and Λ_k^j are defined up to multiplication by ± 1.[15] But if $j \le M_0$, then $\mathcal{L}_f^j = \{b_j\}$ and $\mathcal{H}^j = \mathcal{H}^{jd}$, so the spectrum of this operator is $\{\pm\mathbf{i}\mu(b_j, \rho)\}$, where $\mu(b_j, \rho)$ is a well defined analytic function of ρ, given by the explicit formula (5.11). In this case we specify the choice of Λ_1^j:

$$\text{if} \quad \mathcal{L}_f^j = \{b_j\}, \text{ we choose } \Lambda_1^j(\rho) = \mu(b_j, \rho). \tag{5.21}$$

So for $j \le M_0$, $\Lambda_j(\rho) = \mu(b_j, \rho)$ is a polynomial of ρ, which depends only on $|b_j|$ and ρ.

Since the norm of the operator $K(\rho)$ satisfies (4.17), then

$$|\Lambda_r^j(\rho)| \le C_2 \quad \forall\, \rho,\ \forall\, r,\ \forall\, j. \tag{5.22}$$

Example 5.3. In view of (5.18), if $\mathcal{A} = \{a_*\}$, then all sets $|\mathcal{L}_f^j|$ are one-point. So $M_0 = M = \mathbf{N}$ and

$$\{\pm\Lambda_1(\rho), \dots, \pm\Lambda_{\mathbf{N}}(\rho)\} = \{\pm\mu(a, \rho) \mid a \in \mathbb{Z}^d, |a| = |a_*|, a \ne a_*\}.$$

In this case the spectrum of the Hamiltonian operator $\mathcal{H}(\rho)$ is pure imaginary and multiple. It analytically depends on ρ.

Let $1 \le j_* \le n$ and $\mathcal{D}_0^{j_*}$ be the set

$$\mathcal{D}_0^{j_*} = \{\rho = (\rho_1, \dots, \rho_n) \mid c_* \le \rho_l \le c_0 \quad \text{if} \quad l \ne j_* \quad \text{and} \quad 1 - c_0 \le \rho_{j_*} \le 1\}, \tag{5.23}$$

where $0 < c_* \le \frac{1}{2}c_0 < 1/4$. Its measure satisfies

$$\operatorname{meas} \mathcal{D}_0^{j_*} \ge \tfrac{1}{2}c_0^n.$$

This is a subset of $\mathcal{D} = [c_*, 1]^n$ which lies in the (Const c_0)-vicinity of the point $\rho_* = (0, \dots, 1, \dots, 0)$ in $[0, 1]^n$, where 1 stands on the j_*-th place. Since $K^{n/d}(\rho_*) = 0$, then $K(\rho_*) = K^d(\rho_*)$. Consider any equivalence class $\mathcal{L}_f^j$ and enumerate its elements

[15] More precisely, if Λ_k is not real, then well defined is the quadruple $\{\pm\Lambda_k, \pm\bar{\Lambda}_k\}$; see below Sect. 5.3.

as $b^j_1, \dots, b^j_{n_j}$ ($n_j \le n$). For $\rho = \rho_*$ the matrix $\mathcal{H}^j(\rho_*)$ is diagonal with the eigenvalues $\pm\mathbf{i}\mu(b^j_r, \rho_*), 1 \le r \le n_j$. This suggests that for c_0 sufficiently small we may uniquely numerate the eigenvalues $\{\pm\mathbf{i}\Lambda^j_r(\rho)\}$ ($\rho \in \mathcal{D}^{j_*}_0$) of the matrix $\mathcal{H}^j(\rho)$ in such a way that $\Lambda^j_r(\rho)$ is close to $\mu(b^j_r, \rho_*)$. Below we justify this possibility.

Take any $b \in \mathcal{L}_f$ and denote $\ell(b) = a_b \in \mathcal{A}$. If $a_b = a_{j_*}$, then

$$\mu(b, \rho_*) = C_* \left(\frac{3}{2}\lambda^{-2}_{a_{j_*}} - \lambda^{-2}_{a_{j_*}} \right) = \tfrac{1}{2} C_* \lambda^{-2}_{a_{j_*}}. \tag{5.24}$$

If $a_b \ne a_{j_*}$, then

$$\mu(b, \rho_*) = -C_* \lambda^{-1}_{a(b)} \lambda^{-1}_{a_{j_*}}. \tag{5.25}$$

If $m \in [1, 2]$ is different from $4/3$ and $5/3$, then it is easy to see that $2\lambda_a \ne \pm\lambda_{a'}$ for any $a, a' \in \mathcal{A}$. By (3.14) this implies that for $m \in [1, 2]\backslash\mathcal{C}$ and for $b, b' \in \mathcal{L}_f$ such that $|b| \ne |b'|$ we have

$$|\mu(b, \rho_*)| \ge 2c^{\#}(m) > 0, \quad |\mu(b, \rho_*) \pm \mu(b', \rho_*)| \ge 2c^{\#}(m),$$

and

$$|\mu(b, \rho)| \ge c^{\#}(m) > 0, \quad |\mu(b, \rho) \pm \mu(b', \rho)| \ge c^{\#}(m) \quad \text{for} \quad \rho \in \mathcal{D}^{j_*}_0, \tag{5.26}$$

if c_0 is small. In particular, for each j the spectrum $\pm\mathbf{i}\mu(b^j_r, \rho_*), 1 \le r \le n_j$ of the matrix $\mathcal{H}^j(\rho_*)$ is simple.

LEMMA 5.4. *If $c_0 \in (0, 1/2)$ is sufficiently small,*[16] *then there exists $c^o = c^o(m) > 0$ such that for each r and j, $\Lambda^j_r(\rho)$ is a real analytic function of $\rho \in \mathcal{D}^{j_*}_0$, satisfying*

$$|\Lambda^j_r(\rho) - \mu(b^j_r, \rho)| \le C\sqrt{c_0} \quad \forall\, \rho \in D^{j_*}_0, \tag{5.27}$$

and

$$|\Lambda^j_r(\rho)| \ge c^o(m) > 0 \ \text{ and } \ |\Lambda^j_r(\rho) \pm \Lambda^j_l(\rho)| \ge c^o(m) \quad \forall\, r \ne l, \quad \forall\, j, \quad \forall\, \rho \in D^{j_*}_0, \tag{5.28}$$

$$|\Lambda^{j_1}_{r_1}(\rho) + \Lambda^{j_2}_{r_2}(\rho)| \ge c^0(m) \quad \forall\, j_1, j_2, r_1, r_2 \quad \text{and} \quad \rho \in D^{j_*}_0. \tag{5.29}$$

In particular,

$$\Lambda^j_r \not\equiv 0 \quad \forall r; \quad \Lambda^j_r \not\equiv \pm\Lambda^j_l \quad \forall\, r \ne l. \tag{5.30}$$

The estimate (5.27) assumes that for $\rho \in \mathcal{D}^{j_*}_0$ we fix the sign of the function Λ^j_r by the following agreement:

$$\Lambda^r_j(\rho) \in \mathbb{R} \quad \text{and} \quad \operatorname{sign} \Lambda^j_r(\rho) = \operatorname{sign} \mu(b^j_r, \rho) \ \ \forall \rho \in \mathcal{D}^{j_*}_0, \ \forall 1 \le j_* \le n, \ \forall\, r, j, \tag{5.31}$$

see (5.24), (5.25).

Below we fix any $c_0 = c_0(\mathcal{A}, m, g(\cdot)) \in (0, 1/2)$ such that the lemma's assertion holds, but the parameter $c_* \in (0, \frac{1}{2}c_0]$ will vary at the last stage of our proof, in Sect. 10.2.

[16] Its smallness only depends on $\mathcal{A}, m$ and $g(\cdot)$.

Proof. Since the spectrum of $\mathcal{H}^j(\rho_*)$ is simple and the matrix $\mathcal{H}^j(\rho)$ and the numbers $\mu(b_r^j,\rho)$ are polynomials of $\sqrt{\rho}$, then the basic perturbation theory implies that the functions $\Lambda_r^j(\rho)$ are real analytic in $\sqrt{\rho}$ in the vicinity of ρ_* and we have

$$|\mu(b_r^j,\rho_*)-\mu(b_r^j,\rho)|\le C\sqrt{c_0},\quad |\Lambda_r^j(\rho_*)-\Lambda_r^j(\rho)|\le C\sqrt{c_0}.$$

So (5.27) holds. It is also clear that the functions $\Lambda_r^j(\rho)$ are analytic in $\rho\in\mathcal{D}_0^{j_*}$. Relations (5.27) and (5.26) (and the fact that $\mu(b,\rho)$ depends only on $|b|$ and ρ) imply (5.28) and (5.29) if $c_0>0$ is sufficiently small. □

REMARK 5.5. The differences $|2\lambda_a-\lambda_b|$ can be estimated from below uniformly in a,b in terms of the distance from $m\in[1,2]$ to the points $4/3$ and $5/3$. So the constants $c^\#$ and c^o depend only on this distance, and they can be chosen independent from m if the latter belongs to the smaller segment $[1,5/4]$.

Contrary to (5.29), in general a difference of two eigenvalues $\Lambda_{r_1}^{j_1}-\Lambda_{r_2}^{j_2}$ may vanish identically. Indeed, if $j,k\le M_0$, then $\mathcal{L}_f^k$ and $\mathcal{L}_f^j$ are one-point sets, $\mathcal{L}_f^k=\{b_k\}$ and $\mathcal{L}_f^j=\{b_j\}$, and $\Lambda_1^j=\mu(b_j,\cdot)$, $\Lambda_1^k=\mu(b_k,\cdot)$. So if $|b_j|=|b_k|$, then $\Lambda_1^j\equiv\Lambda_1^k$ due to (5.12). In particular, in view of Example 5.3, if $n=1$ then each $\mathcal{L}_f^j$ is a one-point set, corresponding to some point b_j of the same length. In this case all functions $\Lambda_k(\rho)$ coincide identically. But if $j\le M_0<k$, or if $\max j,k>M_0$ and the set $\mathcal{A}$ is strongly admissible (recall that everywhere in this section it is assumed to be admissible), then $\Lambda_{r_1}^{j_1}-\Lambda_{r_2}^{j_2}\not\equiv 0$. This is the assertion of the non-degeneracy lemma below, proved in Sect. 5.4.

LEMMA 5.6. *Consider any two spaces $Y^{f\,r_1}$ and $Y^{f\,r_2}$ such that $r_1\le r_2$ and $r_2>M_0$. Then*

$$\Lambda_j^{r_1}\not\equiv\pm\Lambda_k^{r_2}\quad\forall\,(r_1,j)\ne(r_2,k),\tag{5.32}$$

provided that either $r_1\le M_0$, or the set $\mathcal{A}$ is strongly admissible.

We recall that for $d\le 2$ all admissible sets are strongly admissible. For $d\ge 3$ non-strongly admissible sets exist. In "Appendix B" we give an example (B.2) of such a set for $d=3$ and show that for it the relation (5.32) does not hold.

5.2 Removing singular values of the parameter ρ. We recall that the hamiltonian operator $\mathcal{H}(\rho)$ equals $\mathbf{i}JK(\rho)$; so $\{\Lambda_l^j(\rho)\}$ are the eigenvalues of the real matrix $JK(\rho)$. Accordingly, the numbers $\{\Lambda_l^j(\rho),1\le l\le n_j\}$, are eigenvalues of the real matrix $\frac{1}{\mathbf{i}}\mathcal{H}^j(\rho)=:L^j(\rho)$. Due to Lemma 5.4 we know that for each j the eigenvalues $\{\pm\Lambda_k^j(\rho),\,k\le n_j\}$, do not vanish identically in ρ and do not identically coincide. Now our goal is to quantify these statements by removing certain singular values of the parameter ρ. To do this let us first denote $P^j(\rho)=(\prod_l\Lambda_l^j(\rho))^2=\pm\det L^j(\rho)$ and consider the determinant

$$P(\rho)=\prod_j P^j(\rho)=\pm\det JK(\rho).$$

Recall that for an $R \times R$-matrix with eigenvalues $\kappa_1, \dots, \kappa_R$ (counted with their multiplicities) the discriminant of the determinant of this matrix equals the product $\prod_{i \neq j}(\kappa_1 - \kappa_j)$. This is a polynomial of the matrix' elements.

Next we define a "poly-discriminant" $D(\rho)$, which is another polynomial of the matrix elements of $JK(\rho)$. Its definition is motivated by Lemma 5.6, and it is different for the admissible and strongly admissible sets $\mathcal{A}$. Namely, if $\mathcal{A}$ is strongly admissible, then

- for $r = 1, \dots, M_0$ define $D^r(\rho)$ as the discriminant of the determinant of the matrix $L^r(\rho) \oplus L^{M_0+1}(\rho) \oplus \cdots \oplus L^M(\rho)$;
- set $D(\rho) = D^1(\rho) \cdot \dots \cdot D^{M_0}(\rho)$.

This is a polynomial in the matrix coefficients of $JK(\rho)$, so a polynomial of $\sqrt{\rho}$. It vanishes if and only if $\Lambda^r_m(\rho)$ equals $\pm\Lambda^l_k(\rho)$ for some r, l, m and k, where either $r, l \geq M_0 + 1$ and $m \neq k$ if $r = l$, or $r \leq M_0$ and $m = 1$.

If $\mathcal{A}$ is admissible, then we:

- for $l \leq M_0, r \geq M_0 + 1$ define $D^{l,r}(\rho)$ as the discriminant of the determinant of the matrix $L^l(\rho) \oplus L^r(\rho)$;
- set $D(\rho) = \prod_{l \leq M_0, r \geq M_0+1} D^{l,r}(\rho)$.

This is a polynomial in the matrix coefficients of $JK(\rho)$, so a polynomial in $\sqrt{\rho}$. It vanishes if and only if $\Lambda^r_1(\rho)$ equals $\pm\Lambda^l_k(\rho)$ for some $r \leq M_0$, some $l \geq M_0 + 1$ and some k, or if $\Lambda^l_k(\rho)$ equals $\pm\Lambda^l_m(\rho)$ for some $l \geq M_0 + 1$ and some $k \neq m$.

Finally, in the both cases we set

$$M(\rho) = \prod_{b \in \mathcal{L}_f} \mu(b, \rho) \prod_{\substack{b, b' \in \mathcal{L}_f \\ |b| \neq |b'|}} \big(\mu(b, \rho) - \mu(b', \rho)\big).$$

This also is a polynomial in $\sqrt{\rho}$ which does not vanish identically due to (5.26).

The set

$$X = \{\rho \mid P(\rho)\, D(\rho)\, M(\rho) = 0\}$$

is an algebraic variety, if written in the variable $\sqrt{\rho}$ (analytically diffeomorphic to the variable $\rho \in [c_*, 1]^{\mathcal{A}}$), and is non-trivial by Lemma 5.4. The open set $\mathcal{D} \backslash X$ is dense in $\mathcal{D}$ and is formed by finitely many connected components. Denote them $Q_1, \dots, Q_L$. For any component Q_l its boundary is a stratified analytic manifold with finitely many smooth analytic components of dimension $< n$, see [KP02]. The eigenvalues $\Lambda_j(\rho)$ and the corresponding eigenvectors are locally analytic functions on the domains Q_l, but since some of these domains may be not simply connected, then the functions may have non-trivial monodromy, which would be inconvenient for us. But since each Q_l is a domain with a regular boundary, then by removing from it finitely many smooth closed hyper-surfaces we cut Q_l to a finite system of simply connected domains $Q^1_l, \dots, Q^{\hat{n}_l}_l$ such that their union has the same measure

as Q_l and each domain Q_l^μ lies on one side of its boundary.[17] We may realise these cuts (i.e. the hyper-surfaces) as the zero-sets of certain polynomial functions of ρ. Denote by $R_1(\rho)$ the product of the polynomials, corresponding to the cuts made, and remove from $\tilde{Q}_l\backslash X$ the zero-set of R_1. This zero-set contains all the cuts we made (it may be bigger than the union of the cuts), and still has zero measure. Again, $(\tilde{Q}_l\backslash X)\backslash\{\text{zero-set of } R_1\}$ is a finite union of domains, where each one lies in some domain Q_l^r.

Intersections of these new domains with the sets $\mathcal{D}_0^{j_*}$ (see (5.23)) will be important for us by virtue of Lemma 5.4, and any fixed set $\mathcal{D}_0^{j_*}$, say $\mathcal{D}_0^1$, will be sufficient for our analysis. To agree the domains with $\mathcal{D}_0^1$ we note that the boundary of $\mathcal{D}_0^1$ in $\mathcal{D}$ is the zero-set of the polynomial

$$R_2(\rho) = (\rho_1 - (1 - c_0))(\rho_2 - c_0)\dots(\rho_n - c_0),$$

and modify the set X above to the set $\tilde{X}$,

$$\tilde{X} = \{\rho \in \mathcal{D} \mid \mathcal{R}(\rho) = 0\}, \quad \mathcal{R}(\rho) = P(\rho)D(\rho)M(\rho)R_1(\rho)R_2(\rho).$$

As before, $\mathcal{D}\backslash\tilde{X}$ is a finite union of open domains with regular boundary. We still denote them Q_l:

$$\mathcal{D}\backslash\tilde{X} = Q_1 \cup \dots \cup Q_{\mathbb{J}}, \quad \mathbb{J} < \infty. \tag{5.33}$$

A domain Q_j in (5.33) may be non simply connected, but since each Q_j belongs to some domain Q_l^r, then the eigenvalues $\Lambda_a(\rho)$ and the corresponding eigenvectors define in these domains single-valued analytic functions. Since every domain Q_l lies either in $\mathcal{D}_0^1$ or in its complement, we may enumerate the domains Q_l in such a way that

$$\mathcal{D}_0^1\backslash\tilde{X} = Q_1 \cup \dots \cup Q_{\mathbb{J}_1}, \quad 1 \le \mathbb{J}_1 \le \mathbb{J}. \tag{5.34}$$

The domains Q_l with $l \le \mathbb{J}_1$ will play a special role in our argument.

Let us take $c_1 = \frac{1}{2}c_*$ and consider the complex vicinity $\mathcal{D}_{c_1}$ of $\mathcal{D}$,

$$\mathcal{D}_{c_1} = \{\rho \in \mathbb{C}^{\mathcal{A}} \mid |\Im\rho_j| < c_1,\ c_* - c_1 < \Re\rho_j < 1 + c_1\ \forall j \in \mathcal{A}\}. \tag{5.35}$$

We naturally extend $\tilde{X}$ to a complex-analytic subset $\tilde{X}^c$ of $\mathcal{D}_{c_1}$ (so $\tilde{X} = \tilde{X}^c \cap \mathcal{D}$), consider the set $\mathcal{D}_{c_1}\backslash\tilde{X}^c$, and for any $\delta > 0$ consider its open sub-domain $\mathcal{D}_{c_1}(\delta)$,

$$\mathcal{D}_{c_1}(\delta) = \{\rho \in \mathcal{D}_{c_1} \mid |\mathcal{R}(\rho)| > \delta\} \subset \mathcal{D}_{c_1}\backslash\tilde{X}^c.$$

Since the factors, forming $\mathcal{R}$, are polynomials with bounded coefficients, then they are bounded in $\mathcal{D}_{c_1}$:

$$\|P\|_{C^1(\mathcal{D}_{c_1})} \le C_1, \dots, \|R_2\|_{C^1(\mathcal{D}_{c_1})} \le C_1. \tag{5.36}$$

[17] For example, if $n = 2$ and $\tilde{Q}_l$ is the annulus $A = \{1 < \rho_1^2 + \rho_2^2 < 2\}$, then we remove from A not the interval $\{\rho_2 = 0, 1 < \rho_1 < 2\} =: J$ (this would lead to a simply connected domain which lies on both parts of the boundary J), but two intervals, J and $-J$.

So in the domain $\mathcal{D}_{c_1}(\delta)$ the norms of the factors $P,\dots,R_2$, making $\mathcal{R}$, are bounded from below by $C_2\delta$, and similar estimates hold for the factors, making P, D and M. Therefore, by the Kramer rule

$$\|(JK)^{-1}(\rho)\| \le C_1\delta^{-1} \quad \forall \rho \in \mathcal{D}_{c_1}(\delta). \tag{5.37}$$

Similar for $\rho \in \mathcal{D}_{c_1}(\delta)$ we have

$$|\Lambda_k^j(\rho)| \ge C^{-1}\delta \quad \forall j,k, \tag{5.38}$$

$$|\mu(b,\rho)| \ge C^{-1}\delta, \quad |\mu(b,\rho)-\mu(b',\rho)| \ge C^{-1}\delta \quad \text{if} \quad b,b' \in \mathcal{L}_f \quad \text{and } |b| \ne |b'|\ . \tag{5.39}$$

Besides,

$$|\Lambda_{k_1}^j(\rho) \pm \Lambda_{k_2}^r(\rho)| \ge C^{-1}\delta \quad \text{where} \quad (j,k_1) \ne (r,k_2) \tag{5.40}$$

and if the set $\mathcal{A}$ is strongly admissible, then the index j is any and $r \ge M_0+1$, while if $\mathcal{A}$ is admissible, then either $j \le M_0$ (and so $k_1=1$) and $r \ge M_0+1$, or $j=r\ge M_0+1$. The functions $\Lambda_k^j(\rho)$ are algebraic functions on the complex domain $\mathcal{D}_{c_1}(\delta)$, but their restrictions to the real parts of these domains split to branches which are well defined analytic functions.

We have

$$\operatorname{meas}(\mathcal{D}\backslash\mathcal{D}_{c_1}(\delta)) \le C\delta^{\beta_4}, \tag{5.41}$$

for some positive C and β_4—this follows easily from Lemma D.1 and Fubini since $\mathcal{R}$ is a polynomial in $\sqrt{\rho}$ (also see Lemma D.1 in [EGK14]). Denote $c_2=c_1/2$, define set $\mathcal{D}_{c_2}$ as in (5.35) but replacing there c_1 with c_2, and denote $\mathcal{D}_{c_2}(\delta)=\mathcal{D}_{c_1}(\delta)\cap\mathcal{D}_{c_2}$. Obviously,

$$\text{the set } \mathcal{D}_{c_2}(2\delta) \text{ lies in } \mathcal{D}_{c_1}(\delta) \text{ with its } C^{-1}\delta\text{-vicinity.} \tag{5.42}$$

Consider the eigenvalues $\pm\mathbf{i}\Lambda_k(\rho)$. They analytically depend on $\rho \in \mathcal{D}_{c_1}(\delta)$, where $|\Lambda_k| \le C_2$ for each $k \le \mathbf{N}$ by (5.22). In view of (5.42),

$$\Big|\frac{\partial^l}{\partial\rho^l}\Lambda_k(\rho)\Big| \le C_l\delta^{-l} \quad \forall\, \rho \in \mathcal{D}_{c_2}(2\delta),\ l\ge 0,\ k \le \mathbf{N}, \tag{5.43}$$

by the Cauchy estimate.

5.3 Block-diagonalising and the end of the proof of theorem 5.1. We shall block-diagonalise the operator $\mathbf{i}JK(\rho)$ for $\rho \in \mathcal{D}_{c_1}(\delta)$. By (5.19) this operator is a direct sum of operators, each of which has a simple spectrum with eigenvalues that are separated by $\ge C^{-1}\delta$. Let us denote one of these blocks by $\mathbf{i}JK_1(\rho)$. Let its dimension be $2N$ and let $I(\xi,\eta)=(\bar\eta,\bar\xi)$. Notice that since $\mathbf{i}JK_1(\rho)$ is "conjugate-real" we have

$$\mathbf{i}JK_1(\rho)I(z) = I(\mathbf{i}JK_1(\rho)z).$$

Fix now a $\rho_0 \in \mathcal{D}_{c_1}(\delta)$. Then, by (5.42) with δ replaced by $\delta/2$, for $|\rho-\rho_0| \le C^{-1}\delta^{4N}$ the operator $\mathbf{i}JK_1(\rho)$ has a single spectrum. Consider a (complex) matrix

$$U(\rho) = \big(z_1(\rho),\dots,z_{2N}(\rho)\big),$$

whose column vectors $\|z_j(\rho)\| = 1$ are eigenvectors of $\mathbf{i}JK_1(\rho)$. It diagonalises $\mathbf{i}JK_1$:

$$U(\rho)^{-1}\big(\mathbf{i}JK_1(\rho)\big)U(\rho) = \mathbf{i}\,\mathrm{diag}\,\{\pm\Lambda_1(\rho),\dots,\pm\Lambda_N(\rho)\}. \tag{5.44}$$

The operator U is smooth in ρ with estimates

$$\sup_\rho\big(\|\partial_\rho^j U(\rho)\| + \|\partial_\rho^j U(\rho)^{-1}\|\big) \le C_j\delta^{-\beta(j)} \quad \forall\, j \ge 0, \tag{5.45}$$

and

$$\inf_\rho |\det(U(\rho))| \ge \frac{1}{C_0}\delta^{\beta(0)}, \tag{5.46}$$

for some $0 < \beta(0) \le \beta(1) \le \cdots$. See Lemma A.6 in [Eli98] and Lemma C.1 in [EGK14].

Since the spectrum is simple, then the pairing $\langle \mathbf{i}Jz_k(\rho), z_l(\rho)\rangle = {}^t z_l(\rho)\big(\mathbf{i}J\big)z_k(\rho)$ is zero unless the eigenvalues of $z_k(\rho)$ and $z_l(\rho)$ are equal but of opposite sign. We therefore enumerate the eigenvectors so that $z_{2j-1}(\rho)$ and $z_{2j}(\rho)$ correspond to eigenvalues of opposite sign. If now $\pi_j(\rho) = \langle \mathbf{i}Jz_{2j-1}(\rho), z_{2j}(\rho)\rangle$, then, for each j,

$$\frac{1}{C_0}\delta^{\beta(0)} \le |\det(U)| = \sqrt{|\det({}^tU\mathbf{i}JU)|} = \prod_l |\pi_l| \le |\pi_j| \le 1,$$

since the matrix elements of ${}^tU\mathbf{i}JU$ are $\langle \mathbf{i}Jz_k(\rho), z_l(\rho)\rangle$.

Replacing each eigenvector z_{2j} by $\frac{1}{\pi_{2j}}z_{2j}$, we can assume without restriction that U verifies

$$\frac{1}{C_0}\delta^{\beta(0)} \le \|z_j(\rho)\| \le C_0\delta^{-\beta(0)} \tag{5.47}$$

and (5.44)–(5.46) (for some choice of constants) and, moreover,

$${}^tU\big(\mathbf{i}J\big)U = J. \tag{5.48}$$

Suppose now that some Λ_j, Λ_1 say, is real. Then z_2 and $I(z_1)$ are parallel, so $z_2 = \mathbf{i}\alpha I(z_1)$ for some complex number $\alpha \in \mathbb{C}^*$ satisfying the bound (5.47) (for some choice of constants). Since $\langle \mathbf{i}Jz_1, z_2\rangle = 1$, we have that $\alpha = \langle Jz_1, I(z_1)\rangle^{-1}$ is real, and, by eventually interchanging z_1 and z_2, we can assume that $\alpha = \beta^2 > 0$. Replacing now z_1, z_2 by $\beta z_1, \frac{1}{\beta}z_2$ we can assume without restriction that U verifies (5.44)–(5.48) (for some choice of constants), and $z_2 = \mathbf{i}I(z_1)$.

Suppose then that some Λ_j, Λ_1 again say, is purely imaginary. Then z_1 and $I(z_1)$ are parallel, so $z_1 = \alpha I(z_1)$ for some unit α. Similarly, $z_2 = \beta I(z_2)$ for some unit β. Since $\langle \mathbf{i}Jz_1, z_2\rangle = 1$, we have that $1 = \alpha\beta\langle \mathbf{i}JI(z_1), I(z_2)\rangle = \alpha\beta$. Let now $\alpha = \gamma^2$, and by replacing z_1, z_2 by $\bar\gamma z_1, \frac{1}{\bar\gamma}z_2$ we can assume without restriction that U verifies (5.44)–(5.48) (for some choice of constants), and $z_1 = I(z_1)$ and $z_2 = I(z_2)$.

Suppose finally that some Λ_j, Λ_1 say, is neither real nor purely imaginary. Then $-\mathbf{i}\overline{\Lambda_1}$ also is an eigenvalue,[18] and, hence, equals to $\pm\mathbf{i}\Lambda_2$ say. Let us assume it is $\mathbf{i}\Lambda_2$,

[18] An example, considered in "Appendix B", shows that quadruples of eigenvalues $\{\pm\mathbf{i}\Lambda, \pm\mathbf{i}\bar\Lambda\}$ indeed may occur in the spectra of operators $\mathbf{i}JK$.

the other case being similar. Then $z_3 = \alpha I(z_1)$ for some unit α, and $z_2 = \beta I(z_4)$ for some $\beta \in \mathbb{C}^*$, both satisfying the bound (5.47) (for some choice of constants). Since $\langle \mathbf{i}Jz_1, z_2\rangle = \langle \mathbf{i}Jz_3, z_4\rangle = 1$, $\alpha\beta$ must be $= 1$. Let now $\alpha = \gamma^2$, and by replacing z_1, z_3 by $\bar\gamma z_1, \bar\gamma z_3$ and z_2, z_4 by $\frac{1}{\bar\gamma} z_2, \frac{1}{\bar\gamma} z_4$ we can assume without restriction that U verifies (5.44)–(5.48) (for some choice of constants), and $z_3 = I(z_1)$ and $z_4 = I(z_2)$.

Now we define a new matrix

$$\tilde U(\rho) = \big(p_1(\rho)\ q_1(\rho) \dots p_N(\rho)\ p_N(\rho)\big)$$

in the following way. If Λ_1 is real, then we take

$$p_1 = -\frac{\mathbf{i}}{\sqrt{2}}(z_1 + \mathbf{i}z_2), \quad q_1 = -\frac{1}{\sqrt{2}}(z_1 - \mathbf{i}z_2),$$

so that $I(p_1) = p_1$, $I(q_1) = q_1$ and $\langle \mathbf{i}Jp_1, q_1\rangle = 1$. We do similarly for all Λ_j real. If Λ_1 is purely imaginary, then we take $p_1 = z_1$ and $q_1 = z_2$, and similarly for all Λ_j purely imaginary. If Λ_1 is neither real nor purely imaginary, and $z_1 = I(z_3)$ and $z_2 = I(z_4)$, then

$$p_1 = -\frac{\mathbf{i}}{\sqrt{2}}(z_1 + \mathbf{i}z_3), \quad p_2 = -\frac{1}{\sqrt{2}}(z_1 - \mathbf{i}z_3)$$

and

$$q_1 = -\frac{\mathbf{i}}{\sqrt{2}}(z_2 + \mathbf{i}z_4), \quad q_2 = -\frac{1}{\sqrt{2}}(z_2 - \mathbf{i}z_4),$$

similarly for all Λ_j neither real nor purely imaginary.

Then the matrix $\tilde U(\rho)$ verifies (5.45)–(5.48) (for some choice of constants) and the mapping

$$w \mapsto \tilde U(\rho)w$$

takes any real vector w into the subspace $\{I(w) = w\}$. By doing this for each "component" $\mathbf{i}JK_1(\rho)$ of the operator (5.19) and taking the direct sum we find a matrix $\hat U(\rho)$ which transforms the Hamiltonian of $\mathbf{i}JK(\rho)$ to the form

$$\frac{1}{2}\sum_{j=1}^{M_0} \mu(b_j,\rho)\Big(p_{b_j}^2 + q_{b_j}^2\Big) + \frac{1}{2}\sum_{j=M_0+1}^{M_{00}} \Lambda_j(\rho)\Big(p_{b_j}^2 + q_{b_j}^2\Big) + \frac{1}{2}\langle \widehat K(\rho)\zeta_h, \zeta_h\rangle\,, \tag{5.49}$$

where ζ_h denotes the the remaining $\{(p_{b_j}, q_{b_j}) : M_{00}+1 \le j \le \mathbf{N}\}$. The Hamiltonian operator $J\widehat K(\rho)$ is formed by the hyperbolic eigenvalues of the operator $\mathbf{i}J\widetilde K(\rho)$.

Since $\Lambda_a(\rho)\xi_a\eta_a$ is transformed to $\frac{1}{2}\Lambda_a(\rho)\Big(p_a^2 + q_a^2\Big)$ by a matrix $\hat U_a$, independent of ρ, that verifies ${}^t\tilde U_a(iJ_a)\tilde U_a = J_a = J$ (see (1.7)) , the full Hamiltonian (4.5) gets transformed to

$$\begin{aligned}&\langle \Omega(\rho), r\rangle + \tfrac{1}{2}\textstyle\sum_{a\in\mathcal{L}_\infty} \Lambda_a(\rho)\Big(p_a^2 + q_a^2\Big) + \tfrac{1}{2}\sum_{j=1}^{M_0} \mu(b_j,\rho)\Big(p_{b_j}^2 + q_{b_j}^2\Big)\\ &\quad + \tfrac{1}{2}\textstyle\sum_{j=M_0+1}^{M_{00}} \Lambda_j(\rho)\Big(p_{b_j}^2 + q_{b_j}^2\Big) + \tfrac{1}{2}\langle \widehat K(\rho)\zeta_h, \zeta_h\rangle\end{aligned} \tag{5.50}$$

plus the error term $\tilde f(r,\theta,p_{\mathcal{L}},q_{\mathcal{L}};\rho) = f(r,\theta,\xi_{\mathcal{L}},\eta_{\mathcal{L}};\rho)$.

Note that in difference with the normal form (4.5), the variable ζ_h belongs to a subspace of the linear space, formed by the vectors $\{(p_a, q_a), a \in \mathcal{L}_f\}$, with the usual reality condition.

We choose any subset $\mathcal{F} \subset \mathcal{L}_f$ of cardinality $|\mathcal{F}| = \mathbf{N} - M_{00}$, and identify the space, where acts the operator $\hat{K}(\rho)$, with the space $\mathcal{L}_{\mathcal{F}} = \{\zeta_{\mathcal{F}} = \{(p_a, q_a), a \in \mathcal{F}\}\}$. We denote the operator $\hat{K}(\rho)$, re-interpreted as an operator in $\mathcal{L}_{\mathcal{F}}$, as $K(\rho)$. Finally, we identify the set of nodes $\{1, \dots, M_{00}\}$ with $\mathcal{L}_{\mathcal{F}}\backslash\mathcal{F}$, and write the collection of frequencies $\{\mu(b_j, \rho), 1 \le j \le M_0\} \cup \{\Lambda_j(\rho), M_0 + 1 \le \rho \le M_{00}\}$ as $\{\Lambda_b(\rho), b \in \mathcal{L}_{\mathcal{F}}\backslash\mathcal{F}\}$. After that the Hamiltonian (5.50) takes the form (5.4), required by Theorem 5.1. We denote by $\hat{\mathbf{U}}_\rho$ the constructed linear symplectic change of variables which transforms the Hamiltonian (4.5) to (5.4)

For convenience we denote

$$\bar{c} = 1/\beta_4 \quad \text{and} \quad \hat{c} = \beta(0)\bar{c}. \tag{5.51}$$

With an eye on the relation (5.41), for $\beta_\# > 0$ and any $\nu > 0$ we choose $\delta = \delta(\nu) = C^{\bar{c}}\nu^{\bar{c}\beta_\#}$. Then

$$C\delta^{\beta_4} = \nu^{\beta_\#}. \tag{5.52}$$

For any $\nu > 0$ we set

$$Q(c_*, \beta_\#, \nu) = \mathcal{D} \cap \mathcal{D}_{c_1}(\delta(\nu)).$$

This is a monotone in ν system of subdomains of $\mathcal{D}$, and $Q(c_*, \beta_\#, \nu) \nearrow (\mathcal{D}\backslash\tilde{X})$ as $\nu \to 0$. In view of (5.41) the measures of these domains satisfy (5.1).

For $\rho \in Q(c_*, \beta_\#, \nu)$ the operator $\tilde{\Phi}_\rho = \Phi_\rho \circ \hat{\mathbf{U}}_\rho$ transforms the hamiltonian $\nu^{-1}h$ to (5.4). Re-denoting this transformation back to Φ_ρ, we see that the constructed objects satisfy the assertions (i)-(v) and (vii) of the theorem. To prove (vi) we recall (see (5.44)) that the operator $U(\rho)$ (complex-)diagonalises one block of those, forming the operator $\mathbf{i}JK(\rho)$. Denote by $\mathbf{U}(\rho)$ the direct sum of the operators $U(\rho)$, corresponding to all blocks of $\mathbf{i}JK(\rho)$. It diagonalises the whole operator $\mathbf{i}JK(\rho)$. Accordingly, the operator $\mathbf{U}(\rho) \circ \hat{\mathbf{U}}^{-1}(\rho)$ diagonalises $JK(\rho)$. Denoting it $U(\rho)$ we see that this operator satisfies the assertion (vi)

5.4 Proof of the non-degeneracy lemma 5.6. Consider the decomposition (5.19) of the Hamiltonian operator $\mathcal{H}(\rho)$. To simplify notation, in this section we suspend the agreement that $|L_f^r| = 1$ for $r \le M_0$, and changing the order of the direct summands achieve that the indices r_1 and r_2, involved in (5.32), are $r_1 = 1$ and $r_2 = 2$. For $r = 1, 2$ we will write elements of the set $\mathcal{L}_f^r$ as $a_j^r, 1 \le j \le n_r$, and vectors of the space Y^{fr} as

$$\zeta = \big(\zeta_{a_j^r} = (\xi_{a_j^r}, \eta_{a_j^r}), 1 \le j \le n_r\big) = \big((\xi_{a_1^r}, \eta_{a_1^r}), \dots, (\xi_{a_{n_r}^r}, \eta_{a_{n_r}^r})\big). \tag{5.53}$$

Using (4.8) and abusing notation, we will regard the mapping $\ell : \mathcal{L}_f \to \mathcal{A}$ also as a mapping $\ell : \mathcal{L}_f \to \{1, \dots, n\}$. Consider the points $\ell(a_1^1), \dots, \ell(a_{n_1}^1)$ (they are different by (5.16)). Changing if needed the labelling (4.8) we achieve that

$$\{\ell(a_1^1), \dots, \ell(a_{n_1}^1)\} \ni 1. \tag{5.54}$$

We write the operator $\mathcal{H}^r$ as $\mathcal{H}^r = \mathbf{i}M^r$, where

$$M^r(\rho) = JK^r(\rho) = JK^{r\,d}(\rho) + JK^{r\,n/d}(\rho) =: M^{r\,d}(\rho) + M^{r\,n/d}(\rho),$$

and the real block-matrices $M^{r\,d} = \mathbf{i}^{-1}\mathcal{H}^{r,\,d}$, $M^{r\,n/d} = \mathbf{i}^{-1}\mathcal{H}^{r,\,n/d}$ are given by (5.13). Then $\{\pm\Lambda_j^r(\rho)\}$ are the eigenvalues of $M^r(\rho)$, and

$$M^{r\,d}(\rho) = \operatorname{diag}\left(\begin{pmatrix}\mu(a_j^r,\rho) & 0\\ 0 & -\mu(a_j^r,\rho)\end{pmatrix},\ 1\le j\le n_r\right),$$

where $\mu(a_j^r,\rho)$ is given by (5.11).

Renumerating the eigenvalues we achieve that in (5.32) (with $r_1 = 1, r_2 = 2$) we have $\Lambda_j^1 = \Lambda_1^1$ and $\Lambda_k^2 = \Lambda_1^2$. As in the proof of Lemma 5.4, consider the vector $\rho_* = (1, 0, \dots, 0)$. Let us abbreviate

$$\mu(a, \rho_*) = \mu(a) \quad \forall\, a,$$

where $\mu(a)$ depends only on $|a|$ by (5.12). In view of (5.13) $M^r(\rho_*) = M^{r\,d}(\rho_*)$ and thus $\Lambda_1^1(\rho_*) = \mu(a_1^1)$ and $\Lambda_1^2(\rho_*) = \mu(a_1^2)$, if we numerate the elements of $\mathcal{L}_f^1$ and $\mathcal{L}_f^2$ accordingly. As in the proof of Lemma 5.4, $\mu(|a_1^r|)$ equals $\frac{1}{2}C_*\lambda_{a_1^r}^{-2}$ or $-C_*\lambda_{\ell(a_1^r)}^{-1}\lambda_{a_1^r}^{-1}$. Therefore the relation $\mu(a_1^1) = \pm\mu(a_1^2)$ is possible only if the sign is "+" and $|a_1^1| = |a_1^2|$. So it remains to verify that under the lemma's assumption

$$\Lambda_1^1(\rho) \not\equiv \Lambda_1^2(\rho) \quad \text{if} \quad |a_1^1| = |a_1^2|. \tag{5.55}$$

Since $|a_1^1| = |a_1^2|$, then

$$\ell(a_1^1) = \ell(a_1^2) =: a_{j_\#} \in \mathcal{A} \quad \text{and} \quad \Lambda_1^1(\rho_*) = \Lambda_1^2(\rho_*) =: \Lambda.$$

To prove that $\Lambda_1^1(\rho) \not\equiv \Lambda_1^2(\rho)$ we compare variations of the two functions around $\rho = \rho_*$. To do this it is convenient to pass from ρ to the new parameter $y = (y_1, \dots, y_n)$, defined by

$$y_j = \sqrt{\rho_j}, \quad j = 1, \dots, n.$$

Abusing notation we will sometime write y_{a_j} instead of y_j. Take any vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, where $x_1 = 0$ and $x_j > 0$ if $j \ge 2$, and consider the following variation $y(\varepsilon)$ of $y_* = (1, 0, \dots, 0)$:

$$y_j(\varepsilon) = \begin{cases} 1 & \text{if } \ j = 1,\\ \varepsilon x_j & \text{if } \ j \ge 2. \end{cases} \tag{5.56}$$

By (5.28), for small ε the real matrix $M^r(\varepsilon) := M^r(\rho(\varepsilon))$ $(r = 1, 2)$ has a simple eigenvalue $\Lambda_1^r(\varepsilon)$, close to Λ. We will show that for a suitable choice of vector x the functions $\Lambda_1^1(\varepsilon)$ and $\Lambda_1^2(\varepsilon)$ are different. More specifically, that their jets at zero of sufficiently high order are different.

Let r be 1 or 2. We denote $\Lambda(\varepsilon) = \Lambda_1^r(\rho(\varepsilon))$, $M(\varepsilon) = M^r(\rho(\varepsilon))$ and denote by $M^d(\varepsilon)$ and $M^{n/d}(\varepsilon)$ the diagonal and non-diagonal parts of $M(\varepsilon)$. The matrix $M^{n/d}(\varepsilon)$ is formed by 2×2-blocks

$$\left(M^{n/d}(\varepsilon)\right)_{a_k^r}^{a_j^r} = C_* \frac{y_{\ell(a_k^r)} y_{\ell(a_j^r)}}{\lambda_{a_k^r} \lambda_{a_j^r}} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \chi^+(a_k^r, a_j^r) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \chi^-(a_k^r, a_j^r) \right), \tag{5.57}$$

(note that if $j = k$, then the block vanishes).

For $\varepsilon = 0$, $M(0) = M^{rd}(0)$ is a matrix with the single eigenvalue $\Lambda(0) = \mu(a_1^r, \rho_*)$, corresponding to the eigen-vector $\zeta(0) = (1, 0, \dots, 0)$. For small ε it analytically extends to a real eigenvector $\zeta(\varepsilon)$ of $M(\varepsilon)$ with the eigenvalue $\Lambda(\varepsilon)$, i.e.

$$M(\varepsilon)\zeta(\varepsilon) = \Lambda(\varepsilon)\zeta(\varepsilon), \quad |\zeta(\varepsilon)| \equiv 1.$$

We abbreviate $\zeta = \zeta(0), M = M(0)$ and define similar $\dot\zeta, \ddot\zeta, \Lambda, \dot\Lambda \dots$ etc, where the upper dot stands for $d/d\varepsilon$. We have

$$M = M^d = \operatorname{diag}\left(\mu(a_1^r), -\mu(a_1^r), \dots, -\mu(a_{n_r}^r)\right), \tag{5.58}$$

$$\dot M^d = 0. \tag{5.59}$$

Since $(M(\varepsilon) - \Lambda(\varepsilon))\zeta(\varepsilon) \equiv 0$, then

$$(M(\varepsilon) - \Lambda(\varepsilon))\dot\zeta(\varepsilon) = -\dot M(\varepsilon)\zeta(\varepsilon) + \dot\Lambda(\varepsilon)\zeta(\varepsilon). \tag{5.60}$$

Jointly with (5.58) and (5.59) this relation with $\varepsilon = 0$ implies that

$$(M^d - \Lambda)\dot\zeta = -\dot M^{n/d}\zeta + \dot\Lambda\zeta. \tag{5.61}$$

In view of (5.58) we have $\langle (M^d - \Lambda)\dot\zeta, \zeta\rangle = 0$. We derive from here and from (5.61) that

$$\dot\Lambda = \langle \dot M^{n/d}\zeta, \zeta\rangle = 0. \tag{5.62}$$

Let us denote by π the linear projection $\pi : \mathbb{R}^{2n_r} \to \mathbb{R}^{2n_r}$ which makes zero the first component of a vector to which it applies. Then $M^d - \Lambda$ is an isomorphism of the space $\pi\mathbb{R}^{2n_r}$, and the vectors $\dot\zeta$ and $-\dot M\zeta + \dot\Lambda\zeta = \dot M^{n/d}\zeta$ belong to $\pi\mathbb{R}^{2n_r}$. So we get from (5.61) that

$$\dot\zeta = -(M^d - \Lambda)^{-1}\dot M^{n/d}\zeta, \tag{5.63}$$

where the equality holds in the space $\pi\mathbb{R}^{2n_r}$. Differentiating (5.60) we find that

$$(M(\varepsilon) - \Lambda(\varepsilon))\ddot\zeta(\varepsilon) = -\ddot M(\varepsilon)\zeta(\varepsilon) - 2\dot M(\varepsilon)\dot\zeta(\varepsilon) + \ddot\Lambda(\varepsilon)\zeta(\varepsilon) + 2\dot\Lambda(\varepsilon)\dot\zeta(\varepsilon). \tag{5.64}$$

Similar to the derivation of (5.62) (and using that $\langle\zeta, \dot\zeta\rangle = 0$ since $|\zeta(\varepsilon)| \equiv 1$), we get from (5.64) and (5.62) that

$$\ddot\Lambda = \langle \ddot M\zeta, \zeta\rangle + 2\langle \dot M\dot\zeta, \zeta\rangle = \langle \ddot M\zeta, \zeta\rangle + 2\langle (M - \Lambda)^{-1}\dot M^{n/d}\zeta, {}^t(\dot M)\zeta\rangle. \tag{5.65}$$

Since for each ε and every j

$$\frac{d^2}{d\varepsilon^2}\rho_j(\varepsilon) = \frac{d^2}{d\varepsilon^2}y_j^2(\varepsilon) = 2x_j^2, \quad \frac{d^2}{d\varepsilon^2}y_1(\varepsilon)y_j(\varepsilon) = 0,$$

and since $\langle \ddot{M}\zeta,\zeta\rangle = \langle \ddot{M}^d\zeta,\zeta\rangle$, then

$$\langle \ddot{M}\zeta,\zeta\rangle = \frac{d^2}{d\varepsilon^2}\mu(a_1^r,\rho(\varepsilon))\,|_{\varepsilon=0} = C_*\lambda_{a_{j_\#}}^{-1}\left(3\lambda_{a_{j_\#}}^{-1}x_{j_\#}^2 - 2\sum_{j=2}^{n} x_j^2\lambda_{a_j}^{-1}\right) =: k_1. \quad (5.66)$$

Note that k_1 does not depend on r.

Now consider the second term in the r.h.s. (5.65). For any $a,b\in\mathcal{L}_f^r$ we see that $\frac{d}{d\varepsilon}(y_{\ell(a)}(\varepsilon)y_{\ell(b)}(\varepsilon))\,|_{\varepsilon=0}$ is non-zero if exactly one of the numbers $\ell(a),\ell(b)$ is a_1, and this derivative equals $x_{\ell(c)}$, where $c\in\{a,b\}$, $\ell(c)\neq a_1$. Therefore, by (5.57),

$$\begin{aligned}&(\dot{M}^{n/d}\zeta)_{a_j^r} = \tfrac{C_*}{\lambda_{a_{j_\#}}}\left(\xi^o_{a_j^r},-\eta^o_{a_j^r}\right), \quad a_j^r\in\mathcal{L}_f^r,\\ &\xi^o_{a_j^r} = \tfrac{\varphi(a_1^r,a_j^r)}{\lambda_{a_j^r}}\chi^-(a_1^r,a_j^r), \quad \eta^o_{a_j^r} = \tfrac{\varphi(a_1^r,a_j^r)}{\lambda_{a_j^r}}\chi^+\left(a_1^r,a_j^r\right),\end{aligned} \quad (5.67)$$

where $\varphi(a_1^r,a_1^r)=0$ and for $j\neq 1$

$$\varphi(a_1^r,a_j^r) = \begin{cases} x_{\ell(a_j^r)} & \text{if } \ j_\#=1,\\ x_{j_\#} & \text{if } \ \ell(a_j^r)=a_1,\\ 0 & \text{if } \ j_\#\neq 1,\ \ell(a_j^r)\neq a_1.\end{cases}$$

Since $\chi^\pm(a_1^r,a_1^r)=0$, then $\xi^o_{a_1^r}=\eta^o_{a_1^r}=0$.

In view of (4.31), at most one of the numbers $\xi^o_{a_j^r},\eta^o_{a_j^r}$ is non-zero. By (5.67),

$$\left((M-\Lambda)^{-1}\dot{M}^{n/d}\zeta\right)_{a_j^r} = \frac{C_*}{\lambda_{a_{j_\#}}}\left(\xi^{oo}_{a_j^r},\,\eta^{oo}_{a_j^r}\right), \quad (5.68)$$

where $\xi^{oo}_{a_j^r}=\eta^{oo}_{a_j^r}=0$ if $j=1$, and otherwise

$$\xi^{oo}_{a_j^r} = \frac{\varphi\left(a_1^r,a_j^r\right)\chi^-\left(a_1^r,a_j^r\right)}{\lambda_{a_j^r}(\mu(a_j^r)-\mu(a_1^r))}, \quad \eta^{oo}_{a_j^r} = \frac{\varphi\left(a_1^r,a_j^r\right)\chi^+\left(a_1^r,a_j^r\right)}{\lambda_{a_j^r}\left(\mu(a_j^r)+\mu(a_1^r)\right)}.$$

Here $\mu(a_j^r)=\frac{1}{2}C_*\lambda_{a_1}^{-2}$ if $\ell(a_j^r)=a_1$ and $\mu(a_j^r)=-C_*\lambda_{a_l^r}^{-1}\lambda_{a_1}^{-1}$ if $\ell(a_j^r)\neq a_1$.

Similar,

$$({}^t\dot{M}\zeta)_{a_j^r} = \frac{C_*}{\lambda_{a_{j_\#}}}\left(\xi^o_{a_j^r},\eta^o_{a_j^r}\right),$$

so the second term in the r.h.s. of (5.65) equals

$$\frac{C_*^2}{\lambda_{a_{j_\#}}^2}\sum_{j=2}^{n_r}\frac{\varphi\left(a_1^r,a_j^r\right)^2}{\lambda_{a_j^r}^2}\left(\frac{\chi^-\left(a_1^r,a_j^r\right)}{\mu\left(a_j^r\right)-\mu\left(a_1^r\right)}+\frac{\chi^+\left(a_1^r,a_j^r\right)}{\mu\left(a_j^r\right)+\mu\left(a_1^r\right)}\right) =: k_2(r)\ . \quad (5.69)$$

Finally, we have seen that

$$\Lambda_1^r(\rho(\varepsilon)) = \Lambda_1^1(\rho_*) + \tfrac{1}{2}\varepsilon^2 k_1 + \tfrac{1}{2}\varepsilon^2 k_2(r) + O(\varepsilon^3), \quad r = 1, 2,$$

where k_1 does not depend on r. Since $a_1^r \sim a_j^r$ for each r and each j (see (5.15)), then for $j > 1$ at least one of the coefficients $\chi^{\pm}(a_1^r, a_j^r)$ is non-zero. As $\chi^+ \cdot \chi^- \equiv 0$, then

$$\frac{\chi^-\left(a_1^r, a_j^r\right)}{\mu\left(a_j^r\right) - \mu\left(a_1^r\right)} + \frac{\chi^+\left(a_1^r, a_j^r\right)}{\mu\left(a_j^r\right) + \mu\left(a_1^r\right)} \neq 0 \quad \forall\, r, \quad \forall\, j > 1. \tag{5.70}$$

We see that the sum, defining $k_2(r)$, is a non-trivial quadratic polynomial of the quantities $\varphi(a_1^r, a_j^r)$ if $n_r \geq 2$, and vanishes if $n_r = 1$.

The following lemma is crucial for the proof.

LEMMA 5.7. *If the set $\mathcal{A}$ is strongly admissible and $|a| = |b|$, $a \neq b$, and $\chi^+(a, a') \neq 0$, $\chi^+(b, b') \neq 0$, or $\chi^-(a, a') \neq 0$, $\chi^-(b, b') \neq 0$, then $|a'| \neq |b'|$.*

Proof. Let first consider the case when $\chi^+ \neq 0$.
We know that $\ell(a) = \ell(b) =: a_{j_\#}$. Assume that $|a'| = |b'|$. Then $\ell(a') = \ell(b') =: a_{j_\flat} \in \mathcal{A}$. Denote $a_{j_\#} + a_{j_\flat} = c$. Then $c \neq 0$ since the set $\mathcal{A}$ is admissible. As $(a, a'), (b, b') \in (\mathcal{L}_f \times \mathcal{L}_f)_+$, then we have $|a_{j_\#} - c| = |a - c| = |b - c|$. As $|a_{j_\#}| = |a| = |b|$, then the three points $a_{j_\#}, a$ and b lie in the intersection of two circles, one centred in the origin and another centred in $c = a_{j_\#} + a_{j_\flat}$. Since $\mathcal{A}$ is strongly admissible, then $a_{j_\#} \angle c$ (see (1.19)). So among the three point two are equal, which is a contradiction. Hence, $|a'| \neq |b'|$ as stated.

The case $\chi^- \neq 0$ is similar. □

We claim that this lemma implies that

$$\Lambda_1^1(\rho(\varepsilon)) \not\equiv \Lambda_1^2(\rho(\varepsilon)) \quad \text{for a suitable choice of the vector } x \text{ in (5.56)}, \tag{5.71}$$

so (5.55) is valid and Lemma 5.6 holds. To prove (5.71) we consider two cases.

Case 1: $j_\# = 1$. Then $\varphi(a_1^r, a_j^r) = x_{\ell(a_j^r)}$. Denoting $\frac{C_*^2}{\lambda_{a_1}^2} \frac{x_{\ell(a_j^r)}^2}{\lambda_{a_j^r}^2} =: z_{\ell(a_j^r)}$ we see that $k_2(1)$ and $k_2(2)$ are linear functions of the variables $z_{a_1}, \ldots, z_{\ell_n}$.

(i) Assume that $\chi^-(a_1^r, a_j^r) = 1$ for some $r \in \{1, 2\}$ and some $j > 1$. Denote $\ell(a_j^r) = a_{j_*}$. Then $j_* \neq j_\#$ and

$$k_2(r) = \frac{z_{a_{j_*}}}{\mu(a_j^r) - \mu(a_1^r)} + \cdots,$$

where $\ldots$ is independent from z_{j_*}. Now let $r' = \{1, 2\} \backslash \{r\}$, and find j' such that $\ell(a_{j'}^{r'}) = a_{j_*}$. If such j' does not exist, then $k_2(r')$ does not depend on z_{j_*}. Accordingly, for a suitable x we have $k_2(r) \neq k_2(r')$, and (5.71) holds. If $n_2 = 1$, then $r = 1$ and $r' = 2$. So j' does not exists and (5.71) is established.

If j' exists, then $n_1, n_2 \geq 2$, so the set $\mathcal{A}$ is strongly admissible. By Lemma 5.7 $\chi^-(a_1^{r'}, a_{j'}^{r'}) = 0$ since $\chi^-(a_1^r, a_j^r) = 1$ and

$$|a_1^r| = |a_1^{r'}|, \quad |a_j^r| = |a_{j'}^{r'}|. \tag{5.72}$$

So

$$k_2(r') = z_{j_*} \frac{\chi^+(a_1^{r'}, a_{j'}^{r'})}{\mu(a_j^{r'}) + \mu(a_1^{r'})} + \cdots .$$

Since χ^+ equals 1 or 0, then using again (5.72) and the fact that $\mu(a)$ only depends on $|a|$, we see that $k_2(r) \neq k_2(r')$ for a suitable x, so (5.71) again holds.

(ii) If $\chi^-(a_1^r, a_j^r) = 0$ for all j and r, then $\chi^+(a_1^r, a_j^r) = 1$ for some r and j. Define z_{j_*} as above. Then the coefficient in $k_2(r)$ in front of z_{j_*} is non-zero, while for $k_2(r')$ it vanishes. This is obvious if $n_{r'} = 1$. Otherwise $\mathcal{A}$ is strongly admissible and it holds by Lemma 5.7 (and since $\chi^- \equiv 0$). So (5.71) again holds.
Case 2: $j_\# \neq 1$. Then by (5.54) there exists $a_j^1 \in \mathcal{L}_f^r$ such that $\ell(a_j^r) = a_1$. So $\chi^+(a_1^1, a_j^1) \neq 0$ or $\chi^-(a_1^1, a_j^1) \neq 0$. Then $\varphi(a_1^1, a_j^1) = x_{a_{j_\#}}$, the sum in (5.69) is non-trivial and for the same reason as in Case 1 (5.71) holds.

This completes the proof of Lemma 5.6. □

PART III. A KAM THEOREM

6 KAM Normal Form Hamiltonians

6.1 Block decomposition, normal form matrices. In this subsection we recall two notions introduced in [EK10] for the nonlinear Schrödinger equation. They are essential to overcome the problems of small divisors in a multidimensional context. Since the structure of the spectrum for the beam equation, $\{\sqrt{|a|^4 + m},\ a \in \mathbb{Z}^d\}$, is similar to that for the NLS equation, $\{|a|^2 + \hat{V}_a,\ a \in \mathbb{Z}^d\}$, then to study the beam equation we will use tools, similar to those used to study the NLS equation.

6.1.1 Partitions. For any $\Delta \in \mathbb{N} \cup \{\infty\}$ we define an equivalence relation on $\mathbb{Z}^d$, generated by the pre-equivalence relation

$$a \sim b \Longleftrightarrow \begin{cases} |a| = |b|, \\ [a - b] \leq \Delta. \end{cases}$$

(see (2.4)). Let $[a]_\Delta$ denote the equivalence class of a—the *block* of a. For further references we note that

$$|a| = |b| \quad \text{and} \quad [a]_\Delta \neq [b]_\Delta \Rightarrow [a - b] \geq \Delta. \tag{6.1}$$

The crucial fact is that the blocks have a finite maximal "diameter"

$$d_\Delta = \max_{[a]=[b]} [a - b]$$

which do not depend on a but only on Δ. This is the content of

PROPOSITION 6.1.

$$d_\Delta \le C\Delta^{\frac{(d+1)!}{2}}. \tag{6.2}$$

The constant C only depends on d.

Proof. In [EK10] it was considered the equivalence relation on $\mathbb{Z}^d$, generated by the pre-equivalence

$$a \approx b \quad \text{if} \quad |a| = |b| \quad \text{and} \quad |a-b| \le \Delta.$$

Denote by $[a]^o_\delta$ and d^o_Δ the corresponding equivalence class and its diameter (with respect to the usual distance). Since $a \sim b$ if and only if $a \approx b$ or $a \approx -b$, then

$$[a]_\Delta = [a]^o_\Delta \cup -[a]^o_\Delta, \tag{6.3}$$

provided that the union in the r.h.s. is disjoint. It is proved in [EK10] that $d^o_\Delta \le D_\Delta =: C\Delta^{\frac{(d+1)!}{2}}$. Accordingly, if $|a| \ge D_\Delta$, then the union above is disjoint, (6.3) holds and diameter of $[a]_\Delta$ satisfies (6.2). If $|a| < D_\Delta$, then $[a]_\Delta$ is contained in a sphere of radius $< D_\Delta$. So the block's diameter is at most $2D_\Delta$. This proves (6.2) if we replace there C_d by $2C_d$. □

If $\Delta = \infty$ then the block of a is the sphere $\{b : |b| = |a|\}$. Each block decomposition is a sub-decomposition of the trivial decomposition formed by the spheres $\{|a| = \text{const}\}$.

6.1.2 Normal form matrices. On $\mathcal{L}_\infty \subset \mathbb{Z}^d$ we define the partition

$$[a]_\Delta = \begin{cases} [a]_\Delta \cap \mathcal{L}_\infty & \text{if } a \in \mathcal{L}_\infty \text{ and } |a| > c, \\ \{b \in \mathcal{L}_\infty : |b| \le c\} & \text{if } a \in \mathcal{L}_\infty \text{ and } |a| \le c, \end{cases}$$

for some positive constant c.

On $\mathcal{L} = \mathcal{F} \sqcup \mathcal{L}_\infty$ we define the partition, denoted $\mathcal{E}_\Delta$,

$$[a] = [a]_\Delta = \begin{cases} [a]_\Delta \cap \mathcal{L}_\infty & a \in \mathcal{L}_\infty, \\ \mathcal{F} & a \in \mathcal{F}. \end{cases} \tag{6.4}$$

REMARK 6.2. Now the diameter of each block $[a]$ is bounded as in (6.1) if we just let $C \gtrsim \max(\#\mathcal{F}, c^d)$.

If $A : \mathcal{L} \times \mathcal{L} \to gl(2, \mathbb{C})$ we define its *block components*

$$A^{[b]}_{[a]} : [a] \times [b] \to gl(2, \mathbb{C})$$

to be the restriction of A to $[a] \times [b]$. A is *block diagonal* over $\mathcal{E}_\Delta$ if, and only if, $A^{[a]}_{[b]} = 0$ if $[a] \ne [b]$. Then we simply write $A_{[a]}$ for $A^{[a]}_{[a]}$.

On the space of 2×2 complex matrices we introduce a projection

$$\Pi : gl(2, \mathbb{C}) \to \mathbb{C}I + \mathbb{C}J,$$

orthogonal with respect to the Hilbert-Schmidt scalar product. Note that $\mathbb{C}I + \mathbb{C}J$ is the space of matrices, commuting with the symplectic matrix J.

DEFINITION 6.3. We say that a matrix $A : \mathcal{L} \times \mathcal{L} \to gl(2, \mathbb{C})$ is on normal form with respect to Δ, $\Delta \in \mathbb{N} \cup \{\infty\}$, and write $A \in \mathcal{NF}_\Delta$, if

(i) A is real valued,
(ii) A is symmetric, i.e. $A_b^a \equiv {}^t A_a^b$,
(iii) A is block diagonal over $\mathcal{E}_\Delta$,
(iv) A satisfies $\Pi A_b^a \equiv A_b^a$ for all $a, b \in \mathcal{L}_\infty$.

Any real quadratic form $\mathbf{q}(w) = \frac{1}{2}\langle w, Aw\rangle$, $w = (p, q)$, can be written as

$$\frac{1}{2}\langle p, A_{11}p\rangle + \langle p, A_{12}q\rangle + \frac{1}{2}\langle q, A_{22}q\rangle + \frac{1}{2}\langle w_{\mathcal{F}}, H(\rho)w_{\mathcal{F}}\rangle$$

where A_{11}, A_{22} and H are real symmetric matrices and A_{12} is a real matrix. We now pass from the real variables $w_a = (p_a, q_a)$ to the complex variables $z_a = (\xi_a, \eta_a)$ by the transformation $w = Uz$ defined through

$$\xi_a = \frac{1}{\sqrt{2}}(p_a + \mathbf{i}q_a), \quad \eta_a = \frac{1}{\sqrt{2}}(p_a - \mathbf{i}q_a), \tag{6.5}$$

for $a \in \mathcal{L}_\infty$, and acting like the identity on $(\mathbb{C}^2)^{\mathcal{F}}$. Then we have

$$\mathbf{q}(Uz) = \frac{1}{2}\langle \xi, P\xi\rangle + \frac{1}{2}\langle \eta, \overline{P}\eta\rangle + \langle \xi, Q\eta\rangle + \frac{1}{2}\langle z_{\mathcal{F}}, H(\rho)z_{\mathcal{F}}\rangle,$$

where

$$P = \frac{1}{2}\left((A_{11} - A_{22}) - \mathbf{i}(A_{12} + {}^t A_{12})\right)$$

and

$$Q = \frac{1}{2}\left((A_{11} + A_{22}) + \mathbf{i}(A_{12} - {}^t A_{12})\right).$$

Hence P is a complex symmetric matrix and Q is a Hermitian matrix. If A is on normal form, then $P = 0$.

Notice that this change of variables is not symplectic but changes the symplectic form slightly:

$$U^*\Omega = \mathbf{i}\sum_{a\in\mathcal{L}} d\xi_a \wedge d\eta_a + \sum_{a\in\mathcal{F}} d\xi_a \wedge d\eta_a.$$

6.2 The unperturbed Hamiltonian. Let $h_{\mathrm{up}}(r, w, \rho)$ be a function of the form

$$\langle r, \Omega_{\rm up}(\rho)\rangle + \frac{1}{2}\langle w, A_{\rm up}(\rho)w\rangle = \langle r, \Omega_{\rm up}(\rho)\rangle + \frac{1}{2}\langle w_{\mathcal{F}}, H_{\rm up}(\rho)w_{\mathcal{F}}\rangle + \frac{1}{2}\sum_{a\in\mathcal{L}_\infty}\Lambda_a\left(p_a^2+q_a^2\right), \tag{6.6}$$

where $w_a = (p_a, q_a)$ and

$$\begin{cases} \Omega_{\rm up} : \mathcal{D} \to \mathbb{R}^{\mathcal{A}} & \\ \Lambda_a : \mathcal{D} \to \mathbb{R}, & a \in \mathcal{L}_\infty, \\ H_{\rm up} : \mathcal{D} \to gl(\mathbb{R}^{\mathcal{F}} \times \mathbb{R}^{\mathcal{F}}), & {}^tH_{\rm up} = H_{\rm up}, \end{cases} \tag{6.7}$$

are $\mathcal{C}^{s_*}$-functions, $s_* \geq 1$. $\mathcal{D}$ is an open ball or a cube of diameter at most 1 in the space $\mathbb{R}^{\mathcal{P}}$, parametrised by some finite subset $\mathcal{P}$ of $\mathbb{Z}^d$.

We can write

$$\langle w, A_{\rm up}(\rho)w\rangle = \langle w_{\mathcal{F}}, H_{\rm up}(\rho)w_{\mathcal{F}}\rangle + \frac{1}{2}\big(\langle p_\infty, Q_{\rm up}(\rho)p_\infty\rangle + \langle q_\infty, Q_{\rm up}(\rho)q_\infty\rangle\big)$$

and

$$Q_{\rm up}(\rho) = \operatorname{diag}\{\Lambda_a(\rho) : a \in \mathcal{L}_\infty\}.$$

Definition 6.4. A function $h_{\rm up}$ of the form (6.6) + (6.7) will be called un *unperturbed Hamiltonian* if it verifies Assumptions A1–3 (given below) described by the positive constants

$$c', c, \delta_0, \beta = (\beta_1, \beta_2, \beta_3), \tau.$$

To formulate these assumptions we shall use the partition $[a] = [a]_\infty$ of $\mathcal{F} \sqcup \mathcal{L}_\infty$ defined in (6.4). Notice that this partition depend on a (possibly quite large) constant c.

6.2.1 A1—spectral asymptotics. There exist a constant $0 < c' \leq c$ and exponents $\beta_1 \geq 0, \beta_2 > 0$ such that for all $\rho \in \mathcal{D}$:

$$|\Lambda_a(\rho) - |a|^2| \leq c\frac{1}{\langle a\rangle^{\beta_1}} \quad a \in \mathcal{L}_\infty; \tag{6.8}$$

$$|(\Lambda_a(\rho) - \Lambda_b(\rho)) - (|a|^2 - |b|^2)| \leq c\max\left(\frac{1}{\langle a\rangle^{\beta_2}}, \frac{1}{\langle b\rangle^{\beta_2}}\right), \quad a, b \in \mathcal{L}_\infty; \tag{6.9}$$

$$\begin{cases} |\Lambda_a(\rho)| \geq c', & a \in \mathcal{L}_\infty, \\ ||(JH_{\rm up}(\rho))^{-1}|| \leq \frac{1}{c'}; \end{cases} \tag{6.10}$$

$$|\Lambda_a(\rho) + \Lambda_b(\rho)| \geq c', \quad a, b \in \mathcal{L}_\infty, \tag{6.11}$$

$$\begin{cases} |(\Lambda_a(\rho) - \Lambda_b(\rho)))| \geq c', & a, b \in \mathcal{L}_\infty,\ [a] \neq [b], \\ ||(\Lambda_a(\rho)I - \mathbf{i}JH_{\rm up}(\rho))^{-1}|| \leq \frac{1}{c'}, & a \in \mathcal{L}_\infty. \end{cases} \tag{6.12}$$

Notice that if $\beta_1 \geq \beta_2$, then (6.8) implies (6.9) (if c is large enough).

6.2.2 A2—transversality. Denote by $(Q_{\rm up})_{[a]}$ the restriction of the matrix $Q_{\rm up}$ to $[a]\times[a]$ and let $(Q_{\rm up})_{[\emptyset]}=0$. Let also $JH_{\rm up}(\rho)_{[\emptyset]}=0$.

There exists a $1\geq\delta_0>0$ such that for all $\mathcal{C}^{s_*}$-functions

$$\Omega:\mathcal{D}\to\mathbb{R}^n,\quad |\Omega-\Omega_{\rm up}|_{\mathcal{C}^{s_*}(\mathcal{D})}<\delta_0, \tag{6.13}$$

and for all $k\in\mathbb{Z}^n\backslash 0$ there exists a unit vector $\mathfrak{z}$ such that

$$|\partial_{\mathfrak{z}}\langle k,\Omega(\rho)\rangle|\geq\delta_0,\quad\forall\rho\in\mathcal{D}$$

[19] and the following dichotomies hold for each $k\in\mathbb{Z}^n\backslash 0$:

(i) for any $a,b\in\mathcal{L}_\infty\cup\{\emptyset\}$ let

$$L(\rho):X\mapsto\langle k,\Omega(\rho)\rangle X+(Q_{\rm up})_{[a]}(\rho)X\pm X(Q_{\rm up})_{[b]}:$$

then either $L(\rho)$ is δ_0-*invertible* for all $\rho\in\mathcal{D}$, i.e.

$$\left\|L(\rho)^{-1}\right\|\leq\frac{1}{\delta_0}\quad\forall\rho\in\mathcal{D}, \tag{6.14}$$

or there exists a unit vector $\mathfrak{z}$ such that

$$|\langle v,\partial_{\mathfrak{z}}L(\rho)v\rangle|\geq\delta_0,\quad\forall\rho\in\mathcal{D}$$

and for any unit-vector v in the domain of $L(\rho)$;[20]

(ii) let

$$L(\rho,\lambda):X\mapsto\langle k,\Omega(\rho)\rangle X+\lambda X+\mathbf{i}XJH_{\rm up}(\rho)$$

and

$$P_{\rm up}(\rho,\lambda)=\det L(\rho,\lambda):$$

then either $L(\rho,\Lambda_a(\rho))$ is δ_0-invertible for all $\rho\in\mathcal{D}$ and $a\in[a]_\infty$, or there exists a unit vector $\mathfrak{z}$ such that, with $m=2\#\mathcal{F}$,

$$\begin{aligned}&|\partial_{\mathfrak{z}}P_{\rm up}(\rho,\Lambda_a(\rho))+\partial_\lambda P_{\rm up}(\rho,\Lambda_a(\rho))\langle v,\partial_{\mathfrak{z}}Q_{\rm up}(\rho)v\rangle|\\&\geq\delta_0\left\|L(\cdot,\Lambda_a(\cdot))\right\|_{\mathcal{C}^1(\mathcal{D})}\left\|L(\cdot,\Lambda_a(\cdot))\right\|^{m-2}_{\mathcal{C}^0(\mathcal{D})}\end{aligned}$$

for all $\rho\in\mathcal{D}$, $a\in[a]_\infty$ and for any unit-vector $v\in(\mathbb{C}^2)^{[a]}$;[21]

(iii) for any $a,b\in\mathcal{F}\cup\{\emptyset\}$ let

$$L(\rho):X\mapsto\langle k,\Omega(\rho)\rangle X-\mathbf{i}JH_{\rm up}(\rho)_{[a]}X+\mathbf{i}XJH_{\rm up}(\rho)_{[b]}:$$

then either $L(\rho)$ is δ_0-invertible for all $\rho\in\mathcal{D}$, or there exists a unit vector $\mathfrak{z}$ and an integer $1\leq j\leq s_*$ such that

$$\left|\partial^j_{\mathfrak{z}}\det L(\rho)\right|\geq\delta_0\left\|L\right\|_{\mathcal{C}^j(\mathcal{D})}\left\|L\right\|^{m^2-2}_{\mathcal{C}^0(\mathcal{D})},\quad\forall\rho\in\mathcal{D}, \tag{6.15}$$

[19] $\partial_{\mathfrak{z}}$ denotes here the directional derivative in the direction $\mathfrak{z}\in\mathbb{R}^p$.

[20] L is a linear operator acting on $([a]\times[b])$-matrices.

[21] L is a linear operator acting on $(1\times m)$-matrices.

where $m^2 = (2\#\mathcal{F})^2$ if both $[a]$ and $[b]$ are $\neq \emptyset$ and $m^2 = 2\#\mathcal{F}$ if one of $[a]$ and $[b] = \emptyset$.[22]

REMARK 6.5. The dichotomy in A2 is imposed not only on Ω_{up} but also on $\mathcal{C}^{s_*}$-perturbations of Ω_{up}, because, in general, the dichotomy for Ω_{up} does not imply that for perturbations.

If, however, any $\mathcal{C}^{s_*}$ perturbation of Ω_{up} can be written as $\Omega_{up} \circ f$ for some diffeomorphism $f = id + \mathcal{O}(\delta_0)$—this is for example the case when $\Omega(\rho) = \rho$—then the dichotomy on Ω implies a dichotomy on $\mathcal{C}^{s_*}$-perturbations.

6.2.3 A3—a Melnikov condition. There exist constants $\beta_3, \tau > 0$ such that

$$|\langle k, \Omega(0)\rangle - (\Lambda_a(0) - \Lambda_b(0)))| \geq \frac{\beta_3}{|k|^\tau} \tag{6.16}$$

for all $k \in \mathbb{Z}^{\mathcal{P}} \backslash 0$ and all $a, b \in \mathcal{L}_\infty \backslash [0]$.

6.3 KAM normal form Hamiltonians. Consider now an unperturbed Hamiltonian h_{up} defined on the set $\mathcal{D}$ (see Definition 6.4). The essential properties of this function are described by the positive constants

$$c', c, \delta_0, \beta = (\beta_1, \beta_2, \beta_3), \tau$$

(occurring in assumptions A1–3), and by the constant

$$\chi = |\nabla_\rho \Omega_{up}|_{\mathcal{C}^{s_*-1}(\mathcal{D})} + \sup_{a \in \mathcal{L}_\infty} |\nabla_\rho \Lambda_a|_{\mathcal{C}^{s_*-1}(\mathcal{D})} + ||\nabla_\rho H_{up}||_{\mathcal{C}^{s_*-1}(\mathcal{D})}. \tag{6.17}$$

Notice that, by Assumption A2, $\chi \geq \delta_0$, and in order to simplify the estimates a little we shall assume that

$$0 < c' \leq \delta_0 \leq \chi \leq c. \tag{6.18}$$

We shall consider a somewhat larger class of functions.

DEFINITION 6.6. A function of the form

$$h(r, w, \rho) = \langle \Omega(\rho), r\rangle + \frac{1}{2}\langle w, A(\rho)w\rangle \tag{6.19}$$

is said to be on *KAM normal form* with respect to the unperturbed Hamiltonian h_{up}, satisfying (6.18), if
(**Hypothesis** Ω) Ω is of class $\mathcal{C}^{s_*}$ on $\mathcal{D}$ and

$$|\Omega - \Omega_{up}|_{\mathcal{C}^{s_*}(\mathcal{D})} \leq \delta. \tag{6.20}$$

(**Hypothesis B**) $A - A_{up} : \mathcal{D} \to \mathcal{M}^b_{(0, m_* + \varkappa), \varkappa}$ is of class $\mathcal{C}^{s_*}$, $A(\rho)$ is on normal form $\in \mathcal{NF}_\Delta$ for all $\rho \in \mathcal{D}$ and

$$||\partial^j_\rho (A(\rho) - A_{up}(\rho))_{[a]}|| \leq \delta \frac{1}{\langle a\rangle^\varkappa} \tag{6.21}$$

[22] in the first case L is a linear operator acting on $(m \times m)$-matrices, and in the second case L is a linear operator acting on $(1 \times m)$-matrices or $(m \times 1)$-matrices.

for $|j| \le s_*$, $a \in \mathcal{L}$ and $\rho \in \mathcal{D}$.[23] Here we require that

$$0 < \varkappa. \tag{6.22}$$

We denote this property by

$$h \in \mathcal{NF}_\varkappa(h_{\rm up}, \Delta, \delta).$$

Since the unperturbed Hamiltonian h_{up} will be fixed in Part III we shall often suppress it, writing simply $h \in \mathcal{NF}_\varkappa(\Delta, \delta)$.

6.4 The KAM theorem. In this section we state an abstract KAM result for perturbations of a certain KAM normal form Hamiltonians.

Let

$$h_{\rm up} = h_{{\rm up},\chi,c',\delta_0,c}$$

be a fixed unperturbed Hamiltonian satisfying (6.18). ($h_{\rm up}$ also depends on β, τ but we shall not track this dependence.)

Let h be a KAM normal form Hamiltonian,

$$h \in \mathcal{NF}_\varkappa(h_{{\rm up},\chi,c',\delta_0,c}, \Delta, \delta),$$

and recall (6.22). We shall also assume $\Delta \ge 1$.

The perturbation will belong to $\mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$ with

$$0 < \sigma, \mu, \gamma_1 \le 1$$

and (recall (2.10))

$$\gamma = (\gamma_1, m_* + \varkappa) > \gamma_* = (0, m_* + \varkappa).$$

These bounds will be, often implicitly, assumed in the rest of Part III.

Theorem 6.7. *There exist positive constants C, α and* exp *such that, for any $h \in \mathcal{NF}_{\varkappa,h_{up}}(\Delta,\delta)$ and for any $f \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$,*

$$\varepsilon = \left|f^T\right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}} \quad \text{and} \quad \xi = |f|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}},$$

if

$$\delta \le \frac{1}{2C} c'$$

and

$$\varepsilon (\log \frac{1}{\varepsilon})^{\rm exp} \le \frac{1}{C} \left(\frac{\sigma\mu}{\max(\gamma_1^{-1}, d_\Delta)} \frac{c'}{\chi + \xi} \right)^{\rm exp} c', \tag{6.23}$$

[23] Here it is important that $||\cdot||$ is the matrix operator norm.

then there exist a closed subset $\mathcal{D}' = \mathcal{D}'(h, f) \subset \mathcal{D}$,

$$\operatorname{meas}(\mathcal{D}\backslash\mathcal{D}') \leq C\left(\log \frac{1}{\varepsilon}\frac{\max(\gamma_1^{-1}, d_\Delta)}{\sigma\mu}\right)^{\exp} \frac{\chi}{\delta_0}((\chi+\xi)\frac{\varepsilon}{\chi})^{\alpha}, \tag{6.24}$$

and a $\mathcal{C}^{s_*}$ *mapping*

$$\Phi : \mathcal{O}_{\gamma_*}(\sigma/2, \mu/2) \times \mathcal{D} \to \mathcal{O}_{\gamma_*}(\sigma, \mu),$$

real holomorphic and symplectic for each parameter $\rho \in \mathcal{D}$, *such that*

$$(h+f)\circ\Phi = h' + f'$$

with

(i)

$$h' \in \mathcal{NF}_{\varkappa}(\infty, \delta'), \quad \delta' \leq \frac{c'}{2},$$

and

$$\left|h' - h\right|_{\substack{\sigma/2,\mu/2\\ \gamma_*,\varkappa,\mathcal{D}}} \leq C;$$

(ii) *for any* $x \in \mathcal{O}_{\gamma_*}(\sigma/2, \mu/2)$, $\rho \in \mathcal{D}$ *and* $|j| \leq s_*$

$$||\partial_\rho^j(\Phi(x,\rho) - x)||_{\gamma_*} + \left\|\partial_\rho^j(d\Phi(x,\rho) - I)\right\|_{\gamma_*,\varkappa} \leq C$$

and $\Phi(\cdot, \rho)$ *equals the identity for* ρ *near the boundary of* $\mathcal{D}$;

(iii) *for* $\rho \in \mathcal{D}'$ *and* $\zeta = r = 0$

$$d_r f' = d_\theta f' = d_\zeta f' = d_\zeta^2 f' = 0.$$

Moreover,

(iv) *if* $\tilde{\rho} = (0, \rho_2, \ldots, \rho_p)$ *and* $f^T(\cdot, \tilde{\rho}) = 0$ *for all* $\tilde{\rho}$, *then* $h' = h$ *and* $\Phi(x, \cdot) = x$ *for all* $\tilde{\rho}$.

The exponent α *is a positive constant only depending on* $d, s_*, \varkappa$ *and* β_2. *The exponent* exp *only depends on* d, $\#\mathcal{A}$ *and* $\tau, \beta_2, \varkappa$. C *is an absolute constant that depends on* $c, \tau, \beta_2, \beta_3$ *and* $\varkappa$. C *also depend on* $\sup_{\mathcal{D}} |\Omega_{up}|$ *and* $\sup_{\mathcal{D}} |H_{up}|$, *but stays bounded when these do.*

The condition on Φ and $h' - h$ may look bad but it is not.

Corollary 6.8. *Under the assumption of Theorem 6.7, let* ε_* *be the largest positive number such that (6.23) holds. Then, for any* $\rho \in \mathcal{D}$ *and* $|j| \leq s_* - 1$,

(i)'

$$\left|\partial_\rho^j(h'(\cdot,\rho) - h(\cdot,\rho))\right|_{\substack{\sigma/2,\mu/2\\ \gamma_*,\varkappa,}} \leq \frac{C}{\varepsilon_*}\left|f^T\right|_{\substack{\sigma,\mu\\ \gamma,\varkappa,\mathcal{D}}};$$

(ii)'

$$||\partial_\rho^j(\Phi(x,\rho)-x)||_{\gamma_*} + \left\|\partial_\rho^j(d\Phi(x,r)-I)\right\|_{\gamma^*,\varkappa} \le \frac{C}{\varepsilon_*}\left|f^T\right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}},$$

for any $x \in \mathcal{O}_{\gamma_*}(\sigma/2,\mu/2)$.

Proof. Let us denote ρ here by ρ_1. If $|f^T|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}} \le \varepsilon_*$, then we can apply the theorem to εf for any $|\varepsilon| \le 1$. Let now $\rho = (\varepsilon,\rho_1)$ and consider $h_{\rm up}$, h and f as functions depending on this new parameter ρ—they will still verify the assumptions of the theorem, which will provide us with a mapping Φ with a $\mathcal{C}^{s_*}$ dependence in $\rho=(\varepsilon,\rho_1)$ and equal to the identity when $\varepsilon = 0$. The bound on the derivative together with assertion (iv) now implies that

$$||\Phi(x,\varepsilon,\tilde\rho)-x||_{\gamma_*} \le C\varepsilon \le \frac{C}{\varepsilon_*}|f^T|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}$$

for any $x \in \mathcal{O}_{\gamma_*}(\sigma/2,\mu/2)$. The same estimate holds for all derivatives with respect to $\tilde\rho$ up to order s_*-1. Take now $\varepsilon = 1$ and we get $(ii)'$.

The argument for $h'-h$ is the same. □

A special case that will interest us in particular is the following.

COROLLARY 6.9. *Let* $h_{up} = h_{up,\chi,c',\delta_0,c}$ *be an unperturbed Hamiltonian, satisfying*

$$\text{(a)} \qquad \delta_0^{1+\aleph} \le c' \le \delta_0 \le \chi \le C'\delta_0^{1-\aleph} \le c,$$

and let $f \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$ *with*

$$\text{(b)} \qquad \xi = |f|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}} \le C'\delta_0^{1-\aleph}.$$

for some $1 > \aleph > 0$ *and* $C' > 0$.

Then there exist constants $\varepsilon_0 > 0$, α *and* κ*— independent of* c', δ_0, χ *and* $\aleph$*—such that if* $\varepsilon = \left|f^T\right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}$ *satisfies*

$$\varepsilon\left(\log\frac{1}{\varepsilon}\right)^\kappa \le \varepsilon_0\delta_0^{1+\aleph\kappa}, \tag{6.25}$$

then there exist a closed subset $\mathcal{D}' = \mathcal{D}'(h,f) \subset \mathcal{D}$,

$$\operatorname{meas}(\mathcal{D}\backslash\mathcal{D}') \le \frac{1}{\varepsilon_0}\delta_0^{-\aleph\kappa}\varepsilon^\alpha, \tag{6.26}$$

and a $\mathcal{C}^{s_*}$ *mapping* Φ

$$\Phi : \mathcal{O}_{\gamma_*}(\sigma/2,\mu/2)\times\mathcal{D} \to \mathcal{O}_{\gamma_*}(\sigma,\mu),$$

real holomorphic and symplectic for each parameter $\rho \in \mathcal{D}$, *such that*

$$(h_{up}+f)\circ\Phi(r,w,\rho) = \langle\Omega'(\rho),r\rangle + \frac{1}{2}\langle w, A'(\rho)w\rangle + f'(r,w,\rho)$$

with

(i) *the frequency vector* Ω' *satisfies*

$$|\Omega' - \Omega_{up}|_{\mathcal{C}^{s_*-1}(\mathcal{D})} \leq c'$$

and, for each $|j| \leq s_*$ *and* $\rho \in \mathcal{D}$, *the matrix*

$$A'(\rho) = A'_\infty(\rho) \oplus H'(\rho) \in \mathcal{NF}_\infty$$

and satisfies

$$||\partial^j_\rho(H'(\rho) - H_{up}(\rho)|| \leq c';$$

(ii)$'$ *for any* $x \in \mathcal{O}_{\gamma_*}(\sigma/2, \mu/2)$, $\rho \in \mathcal{D}$ *and* $|j| \leq s_* - 1$,

$$||\partial^j_\rho(\Phi(x,\rho) - x)||_{\gamma_*} + \left\|\partial^j_\rho(d\Phi(x,r) - I)\right\|_{\gamma^*,\varkappa} \leq \frac{1}{\varepsilon_0}\frac{\varepsilon}{\delta_0^{1+\aleph\kappa}}(\log\frac{1}{\delta_0})^\kappa$$

and $\Phi(\cdot,\rho)$ *equals the identity for* ρ *near the boundary of* $\mathcal{D}$;

(iii) *for* $\rho \in \mathcal{D}'$ *and* $\zeta = r = 0$

$$d_r f' = d_\theta f' = d_\zeta f' = d^2_\zeta f' = 0.$$

The exponent α *is a positive constant only depending on* $d, s_*, \varkappa$ *and* β_2. *The exponent* κ *also depends on* $\#\mathcal{A}$ *and* τ. *The constant* ε_0 *depends on everything except, as already said,* c', δ_0, χ *and* $\aleph$.

Proof. We apply the theorem with $h = h_{up}$, i.e. $\delta = 0$ and $\Delta = 1$. The condition (6.23) is implied by

$$\varepsilon\left(\log\frac{1}{\varepsilon}\right)^{\exp} \leq \frac{1}{C''}\left(\frac{c'}{\chi+\xi}\right)^{\exp} c'$$

for some C'' depending on C, γ_1, σ, μ. With the choice of c', ξ, χ this is now implied by (6.25) if $\kappa \geq 1 + 2\exp$.

The estimate of the measure becomes, from (6.24),

$$\frac{1}{\varepsilon_0}\left(\log\frac{1}{\varepsilon}\right)^{\exp}\delta_0^{-\aleph(1+\alpha)}\varepsilon^\alpha \leq \frac{1}{\varepsilon_0}\delta_0^{-\aleph(1+\alpha)}\varepsilon^{\frac{\alpha}{2}},$$

which is what is claimed if we replace $\frac{\alpha}{2}$ by α, and take $\kappa \geq (1+\alpha)$.

(i) is just a consequence of $h' \in \mathcal{NF}(\infty, c')$. The bound in (ii) follows from the bound $(ii)'$ in Corollary 6.8 plus an easy estimate of ε_*. □

7 Small Divisors

Control of the small divisors is essential for solving the homological equation (next section). In this section we shall control these divisors for $k \neq 0$ using Assumptions A2 and A3.

For a mapping $L : \mathcal{D} \to gl(\dim, \mathbb{R})$ define, for any $\kappa > 0$,

$$\Sigma(L, \kappa) = \{\rho \in \mathcal{D} : ||L^{-1}(\rho)|| > \frac{1}{\kappa}\}.$$

Let

$$h(r, w, \rho) = \langle r, \Omega(\rho)\rangle + \frac{1}{2}\langle w, A(\rho)w\rangle$$

be a normal form Hamiltonian in $\mathcal{NF}_{\varkappa}(\Delta, \delta)$. Recall the convention (6.18) and assume $\varkappa > 0$ and

$$\delta \leq \frac{1}{C}c', \tag{7.1}$$

where C is to be determined.

LEMMA 7.1. *Let*

$$L_k = \langle k, \Omega(\rho)\rangle.$$

There exists a constant C such that if (7.1) holds, then

$$\text{meas}\left(\bigcup_{0<|k|\leq N} \Sigma(L_k, \kappa)\right) \leq CN^{\exp}\frac{\kappa}{\delta_0}$$

and

$$\text{dist}\left(\mathcal{D}\backslash\Sigma(L_k, \kappa), \Sigma\left(L_k, \frac{\kappa}{2}\right)\right) > \frac{1}{C}\frac{\kappa}{N^{\chi}}$$

[24] *for any $\kappa > 0$.*

(The exponent exp *only depends on $\#\mathcal{A}$. C is an absolute constant.)*

Proof. We only need to consider $\kappa \leq \delta_0$ since otherwise the result is trivial. Since $\delta \leq \delta_0$, using Assumption A2(i), with $a = b = \emptyset$, we have, for each $k \neq 0$, either that

$$|\langle \Omega(\rho), k\rangle| \geq \delta_0 \geq \kappa \quad \forall \rho \in \mathcal{D}$$

or that

$$\partial_{\mathfrak{z}}\langle \Omega(\rho), k\rangle \ \geq \delta_0 \quad \forall \rho \in \mathcal{D}$$

(for some suitable choice of a unit vector $\mathfrak{z}$). The first case implies $\Sigma(L_k, \kappa) = \emptyset$. The second case implies that $\Sigma(L_k, \kappa)$ has Lebesgue measure $\lesssim \frac{\kappa}{\delta_0}$. Summing up over all

[24] This is assumed to be fulfilled if $\Sigma_{L_k}(\frac{\kappa}{2}) = \emptyset$.

$0 < |k| \leq N$ gives the first statement. The second statement follows from the mean value theorem and the bound

$$|\nabla_\rho L_k(\rho)| \leq N(\chi + \delta). \qquad \square$$

LEMMA 7.2. *Let*

$$L_{k,[a]} = (\langle k, \Omega\rangle I - \mathbf{i}JA)_{[a]}\,.$$

There exists a constant C such that if (7.1) holds, then,

$$\operatorname{meas}\left(\bigcup_{\substack{0<|k|\leq N\\ [a]}} \Sigma(L_{k,[a]}(\kappa)\right) \leq CN^{\exp}(\frac{\kappa}{\delta_0})^{\frac{1}{s_*}}$$

and

$$\operatorname{dist}\left(\mathcal{D}\backslash\Sigma(L_{k,[a]},\kappa), \Sigma\left(L_{k,[a]}, \frac{\kappa}{2}\right)\right) > \frac{1}{C}\frac{\kappa}{N\chi},$$

for any $\kappa > 0$.

(The exponent exp *only depends on d and $\#\mathcal{A}$. C is an absolute constant that depends on c. C also depend on $\sup_{\mathcal{D}} |\Omega_{up}|$ and $\sup_{\mathcal{D}} |H_{up}|$, but stays bounded when these do.)*

Proof. Consider first $a \in \mathcal{L}_\infty$. Then $L_{k,[a]}$ is conjugate to a sum of two Hermitian operators of the form

$$L = \langle k, \Omega\rangle I + Q_{[a]},$$

where $Q_{[a]}$ is the restriction of Q to $[a] \times [a]$ (see the discussion in Sect. 6.1.2).

If we let

$$L_{\mathrm{up}} = \langle k, \Omega\rangle I + (Q_{\mathrm{up}})_{[a]},$$

where Q_{up} comes from the unperturbed Hamiltonian, then it follows, from (6.21) and (7.1), that

$$\|L - L_{\mathrm{up}}\|_{\mathcal{C}^1(\mathcal{D})} \leq \delta \leq \mathrm{ct}.\delta_0.$$

If now L_{up} is δ_0-invertible, then this implies that L is $\frac{\delta_0}{2}$-invertible.

Otherwise, by assumption A2(i), there exists a unit vector $\mathfrak{z}$ such that

$$|\langle v, \partial_{\mathfrak{z}} L_{\mathrm{up}}(\rho)v\rangle| \geq \delta_0$$

for any unit vector v. Since $Q_{[a]}$ is Hermitian we have, for any eigenvalue $\Lambda(\rho)$, $\mathcal{C}^1$ in the direction $\mathfrak{z}$, and any associated unit eigenvector $v(\rho)$,

$$\partial_{\mathfrak{z}}\big(\langle k, \Omega(\rho)\rangle + \Lambda(\rho)\big) = \langle v(\rho), \partial_{\mathfrak{z}} L(\rho)v(\rho)\rangle = \langle v(\rho), \partial_{\mathfrak{z}} L_{\mathrm{up}}(\rho)v(\rho)\rangle + \mathcal{O}(\delta).$$

Hence

$$\left|\partial_{\mathfrak{z}}\big(\langle k,\Omega(\rho)\rangle+\Lambda(\rho)\big)\right|\geq\delta_0-\mathrm{Ct}.\delta\geq\frac{\delta_0}{2},$$

which implies that $|\langle k,\Omega(\rho)\rangle+\Lambda(\rho)|$ is larger than κ outside a set of Lebesgue measure $\lesssim\frac{\kappa}{\delta_0}$. Since $L(\rho)$ is Hermitian this implies that

$$\operatorname{meas}\Sigma(L,\kappa))\lesssim|a|^d\,\frac{\kappa}{\delta_0}$$

– the dimension of L is $\lesssim|a|^d$. (This argument is valid if $\Lambda(\rho)$ is $\mathcal{C}^1$ in the direction $\mathfrak{z}$ which can always be assumed when Q is analytic in ρ. The non-analytic case follows by analytical approximation.)

We still have to sum up over, a priori, infinitely many $[a]$'s. However, since $|\langle k,\Omega(\rho)\rangle|\lesssim|k|\lesssim N$, it follows, by (6.8), that

$$|\langle k,\Omega(\rho)\rangle+\Lambda(\rho)|\geq|\Lambda_a(\rho)|-\delta-\mathrm{Ct}.\,|k|\geq|a|^2-c\langle a\rangle^{-\beta 2}-\delta-\mathrm{Ct}.\,|k|$$

for some appropriate $a\in[a]$. Hence $|\langle k,\Omega(\rho)\rangle+\Lambda(\rho)|$ is larger than κ for $|a|\gtrsim N^{\frac12}$. Summing up over all $0<|k|\leq N$ and all $|a|\lesssim N^{\frac12}$ gives a set whose complement Σ verifies the estimate.

Consider now $a\in\mathcal{F}$ and let $L(\rho)=\big(\langle k,\Omega\rangle I-\mathbf{i}JH\big)$. It follows, by (6.21) and (7.1), that

$$\|L-L_{\mathrm{up}}\|_{\mathcal{C}^{s_*}}\leq\delta\leq\frac12\delta_0,$$

where $L_{\mathrm{up}}(\rho)=\big(\langle k,\Omega\rangle I-\mathbf{i}JH_{\mathrm{up}}\big)$—now we are not dealing with an Hermitian operator.

If now L_{up} is δ_0-invertible, then L will be $\frac{\delta_0}{2}$-invertible. Otherwise, by assumption A2(iii), there exists a unit vector $\mathfrak{z}$ and an integer $1\leq j\leq s_*$ such that

$$\left|\partial_{\mathfrak{z}}^j\det L_{\mathrm{up}}(\rho)\right|\geq\delta_0\,\|L_{\mathrm{up}}\|_{\mathcal{C}^j(\mathcal{D})}\,\|L_{\mathrm{up}}\|_{\mathcal{C}^0(\mathcal{D})}^{m-2},\quad\forall\rho\in\mathcal{D}.$$

Since, by convexity estimates (see [Hör76]),

$$\left|\partial_{\mathfrak{z}}^j\det L_{\mathrm{up}}(\rho)\right|\leq\mathrm{Ct}.\,\|L_{\mathrm{up}}\|_{\mathcal{C}^j(\mathcal{D})}\,\|L_{\mathrm{up}}\|_{\mathcal{C}^0(\mathcal{D})}^{m-1}$$

and

$$\left|\partial_{\mathfrak{z}}^j(\det L(\rho)-\det L_{\mathrm{up}}(\rho))\right|\leq\mathrm{Ct}.\delta\big(\,\|L_{\mathrm{up}}\|_{\mathcal{C}^j}+\delta\big)(\|L\|_{\mathcal{C}^0(\mathcal{D})}+\delta)^{m-2},$$

this implies that

$$\left|\partial_{\mathfrak{z}}^j\det L(\rho)\right|\geq(\delta_0-\mathrm{Ct}.\delta)\,\|L_{\mathrm{up}}\|_{\mathcal{C}^1(\mathcal{D})}\,\|L_{\mathrm{up}}\|_{\mathcal{C}^0(\mathcal{D})}^{m-1},\quad\forall\rho\in\mathcal{D},$$

which is $\geq\frac{\delta_0}{2}$ if δ is sufficiently small.

Then, by Lemma D.1,

$$|\det L(\rho)| \geq \kappa \, \|L\|_{\mathcal{C}^j}^{m-1},$$

outside a set of Lebesgue measure

$$\leq \text{Ct.} \left(\frac{\kappa}{\delta_0}\right)^{\frac{1}{j}}.$$

Hence, by Cramer's rule,

$$\operatorname{meas} \Sigma(L, \kappa) \leq \text{Ct.} \left(\frac{\kappa}{\delta_0}\right)^{\frac{1}{j}} \leq \text{Ct.} \left(\frac{\kappa}{\delta_0}\right)^{\frac{1}{j}}.$$

Summing up over all $|k| \leq N$ gives the first estimate.

The second estimate follows from the mean value theorem and the bound

$$\left|\nabla_\rho L_{k,[a]}(\rho)\right| \leq N(\chi + \delta). \qquad \square$$

LEMMA 7.3. *Let*

$$L_{k,[a],[b]} = (\langle k, \Omega\rangle I - \mathbf{i} \operatorname{ad}_{JA})_{[a]}^{[b]}.$$

There exists a constant C such that if (7.1) holds, then,

$$\bigcup_{\substack{0<|k|\leq N \\ [a],[b]}} \Sigma(L_{k,[a],[b]}, \kappa) \leq C(N\Delta)^{\exp} \left(\frac{\kappa}{\delta_0}\right)^{\alpha} \left(\frac{\chi}{\delta_0}\right)^{1-\alpha}$$

and

$$\operatorname{dist}\left(\mathcal{D} \backslash \Sigma(L_{k,[a],[b]}, \kappa), \Sigma\left(L_{k,[a],[b]}, \frac{\kappa}{2}\right)\right) > \frac{1}{C} \frac{\kappa}{\Delta^{\exp}} N\chi,$$

for any $\kappa > 0$. Here

$$\alpha = \min\left(\frac{\beta_2 \varkappa}{\beta_2 \varkappa + 2d(\beta_2 + \varkappa)}, \frac{1}{s_*}\right).$$

(The exponent exp *only depends on d, $\#\mathcal{A}$ and $\tau, \beta_2, \varkappa$. C is an absolute constant that depends on $c, \tau, \beta_2, \beta_3$ and $\varkappa$. C also depend on $\sup_{\mathcal{D}} |\Omega_{up}|$ and $\sup_{\mathcal{D}} |H_{up}|$, but stays bounded when these do.)*

Proof. Consider first $a, b \in \mathcal{F}$. This case is treated as the operator $L(\rho) = \big(\langle k, \Omega\rangle I - \mathbf{i}JH\big)$ in the previous lemma.

Consider then $a \in \mathcal{L}_\infty$ and $b \in \mathcal{F}$. Then $L_{k,[a]}$ is conjugate to a sum of two operators of the form

$$X \mapsto \langle k, \Omega(\rho)\rangle X + Q_{[a]}(\rho)X + X\mathbf{i}JH(\rho)$$

(see the discussion in Sect. 6.1.2). This operator in not Hermitian, but only "partially" Hermitian: it decomposes as an orthogonal sum of operators of the form $L(\rho, \Lambda(\rho))$, where

$$L(\rho,\lambda) : X \mapsto \langle k, \Omega(\rho)\rangle X + \lambda X + \mathbf{i} X J H(\rho),$$

and $\Lambda(\rho)$ is an eigenvalue of $Q_{[a]}(\rho)$.

If we let

$$L_{up}(\rho,\lambda) : X \mapsto \langle k, \Omega(\rho)\rangle X + \lambda X + X\mathbf{i} J H_{up}(\rho),$$

then it follows, from (6.21) and (7.1), that

$$\|L(\cdot,\lambda) - L_{up}(\cdot,\lambda)\|_{\mathcal{C}^1(\mathcal{D})} \le \delta \le \text{ct.}\delta_0.$$

If $L_{up}(\rho, \Lambda_a(\rho))$ is δ_0-invertible for all $a \in [a]$, then this implies that, for any eigenvalue $\Lambda(\rho)$ of $Q_{[a]}(\rho)$, $L(\rho, \Lambda(\rho))$ is $\frac{\delta_0}{2}$-invertible.

Otherwise, by Assumption A2(ii), there exists a unit vector $\mathfrak{z}$ such that

$$|\partial_{\mathfrak{z}} P_{up}(\rho, \Lambda_a(\rho)) + \partial_\lambda P_{up}(\rho, \Lambda_a(\rho))\langle v, \partial_{\mathfrak{z}} Q_{up}(\rho) v\rangle| \ge \delta_0 \|L_{up}\|_{\mathcal{C}^1(\mathcal{D})} \|L_{up}\|^{m-2}_{\mathcal{C}^0(\mathcal{D})}$$

for all $\rho \in \mathcal{D}$, all $a \in [a]$ and for any unit-vector $v \in (\mathbb{C}^2)^{[a]}$. If now

$$P(\rho,\lambda) = \det L(\rho,\lambda),$$

then, for any eigenvalue $\Lambda(\rho)$, $\mathcal{C}^1$ in the direction $\mathfrak{z}$, and any associated unit eigenvector $v(\rho)$,

$$\begin{aligned}\frac{d}{d_{\mathfrak{z}}} P(\rho, \Lambda(\rho)) &= \partial_{\mathfrak{z}} P(\rho, \Lambda(\rho)) + \partial_\lambda P(\rho, \Lambda(\rho))\langle v(\rho), \partial_{\mathfrak{z}} Q(\rho) v(\rho)\rangle \\ &= \partial_{\mathfrak{z}} P_{up}(\rho, \Lambda_a(\rho)) + \partial_\lambda P_{up}(\rho, \Lambda_a(\rho))\langle v(\rho), \partial_{\mathfrak{z}} Q_{up}(\rho) v(\rho)\rangle \\ &\quad + \mathcal{O}\left(\delta \|L_{up}\|_{\mathcal{C}^1(\mathcal{D})} \|L_{up}\|^{m-1}_{\mathcal{C}^0(\mathcal{D})}\right).\end{aligned}$$

Hence

$$\left|\frac{d}{d_{\mathfrak{z}}} P(\rho, \Lambda(\rho))\right| \ge \frac{\delta_0}{2} \|L_{up}\|_{\mathcal{C}^1(\mathcal{D})} \|L_{up}\|^{m-2}_{\mathcal{C}^0(\mathcal{D})}.$$

Then

$$\left|\frac{P(\rho, \Lambda(\rho))}{||L||^{m-1}_{\mathcal{C}^0(\mathcal{D})}}\right| \ge \kappa$$

outside a set of Lebesgue measure $\lesssim \frac{\kappa}{\delta_0}$. Hence, by Cramer's rule,

$$\text{meas}\, \Sigma(L, \kappa) \le \text{Ct.}\frac{\kappa}{\delta_0}.$$

Since $|\langle k, \Omega(\rho)\rangle| \lesssim |k| \lesssim N$, it follows, by (6.8), that for any eigenvalue $\alpha(\rho)$ of $JH(\rho)$,

$$|\langle k, \Omega(\rho)\rangle + \Lambda(\rho) + \alpha(\rho)| \geq |\Lambda_a(\rho)| - \delta - \text{Ct.}\,|k| \geq |a|^2 - c\langle a\rangle^{-\beta_1} - \delta - \text{Ct.}\,|k|$$

for some appropriate $a \in [a]$. Hence, $\Sigma(L, \kappa) = \emptyset$ for $|a| \gtrsim N^{\frac{1}{2}}$.

Summing up over all $0 < |k| \leq N$ and all $|a| \lesssim N^{\frac{1}{2}}$ gives the first estimate.

Consider finally $a, b \in \mathcal{L}_\infty$. Then $L_{k,[a],[b]}$ is conjugate to a sum of four operators of the forms

$$X \mapsto \langle k, \Omega\rangle X + Q_{[a]}X + X^t Q_{[b]}$$

and

$$X \mapsto \langle k, \Omega\rangle X + Q_{[a]}X - XQ_{[b]}.$$

These operators are Hermitian with respect to the Hilbert–Schmidt norm on the space of matrices X. Changing from the operator norm to the Hilbert–Schmidt norm (and conversely) changes any estimate by a factor that depends on the dimension of the space of matrices X, which, we recall, is bounded by some power of Δ.

With this modification, the first operator is treated exactly as the operator $X \mapsto \langle k, \Omega\rangle X + Q_{[a]}X$ in the previous lemma, so let us concentrate on the second one, which we shall call $L = L_{k,[a],[b]}$. It follows as in the previous lemma that the Lebesgue measure of $\Sigma(L, \kappa)$ is $\lesssim (|a|\,|b|)^d \frac{\kappa}{\delta_0}$—recall that the operator is of dimension $\lesssim (|a|\,|b|)^{2d}$.

The problem now is the measure estimate of $\bigcup \Sigma(L_{k,[a],[b]}, \kappa)$ since, a priori, there may be infinitely many $\Sigma(L_{k,[a],[b]}, \kappa)$ that are non-void. We can assume without restriction that $|a| \leq |b|$. Since $|\langle k, \Omega(\rho)\rangle| \leq \text{Ct.}\,|k| \leq \text{Ct.}N$, it is enough to consider $|b| - |a| \leq \text{Ct.}N$.

Suppose first that $[a]$ and $[b]$ are $\neq [0]$. Let $\alpha(\rho)$ and $\beta(\rho)$ be eigenvalues of $Q_{[a]}(\rho)$ and $Q_{[b]}(\rho)$ respectively, and chose a, b such that

$$|\alpha(\rho) - \Lambda_a(\rho)| \leq \delta \frac{1}{\langle a\rangle^{\varkappa}}, \quad |\beta(\rho) - \Lambda_b(\rho)| \leq \delta \frac{1}{\langle b\rangle^{\varkappa}}.$$

Using Assumption A3 now gives

$$\begin{aligned} |\langle k, \Omega(\rho)\rangle \ + \alpha(\rho) - \beta(\rho)| &\geq |\langle k, \Omega_{\text{up}}(\rho)\rangle \ + \Lambda_a(\rho) - \Lambda_b(\rho)| - |k|\,\delta - 2\delta \frac{1}{\langle a\rangle^{\varkappa}} \\ &\geq |\langle k, \Omega_{\text{up}}(0)\rangle \ + \Lambda_a(0) - \Lambda_b(0)| - \chi(|k| + 2) \\ &\quad -\delta(|k| + 2) \geq \frac{\beta_4}{|k|^\tau} - 6\,|k|\,\chi, \end{aligned}$$

and this is $\geq \kappa$ unless

$$|k| \geq K \approx (\frac{\beta_3}{\chi})^{\frac{1}{\tau+1}}.$$

Recall that $\chi \geq \delta_0$, by convention, and that $\kappa \leq \delta_0$, because otherwise the lemma is trivial.

From now on we only consider $K \leq |k| \leq N$. By Assumption A2, there exists a unit vector $\mathfrak{z}$ such that

$$|\partial_{\mathfrak{z}} \langle k, \Omega(\rho)\rangle| \geq \delta_0.$$

Since $|k| \leq N$ and $|a|^2 - |b|^2$ are integers, it follows that (for any κ')

$$|\langle k, \Omega(\rho)\rangle \ + |a|^2 - |b|^2 \,| \geq 2\kappa'$$

for all a, b and all ρ outside a set of Lebesgue measure $\lesssim N^{\frac{\kappa'}{\delta_0}}$. Summing up over all $K \leq |k| \leq N$ gives a set Σ_1 of Lebesgue measure

$$\lesssim N^{\exp} \frac{\kappa'}{\delta_0}.$$

By (6.9) it follows that, for ρ outside of Σ_1,

$$|\langle k, \Omega(\rho)\rangle \ + \Lambda_a(\rho) - \Lambda_b(\rho)| \geq \kappa',$$

if just

$$|a|^{\beta_2} \geq 2\frac{c}{\kappa'}.$$

Then

$$|\langle k, \Omega(\rho)\rangle \ + \alpha(\rho) - \beta(\rho)| \geq \kappa' - 2\delta \frac{1}{\langle a\rangle^{\varkappa}}$$

which is $\geq \kappa$ if $\kappa' \geq 2\kappa$ and

$$|a|^{\varkappa} \geq 2(\frac{\delta}{\kappa'}).$$

Let

$$M = 2 \max\left(\left(\frac{c}{\kappa'}\right)^{\frac{1}{\beta_2}}, \left(\frac{\delta_0}{\kappa'}\right)^{\frac{1}{\varkappa}} \right).$$

Then it only remains to consider $[a]$ and $[b]$ with $|a| \leq M$ and $|b| \leq M + \mathrm{Ct}.N$. We have seen above that the the Lebesgue measure of each $\Sigma(L_{k,[a],[b]}, \kappa)$ is $\lesssim (|a|\,|b|)^d \frac{\kappa}{\delta_0}$. Summing up over all these a and b gives a set Σ_2 of Lebesgue measure

$$\lesssim N^{\exp} M^{2d} \frac{\kappa}{\delta_0}.$$

Suppose now that $[a]$ or $[b]$ is $= [0]$. Then $|a|$ and $|b|$ are $\lesssim c + N \lesssim N$. Summing up over all these a and b gives a set Σ_3 of Lebesgue measure

$$\lesssim N^{\exp} \frac{\kappa}{\delta_0}.$$

The union of Σ_1, Σ_2 and Σ_3 has Lebesgue measure

$$\lesssim N^{\exp}\left(\frac{\kappa'}{\delta_0}+M^{4d}\frac{\kappa}{\delta_0}\right)\lesssim N^{\exp}\left(\frac{\kappa'}{\delta_0}+\left(\frac{1}{\kappa'}\right)^{\theta}\frac{\kappa}{\delta_0}\right)\quad \theta=4d\left(\frac{1}{\beta_2}+\frac{1}{\varkappa}\right).$$

Take now $\kappa'=\kappa^{\frac{1}{1+\theta}}$ and observe that $N\chi^{\frac{1}{\tau}}\gtrsim 1$ (because $N\geq K$). Then the bound becomes

$$\lesssim N^{\exp}\left(\frac{\kappa}{\delta_0}\right)^{\frac{1}{1+\theta}}\left(\frac{\chi}{\delta_0}\right)^{\frac{\theta}{1+\theta}}$$

(with a new and larger exponent exp). □

8 Homological Equation

Let h be a normal form Hamiltonian (6.19),

$$h(r,w,\rho)=\langle\Omega(\rho),r\rangle+\frac{1}{2}\langle w,A(\rho)w\rangle\in\mathcal{NF}_{\varkappa}(\Delta,\delta)$$

– recall the convention (6.18)—and assume $\varkappa>0$ and

$$\delta\leq\frac{1}{C}c', \tag{8.1}$$

where C is to be determined. Let

$$\gamma=(\gamma,m_*+\varkappa)\geq\gamma_*=(0,m_*+\varkappa).$$

Remark 8.1. Notice the abuse of notations here. It will be clear from the context when γ is a two-vector, like in $\|\cdot\|_{\gamma,\varkappa}$, and when it is a scalar, like in $e^{\gamma d}$.

Let $f\in\mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$. In this section we shall construct a jet-function S that solves the *non-linear*[25] *homological equation*

$$\{h,S\}+\{f-f^T,S\}^T+f^T=0 \tag{8.2}$$

as good as possible—the reason for this will be explained in the beginning of the next section. In order to do this we shall start by analysing the *homological equation*

$$\{h,S\}+f^T=0. \tag{8.3}$$

We shall solve this equation modulo some "cokernel" and modulo an "error".

[25] "Non-linear" because the solution depends non-linearly on f.

8.1 Three components of the homological equation. Let us write

$$f^T(\theta, r, w) = f_r(r, \theta) + \langle f_w(\theta), w\rangle + \frac{1}{2}\langle f_{ww}(\theta)w, w\rangle$$

and recall that, by Proposition 2.8, $f^T \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma, \mu)$. Let

$$S(\theta, r, w) = S_r(r, \theta) + \langle S_w(\theta), w\rangle + \frac{1}{2}\langle S_{ww}(\theta)w, w\rangle,$$

where f_r and S_r are affine functions in r—here we have not indicated the dependence on ρ.

Then the Poisson bracket $\{h, S\}$ equals

$$\begin{aligned} -\big(&\partial_\Omega S_r(r, \theta) + \langle \partial_\Omega S_w(\theta), w\rangle + \frac{1}{2}\langle \partial_\Omega S_{ww}(\theta), w\rangle \\ &+\langle AJS_w(\theta), w\rangle + \frac{1}{2}\langle AJS_{ww}(\theta)w, w\rangle - \frac{1}{2}\langle S_{ww}(\theta)JAw, w\rangle \end{aligned}$$

where ∂_Ω denotes the derivative of the angles θ in direction Ω. Accordingly the homological Eq. (8.3) decomposes into three linear equations:

$$\begin{cases} \partial_\Omega S_r(r, \theta) = f_r(r, \theta), \\ \partial_\Omega S_w(\theta) - AJS_w(\theta) = f_w(\theta), \\ \partial_\Omega S_{ww}(\theta) - AJS_{ww}(\theta) + S_{ww}(\theta)JA = f_{ww}(\theta). \end{cases}$$

8.2 The first equation

LEMMA 8.2. *There exists constant C such that if (8.1) holds, then, for any $N \geq 1$ and $\kappa > 0$, there exists a closed set $\mathcal{D}_1 = \mathcal{D}_1(h, \kappa, N) \subset \mathcal{D}$, satisfying*

$$\operatorname{meas}(\mathcal{D}\backslash\mathcal{D}_1) \leq CN^{\exp}\frac{\kappa}{\delta_0}$$

and there exist $\mathcal{C}^{s_*}$ functions S_r and R_r on $\mathbb{C}^{\mathcal{A}} \times \mathbb{T}^{\mathcal{A}} \times \mathcal{D} \to \mathbb{C}$, real holomorphic in r, θ, such that for all $\rho \in \mathcal{D}_1$

$$\partial_{\Omega(\rho)} S_r(r, \theta, \rho) = f_r(r, \theta, \rho) - \hat{f}_r(r, 0, \rho) - R_r(\theta, \rho) \quad [26] \tag{8.4}$$

and for all $(r, \theta, \rho) \in \mathbb{C}^{\mathcal{A}} \times \mathbb{T}^{\mathcal{A}}_{\sigma'} \times \mathcal{D}$, $|r| < \mu$, $\sigma' < \sigma$, and $|j| \leq s_*$

$$|\partial^j_\rho S_r(r, \theta, \rho)| \leq C\frac{1}{\kappa(\sigma - \sigma')^n}\big(N\frac{\chi}{\kappa}\big)^{|j|}|f^T|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}, \tag{8.5}$$

$$|\partial^j_\rho R_r(r, \theta, \rho)| \leq C\frac{e^{-(\sigma-\sigma')N}}{(\sigma - \sigma')^n}|f^T|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}. \tag{8.6}$$

Moreover, $S_r(\cdot, \rho) = 0$ for ρ near the boundary of $\mathcal{D}$.

(The exponent exp *only depends on $n = \#\mathcal{A}$, and C is an absolute constant.)*

[26] $\hat{f}_r(r, 0, \rho)$ is the 0:th Fourier coefficient, or the mean value, of the function $\theta \mapsto f_r(r, \theta, \rho)$.

Proof. Written in Fourier components the Eq. (8.4) then becomes, for $k \in \mathbb{Z}^{\mathcal{A}}$,

$$L_k(\rho)\hat{S}(k) =: \langle k, \Omega(\rho)\rangle \hat{S}(k) = -\mathbf{i}(\hat{F}(k) - \hat{R}(k))$$

where we have written S, F and R for $S_r, (f_r - \hat{f}_r)$ and R_r respectively. Therefore (8.4) has the (formal) solution

$$S(r,\theta,\rho) = \sum \hat{S}(r,k,\rho)e^{\mathbf{i}\langle k,\theta\rangle} \quad \text{and} \quad R(r,\theta,\rho) = \sum \hat{F}(r,k,\rho)e^{\mathbf{i}\langle k,\theta\rangle}$$

with

$$\hat{S}(r,k,\rho) = \begin{cases} -L_k(\rho)^{-1}\mathbf{i}\hat{F}(r,k,\rho) & \text{if } 0 < |k| \le N, \\ 0 & \text{if not,} \end{cases}$$

and

$$\hat{R}(r,k,\rho) = \begin{cases} \hat{F}(r,k,\rho) & \text{if } |k| > N, \\ 0 & \text{if not.} \end{cases}$$

By Lemma 7.1

$$||(L_k(\rho))^{-1}|| \le \frac{1}{\kappa}$$

for all ρ outside some set $\Sigma(L_k, \kappa)$ such that

$$\operatorname{dist}(\mathcal{D}\backslash\Sigma(L_k,\kappa), \Sigma(L_k, \frac{\kappa}{2})) \ge \text{ct.}\frac{\kappa}{N\chi}$$

and

$$\mathcal{D}_1 = \mathcal{D}\backslash \bigcup_{0<|k|\le N} \Sigma(L_k,\kappa)$$

fulfils the estimate of the lemma.

For $\rho \notin \Sigma(L_k, \frac{\kappa}{2})$ we get

$$|\hat{S}(r,k,\rho)| \le \text{Ct.}\frac{1}{\kappa}|\hat{F}(r,k,\rho)|.$$

Differentiating the formula for $\hat{S}(r,k,\rho)$ once we obtain

$$\partial_\rho^j \hat{S}(r,k,\rho) = \left(-\frac{\mathbf{i}}{\langle\Omega,k\rangle}\partial_\rho^j \hat{F}(r,k,\rho) + \frac{\mathbf{i}}{\langle\Omega,k\rangle^2}\langle\partial_\rho^j\Omega,k\rangle\hat{F}(r,k,\rho)\right)$$

which gives, for $\rho \notin \Sigma(L_k, \frac{\kappa}{2})$,

$$|\partial_\rho^j \hat{S}(r,k,\rho)| \le \text{Ct.}\frac{1}{\kappa}\left(N\frac{\chi}{\kappa}\right)\max_{0\le l\le j}|\partial_\rho^l \hat{F}(r,k,\rho)|.$$

(Here we used that $|\partial_\rho\Omega(\rho)| \le \chi + \delta$.) The higher order derivatives are estimated in the same way and this gives

$$|\partial_\rho^j \hat{S}(r,k,\rho)| \le \text{Ct.}\frac{1}{\kappa}(N\frac{\chi}{\kappa})^{|j|}\max_{0\le l\le j}|\partial_\rho^l \hat{F}(r,k,\rho)|$$

for any $|j| \leq s_*$, where Ct. is an absolute constant.

By Lemma D.2, there exists a $\mathcal{C}^\infty$-function $g_k : \mathcal{D} \to \mathbb{R}$, being $= 1$ outside $\Sigma(L_k, \kappa)$ and $= 0$ on $\Sigma(L_k, \frac{\kappa}{2})$ and such that for all $j \geq 0$

$$|g_k|_{\mathcal{C}^j(\mathcal{D})} \leq (\mathrm{Ct.}\frac{N\chi}{\kappa})^j.$$

Multiplying $\hat{S}(r,k,\rho)$ with $g_k(\rho)$ gives a $\mathcal{C}^{s_*}$-extension of $\hat{S}(r,k,\rho)$ from $\mathcal{D}\backslash\Sigma(L_k,\kappa)$ to $\mathcal{D}$ satisfying the same bound (8.5).

It follows now, by a classical argument, that the formal solution converges and that $|\partial_\rho^j S(r,\theta,\rho)|$ and $|\partial_\rho^j R(r,\theta,\rho)|$ fulfils the estimates of the lemma. When summing up the series for $|\partial_\rho^j R(r,\theta,\rho)|$ we get a term $e^{-\frac{1}{C}(\sigma-\sigma')N}$ (because of truncation of Fourier modes), but the factor $\frac{1}{C}$ disappears by replacing N by CN.

By construction S and R solve Eq. (8.4) for any $\rho \in \mathcal{D}_1$.

If we multiply $\hat{S}(r,k,\rho)$ by a second $\mathcal{C}^\infty$ cut-off function $h_k : \mathcal{D} \to \mathbb{R}$—which is $= 1$ at a distance $\geq \frac{\kappa}{N\chi}$ from the boundary of $\mathcal{D}$ and $= 0$ near this boundary—then the new function will satisfy the bound (8.5), it will solve the Eq. (8.4) on a new domain, smaller but still satisfying the measure bound of the Lemma, and it will vanish near the boundary of $\mathcal{D}$. □

8.3 The second equation. Concerning the second component of the homological equation we have

LEMMA 8.3. *There exists an absolute constant C such that if (8.1) holds, then, for any $N \geq 1$ and*

$$0 < \kappa \leq c',$$

there exists a closed set $\mathcal{D}_2 = \mathcal{D}_2(h,\kappa,N) \subset \mathcal{D}$, satisfying

$$\mathrm{meas}(\mathcal{D}\backslash\mathcal{D}_2) \leq CN^{\exp}\left(\frac{\kappa}{\delta_0}\right)^{\frac{1}{s_*}},$$

and there exist $\mathcal{C}^{s_*}$-functions S_w and $R_w : \mathbb{T}^{\mathcal{A}} \times \mathcal{D} \to Y_\gamma$, real holomorphic in θ, such that for $\rho \in \mathcal{D}_2$

$$\partial_{\Omega(\rho)} S_w(\theta,\rho) - A(\rho) J S_w(\theta,\rho) = f_w(\theta,\rho) - R_w(\theta,\rho) \tag{8.7}$$

and for all $(\theta,\rho) \in \mathbb{T}^{\mathcal{A}}_{\sigma'} \times \mathcal{D}$, $\sigma' < \sigma$, and $|j| \leq s_*$

$$||\partial_\rho^j S_w(\theta,\rho)||_\gamma \leq C\frac{1}{\kappa(\sigma-\sigma')^n}\left(N\frac{\chi}{\kappa}\right)^{|j|}|f^T|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}, \tag{8.8}$$

$$||\partial_\rho^j R_w(\theta,\rho)||_\gamma \leq C\frac{e^{-(\sigma-\sigma')N}}{(\sigma-\sigma')^n}|f^T|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}. \tag{8.9}$$

Moreover, $S_w(\cdot,\rho) = 0$ for ρ near the boundary of $\mathcal{D}$.

(The exponent exp *only depends on d and $\#\mathcal{A}$. C is an absolute constant that depends on c. C also depend on $\sup_{\mathcal{D}}|\Omega_{up}|$ and $\sup_{\mathcal{D}}|H_{up}|$, but stays bounded when these do.)*

Proof. Let us re-write (8.7) in the complex variables $, z = (\xi\eta)$ described in Sect. 6.2. The quadratic form $(1/2)\langle w, A(\rho)w\rangle$ gets transformed, by $w = Uz$, to

$$\langle \xi, Q(\rho)\eta\rangle + \frac{1}{2}\langle z_{\mathcal{F}}, H'(\rho)z_{\mathcal{F}}\rangle,$$

where Q' is a Hermitian matrix and H' is a real symmetric matrix. Then we make in (8.7) the substitution $S = {}^tUS_w$, $R = {}^tUR_w$ and $F = {}^tUf_w$, where $S = {}^t(S_\xi, S_\eta, S_{\mathcal{F}})$, etc. In this notation Eq. (8.7) decouples into the equations

$$\begin{aligned}
\partial_\Omega S_\xi + \mathbf{i}QS_\xi &= F_\xi - R_\xi,\\
\partial_\Omega S_\eta - \mathbf{i}{}^tQS_\eta &= F_\eta - R_\eta,\\
\partial_\Omega S_{\mathcal{F}} - HJS_{\mathcal{F}} &= F_{\mathcal{F}} - R_{\mathcal{F}}.
\end{aligned}$$

Let us consider the first equation. Written in the Fourier components it becomes

$$(\langle k, \Omega(\rho)\rangle I + Q)\hat{S}_\xi(k) = -\mathbf{i}(\hat{F}_\xi(k) - \hat{R}_\xi(k)). \tag{8.10}$$

This equation decomposes into its "components" over the blocks $[a] = [a]_\Delta$ and takes the form

$$L_{k,[a]}(\rho)\hat{S}_{[a]}(k) =: (\langle k, \Omega(\rho)\rangle + Q_{[a]})\hat{S}_{[a]}(k) = -\mathbf{i}(\hat{F}_{[a]}(k) - \hat{R}_{[a]}(k)) \tag{8.11}$$

– the matrix $Q_{[a]}$ being the restriction of Q_ξ to $[a] \times [a]$, the vector $F_{[a]}$ being the restriction of F_ξ to $[a]$ etc.

Equation (8.11) has the (formal) solution

$$\hat{S}_{[a]}(k, \rho) = \begin{cases} -(L_{k,[a]}(\rho))^{-1}\mathbf{i}\hat{F}_{[a]}(k, \rho) & \text{if } |k| \le N,\\ 0 & \text{if not} \end{cases}$$

and

$$\hat{R}_a(k, \rho) = \begin{cases} \hat{F}_a(k, \rho) & \text{if } |k| > N,\\ 0 & \text{if not.} \end{cases}$$

For $k \neq 0$, by Lemma 7.2,

$$||(L_{k,[a]}(\rho))^{-1}|| \le \frac{1}{\kappa}$$

for all ρ outside some set $\Sigma(L_{k,[a]}, \kappa)$ such that

$$\operatorname{dist}\left(\mathcal{D}\backslash\Sigma\left(L_{k,[a]}, \kappa\right), \Sigma\left(L_{k,[a]}, \frac{\kappa}{2}\right)\right) \ge \text{ct.}\frac{\kappa}{N\chi}$$

and

$$\mathcal{D}_2 = \mathcal{D}\backslash \bigcup_{\substack{0<|k|\le N\\ [a]}} \Sigma_{k,[a]}(\kappa),$$

fulfils the required estimate.

For $k=0$, it follows by (8.1) and (6.10) that

$$||(L_{k,[a]}(\rho))^{-1}|| \leq \frac{1}{c'} \leq \frac{2}{\kappa}\,.$$

We then get, as in the proof of Lemma 8.2, that $\hat{S}_{[a]}(k,\cdot)$ and $\hat{R}_{[a]}(k,\cdot)$ have $\mathcal{C}^{s_*}$-extension to $\mathcal{D}$ satisfying

$$||\partial_\rho^j \hat{S}_{[a]}(k,\rho)|| \leq \text{Ct.}\frac{1}{\kappa}\left(N\frac{\chi}{\kappa}\right)^{|j|} \max_{0\leq l\leq j} ||\partial_\rho^l \hat{F}_{[a]}(k,\rho)||$$

and

$$||\partial_\rho^j R_{[a]}(k,\rho)|| \leq \text{Ct.}||\partial_\rho^j \hat{F}_{[a]}(k,\rho)||,$$

and satisfying (8.11) for $\rho\in\mathcal{D}_2$.

These estimates imply that

$$||\partial_\rho^j \hat{S}_\xi(k,\rho)||_\gamma \leq \text{Ct.}\frac{1}{\kappa}\left(N\frac{\chi}{\kappa}\right)^{|j|} \max_{0\leq l\leq j} ||\partial_\rho^l \hat{F}_\xi(k,\rho)||_\gamma$$

and

$$||\partial_\rho^j R_\xi(k,\rho)||_\gamma \leq \text{Ct.}||\partial_\rho^j F_\xi(k,\rho)||_\gamma.$$

Summing up the Fourier series, as in Lemma 8.2, we get

$$||\partial_\rho^j S_\xi(\theta,\rho)||_\gamma \leq \text{Ct.}\frac{1}{\kappa(\sigma-\sigma')^n}\left(N\frac{\chi}{\kappa}\right)^{|j|} \max_{0\leq l\leq j} \sup_{|\Im\theta|<\sigma} ||\partial_\rho^l F_\xi(\cdot,\rho)||_\gamma$$

and

$$||\partial_\rho^j R_\xi(\theta,\rho)||_\gamma \leq \text{Ct.}\frac{e^{-\frac{1}{\text{Ct.}}(\sigma-\sigma')N}}{(\sigma-\sigma')^n} \sup_{|\Im\theta|<\sigma} ||\partial_\rho^j F_\xi(\cdot,\rho)||_\gamma$$

for $(\theta,\rho)\in\mathbb{T}^{\mathcal{A}}_{\sigma'}\times\mathcal{D}$, $0<\sigma'<\sigma$, and $|j|\leq s_*$. This implies the estimates (8.8) and (8.9)—the factor $\frac{1}{\text{Ct.}}$ disappears by replacing N by $\text{Ct.}N$.

The other two equations are treated in exactly the same way. □

8.4 The third equation. Concerning the third component of the homological equation, (8.3), we have the following result.

LEMMA 8.4. *There exists an absolute constant C such that if (8.1) holds, then, for any $N\geq 1$, $\Delta'\geq\Delta\geq 1$, and*

$$\kappa \leq \frac{1}{C}c',$$

there exist a closed subset $\mathcal{D}_3=\mathcal{D}_3(h,\kappa,N)\subset\mathcal{D}$, satisfying

$$\text{meas}(\mathcal{D}\backslash\mathcal{D}_3) \leq C(\Delta N)^{\exp_1}\left(\frac{\kappa}{\delta_0}\right)^{\alpha}\left(\frac{\chi}{\delta_0}\right)^{1-\alpha}$$

and there exist real $\mathcal{C}^{s_*}$-functions $B_{ww} : \mathcal{D} \to \mathcal{M}_{\gamma,\varkappa} \cap \mathcal{NF}_{\Delta'}$ and S_{ww}, $R_{ww} = R^F_{ww} + R^s_{ww} : \mathbb{T}^{\mathcal{A}} \times \mathcal{D} \to \mathcal{M}_{\gamma,\varkappa}$, real holomorphic in θ, such that for all $\rho \in \mathcal{D}_3$

$$\begin{aligned} &\partial_{\Omega(\rho)} S_{ww}(\theta,\rho) - A(\rho) J S_{ww}(\theta,\rho) + S_{ww}(\theta,\rho) J A(\rho) \\ &\quad = f_{ww}(\theta,\rho) - B_{ww}(\rho) - R_{ww}(\theta,\rho) \end{aligned} \tag{8.12}$$

and for all $(\theta,\rho) \in \mathbb{T}^{\mathcal{A}}_{\sigma'} \times \mathcal{D}$, $\sigma' < \sigma$, and $|j| \le s_*$

$$\left\|\partial^j_\rho S_{ww}(\theta,\rho)\right\|_{\gamma,\varkappa} \le C\Delta' \frac{\Delta^{\exp_2} e^{2\gamma d_\Delta}}{\kappa(\sigma-\sigma')^n} \left(N\frac{\chi+\delta}{\kappa}\right)^{|j|} \left|f^T\right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}, \tag{8.13}$$

$$\left\|\partial^j_\rho B_{ww}(\rho)\right\|_{\gamma',\varkappa} \le C\Delta'\Delta^{\exp_2} \left|f^T\right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}, \tag{8.14}$$

and

$$\begin{cases} \left\|\partial^j_\rho R^F_{ww}(\theta,\rho)\right\|_{\gamma,\varkappa} \le C\Delta'\Delta^{\exp_2} \left(\frac{e^{-(\sigma-\sigma')N}}{(\sigma-\sigma')^n}\right) \left|f^T\right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}} \\ \left\|\partial^j_\rho R^s_{ww}(\theta,\rho)\right\|_{\gamma',\varkappa} \le C\Delta'\Delta^{\exp_2} e^{-(\gamma-\gamma')\Delta'} \left|f^T\right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}} \end{cases}, \tag{8.15}$$

for any $\gamma_* \le \gamma' \le \gamma$.

Moreover, $S_{ww}(\cdot,\rho) = 0$ for ρ near the boundary of $\mathcal{D}$.

The exponent α is a positive constant only depending on $d, s_*, \varkappa$ and β_2.[27] .

(The exponent $\exp_1$ only depends on d, $n = \#\mathcal{A}$ and $\tau, \beta_2, \varkappa$. The exponent $\exp_2$ only depends on d, m_*, s_*. C is an absolute constant that depends on $c, \tau, \beta_2, \beta_3$ and $\varkappa$. C also depend on $\sup_{\mathcal{D}} |\Omega_{up}|$ and $\sup_{\mathcal{D}} |H_{up}|$, but stays bounded when these do.)

Proof. It is also enough to find complex solutions S_{ww}, R_{ww} and B_{ww} verifying the estimates, because then their real parts will do the job.

As in the previous section, and using the same notation, we re-write (8.12) in complex variables. So we introduce $S = {}^tUS_{\zeta,\zeta}U$, $R = {}^tUR_{\zeta,\zeta}U$, $B = {}^tUB_{\zeta,\zeta}U$ and $F = {}^tUJf_{\zeta,\zeta}U$. In appropriate notation (8.12) decouples into the equations

$$\begin{aligned} &\partial_\Omega S_{\xi\xi} + \mathbf{i}QS_{\xi\xi} + \mathbf{i}S_{\xi\xi}\ {}_tQ = F_{\xi\xi} - B_{\xi\xi} - R_{\xi\xi}, \\ &\partial_\Omega S_{\xi\eta} + \mathbf{i}QS_{\xi\eta} - \mathbf{i}S_{\xi\eta}Q = F_{\xi\eta} - B_{\xi\eta} - R_{\xi\eta}, \\ &\partial_\Omega S_{\xi z_\mathcal{F}} + \mathbf{i}QS_{\xi z_\mathcal{F}} + S_{\xi z_\mathcal{F}}\ JH = F_{\xi z_\mathcal{F}} - B_{\xi\xi} - R_{\xi z_\mathcal{F}}, \\ &\partial_\Omega S_{z_\mathcal{F} z_\mathcal{F}} + HJS_{z_\mathcal{F} z_\mathcal{F}} - S_{z_\mathcal{F} z_\mathcal{F}} JH = F_{z_\mathcal{F} z_\mathcal{F}} - B_{z_\mathcal{F} z_\mathcal{F}} - R_{z_\mathcal{F} z_\mathcal{F}}, \end{aligned}$$

and equations for $S_{\eta\eta}, S_{\eta\xi}, S_{z_\mathcal{F}\xi}, S_{\eta z_\mathcal{F}}, S_{z_\mathcal{F}\eta}$. Since those latter equations are of the same type as the first four, we shall concentrate on these first.

First equation. Written in the Fourier components it becomes

$$(\langle k, \Omega(\rho)\rangle I + Q)\hat{S}_{\xi\xi}(k) + \hat{S}_{\xi\xi}(k)^t Q = -\mathbf{i}(\hat{F}_{\xi\xi}(k) - \delta_{k,0}B - \hat{R}_{\xi\xi}(k)). \tag{8.16}$$

[27] α is the exponent of Lemma 7.3

This equation decomposes into its "components" over the blocks $[a] \times [b]$, $[a] = [a]_\Delta$, and takes the form

$$L(k,[a],[b],\rho)\hat{S}^{[b]}_{[a]}(k) =: \langle k, \Omega(\rho)\rangle\ \hat{S}^{[b]}_{[a]}(k) + Q_{[a]}(\rho)\hat{S}^{[b]}_{[a]}(k) + \hat{S}^{[b]}_{[a]}(k)\ {}^tQ_{[b]}(\rho) = -\mathbf{i}(\hat{F}^{[b]}_{[a]}(k,\rho) - \hat{R}^{[b]}_{[a]}(k) - \delta_{k,0}B^{[b]}_{[a]}), \quad (8.17)$$

– the matrix $Q_{[a]}$ being the restriction of $Q_{\xi\xi}$ to $[a] \times [a]$, the vector $F^{[b]}_{[a]}$ being the restriction of $F_{\xi\xi}$ to $[a] \times [b]$ etc.

Equation (8.17) has the (formal) solution:

$$\hat{S}^{[b]}_{[a]}(k,\rho) = \begin{cases} -L(k,[a],[b],\rho)^{-1}\mathbf{i}\hat{F}^{[b]}_{[a]}(k,\rho) & \text{if } \operatorname{dist}([a],[b]) \le \Delta' \text{ and } |k| \le N, \\ 0 & \text{if not,} \end{cases}$$

$B^{[b]}_{[a]} = 0$ and

$$\hat{R}^b_a(k,\rho) = \begin{cases} \hat{F}^b_a(k,\rho) & \text{if } \operatorname{dist}([a],[b]) \ge \Delta' \text{ or } |k| > N, \\ 0 & \text{if not.} \end{cases}$$

We denote $\hat{R}^b_a(k,\rho)$ by $\widehat{(R^s)}^b_a(k,\rho)$ if $\operatorname{dist}([a],[b]) \ge \Delta'$—truncation off "diagonal" in space modes—and by $\widehat{(R^F)}^b_a(k,\rho)$ if $|k| > N$ —truncation in Fourier modes.

For $k \neq 0$, by Lemma 7.3,

$$||(L_{k,[a],[b]}(\rho))^{-1}|| \le \frac{1}{\kappa}$$

for all ρ outside some set $\Sigma_{k,[a],[b]}(\kappa)$ such that

$$\operatorname{dist}(\mathcal{D}\backslash\Sigma_{k,[a],[b]}(\kappa), \Sigma_{k,[a],[b]}(\frac{\kappa}{2})) \ge \text{ct.}\frac{\kappa}{N\chi},$$

and

$$\mathcal{D}_3 = \mathcal{D}\backslash \bigcup_{\substack{0<|k|\le N \\ [a],[b]}} \Sigma_{k,[a],[b]}(\kappa)$$

fulfils the required estimate. For $k = 0$, it follows by (8.1) and (6.11) that

$$||(L_{k,[a],[b]}(\rho))^{-1}|| \le \frac{1}{c'} \le \frac{1}{\kappa}.$$

We then get, as in the proof of Lemma 8.2, that $\hat{S}^{[b]}_{[a]}(k,\cdot)$ and $\hat{R}^{[b]}_{[a]}(k,\cdot)$ have $\mathcal{C}^{s_*}$-extension to $\mathcal{D}$ satisfying

$$||\partial^j_\rho \hat{S}^{[b]}_{[a]}(k,\rho)|| \le \text{Ct.}\frac{1}{\kappa}\left(N\frac{\chi}{\kappa}\right)^{|j|} \max_{0\le l\le j} ||\partial^l_\rho \hat{F}^{[b]}_{[a]}(k,\rho)||$$

and

$$||\partial_\rho^j R_a^b(k,\rho)|| \leq \text{Ct.}||\partial_\rho^j \hat{F}_a^b(k,\rho)||,$$

and satisfying (8.17) for $\rho \in \mathcal{D}_3$.

These estimates imply that, for any $\gamma_* \leq \gamma' \leq \gamma$,

$$||\partial_\rho^j \hat{S}_{\xi\xi}(k,\rho)||_{\mathcal{B}(Y_{\gamma'},Y_{\gamma'})} \leq \text{Ct.}\Delta' \frac{\Delta^{\exp} e^{2\gamma d_\Delta}}{\kappa} \left(N\frac{\chi}{\kappa}\right)^{|j|} \max_{0\leq l\leq j} ||\partial_\rho^l \hat{F}_{\xi\xi}(k,\rho)||_{\mathcal{B}(Y_{\gamma'},Y_{\gamma'})}$$

and

$$||\partial_\rho^j \hat{R}_{\xi\xi}(k,\rho)||_{\mathcal{B}(Y_{\gamma'},Y_{\gamma'})} \leq \text{Ct.}\Delta'\Delta^{\exp}||\partial_\rho^j \hat{F}_{\xi\xi}(k,\rho)||_{\mathcal{B}(Y_{\gamma'},Y_{\gamma'})}.$$

The factor $\Delta^{\exp} e^{2\gamma d_\Delta}$ occurs because the diameter of the blocks $\leq d_\Delta$ interferes with the exponential decay and influences the equivalence between the l^1-norm and the operator-norm. The factor $\Delta'\Delta^{\exp}$ occurs because the truncation $\lesssim \Delta' + d_\Delta$ of diagonal influences the equivalence between the sup-norm and the operator-norm.

The estimates of the "block components" also gives estimates for the matrix norms and, for any $\gamma_* \leq \gamma' \leq \gamma$,

$$||\partial_\rho^j \hat{S}_{\xi\xi}(k,\rho)||_{\gamma,\varkappa} \leq \text{Ct.}\Delta' \frac{\Delta^{\exp} e^{2\gamma d_\Delta}}{\kappa} \left(N\frac{\chi}{\kappa}\right)^{|j|} \max_{0\leq l\leq j} ||\partial_\rho^l \hat{F}_{\xi\xi}(k,\rho)||_{\gamma,\varkappa}$$

and

$$||\partial_\rho^j R_{\xi\xi}(k,\rho)||_{\gamma,\varkappa} \leq \text{Ct.}||\partial_\rho^j F_{\xi\xi}(k,\rho)||_{\gamma,\varkappa}.$$

Summing up the Fourier series, as in Lemma 8.3, we get that $S_{\xi\xi}(\theta,\rho)$ satisfies the estimate (8.13). $R_{\xi\xi}(\theta,\rho)$ decompose naturally into a sum of a factor $R^F_{\xi\xi}(\theta,\rho)$, which is truncated in Fourier modes and therefore satisfies the first estimate of (8.15), and a factor $R^s_{\xi\xi}(\theta,\rho)$, which is truncated in off "diagonal" in space modes and therefore satisfies the second estimate of (8.15).

The third equation. We write the equation in Fourier components and decompose it into its "components" on each product block $[a]\times[b]$, $[b] = \mathcal{F}$:

$$\begin{aligned} L(k,[a],[b],\rho)\hat{S}^{[b]}_{[a]}(k) &:= \langle k,\Omega(\rho)\rangle\ \hat{S}^{[b]}_{[a]}(k) + Q_{[a]}(\rho)\hat{S}^{[b]}_{[a]}(k) \\ -\mathbf{i}\hat{S}^{[b]}_{[a]}(k)JH(\rho) &= -\mathbf{i}(\hat{F}^{[b]}_{[a]}(k,\rho) - \delta_{k,0}B^{[b]}_{[a]} - \hat{R}^{[b]}_{[a]}(k)) \end{aligned}$$

– here we have suppressed the upper index $\xi z_{\mathcal{F}}$.

The formal solution is the same as in the previous case and it converges to functions verifying (8.13) and (8.15), by Lemma 7.3, and by (6.12).

The Fourth Equation We write the equation in Fourier components:

$$\begin{aligned} L(k,[a],[b],\rho)\hat{S}^{[b]}_{[a]}(k) &:= \langle k,\Omega(\rho)\rangle\ \hat{S}^{[b]}_{[a]}(k) - \mathbf{i}HJ(\rho)\hat{S}^{[b]}_{[a]}(k) \\ +\mathbf{i}\hat{S}^{[b]}_{[a]}(k)JH(\rho) &= -\mathbf{i}(\hat{F}^{[b]}_{[a]}(k,\rho) - \delta_{k,0}B^{[b]}_{[a]} - \hat{R}^{[b]}_{[a]}(k)), \end{aligned}$$

where $[a] = [b] = \mathcal{F}$—here we have suppressed the upper index $z_{\mathcal{F}} z_{\mathcal{F}}$.

The equation is solved (formally) by

$$\hat{S}^{[b]}_{[a]}(k,\rho) = \begin{cases} -L(k,[a],[b],\rho)^{-1}\mathbf{i}\hat{F}^{[b]}_{[a]}(k,\rho) & \text{if } 0 < |k| \le N, \\ 0 & \text{if not,} \end{cases}$$

$$\hat{R}^{[b]}_{[a]}(k,\rho) = \begin{cases} \hat{F}^{[b]}_{[a]}(k,\rho) & \text{if } |k| > N, \\ 0 & \text{if not;} \end{cases}$$

and

$$B^{[b]}_{[a]}(\rho) = \hat{F}^{[b]}_{[a]}(0,\rho).$$

The formal solution now converges to a solution verifying (8.13), (8.14) and (8.15) by Lemma 7.3. The factor R^s is here $= 0$.

The Second Equation We write the equation in Fourier components and decompose it into its "components" on each product block $[a] \times [b]$:

$$\begin{aligned} L(k,[a],[b],\rho)\hat{S}^{[b]}_{[a]}(k) =: \langle k, \Omega(\rho)\rangle\ \hat{S}^{[b]}_{[a]}(k) + Q_{[a]}(\rho)\hat{S}^{[b]}_{[a]}(k) \\ -\hat{S}^{[b]}_{[a]}(k)Q_{[b]}(\rho) = -\mathbf{i}\left(\hat{F}^{[b]}_{[a]}(k,\rho) - \hat{R}^{[b]}_{[a]}(k) - \delta_{k,0}B^{[b]}_{[a]}\right) \end{aligned}$$

– here we have suppressed the upper index $\xi\eta$. This equation is now solved (formally) by

$$S^{[b]}_{[a]}(\theta,\rho) = \sum \hat{S}^{[b]}_{[a]}(k,\rho)e^{\mathbf{i}k\cdot\theta} \quad \text{and} \quad R^{[b]}_{[a]}(\theta,\rho) = \sum \hat{R}^{[b]}_{[a]}(k,\rho)e^{\mathbf{i}k\cdot\theta},$$

with

$$\hat{S}^{[b]}_{[a]}(k,\rho) = \begin{cases} L(k,[a],[b],\rho)^{-1}\mathbf{i}\hat{F}^{[b]}_{[a]}(k,\rho) & \text{if } \operatorname{dist}([a],[b]) \le \Delta' \text{ and } 0 < |k| \le N, \\ 0 & \text{if not,} \end{cases}$$

$$B^{b}_{a}(\rho) = \begin{cases} \hat{F}^{b}_{a}(0,\rho) & \text{if } \operatorname{dist}([a],[b]) \le \Delta' \text{ and } k = 0, \\ 0 & \text{if not,} \end{cases}$$

and

$$\hat{R}^{b}_{a}(k,\rho) = \begin{cases} \hat{F}^{b}_{a}(k,\rho) & \text{if } \operatorname{dist}([a],[b]) \ge \Delta' \text{ or } |k| > N, \\ 0 & \text{if not.} \end{cases}$$

We denote again $\hat{R}^{b}_{a}(k,\rho)$ by $\widehat{(R^s)}^{b}_{a}(k,\rho)$ if $\operatorname{dist}([a],[b]) \ge \Delta'$ and by $\widehat{(R^F)}^{b}_{a}(k,\rho)$ if $|k| > N$.

We have to distinguish two cases, depending on when $k = 0$ or not.

The case $k \neq 0$.

We have, by Lemma 7.3,

$$||(L_{k,[a],[b]}(\rho))^{-1}|| \leq \frac{1}{\kappa}$$

for all ρ outside some set $\Sigma_{k,[a],[b]}(\kappa)$ such that

$$\operatorname{dist}\left(\mathcal{D}\backslash\Sigma_{k,[a],[b]}(\kappa), \Sigma_{k,[a],[b]}\left(\frac{\kappa}{2}\right)\right) \geq \text{ct.}\frac{\kappa}{N\chi},$$

and

$$\mathcal{D}_3 = \mathcal{D}\backslash \bigcup_{\substack{0<|k|\leq N\\ [a],[b]}} \Sigma_{k,[a],[b]}(\kappa)$$

fulfils the required estimate.

The case $k = 0$. In this case we consider the block decomposition $\mathcal{E}_{\Delta'}$ and we distinguish whether $|a| = |b|$ or not.

If $|a| > |b|$, we use (8.1) and (6.12) to get

$$|\alpha(\rho) - \beta(\rho)| \geq c' - \frac{\delta}{\langle a\rangle^{\varkappa}} - \frac{\delta}{\langle b\rangle^{\varkappa}} \geq \frac{c'}{2} \geq \kappa.$$

This estimate allows us to solve the equation by choosing

$$B^{[b]}_{[a]} = \hat{R}^{[b]}_{[a]}(0) = 0$$

and

$$\hat{S}^{[b]}_{[a]}(0,\rho) = L(0,[a],[b],\rho)^{-1}\hat{F}^{[b]}_{[a]}(0,\rho)$$

with

$$||\partial^j_\rho \hat{S}^{[b]}_{[a]}(0,\rho)|| \leq \text{Ct.}\frac{1}{\kappa}\left(N\frac{\chi}{\kappa}\right)^{|j|}\max_{0\leq l\leq j}\left\|\partial^l_\rho \hat{F}^{[b]}_{[a]}(0,\rho)\right\|,$$

which implies (8.13).

If $|a| = |b|$, we cannot control $|\alpha(\rho) - \beta(\rho)|$ from below, so then we define

$$\hat{S}^{[b]}_{[a]}(0) = 0$$

and

$$\begin{aligned} B^b_a(\rho) = \hat{F}^b_a(0,\rho)), \quad \hat{R}^b_a(0) = 0 \quad &\text{for } [a]_{\Delta'} = [b]_{\Delta'}\\ \hat{R}^b_a(0,\rho) = \hat{F}^b_a(0,\rho) \quad B^b_a = 0, \quad &\text{for } [a]_{\Delta'} \neq [b]_{\Delta'}. \end{aligned}$$

Clearly R and B verify the estimates (8.15) and (8.14).

Hence, the formal solution converges to functions verifying (8.13), (8.14) and (8.15) by Lemma 7.3. Moreover, for $\rho \in \mathcal{D}'$, these functions are a solution of the fourth equation. □

8.5 The homological equation. For simplicity we shall restrict ourselves here to $\sigma, \mu, \gamma \leq 1$.

LEMMA 8.5. *There exists a constant C such that if (8.1) holds, then, for any $N \geq 1$, $\Delta' \geq \Delta \geq 1$ and*

$$\kappa \leq \frac{1}{C} c',$$

there exists a closed subset $\mathcal{D}' = \mathcal{D}(h, \kappa, N) \subset \mathcal{D}$, satisfying

$$\operatorname{meas}(\mathcal{D} \backslash \mathcal{D}') \leq C(\Delta N)^{\exp_1} \left(\frac{\kappa}{\delta_0}\right)^{\alpha} \left(\frac{\chi}{\delta_0}\right)^{1-\alpha}$$

and there exist real jet-functions $S, R = R^F + R^s \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma, \mu)$ and h_+ verifying, for $\rho \in \mathcal{D}'$,

$$\{h, S\} + f^T = h_+ + R, \tag{8.18}$$

and such that

$$h + h_+ \in \mathcal{NF}_{\varkappa}(\Delta', \delta_+)$$

and, for all $0 < \sigma' < \sigma$,

$$|h_+|^{\sigma',\mu}_{\gamma,\varkappa,\mathcal{D}} \leq X \left|f^T\right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}, \tag{8.19}$$

$$|S|^{\sigma',\mu}_{\gamma,\varkappa,\mathcal{D}} \leq \frac{1}{\kappa} X \left(N \frac{\chi}{\kappa}\right)^{s_*} \left|f^T\right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}, \tag{8.20}$$

and

$$\begin{cases} \left|R^F\right|^{\sigma',\mu}_{\gamma,\varkappa,\mathcal{D}} \leq X e^{-(\sigma-\sigma')N} \left|f^T\right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}, \\ |R^s|^{\sigma',\mu}_{\gamma',\varkappa,\mathcal{D}} \leq X e^{-(\gamma-\gamma')\Delta'} \left|f^T\right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}, \end{cases} \tag{8.21}$$

for $\gamma_* \leq \gamma' \leq \gamma$, where

$$X = C\Delta' \left(\frac{\Delta}{\sigma - \sigma'}\right)^{\exp_2} e^{2\gamma d_\Delta}.$$

Moreover, $S_r(\cdot, \rho) = 0$ for ρ near the boundary of $\mathcal{D}$.

The exponent α is a positive constant only depending on $d, s_*, \varkappa$ and β_2.

(The exponent $\exp_1$ only depends on d, $n = \#\mathcal{A}$ and $\tau, \beta_2, \varkappa$. The exponent $\exp_2$ only depends on d, m_*, s_*. C is an absolute constant that depends on $c, \tau, \beta_2, \beta_3$ and $\varkappa$. C also depend on $\sup_{\mathcal{D}} |\Omega_{up}|$ and $\sup_{\mathcal{D}} |H_{up}|$, but stays bounded when these do.)

REMARK 8.6. The estimates (8.19) provides an estimate of δ_+. Indeed, let $\frac{1}{2}\langle w, Bw\rangle$ denote the quadratic part of h_+. Then, for any $a, b \in [a]_{\Delta'}$,

$$\left|\partial_\rho^j B_a^b\right| \leq \frac{1}{C}||\partial_\rho^j B||_{(\gamma,m_*),\varkappa} e_{(\gamma,\varkappa),\varkappa}(a,b)^{-1} \leq \text{Ct.}(\Delta')^\varkappa \left|f^T\right|_{\gamma,\varkappa,\mathcal{D}}^{\sigma,\mu} \frac{1}{\langle a\rangle^\varkappa}$$

– recall the definition of the matrix norm (2.8) and of the exponential weight (2.5). By (8.19) this is

$$\leq \text{Ct.}(\Delta')^\varkappa \left|f^T\right|_{\gamma,\varkappa,\mathcal{D}}^{\sigma,\mu} \frac{1}{\langle a\rangle^\varkappa}.$$

Since $\#[a]_{\Delta'} \lesssim (\Delta')^{\exp}$ we get

$$||\partial_\rho^j B(\rho)_{[a]_{\Delta'}}|| \leq \text{Ct.}(\Delta')^{\exp} \left|f^T\right|_{\gamma,\varkappa,\mathcal{D}}^{\sigma,\mu} \frac{1}{\langle a\rangle^\varkappa}.$$

This gives the estimate

$$\delta_+ - \delta \leq \text{Ct.}(\Delta')^{\exp} \left|f^T\right|_{\gamma,\varkappa,\mathcal{D}}^{\sigma,\mu}.$$

Proof. The set $\mathcal{D}'$ will now be given by the intersection of the sets in the three previous lemmas of this section. We set

$$h_+(r,w) = \hat{f}_r(r,0) + \frac{1}{2}\langle w, Bw\rangle,$$

$$S(r,\theta,w) = S_r(\theta,r) + \langle S_w(\theta)w\rangle + \frac{1}{2}\langle S_{ww}(\theta)w, w\rangle,$$

and

$$R(r,\theta,w) = R_r(r,\theta) + \langle R_w(\theta), w\rangle + \frac{1}{2}\langle R_{ww}(\theta)w, w\rangle,$$

with $R_{ww} = R_{ww}^F + R_{ww}^s$. These functions also depend on $\rho \in \mathcal{D}$ and they verify equation (8.18) for $\rho \in \mathcal{D}'$.

If $x = (r,\theta,w) \in \mathcal{O}_{\gamma_*}(\sigma,\mu)$, then

$$|h_+(x)| \leq \left|f^T\right|_{\gamma,\varkappa,\mathcal{D}}^{\sigma,\mu} + \frac{1}{2}||Bw||_{\gamma_*}||w||_{\gamma_*}.$$

Since

$$\|B\|_{\gamma,\varkappa} \geq \|B\|_{\gamma_*,\varkappa} \geq \|B\|_{\mathcal{B}(Y_{\gamma_*},Y_{\gamma_*})}$$

it follows that

$$|h_+(x)| \leq \text{Ct.}\left|f^T\right|_{\gamma,\varkappa,\mathcal{D}}^{\sigma,\mu}.$$

We also have for any $x = (r,\theta,w) \in \mathcal{O}_{\gamma'}(\sigma,\mu)$, $\gamma_* \leq \gamma' \leq \gamma$,

$$||Jdh_+(x)||_{\gamma'} \leq \text{Ct.}\left|f^T\right|_{\gamma,\varkappa,\mathcal{D}}^{\sigma,\mu} + ||Bw||_{\gamma'}.$$

Since

$$\|B\|_{\gamma,\varkappa} \geq \|B\|_{\gamma',\varkappa} \geq \|B\|_{\mathcal{B}(Y_{\gamma'},Y_{\gamma'})}$$

it follows that

$$||Jdh_+(x)||_{\gamma'} \leq \text{Ct.} \left|f^T\right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}.$$

Finally $Jd^2h_+(x)$ equals JB which satisfies the required bound.

The estimates of the derivatives with respect to ρ are the same and obtained in the same way.

The functions $S(\theta,r,\zeta)$, $R^F(\theta,r,\zeta)$ and $R^s(\theta,r,\zeta)$ are estimated in the same way. □

8.6 The non-linear homological equation. The Eq. (8.2) can now be solved easily. We restrict ourselves again to $\sigma,\mu,\gamma \leq 1$.

PROPOSITION 8.7. *There exists a constant C such that for any*

$$h \in \mathcal{NF}_\varkappa(\Delta,\delta), \quad \delta \leq \frac{1}{C}c',$$

and for any

$$N \geq 1, \quad \Delta' \geq \Delta \geq 1, \quad \kappa \leq \frac{1}{C}c'$$

there exists a closed subset $\mathcal{D}' = \mathcal{D}(h,\kappa,N) \subset \mathcal{D}$, satisfying

$$\text{meas}(\mathcal{D}\backslash\mathcal{D}') \leq C(\Delta N)^{\exp_1}\left(\frac{\kappa}{\delta_0}\right)^{\alpha}\left(\frac{\chi}{\delta_0}\right)^{1-\alpha},$$

and, for any $f \in \mathcal{T}_{\gamma,\varkappa}(\sigma,\mu,\mathcal{D})$

$$\varepsilon = \left|f^T\right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}} \quad \text{and} \quad \xi = |f|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}},$$

there exist real jet-functions $S, R = R^F + R^s \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$ and h_+ verifying, for $\rho \in \mathcal{D}'$,

$$\{h,S\} + \{f - f^T, S\}^T + f^T = h_+ + R \tag{8.22}$$

and such that

$$h + h_+ \in \mathcal{NF}_\varkappa(\Delta',\delta_+)$$

and, for all $\sigma' < \sigma$ and $\mu' < \mu$,

$$|h_+|^{\sigma',\mu'}_{\gamma,\varkappa,\mathcal{D}} \leq CXY\varepsilon, \tag{8.23}$$

$$|S|^{\sigma',\mu'}_{\gamma,\varkappa,\mathcal{D}} \leq C\frac{1}{\kappa}XY\varepsilon \tag{8.24}$$

and

$$\begin{cases} |R^F|^{\sigma',\mu'}_{\gamma,\varkappa,\mathcal{D}} \le Ce^{-(\sigma-\sigma')N}XY\varepsilon, \\ |R^s|^{\sigma',\mu'}_{\gamma',\varkappa,\mathcal{D}} \le Ce^{-(\gamma-\gamma')\Delta'}XY\varepsilon, \end{cases} \tag{8.25}$$

for $\gamma_* \le \gamma' \le \gamma$, *where*

$$X = \left(\frac{N\Delta' e^{\gamma d_\Delta}}{(\sigma-\sigma')(\mu-\mu')}\right)^{\exp_2}$$

and

$$Y = \left(\frac{\chi+\xi}{\kappa}\right)^{4s_*+3}.$$

Moreover, $S_r(\cdot,\rho) = 0$ *for* ρ *near the boundary of* $\mathcal{D}$.

Moreover, if $\tilde\rho = (0,\rho_2,\dots,\rho_p)$ *and* $f^T(\cdot,\tilde\rho) = 0$ *for all* $\tilde\rho$, *then* $S = R = 0$ *and* $h_+ = h$ *for all* $\tilde\rho$.

The exponent α *is a positive constant only depending on* $d, s_*, \varkappa$ *and* β_2.

(The exponent $\exp_1$ *only depends on* d, $n = \#\mathcal{A}$ *and* $\tau,\beta_2,\varkappa$. *The exponent* $\exp_2$ *only depends on* d, m_*, s_*. C *is an absolute constant that depends on* c,τ,β_2,β_3 *and* $\varkappa$. C *also depend on* $\sup_{\mathcal{D}}|\Omega_{up}|$ *and* $\sup_{\mathcal{D}}|H_{up}|$, *but stays bounded when these do.)*

REMARK 8.8. Notice that the "loss" of S with respect to κ is of "order" $4s_*+4$. However, if χ, δ and $\xi = |f|^{\sigma,\mu}_{\gamma',\varkappa,\mathcal{D}}$ are of size $\lesssim \kappa$, then the loss is only of "order" 1.

Proof. Let $S = S_0+S_1+S_2$ be a jet-function such that S_1 starts with terms of degree 1 in r,w and S_2 starts with terms of degree 2 in r,w—jet functions are polynomials in r,w and we give (as is usual) w degree 1 and r degree 2.

Let now $\sigma' = \sigma_5 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < \sigma_0 = \sigma$ be a (finite) arithmetic progression, i.e. $\sigma_j - \sigma_{j+1}$ do not depend on j, and let and $\mu' = \mu_5 < \mu_4 < \mu_3 < \mu_2 < \mu_1 < \mu_0 = \mu$ be another arithmetic progressions.

Then $\{h',S\} + \{f-f^T,S\}^T + f^T = h_+ + R$ decomposes into three homological equations

$$\{h',S_0\} + f^T = (h_+)_0 + R_0,$$

$$\{h',S_1\} + f_1^T = (h_+)_1 + R_1, \quad f_1 = \{f-f^T,S_0\},$$

$$\{h',S_2\} + f_2^T = (h_+)_2 + R_2, \quad f_2 = \{f-f^T,S_1\}.$$

By Lemma 8.5 we have for the first equation

$$|(h_+)_0|^{\sigma_1,\mu}_{\gamma,\varkappa,\mathcal{D}} \le X\varepsilon, \quad |S_0|^{\sigma_1,\mu}_{\gamma,\varkappa,\mathcal{D}} \le \frac{1}{\kappa}XY\varepsilon$$

where

$$X = C\Delta' \left(\frac{5\Delta}{\sigma - \sigma'} \right)^{\exp} e^{2\gamma_1 d_\Delta}.$$

and where Y, Z are defined by the right hand sides in the estimates (8.20) and (8.21).

By Proposition 2.9 we have

$$\xi_1 = |f_1|^{\sigma_2,\mu_2}_{\gamma,\varkappa,\mathcal{D}} \leq \frac{1}{\kappa} XYW\xi\varepsilon$$

where

$$W = C \left(\frac{5}{(\sigma - \sigma')} + \frac{5}{(\mu - \mu')} \right).$$

By Proposition 2.8 $\varepsilon_1 = \left|f_1^T\right|^{\sigma_2,\mu_2}_{\gamma,\varkappa,\mathcal{D}}$ satisfies the same bound as ξ_1.

By Lemma 8.5 we have for the second equation

$$|(h_+)_1|^{\sigma_3,\mu_2}_{\gamma,\varkappa,\mathcal{D}} \leq X\varepsilon_1, \quad |S_1|^{\sigma_3,\mu_2}_{\gamma,\varkappa,\mathcal{D}} \leq \frac{1}{\kappa} XY\varepsilon_1.$$

By Propositions 2.8 and 2.9 we have

$$\xi_2 = |f_2|^{\sigma_4,\mu_4}_{\gamma,\varkappa,\mathcal{D}} \leq \frac{1}{\kappa} XYW\xi_1\varepsilon_1,$$

and $\varepsilon_2 = \left|f_2^T\right|^{\sigma_4,\mu_4}_{\gamma,\varkappa,\mathcal{D}}$ satisfies the same bound.

By Lemma 8.5 we have for the third equation

$$|(h_+)_2|^{\sigma_5,\mu_4}_{\gamma,\varkappa,\mathcal{D}} \leq X\varepsilon_2, \quad |S_2|^{\sigma_5,\mu_4}_{\gamma,\varkappa,\mathcal{D}} \leq \frac{1}{\kappa} XY\varepsilon_2.$$

Putting this together we find that

$$\varepsilon + \varepsilon_1 + \varepsilon_2 \leq (1 + \frac{1}{\kappa} XYW\xi)^3 \varepsilon = T\varepsilon$$

and

$$|h_+|^{\sigma',\mu'}_{\gamma,\varkappa,\mathcal{D}} \leq XT\varepsilon, \quad |S|^{\sigma',\mu'}_{\gamma,\varkappa,\mathcal{D}} \leq \frac{1}{\kappa} XYT\varepsilon.$$

Renaming X and Y gives now the estimates for h_+ and S. $R = R_0 + R_1 + R_2$ and its estimates follows immediately from the homological equation.

The final statement does not follow from Lemma 8.5. However, if one follows the whole construction through the proofs of Lemmas 8.2 to 8.5 one sees that it holds. For example in Lemma 8.2 it is seen immediately that this holds for $\tilde{\rho} \notin \Sigma(L_k, \frac{\kappa}{2})$. The only arbitrariness in the construction is the extension, but we have chosen it so that S_r and R_r are $= 0$ on $\Sigma(L_k, \frac{\kappa}{2})$. The construction Lemmas 8.3 and (8.4) displays the same feature. □

9 Proof of the KAM Theorem

Theorem 6.7 is proved by an infinite sequence of change of variables typical for KAM-theory. The change of variables will be done by the classical Lie transform method which is based on a well-known relation between composition of a function with a Hamiltonian flow Φ_S^t and Poisson brackets:

$$\frac{d}{dt} f \circ \Phi_S^t = \{f, S\} \circ \Phi_S^t$$

from which we derive

$$f \circ \Phi_S^1 = f + \{f, S\} + \int_0^1 (1-t)\{\{f, S\}, S\} \circ \Phi_S^t \, \mathrm{d}t.$$

Given now three functions h, k and f. Then

$$(h+k+f) \circ \Phi_S^1 = h + k + f + \{h+k+f, S\} + \int_0^1 (1-t)\{\{h+k+f, S\}, S\} \circ \Phi_S^t \, \mathrm{d}t.$$

If now S is a solution of the equation

$$\{h, S\} + \{f - f^T, S\}^T + f^T = h_+ + R^F + R^s, \tag{9.1}$$

then

$$(h+k+f) \circ \Phi_S^1 = h + k + h_+ + f_+ + R^s$$

with

$$f_+ = R^F + (f - f^T) + \{k + f^T, S\} + \{f - f^T, S\} - \{f - f^T, S\}^T + \int_0^1 (1-t)\{\{h+k+f, S\}, S\} \circ \Phi_S^t \, \mathrm{d}t \tag{9.2}$$

and

$$f_+^T = R^F + \{k + f^T, S\}^T + \left(\int_0^1 (1-t)\{\{h+k+f, S\}, S\} \circ \Phi_S^t \, \mathrm{d}t \right)^T. \tag{9.3}$$

If we assume that S and R^F are "small as" f^T, then f_+^T is is "small as" kf^T—this is the basis of a linear iteration scheme with (formally) linear convergence.[28] But if also k is of the size f^T, then f^+ is "small as" the square of f^T—this is the basis of a quadratic iteration scheme with (formally) quadratic convergence. We shall combine both of them.

First we shall give a rigorous version of the change of variables described above. We restrict ourselves to the case when $\sigma, \mu, \gamma \leq 1$.

[28] It was first used by Poincaré, credited by him to the astronomer Delauney, and it has been used many times since then in different contexts.

9.1 The basic step. Let $h \in \mathcal{NF}_{\varkappa}(\Delta, \delta)$ and assume $\varkappa > 0$ and

$$\delta \leq \frac{1}{C} c'. \tag{9.4}$$

Let

$$\gamma = (\gamma, m_* + \varkappa) \geq \gamma_* = (0, m_* + \varkappa)$$

and recall Remark 8.1 and the convention (6.18). Let $N \geq 1$, $\Delta' \geq \Delta \geq 1$ and

$$\kappa \leq \frac{1}{C} c'.$$

The constant C is to be determined.

Proposition 8.7 then gives, for any $f \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma, \mu)$,

$$\varepsilon = \left|f^T\right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}} \quad \text{and} \quad \xi = |f|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}},$$

a set $\mathcal{D}' = \mathcal{D}'(h, \kappa, N) \subset \mathcal{D}$ and functions $h_+, S, R = R^F + R^s$, satisfying (8.23)+(8.24)+(8.25) and solving the equation (9.1),

$$\{h, S\} + \{f - f^T, S\}^T + f^T = h_+ + R,$$

for any $\rho \in \mathcal{D}'$. Let now $0 < \sigma' = \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < \sigma_0 = \sigma$ and $0 < \mu' = \mu_4 < \mu_3 < \mu_2 < \mu_1 < \mu_0 = \mu$ be (finite) arithmetic progressions.

The Flow Φ^t_S. We have, by (8.24),

$$|S|^{\sigma_1,\mu_1}_{\gamma,\varkappa,\mathcal{D}} \leq \text{Ct.} \frac{1}{\kappa} XY\varepsilon$$

where X, Y and Ct. are given in Proposition 8.7, i.e.

$$X = \left(\frac{\Delta' e^{\gamma d_\Delta} N}{(\sigma_0 - \sigma_1)(\mu_0 - \mu_1)}\right)^{\exp_2} = \left(\frac{4^2 \Delta' e^{\gamma d_\Delta} N}{(\sigma - \sigma')(\mu - \mu')}\right)^{\exp_2}, \quad Y = \left(\frac{\chi + \xi}{\kappa}\right)^{4s_* + 3}$$

– we can assume without restriction that $\exp_2 \geq 1$.

If

$$\varepsilon \leq \frac{1}{C} \frac{\kappa}{X^2 Y}, \tag{9.5}$$

and C is sufficiently large, then we can apply Proposition 2.11(i). By this proposition it follows that for any $0 \leq t \leq 1$ the Hamiltonian flow map Φ^t_S is a $\mathcal{C}^{s_*}$-map

$$\mathcal{O}_{\gamma'}(\sigma_{i+1}, \mu_{i+1}) \times \mathcal{D} \to \mathcal{O}_{\gamma'}(\sigma_i, \mu_i), \quad \forall \gamma_* \leq \gamma' \leq \gamma, \quad i = 1, 2, 3,$$

real holomorphic and symplectic for any fixed $\rho \in \mathcal{D}$. Moreover,

$$||\partial^j_\rho (\Phi^t_S(x, \cdot) - x)||_{\gamma'} \leq \text{Ct.} \frac{1}{\kappa} XY\varepsilon$$

and

$$\left\|\partial_\rho^j(d\Phi_S^t(x,\cdot)-I)\right\|_{\gamma',\varkappa}\le \text{Ct.}\frac{1}{\kappa}XY\varepsilon$$

for any $x\in\mathcal{O}_{\gamma'}(\sigma_2,\mu_2)$, $\gamma_*\le\gamma'\le\gamma$, and $0\le|j|\le s_*$.

A transformation. Let now $k\in\mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$ and set

$$\eta=|k|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}.$$

Then we have

$$(h+k+f)\circ\Phi_S^1=h+k+h_++f_++R$$

where f_+ is defined by (9.2), i.e.

$$f_+=(f-f^T)+\{k+f^T,S\}+\{f-f^T,S\}-\{f-f^T,S\}^T+\int_0^1(1-t)\{\{h+k+f,S\},S\}\circ\Phi_S^t\,\mathrm{d}t.$$

The integral term is the sum

$$\int_0^1(1-t)\{h_++R-f^T,S\}\circ\Phi_S^t\,\mathrm{d}t+\int_0^1(1-t)\{\{k+f,S\}-\{f-f^T,S\}^T,S\}\circ\Phi_S^t\,\mathrm{d}t.$$

The estimates of $\{k+f^T,S\}$ *and* $\{f-f^T,S\}$. By Proposition 2.9(i)

$$\left|\{k+f^T,S\}\right|^{\sigma_2,\mu_2}_{\gamma,\alpha,\mathcal{D}}\le\text{Ct.}X\,|S|^{\sigma_1,\mu_1}_{\gamma,\varkappa,\mathcal{D}}\,\left|k+f^T\right|^{\sigma_1,\mu_1}_{\gamma,\alpha,\mathcal{D}}.$$

Hence

$$\left|\{k+f^T,S\}\right|^{\sigma_2,\mu_2}_{\gamma,\alpha,\mathcal{D}}\le\text{Ct.}\frac{1}{\kappa}X^2Y(\eta+\varepsilon)\varepsilon.\tag{9.6}$$

Similarly,

$$\left|\{f-f^T,S\}\right|^{\sigma_2,\mu_2}_{\gamma,\alpha,\mathcal{D}}\le\text{Ct.}\frac{1}{\kappa}X^2Y\xi\varepsilon.\tag{9.7}$$

The estimate of $\{h_+-f^T,S\}\circ\Phi_S^t$. The estimate of h_+ is given by (8.23):

$$|h_+|^{\sigma_1,\mu_1}_{\gamma,\varkappa,\mathcal{D}}\le\text{Ct.}XY\varepsilon.$$

This gives, again by Proposition 2.9(i),

$$\left|\{h_+-f^T,S\}\right|^{\sigma_2,\mu_2}_{\gamma,\alpha,\mathcal{D}}\le\text{Ct.}\frac{1}{\kappa}X^3Y^2\varepsilon^2.$$

Let now $F=\{h_+-f^T,S\}$. If ε verifies (9.5) for a sufficiently large constant C, then we can apply Proposition 2.11(ii). By this proposition, for $|t|\le1$, the function $F\circ\Phi_S^t\in\mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma_3,\mu_3)$ and

$$\left|\{h_+-f^T,S\}\circ\Phi_S^t\right|^{\sigma_3,\mu_3}_{\gamma,\varkappa,\mathcal{D}}\le\text{Ct.}\frac{1}{\kappa}X^3Y^2\varepsilon^2.\tag{9.8}$$

The estimate of $\{R,S\}\circ\Phi_S^t$. The estimate of R is given by (8.25). It implies that

$$|R|^{\sigma_1,\mu_1}_{\gamma,\varkappa,\mathcal{D}} \le \mathrm{Ct}.XY\varepsilon.$$

Then, as in the previous case,

$$\big|\{R,S\}\circ\Phi_S^t\big|^{\sigma_3,\mu_3}_{\gamma,\varkappa,\mathcal{D}} \le \mathrm{Ct}.\frac{1}{\kappa}X^3Y^2\varepsilon^2. \tag{9.9}$$

The estimate of $\{\{k+f,S\}-\{f-f^T,S\}^T,S\}\circ\Phi_S^t$. This function is estimated as above. If $F=\{\{k+f,S\}-\{f-f^T,S\}^T,S\}$, then, by Propositions 2.8 and 2.9(i),

$$|F|^{\sigma_3,\mu_3}_{\gamma,\alpha,\mathcal{D}} \le \mathrm{Ct}.(\frac{1}{\kappa}X^2Y)^2(\eta+\xi)\varepsilon^2$$

and by Proposition 2.11(ii)

$$\big|\{\{k+f,S\}-\{f-f^T\}^T,S\}\circ\Phi_S^t\big|^{\sigma_4,\mu_4}_{\gamma,\varkappa,\mathcal{D}} \le \mathrm{Ct}.(\frac{1}{\kappa}X^2Y)^2(\eta+\xi)\varepsilon^2. \tag{9.10}$$

The estimates of R^F *and* R^s. These estimates are given by (8.25):

$$\big|R^F\big|^{\sigma_1,\mu_1}_{\gamma,\varkappa,\mathcal{D}} \le \mathrm{Ct}.XYe^{-(\sigma-\sigma')N}\varepsilon$$

and

$$|R^s|^{\sigma_1,\mu_1}_{\gamma,\varkappa,\mathcal{D}} \le \mathrm{Ct}.XYe^{-(\gamma-\gamma')\Delta'}\varepsilon.$$

Renaming now X and Y and denoting R^s by R_+ gives the following lemma.

LEMMA 9.1. *There exists an absolute constant* C_1 *such that, for any*

$$h\in\mathcal{NF}_\varkappa(\Delta,\delta),\quad \varkappa>0,\quad \delta\le\frac{1}{C_1}c',$$

and for any

$$N\ge 1,\quad \Delta'\ge\Delta\ge 1,\quad \kappa\le\frac{1}{C_1}c',$$

there exists a closed subset $\mathcal{D}'=\mathcal{D}(h,\kappa,N)\subset\mathcal{D}$, *satisfying*

$$\mathrm{meas}(\mathcal{D}\backslash\mathcal{D}')\le C_1(\Delta N)^{\exp_1}\left(\frac{\kappa}{\delta_0}\right)^{\alpha}\left(\frac{\chi}{\delta_0}\right)^{1-\alpha}$$

and, for any $f\in\mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$,

$$\varepsilon=\big|f^T\big|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}\quad\text{and}\quad \xi=[f]^{\gamma,\varkappa}_{\sigma,\mu,\mathcal{D}}.$$

satisfying

$$\varepsilon \le \frac{1}{C_1}\frac{\kappa}{XY}, \quad \begin{cases} X = (\frac{N\Delta' e^{\gamma d_\Delta}}{(\sigma-\sigma')(\mu-\mu')})^{\exp_1}, & \sigma' < \sigma,\ \mu' < \mu, \\ Y = \left(\frac{\chi+\xi}{\kappa}\right)^{\exp_1}, \end{cases}$$

and for any $k \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$,

$$\eta = |k|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}},$$

there exists a $\mathcal{C}^{s_*}$ *mapping*

$$\Phi : \mathcal{O}_{\gamma'}(\sigma',\mu') \times \mathcal{D} \to \mathcal{O}_{\gamma'}(\sigma - \frac{\sigma-\sigma'}{2}, \mu - \frac{\mu-\mu'}{2}), \quad \forall \gamma_* \le \gamma' \le \gamma,$$

real holomorphic and symplectic for each fixed parameter $\rho \in \mathcal{D}$, *and functions* $f_+, R_+ \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma',\mu')$ *and*

$$h + h_+ \in \mathcal{NF}_\varkappa(\Delta', \delta_+),$$

such that

$$(h + k + f) \circ \Phi = h + k + h_+ + f_+ + R_+, \quad \forall \rho \in \mathcal{D}',$$

and

$$|h_+|^{\sigma',\mu'}_{\gamma,\varkappa,\mathcal{D}} \le C_1 XY\varepsilon,$$

$$|f_+ - f|^{\sigma',\mu'}_{\gamma,\varkappa,\mathcal{D}} \le C_1 XY(1+\eta+\xi)\varepsilon,$$

$$\left|f_+^T\right|^{\sigma',\mu'}_{\gamma,\varkappa,\mathcal{D}} \le C_1 \frac{1}{\kappa} XY(\eta + \kappa e^{-(\sigma-\sigma')N} + \varepsilon)\varepsilon,$$

and

$$|R_+|^{\sigma',\mu'}_{\gamma',\varkappa,\mathcal{D}} \le C_1 XY e^{-(\gamma-\gamma')\Delta'}\varepsilon$$

for any $\gamma_* \le \gamma' \le \gamma$.

Moreover,

$$||\partial_\rho^j(\Phi(x,\rho) - x)||_{\gamma'} + \left\|\partial_\rho^j(d\Phi(x,\rho) - I)\right\|_{\gamma',\varkappa} \le C_1 \frac{1}{\kappa} XY\varepsilon$$

for any $x \in \mathcal{O}_{\gamma'}(\sigma',\mu')$, $\gamma_* \le \gamma' \le \gamma$, $|j| \le s_*$ *and* $\rho \in \mathcal{D}$, *and* $\Phi(\cdot,\rho)$ *equals the identity for* ρ *near the boundary of* $\mathcal{D}$.

Finally, if $\tilde\rho = (0, \rho_2, \dots, \rho_p)$ *and* $f^T(\cdot,\tilde\rho) = 0$ *for all* $\tilde\rho$, *then* $f_+ - f = R_+ = h_+ = 0$ *and* $\Phi(x,\cdot) = x$ *for all* $\tilde\rho$.

REMARK 9.2. The exponent α is a positive constant only depending on $d, s_*, \varkappa$ and β_2. The exponent $\exp_1$ only depends on d, $n = \#\mathcal{A}, s_*$ and $\tau, \beta_2, \varkappa$. C_1 is an absolute constant that depends on $c, \tau, \beta_2, \beta_3$ and $\varkappa$. C_1 also depend on $\sup_{\mathcal{D}} |\Omega_{up}|$ and $\sup_{\mathcal{D}} |H_{up}|$, but stays bounded when these do.

9.2 A finite induction. We shall first make a finite iteration without changing the normal form in order to decrease strongly the size of the perturbation. We shall restrict ourselves to the case when $N = \Delta'$.

LEMMA 9.3. *There exists a constant C_2 such that, for any*

$$h \in \mathcal{NF}_{\varkappa}(\Delta, \delta), \quad \varkappa > 0, \quad \delta \leq \frac{1}{C_2} c',$$

and for any

$$\Delta' \geq \Delta \geq 1, \quad \kappa \leq \frac{1}{C_2} c',$$

there exists a closed subset $\mathcal{D}' = \mathcal{D}(h, \kappa, \Delta') \subset \mathcal{D}$, satisfying

$$\operatorname{meas}(\mathcal{D} \backslash \mathcal{D}') \leq C_2 (\Delta')^{\exp_2} \left(\frac{\kappa}{\delta_0} \right)^{\alpha} (\frac{\chi}{\delta_0})^{1-\alpha}$$

and, for any $f \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma, \mu)$,

$$\varepsilon = \left| f^T \right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}} \quad \text{and} \quad \xi = [f]^{\gamma,\varkappa}_{\sigma,\mu,\mathcal{D}},$$

satisfying

$$\varepsilon \leq \frac{1}{C_2} \frac{\kappa}{XY}, \quad \begin{cases} X = \left(\frac{\Delta' e^{\gamma^d \Delta}}{(\sigma - \sigma')(\mu - \mu')} \log \frac{1}{\varepsilon} \right)^{\exp_2}, & \sigma' < \sigma, \ \mu' < \mu, \\ Y = \left(\frac{\chi + \xi}{\kappa} \right)^{\exp_2}, \end{cases}$$

there exists a $\mathcal{C}^{s_*}$ mapping

$$\Phi : \mathcal{O}_{\gamma'}(\sigma', \mu') \times \mathcal{D} \to \mathcal{O}_{\gamma'} \left(\sigma - \frac{\sigma - \sigma'}{2}, \mu - \frac{\mu - \mu'}{2} \right), \quad \forall \gamma_* \leq \gamma' \leq \gamma,$$

real holomorphic and symplectic for each fixed parameter $\rho \in \mathcal{D}$, and functions $f' \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma', \mu')$ and

$$h' \in \mathcal{NF}_{\varkappa}(\Delta', \delta'),$$

such that

$$(h + f) \circ \Phi = h' + f', \quad \forall \rho \in \mathcal{D}',$$

and

$$\left| h' - h \right|^{\sigma',\mu'}_{\gamma,\varkappa,\mathcal{D}} \leq C_2 XY \varepsilon,$$

$$\xi' = \left| f' \right|^{\sigma',\mu'}_{\gamma',\varkappa,\mathcal{D}} \leq \xi + C_2 XY (1 + \xi) \varepsilon,$$

and

$$\varepsilon' = \left|(f')^T\right|^{\sigma',\mu'}_{\gamma',\varkappa,\mathcal{D}} \le C_2 XY(e^{-(\sigma-\sigma')\Delta'} + e^{-(\gamma-\gamma')\Delta'})\varepsilon,$$

for any $\gamma_* \le \gamma' \le \gamma$.

Moreover,

$$||\partial^j_\rho(\Phi(x,\rho)-x)||_{\gamma'} + \left\|\partial^j_\rho(d\Phi(x,\rho)-I)\right\|_{\gamma',\varkappa} \le C_2\frac{1}{\kappa}XY\varepsilon$$

for any $x \in \mathcal{O}_{\gamma'}(\sigma',\mu')$, $\gamma_* \le \gamma' \le \gamma$, $|j| \le s_*$, *and* $\rho \in \mathcal{D}$, *and* $\Phi(\cdot,\rho)$ *equals the identity for* ρ *near the boundary of* $\mathcal{D}$.

Finally, if $\tilde\rho = (0,\rho_2,\dots,\rho_p)$ *and* $f^T(\cdot,\tilde\rho) = 0$ *for all* $\tilde\rho$, *then* $f' - f = h' = 0$ *and* $\Phi(x,\cdot) = x$ *for all* $\tilde\rho$.

(The exponents α, $\exp_2$ *and the constant* C_2 *have the same properties as those in Remark* 9.2*).*

Proof. Let $N = \Delta'$ and $\kappa \le \frac{c'}{C_1}$. Let $\sigma_1 = \sigma - \frac{\sigma-\sigma'}{2}$, $\mu_1 = \mu - \frac{\mu-\mu'}{2}$ and $\sigma_{K+1} = \sigma'$, $\mu_{K+1} = \mu'$, and let $\{\sigma_j\}_1^{K+1}$ and $\{\mu_j\}_1^{K+1}$ be arithmetical progressions. Let

$$(\sigma-\sigma')\Delta' \le K \le (\sigma-\sigma')\Delta'\left(\log\frac{\kappa}{\varepsilon}\right)^{-1}.$$

This implies that

$$\kappa e^{-(\sigma_j-\sigma_{j+1})N} \le \varepsilon.$$

We let $f_1 = f$ and $k_1 = 0$, and we let $\varepsilon_1 = [f_1^T]^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}} = \varepsilon$, $\xi_1 = [f_1]^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}} = \xi$, $\delta_1 = \delta$ and $\eta_1 = [k_1]^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}} = 0$.

Define now

$$\varepsilon_{j+1} = C_1\frac{1}{\kappa}X_jY_j(\eta_j+\varepsilon_1+\varepsilon_j)\varepsilon_j,$$

$$\xi_{j+1} = \xi_j + C_1X_jY_j(1+\eta_j+\xi_j)\varepsilon_j, \quad \eta_{j+1} = \eta_j + C_1X_jY_j\varepsilon_j,$$

with

$$X_j = \left(\frac{N\Delta' e^{\gamma d_\Delta}}{(\sigma_j-\sigma_{j+1})(\mu_j-\mu_{j+1})}\right)^{\exp_1}, \quad Y_j = \left(\frac{\chi+\xi_j}{\kappa}\right)^{\exp_1},$$

where $C_1, \exp_1$ are given in Lemma 9.1. Notice that $X_j = X_1$.

Sublemma. *If*

$$\varepsilon_1 \le \frac{1}{C_2}\frac{\kappa}{X_1^2Y_1^2}, \quad C_2 = 3eC_1 2^{\exp_1},$$

then, for all $j \geq 1$,

$$\varepsilon_j \leq \frac{1}{C_1}\frac{\kappa}{X_j^2 Y_j^2} \quad \textit{and} \quad \varepsilon_j \leq \left(\frac{C_2}{2}\frac{X_1^2 Y_1^2}{\kappa}\varepsilon_1\right)^{j-1}\varepsilon_1 \leq e^{-(j-1)}\varepsilon_1,$$

$$\xi_j - \xi_1 \leq 2C_1 X_1 Y_1 (1+\xi_1)\varepsilon_1 \quad \textit{and} \quad \eta_j \leq 2C_1 X_1 Y_1 \varepsilon_1.$$

This sublemma shows that we can apply Lemma 9.1 K times to get a sequence of mappings

$$\Phi_j : \mathcal{O}_{\gamma'}(\sigma_{j+1}, \mu_{j+1}) \times \mathcal{D}' \to \mathcal{O}_{\gamma'}\left(\sigma_j - \frac{\sigma_j - \sigma_{j+1}}{2}, \mu_j - \frac{\mu_j - \mu_{j+1}}{2}\right), \quad \gamma_* \leq \gamma' \leq \gamma_j$$

and functions f_{j+1} and R_{j+1} such that, for $\rho \in \mathcal{D}'$,

$$(h + k_j + f_j) \circ \Phi_j = h + k_{j+1} + f_{j+1}$$

with $k_{j+1} = k_j + h_{j+1} + R_{j+1}$.

Let $f' = f_{K+1} + R_1 + \cdots + R_{K+1}$ and $h' = h_1 + \cdots + h_{K+1}$. Then

$$\left|h' - h\right|^{\sigma',\mu'}_{\gamma,\varkappa,\mathcal{D}} \leq C_1 \sum X_j Y_j \varepsilon_j \leq \eta_{K+1} \leq 2C_1 X_1 Y_1 \varepsilon_1,$$

$$\left|f' - f\right|^{\sigma',\mu'}_{\gamma,\varkappa,\mathcal{D}} \leq C_1 \sum X_j Y_j (1 + \xi_j + \eta_j)\varepsilon_j \leq 4C_1 X_1 Y_1 (1+\xi_1)\varepsilon_1,$$

and

$$\begin{aligned}\left|(f')^T\right|^{\sigma',\mu'}_{\gamma,\varkappa,\mathcal{D}} &\leq \varepsilon_{K+1} + C_1 \sum X_j Y_j e^{(\gamma-\gamma')\Delta'}\varepsilon_j \\ &\leq e^{-K}\varepsilon_1 + 2C_1 X_1 Y_1 e^{(\gamma-\gamma')\Delta'}\varepsilon_1 \leq e^{(\sigma-\sigma')\Delta'}\varepsilon_1 + 2C_1 X_1 Y_1 e^{(\gamma-\gamma')\Delta'}\varepsilon_1.\end{aligned}$$

We then take $\Phi = \Phi_1 \circ \cdots \circ \Phi_K$. For the estimates of Φ, write $\Psi_j = \Phi_j \circ \cdots \circ \Phi_K$ and $\Psi_{K+1} = id$. For $(x, \rho) \in \mathcal{O}_{\gamma'}(\sigma', \mu') \times \mathcal{D}$ we then have

$$||\Phi(x,\rho) - x||_{\gamma'} \leq \sum_{j=1}^{K} ||\Psi_j(x,\rho) - \Psi_{j+1}(x,\rho)||_{\gamma'}.$$

Then

$$||\Psi_j(x,\rho) - \Psi_{j+1}(x,\rho)||_{\gamma'} = ||\Phi_j(\Psi_{j+1}(x,\rho),\rho) - \Psi_{j+1}(x,\rho)||_{\gamma'}$$

is

$$\leq C_1 \frac{1}{\kappa} X_j Y_j \varepsilon_j.$$

It follows that

$$||\Phi(x,\rho)-x||_{\gamma'} \le 2C_1\frac{1}{\kappa}X_1Y_1\varepsilon_1.$$

The estimate of $||d\Phi(x,\rho)-I||_{\gamma'}$ is obtained in the same way.

The derivatives with respect to ρ depends on higher order differentials which can be estimated by Cauchy estimates.

The result now follows if we take C_2 sufficiently large and increases the exponent $\exp_1$. □

Proof of Sublemma. The estimates are true for $j=1$ so we proceed by induction on j. Let us assume the estimates hold up to j. Then, for $k \le j$,

$$Y_k \le (\frac{\chi+\xi_1+2C_1X_1Y_1(1+\xi_1)\varepsilon_1}{\kappa})^{\exp_1} = 2^{\exp_1}Y_1$$

and

$$\varepsilon_{j+1} \le 2^{\exp_1}\frac{X_1Y_1}{\kappa}[2C_1X_1Y_1\varepsilon_1+\varepsilon_1+\varepsilon_1]\varepsilon_j \le C'\frac{X_1^2Y_1^2}{\kappa}\varepsilon_1\varepsilon_j,$$

$C'=3C_12^{\exp_1}$. Then

$$\begin{aligned}\xi_{j+1}-\xi_1 &\le 2^{\exp_1}X_1Y_1(1+\xi_1+4C_1X_1Y_1(1+\xi_1)\varepsilon_1)(\varepsilon_1+\cdots+\varepsilon_{j+1})\\ &\le 2^{\exp_1}X_1Y_1(1+\xi_1)(1+4C_1X_1Y_1\varepsilon_1)2\varepsilon_1 \le 2^{\exp_1}4X_1Y_1(1+\xi_1)\varepsilon_1,\end{aligned}$$

if $4C_1X_1Y_1\varepsilon_1 \le 1$ and $C'\frac{X_1^2Y_1^2}{\kappa}\varepsilon_1 \le \frac{1}{e} \le \frac{1}{2}$—and similarly for η_{j+1}.

9.3 The infinite induction. We are now in position to prove our main result, Theorem 6.7.

Let h be a normal form Hamiltonian in $\mathcal{NF}_{\varkappa}(\Delta,\delta)$ and let $f \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$ be a perturbation such that

$$0<\varepsilon=\left|f^T\right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}, \quad \xi=|f|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}.$$

We construct the transformation Φ as the composition of infinitely many transformations Φ as in Lemma 9.3. We first specify the choice of all the parameters for $j \ge 1$.

Let $C_2, \exp_2$ and α be the constants given in Lemma 9.3.

9.3.1 Choice of parameters. We have assumed $\gamma, \sigma, \mu \leq 1$ and we take $\Delta \geq 1$. By decreasing γ or increasing Δ we can also assume $\gamma = (d_\Delta)^{-1}$.

We choose for $j \geq 1$

$$\mu_j = \left(\frac{1}{2} + \frac{1}{2^j}\right)\mu \quad \text{and} \quad \sigma_j = \left(\frac{1}{2} + \frac{1}{2^j}\right)\sigma.$$

We define inductively the sequences ε_j, Δ_j, δ_j and ξ_j by

$$\begin{cases} \varepsilon_{j+1} = \varepsilon^{K_j}\varepsilon, & \varepsilon_1 = \varepsilon, \\ \Delta_{j+1} = 4K_j \max\left(\frac{1}{\sigma_j - \sigma_{j+1}}, d_{\Delta_j}\right) \log\frac{1}{\varepsilon}, & \Delta_1 = \Delta, \\ \gamma_{j+1} = (d_{\Delta_{j+1}})^{-1}, & \gamma_1 = \gamma, \\ \delta_{j+1} = \delta_j + C_2 X_j Y_j \varepsilon_j, & \delta_1 = \delta \geq 0, \\ \xi_{j+1} = \xi_j + C_2 X_j Y_j (1+\xi_j)\varepsilon_j, & \xi_1 = \xi, \end{cases} \tag{9.11}$$

where

$$\begin{cases} X_j = \left(\frac{\Delta_{j+1} e^{\gamma_j d_{\Delta_j}}}{(\sigma_j - \sigma_{j+1})(\mu_j - \mu_{j+1})} \log\frac{1}{\varepsilon_j}\right)^{\exp_2} = \left(\frac{K_j \Delta_{j+1} e 4^{j+1}}{\sigma\mu} \log\frac{1}{\varepsilon}\right)^{\exp_2}, \\ Y_j = \left(\frac{\chi + \xi_j}{\kappa_j}\right)^{\exp_2} \end{cases}$$

–for d_Δ see (6.2).The κ_j is defined implicitly by

$$2^j \varepsilon_j = \frac{1}{C_2} \frac{\kappa_j}{X_j Y_j},$$

These sequences depend on the choice of K_j. We shall let K_j increase like

$$K_j = K^j$$

for some K sufficiently large.

LEMMA 9.4. *There exist constants C' and $\exp'$ such that, if*

$$K \geq C'$$

and

$$\varepsilon \left(\log\frac{1}{\varepsilon}\right)^{\exp'} \leq \frac{1}{C'} \left(\frac{\sigma\mu}{(\chi+\xi)K\Delta}\right)^{\exp'},$$

then

(i) $$\delta_j - \delta, \quad \xi_j - \xi, \quad \kappa_j \ \leq \ 2C_2 X_1 Y_1 \varepsilon;$$

(ii) $$\varepsilon_{j+1} \geq C_2 X_j Y_j (e^{-\frac{1}{2}(\sigma_j - \sigma_{j+1})\Delta_{j+1}} + e^{-\frac{1}{2}(\gamma_j - \gamma_{j+1})\Delta_{j+1}})\varepsilon_j;$$

(iii) $$\sum_{j\geq 1} \Delta_{j+1}^{\exp_2} \kappa_j^\alpha \leq 2\Delta_2^{\exp_2} \kappa_1^\alpha \leq C' \left(\frac{K d_\Delta \log\frac{1}{\varepsilon}}{\sigma\mu}\right)^{\exp_2} ((\chi+\xi)\varepsilon)^\alpha.$$

(The exponents α, $\exp'$ and the constant C' has the same properties as those in Remark 9.2).

Proof. Δ_{j+1} is equal to

$$4K_j \max\left(\frac{1}{\sigma_j - \sigma_{j+1}}, d_{\Delta_j}\right) \log\frac{1}{\varepsilon} \leq \left(\mathrm{Ct.}\frac{1}{\sigma}\log\frac{1}{\varepsilon}\right)(2K)^j \Delta_j^a,$$

where a is some exponent depending on d. By a finite induction one sees that this is

$$\leq \left(\mathrm{Ct.}\frac{1}{\sigma}\log\frac{1}{\varepsilon}\right)(2K)\Delta)^{a^j},$$

if, as we shall assume, $a \geq 2$. Now X_j equals

$$\left(\frac{K_j\Delta_{j+1}e4^{j+1}}{\sigma\mu}\log\frac{1}{\varepsilon}\right)^{\exp_2} \leq \left((\mathrm{Ct.}\frac{1}{\sigma\mu}\log\frac{1}{\varepsilon})(4K)^{j^2}\Delta_j^a\right)^{2\exp_2}.$$

which, by assumption on ε, is

$$\leq \left(\left(\mathrm{Ct.}\frac{1}{\sigma\mu}\log\frac{1}{\varepsilon}\right)K\Delta\right)^{4\exp_2 a^j} \leq \left(\frac{1}{\varepsilon}\right)^{4\exp_2 a^j},$$

if, as we shall assume, $a \geq 3$.

(i) holds trivially for $j = 1$, (i) , so assume it holds up to $j - 1 \geq 1$. Then $\delta_j \leq \delta + 2C_2X_1Y_1\varepsilon$ and $\xi_j \leq \xi + 2C_2X_1Y_1\varepsilon$, and hence

$$Y_j \leq \left(\frac{\chi + \xi + 2C_2X_1Y_1\varepsilon}{\kappa_j}\right)^{\exp_2} \leq 2^{\exp_2}Y_1\left(\frac{\kappa_1}{\kappa_j}\right)^{\exp_2}.$$

By definition of κ_j,

$$\kappa_j^{1+\exp_2} = 2^jC_2X_jY_j\varepsilon_j\kappa_j^{\exp_2} \leq 2^{\exp_2}C_2Y_1\kappa_1^{\exp_2}2^jX_j\varepsilon_j \leq 2^jX_j\varepsilon^{K_{j-1}}$$

by assumption on ε. Hence

$$2^jC_2X_jY_j\varepsilon_j = \kappa_j \leq 2^jX_j\varepsilon^{2bK_{j-1}} \leq \varepsilon^{2bK_{j-1}-4\exp_2 a^j - j\log 2}, \quad b = \frac{1}{2(1+\exp_2)}.$$

If K is large enough—notice that $j \geq 2$—this is $\leq \varepsilon^{bK_{j-1}}$.

Hence

$$\kappa_j \leq \varepsilon^{bK_{j-1}} \leq \varepsilon^{bK} \leq \varepsilon \leq 2C_2X_1Y_1\varepsilon,$$

if K is large enough. Moreover

$$\delta_j - \delta = \sum_{k=2}^{j} C_2X_kY_k\varepsilon_k \leq \varepsilon^{bK_1} \leq 2C_2X_1Y_1\varepsilon_1$$

if K is large enough. From these estimates one also obtains the required bound for $\xi_j - \xi$ if K is large enough. This concludes the proof of (i).

To see (ii), notice that

$$e^{-(\sigma_j-\sigma_{j+1})\Delta_{j+1}} \leq e^{-4K_j \log \frac{1}{\varepsilon}} \leq \varepsilon^{K_j}\varepsilon.$$

Notice also that Δ_{j+1} is much larger then Δ_j so that γ_{j+1} is much smaller than γ_j and, hence,

$$e^{-(\gamma_j-\gamma_{j+1})\Delta_{j+1}} \leq e^{-4K_j \frac{\gamma_j-\gamma_{j+1}}{\gamma_j} \log \frac{1}{\varepsilon}} \leq \varepsilon^{K_j}\varepsilon.$$

This implies that

$$C_2 X_j Y_j \left(e^{-\frac{1}{2}(\sigma_j-\sigma_{j+1})\Delta_{j+1}} + e^{-\frac{1}{2}(\gamma_j-\gamma_{j+1})\Delta_{j+1}}\right)\varepsilon_j \leq \varepsilon^{K_j}\varepsilon = \varepsilon_{j+1}.$$

To see (iii) we have for $j \geq 2$

$$\Delta_{j+1}^{\exp_2}\kappa_j^\alpha \leq X_j^{\exp_2}\kappa_j^\alpha \leq \left(\frac{1}{\varepsilon}\right)^{4\exp_2^2 a^j} \kappa_j^\alpha \leq e^{-4\exp_2^2 a^j \log\frac{1}{\varepsilon}} e^{\alpha b K_{j-1}\log\frac{1}{\varepsilon}}$$

which is

$$\leq \varepsilon^{\frac{1}{2}bK_{j-1}\alpha} \leq 2^{-j}\varepsilon,$$

if K is large enough (depending on α). This implies the first inequality in (iii). The second one is a simple computation. □

9.3.2 The iteration

PROPOSITION 9.5. *There exist positive constants C_3, α and $\exp_3$ such that, for any $h \in \mathcal{NF}_\varkappa(\Delta,\delta)$ and for any $f \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma,\mu)$,*

$$\varepsilon = \left|f^T\right|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}}, \quad \xi = |f|^{\sigma,\mu}_{\gamma,\varkappa,\mathcal{D}},$$

if

$$\delta \leq \frac{1}{C_3}c'$$

and

$$\varepsilon(\log\frac{1}{\varepsilon})^{\exp_3} \leq \frac{1}{C_3}\left(\frac{\sigma\mu}{(\chi+\xi)\max(\frac{1}{\gamma},d_\Delta)}c'\right)^{\exp_3} c',$$

then there exist a closed subset $\mathcal{D}' = \mathcal{D}'(h,f) \subset \mathcal{D}$,

$$\operatorname{meas}(\mathcal{D}\backslash\mathcal{D}') \leq C_3\left(\frac{\max(\frac{1}{\gamma},d_\Delta)\log\frac{1}{\varepsilon}}{\sigma\mu}\right)^{\exp_3}\frac{\chi}{\delta_0}\left((\chi+\xi)\frac{\varepsilon}{\chi}\right)^\alpha$$

and a $\mathcal{C}^{s_*}$ *mapping*

$$\Phi : \mathcal{O}_{\gamma_*}(\sigma/2, \mu/2) \times \mathcal{D} \to \mathcal{O}_{\gamma_*}(\sigma, \mu),$$

real holomorphic and symplectic for given parameter $\rho \in \mathcal{D}$, *and*

$$h' \in \mathcal{NF}_{\varkappa}(\infty, \delta'), \quad \delta' \leq \frac{c'}{2},$$

such that

$$(h + f) \circ \Phi = h' + f'$$

verifies

$$\left|f' - f\right|_{\substack{\sigma/2,\mu/2 \\ \gamma_*,\varkappa,\mathcal{D}}} \leq C_3$$

and, for $\rho \in \mathcal{D}'$, $(f')^T = 0$.

Moreover,

$$\left|h' - h\right|_{\substack{\sigma/2,\mu/2 \\ \gamma_*,\varkappa,\mathcal{D}}} \leq C_3$$

and

$$||\partial_\rho^j(\Phi(x, \cdot) - x)||_{\gamma_*} + \left\|\partial_\rho^j(d\Phi(x, \cdot) - I)\right\|_{\gamma_*,\varkappa} \leq C_3$$

for any $x \in \mathcal{O}_{(0,m_*)}(\sigma', \mu')$, $|j| \leq s_*$, *and* $\rho \in \mathcal{D}$, *and* $\Phi(\cdot, \rho)$ *equals the identity for* ρ *near the boundary of* $\mathcal{D}$.

Finally, if $\tilde{\rho} = (0, \rho_2, \ldots, \rho_p)$ *and* $f^T(\cdot, \tilde{\rho}) = 0$ *for all* $\tilde{\rho}$, *then* $h' = h$ *and* $\Phi(x, \cdot) = x$ *for all* $\tilde{\rho}$.

(The exponents α, $\exp_3$ *and the constant* C_3 *have the same properties as those in Remark* 9.2*).*

Proof. Assume first that $\gamma{=}d_\Delta^{-1}$. Choose the number $\mu_j, \sigma_j, \varepsilon_j, \Delta_j, \gamma_j, \delta_j, \xi_j, X_j, Y_j, \kappa_j$ as above in Lemma 9.4 with $K = C'$. Let $h_1 = h$, $f_1 = f$.

Since

$$\kappa_j, \ \delta_j - \delta \ \leq 2C_2X_1Y_1\varepsilon \leq \frac{1}{2C_2}c'$$

by Lemma 9.4 and by assumption on ε we can apply Lemma 9.3 iteratively. It gives, for all $j \geq 1$, a set $\mathcal{D}_j \subset \mathcal{D}$,

$$\text{meas}(\mathcal{D} \backslash \mathcal{D}_j) \leq C_2\Delta_{j+1}^{\exp_2}\left(\frac{\kappa_j}{\delta_0}\right)^{\alpha}\left(\frac{\chi}{\delta_0}\right)^{1-\alpha},$$

and a $\mathcal{C}^{s_*}$ mapping

$$\Phi_{j+1} : \mathcal{O}^{\gamma'}(\sigma_{j+1}, \mu_{j+1}) \times \mathcal{D}_{j+1} \to \mathcal{O}^{\gamma'}\left(\sigma_j - \frac{\sigma_j - \sigma_{j+1}}{2}, \mu_j - \frac{\mu_j - \mu_{j+1}}{2}\right),$$

$$\forall \gamma_* \leq \gamma' \leq \gamma_{j+1},$$

real holomorphic and symplectic for each fixed parameter ρ, and functions $f_{j+1} \in \mathcal{T}_{\gamma,\varkappa,\mathcal{D}}(\sigma_{j+1},\mu_{j+1})$ and

$$h_{j+1} \in \mathcal{NF}_{\varkappa}(\Delta_{j+1},\delta_{j+1})$$

such that

$$(h_j + f_j)\circ\Phi_{j+1} = h_{j+1} + f_{j+1}, \quad \forall \rho \in \mathcal{D}_{j+1},$$

with

$$\big|f_{j+1}^T\big|^{\sigma_{j+1},\mu_{j+1}}_{\gamma_{j+1},\varkappa,\mathcal{D}} \le \varepsilon_{j+1}$$

and

$$|f_{j+1}|^{\sigma_{j+1},\mu_{j+1}}_{\gamma_{j+1},\varkappa,\mathcal{D}} \le \xi_{j+1}.$$

Moreover,

$$|h_{j+1} - h_j|^{\sigma_{j+1},\mu_{j+1}}_{\gamma_{j+1},\varkappa,\mathcal{D}} \le C_2 X_j Y_j \varepsilon_j$$

and

$$||\partial_\rho^l(\Phi_{j+1}(x,\cdot) - x)||_{\gamma'} + \Big\|\partial_\rho^l(d\Phi_{j+1}(x,\cdot) - I)\Big\|_{\gamma',\varkappa} \le C_2\frac{1}{\kappa_j}X_jY_j\varepsilon_j$$

for any $x \in \mathcal{O}_{\gamma'}(\sigma_{j+1},\mu_{j+1})$, $\gamma_* \le \gamma' \le \gamma_{j+1}$ and $|l| \le s_*$.

We let $h' = \lim h_j$, $f' = \lim f_j$ and $\Phi = \Phi_2\circ\cdots\circ\Phi_3\circ\ldots$. Then $(h+f)\circ\Phi = h'+f'$ and h' and f' verify the statement. The convergence of Φ and its estimates follows as in the proof of Lemma 9.3.

Let $\mathcal{D}' = \bigcup \mathcal{D}_j$. Then, by Lemma 9.4,

$$\operatorname{meas}(\mathcal{D}\backslash\mathcal{D}') \le C_2\frac{\chi^{1-\alpha}}{\delta_0}\sum_j \Delta_{j+1}^{\exp_2}\kappa_j^\alpha \le C_3\frac{\chi^{1-\alpha}}{\delta_0}\Big(\frac{d_\Delta \log\frac{1}{\varepsilon}}{\sigma\mu}\Big)^{\exp_2}((\chi+\xi)\varepsilon)^\alpha.$$

The last statement is obvious.

If $\gamma < (d_\Delta)^{-1}$, then we increase Δ and we obtain the same result. If $\gamma > (d_\Delta)^{-1}$, then we can just decrease γ and we obtain the same result. □

Theorem 6.7 now follows from this proposition.

PART IV. SMALL AMPLITUDE SOLUTIONS

10 Proofs of Theorems 1.3, 1.4

We shall now treat the beam equation by combining the Birkhoff normal form theorem 5.1 and the KAM theorem 6.7 or, more precisely, its Corollary 6.9. In order to apply Corollary 6.9 we need to verify, first that the quadratic part of the Hamiltonian (5.4) is a KAM normal form Hamiltonian and, second that the perturbation f is sufficiently small.

We recall the agreement about constants made in the introduction.

10.1 A KAM normal form Hamiltonian. Let h be the Hamiltonian (1.11)+(1.12).

Theorem 10.1. *There exists a zero-measure Borel set $\mathcal{C} \subset [1,2]$ such that for any strongly admissible set $\mathcal{A}$ and any $m \notin \mathcal{C}$ there exist real numbers $\gamma_g > \gamma_* = (0, m_* + 2)$ and $\beta_0, \nu_0, c_0 > 0$, where c_0, β_0, ν_0 depend on m, such that, for any $0 < c_* \le c_0$, $0 < \beta_\# \le \beta_0$ and $0 < \nu \le \nu_0$ there exists an open set $Q = Q(c_*, \beta_\#, \nu) \subset [c_*, 1]^{\mathcal{A}}$, increasing as $\nu \to 0$ and satisfying*

$$\operatorname{meas}([c_*, 1]^{\mathcal{A}} \backslash Q) \le C\nu^{\beta_\#}, \tag{10.1}$$

with the following property:

For any $\rho \in Q$ there exists a real holomorphic diffeomorphism (onto its image)

$$\Psi_\rho : \mathcal{O}_{\gamma_*}\left(\tfrac{1}{2}, \mu_*^2\right) \to \mathbf{T}_\rho(\nu, 1, 1, \gamma_*), \quad \mu_* = \tfrac{c_*}{2\sqrt{2}}, \tag{10.2}$$

satisfying (5.5), such that

$$\Psi_\rho^*(dp \wedge dq) = \nu dr_{\mathcal{A}} \wedge d\theta_{\mathcal{A}} \ + \ \nu du_{\mathcal{L}} \wedge dv_{\mathcal{L}},$$

and such that, up to a constant,

$$\frac{1}{\nu}(h \circ \Psi_\rho) = h_{\mathrm{up}} + f,$$

$$h_{\mathrm{up}}(r, \theta, p_{\mathcal{L}}, q_{\mathcal{L}}) = \langle \Omega(\rho), r\rangle + \frac{1}{2}\sum_{a \in \mathcal{L}_\infty} \Lambda_a(\rho)(p_a^2 + q_a^2) + \nu\langle K(\rho)\zeta_{\mathcal{F}}, \zeta_{\mathcal{F}}\rangle, \tag{10.3}$$

where $\mathcal{F} = \mathcal{F}_\rho \subset \mathcal{L}_f$, with the following properties:

(i) *Ψ_ρ depends smoothly on ρ and*

$$\Psi_\rho\left(\mathcal{O}_\gamma\left(\tfrac{1}{2}, \mu_*^2\right)\right) \subset \mathbf{T}_\rho(\nu, 1, 1, \gamma), \quad \gamma_* \le \gamma \le \gamma_g;$$

(ii) *h_{up} satisfies, on any ball (or cube) $\mathcal{D} \subset Q$, the Hypotheses A1-A3 of Sect. 6.2 for some constants $c', c, \delta_0, \beta, \tau$ such that*

$$c' \ge \nu^{1+\beta_\#}, \quad c = 2\max\{\langle a\rangle^3, a \in \mathcal{A}\}, \quad \beta_1 = \beta_2 = 2, \tag{10.4}$$

$$\delta_0 \ge \nu^{1+\beta_\#}, \quad s_* = 4\,(\#\mathcal{F})^2, \tag{10.5}$$

$$\beta_3 = \beta_3(m) > 0, \quad \tau = \tau(m) > 0\,; \tag{10.6}$$

(iii)

$$\chi = |\nabla_\rho \Omega|_{\mathcal{C}^{s_*-1}(\mathcal{D})} + \sup_{a \in \mathcal{L}_\infty} |\nabla_\rho \Lambda_a|_{\mathcal{C}^{s_*-1}(\mathcal{D})} + ||\nu \nabla_\rho K||_{\mathcal{C}^{s_*-1}(\mathcal{D})} \le C\nu^{1-\beta_\#};$$

(iv) *f belongs to $\mathcal{T}_{\gamma,\varkappa=2,Q}(\frac{1}{2},\mu_*^2)$ and satisfies*

$$\xi = |f|_{\substack{1/2,\mu_*^2\\ \gamma_g,2,\mathcal{D}}} \le C\nu^{1-\beta_\#}, \quad \varepsilon = |f^T|_{\substack{1/2,\mu_*^2\\ \gamma_g,2,\mathcal{D}}} \le C\nu^{3/2-\beta_\#}.$$

If $\mathcal{A}$ is admissible but not strongly admissible, then the same thing is true with the difference that (ii) *only holds for balls (or cubes) $\mathcal{D} \subset Q \cap \mathcal{D}_0$, where $\mathcal{D}_0 \subset [0,1]^{\mathcal{A}}$ is an open set, independent of $c_*, \beta_\#$ and ν, such that*

$$\operatorname{meas}(\mathcal{D}_0) \ge \tfrac{1}{2}\, c_0^{\#\mathcal{A}}. \tag{10.7}$$

The constant C depends on $m, c_*, \beta_\#$, but not on ν.

Proof. We apply Theorem 5.1 and denote the constructed there symplectic transformation by Ψ. We let $\mathcal{L}_\infty = \mathcal{L}\backslash\mathcal{F} = (\mathcal{L}\backslash\mathcal{L}_f) \cup (\mathcal{L}_f\backslash\mathcal{F})$ (this is a slight abuse of notation since in Part II we denoted by $\mathcal{L}_\infty$ the set $\mathcal{L}\backslash\mathcal{L}_f$). For β_0, ν_0 and ε_0 we take the same constants as in Theorem 5.1. If $\mathcal{A}$ is only admissible, we take for $\mathcal{D}_0$ the set $\mathcal{D}_0 = \mathcal{D}_0^1$, see (5.23).

The assertion (i) of the theorem holds by Theorem 5.1.

To prove (ii) and (iii) we will first verify (ii) for a smaller c',

$$c' \ge \nu^{1+2\beta_\#(\beta(0)+\bar{c})}, \tag{10.8}$$

and in (iii) will replace the exponent for ν by a bigger number.

By (4.44), (4.45), (5.6) and (5.7) we have that

$$\chi = |\nabla_\rho \Omega|_{\mathcal{C}^{s_*-1}(Q)} + \sup_{a\in\mathcal{L}_\infty} |\nabla_\rho \Lambda_a|_{\mathcal{C}^{s_*-1}(Q)} + ||\nu\nabla_\rho K||_{\mathcal{C}^{s_*-1}(Q)} \le \text{ct.}\nu^{1-\beta_\#\beta(s_*-1)},$$

which implies (iii) with a modified exponent.

Now let us consider (ii). We will check the validity of the three hypotheses A1–A3 (with c' as in (10.8)).

First we note that using (4.45), (3.4), (5.22), (5.38) and (5.52) we get

$$\tfrac{1}{2} + \tfrac{1}{2}|a|^2 \le \Lambda_a \le 2|a|^2 + 1, \quad |\Lambda_a - \lambda_a|_{C^j(\mathcal{D}_0)} \le C_3\nu|a|^{-2} \quad \forall\, j \ge 1,\ \forall\, a \in \mathcal{L}\backslash\mathcal{L}_f, \tag{10.9}$$

$$C_1\nu^{1+\bar{c}\beta_\#} \le |\Lambda_a| \le C_2\nu \quad \forall\, a \in \mathcal{L}_f\backslash\mathcal{F}. \tag{10.10}$$

It is convenient to re-denote

$$\lambda_a =: 0 \quad \text{if} \quad a \in \mathcal{L}_f\backslash\mathcal{F}\,; \tag{10.11}$$

then the second relation in (10.9) holds for all a. We recall that the numbers $\{\pm\Lambda_a, a \in \mathcal{F}\}$ are the eigenvalues of the operator JK. They satisfy the estimates (5.8).

The vector–function $\Omega(\rho) \in \mathbb{R}^n$ is defined in (4.44), so

$$\Omega(\rho) = \omega + \nu M\rho, \quad \det M \ne 0, \tag{10.12}$$

and K is a symmetric real linear operator in the space $Y_{\mathcal{F}}$. Its norm satisfies

$$\|\nu K(\rho)\|_{C^j} \le C_j \nu^{1-\beta_{\#}\beta(j)}, \quad j \ge 0. \tag{10.13}$$

See Theorem 5.1, items (ii)–(iv).

Hypothesis A1. Relations (6.8) and (6.9) and the first relation in (6.10) immediately follow from (10.9) and (10.10).

To prove the second relation in (6.10), with $H_{\rm up} = K$, note that by Theorem 5.1 the operator U conjugates JK with the diagonal operator with the eigenvalues $\pm\mathbf{i}\Lambda_a(\rho)$, $a \in \mathcal{F}$. So by (5.8) and (5.9) the norm of $(JK)^{-1}$ is bounded by $C\nu^{-1-\beta_{\#}(\bar{c}+2\beta(0)}$, and the required estimate follows from (10.8). The second relation in (6.12) follows by the same argument from (5.8), which implies that the norms of the eigenvalues of $\Lambda_a I - \mathbf{i}JK$ are $\ge C^{-1}\nu^{\bar{c}\beta_{\#}}$. The first relation in (6.12) is a consequence of (10.9), (10.10) and (6.4).

Now consider (6.11).[29] If $a \in \mathcal{L}_\infty$ and $b \in \mathcal{L}\backslash\mathcal{L}_f$, then again the relation follows from (10.9) and (10.10). Next, let $a, b \in \mathcal{L}_f\backslash\mathcal{F}$. Let us write Λ_a and Λ_b as Λ_r^j and Λ_m^k, $j \le k$. If $j = k$, then the condition follows from (5.40), (5.52) (from (5.38) if $m = r$). If $j \le M_0 < k$, then again it follows from (5.40). If $j, k \le M_0$, then $\Lambda_r^j = \Lambda_1^j = \mu(b_j, \rho)$ and $\Lambda_m^k = \mu(b_m, \rho)$, so the relation follows from (5.39). Finally, let $j, k > M_0$. Then if the set $\mathcal{A}$ is strongly admissible, the required relation follows from (5.40), while if $\rho \in \mathcal{D}_0 = \mathcal{D}_0^1$, then it follows from (5.29).

Hypothesis A2. By (10.12), $\partial_{\mathfrak{z}}\Omega(\rho) = \nu M_{\mathfrak{z}}$. Choosing

$$\mathfrak{z} = \frac{{}^tMk}{|{}^tMk|} \tag{10.14}$$

and using that $|\Omega' - \Omega|_{C^{s_*}} \le \delta_0$ we achieve that $\partial_{\mathfrak{z}}\langle k, \Omega'(\rho)\rangle \ge C\nu$, so (6.13) holds.

To verify (i) we restrict ourselves to the more complicated case when $a, b \ne \emptyset$. Then $L(\rho)$ is a diagonal operator with the eigenvalues

$$\lambda_{ab}^k := \langle k, \Omega'(\rho)\rangle + \Lambda_a(\rho) \pm \Lambda_b(\rho), \quad a \in [a],\ b \in [b].$$

Clearly

$$|\lambda_{ab}^k - (\langle k, \omega\rangle + \lambda_a \pm \lambda_b)| \le C\nu|k|.$$

(we recall (10.11)). Therefore by Propositions 3.6 and 3.7 the first alternative in (i) holds, unless

$$|k| \ge C\nu^{-\bar{\beta}} \tag{10.15}$$

for some (fixed) $\bar{\beta} > 0$. But if we choose $\mathfrak{z}$ as in (10.14), then $\partial_{\mathfrak{z}}L(\rho)$ becomes a diagonal matrix with the diagonal elements bigger than $|{}^tMk| - C\nu|k| - C_1\nu$. So if k satisfies (10.15), then the second alternative in (i) holds.

[29] This is the only condition of Theorem 6.7 which we cannot verify for any $\rho \in Q$ without assuming that the set $\mathcal{A}$ is strongly admissible.

To verify (ii) we write $L(\rho,\Lambda_a)$ as the multiplication from the right by the matrix

$$L=(\langle k,\Omega'\rangle+\Lambda_a(\rho))I+\mathbf{i}\nu J\widehat{K}.$$

The transformation U conjugates L with the diagonal operator with the eigenvalues $\lambda^k_{aj}=:\langle k,\Omega'\rangle+\Lambda_a(\rho)\pm\nu\mathbf{i}\Lambda^h_j$. In view of (5.8), $|\lambda^k_{aj}|\ge|\Im\lambda^k_{aj}|\ge C^{-1}\nu^{1+\bar c\beta_\#}$. This implies (ii) by (5.9) and (10.8).

It remains to verify (iii). As before, we restrict ourselves to the more complicated case $a,b\in\mathcal{F}$. Let us denote

$$\lambda(\rho):=\langle k,\Omega'(\rho)\rangle=\langle k,\omega\rangle+\nu\langle k,M\rho\rangle+\langle k,(\Omega'-\Omega)(\rho)\rangle,$$

and write the operator $L(\rho)$ as

$$L(\rho)=\lambda(\rho)I+L^0(\rho),\quad L^0(\rho)X=[X,iJ(\nu K)(\rho)].$$

In view of (10.13),

$$\|L^0\|_{C^j}\le C_j\nu^{1-\beta(j)\beta_\#}\quad\text{for}\quad j\ge0. \tag{10.16}$$

Now it is easy to see that if $|\langle k,\omega\rangle|\ge C(\nu^{1-\beta(0)\beta_\#}+\nu|k|)$ with a sufficiently big C, then the first alternative in (iii) holds.

So it remains to consider the case when

$$|\langle k,\omega\rangle|\le C(\nu^{1-\beta(0)\beta_\#}+\nu|k|). \tag{10.17}$$

By Proposition 3.6 the l.h.s. is bigger than $\kappa|k|^{-n^2}$. Assuming that $\beta_0\ll1$, we derive from this and (10.17) that

$$|k|\ge C\nu^{-1/(1+n^2)}. \tag{10.18}$$

In view of (10.16)-(10.18), again if $\beta_0\ll1$, we have:

$$|\lambda(\rho)|\le C\nu(\nu^{-\beta(0)\beta_\#}+|k|)\le C_1\nu|k|, \tag{10.19}$$

$$|(\partial_\rho)^j\lambda(\rho)|\le C_j|k|\delta_0,\quad 2\le j\le s_*, \tag{10.20}$$

$$\|L\|_{C^j}\le C\nu(\nu^{-\beta(j)\beta_\#}+|k|)+C_j|k|\delta_0,\quad j\ge0. \tag{10.21}$$

Denote $\det L(\rho)=D(\rho)$. Then

$$D(\rho)=\prod_{a,b\in\mathcal{F}}\prod_{\sigma_1,\sigma_2=\pm}\Lambda(\rho;a,b,\sigma_1,\sigma_2),$$

where $\Lambda(\rho;a,b,\sigma_1,\sigma_2)=\lambda(\rho)+\sigma_1\nu\Lambda_a(\rho)-\sigma_2\nu\Lambda_b(\rho)$. Choosing $\mathfrak{z}$ as in (10.14) we get

$$|\Lambda|\le C\nu|k|,\quad |\partial_{\mathfrak{z}}\Lambda|\ge C^{-1}|k|\nu-|k|\delta_0\ge\tfrac12C^{-1}|k|\nu,\quad |\partial^j_{\mathfrak{z}}\Lambda|\le C_j|k|\delta_0\ \text{ if }\ j\ge2$$

(that is, these relations hold for all values of the arguments $\rho, a, b, \sigma_1, \sigma_2$). Recall that $2\,|\mathcal{F}| = m$; then $s_* = m^2$. Chose in (6.15) $j = s_* = m^2$. Then, in view of the relations above, we get:

$$|\partial_{\mathfrak{z}}^{s_*} D(\rho)| \geq m^2! \left(C^{-1}|k|\nu\right)^{m^2} - C_1(|k|\nu)^{m^2-1}(|k|\delta_0) \geq \tfrac{1}{2} m^2! \left(C^{-1}|k|\nu\right)^{m^2}.$$

In the same time, by (10.21) the r.h.s. of (6.15) is bounded from above by

$$C_m \delta_0 \left(\nu^{(m^2-1)(1-\beta(m^2)\beta_\#)} + \nu^{m^2-1}|k|^{m^2-1}\right).$$

In view of (10.8), (10.5) this implies the relation (6.15) if we choose $\beta_\# < (\beta(m^2)(1+n^2))^{-1}$ (as always, we decrease ν_0, if needed).

Hypothesis A3. The required inequality follows from Proposition 3.7 since the divisor, corresponding to (6.16) where $a, b \notin \mathcal{L}_f$, cannot be resonant.

Finally, let us denote

$$\beta_\#^0 = \beta_\# \max(1, \hat{c}, 2(\beta(0) + \bar{c}), \beta(s_* - 1)).$$

Our argument shows that the assertions (ii), (iii) of the theorem hold with $\beta_\#$ replaced by $\beta_\#^0$. The assertion (iv) with $\beta_\# =: \beta_\#^0$ follows from (5.10). Now it remains to re-denote $\beta_\#^0$ by $\beta_\#$. □

10.2 The main result. We have c_0, β_0, ν_0 so small that Theorem 10.1 applies. Now we shall make them even smaller.

Theorem 10.2. *There exists a zero-measure Borel set $\mathcal{C} \subset [1,2]$ such that for any strongly admissible set $\mathcal{A}$ and any $m \notin \mathcal{C}$ there exist real numbers $c_0, \beta_0 > 0$, depending only on $\mathcal{A}$, m and G, such that, for any $0 < c_* \leq c_0$ and $0 < \beta_\# \leq \beta_0$ the following hold.*

There exists a ν_0 such that if $\nu \leq \nu_0$, then there exist a closed set $Q' = Q'(c_*, \beta_\#, \nu) \subset Q = Q(c_*, \beta_\#, \nu)$, and a $\mathcal{C}^{s_*}$-mapping Φ

$$\Phi : \mathcal{O}_{\gamma_*}(1/4, \mu_*^2/2) \times Q \to \mathcal{O}_{\gamma_*}(1/2, \mu_*^2), \quad \mu_* = \tfrac{c_*}{2\sqrt{2}}, \quad \gamma_* = (0, m_* + 2),$$

real holomorphic and symplectic for each parameter $\rho \in Q$, such that

$$(h_{\rm up} + f) \circ \Phi(r, w, \rho) = \langle \Omega'(\rho), r\rangle + \frac{1}{2}\langle w, A'(\rho)w\rangle + f'(r, w, \rho)$$

with the following properties:

(i) *the frequency vector Ω' satisfies*

$$|\Omega' - \Omega|_{\mathcal{C}^{s_*-1}(Q)} \leq \nu^{1+\aleph},$$

and the matrix

$$A'(\rho) = A'_\infty(\rho) \oplus H'(\rho) \in \mathcal{NF}_\infty$$

satisfies

$$||\partial_\rho^j (H'(\rho) - \nu K(\rho))|| \leq \nu^{1+\aleph},$$

for $|j| \leq s_*$ and $\rho \in Q$;

(ii) *for any* $x \in \mathcal{O}_{\gamma_*}(1/4, \mu_*^2/2)$, $\rho \in Q$ *and* $|j| \le s_* - 1$,

$$||\partial_\rho^j(\Phi(x,\rho) - x)||_{\gamma_*} + \left\|\partial_\rho^j(d\Phi(x,r) - I)\right\|_{\gamma^*,\varkappa} \le \nu^{\frac{1}{2}-\aleph(\kappa+2)};$$

(iii) *for* $\rho \in Q'$ *and* $\zeta = r = 0$

$$d_r f' = d_\theta f' = d_\zeta f' = d_\zeta^2 f' = 0;$$

(iv) *if* $\mathcal{A}$ *is strongly admissible, then*

$$\lim_{\nu\to 0} \operatorname{meas} Q'(c_*, \beta_\#, \nu) = (1 - c_*)^{\#\mathcal{A}}.$$

If $\mathcal{A}$ *is admissible but not strongly admissible, then*

$$\liminf_{\nu\to 0} \operatorname{meas} Q'(c_*, \beta_\#, \nu) \ge \tfrac{1}{2} c_0^{\#\mathcal{A}}.$$

The exponent $\aleph$ *is defined by* $\aleph(\kappa+2) = \min(\frac{1}{8}, \alpha)$ *where* α *and* κ *are given in Corollary 6.9.*

Proof. By Proposition 10.1 we know that the Hamiltonian h_{up} of (10.3) satisfies the Hypotheses A1–A3 of Sect. 6.2 with the choice of parameters (10.4)–(10.6)—c', δ_0 are here still to be determined—on any ball $\mathcal{D} \subset Q(c_*, \beta_\#, \nu) \subset [c_*, 1]^{\mathcal{A}}$ with

$$\operatorname{meas}([c_*, 1]^{\mathcal{A}} \backslash Q(c_*, \beta_\#, \nu)) \le C\nu^{\beta_\#}. \tag{10.22}$$

In order to apply Corollary 6.9 to the Hamiltonian $h_{up} + f$ it remains to verify the assumptions (a), (b) of that corollary, and (6.25).

Choose $\aleph$ so that $\aleph(\kappa+2) = \min(\frac{1}{8}, \alpha)$. (Here κ and α are given in Corollary 6.9.) If we take $\beta_0 \le \aleph^2$, then

$$\chi,\ \xi \le \text{Ct.}\nu^{1-\aleph^2} \quad \text{and} \quad \varepsilon \le \text{Ct.}(\nu^{1-\aleph^2})^{\frac{3}{2}}$$

for any $\beta_\# \le \beta_0$. By (10.4) and (10.5) we have

$$c' = \delta_0 \ge \nu^{1+\aleph}.$$

Then a) and b) are fulfilled.

The smallness condition (6.25) in Corollary 6.9, is now easily seen hold, by the assumption on $\aleph$, if we take ν sufficiently small. (Notice that this bound on ν depends on c_* through μ_*.) We can therefore apply this corollary: there exists a closed subset $\mathcal{D}'(\nu) \subset \mathcal{D}$, with the measure bound (6.26) becomes

$$\operatorname{meas}(\mathcal{D}\backslash\mathcal{D}'(\nu)) \le \frac{1}{\varepsilon_0}\delta_0^{-\aleph\kappa}\varepsilon^\alpha \le \nu^\aleph,$$

(by the assumption on $\aleph$); the bound in (ii) follows since $c' \ge \nu^{1+\aleph}$; the bound in (iii) holds if ν_0 is small enough. The diffeomorphism Φ trivially extends from $\mathcal{D}$ to Q since it equals the identity near the boundary of $\mathcal{D}$.

In order to prove (iv), assume first that $\mathcal{A}$ is strongly admissible. Then for any c_* the sets $Q_\nu = Q(c_*, \beta_\#, \nu)$, form an increasing system of open sets in $[c_*, 1]^{\mathcal{A}}$ such that their union is of full measure. So for any $\epsilon > 0$ we can find $\nu_\epsilon > 0$ such that $\operatorname{meas} Q_\nu \geq (1-\epsilon)(1-c_*)^n$ $(n = \#\mathcal{A})$ if $\nu \leq \nu_\epsilon$. Since Q_{ν_ϵ} is open there is a finite disjoint union $\cup_{j=1}^N \mathcal{D}_j \subset Q_{\nu_\epsilon}$ of open balls (or cubes) whose measure differ from that of Q_{ν_ϵ} by at most $\epsilon(1-c_*)^n$. [Use for example the Vitali covering theorem.]

For any $j \geq 1$ we construct a closed set $\mathcal{D}'_j(\nu)$ as above. Then

$$\mathcal{D}'_j(\nu) \subset \mathcal{D}_j \subset Q_{\nu_\epsilon} \subset Q_\nu$$

for any $0 < \nu \leq \nu_\epsilon$, and $\operatorname{meas}(\mathcal{D}_j \backslash \mathcal{D}'_j(\nu)) \leq \nu^{\aleph}$. If now $Q'_\nu = \cup_{j=1}^N \mathcal{D}'_j(\nu)$ and $\nu'_\epsilon \in [0, \nu_\epsilon]$ is sufficiently small, then $\operatorname{meas}(Q_\nu \backslash Q'_\nu) \leq 2\epsilon(1-c_*)^n$ for all $0 < \nu \leq \nu'_\epsilon$. This implies the first assertion in (iv). To prove the second we simply replace in the argument above the cube $[0,1]^{\mathcal{A}}$ by the set $\mathcal{D}_0$ as in (10.7). □

10.2.1 Proof of theorems 1.3 and 1.4

Proof. Given $\beta_\#$. For any c_* and ν, let $Q'(c_*, \nu) \subset Q(c_*, \beta_\#, \nu)$ be the closed set defined in Theorem 10.2. Then, for any $c_* > 0$,

$$\bigcup_{\nu \in \mathbb{Q}^*} Q'(c_*, \nu)$$

is of Lebesgue measure: $= (1-c_*)^{\#\mathcal{A}}$ when $\mathcal{A}$ is strongly admissible; $\geq c_0^{\#\mathcal{A}}$ when $\mathcal{A}$ is admissible. It follows that the set

$$\tilde{\mathfrak{J}} = \left\{ I = \nu\rho : \rho \in \bigcup_{\substack{c_*, \nu \in \mathbb{Q}^* \\ \nu^{\beta_\#} \leq c_*}} Q'(c_*, \nu) \right\}$$

at $I = 0$ has: density $= 1$ when $\mathcal{A}$ is strongly admissible; positive density when $\mathcal{A}$ is admissible.

Chose an enumeration $\{(c_j, \nu_j)\}_j$ of $\mathbb{Q}^* \times \mathbb{Q}^*$ and let $\tilde{\mathfrak{J}}_j = \nu_j Q'(c_j, \nu_j)$ so that $\tilde{\mathfrak{J}} = \bigcup_j \tilde{\mathfrak{J}}_j$.

Now we fix j and let $\nu = \nu_j$. We define for any $I \in \tilde{\mathfrak{J}}_j$,

$$U'_j(\theta_{\mathcal{A}}, I = \nu\rho) = \Psi_\rho \circ \Phi(r_{\mathcal{A}} = 0, \theta_{\mathcal{A}}, \zeta_{\mathcal{L}} = 0, \rho).$$

We have, by Theorem 10.2,

$$||\Phi(x, \rho) - x||_{\gamma_*} \leq \nu^{\frac{1}{2} - \aleph(\kappa+2)};$$

for any $x \in \mathcal{O}_{\gamma_*}(1/4, \mu_*^2/2)$, $\rho \in Q(c_j, \beta_\#, \nu_j)$, and, by Theorem 10.1,

$$\begin{aligned} || \Psi_\rho(r, \theta, \xi_{\mathcal{L}}, \eta_{\mathcal{L}}) - (\sqrt{\nu\rho}\cos(\theta), \sqrt{\nu\rho}\sin(\theta), \sqrt{\nu\rho}\xi_{\mathcal{L}}, \sqrt{\nu\rho}\eta_{\mathcal{L}}) \, ||_{\gamma_*} \\ \leq C(\sqrt{\nu}\,|r| + \sqrt{\nu}\,\|(\xi_{\mathcal{L}}, \eta_{\mathcal{L}})\|_{\gamma_*} + \nu^{\frac{3}{2}})\nu^{-\tilde{c}\beta_\#} \end{aligned}$$

for all $(r,\theta,\xi_{\mathcal{L}},\eta_{\mathcal{L}}) \in \mathcal{O}_\gamma(\frac{1}{2},\mu_*^2) \cap \{\theta \text{ real}\}$. Therefore

$$|| U_j'(\theta_{\mathcal{A}},\nu\rho) - (\sqrt{\nu\rho}\cos(\theta),\sqrt{\nu\rho}\sin(\theta),0,0) ||_{\gamma_*} \\ \leq C(\sqrt{\nu}\nu^{\frac{1}{2}-\aleph(\kappa+2)}+\nu^{\frac{3}{2}})\nu^{-\tilde{c}\beta_\#} \leq C\nu^{1-\aleph(\kappa+2)-\tilde{c}\beta_\#} \leq CI^{1-\aleph(\kappa+2)-\tilde{c}\beta_\#-\beta_\#}$$

which is $\leq CI^{1-\aleph(\kappa+3)}$ if $\beta_\#$ is small enough. Thus U_j' verifies (1.26).

Also, by Theorem 10.2, the frequency vector Ω_j' satisfies

$$|\Omega_j'(\rho) - \Omega(\rho)| \leq \nu^{1+\aleph} \leq CI^{1+\aleph-\beta_\#} \leq CI^{1+\frac{\aleph}{2}}$$

for $\rho \in Q(c_*,\beta_\#,\nu)$, and, by Theorem 5.1,

$$\Omega(\rho) = \omega_{\mathcal{A}} + \nu M\rho.$$

Therefore the vector $\Omega_{\mathcal{A},j}'(\nu\rho) = \Omega_j'(\rho)$ will satisfy (1.27).

Part (i), for $\rho \in \tilde{\mathfrak{J}}_j$ is clear by construction.

If ρ is such that $\mathcal{F} = \mathcal{F}_\rho$ is non-void, then the eigenvalues $\{\pm\mathbf{i}\Lambda_a(\rho), a \in \mathcal{F}\}$ of $JK(\rho)$ verifies (see (5.8))

$$|\Im\Lambda_a(\rho)| \geq C^{-1}\nu^{\tilde{c}\beta_\#}, \quad \forall a \in \mathcal{F}.$$

Since, by Theorem 10.2,

$$||\frac{1}{\nu}JH'(\rho) - JK(\rho)|| \leq \nu^{\aleph},$$

it follows (see for example Lemma A2 in [Eli98] and Lemma C.2 in [EGK14]) that the eigenvalues of the matrix $\frac{1}{\nu}JH'(\rho)$, hence those of $JH'(\rho)$, have real parts bounded away from 0 when $\tilde{c}\beta_\# < \aleph$ and ν is small enough. This proves (iii).

If the $\tilde{\mathfrak{J}}_j$'s were mutually disjoint, the mappings U_j' would extend to a mapping U' on $\tilde{\mathfrak{J}}$. But they are not. However there are closed subsets $\mathfrak{J}_j$ of $\tilde{\mathfrak{J}}_j$, mutually disjoint, such that the density of the set $\mathfrak{J} = \bigcup_j \mathfrak{J}_j$ at $I = 0$ is the same as that of the set $\tilde{\mathfrak{J}}$. Now we just restrict each U_j' to $\mathfrak{J}_j$, and these restrictions give a mapping U' on $\mathfrak{J}$.

[To see the existence of the sets $\mathfrak{J}_j$ we construct, by induction, Borel subsets $\mathfrak{J}_j'$ of $\tilde{\mathfrak{J}}_j$, mutually disjoint, such that $\bigcup_j \mathfrak{J}_j' = \tilde{\mathfrak{J}}$. The set $\mathfrak{J}_j'$ are not closed, but each has a closed subset $\mathfrak{J}_j$ such that $\text{meas}(\mathfrak{J}_j' \backslash \mathfrak{J}_j) < 2^{-j}\,\text{meas}(\mathfrak{J}_j')$ by the regularity of the Lebesgue measure. Since each $\tilde{\mathfrak{J}}_j$ is separated from $I = 0$, it follows that the density of $\mathfrak{J} = \bigcup_j \mathfrak{J}_j$ at 0 is the same as that of $\tilde{\mathfrak{J}}$.] □

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Appendix A. Proofs of Lemmas 2.7 and 4.5

For any $\gamma = (\gamma_1, \gamma_2)$ let us denote by Z_γ the space of complex sequences $v = (v_s, s \in \mathbb{Z}^d)$ with the finite norm $\|v\|_\gamma$, defined by the same relation as the norm in the space Y_γ. By $M_{\gamma,0}$ we denote the space of complex $\mathbb{Z}^d \times \mathbb{Z}^d$—matrices, given a norm, defined by the same formula as the norm in $\mathcal{M}_{\gamma,0}$, but with $[a-b]$ replaced by $|a-b|$.

For any vector $v \in Z_\varrho$, $\varrho \geq 0$, we will denote by $\mathcal{F}(v)$ its Fourier-transform:

$$\mathcal{F}(v) = u(x) \;\Leftrightarrow\; u(x) = \sum v_a e^{\mathbf{i}\langle a,x\rangle}.$$

By Example 2.1 if $u(x)$ is a bounded real holomorphic function with the radius of analyticity $\varrho' > 0$, then $\mathcal{F}^{-1}u \in Z_\varrho$ for $\varrho < \varrho'$. Finally, for a Banach space X and $r > 0$ we denote by $B_r(X)$ the open ball $\{x \in X \mid |x|_X < r\}$.

Let F be the Fourier-image of the nonlinearity g, regarded as the mapping $u(x) \mapsto g(x, u(x))$, i.e. $F(v) = \mathcal{F}^{-1} g(x, \mathcal{F}(v)(x))$.

Lemma A.1. *For sufficiently small $\mu_g > 0$, $\gamma_{g1} > 0$ and for $\gamma_g = (\gamma_{g1}, \gamma_{g2})$, where $\gamma_{g2} \geq m_* + \varkappa$ we have:*

(i) *F defines a real holomorphic mapping $B_{\mu_g}(Z_{\gamma_g}) \to Z_{\gamma_g}$,*
(ii) *dF defines a real holomorphic mapping $B_{\mu_g}(Z_{\gamma_g}) \to M^b_{\gamma',0}$, where $\gamma' = (\gamma_{g1}, \gamma_{g2} - m_*)$.*

Proof. (i) For sufficiently small $\varrho', \mu > 0$ the nonlinearity g defines a real holomorphic function $g : \mathbb{T}^d_{\varrho'} \times B_\mu(\mathbb{C}) \to \mathbb{C}$ and the norm of this function is bounded by some constant M. We may write it as $g(x,u) = \sum_{r=3}^\infty g_r(x) u^r$, where $g_r(x) = \frac{1}{r!} \frac{\partial^r}{\partial u^r} g(x,u)\,|_{u=0}$. So $g_r(x)$ is holomorphic in $x \in \mathbb{T}^d_{\varrho'}$ and by the Cauchy estimate $|g_r| \leq M\mu^{-r}$ for all $x \in \mathbb{T}^d_{\varrho'}$. Accordingly,

$$\|\mathcal{F}^{-1} g_r\|_{\gamma_g} \leq C_\varrho M \mu^{-r} \quad \text{if} \quad 0 \leq \gamma_{g1} \leq \varrho,$$

for any $\varrho < \varrho'$; cf. Example 2.1. We may write $F(v)$ as

$$F(v) = \sum_{r=3}^\infty (\mathcal{F}^{-1} g_r) \star \underbrace{v \star \cdots \star v}_{r} =: \sum_{r=3}^\infty F_r(v). \tag{A.1}$$

Since the space Z_{γ_g} is an algebra with respect to the convolution (see Lemma 1.1 in [EK08]), the r-th term of the sum is a mapping from Z_{γ_g} to itself, whose norm is bounded as follows:

$$\|(\mathcal{F}^{-1} g_r) \star \underbrace{v \star \cdots \star v}_{r}\|_{\gamma_g} \leq C_1 C^{r+1} \mu^{-r} \|v\|^r_{\gamma_g}. \tag{A.2}$$

This implies the assertion with a suitable $\mu_g > 0$.

(ii) The assertion (i) and the Cauchy estimate imply that the operator-norm of $dF(v)$ is bounded if $|v|_\gamma < \mu_g$. To estimate $|dF(v)|_{\gamma',0}$, for $r \geq 3$ consider the term $F_r(v)$ in (A.1). This is the Fourier transform of the mapping $u(x) \mapsto g_r(x) u(x)^r$, and its differential $dF_r(v)$ is a linear operator in Z_{γ_g} which is the Fourier-image of the operator of multiplication by the function $r g_r(x) u^{r-1}(x)$. So the matrix $\big(dF_r(v)^b_a,\, a, b \in \mathbb{Z}^d\big)$ of the former operator is nothing but the matrix of the latter operator, written in the trigonometric basis $\{e^{\mathbf{i}\langle a,x\rangle}\}$. Therefore

$$(dF_r(v))^b_a = (2\pi)^{-d} \int e^{-\mathbf{i}\langle b,x\rangle} r g_r(x) u^{r-1} e^{\mathbf{i}\langle a,x\rangle}\, dx.$$

That is, $(dF_r(v))_a^b = G_r(b-a)$, where $G_r(a)$ is the Fourier transform of the function $rg_r(x)u^{r-1}$. So

$$\begin{aligned}|dF_r(v)|_{\gamma',0} &= \sup_a C\sum_b |(|dF_r(v)_a^b|e^{\gamma_{g1}|a-b|}\langle a-b\rangle^{\gamma_{g2}-m_*}\\ &= \sup_a C\sum_b |(|G_r(a-b)|e^{\gamma_{g1}|a-b|}\langle a-b\rangle^{\gamma_{g2}}\langle a-b\rangle^{-m_*} \le C'|G_r(\cdot)|_{\gamma_g}\\ &\le C\left(\sum_c |G_r(c)|^2 e^{2\gamma_{g1}|c|}\langle c\rangle^{2(\gamma_{g2}-m_*)}\right)^{1/2}\left(\sum_c \langle c\rangle^{-2m_*}\right)^{1/2} = C'|G_r|_{\gamma_g}\end{aligned}$$

(we recall that $m_* > d/2$). Applying (A.2) with r convolutions instead of $r+1$, we see that $|G_r(\cdot)|_{\gamma_g} \le C_2C^r\mu^{-r}\|v\|_{\gamma_g}^{r-1}$. So

$$|(dF_r(v))|_{\gamma',0} \le C_3C^r\mu^{-r}\|v\|_{\gamma_g}^{r-1}.$$

Since $dF(v) = \sum_{r\ge3} dF_r(v)$, then the assertion ii) follows, if we replace μ_g by a smaller positive number. □

Proof of Lemma 2.7. Let us consider the functional $h_{\ge4}(\zeta)$ as in (1.12), and write it as $h_{\ge4}(\zeta) = \mathbf{G}\circ\Upsilon\circ D^-\zeta$. Here D^- is defined in (4.14), Υ is the operator

$$\Upsilon : Y_\gamma \to Z_\gamma, \quad \zeta \to v, \quad v_a = (\xi_a + \eta_{-a})/\sqrt{2}\ \ \forall\, a,$$

and $\mathbf{G}(v) = \int g(x,(\mathcal{F}^{-1}v)(x))\,dx$. Lemma A.1 with F replaced by $\mathbf{G}$ immediately implies that p is a real holomorphic function on $B_{\mu_g}(Y_\gamma)$ with a suitable $\mu_g > 0$. Next, since

$$\nabla h_{\ge4}(\zeta) = D^-\circ{}^t\Upsilon\circ\nabla\mathbf{G}(\Upsilon\circ D^-\zeta),$$

where $\nabla\mathbf{G} = F$ is the map in Lemma A.1, then $\nabla h_{\ge4}$ defines a real holomorphic mapping $B_{\mu_g}(Y_\gamma)\to Y_\gamma$, bounded uniformly in $\gamma_* \le \gamma \le \gamma_g$.

By the Cauchy estimate, for any $0 < \mu_g' < \mu$ the Hessian of $h_{\ge4}$ defines an analytic mapping

$$\nabla^2 h_{\ge4} : B_{\mu_g'}(Y_\gamma) \to \mathcal{B}(Y_\gamma, Y_\gamma), \tag{A.3}$$

and $\nabla^2 h_{\ge4}(\zeta)$ is the linear operator

$$\nabla^2 h_{\ge4}(\zeta) = D^-({}^t\Upsilon\ \nabla^2\mathbf{G}(\Upsilon\circ D^-\zeta)\ \Upsilon)D^-.$$

Note that for any infinite matrix A the matrix ${}^t\Upsilon A\Upsilon$ is formed by 2×2–blocks and satisfies

$$|({}^t\Upsilon A\Upsilon)_a^b| \le \frac{1}{2}\sum_{a'=\pm a,\,b'=\pm b}|A_{a'}^{b'}|.$$

Noting also that for $a' = \pm a$, $b' = \pm b$ we have $[a-b] \le |a'-b'|$, and that $\min(r_1,r_2)^2 r_1^{-1}r_2^{-1} \le 1$ if $r_1, r_2 \ge 1$, we find that the first term which enters the definition of $|\nabla^2 h_{\ge4}|_{\gamma',2}$ estimates as follows:

$$\begin{aligned}&\sup_{a\in\mathbb{Z}^d}\textstyle\sum_{b\in\mathbb{Z}^d}|\nabla^2 h_{\ge4}|_a^b e^{\gamma_1[a-b]}\max(1,[a-b])^{\gamma_2-m_*}\min(\langle a\rangle,\langle b\rangle)^2\\ &\le \sup_{a\in\mathbb{Z}^d}\tfrac{1}{2}\textstyle\sum_{b\in\mathbb{Z}^d}\sum_{a'=\pm a,b'=\pm b}|\nabla^2\mathbf{G}|_{a'}^{b'}e^{\gamma_1|a'-b'|}\max(1,[a'-b'])^{\gamma_2-m_*}\frac{\min(\langle a'\rangle,\langle b'\rangle)^2}{\langle a'\rangle\langle b'\rangle}\\ &\le \sup_{a'\in\mathbb{Z}^d}2\textstyle\sum_{b'\in\mathbb{Z}^d}|\nabla^2\mathbf{G}|_{a'}^{b'}e^{\gamma_1|a'-b'|}\max(1,[a'-b'])^{\gamma_2-m_*} \le 2|\nabla^2\mathbf{G}|_{\gamma',0}.\end{aligned}$$

The second term which enters the definition of the norm estimates similar, so

$$|\nabla^2 h_{\geq 4}(\zeta)|_{\gamma',2} \leq 2|\nabla^2 \mathbf{G}(v)|_{\gamma'} = 2|dF(v)|_{\gamma'}, \tag{A.4}$$

$v = \Upsilon\zeta$. In view of (A.3) and item ii) of Lemma A.1, the mapping

$$\nabla^2 h_{\geq 4} : B_{\mu'_g}(Y_\gamma) \to \mathcal{M}^b_{\gamma,2},$$

is real holomorphic and is bounded in norm by a γ-independent constant. Jointly with (A.4) and Lemma A.1 this implies the assertion of Lemma 2.7, if we replace μ_g by any smaller positive number. □

Proof of Lemma 4.5. The proof is similar to that of Lemma 2.7 but simpler, and we restrict ourselves to estimating the Hessian of Q^r. Let us start with the Hessian of P^r. For any $\zeta \in \mathcal{O}(1,1,1)$ we have:

$$d^2 P^r(\zeta)(\zeta',\zeta') = 2M \sum_a \sum_\varsigma A^\varsigma_a (\zeta^{\varsigma_1}_{a_1} \dots \zeta^{\varsigma_{r-2}}_{a_{r-2}}) \zeta'^{\varsigma_{r-1}}_{a_{r-1}} \zeta'^{\varsigma_r}_{a_r} + \dots =: R(\zeta)(\zeta',\zeta') + \dots . \tag{A.5}$$

Here the dots ... stand for similar sums, where the pair ζ', ζ' replaces ζ, ζ on other $\binom{r}{2}$ positions. For any $b_1, b_2 \in \mathbb{Z}^d$ the element $(\nabla^2_1 P^r(\zeta))^{b_2}_{b_1}$ of the Hessian $(\nabla^2 P^r(\zeta))^{b_2}_{b_1}$, coming from the component R of $d^2 P^r$, corresponds to the quadratic form $R(\zeta)\Big(1_{b_1}(\xi,\eta), 1_{b_2}(\xi,\eta)\Big)$, where 1_b stands for the δ-function on the lattice $\mathbb{Z}^d$, equal one at b at equal zero outside b.

Denote by $\tilde\zeta$ the vector $\tilde\zeta_a = |\zeta_a| + |\zeta_{-a}|$, $a \in \mathbb{Z}^d$. Then $|\zeta_{(\varsigma^0_j\ a_j)}| \leq |\tilde\zeta_{a_j}|$, and we see from (A.5) that $|\nabla^2_1 P^r(\zeta)^{b_2}_{b_1}|$ is bounded by

$$2^{r-1} M \textstyle\sum_{a_1+\dots+a_{r-2}=-\varsigma^0_{r-1} b_{r-1} - \varsigma^0_r b_r} \tilde\zeta_{a_1} \dots \tilde\zeta_{a_{r-2}} = 2^{r-1} M (\tilde\zeta \star \dots \star \tilde\zeta)(-\varsigma^0_{r-1} b_1 - \varsigma^0_r b_2).$$

Since the space Y_γ is an algebra with respect to convolution, then

$$|\tilde\zeta \star \dots \star \tilde\zeta|_\gamma \leq C^{r-3} |\tilde\zeta|^{r-2}_\gamma. \tag{A.6}$$

As in the proof of Lemma 2.7, $|\nabla^2 Q^r(\zeta)^{b_2}_{b_1}| \leq \langle b_1\rangle^{-1} \langle b_2\rangle^{-1} |\nabla^2 P^r(D^-\zeta)^{b_2}_{b_1}|$. Denoting by $\nabla^2_1 Q^r$ the component of $\nabla^2 Q^r$, corresponding to $\nabla^2_1 P^r$, denoting $b'_1 = -\varsigma^0_{r-1} b_1$, $b'_2 = \varsigma^0_r b_2$, and using that $[b_1 - b_2] \leq |b'_1 - b'_2|$, we find :

$$\begin{aligned}
&\sup_{b_1} \textstyle\sum_{b_2} |(\nabla^2_1 Q^r)^{b_2}_{b_1}| e^{\gamma_1 [b_1 - b_2]} \max(1, [b_1 - b_2])^{\gamma_2 - m_*} \min(\langle b_1\rangle, \langle b_2\rangle)^2 \\
&\leq C^r M \sup_{b_1} \textstyle\sum_{b_2} (\tilde\zeta \star \dots \star \tilde\zeta)(b'_1 - b'_2) e^{\gamma_1 [b_1 - b_2]} \max(1, [b_1 - b_2])^{\gamma_2 - m_*} \frac{\min(\langle b_1\rangle, \langle b_2\rangle)^2}{\langle b_1\rangle \langle b_2\rangle} \\
&\leq C^r M \sup_{b'_1} \textstyle\sum_{b'_2} (\tilde\zeta \star \dots \star \tilde\zeta)(b'_1 - b'_2) e^{\gamma_1 [b_1 - b_2]} \langle b_1 - b_2\rangle^{\gamma_2 - m_*} \leq C'^r M |\tilde\zeta|^{r-2}_\gamma \leq C^r M
\end{aligned}$$

(since $|\zeta|_\gamma \leq 1$). This implies the estimate for $\nabla^2_1 Q^r$, required by the lemma. Other components of $\nabla^2 Q^r$, corresponding to the dots in (A.5), may be estimated in the same way. □

Appendix B. Examples

In this appendix we discuss some examples of Hamiltonian operators $\mathcal{H}(\rho) = \mathbf{i}JK(\rho)$ defined in (5.13), corresponding to various dimensions d and sets $\mathcal{A}$. In particular we are interested in examples which give rise to partially hyperbolic KAM solutions.

Examples with $(\mathcal{L}_f \times \mathcal{L}_f)_+ = \emptyset$.

As we noticed in (5.14), if $(\mathcal{L}_f \times \mathcal{L}_f)_+ = \emptyset$ then $\mathcal{H}$ is Hermitian, so the constructed KAM-solutions are linearly stable. This is always the case when $d = 1$.

When $d = 2$ and $\mathcal{A} = \{(k, 0), (0, \ell)\}$ with the additional assumption that neither k^2 nor ℓ^2 can be written as the sum of squares of two natural numbers, we also have $(\mathcal{L}_f \times \mathcal{L}_f)_+ = \emptyset$. Similar examples can be constructed in higher dimension, for instance for $d = 3$ we can take $\mathcal{A} = \{(1, 0, 0), (0, 2, 0)\}$ or $\mathcal{A} = \{(1, 0, 0), (0, 2, 0), (0, 0, 3)\}$.

We note that in [GY06] the authors perturb solutions (1.5), corresponding to sets $\mathcal{A}$ for which $(\mathcal{L}_f \times \mathcal{L}_f)_+ = \emptyset$ and $(\mathcal{L}_f \times \mathcal{L}_f)_- = \emptyset$. This significantly simplifies the analysis since in that case there is no matrix K in the normal form (4.5) and the unperturbed quadratic Hamiltonian is diagonal.

Examples with $(\mathcal{L}_f \times \mathcal{L}_f)_+ \neq \emptyset$.

In this case hyperbolic directions may appear as we show below.

The choice $\mathcal{A} = \{(j, k), (0, -k)\}$ leads to $((j, -k), (0, k)) \in (\mathcal{L}_f \times \mathcal{L}_f)_+$.

Note that this example can be plunged in higher dimensions, e.g. the 3d-set $\mathcal{A} = \{(j, k, 0), (0, -k, 0)\}$ leads to a non trivial $(\mathcal{L}_f \times \mathcal{L}_f)_+$.

Examples with hyperbolic directions

Here we give examples of normal forms with hyperbolic eigenvalues, first in dimension two, then—in higher dimensions. That is, for the beam equation (1.1) we will find admissible sets $\mathcal{A}$ such that the corresponding matrices $\mathbf{i}JK(\rho)$ in the normal form (4.5) have unstable directions. Then the set $\mathcal{F}$ in (1.24) is non-empty and by Theorem 1.4 the time-quasiperiodic solutions of (1.1), constructed in the theorem, are linearly unstable.

We begin with dimension $d = 2$. Let

$$\mathcal{A} = \{(0, 1), (1, -1)\}.$$

We easily compute using (4.30), (4.30) that

$$\mathcal{L}_f = \{(0, -1), (1, 0), (-1, 0), (1, 1), (-1, 1), (-1, -1))\},$$

and

$$(\mathcal{L}_f \times \mathcal{L}_f)_+ = \{\big((0, -1), (1, 1)\big); \big((1, 1), (0, -1)\big)\}, \quad (\mathcal{L}_f \times \mathcal{L}_f)_- = \emptyset.$$

So in this case the decomposition (5.19) of the Hamiltonian operator $\mathcal{H}(\rho) = \mathbf{i}JK(\rho)$ reads

$$\mathcal{H}(\rho) = \mathcal{H}_1(\rho) \oplus \mathcal{H}_2(\rho) \oplus \mathcal{H}_3(\rho) \oplus \mathcal{H}_4(\rho) \oplus \mathcal{H}_5(\rho),$$

where $\mathcal{H}_1(\rho) \oplus \mathcal{H}_2(\rho) \oplus \mathcal{H}_3(\rho) \oplus \mathcal{H}_4(\rho)$ is a diagonal operator with purely imaginary eigenvalues and $\mathcal{H}_5(\rho)$ is an operator in $\mathbb{C}^4$ which may have hyperbolic eigenvalues. That is, now $M = 5$ and $M_0 = 4$.

Let us denote $\zeta_1 = (\xi_1, \eta_1)$ (reps. $\zeta_2 = (\xi_2, \eta_2)$) the (ξ, η)-variables corresponding to the mode $(0, -1)$ (reps. $(1, 1)$). We also denote $\rho_1 = \rho_{(1,0)}$, $\rho_2 = \rho_{(1,-1)}$, $\lambda_1 = \sqrt{1 + m}$ and $\lambda_2 = \sqrt{4 + m}$. By construction $\mathcal{H}_5(\rho)$ is the restriction of the Hamiltonian $\langle K(m, \rho)\zeta_f, \zeta_f\rangle$ to the modes (ξ_1, η_1) and (ξ_2, η_2). We calculate using (4.47) that

$$\langle \mathcal{H}_5(\rho)(\zeta_1, \zeta_2), (\zeta_1, \zeta_2)\rangle = \beta(\rho)\xi_1\eta_1 + \gamma(\rho)\xi_2\eta_2 + \alpha(\rho)(\eta_1\eta_2 + \xi_1\xi_2), \tag{B.1}$$

where

$$\alpha(\rho) = \frac{6}{4\pi^2}\frac{\sqrt{\rho_1\rho_2}}{\lambda_1\lambda_2}, \quad \beta(\rho) = \frac{3}{4\pi^2}\frac{1}{\lambda_1}\Big(\frac{\rho_1}{\lambda_1} - \frac{2\rho_2}{\lambda_2}\Big), \quad \gamma(\rho) = \frac{3}{4\pi^2}\frac{1}{\lambda_2}\Big(\frac{\rho_2}{\lambda_2} - \frac{2\rho_1}{\lambda_1}\Big).$$

Thus the linear Hamiltonian system, governing the two modes, reads[30]

$$\begin{cases} \dot\xi_1 = -\mathbf{i}(\beta\xi_1 + \alpha\eta_2), \\ \dot\eta_1 = \mathbf{i}(\beta\eta_1 + \alpha\xi_2), \\ \dot\xi_2 = -\mathbf{i}(\gamma\xi_2 + \alpha\eta_1), \\ \dot\eta_2 = \mathbf{i}(\gamma\eta_2 + \alpha\xi_1). \end{cases}$$

So the Hamiltonian operator $\mathcal{H}_5$ has the matrix $\mathbf{i}L$, where

$$L = \begin{pmatrix} -\beta & 0 & 0 & -\alpha \\ 0 & \beta & \alpha & 0 \\ 0 & -\alpha & -\gamma & 0 \\ \alpha & 0 & 0 & \gamma \end{pmatrix}.$$

We calculate the characteristic polynomial of L and obtain after a factorisation that

$$\det(L - \lambda I) = \left(\lambda^2 + (\gamma - \beta)\lambda - \beta\gamma + \alpha^2\right)\left(\lambda^2 - (\gamma - \beta)\lambda - \beta\gamma + \alpha^2\right).$$

Both quadratic polynomials which are the factors in the r.h.s. have the same discriminant $\Delta = (\beta + \gamma)^2 - 4\alpha^2$. If $\rho_1 \sim 1$ and $0 < \rho_2 \ll 1$, then $\Delta > 0$. So all eigenvalues of L are real, while the eigenvalues of $\mathcal{H}_5$ and $\mathcal{H}$ are pure imaginary (in agreement with Lemma 5.4). But if $\rho_1 = \rho_2 = \rho$, then

$$\gamma - \beta = \frac{3\rho}{4\pi^2}\left(\frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2}\right), \quad \beta + \gamma = \frac{3\rho}{4\pi^2}\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} - \frac{4}{\lambda_1\lambda_2}\right), \quad \alpha = \frac{6\rho}{4\pi^2}\frac{1}{\lambda_1\lambda_2},$$

and

$$\Delta = \frac{9\rho}{(2\pi)^4}\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right)\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} - \frac{8}{\lambda_1\lambda_2}\right) \le \frac{9\rho}{(2\pi)^4}\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right)\left(\frac{1}{\lambda_1^2} - \frac{7}{\lambda_2^2}\right).$$

Thus, $\Delta < 0$ for all $m \in [1, 2]$. Since the eigenvalues of the matrix $L = (1/\mathbf{i})\mathcal{H}_5$ are $\pm(\gamma - \beta) \pm \sqrt{\Delta}$, then all four of them have nontrivial imaginary parts for all values of the parameter $m \in [1, 2]$, and accordingly the operator $\mathcal{H}$ has 4 hyperbolic directions. By analyticity, for all $m \in [1, 2]$ with a possible exception of finitely many points, the real parts of the eigenvalues also are non-zero. In this case the operator $\mathcal{H}$ has a quadruple of hyperbolic eigenvalues.

This example can be generalised to any dimension $d \ge 3$. Let us do it for $d = 3$. Let

$$\mathcal{A} = \{(0, 1, 0), (1, -1, 0)\}. \tag{B.2}$$

We verify that $\mathcal{L}_f$ contains 16 points, that $(\mathcal{L}_f \times \mathcal{L}_f)_- = \emptyset$ and

$$\begin{aligned}(\mathcal{L}_f \times \mathcal{L}_f)_+ = \{&((0, -1, 0), (1, 1, 0)); ((1, 1, 0), (0, -1, 0)); \\ &((1, 0, -1), (0, 0, 1)); ((0, 0, 1), (1, 0, -1)); \\ &((1, 0, 1), (0, 0, -1)); ((0, 0, -1), (1, 0, 1))\}.\end{aligned}$$

I.e. $(\mathcal{L}_f \times \mathcal{L}_f)_+$ contains three pairs of symmetric couples $(a, b), (b, a)$ which give rise to three non trivial 2×2-blocks in the matrix $\mathcal{H}$. Now $M = 13$, $M_0 = 10$ and the decomposition (5.19) reads

$$\mathcal{H}(\rho) = \mathcal{H}_1(\rho) \oplus \cdots \oplus \mathcal{H}_{13}(\rho).$$

[30] Recall that the symplectic two-form is: $-\mathbf{i}\sum d\xi \wedge d\eta$.

Here $\mathcal{H}_1(\rho)\oplus\cdots\oplus\mathcal{H}_{10}(\rho)$ is the diagonal part of $\mathcal{H}$ with purely imaginary eigenvalues, while the operators $\mathcal{H}_{11}(\rho)$, $\mathcal{H}_{12}(\rho)$, $\mathcal{H}_{13}(\rho)$ correspond to non-diagonal 4×4-matrices.

Denoting $\rho_1=\rho_{(0,1,0)}$ and $\rho_2=\rho_{(1,-1,0)}$ we find that the restriction of the Hamiltonian $\langle K(m,\rho)\zeta_f,\zeta_f\rangle$ to the modes $(\xi_1,\eta_1):=(\xi_{(0,-1,0)},\eta_{(0,-1,0)})$ and $(\xi_2,\eta_2):=(\xi_{(1,1,0)},\eta_{(1,1,0)})$ is governed by the Hamiltonian (B.1), as in the 2d case. Similarly the restrictions of the Hamiltonian $\langle K(m,\rho)\zeta_f,\zeta_f\rangle$ to the pair of modes $(\xi_{(1,0,-1)},\eta_{(1,0,-1)})$ and $(\xi_{(0,0,1)},\eta_{(0,0,1)})$ and to the pair of modes $(\xi_{(1,0,1)},\eta_{(1,0,1)})$ and $(\xi_{(0,0,-1)},\eta_{(0,0,-1)})$ are given by the same Hamiltonian (B.1). So $\mathcal{H}_{11}(\rho)\equiv\mathcal{H}_{12}(\rho)\equiv\mathcal{H}_{13}(\rho)$ and for $\rho_1=\rho_2$ we have 3 hyperbolic directions, one in each block Y^{f11}, Y^{f12} and Y^{f13} (see (5.17)) with the same eigenvalues.

We notice that the eigenvalues are identically the same for all three blocks, thus the relation (5.32) is violated. This does not contradict Lemma 5.6 since the set (B.2) is not strongly admissible. Indeed, denoting $a=(0,1,0)$, $b=(1,-1,0)$ we see that $c:=a+b=(1,0,0)$. So three points $(0,-1,0),(0,0,\pm1)\in\{x\mid |x|=|a|\}$ all lie at the distance $\sqrt{2}$ from c. Hence, it is not true that $a\measuredangle b$.

Appendix C. Admissible and Strongly Admissible Random *R*-sets

Given d and n, let $B(R)$ be the (round) ball of radius R in $\mathbb{R}^d$, and $\mathbf{B}(R)=B(R)\cap\mathbb{Z}^d$. The family $\Omega=\Omega(R)$ of n-sets $\{a_1,\dots,a_n\}$ in $\mathbf{B}(R)$, $\Omega=\mathbf{B}\times\cdots\times\mathbf{B}$ (n times) has cardinality of order CR^{nd}.

The family on n-sets $\{a_1,\dots,a_n\}$ in Ω such that $|a_j|=|a_k|$ for some $j\neq k$ has cardinality $\le C'R^{nd-1}$ (the constant C' as well as all other constants in this section depend, without saying, on n,d). Its complement in Ω is the set $\Omega_{\rm adm}=\Omega_{\rm adm}(R)$ of admissible n-sets in $\mathbf{B}(R)$. Hence

$$\frac{\#\Omega_{\rm adm}(R)}{\#\Omega(R)}=1-O(R^{-1}),\quad R\to\infty.$$

We provide the set Ω with the uniform probability measure $\mathbb{P}$ and will call elements of Ω *n-points random R-sets*. The calculation above shows that

$$\mathbb{P}\,(\Omega_{\rm adm})\to 1\quad\text{as}\quad R\to\infty. \tag{C.1}$$

That is, admissible n-points random R-sets with large R are typical.

To consider strongly admissible sets, let $S(R)$ be the sphere of radius R in $\mathbb{R}^d$, i.e. the boundary of $B(R)$, and let $\mathbf{S}(R)=S(R)\cap\mathbb{Z}^d$ (this set is non-empty only if R^2 is an integer). We have that, for any $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$\Gamma_{R,d}:=|\mathbf{S}(R)|\le C_\varepsilon R^{d-2+\varepsilon}\quad\forall\,R>0. \tag{C.2}$$

Indeed, for $d=2$ this is a well-known result from number theory (see [HW08], Theorem 338). For $d\ge 3$ it follows by induction and an easy integration argument. For example for $d=3$, then

$$\mathbf{S}(R)=\{a\in\mathbb{Z}^3:|a|^2=R^2\}=\bigcup_{a_3^2\le R^2}\{a=(a_1,a_2,a_3)\in\mathbb{Z}^3:a_1^2+a_2^2=R^2-a_3^2\}$$

so

$$\Gamma_{R,3}=\sum_{n^2\le R^2}\Gamma_{\sqrt{R^2-n^2},2}\le C_\varepsilon\sum_{n^2\le R^2}(R^2-n^2)^{\varepsilon/2}\le C_\varepsilon R^\varepsilon\sum_{n^2\le R^2}\Big(1-(\frac{n}{R})^2\Big)^{\varepsilon/2},$$

which is $\le C_\varepsilon R^\varepsilon(2R+1)\le C'_\varepsilon R^{1+\varepsilon}$.

For vectors $a, b \in \mathbb{Z}^d$ we will write $a \not\angle\!\!\!\angle b$ iff $a \not\angle a + b$. Consider again the ensemble $\Omega = \Omega(R)$ of n-points random R-sets, $\Omega = \{\omega = (a_1, \dots, a_n)\}$, and for $j = 1, \dots, n$ define the random variable ξ_j as $\xi_j(\omega) = a_j$. Consider the event

$$\Omega_{\not\angle\!\!\!\angle} = \{\xi^i \not\angle\!\!\!\angle \xi^j \quad \text{for all} \quad i \neq j\}.$$

Then $\Omega_{\text{s\_adm}} = \Omega_{\text{adm}} \cap \Omega_{\not\angle\!\!\!\angle}$ is the collection of strongly admissible sets. Clearly

$$\mathbb{P}\,(\Omega \backslash \Omega_{\not\angle\!\!\!\angle}) \leq n(n-1)(1 - \mathbb{P}\,\{\xi^1 \not\angle\!\!\!\angle \xi^2\}). \tag{C.3}$$

So if we prove that

$$1 - \mathbb{P}\,\{\xi^1 \not\angle\!\!\!\angle \xi^2\} \leq CR^{-\kappa}, \tag{C.4}$$

then, in view of (C.1), we would show that

$$\mathbb{P}\,(\Omega_{\text{s\_adm}}) \to 1 \quad \text{as} \quad R \to \infty. \tag{C.5}$$

Below we restrict ourselves to the case $d = 3$ since for higher dimension the argument is similar, but more cumbersome. We have that

$$1 - \mathbb{P}\,\{\xi^1 \not\angle\!\!\!\angle \xi^2\} = |\mathbf{B}(R)|^{-2} C^{**}, \quad C^{**} = \#\{(a, b) \in \mathbf{B}(R) \times \mathbf{B}(R) \mid \text{ not } a \not\angle\!\!\!\angle b\}, \tag{C.6}$$

and, denoting $a + b = c$, that

$$C^{**} \leq \#\{(a, c) \in \mathbf{B}(2R) \times \mathbf{B}(2R) \mid \text{ not } a \not\angle c\}. \tag{C.7}$$

Now we will estimate the r.h.s. of (C.7), re-denoting $2R$ back to R. That is, will estimate the cardinality of the set

$$X = \{(a, b) \in \mathbf{B}(R) \times \mathbf{B}(R) \mid \text{ not } a \not\angle b\}.$$

It is clear that $(a, b) \in X$, $a \neq 0$, iff there exist points $a', a'' \in \mathbf{S}(|a|)$ such that b lies in the line $\Pi_{a,a',a''}$, which is perpendicular to the triangle (a, a', a'') and passes through its centre, so it also passes through the origin. Let $v = v_{a,a',a''}$ be a primitive integer vector in the direction of $\Pi_{a,a',a''}$. For any $a \in \mathbb{Z}^d, a \neq 0$, denote

$$\Delta(a) = \big\{\, \{a', a''\} \subset \mathbf{S}(|a|) \backslash \{a\} \mid a' \neq a'' \big\}.$$

Then

$$|\Delta(a)| < \Gamma^2_{|a|,3} \leq C_\theta^2 R^{2\theta}, \quad \theta = \theta_3,$$

see (C.2). For a fixed $a \in \mathbf{B}(R) \backslash \{0\}$ consider the mapping

$$\Delta(a) \ni \{a', a''\} \mapsto v = v_{a,a',a''}.$$

It is clear that each direction $v = v_{a,a',a''}$ gives rise to at most $2R|v|^{-1}$ points b such that $(a, b) \in X$. So, denoting

$$X_a = \{b \in \mathbf{B}(R) \mid (a, b) \in X\},$$

we have

$$|X_a| \leq 2R \sum |v_{a,a',a''}|^{-1}, \quad \text{if } a \neq 0,$$

where the summation goes through all different vectors v, corresponding to various $\{a', a''\} \in \Delta(a)$. As $|v|^{-1}$ is the bigger the smaller $|v|$ is, we see that the r.h.s. is $\leq 2R\sum_{v\in\mathbf{B}(R')\setminus\{0\}} |v|^{-1}$, where R' is any number such that $|\mathbf{B}(R')| \geq |\Delta(a)|$. Since $|\Delta(a)| \leq \Gamma^2_{|a|,3}$, then choosing $R' = R'_a = C\Gamma^{2/3}_{|a|,3}$ we get for any $a \in \mathbf{B}(R)\setminus\{0\}$ that

$$|X_a| \leq 2CR\sum_{\mathbf{B}(R'_a)\setminus\{0\}} |v|^{-1} \leq C_1 R\int_{B(R'_a)} |x|^{-1}\,dx \leq C_2 R(R'_a)^2 = C_3 R\,\Gamma^{4/3}_{|a|,3}.$$

Since $0 \angle b$ for any b, then $X_0 = \{0\}$ and

$$|X| = \sum_{a\in\mathbf{B}(R)} |X_a| \leq 1 + CR \sum_{a\in\mathbf{B}(R)\setminus\{0\}} \Gamma^{4/3}_{|a|,3}.$$

Evoking the estimate (C.2) we finally get that

$$|X| \leq C_1 R\sum_{a\in\mathbf{B}(R)\setminus\{0\}} |a|^{\frac{4}{3}\theta_3} \leq C_2 R\int_{B(R)} |x|^{\frac{4}{3}\theta_3}\,dx \leq C_3 R^{1+3+\frac{4}{3}\theta_3} = C_3 R^{5+1/3+\varepsilon'},$$

with any positive ε'. Jointly with (C.6), (C.7) and the definition of the set X this implies the required relation (C.4) with $\kappa = 2/3 - \varepsilon'$, and (C.5) follows. That is, n-points random R-sets with large R are typical, for any d and any n.

Appendix D. Two Lemmas

D.0.2 Transversality

LEMMA D.1. *Let I be an open interval and let $f : I \to \mathbb{R}$ be a $\mathcal{C}^j$-function whose j:th derivative satisfies*

$$\left|f^{(j)}(x)\right| \geq \delta, \quad \forall x \in I.$$

Then,

$$\text{meas}\{x \in I : |f(x)| < \varepsilon\} \leq C(\frac{\varepsilon}{\delta_0})^{\frac{1}{j}}.$$

C is a constant that only depends on $|f|_{\mathbb{C}^j(I)}$.

Proof. It is enough to prove this for $\varepsilon < 1$. Let $I_1 = I$, $\delta_1 = \delta$ and $g_k = f^{(j-k)}$, $k = 1, \dots j$. Let $\delta_1, \delta_2, \dots \delta_{j+1}$ be a deceasing sequence of positive numbers.

Since $g'_1 = f^{(j)}$ we have $|g'_1(x)| \geq \delta_1$ for all $x \in I_1$ and, hence, the set

$$E_1 = \{x \in I_1 : |g_1(x)| < \delta_2\}$$

has Lebesgue measure $\lesssim \frac{\delta_2}{\delta_1}$. On $I_2 = I_1\setminus E_1$ we have $|g'_2(x)| \geq \delta_2$ for all $x \in I_2$ and, hence, the set

$$E_2 = \{x \in I_2 : |g_2(x)| < \delta_3\}$$

has Lebesgue measure $\lesssim \frac{\delta_3}{\delta_2}$. Continue this j steps. On $I_j = I_{j-1}\setminus E_{j-1}$ we have $\left|g'_j(x)\right| \geq \delta_j$ for all $x \in I_j$ and, hence, the set

$$E_j = \{x \in I_j : |g_j(x)| < \delta_{j+1}\}$$

has Lebesgue measure $\lesssim \frac{\delta_{j+1}}{\delta_j}$.

Now the set $\{x \in I : |f(x)| < \delta_{j+1}\}$ is contained in the union of the sets E_k which has measure

$$\lesssim \frac{\delta_2}{\delta_1} + \cdots + \frac{\delta_{j+1}}{\delta_j}.$$

Take now $\delta_k = \eta^{k-1}\delta$. Then this measure is $\lesssim \eta$ and $\delta_{j+1} = \eta^j \delta$. Chose finally η so that $\eta^j \delta = \varepsilon$. □

D.0.3 Extension

LEMMA D.2. *Let $X \subset Y$ be subsets of $\mathcal{D}_0$ such that*

$$\operatorname{dist}(\mathcal{D}_0 \backslash Y, X) \geq \varepsilon,$$

then there exists a $\mathcal{C}^\infty$-function $g : \mathcal{D}_0 \to \mathbb{R}$, being $= 1$ on X and $= 0$ outside Y and such that for all $j \geq 0$

$$|g|_{\mathcal{C}^j(\mathcal{D}_0)} \leq C \left(\frac{C}{\varepsilon}\right)^j.$$

C is an absolute constant.

Proof. This is a classical result obtained by convoluting the characteristic function of X with a $\mathcal{C}^\infty$-approximation of the Dirac-delta supported in a ball of radius $\leq \frac{\varepsilon}{2}$. □

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L. Hakan Eliasson, Université Paris Diderot, Sorbonne Paris Cité, Institut de, Mathémathiques de Jussieu-Paris Rive Gauche, UMR 7586, CNRS, Sorbonne Universités, UPMC Univ. Paris 06, 75013, Paris, France

hakan.eliasson@imj-prg.fr

Benoît Grébert, Laboratoire de Mathématiques Jean Leray, Université de Nantes, UMR CNRS 6629, 2, Rue de la Houssinière, 44322 Nantes Cedex 03, France

benoit.grebert@univ-nantes.fr

Sergei B. Kuksin, CNRS, Institut de Mathémathiques de Jussieu-Paris Rive Gauche, UMR 7586, Université Paris Diderot, Sorbonne Paris Cité, Sorbonne Universités, UPMC Univ. Paris 06, 75013, Paris, France

kuksin@gmail.com, sergei.kuksin@imj-prg.fr

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