Geom. Funct. Anal. Vol. 26 (2016) 1245–1254
DOI: 10.1007/s00039-016-0386-3
Published online October 12, 2016

GAFA Geometric And Functional Analysis

A C^∞ CLOSING LEMMA FOR HAMILTONIAN DIFFEOMORPHISMS OF CLOSED SURFACES

Masayuki Asaoka and Kei Irie

Abstract. We prove a C^∞ closing lemma for Hamiltonian diffeomorphisms of closed surfaces. This is a consequence of a C^∞ closing lemma for Reeb flows on closed contact three-manifolds, which was recently proved as an application of spectral invariants in embedded contact homology. A key new ingredient of this paper is an analysis of an area-preserving map near its fixed point, which is based on some classical results in Hamiltonian dynamics: existence of KAM invariant circles for elliptic fixed points, and convergence of the Birkhoff normal form for hyperbolic fixed points.

1 Introduction

The aim of this paper is to prove a C^∞ closing lemma for Hamiltonian diffeomorphisms of closed surfaces. Let us first introduce some notations. For any closed surface (i.e., C^∞ two-manifold) S, let $\mathrm{Diff}\,(S)$ denote the group of all C^∞ diffeomorphisms of S, equipped with the C^∞ topology. For any $\varphi \in \mathrm{Diff}\,(S)$, let $\mathrm{Fix}(\varphi)$ denote the set of fixed points of φ, and $\mathcal{P}(\varphi)$ denote the set of periodic points of φ:

$$\mathrm{Fix}(\varphi) := \{x \in S \mid \varphi(x) = x\}, \qquad \mathcal{P}(\varphi) := \bigcup_{m=1}^{\infty} \mathrm{Fix}(\varphi^m).$$

Also, the closure of $\{x \in S \mid \varphi(x) \neq x\}$ is called the support of φ, and denoted as $\mathrm{supp}\,\varphi$.

When S is equipped with an area form (i.e., nowhere vanishing 2-form) ω, let

$$\mathrm{Diff}\,(S,\omega) := \{\varphi \in \mathrm{Diff}\,(S) \mid \varphi^*\omega = \omega\},$$

which is the group of area-preserving diffeomorphisms. For any $h \in C^\infty(S)$, we define its Hamiltonian vector field X_h by $i_{X_h}\omega = -dh$. Our convention for the interior product i is $i_{X_h}\omega(\,\cdot\,) = \omega(X_h, \,\cdot\,)$. For any $H \in C^\infty([0,1] \times S)$ and $t \in [0,1]$, we define $H_t \in C^\infty(S)$ by $H_t(x) := H(t,x)$, and $(\varphi^t_H)_{t\in[0,1]}$ denotes the isotopy on S defined by $\varphi^0_H = \mathrm{id}_S$ and $\partial_t\varphi^t_H = X_{H_t}(\varphi^t_H)$. Then we define

$$\mathrm{Ham}\,(S,\omega) := \{\varphi^1_H \mid H \in C^\infty([0,1] \times S)\},$$

Mathematics Subject Classification: 37J10, 37E30, 37J45 which is the group of Hamiltonian diffeomorphisms. It is known that $\mathrm{Ham}\,(S,\omega) = \mathrm{Diff}\,(S,\omega)$ when S is homeomorphic to the two-sphere.

Throughout this paper, $\mathrm{Ham}\,(S,\omega)$ and $\mathrm{Diff}\,(S,\omega)$ are equipped with topologies induced from the C^∞ topology on $\mathrm{Diff}\,(S)$. Now we can state our main result as follows:

Theorem 1.1. *Let S be any closed, oriented surface, ω be any area form on S, and $\varphi \in \mathrm{Ham}\,(S,\omega)$. For any nonempty open set $U \subset S$, there exists a sequence $(\varphi_j)_{j\geq 1}$ in $\mathrm{Ham}\,(S,\omega)$ such that $\mathcal{P}(\varphi_j) \cap U \neq \emptyset$ for every $j \geq 1$ and $\lim_{j\to\infty} \varphi_j = \varphi$.*

Using standard arguments, we can prove a C^∞ general density theorem for Hamiltonian diffeomorphisms.

COROLLARY 1.2. *For any (S,ω) as in Theorem 1.1,*

$$\{\varphi \in \mathrm{Ham}\,(S,\omega) \mid \mathcal{P}(\varphi) \textit{ is dense in } S\}$$

is residual (i.e., contains a countable intersection of open and dense sets) in $\mathrm{Ham}\,(S,\omega)$.

Proof. For any nonempty open set $U \subset S$, let $\mathcal{H}_U$ denote the set consisting of $\varphi \in \mathrm{Ham}\,(S,\omega)$ such that there exists a nondegenerate periodic orbit of φ which intersects U. Then, $\mathcal{H}_U$ is obviously open in $\mathrm{Ham}\,(S,\omega)$, and dense by Theorem 1.1. Let $(U_i)_{i\in I}$ be any countable basis of open sets in S. Then $\bigcap_{i\in I} \mathcal{H}_{U_i}$ is residual in $\mathrm{Ham}\,(S,\omega)$, and if $\varphi \in \bigcap_{i\in I} \mathcal{H}_{U_i}$ then $\mathcal{P}(\varphi)$ is dense in S. $\square$

We can also prove a C^r general density theorem ($1 \leq r \leq \infty$) for area-preserving diffeomorphisms of the two-sphere.

COROLLARY 1.3. *Let r be a positive integer or ∞, and let $\mathrm{Diff}_r(S,\omega)$ denote the set of C^r diffeomorphisms of S preserving ω. When S is homeomorphic to the two-sphere,*

$$\{\varphi \in \mathrm{Diff}_r(S,\omega) \mid \mathcal{P}(\varphi) \textit{ is dense in } S\}$$

is residual in $\mathrm{Diff}_r(S,\omega)$ with the C^r topology.

Proof. The case $r = \infty$ is immediate from Corollary 1.2, since $\mathrm{Diff}_\infty(S,\omega) = \mathrm{Ham}\,(S,\omega)$. The case $1 \leq r < \infty$ follows from the case $r = \infty$ and the fact that $\mathrm{Diff}_\infty(S,\omega)$ is dense in $\mathrm{Diff}_r(S,\omega)$ with the C^r topology, which is proved in [Zeh77]. $\square$

REMARK 1.4 (Historical remarks). The C^1 closing lemma (and general density theorem) was first proved for nonconservative dynamics by Pugh [Pug67a, Pug67b], and later proved for conservative dynamics by Pugh–Robinson [PR83]. In particular, [PR83] established the C^1 closing lemma for symplectic and volume-preserving diffeomorphisms in arbitrary dimensions. On the other hand, a C^r closing lemma for $r \geq 2$ is not established except for a few cases (see [AZ12] Section 5), and it has been considered as an important open problem in the theory of dynamical systems (in particular, see Smale [Sma98] Problem 10).

As for the C^r general density theorem for area-preserving diffeomorphisms of closed surfaces (which is completely settled for the two-sphere by Corollary 1.3), as far as the authors know the only affirmative result so far is that the union of the (un)stable manifolds of hyperbolic periodic points are dense in the surface for a C^r generic area-preserving diffeomorphism of a closed surface and $1 \leq r \leq \infty$. This result was proved by Franks–Le Calvez [FL03] for the two-sphere and by Xia ([Xia06]), supplemented by Koropecki–Le Calvez–Nassiri ([KLN15], Section 8.5), for a general surface.

Theorem 1.1 is a consequence of a C^∞ closing lemma for Reeb flows on closed contact three-manifolds (Lemma 2.1), which was proved in [Iri15]. For the convenience of the reader, we sketch its proof in Section 2. The proof uses recent developments in quantitative aspects of embedded contact homology, in particular the result in [CHR15] by Cristofaro-Gardiner, Hutchings and Ramos.

In Section 3, we prove a C^∞ closing lemma for area-preserving diffeomorphisms of a surface with boundary, which satisfy some technical conditions (Lemma 3.1). The idea of the proof is to regard an area-preserving map as a "return map" of a certain Reeb flow, which is inspired by a recent paper [Hut15] by Hutchings.

In Section 4, we prove Theorem 1.1 using Lemma 3.1 and an analysis of an area-preserving map near its fixed point. We exploit some classical results in Hamiltonian dynamics: existence of KAM invariant circles for elliptic fixed points, and convergence of the Birkhoff normal form for hyperbolic fixed points.

2 Reeb flows on contact three-manifolds

Let (M, λ) be a contact manifold, where λ denotes the contact form, with the contact distribution $\xi_\lambda := \ker \lambda$. The Reeb vector field R_λ is defined by equations $\lambda(R_\lambda) = 1$, $d\lambda(R_\lambda, \cdot\,) = 0$. Let $\mathcal{P}(M, \lambda)$ denote the set of periodic orbits of R_λ, namely

$$\mathcal{P}(M, \lambda) := \{\gamma : \mathbb{R}/T_\gamma\mathbb{Z} \to M \mid T_\gamma > 0,\, \dot{\gamma} = R_\lambda(\gamma)\}.$$

Lemma 2.1 below is proved as a claim in the proof of [Iri15] Lemma 3.1 (in a slightly weaker form). The aim of this section is to sketch its proof, referring to [Iri15] for details.

Lemma 2.1 ([Iri15]). *Let (M, λ) be a closed contact three-manifold. For any $h \in C^\infty(M, \mathbb{R}_{\geq 0}) \setminus \{0\}$, there exist $t \in [0, 1]$ and $\gamma \in \mathcal{P}(M, (1 + th)\lambda)$ which intersects* $\operatorname{supp} h$.

Our proof of Lemma 2.1 is based on embedded contact homology (ECH). For any closed contact three-manifold (M, λ) and $\Gamma \in H_1(M : \mathbb{Z})$, this theory assigns a $\mathbb{Z}/2$-vector space[1] $\mathrm{ECH}(M, \xi_\lambda, \Gamma)$, which is relatively $\mathbb{Z}$-graded if $c_1(\xi_\lambda) + 2\mathrm{PD}(\Gamma) \in$

[1] ECH can be defined with $\mathbb{Z}$ coefficients, however $\mathbb{Z}/2$ coefficients are sufficient for our purpose.

$H^2(M:\mathbb{Z})$ is torsion (c_1 denotes the first Chern class, and PD denotes the Poincaré dual). It is easy to see that such Γ exists for any (M, ξ_λ).

For each $\sigma \in \mathrm{ECH}(M, \xi_\lambda, \Gamma)\backslash\{0\}$, one can assign $c_\sigma(M, \lambda) \in \mathbb{R}_{\geq 0}$ (see [Hut11] Section 4.1), the associated *ECH spectral invariant* (this term is not used in [Hut11]). One can prove the following properties:

(a): $c_\sigma(M,\lambda) \in \{0\} \cup \left\{ \sum_{i=1}^m T_{\gamma_i} \middle| m \geq 1,\, \gamma_1, \dots, \gamma_m \in \mathcal{P}(M,\lambda) \right\}$.

(b): If a sequence $(f_j)_{j\geq 1}$ in $C^\infty(M, \mathbb{R}_{>0})$ satisfies $\lim_{j\to\infty} \|f_j - 1\|_{C^0} = 0$, then $\lim_{j\to\infty} c_\sigma(M, f_j\lambda) = c_\sigma(M,\lambda)$.

(c): Let $\Gamma \in H_1(M:\mathbb{Z})$ be such that $c_1(\xi_\lambda) + 2\mathrm{PD}(\Gamma)$ is torsion, and let I denote an arbitrary absolute $\mathbb{Z}$-grading on $\mathrm{ECH}(M, \xi_\lambda, \Gamma)$.

(i): $\mathrm{ECH}(M, \xi_\lambda, \Gamma)$ is unbounded from above with this $\mathbb{Z}$-grading.

(ii): If M is connected, for any sequence $(\sigma_k)_{k\geq 1}$ of nonzero homogeneous elements in $\mathrm{ECH}(M, \xi_\lambda, \Gamma)$ satisfying $\lim_{k\to\infty} I(\sigma_k) = \infty$, there holds

$$\lim_{k\to\infty} \frac{c_{\sigma_k}(M,\lambda)^2}{I(\sigma_k)} = \int_M \lambda \wedge d\lambda =: \mathrm{vol}(M,\lambda).$$

(iii): For any $h \in C^\infty(M, \mathbb{R}_{\geq 0})\backslash\{0\}$, there exists $\sigma \in \mathrm{ECH}(M, \xi_\lambda, \Gamma)$ such that $c_\sigma(M, (1+h)\lambda) > c_\sigma(M,\lambda)$.

(a) is [Iri15] Lemma 2.4 (a special case is proved in [CH16] Lemma 3.1). (b) is explained in [CH16] Section 2.5 as the "Continuity axiom". (c)-(i) follows from Seiberg–Witten Floer theory (see [CH16] Section 2.6). (c)-(ii) is [CHR15] Theorem 1.3. (c)-(iii) follows from (i) and (ii), since $\mathrm{vol}(M, (1+h)\lambda) > \mathrm{vol}(M,\lambda)$.

We also need Lemma 2.2 below, which is proved by elementary arguments using Sard's theorem (see [Iri15] Section 2.1 for details).

LEMMA 2.2 ([Iri15] Lemma 2.2). *For any closed contact manifold (M,λ),*

$$\mathcal{A}(M,\lambda)_+ := \{0\} \cup \left\{ \sum_{i=1}^m T_{\gamma_i} \middle| m \geq 1,\, \gamma_1, \dots, \gamma_m \in \mathcal{P}(M,\lambda) \right\}$$

is a null (i.e., Lebesgue measure zero) set.

Proof of Lemma 2.1. We may assume that M is connected, and we set $\lambda_t := (1+th)\lambda$ for any $t \in [0,1]$. Suppose that the lemma does not hold, i.e., $\gamma \in \mathcal{P}(M,\lambda_t) \implies \mathrm{Im}\,\gamma \cap \mathrm{supp}\,h = \emptyset$ for any $t \in [0,1]$. Then $\mathcal{P}(M,\lambda_t) = \mathcal{P}(M,\lambda)$ for any $t \in [0,1]$, since $R_{\lambda_t} = R_\lambda$ on $M\backslash\mathrm{supp}\,h$. Hence $\mathcal{A}(M,\lambda_t)_+ = \mathcal{A}(M,\lambda)_+$ for any $t \in [0,1]$.

For any $\Gamma \in H_1(M:\mathbb{Z})$, $\sigma \in \mathrm{ECH}(M,\xi_\lambda,\Gamma)\backslash\{0\}$ and $t \in [0,1]$, (a) shows that

$$c_\sigma(M,\lambda_t) \in \mathcal{A}(M,\lambda_t)_+ = \mathcal{A}(M,\lambda)_+.$$

(b) shows that $c_\sigma(M,\lambda_t)$ is continuous on $t \in [0,1]$. On the other hand, $\mathcal{A}(M,\lambda)_+$ is a null set (Lemma 2.2). Thus $c_\sigma(M,\lambda_t)$ is constant on $t \in [0,1]$, in particular we obtain $c_\sigma(M,(1+h)\lambda) = c_\sigma(M,\lambda)$ for any σ, which contradicts (c)-(iii). $\square$

3 Return maps of Reeb flows

The aim of this section is to prove Lemma 3.1 below. Ω^1 denotes the set of C^∞ 1-forms.

LEMMA 3.1. *Let S be any compact, connected surface with boundary such that ∂S is diffeomorphic to S^1. Let ω be any area form on S and $\varphi \in \mathrm{Diff}\,(S, \omega)$ such that:*

- *$\varphi \equiv \mathrm{id}_S$ near ∂S.*
- *For any $\beta \in \Omega^1(S)$ such that $d\beta = \omega$, $\varphi^*\beta - \beta$ is exact.*

Then, for any nonempty open set U in $\mathrm{int}\, S := S \backslash \partial S$, there exists a sequence $(\varphi_j)_{j \geq 1}$ in $\mathrm{Diff}\,(S, \omega)$ such that $\lim_{j\to\infty} \varphi_j = \varphi$, and for every $j \geq 1$ there holds

$$\mathcal{P}(\varphi_j) \cap U \neq \emptyset, \qquad \mathrm{supp}\,(\varphi^{-1} \circ \varphi_j) \subset U.$$

Our idea to prove Lemma 3.1 is to realize $\varphi|_{\mathrm{int}\, S}$ as a "return map" of a certain Reeb flow. First we recall the following notion from [HWZ98].

DEFINITION 3.2. Let (M, λ) be a closed contact three-manifold, and let $(\varphi^t)_{t\in\mathbb{R}}$ denote the flow on M generated by R_λ, i.e., $\varphi^0 = \mathrm{id}_M$ and $\partial_t \varphi^t = R_\lambda(\varphi^t)$. A *global surface of section* in (M, λ) is a compact surface Σ with boundary, which is embedded in M and satisfies the following conditions:

- Each connected component of $\partial\Sigma$ is a (image of) periodic orbit of R_λ.
- $\mathrm{int}\,\Sigma$ is transversal to R_λ.
- For any $p \in M \backslash \Sigma$, there exist $t_-(p) \in \mathbb{R}_{<0}$, $t_+(p) \in \mathbb{R}_{>0}$ such that $\varphi^{t_-(p)}(p)$, $\varphi^{t_+(p)}(p) \in \Sigma$ and $t \in (t_-(p), t_+(p)) \implies \varphi^t(p) \notin \Sigma$.

Let us define $\pi_\pm : M\backslash\Sigma \to \mathrm{int}\,\Sigma$ by $\pi_\pm(p) := \varphi^{t_\pm(p)}(p)$. We also define the *return map* $\rho_{M,\lambda,\Sigma} : \mathrm{int}\,\Sigma \to \mathrm{int}\,\Sigma$ so that $\rho_{M,\lambda,\Sigma}(\pi_-(p)) = \pi_+(p)$ for any $p \in M\backslash\Sigma$.

It is easy to see that $\rho_{M,\lambda,\Sigma}$ preserves $d\lambda|_{\mathrm{int}\,\Sigma}$. We abbreviate $\rho_{M,\lambda,\Sigma}$ as ρ_λ when there is no risk of confusion.

The next lemma is a small variation of [Hut15] Proposition 2.1.

LEMMA 3.3. *For any (S, ω, φ) which satisfies the assumptions in Lemma 3.1, there exists (M, λ, Σ) such that $(\mathrm{int}\, S, \omega, \varphi|_{\mathrm{int}\, S})$ is C^∞ conjugate to $(\mathrm{int}\,\Sigma, d\lambda, \rho_{M,\lambda,\Sigma})$.*

Proof. Let us take a Liouville vector field V on (S, ω), i.e., $d(i_V\omega) = \omega$ and V is outer normal to ∂S. We set $\beta := i_V\omega$. There exists a local chart (r, θ) ($\sqrt{1-\varepsilon^2} \leq r \leq 1$, $\theta \in \mathbb{R}/\mathbb{Z}$) near ∂S such that $\partial S = \{r = 1\}$ and $\beta = ar^2 d\theta$, where $a := \int_S \omega$.

Let $Y := [0, 1] \times S/\sim$, where $\sim$ is defined as $(1, x) \sim (0, \varphi(x))$ $(x \in S)$. For any $t \in [0, 1]$, we define an embedding $e_t : S \to Y$ by $e_t(x) := [(t, x)]$. Then, there exists a contact form λ_Y on Y such that:

- There exists $h \in \mathbb{Z}_{>0}$ such that $\lambda_Y = a(hdt + r^2 d\theta)$ near $\partial Y = [0, 1] \times \partial S/\sim$.
- The Reeb vector field R_{λ_Y} is parallel to ∂_t.
- $e_t^* d\lambda_Y = \omega$ for any $t \in [0, 1]$.

λ_Y is defined as follows: since $\varphi^*\beta - \beta$ is exact, there exists $f \in C^\infty(S)$ such that $\varphi^*\beta - \beta = df$. f is constant near ∂S, since $\varphi \equiv \mathrm{id}_S$ near ∂S. By adding a constant, we may assume that $\min f > 0$ and $f \equiv ah$ near ∂S for some $h \in \mathbb{Z}_{>0}$. Now we can proceed in exactly the same way as the proof of [Hut15] Proposition 2.1.

Let $Z = \mathbb{R}/\mathbb{Z} \times \{z \in \mathbb{C} \mid |z| < \varepsilon\}$, and $C := \mathbb{R}/\mathbb{Z} \times \{0\} \subset Z$. We define $M := \mathrm{int}\, Y \sqcup Z/\sim$, where $\sim$ is defined as

$$(t, r, \theta) \sim (\tau, z = \rho e^{\sqrt{-1}\psi}) \iff r^2 + \rho^2 = 1,\ \psi = 2\pi t,\ \theta = \tau - ht.$$

Then, it is easy to see that $\lambda_Y|_{\mathrm{int}\, Y}$ extends to a C^∞ contact form λ on M, such that C is a periodic orbit of R_λ. Finally, $\Sigma := \{0\} \times \mathrm{int}\, S \cup C$ is a global surface of section in (M, λ), and $(\mathrm{int}\, \Sigma, d\lambda, \rho_{M,\lambda,\Sigma})$ is conjugate to $(\mathrm{int}\, S, \omega, \varphi|_{\mathrm{int}\, S})$ via $e_0|_{\mathrm{int}\, S}$. $\square$

Proof of Lemma 3.1. By Lemma 3.3, there exists (M, λ, Σ) such that $(\mathrm{int}\, \Sigma, d\lambda, \rho_{M,\lambda,\Sigma})$ is conjugate to $(\mathrm{int}\, S, \omega, \varphi|_{\mathrm{int}\, S})$ via a diffeomorphism $F : \mathrm{int}\, S \to \mathrm{int}\, \Sigma$.

Let us take $h \in C^\infty(M, \mathbb{R}_{\geq 0}) \backslash \{0\}$ such that $\mathrm{supp}\, h \subset \pi_-^{-1}(F(U))$. By Lemma 2.1, there exists a sequence $(t_j)_{j \geq 1}$ in $\mathbb{R}_{>0}$ such that $\lim_{j\to\infty} t_j = 0$ and for every j there exists $\gamma_j \in \mathcal{P}(M, (1 + t_j h)\lambda)$ which intersects $\mathrm{supp}\, h$.

Let $\lambda_j := (1 + t_j h)\lambda$. Then Σ is a global surface of section in (M, λ_j) for any sufficiently large j, and $\lim_{j\to\infty} \rho_{\lambda_j} = \rho_\lambda$ in the C^∞ topology. Let $K := \pi_-(\mathrm{supp}\, h) \subset F(U)$. For any such j, there holds $\mathrm{supp}\, (\rho_\lambda^{-1} \circ \rho_{\lambda_j}) \subset K$ and $\mathcal{P}(\rho_{\lambda_j}) \cap K \neq \emptyset$. Also, ρ_{λ_j} preserves $d\lambda_j|_{\mathrm{int}\, \Sigma} = d\lambda|_{\mathrm{int}\, \Sigma}$. Moreover, $F^{-1} \circ \rho_{\lambda_j} \circ F \in \mathrm{Diff}\, (\mathrm{int}\, S, \omega)$ extends to $\varphi_j \in \mathrm{Diff}\, (S, \omega)$ by setting $\varphi_j|_{\partial S} := \mathrm{id}_{\partial S}$, since $\mathrm{supp}\, (\rho_\lambda^{-1} \circ \rho_{\lambda_j})$ is compact. Then $(\varphi_j)_j$ satisfies the requirements in the lemma. $\square$

4 Proof of Theorem 1.1

First we prove the following lemma.

Lemma 4.1. *Let ω be any area form on $A := [0, 1] \times S^1$, and U, V be open neighborhoods of $\{1\} \times S^1$ which are disjoint from $\{0\} \times S^1$. For any diffeomorphism $\psi : U \to V$ which satisfies $\psi^*\omega = \omega$, there exists $\bar{\psi} \in \mathrm{Diff}\, (A, \omega)$ which satisfies $\bar{\psi} \equiv \psi$ near $\{1\} \times S^1$ and $\bar{\psi} \equiv \mathrm{id}$ near $\{0\} \times S^1$.*

Proof. Since any orientation-preserving diffeomorphism on S^1 is smoothly isotopic to id_{S^1}, ψ is smoothly isotopic to id near $\{1\} \times S^1$. Hence there exists $f \in \mathrm{Diff}\, (A)$ such that $f \equiv \psi$ near $\{1\} \times S^1$ and $f \equiv \mathrm{id}$ near $\{0\} \times S^1$.

$(f^*\omega - \omega)|_{\mathrm{int}\, A}$ represents 0 in $H^2_{c,\mathrm{dR}}(\mathrm{int}\, A)$. Thus there exists $\eta \in \Omega^1(A)$, which vanishes near ∂A and satisfies $d\eta = f^*\omega - \omega$. For any $t \in [0, 1]$, let $\omega_t := \omega + t(f^*\omega - \omega)$, and define a vector field X_t by $i_{X_t}\omega_t = \eta$. Then $X_t \equiv 0$ near ∂A. Let $(g_t)_{t\in[0,1]}$ be the isotopy on A defined by $g_0 = \mathrm{id}_A$ and $\partial_t g_t = X_t(g_t)$. Then, $\bar{\psi} := f \circ g_1$ satisfies the requirements in the lemma. $\square$

Let us start the proof of Theorem 1.1. We may assume that S is connected. Let $\varphi \in \mathrm{Ham}\,(S,\omega)$, and take $H \in C^\infty([0,1]\times S)$ such that $\varphi = \varphi^1_H$. By the Arnold conjecture for surfaces (see [Flo86] and the references therein), there exists $q \in \mathrm{Fix}(\varphi)$ such that $(\varphi^t_H(q))_{t\in[0,1]}$ forms a contractible loop on S.

LEMMA 4.2. *For any $\beta \in \Omega^1(S\setminus\{q\})$ such that $d\beta = \omega$, $\varphi^*\beta - \beta$ is exact.*

Proof. It is sufficient to show that any $\gamma : \mathbb{R}/\mathbb{Z} \to S\setminus\{q\}$ satisfies $\int_\gamma \varphi^*\beta - \beta = 0$. Let us define $\Gamma : [0,1]\times\mathbb{R}/\mathbb{Z} \to S$ by $\Gamma(t,\theta) := \varphi^t_H(\gamma(\theta))$, then $\int_\Gamma \omega = 0$. It is sufficient to prove the following claim:

Claim: There exists a smooth family of maps $(\Gamma_s : [0,1]\times\mathbb{R}/\mathbb{Z} \to S)_{s\in[0,1]}$ such that $\Gamma_0 = \Gamma$, $q \notin \mathrm{Im}\,\Gamma_1$, and $\Gamma_s|_{\{0,1\}\times\mathbb{R}/\mathbb{Z}} = \Gamma|_{\{0,1\}\times\mathbb{R}/\mathbb{Z}}$ for any $s\in[0,1]$.

Indeed, once we have established the claim, we can complete the proof by

$$\int_\gamma \varphi^*\beta - \beta = \int_{\Gamma_1} d\beta = \int_{\Gamma_1}\omega = \int_{\Gamma_0}\omega = 0.$$

Now let us prove the above claim. Since $(\varphi^t_H(q))_{t\in[0,1]}$ forms a contractible loop on S, there exists a smooth map $C : [0,1]^2 \to S$ such that

$$C(0,t) = \varphi^t_H(q), \quad C(1,t) = q, \quad C(s,0) = C(s,1) = q \qquad (\forall s,t\in[0,1]).$$

Then there exists a smooth family of vector fields $(\xi_{s,t})_{(s,t)\in[0,1]^2}$, where $\xi_{s,t}$ is a smooth vector field on S for each $(s,t)\in[0,1]^2$, such that

$$\xi_{s,0} = \xi_{s,1} = 0 \quad (\forall s\in[0,1]), \quad \xi_{s,t}(C(s,t)) = \partial_s C(s,t) \quad (\forall s,t\in[0,1]).$$

Let $(\Phi_{s,t})_{(s,t)\in[0,1]^2}$ denote the smooth family of isotopies on S defined by

$$\Phi_{0,t} = \mathrm{id}_S \quad (\forall t\in[0,1]), \quad \partial_s\Phi_{s,t} = \xi_{s,t}(\Phi_{s,t}) \quad (\forall s,t\in[0,1]).$$

Then it is easy to see that

$$C(s,t) = \Phi_{s,t}(\varphi^t_H(q)) \quad (\forall s,t\in[0,1]), \qquad \Phi_{s,0} = \Phi_{s,1} = \mathrm{id}_S \quad (\forall s\in[0,1]).$$

Now let us define Γ_s by $\Gamma_s(t,\theta) := \Phi_{s,t}(\Gamma(t,\theta))$. The properties $\Gamma_0 = \Gamma$ and $\Gamma_s|_{\{0,1\}\times\mathbb{R}/\mathbb{Z}} = \Gamma|_{\{0,1\}\times\mathbb{R}/\mathbb{Z}}\,(\forall s\in[0,1])$ are easy to see. The property $q\notin\mathrm{Im}\,\Gamma_1$ can be confirmed by

$$\Gamma_1(t,\theta) = \Phi_{1,t}(\Gamma(t,\theta)) = \Phi_{1,t}\circ\varphi^t_H(\gamma(\theta)) \neq \Phi_{1,t}\circ\varphi^t_H(q) = C(1,t) = q.$$

The middle inequality follows since $\Phi_{1,t}$ and φ^t_H are bijections and $q\notin\mathrm{Im}\,\gamma$. $\square$

Let U be any nonempty open set in S. We may assume that U is diffeomorphic to $\mathbb{R}^2$ and $\bar{U}$ (the closure of U in S) is disjoint from q. We are going to show that, there exists a sequence $(\varphi_j)_j$ in $\mathrm{Ham}\,(S,\omega)$ which satisfies the requirements in Theorem 1.1.

Let us take a local chart (x,y) near q such that $q=(0,0)$ and $\omega = dx\wedge dy$. By adding a C^∞ small perturbation to φ, we may assume that the following conditions are satisfied:

- The eigenvalues of $d\varphi(q)$ are in $\{z \in \mathbb{C} \mid z^3 \neq 1,\ z^4 \neq 1\}$. In particular q is a nondegenerate fixed point of φ.
- With respect to the local chart (x, y), φ is real-analytic near $(0, 0)$.

By the first condition, the fixed point q is either hyperbolic (i.e., the eigenvalues of $d\varphi(q)$ are in $\mathbb{R}\backslash\{\pm 1\}$) or elliptic (i.e., the eigenvalues are in $\{z \in \mathbb{C} \mid |z| = 1\}\backslash\{\pm 1\}$). We consider the two cases separately.

The case q is hyperbolic

According to the result by Moser ([Mos56] Theorem 1), there exists a local chart (X, Y) defined near q such that $q = (0, 0)$, $\omega = dX \wedge dY$ and

$$\varphi(X, Y) = (u(XY)X, u(XY)^{-1}Y),$$

where $u(t)$ is a real-analytic function defined near $t = 0$. Notice that $u(0)$ is an eigenvalue of $d\varphi(q)$, in particular nonzero.

Let us take sufficiently small $\varepsilon > 0$, and let $U_\varepsilon := \{(X, Y) \mid X^2 + Y^2 < \varepsilon\}$. Let us define a diffeomorphism $F : (0, \varepsilon/2) \times \mathbb{R}/2\pi\mathbb{Z} \to U_\varepsilon\backslash\{q\}$ by

$$F(r, \theta) := \sqrt{2r}(\cos\theta, \sin\theta),$$

then $F^*(dX \wedge dY) = dr \wedge d\theta$, and $F^{-1} \circ \varphi \circ F$ is defined on $(0, \delta) \times \mathbb{R}/2\pi\mathbb{Z}$ for sufficiently small $\delta > 0$. Let us define R and Θ by

$$(R(r, \theta), \Theta(r, \theta)) := F^{-1} \circ \varphi \circ F(r, \theta).$$

Setting $v(r, \theta) := u(r \cdot \sin 2\theta)$, direct computations show

$$\begin{aligned} R(r, \theta) &= r \cdot (v(r, \theta)^2 \cos^2\theta + v(r, \theta)^{-2} \sin^2\theta), \\ \tan\Theta(r, \theta) &= v(r, \theta)^{-2} \tan\theta. \end{aligned}$$

Hence, for sufficiently small $\delta' > 0$, $F^{-1} \circ \varphi \circ F$ uniquely extends to $(-\delta', \delta) \times \mathbb{R}/2\pi\mathbb{Z}$ as a real-analytic map, which we denote by ψ. Let $\omega_0 := dr \wedge d\theta$. Since $\psi^*\omega_0 - \omega_0$ is real-analytic and vanishes on $(0, \delta) \times \mathbb{R}/2\pi\mathbb{Z}$, it vanishes on $(-\delta', \delta) \times \mathbb{R}/2\pi\mathbb{Z}$, thus $\psi^*\omega_0 = \omega_0$. By Lemma 4.1, there exists $\bar\psi \in \mathrm{Diff}([-1, 0] \times \mathbb{R}/2\pi\mathbb{Z}, \omega_0)$ such that $\bar\psi \equiv \psi$ near $\{0\} \times \mathbb{R}/2\pi\mathbb{Z}$, and $\bar\psi \equiv \mathrm{id}$ near $\{-1\} \times \mathbb{R}/2\pi\mathbb{Z}$.

Let $\bar S := ([-1, \delta) \times \mathbb{R}/2\pi\mathbb{Z}) \cup_F (S\backslash\{q\})$. We define an area form $\bar\omega$ on $\bar S$ by

$$\bar\omega|_{[-1,\delta)\times\mathbb{R}/2\pi\mathbb{Z}} = \omega_0, \qquad \bar\omega|_{S\backslash\{q\}} = \omega.$$

We also define $\bar\varphi \in \mathrm{Diff}(\bar S, \bar\omega)$ by $\bar\varphi|_{[-1,0]\times\mathbb{R}/2\pi\mathbb{Z}} = \bar\psi$ and $\bar\varphi|_{S\backslash\{q\}} = \varphi$.

By Lemma 4.2, one can apply Lemma 3.1 for $(\bar S, \bar\omega, \bar\varphi)$. Then there exists a sequence $(\bar\varphi_j)_j$ in $\mathrm{Diff}(\bar S, \bar\omega)$ such that $\mathrm{supp}(\bar\varphi^{-1} \circ \bar\varphi_j) \subset U$, $\mathcal{P}(\bar\varphi_j) \cap U \neq \emptyset$ for every j, and $\lim_{j\to\infty} \bar\varphi_j = \bar\varphi$. For each j, $\bar\varphi_j|_{S\backslash\{q\}}$ extends to $\varphi_j \in \mathrm{Diff}(S, \omega)$ by $\varphi_j(q) := q$. Moreover $\varphi_j \in \mathrm{Ham}(S, \omega)$, since $\mathrm{supp}(\varphi^{-1} \circ \varphi_j) \subset U$ and U is diffeomorphic to $\mathbb{R}^2$. Then, the sequence $(\varphi_j)_j$ satisfies the requirements in Theorem 1.1.

The case q is elliptic

We assumed that the eigenvalues of $d\varphi(q)$ are in $\{z \in \mathbb{C} \mid z^3 \neq 1,\ z^4 \neq 1\}$. Then, there exists a local chart (X, Y) near q such that $q = (0, 0)$, $\omega = dX \wedge dY$ and

$$\varphi(X,Y) = (\cos\theta(X,Y)X - \sin\theta(X,Y)Y, \sin\theta(X,Y)X + \cos\theta(X,Y)Y) + O_4(X,Y),$$

where $\theta(X,Y) = \theta_0 + \theta_1(X^2+Y^2)$ (θ_0, θ_1 are real constants), and O_4 is a real-analytic map whose expansion involves terms of order ≥ 4 only (see [SM95] Section 32 and Section 23, pp. 172–173).

By adding a C^∞ small perturbation to φ, we may assume that $\theta_1 \neq 0$. Then, there exists a neighborhood D of q which is diffeomorphic to D^2, preserved by φ and sufficiently close to q such that $D \cap \bar{U} = \emptyset$ (see [SM95] Section 34). ∂D is a so called KAM invariant circle.

Again by Lemmas 4.1 and 4.2, one can apply Lemma 3.1 to conclude that there exists a sequence $(\varphi'_j)_j$ in $\mathrm{Diff}\,(S \setminus D, \omega)$ such that $\mathcal{P}(\varphi'_j) \cap U \neq \emptyset$, $\mathrm{supp}\,(\varphi^{-1} \circ \varphi'_j) \subset U$ for every j, and $\lim_{j \to \infty} \varphi'_j = \varphi|_{S \setminus D}$. Every φ'_j extends to $\varphi_j \in \mathrm{Diff}\,(S, \omega)$ by setting $\varphi_j|_D := \varphi|_D$. $\varphi_j \in \mathrm{Ham}\,(S, \omega)$ since $\mathrm{supp}\,(\varphi^{-1} \circ \varphi_j) \subset U$ and U is diffeomorphic to $\mathbb{R}^2$. The sequence $(\varphi_j)_j$ satisfies the requirements in Theorem 1.1. $\square$

Acknowledgments

We are grateful to the referees for their comments which are very helpful to improve presentations of the paper. We also appreciate Kenji Fukaya, Viktor Ginzburg, Yi-Jen Lee and Kaoru Ono for useful conversations. MA is supported by JSPS Grant-in-Aid for Scientific Research (C) (26400085). KI is supported by JSPS Grant-in-Aid for Young Scientists (B) (25800041) and JSPS Postdoctoral Fellowships for Research Abroad. KI also acknowledges the Simons Center for Geometry and Physics in Stony Brook University, where a part of the revision of this paper took place.

References

[AZ12] D.V. Anosov and E.V. Zhuzhoma. Closing lemmas. *Differential Equations*, (13)**48** (2012), 1653–1699

[CH16] D. Cristofaro-Gardiner and M. Hutchings. From one Reeb orbit to two. *Journal of Differential Geometry*, (1)**102** (2016), 25–36

[CHR15] D. Cristofaro-Gardiner, M. Hutchings and V.G.B. Ramos. The asymptotics of ECH capacities. *Inventiones Mathematicae*, (1)**199** (2015), 187–214

[Flo86] A. Floer. Proof of the Arnold conjecture for surfaces and generalizations to certain Kähler manifolds. *Duke Mathematical Journal*, (1)**53** (1986), 1–32

[FL03] J. Franks and P. Le Calvez. Regions of instability for non-twist maps. *Ergodic Theory and Dynamical Systems*, (1)**23** (2003), 111–141

[HWZ98] H. Hofer, K. Wysocki and E. Zehnder. The dynamics on three-dimensional strictly convex energy surfaces. *Annals of Mathematics (2)*, (1)**148** (1998), 197–289

[Hut11] M. Hutchings. Quantitative embedded contact homology. *Journal of Differential Geometry*, (2)**88** (2011), 231–266

[Hut15] M. Hutchings. Mean action and the Calabi invariant (2015). arXiv:1509.02183v3

[Iri15] K. Irie. Dense existence of periodic Reeb orbits and ECH spectral invariants. *Journal of Modern Dynamics* **9** (2015), 357–363

[KLN15] A. Koropecki, P. Le Calvez and M. Nassiri. Prime ends rotation numbers and periodic points. *Duke Mathematical Journal*, (3)**164** (2015), 403–472

[Mos56] J. Moser. The analytic invariants of an area-preserving mapping near a hyperbolic fixed point. *Communications on Pure and Applied Mathematics* (4)**9** (1956), 673–692

[Pug67a] C.C. Pugh. The closing lemma. *American Journal of Mathematics* (4)**89** (1967), 956–1009

[Pug67b] C.C. Pugh. An improved closing lemma and a General density theorem. *American Journal of Mathematics* (4)**89** (1967), 1010–1021

[PR83] C.C. Pugh and C. Robinson. The C^1 closing lemma, including Hamiltonians. *Ergodic Theory and Dynamical Systems* (2)**3** (1983), 261–313

[SM95] C.L. Siegel and J.K. Moser. *Lectures on celestial mechanics. Classics in Mathematics*. Springer-Verlag, Berlin (1995)

[Sma98] S. Smale. Mathematical problems for the next century. *The Mathematical Intelligencer* (2)**20** (1998), 7–15

[Xia06] Z. Xia. Area-preserving surface diffeomorphisms. *Communications in Mathematical Physics* (3)**263** (2006), 723–735

[Zeh77] E. Zehnder. *Note on smoothing symplectic and volume-preserving diffeomorphisms. Geometry and topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976)*, pp. 828–854. Lecture Notes in Math., Vol. 597, Springer, Berlin (1977)

Masayuki Asaoka, Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan
asaoka@math.kyoto-u.ac.jp

Kei Irie, Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-8502, Japan
and
Simons Center for Geometry and Physics, State University of New York, Stony Brook, NY 11794-3636, USA
iriek@kurims.kyoto-u.ac.jp

Received: April 16, 2016
Accepted: September 8, 2016