

# KÄHLER–EINSTEIN METRICS ALONG THE SMOOTH CONTINUITY METHOD

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**Abstract.** We show that if a Fano manifold  $M$  is K-stable with respect to special degenerations equivariant under a compact group of automorphisms, then  $M$  admits a Kähler–Einstein metric. This is a strengthening of the solution of the Yau–Tian–Donaldson conjecture for Fano manifolds by Chen–Donaldson–Sun (Int Math Res Not (8):2119–2125, 2014), and can be used to obtain new examples of Kähler–Einstein manifolds. We also give analogous results for twisted Kähler–Einstein metrics and Kähler–Ricci solitons.

## 1 Introduction

Let  $M$  be a Fano manifold of dimension  $n$ . A basic problem in Kähler geometry is whether  $M$  admits a Kähler–Einstein metric. The Yau–Tian–Donaldson conjecture [Yau93, Tia97, Don02], confirmed recently by Chen–Donaldson–Sun [CDS14, CDS15a, CDS15b, CDS15c], says that  $M$  admits a Kähler–Einstein metric if and only if it is K-stable. In general it seems to be intractable at present to check K-stability since in principle one must study an infinite number of possible degenerations of  $M$  to  $\mathbf{Q}$ -Fano varieties. One goal of this paper is to study some situations with large symmetry groups, where the problem reduces to checking a finite number of possibilities. This can then be used to yield new examples of Kähler–Einstein manifolds.

Suppose then that a compact group  $G$  acts on  $M$  by holomorphic automorphisms. Our main theorem is the following equivariant version of the result of Chen–Donaldson–Sun.

**Theorem 1.** *Suppose that  $(M, K_M^{-1})$  is K-stable, with respect to special degenerations that are  $G$ -equivariant. Then  $M$  admits a Kähler–Einstein metric.*

Here a  $G$ -equivariant special degeneration is a special degeneration  $X \rightarrow \mathbf{C}$  in the sense of Tian [Tia97], together with a holomorphic  $G$  action which commutes with the  $\mathbf{C}^*$ -action, preserves the fibers, and restricts to the given action of  $G$  on the generic fibers  $X_t = M$  for  $t \neq 0$ . We also obtain an analogous result for Kähler–Ricci solitons, and their twisted versions; see Definition 9 for detailed definitions, and Proposition 10 for the most general result.

An important special case is when  $G$  is a torus. In particular if  $M$  is a toric manifold, and  $G = T^n$  is the  $n$ -torus, then Proposition 10 implies that we only need

to check special degenerations of the form  $X = M \times \mathbf{C}$  to ensure the existence of a Kähler–Einstein metric or Kähler–Ricci soliton on  $M$ . In particular this recovers the result of Wang–Zhu [WZ04] showing that all toric Fano manifolds admit a Kähler–Ricci soliton. In addition we can recover the result of Li [Li11] on the greatest lower bound on the Ricci curvature of toric Fano manifolds.

A more interesting situation is when  $G = T^{n-1}$ , i.e.  $M$  is a complexity-one  $T$ -variety. In this case it is possible, in concrete examples, to check all  $G$ -equivariant special degenerations of  $M$ , and as a consequence we can obtain new examples of threefolds with Kähler–Einstein metrics and Kähler–Ricci solitons. Work of Ilten–Süss [IS] shows that we obtain five new Kähler–Einstein threefolds. To our knowledge these are the first examples where K-stability is used to obtain new Kähler–Einstein manifolds.

Our method of proof of Theorem 1 is to use the classical continuity path

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\alpha \tag{1}$$

for  $t \in [0, 1]$  proposed by Aubin [Aub84], and its analog for Kähler–Ricci solitons studied by Tian–Zhu [TZ00], and to show that if we cannot find a solution for  $t = 1$ , then there must be a  $G$ -equivariant destabilizing special degeneration. In particular we obtain a new proof of the result of Chen–Donaldson–Sun [CDS15c], without using metrics with conical singularities. At the same time our arguments are analogous to those in [CDS15c], using also the adaptation of some of those ideas to the smooth continuity method in [Sze16].

A key advantage of the smooth continuity path is that it allows one to work in a  $G$ -equivariant setting. In contrast, in [CDS15c] one considers Kähler–Einstein metrics singular along a smooth divisor  $D \subset M$ , and such a divisor can not be  $G$ -invariant unless  $G$  is finite (see Song–Wang [SW, Theorem 2.1]). The disadvantage of the smooth continuity path is that in effect one must consider pairs  $(V, \chi)$  of a variety  $V$  together with a possibly singular current  $\chi$ , as opposed to pairs  $(V, D)$  of a variety and a divisor. In [CDS15c] a destabilizing special degeneration is obtained by applying the Luna slice theorem, and for this we must restrict ourselves to a suitable finite dimensional variety rather than the infinite dimensional space of currents. For this the basic idea is to approximate a current  $\chi$  by a sum of currents of integration along divisors.

A brief outline of the paper is as follows. In Section 2 we collect some basic definitions and results on twisted Kähler–Ricci solitons. The proof of the main result, Proposition 10, will then be given in Section 3. We give some examples of the applications of our results to toric manifolds and other manifolds of large symmetry group in Section 4. In Section 5 we discuss how to adapt the methods of [Sze16] and [PSS] to obtain the partial  $C^0$ -estimates along the continuity method for solitons. A crucial point is the reductivity of the automorphism group of the limiting variety. This essentially follows from the work of Berndtsson [Ber15] as used in [CDS15c], but since we did not find the exact statement that we need in the literature, we give a brief exposition in Section 6.

## 2 Twisted Kähler–Ricci Solitons

Suppose that  $W$  is a  $\mathbf{Q}$ -Fano variety, with log terminal singularities. In particular a power  $K_{W_0}^r$  of the canonical bundle on the regular set  $W_0$  extends as a line bundle on  $W$ . We say that a metric  $h$  on  $K_{W_0}^{-1}$  is continuous on  $W$ , if the induced metric on  $K_{W_0}^{-r}$  extends to a continuous metric on  $K_W^{-r}$ . Fixing an open cover  $\{U_i\}$  and local trivializing holomorphic sections  $\sigma_i$  of  $K_V^{-r}|_{U_i \cap W_0}$ , we will write

$$|\sigma_i|_{h^r}^2 = e^{-r\phi_i}, \tag{2}$$

for continuous functions  $\phi_i$  on  $U_i$ . We will write the metric  $h$  simply as  $e^{-\phi}$  following the notation in Berndtsson [Ber15]. In particular  $e^{-\phi}$  defines a volume form on  $W_0$ , given in a local chart  $U_i$  by

$$e^{-\phi} = |\sigma_i|_{h^r}^{2/r} (\sigma_i \wedge \bar{\sigma}_i)^{-1/r}. \tag{3}$$

The log terminal condition says that this volume form has finite volume. We write  $\omega_\phi$  for the curvature current of the metric  $e^{-\phi}$  on  $W_0$ , so in our local charts  $\omega_\phi = \sqrt{-1} \partial \bar{\partial} \phi_i$ . Since the potentials  $\phi_i$  are locally bounded, by Bedford–Taylor [BT76] we can form the wedge product  $\omega_\phi^n$ , which defines a measure on  $W_0$ , and also on  $W$  extending it trivially. The metric  $h_\phi$  is a weak Kähler–Einstein metric if  $\omega_\phi$  is a Kähler current, and we have

$$e^{-\phi} = \omega_\phi^n. \tag{4}$$

Berman and Witt-Nyström [BN] have studied the analogous notion of weak Kähler–Ricci solitons. Suppose that  $v$  is a holomorphic vector field on  $W_0$ , whose imaginary part generates the action of a torus  $T$  on  $W$  (see Berman–Boucksom–Eyssidieux–Guedj–Zeriahi [BBEGZ, Lemma 5.2] to see that one obtains an action on  $W$ ). A Kähler–Ricci soliton on  $(W, v)$  is a  $T$  invariant continuous metric  $e^{-\phi}$ , smooth on  $W_0$  with positive curvature current  $\omega_\phi$  satisfying

$$e^{-\phi} = e^{\theta_v} \omega_\phi^n. \tag{5}$$

Here  $e^{\theta_v} \omega_\phi^n$  is a measure defined in [BN] for general  $\phi$ . If  $\phi$  is smooth, then  $\theta_v$  is simply a Hamiltonian function for the vector field  $v$ , satisfying

$$L_v \omega_\phi = \sqrt{-1} \partial \bar{\partial} \theta_v, \tag{6}$$

with the normalization

$$\int_{W_0} e^{\theta_v} \omega_\phi^n = \int_{W_0} \omega_\phi^n = V. \tag{7}$$

In particular  $\theta_v$  depends on  $\phi$ . For continuous metrics  $h_\phi$  (or more general metrics with positive curvature current), the measure constructed in [BN] still satisfies the

normalization (7). In addition by [BN, Corollary 2.9] we have some fixed constant  $C$  (depending only on  $M, v$ ), such that

$$C^{-1}\omega_\phi^n \leq e^{\theta_v}\omega_\phi^n \leq C\omega_\phi^n. \tag{8}$$

We now use this to define the twisted analogs of Kähler–Ricci solitons, which arise naturally along the continuity method. Suppose that  $e^{-\psi}$  is another metric on  $K_{W_0}^{-1}$  which in our local charts is given by plurisubharmonic functions  $\psi_i \in L^1_{loc}(U_i \cap W_0)$ .

DEFINITION 2. For  $t \in (0, 1)$  we say that the pair  $(W, (1 - t)\psi)$  is *klt*, if in each chart  $U_i \cap W_0$  the function  $e^{-\psi_i}$  is integrable, with respect to the volume form  $(\sigma_i \wedge \bar{\sigma}_i)^{-1/r}$ . We will on occasion write  $(W, (1 - t)\omega_\psi)$  for the pair, where as before  $\omega_\psi$  is the curvature of  $e^{-\psi}$ .

Equivalently we can think of  $e^{-t\phi - (1-t)\psi}$  as a volume form on  $W_0$  with  $e^{-\phi}$  being a continuous metric as above. The *klt* condition is then

$$\int_{W_0} e^{-t\phi - (1-t)\psi} < \infty. \tag{9}$$

DEFINITION 3. A twisted Kähler–Ricci soliton on the triple  $(W, (1 - t)\psi, v)$ , where  $v$  is a holomorphic vector field as above, is a continuous metric  $e^{-\phi}$  such that

$$e^{-t\phi - (1-t)\psi} = e^{\theta_v}\omega_\phi^n. \tag{10}$$

This equation is interpreted as an equality of measures on  $W_0$ , and in particular  $e^{-\phi}$  here need not be smooth on  $W_0$ , so  $e^{\theta_v}\omega_\phi^n$  is the measure defined by Berman–Witt–Nyström [BN]. Note that the existence of such a metric implies that  $(W, (1 - t)\psi)$  is *klt*. When  $t = 1$  or  $v = 0$ , we will simply omit the corresponding term in the triple. So we can talk about a Kähler–Einstein metric on  $W$ , a twisted Kähler–Einstein metric on  $(W, (1 - t)\psi)$ , or a Kähler–Ricci soliton on  $(W, v)$ .

REMARK 4. If  $W, \phi, \psi$  are smooth, then the twisted Kähler–Ricci soliton equation is equivalent (up to adding a constant to  $\phi$ ) to

$$\text{Ric}(\omega_\phi) - L_v\omega_\phi = t\omega_\phi + (1 - t)\omega_\psi, \tag{11}$$

which is the natural continuity path for finding Kähler–Ricci solitons, used by Tian–Zhu [TZ00] for instance.

Even when  $W$  is normal and  $\phi$  is only continuous, it is useful to have an equation for twisted Kähler–Ricci solitons in the form (11). For this the extra condition needed is that the measure  $e^{\theta_v}\omega_\phi^n$  defines a singular metric  $e^{-\tau}$  on  $K_W$ , with  $\tau \in L^1_{loc}$ . Then  $\phi$  defines a twisted Kähler–Ricci soliton on  $(W, (1 - t)\psi, v)$  if

$$\omega_\tau = t\omega_\phi + (1 - t)\omega_\psi, \tag{12}$$

where  $\omega_\tau$  is the curvature of  $e^{-\tau}$ . Note that by an argument similar to Berman–Boucksom–Eyssidieux–Guedj–Zeriahi [BBEGZ, Proposition 3.8], if  $e^{-\tau}$  is only defined outside a closed subset  $S \subset W$  with  $(2n - 2)$ -dimensional Hausdorff measure

$\Lambda_{2n-2}(S) = 0$ , and Equation (11) holds on  $W \setminus S$ , then  $e^{-\phi}$  is a twisted Kähler–Ricci soliton. Indeed in this case  $e^{-\tau}$  extends as a singular metric with positive curvature current over all of  $W$  (see Harvey–Polking [HP, Theorem 1.2], Demailly [Dem85]), and then (12) implies that up to modifying  $\psi$  by a constant, we must have

$$e^{\theta_v \omega_\phi^n} = e^{-\tau} = e^{-t\phi - (1-t)\psi}, \tag{13}$$

since (12) implies that  $f = \tau - t\phi - (1-t)\psi$  is a global  $L^1$  function with  $\sqrt{-1}\partial\bar{\partial}f = 0$  on  $W$ .

We need the following result, generalizing the classical results of Bando–Mabuchi [BM85] and Matsushima [Mat57], which are essentially contained in Berndtsson [Ber15], Boucksom–Berman–Eyssidieux–Guedj–Zeriahi [BBEGZ], Berman–Witt–Nyström [BN] and Chen–Donaldson–Sun [CDS15c] (see also Yi [Yi]). We will give an outline proof in Section 6.

**PROPOSITION 5.** *Suppose that  $e^{-\phi_0}, e^{-\phi_1}$  are two twisted Kähler–Ricci solitons on  $(W, (1-t)\psi, v)$ . Then there exists a holomorphic vector field  $w$  on  $W$ , commuting with  $v$  and satisfying  $\iota_w \omega_\psi = 0$ , such that the biholomorphisms  $F_t : W \rightarrow W$  induced by  $w$  satisfy  $F_1^*(\omega_{\phi_1}) = \omega_{\phi_0}$ . In addition  $L_{\text{Im } w} \omega_\phi = 0$ .*

**DEFINITION 6.** *For any triple  $(W, (1-t)\psi, v)$  we define the Lie algebra*

$$\mathfrak{g}_{W,\psi,v} = \{w \in H^0(TW) : \iota_w \omega_\psi = 0 \text{ and } [v, w] = 0\}. \tag{14}$$

As before, we may omit  $\psi$  or  $v$  from the notation if  $t = 1$  or  $v = 0$ . In particular  $\mathfrak{g}_W = H^0(TW)$ . We will also write  $\mathfrak{g}_{W,\beta} = \mathfrak{g}_{W,\psi}$  if  $\beta = \omega_\psi$  is the curvature of  $e^{-\psi}$ . Using a projective embedding into  $\mathbf{P}^N$ , we can realize  $\mathfrak{g}_{W,\psi,v}$  as a subalgebra of  $\mathfrak{sl}(N+1, \mathbf{C})$ .

Note that for example  $\mathfrak{g}_{W,\psi}$  is trivial if  $\omega_\psi$  is strictly positive and  $t < 1$ . In fact Berndtsson [Ber15, Proposition 8.2] implies that if  $e^{-\psi}$  is integrable, then  $\mathfrak{g}_{W,\psi}$  is trivial. In our application, when  $(W, (1-t)\psi)$  is klt,  $e^{-(1-t)\psi}$  will be integrable, but  $e^{-\psi}$  will typically not be.

Note also that the Lie group with Lie algebra  $\mathfrak{g}_{W,\psi}$  will usually be strictly smaller than the identity component of the group of biholomorphisms of  $W$  preserving  $\omega_\psi$ . The difference comes from the fact that if  $v$  is a real vector field then  $L_v \omega_\psi = 0$  does not imply  $L_{Jv} \omega_\psi = 0$  for the complex structure  $J$ , whereas our Lie algebra above is automatically closed under multiplication by  $\sqrt{-1}$ . On the other hand when  $\omega_\psi = [D]$  is the current of integration along a divisor, then  $\mathfrak{g}_{W,\psi}$  coincides with the vector fields on  $W$  parallel to  $D$ . Indeed  $\iota_v [D] = 0$  is equivalent to  $v$  being parallel to  $D$  along the smooth part of  $D$ .

The following theorem generalizes [CDS15c, Theorem 6], which in turn is a generalization of Matsushima’s theorem [Mat57] on the reductivity of the automorphism group of a Kähler–Einstein manifold. We will give the proof in Section 6.

PROPOSITION 7. *Suppose that  $(W, (1 - t)\psi, v)$  admits a twisted Kähler–Einstein metric  $e^{-\phi}$ . Then  $\mathfrak{g}_{W,\psi,v}$  is reductive. In addition if  $G$  is a group of biholomorphisms of  $W$ , fixing  $\omega_\phi$  and  $v$ , then the centralizer  $(\mathfrak{g}_{W,\psi,v})^G$  is also reductive.*

We finally recall some properties of the “twisted” Futaki invariant, generalizing the log-Futaki invariant in [CDS15c] and the modified Futaki invariant of Tian–Zhu [TZ00]. We will say that a metric  $e^{-\phi}$  is smooth on  $W$  if it is the restriction of a smooth metric under a projective embedding of  $W$ . This is, in particular, a stronger condition than just requiring  $e^{-\phi}$  to be smooth on  $W_0$ . In practice we will simply take the restriction of a suitable scaling of the Fubini–Study metric. For such a smooth metric  $e^{-\phi}$  on  $W$  we define

$$\begin{aligned} \text{Fut}_{(1-t)\psi,v}(W, w) &= \text{Fut}_v(W, w) - \frac{1-t}{V} \left[ \int_W \theta_w (e^{\theta_v} - 1) \omega_\phi^n \right. \\ &\quad \left. + n \int_W \theta_w (\omega_\psi - \omega_\phi) \wedge \omega_\phi^{n-1} \right], \end{aligned} \tag{15}$$

where  $\text{Fut}_v(W, w)$  is Tian–Zhu’s modified Futaki invariant, which we write in the form

$$\text{Fut}_v(W, w) = \frac{1}{V} \int_W \theta_w e^{\theta_v} \omega_\phi^n - \frac{\int_W \theta_w e^{-\phi}}{\int_W e^{-\phi}}. \tag{16}$$

This is shown to be equivalent to Tian–Zhu’s definition in terms of the Ricci potential, by He [He] (see Ding–Tian [DT, Section 1] for properties of the Ricci potential for metrics of the form  $\omega_\phi$  on singular varieties). One can check by direct calculation that our definition of the twisted Futaki invariant is independent of the choice of smooth metric  $e^{-\phi}$ , remembering that  $\iota_w \omega_\psi = 0$ .

We will on occasion write  $\text{Fut}_{(1-t)\omega_\psi,v}$  instead of  $\text{Fut}_{(1-t)\psi,v}$ , when the curvature of  $e^{-\psi}$  is more natural. We will need the following:

PROPOSITION 8. *If  $(W, (1 - t)\psi, v)$  admits a twisted Kähler–Ricci soliton, then*

$$\text{Fut}_{(1-t)\psi,v}(W, w) = 0 \tag{17}$$

for all  $w \in \mathfrak{g}_{W,\psi,v}$ .

If  $W$  were smooth, and the twisted Kähler–Ricci soliton had smooth potential, then this would follow directly from the definitions. In general we obtain the result by relating the twisted Futaki invariant to the twisted Ding functional, and using that the twisted Ding functional is bounded below if there exists a twisted KR-soliton. This is analogous to an argument in [CDS15c], and the proof will be given in Section 6.

*Twisted stability.* Suppose now that  $M$  is a smooth Fano manifold, with a holomorphic vector field  $v$  such that  $\text{Im } v$  generates a torus  $T$ . Suppose that  $G$  is a compact group of automorphisms of  $M$ , containing  $T$ . We embed  $M \subset \mathbf{P}^N$  using  $G$ -invariant

sections of  $K_M^{-m}$  for some  $m$ . Let  $\alpha = \frac{1}{m}\omega_{FS}|_M$ , which we can write as the curvature of a smooth metric  $e^{-\psi_\alpha}$  on  $K_M^{-1}$  in the notation above. This metric will then be  $G$ -invariant. It was shown by Dervan [Der] that twisted K-stability is a necessary condition for the existence of a twisted KE metric on  $(M, (1 - t)\alpha)$ , while a corresponding stability notion for Kähler–Ricci solitons was developed by Berman–Witt–Nyström [BN]. We can combine these ideas to obtain a stability notion for twisted Kähler–Ricci solitons as follows.

The vector field  $v$  on  $M$  is the restriction of a holomorphic vector field on  $\mathbf{P}^N$ , which we will also denote by  $v$ . The imaginary part  $\text{Im } v$  corresponds to a matrix in  $\mathfrak{u}(N + 1)$ , with eigenvalues  $\mu_i$ , so that  $v$  has Hamiltonian function

$$\theta_v = \frac{\sum_i \mu_i |Z_i|^2}{\sum_i |Z_i|^2} \tag{18}$$

for suitable homogeneous coordinates  $Z_i$ . We assume that  $\theta_v$  is normalized as before (i.e.  $e^{\theta_v}$  has average 1 on  $M$ ).

Under our embedding the group  $G$  above can be thought of as a subgroup of  $U(N + 1)$ . Suppose that we have a  $\mathbf{C}^*$ -action  $\lambda \subset GL(N + 1, \mathbf{C})^G$ , generated by a vector field  $w$  on  $\mathbf{P}^N$ , where  $GL(N + 1, \mathbf{C})^G$  denotes the centralizer of  $G$ . Suppose that the central fiber  $W = \lim_{t \rightarrow 0} \lambda(t) \cdot M$  is a  $\mathbf{Q}$ -Fano variety. We can also take the limit

$$\beta = \lim_{t \rightarrow 0} \lambda(t) \cdot \alpha, \tag{19}$$

which is a closed positive current on  $W$ . The  $\mathbf{C}^*$ -action  $\lambda$  defines a special degeneration (in the terminology of Tian [Tia97]), and its twisted Futaki invariant is defined to be

$$\text{Fut}_{(1-t)\alpha, v}(M, w) = \text{Fut}_{(1-t)\beta, v}(W, w), \tag{20}$$

again omitting  $\alpha, \beta$  or  $v$  if  $t = 1$  or  $v = 0$ . Here we are using the metric  $\frac{1}{m}\omega_{FS}$  restricted to  $W$  for defining the twisted Futaki invariant.

**DEFINITION 9.** *We say that the triple  $(M, (1-t)\alpha, v)$  is K-semistable (with respect to  $G$ -equivariant special degenerations), if  $\text{Fut}_{(1-t)\alpha, v}(M, w) \geq 0$  for all  $w$  as above. The triple is K-stable if in addition equality holds only when  $(W, (1-t)\beta)$  is biholomorphic to  $(M, (1-t)\alpha)$ , i.e. the pairs are in the same  $GL^G$ -orbit.*

The terminology more consistent with existing literature would be “twisted modified K-polystable”, but we hope no confusion is caused by simply using the terminology “K-stable”. Dervan [Der] showed that if  $(M, (1 - t)\alpha)$  admits a twisted Kähler–Einstein metric then it is K-stable, while Berman–Witt–Nyström showed that if  $(M, v)$  admits a Kähler–Ricci soliton, then it is K-stable in the sense of the above definition. We expect that one can combine the arguments to show that if the triple  $(M, (1 - t)\alpha, v)$  admits a twisted Kähler–Ricci soliton, then it is K-stable, but we will not pursue that here. Our main result is a result in the converse direction, the proof of which will be given in Section 3.



PROPOSITION 10. *If  $(M, (1 - s)\alpha, v)$  is  $K$ -semistable for all  $G$ -equivariant special degenerations, then  $(M, (1 - t)\alpha, v)$  admits a twisted Kähler–Ricci soliton for all  $t < s$ . In addition if  $(M, v)$  is  $K$ -stable, then  $(M, v)$  admits a Kähler–Ricci soliton.*

Note that we also expect that if  $(M, (1 - t)\alpha, v)$  is  $K$ -stable, then  $(M, (1 - t)\alpha, v)$  admits a twisted Kähler–Ricci soliton, however this does not quite follow from our arguments.

A key ingredient in our arguments is a comparison of the twisted and untwisted Futaki invariants and from (15) it follows that

$$\text{Fut}_{(1-t)\alpha, v}(M, w) = \text{Fut}_v(M, w) - \frac{1-t}{V} \left[ \int_W \theta_w(e^{\theta_v} - 1) \omega_\phi^n + n \int_W \theta_w(\beta - \omega_\phi) \wedge \omega_\phi^{n-1} \right]. \tag{21}$$

Recall here that  $(W, \beta)$  is the limit of the pair  $(M, \alpha)$  under the  $\mathbf{C}^*$ -action generated by  $w$ . The following result builds on work in [LS] and Dervan [Der].

PROPOSITION 11. *Using the same setup as above, we have the formula*

$$\frac{1}{V} \int_W \theta_w \left[ (n + 1)\omega_\phi - n\beta \right] \wedge \omega_\phi^{n-1} = \max_W \theta_w. \tag{22}$$

Here  $\omega_\phi$  is the restriction of  $\frac{1}{m}\omega_{FS}$  to  $W$ , and  $\theta_w$  is the corresponding Hamiltonian function for  $w$ .

We will give the proof below, after Lemma 12. For now note that as a consequence we have

$$\text{Fut}_{(1-t)\alpha, v}(M, w) = \text{Fut}_v(M, w) + \frac{1-t}{V} \int_W (\max_W \theta_w - \theta_w) e^{\theta_v} \omega_\phi^n. \tag{23}$$

In particular the difference is always positive, and is equal to zero only if  $\theta_w$  is constant on  $W$ , i.e. if we had a trivial degeneration. Note also that the right hand side is independent of the choice of metric  $\alpha$  on  $M$ , however as discussed in [Sze13] (and can be seen from the proof below), if one replaces  $\alpha$  by the current of integration along a divisor, leading to the notion of log  $K$ -stability used in [CDS15c], the twisted Futaki invariant might drop for special divisors.

For the proof of Proposition 11, and also for later use we will need to represent  $\alpha$  as an integral of currents of integration along divisors on  $M$ . The formula (22) is invariant under scaling  $\omega_\phi$  and  $\omega_\psi$ , and so to simplify notation we will assume that the cohomology classes  $[\alpha], [\omega_\phi]$  coincide with the classes of the hyperplane divisors  $M \cap H, W \cap H$ . In particular we then have  $V = 1$ . We will also normalize the Fubini–Study metric  $\omega_{FS}$  on  $\mathbf{P}^N$  to represent the same cohomology class as  $[H]$ .

Let us write  $\mathbf{P}^{N*}$  for the dual projective space of hyperplanes. Since  $\alpha$  is the restriction of  $\omega_{FS}$  to  $M$ , we have (see e.g. Shiffman–Zelditch [SZ99])

$$\alpha = \int_{\mathbf{P}^{N*}} [M \cap H] d\mu(H), \tag{24}$$



where  $d\mu$  is simply the Fubini–Study volume form, scaled to have volume 1. It follows that the limit  $\beta = \lim_{t \rightarrow 0} \lambda(t)_* \alpha$  is given by

$$\beta = \int_{\mathbf{P}^{N^*}} [W \cap H_0] d\mu(H), \tag{25}$$

where for each hyperplane  $H$  we wrote

$$H_0 = \lim_{t \rightarrow 0} \lambda(t) \cdot H. \tag{26}$$

In this formula for the limit  $\beta$  it is important that  $W$  is not contained in a hyperplane, otherwise we would not necessarily have the relation

$$\lim_{t \rightarrow 0} \lambda(t) \cdot (M \cap H) = \left( \lim_{t \rightarrow 0} \lambda(t) \cdot M \right) \cap \left( \lim_{t \rightarrow 0} \lambda(t) \cdot H \right), \tag{27}$$

used above. It follows that

$$\int_W \theta_w \beta \wedge \omega_{FS}^{n-1} = \int_{\mathbf{P}^{N^*}} \int_{W \cap H_0} \theta_w \omega_{FS}^{n-1} d\mu(H). \tag{28}$$

A key point is that there is a subspace  $P_w \subset \mathbf{P}^{N^*}$ , depending on  $w$ , such that for all  $H \notin P_w$  the integral

$$\int_{W \cap H_0} \theta_w \omega_{FS}^{n-1} \tag{29}$$

has the same value. The following lemma gives a formula for this integral, and in particular shows this independence. This formula is essentially contained in [LS, proof of Theorem 12], and was made more explicit by Dervan [Der].

LEMMA 12. *Let us normalize the Fubini–Study metric so that  $[\omega_{FS}] = [H]$  in  $H^2(\mathbf{P}^N)$ . Then there is a subspace  $P_w \subset \mathbf{P}^{N^*}$  such that for  $H \notin P_w$  we have*

$$\int_{W \cap H_0} \theta_w \omega_{FS}^{n-1} = \frac{1}{n} \int_W \left[ (n+1)\theta_w - \max_W \theta_w \right] \omega_{FS}^n. \tag{30}$$

*Proof.* Let us write  $R = \bigoplus R_k$  for the graded coordinate ring of  $W$ . In suitable homogeneous coordinates the function  $\theta_w$  on  $\mathbf{P}^N$  is given by

$$\theta_w(Z) = \frac{\sum_i \mu_i |Z_i|^2}{\sum_i |Z_i|^2}, \tag{31}$$

where the  $\mu_i$  are the weights of the  $\mathbf{C}^*$ -action  $\lambda(t)$  induced by  $\theta_w$ , on the linear functions  $R_1$ . For a generic hyperplane  $H$ , the limit  $H_0 = \lim_{t \rightarrow 0} \lambda(t) \cdot H$  has equation  $Z_{max} = 0$ , where  $\mu_{max}$  is the largest weight (if there are several equal largest weights, then  $Z_{max}$  can denote any of the corresponding coordinates). Indeed this is the case for all hyperplanes not passing through the set where  $\theta_w$  achieves its maximum. This can be seen from the fact that the effect of acting by  $\lambda(t)$  as  $t \rightarrow 0$  is the same as flowing along the negative gradient flow of  $\theta_w$ .

Denoting by  $S = \bigoplus S_k$  the graded coordinate ring of  $W \cap H_0$ , we have  $S = R/Z_{max}R$ , i.e.  $S_k = R_k/Z_{max}R_{k-1}$ . Let us write  $w_k$  for the total weight of the action  $\lambda$  on  $R_k$ , and  $w'_k$  for the weight of the action on  $S_k$ . From the equivariant Riemann–Roch theorem we have

$$\dim R_k = k^n \int_W \frac{\omega_{FS}^n}{n!} + O(k^{n-1}), \tag{32}$$

and

$$w_k = k^{n+1} \int_W \theta_w \frac{\omega_{FS}^n}{n!} + ck^n + O(k^{n-1}) \tag{33}$$

for some constant  $c$ . Similarly

$$w'_k = k^n \int_{W \cap H_0} \theta_w \frac{\omega_{FS}^{n-1}}{(n-1)!} + O(k^{n-1}). \tag{34}$$

From the description  $S_k = R_k/Z_{max}R_{k-1}$  we get

$$\begin{aligned} w'_k &= w_k - w_{k-1} - \mu_{max} \dim R_{k-1} \\ &= (n+1)k^n \int_W \theta_w \frac{\omega_{FS}^n}{n!} - \mu_{max} k^n \int_W \frac{\omega_{FS}^n}{n!}. \end{aligned} \tag{35}$$

Combining this with (34) we get

$$\int_{W \cap H_0} \theta_w \omega_{FS}^{n-1} = \frac{1}{n} \int_W \left[ (n+1)\theta_w - \mu_{max} \right] \omega_{FS}^n. \tag{36}$$

The fact that  $W$  is invariant under the action of  $\lambda(t)$  and not contained in a hyperplane implies that  $\max_W u = \mu_{max} = \max_{\mathbf{P}^N} u$ .  $\square$

Proposition 11 follows from this lemma together with the formula (28). Indeed, the lemma together with (28) implies that

$$\int_W \theta_w \beta \wedge \omega_{FS}^{n-1} = \frac{1}{n} \int_W \left[ (n+1)\theta_w - \max_W \theta_w \right] \omega_{FS}^n, \tag{37}$$

since the set of hyperplanes in  $P_w$  has measure zero. At the same time, the metric  $\omega_\phi$  in (22) is taken to be  $\frac{1}{m}\omega_{FS}$ . We then need to scale  $\theta_w$  by the same factor as well, and possibly add a constant to have the correct normalization. Equation (37) remains valid after these scalings, and Proposition 11 follows directly from this.

### 3 Proof of the Main Result

In this section we give the proof of our main result, Proposition 10. The setup is that we have a smooth Fano manifold  $M$  with the holomorphic action of a compact group  $G$ . We have a  $G$ -invariant Kähler metric  $\alpha \in c_1(M)$ , and for simplicity we assume that  $\alpha$  is the restriction of  $\frac{1}{m}\omega_{FS}$  to  $M$ , under an embedding  $M \subset \mathbf{P}^{N_m}$  using a basis of sections of  $K_M^{-m}$ , for some  $m > 0$ . We are also given a vector field  $v$  on  $M$ , invariant under the action of  $G$ . In order to find a Kähler–Ricci soliton on  $(M, v)$  we try to solve the equations

$$\text{Ric}(\omega_t) - L_v\omega_t = t\omega_t + (1 - t)\alpha, \tag{38}$$

for  $t \in [0, 1]$ . From Zhu [Zhu00] we know that there is a solution for  $t = 0$  and by Tian–Zhu [TZ00] the possible values of  $t$  form an open set. We therefore have a solution for  $t \in [0, T)$  and we need to understand the limit of a sequence of solutions as  $t \rightarrow T$ .

**3.1 The case  $T < 1$ .** We first focus on the case  $T < 1$ , and we assume that the triple  $(M, (1 - s)\psi_\alpha, v)$  is K-stable with respect to  $G$ -equivariant special degenerations, for some  $s \in (T, 1]$ . We show that in this case the continuity method cannot blow up at time  $T$ , i.e. we can solve our equation for  $t = T$  as well. The strategy is the same as that in [CDS15c].

We first show that along a sequence  $t_k \rightarrow T$ , the Gromov–Hausdorff limit of  $(M, \omega_{t_k})$  has the structure of a  $\mathbf{Q}$ -Fano variety  $W$ , together with a metric  $\psi$  on  $K_W$ , and a vector field  $v$  such that the triple  $(W, (1 - T)\psi, v)$  admits a twisted Kähler–Ricci soliton. We then need to show that  $W$  is the central fiber of a special degeneration for  $M$ . One difficulty, when comparing this to the analogous result in [CDS15c], is that we are not able to show that the pair  $(W, (1 - T)\psi)$  is the central fiber of a special degeneration for  $(M, (1 - T)\psi_\alpha)$  since we are not able to use the Luna slice theorem on the infinite dimensional space of pairs consisting of a variety and a positive current. Instead we use an argument approximating  $\alpha$  with a convex combination of hyperplane sections.

The key ingredient to understanding the Gromov–Hausdorff limit of a sequence  $(M, \omega_{t_k})$  is the partial  $C^0$ -estimate, first introduced by Tian [Tia90]. This was established in [Sze16] in the case when  $v = 0$ , using the method in Chen–Donaldson–Sun [CDS15b], and it was shown by Phong–Song–Sturm [PSS] for Kähler–Ricci solitons (i.e.  $v$  is non-zero, but  $t = 1$ ), generalizing the work of Donaldson–Sun [DS14]. A modest combination and generalization of these ideas gives the analogous result for the equation (38), and we will give a brief outline of the necessary changes in Section 5.

For each  $t$ , the metric  $\omega_t$  introduces Hermitian inner products on  $H^0(K_M^{-m})$  for all  $m > 0$ , moreover these inner products are  $G$ -invariant (by the uniqueness of solutions to (38) for  $t < 1$ ). The partial  $C^0$ -estimate says that we can find a uniform  $m$ , and  $\kappa > 0$ , independent of  $t$ , such that an orthonormal basis  $\{s_0, \dots, s_{N_m}\}$  of

$H^0(K_M^{-m})$  satisfies

$$\kappa < \sum_{i=0}^{N_m} |s_i|^2(x) < \kappa^{-1} \tag{39}$$

for all  $x \in M$ . Let us write  $N = N_m$  for this choice of  $m$  from now.

Let us now write  $V_t = H^0(K_M^{-m})$  for the unitary  $G$ -representation, with metric induced by  $\omega_t$ . Note that  $V_t$  are equivalent  $G$ -representations, and hence they are unitarily equivalent as well. It follows that we have  $G$ -equivariant unitary maps  $f_t : V_0 \rightarrow V_t$ . In other words if we pick an orthonormal basis  $\{s_0, \dots, s_N\}$  for  $H^0(K_M^{-m})$  with respect to the metric  $\omega_0$ , then for all  $t > 0$  we can find an orthonormal basis  $\{s_0^{(t)}, \dots, s_N^{(t)}\}$  with respect to  $\omega_t$ , by applying the map  $f_t$ . Using these bases, we have embeddings  $F_t : M \rightarrow \mathbf{P}^N$ , such that for  $s \neq t$  we have  $F_s = \rho \circ F_t$  with  $\rho \in GL(N + 1)^G$ , i.e.  $\rho$  commutes with  $G$ . In particular the vector field  $(F_t)_*v$  along the image  $F_t(M)$  is induced by a fixed holomorphic vector field  $v$  on  $\mathbf{P}^N$ , since  $v$  is  $G$ -invariant.

We can choose a subsequence  $t_k \rightarrow T$ , such that  $F_{t_k}(M)$  converges to a limit  $W \subset \mathbf{P}^N$ , and as shown in Donaldson–Sun [DS14], the partial  $C^0$ -estimate implies, up to replacing  $m$  by a multiple, that  $W$  is a normal  $\mathbf{Q}$ -Fano variety, homeomorphic to the Gromov–Hausdorff limit  $Z$  of the sequence  $(M, \omega_{t_k})$ . Moreover the maps  $F_{t_k} : M \rightarrow \mathbf{P}^N$  converge to a Lipschitz map  $F_T : Z \rightarrow \mathbf{P}^N$  under this Gromov–Hausdorff convergence, such that  $F_T : Z \rightarrow W$  is a homeomorphism. Note that by choosing a further subsequence we can assume that the currents  $(F_{t_k})_*\alpha$  converge weakly to a current  $\beta$ , which is necessarily supported on  $W$  and is invariant under the action of  $\text{Im } v$ . Let us write  $\beta$  as the curvature  $\omega_\psi$  of a singular metric  $e^{-\psi}$  on  $K_W^{-1}$ . We can similarly define a weak limit  $\omega_T$  of the metrics  $(F_{t_k})_*(\omega_{t_k})$ , which is also supported on  $W$ . Note that if we write

$$\omega_{t_k} = \frac{1}{m}(F_{t_k})^*\omega_{FS} + \sqrt{-1}\partial\bar{\partial}\phi_k, \tag{40}$$

then the partial  $C^0$ -estimate implies that we have bounds  $|\phi_k|, |\nabla\phi_k|_{\omega_{t_k}} < C$ . This in particular implies that the  $\phi_k$  converge to a Lipschitz function  $\phi_T$  on  $(Z, d_Z)$ , and since  $Z$  is homeomorphic to  $W$ , this means that  $\phi_T$  is continuous on  $W$  (using the topology induced from  $\mathbf{P}^N$ ). This implies that  $\omega_T$  is the curvature of a continuous metric  $e^{-\phi_T}$  on  $K_W^{-1}$  (recall that we might need to take a power  $K_W^{-m}$  here). We need the following.

**PROPOSITION 13.** *The triple  $(W, (1 - T)\psi, v)$  admits a twisted Kähler–Ricci soliton, and in particular  $(W, (1 - T)\psi)$  has klt singularities. In fact the twisted Kähler–Ricci soliton is given by the metric  $e^{-\phi_T}$ .*

*Proof.* Let us decompose the Gromov–Hausdorff limit as  $Z = \mathcal{R} \cup \mathcal{D} \cup \mathcal{S}_2$ . Here  $\mathcal{R}$  is the regular set, and  $\mathcal{D}$  is the set of points which admit a tangent cone of the form  $\mathbf{C}^{n-1} \times \mathbf{C}_\gamma$ , where  $\mathbf{C}_\gamma$  is the standard cone with cone angle  $2\pi\gamma$ . See Section 5 for

more details. From the results of Cheeger–Colding [CC97] and Cheeger–Colding–Tian [CCT02] we know that  $S_2$  is a set of Hausdorff dimension at most  $2n - 4$ . Since  $F_T$  is Lipschitz, we know that  $F_T(S_2)$  is also a set with Hausdorff dimension at most  $2n - 4$ . Let us write  $W' = W_0 \setminus F_T(S_2)$ , where as before  $W_0$  is the regular part of the algebraic variety  $W$ . We will construct the twisted Kähler–Ricci soliton on an open set  $U$  containing  $W'$ . Note that  $W \setminus U$  is then a closed set of Hausdorff dimension at most  $2n - 4$ . As explained in Remark 4, it is enough to show that the measure  $e^{\theta_v} \omega_T^n$  corresponding to the metric  $e^{-\phi_T}$  defines a singular metric  $e^{-\tau}$  on  $K_U$  with  $\tau \in L^1_{loc}$  such that its curvature satisfies

$$\omega_\tau = T\omega_T + (1 - T)\psi. \tag{41}$$

To simplify notation we will identify  $Z$  with  $W$ , and so on  $W$  in addition to the metric  $\omega_{FS}$  induced by the Fubini–Study metric we have the metric  $d_Z$  inducing the same topology. For simplicity let us also write  $d_k$  for the metric on  $M$  induced by  $\omega_{t_k}$ , and  $M_k$  for the metric space  $(M, d_k)$ . Thus we have  $M_k \rightarrow (W, d_Z)$  in the Gromov–Hausdorff sense. The maps  $F_k : M_k \rightarrow \mathbf{P}^N$  are compatible with the convergence in the sense that if  $p_k \rightarrow p$  with  $p_k \in M_k$  and  $p \in W$ , then  $F_k(p_k) \rightarrow p$  in  $\mathbf{P}^N$ .

For each  $p \in W'$  we will obtain the required property of the measure  $e^{\theta_v} \omega_T^n$  on an open neighborhood of  $p$ . If  $p \in W'$ , then either  $p \in \mathcal{R}$  or  $p \in \mathcal{D}$ . We will only deal with the case  $p \in \mathcal{D}$  since the other case is easier. We can write  $p = \lim p_k$  for  $p_k \in M_k$ , such that for a sufficiently small  $r > 0$  the balls  $B_{d_k}(p_k, r)$ , scaled to unit size are very close in the Gromov–Hausdorff sense to the unit ball in a cone  $\mathbf{C}^{n-1} \times \mathbf{C}_\gamma$ , for large  $k$ . As discussed in [Sze16], based on the ideas in [CDS15b], this implies that we have biholomorphisms  $H_k : \Omega_k \rightarrow B^{2n}$ , where  $\Omega_k \subset M_k$  contain a ball around  $p_k$  of a fixed size, such that the metric  $\tilde{\omega}_k = r^{-2} \omega_{t_k}$  on  $B^{2n}$  is well approximated by the standard conical metric on  $B^{2n}$ . More precisely, we have coordinates  $(u, v_1, \dots, v_{n-1})$  such that if we write

$$\eta_\gamma = \sqrt{-1} \frac{du \wedge d\bar{u}}{|u|^{2-2\gamma}} + \sqrt{-1} \sum_{i=1}^{n-1} dv_i \wedge d\bar{v}_i, \tag{42}$$

then for some fixed constant  $C$  (independent of  $k$ )

1.  $\tilde{\omega}_k = \sqrt{-1} \partial\bar{\partial} \phi_k$  with  $0 \leq \phi_k \leq C$ ,  $|r^2 v_k(\phi_k)| < C$ , where  $v_k$  is the soliton vector field in this chart.
2.  $\omega_{Euc} < C \tilde{\omega}_k$ ,
3. Given any  $\delta > 0$  and compact set  $K$  away from  $\{u = 0\}$ , we can assume (by taking  $r$  above smaller and  $k$  larger if necessary), that  $|\tilde{\omega}_k - \eta_\gamma|_{C^{1,\alpha}} < \delta$  on  $K$ .

We will also write  $\alpha_k$  for the form  $\alpha$  in this chart.

It is shown in [CDS15c, Proposition 22] that the biholomorphisms  $H_k : \Omega_k \rightarrow B^{2n}$  converge to a homeomorphism  $H_\infty : \Omega_\infty \rightarrow B^{2n}$ . In addition  $\Omega_\infty$  necessarily contains a ball  $B_{d_Z}(p, \epsilon) \subset W$  for some small  $\epsilon > 0$ , since all the sets  $\Omega_k$  contain balls of a uniform size around  $p_k$ . It follows that  $\Omega_\infty$  also contains a ball  $B$  around  $p$  in

the topology on  $W$  induced from  $\mathbf{P}^N$ , and so  $H_\infty$  defines a holomorphic chart on  $W$  in a neighborhood of  $p$ . These charts can be used to define holomorphic maps  $f_{t_k} : B \rightarrow F_{t_k}(M)$ , biholomorphic onto their image, such that the  $f_{t_k}$  converge to the identity map as  $k \rightarrow \infty$ . In this formulation  $\beta$  is given as the weak limit of  $f_{t_k}^*(F_{t_k})_*\alpha$ , which in terms of our charts amounts to saying that  $\beta$  is the weak limit of the forms  $\alpha_k$ . In the same vein  $\omega_T$  is the weak limit of  $\omega_{t_k} = r^2\tilde{\omega}_k$ .

The metrics  $\tilde{\omega}_k$  satisfy the equations

$$e^{r^2v_k(\phi_k)}(\sqrt{-1}\partial\bar{\partial}\phi_k)^n = e^{-r^2t_k\phi_k - (1-t_k)\psi_k}\omega_{Euc}^n, \tag{43}$$

for suitable local potentials  $\psi_k$  of the forms  $\alpha_k$  restricted to this chart. Note that  $r^2v_k(\phi_k)$  is a Hamiltonian for  $v_k$  with respect to  $\omega_{t_k}$ . The bound  $\omega_{Euc} < C\sqrt{-1}\partial\bar{\partial}\phi_k$  together with  $|\phi_k|, |r^2v_k(\phi_k)| < C$  and (43) implies an upper bound for  $\psi_k$  (note that  $t_k$  is bounded away from 1). In addition since we control  $\tilde{\omega}_k$  on compact sets away from  $\{u = 0\}$ , on any such set we have a lower bound for  $\psi_k$  as well. It follows that up to choosing a further subsequence, we have  $\psi_k \rightarrow \psi_\infty$  in  $L^1_{loc}$ , for some plurisubharmonic  $\psi_\infty$ , and then necessarily  $\beta = \sqrt{-1}\partial\bar{\partial}\psi_\infty$ . In addition we can assume that  $r^2t_k\phi_k$  converge uniformly to  $T\phi_\infty$  for a continuous  $\phi_\infty$  such that  $\omega_T = \sqrt{-1}\partial\bar{\partial}\phi_\infty$ . It follows that

$$-r^2v_k(\phi_k) - \log \frac{(\sqrt{-1}\partial\bar{\partial}\phi_k)^n}{\omega_{Euc}^n} = r^2t_k\phi_k + (1-t_k)\psi_k \tag{44}$$

are plurisubharmonic functions converging in  $L^1_{loc}$ . The bound on  $v(\phi_k)$  implies then that the limit  $\omega_T^n$  gives a singular metric on  $K_{B^{2n}}$  with locally integrable potential, and therefore by (8), we have that  $e^{\theta_v}\omega_T^n$  also defines such a singular metric  $e^{-\tau}$ . The convergence above then shows that (41) holds, which is what we wanted to show.  $\square$

One important conclusion that we need to draw from this is that according to Proposition 7 the Lie algebra  $\mathfrak{g}_{W,\beta,v}$  is reductive. In addition the twisted Futaki invariant vanishes,  $\text{Fut}_{(1-T)\beta,v}(W, w) = 0$ , for any  $w \in \mathfrak{g}_{W,\beta,v}$ .

Let us now identify  $M$  with its image  $F_0(M) \subset \mathbf{P}^N$ , and write  $\alpha = (F_0)_*\alpha$ . From the above discussion, for each  $k$ , we have  $F_{t_k}(M) = \rho_k(M)$  for some  $\rho_k \in GL^G$ , and  $\rho_k(M) \rightarrow W, \rho_k(\alpha) \rightarrow \beta$ . As before, we can write

$$\alpha = \int_{\mathbf{P}^{N*}} [M \cap H] d\mu(H), \tag{45}$$

since  $\alpha$  is a scaling of the restriction of the Fubini–Study metric. Note that

$$\rho_k(\alpha) = \int_{\mathbf{P}^{N*}} [\rho_k(M) \cap \rho_k(H)] d\mu(H), \tag{46}$$

and the following lemma implies that we can choose a subsequence of the  $\rho_k$ , such that the limit

$$\rho_\infty(H) = \lim_{k \rightarrow \infty} \rho_k(H) \tag{47}$$

exists for all  $H \in \mathbf{P}^{N^*}$ . Note that we write  $\rho_\infty(H)$  just as a notation, rather than suggesting that an automorphism  $\rho_\infty$  of  $\mathbf{P}^{N^*}$  exists.

LEMMA 14. *Up to choosing a subsequence, we can assume that  $\rho_k(H)$  converges for all  $H \in \mathbf{P}^{N^*}$ .*

*Proof.* Write  $\mathbf{P}^{N^*} = \mathbf{P}(V)$  for an  $N + 1$ -dimensional vector space  $V$ . Thinking of the  $\rho_k$  as matrices, let us scale each of them in such a way that all entries are in  $\{z : |z| \leq 1\}$ , and at least one entry equals 1. We can choose a subsequence such that as matrices, we have

$$\lim_k \rho_k = \rho, \tag{48}$$

where  $\rho$  is not necessarily invertible. Let  $W_1 = \text{Ker } \rho$ . For any  $x \in \mathbf{P}(V) \setminus \mathbf{P}(W_1)$  we can then take the limit

$$\lim_k \rho_k(x) = \rho(x). \tag{49}$$

Now let us restrict the  $\rho_k$  to  $W_1$ , thinking of them as linear maps  $\rho_k : W_1 \rightarrow V$ . Once again, taking matrix representatives, we can normalize each to have entries in the unit disk, with at least one entry equal to 1. Just as above, up to choosing a further subsequence, we will have a limiting, nonzero linear map  $\rho : W_1 \rightarrow V$  with kernel  $W_2 \subset W_1$ . For  $x \in \mathbf{P}(W_1) \setminus \mathbf{P}(W_2)$  the limit will exist as above.

Repeating this process a finite number of times we will have a subsequence  $\rho_k$  such that  $\rho_k(x)$  converges for all  $x \in \mathbf{P}(V)$ . □

It follows that we have

$$\beta = \int_{\mathbf{P}^{N^*}} [W \cap \rho_\infty(H)] d\mu(H), \tag{50}$$

where as before it is important to note that  $W$  is irreducible and not contained in a hyperplane.

In the spirit of Definition 6, for any current  $\tau$  on  $\mathbf{P}^N$ , let us denote by  $\mathfrak{g}_{W,\tau} \subset \mathfrak{sl}(N + 1, \mathbf{C})$  the space of those holomorphic vector fields  $v$ , which are tangent to  $W$  and satisfy  $\iota_v \tau = 0$ . If  $\tau = [S]$ , the current of integration along a subvariety  $S$ , we will write  $\mathfrak{g}_S = \mathfrak{g}_{[S]}$ . Note that in this case  $\mathfrak{g}_S$  is simply the Lie algebra of the stabilizer of  $S$  in  $SL(N + 1, \mathbf{C})$ .

LEMMA 15. *We can find  $H_1, \dots, H_d$  for some  $d$  such that*

$$\mathfrak{g}_{W,\beta} = \mathfrak{g}_W \cap \bigcap_{i=1}^d \mathfrak{g}_{[W \cap \rho_\infty(H_i)]}. \tag{51}$$



*Proof.* Suppose that  $v$  is a holomorphic vector field, which does not vanish along  $W$ , and let  $\xi = \iota_{\bar{v}}\omega_{FS}^n$ . This is an  $(n, n - 1)$ -form such that  $\iota_v\xi$  is a non-negative  $(n - 1, n - 1)$ -form. If  $A \subset T_p\mathbf{P}^N$  is a complex  $(n - 1)$ -dimensional subspace, then  $\iota_v\xi$  vanishes on  $A$  only if  $v \in A$ .

If  $\iota_v\beta = 0$ , then we have

$$\int_{H \in \mathbf{P}^*} \int_{W \cap \rho_\infty(H)} \iota_v\xi \, d\mu = 0, \tag{52}$$

and so for almost every  $H$  we must have

$$\int_{W \cap \rho_\infty(H)} \iota_v\xi = 0. \tag{53}$$

In particular, for almost every  $H$  we must have  $v \in A$  for all tangent planes  $A = T_p(W \cap \rho_\infty(H))$  at all smooth points  $p \in W \cap \rho_\infty(H)$ . It follows that  $\iota_v[W \cap \rho_\infty(H)] = 0$ , i.e.

$$\mathfrak{g}_\beta \subset \mathfrak{g}_{[W \cap \rho_\infty(H)]}. \tag{54}$$

If we choose one such  $H$ , say  $H_1$ , it may happen that  $\mathfrak{g}_{[W \cap \rho_\infty(H_1)]}$  is too large, i.e. there is a  $w \in \mathfrak{g}_{[W \cap \rho_\infty(H_1)]}$  such that  $\iota_w\beta \neq 0$ . But we have

$$\iota_w\beta = \int_{H \in \mathbf{P}^*} \iota_w[W \cap \rho_\infty(H)] \, d\mu, \tag{55}$$

so we must have a positive measure set of  $H$  for which  $\iota_w[W \cap \rho_\infty(H)] \neq 0$ . We can thus choose an  $H_2$ , so that we still have

$$\mathfrak{g}_\beta \subset \mathfrak{g}_{[W \cap \rho_\infty(H_1)]}, \tag{56}$$

but  $\mathfrak{g}_{[W \cap \rho_\infty(H_1)]} \cap \mathfrak{g}_{[W \cap \rho_\infty(H_2)]}$  is strictly smaller than  $\mathfrak{g}_{[W \cap \rho_\infty(H_1)]}$ . Repeating this a finite number of times, we obtain the required result.  $\square$

It follows from this result that we can choose  $H'_1, \dots, H'_l$  for some  $l$  such that the Lie algebra of the stabilizer of the  $(l + 1)$ -tuple  $(W, W \cap \rho_\infty(H'_1), \dots, W \cap \rho_\infty(H'_l))$  in  $GL^G$ , for the action on a product of Hilbert schemes, is equal to the  $G$ -invariant part of  $\mathfrak{g}_{W,\beta}$ , and so according to Proposition 7 it is reductive. Using a result similar to Luna’s slice theorem [Lun73] as in [Don, Proposition 1] (as in [CDS15c] as well), we can therefore find a  $\mathbf{C}^*$ -subgroup  $\lambda \subset GL^G$  and an element  $g \in GL^G$  such that

$$(W, W \cap \rho_\infty(H'_1), \dots, W \cap \rho_\infty(H'_l)) = \lim_{t \rightarrow 0} \lambda(t)g \cdot (M, M \cap H'_1, \dots, M \cap H'_l). \tag{57}$$

In addition for a subset of  $E \subset \mathbf{P}^{N*}$  of measure zero, if  $H_1, \dots, H_K \notin E$ , then the stabilizer of

$$(W, W \cap \rho_\infty(H'_1), \dots, W \cap \rho_\infty(H'_l), W \cap \rho_\infty(H_1), \dots, W \cap \rho_\infty(H_K)) \tag{58}$$

will still be the same as that of  $(W, \beta)$ , and so we can still find a corresponding  $\mathbf{C}^*$ -subgroup  $\lambda$  and  $g \in GL^G$  which will satisfy (57) as well as

$$\begin{aligned} W &= \lim_{t \rightarrow 0} \lambda(t)g \cdot M \\ W \cap \rho_\infty(H_i) &= \lim_{t \rightarrow 0} \lambda(t)g \cdot (M \cap H_i), \quad \text{for } i = 1, \dots, K. \end{aligned} \tag{59}$$

Note that all of these  $\lambda$  must fix  $W$ , but the  $\lambda$  may vary as we change the collection  $(H_1, \dots, H_K)$ .

Each of the  $\mathbf{C}^*$ -actions  $\lambda$  is generated by a vector field  $w$  commuting with  $v$ , with Hamiltonian function  $\theta_w$ . We will assume that  $\theta_w$  is normalized so that

$$\int_W \theta_w \omega_{FS}^n = 0. \tag{60}$$

Let us write  $\|w\| = \sup_W |\theta_w|$ , although note that any two norms on the finite dimensional space of such  $w$  are equivalent.

Because of (50), for any  $\epsilon > 0$  we can choose  $K$  large, and  $H_1, \dots, H_K \notin E$ , such that no  $N + 1$  of the  $H_i$  lie on a hyperplane in  $\mathbf{P}^{N^*}$ , and for all vector fields  $w$  as above we have

$$\int_W \theta_w \beta \wedge \omega_{FS}^{n-1} \leq \epsilon \|w\| + \frac{1}{K} \sum_{j=1}^K \int_{W \cap \rho_\infty(H_j)} \theta_w \omega_{FS}^{n-1}. \tag{61}$$

Applying this to the  $w$  corresponding to the  $\mathbf{C}^*$ -action  $\lambda$  that we obtain for  $(H_1, \dots, H_K)$ , we have

$$\int_W \theta_w \beta \wedge \omega_{FS}^{n-1} \leq \epsilon \|w\| + \frac{1}{K} \sum_{j=1}^K \lim_{t \rightarrow 0} \int_{\lambda(t) \cdot (M \cap H_j)} \theta_w \omega_{FS}^{n-1}. \tag{62}$$

Using Lemma 12, and the fact that no  $N + 1$  of the  $H_i$  are in a hyperplane, we obtain, using also the normalization of  $\theta_w$ , that

$$\int_W \theta_w \beta \wedge \omega_{FS}^{n-1} \leq \left( \epsilon + \frac{NC}{K} \right) \|w\| - \frac{K - N}{Kn} \int_W \max_W \theta_w \omega_{FS}^n, \tag{63}$$

for some fixed constant  $C$ . Choosing  $K$  sufficiently large (depending on  $\epsilon$ ), we obtain a  $\mathbf{C}^*$ -action generated by a vector field  $w$ , with Hamiltonian function  $\theta_w$  as above, such that

$$\int_W \theta_w \beta \wedge \omega_{FS}^{n-1} \leq 2\epsilon \|w\| - \frac{1}{n} \int_W \max_W \theta_w \omega_{FS}^n. \tag{64}$$

Moreover this  $\mathbf{C}^*$ -action satisfies  $W = \lim_{t \rightarrow 0} \lambda(t)g \cdot M$ , but not necessarily  $\beta = \lim_{t \rightarrow 0} \lambda(t)g \cdot \alpha$ . Nevertheless the vector field  $v$  satisfies  $\iota_v \beta = 0$  by construction.

Since  $(W, (1 - T)\beta)$  admits a twisted Kähler–Ricci soliton, we know that

$$\text{Fut}_{(1-T)\beta, v}(W, w) = 0, \tag{65}$$

and so

$$\text{Fut}_v(M, w) - \frac{1 - T}{V} \left[ \int_W \theta_w e^{\theta_v} \omega_{FS}^n + \int_W \theta_w n \beta \wedge \omega_{FS}^{n-1} \right] = 0. \tag{66}$$

At the same time we are assuming that for some  $s > T$ , the triple  $(M, (1 - s)\psi, v)$  is K-semistable, which, using Proposition 11, implies that we have

$$\text{Fut}_v(M, w) - \frac{1 - s}{V} \left[ \int_W \theta_w e^{\theta_v} \omega_{FS}^n - V \max_W \theta_w \right] \geq 0. \tag{67}$$

Together (66) and (67) imply

$$\frac{s - T}{V} \int_W \theta_w e^{\theta_v} \omega_{FS}^n + (1 - s) \max_W \theta_w + \frac{1 - T}{V} \int_W \theta_w n \beta \wedge \omega_{FS}^{n-1} \geq 0. \tag{68}$$

Using also (64) we then get

$$0 \leq \frac{1 - T}{V} 2n\epsilon \|w\| + \frac{s - T}{V} \int_W (\theta_w - \max_W \theta_w) e^{\theta_v} \omega_{FS}^n. \tag{69}$$

Since  $s > T$  and  $T < 1$ , this is a contradiction if  $\epsilon$  is sufficiently small, unless  $\|w\| = 0$ . For this, note that there is a uniform constant  $c > 0$  such that

$$\int_W (\max_W \theta_w - \theta_w) e^{\theta_v} \omega_{FS}^n \geq c \|w\| \tag{70}$$

for all possible  $w$  that we have, since these form a finite dimensional space.

It follows that we must have  $\|w\| = 0$ , which means that  $\theta_w$  is constant on  $W$ . This implies that the corresponding  $\mathbf{C}^*$ -action  $\lambda$  is trivial, and so in fact by (59) we have

$$(W, W \cap \rho_\infty(H_1), \dots, W \cap \rho_\infty(H_K)) = g \cdot (M, M \cap H_1, \dots, M \cap H_K) \tag{71}$$

for some  $g \in SL^G$ . It follows that

$$\lim_{k \rightarrow \infty} \rho_k(H_i) = \rho_\infty(H_i) = g(H_i). \tag{72}$$

We can assume that  $H_1, \dots, H_{N+1}$  are in general position in  $\mathbf{P}^{N^*}$ , and then each  $\rho_k$  is determined by the hyperplanes  $\rho_k(H_i)$  for  $i = 1, \dots, N + 1$ . In particular (72) then implies that  $\rho_k \rightarrow g$  in  $SL^G$ , which in turn implies that the sequence  $\rho_k \in SL^G$  is bounded. If we write

$$\frac{1}{m} (F_k)_* \omega_{FS} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_k \tag{73}$$

for the pullbacks of the Fubini–Study metrics to  $M$  under our embeddings  $F_k$ , we then have a uniform bound  $|\phi_k| < C$ . The partial  $C^0$ -estimate implies that then we also have

$$\omega_{t_k} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi'_k \tag{74}$$

with  $|\phi'_k| < C'$  for a uniform constant, for the metrics  $\omega_{t_k}$  along the continuity path. It is then standard using the estimates of Yau [Yau78] that we have uniform  $C^{l,\alpha}$  bounds for  $\omega_{t_k}$ , and so we can obtain a solution of Equation (38) for  $t = T$  (see also Zhu [Zhu00] for the  $C^2$ -estimate in the soliton case).

**3.2 The case  $T = 1$ .** Suppose now that  $T = 1$ , i.e. we can solve Equation (38) for all  $t < 1$ . This case is much more similar to the work of Chen–Donaldson–Sun [CDS15c], since the “current part” of the equation disappears as  $t \rightarrow 0$ . The case of Kähler–Ricci solitons was also studied by Jiang–Wang–Zhu [JWZ]. We briefly describe the argument for the sake of completeness. Just as in the case  $T < 1$ , we have embeddings  $F_t : M \rightarrow \mathbf{P}^N$  using suitable orthonormal bases for  $H^0(K_M^{-m})$  with respect to the metric  $\omega_t$ , for some large  $m$ . The partial  $C^0$ -estimate is still valid, in the Kähler–Einstein case by [Sze16] based on the method in [CDS15c], and in the soliton case due to Jiang–Wang–Zhu [JWZ]. It follows that as before, up to increasing  $m$  and choosing a sequence  $t_k \rightarrow 1$  we have the algebraic convergence  $F_{t_k}(M) \rightarrow W \in \mathbf{P}^N$  to a normal  $\mathbf{Q}$ -Fano variety, homeomorphic to the Gromov–Hausdorff limit  $(Z, d_Z)$  of the sequence  $(M, \omega_{t_k})$ . As before, we identify  $(M, \alpha) = (F_0(M), (F_0)_*\alpha)$  and so  $(F_{t_k}(M), (F_{t_k})_*\alpha) = \rho_k \cdot (M, \alpha)$  for  $\rho_k \in SL^G$ . The vector field  $v$  on each  $F_{t_k}(M)$  is induced by a fixed vector field  $v$  on  $\mathbf{P}^N$ , which is also tangent to the limit  $W$ . We can also choose a further subsequence of  $t_k$  if necessary to have a weak limit  $(F_{t_k})_*\omega_{t_k} \rightarrow \omega_1$ . We have the following, see [JWZ, Corollary 1.4]. A proof can also be given in the spirit of the proof of Proposition 13.

**PROPOSITION 16.** *The pair  $(W, v)$  admits a Kähler–Ricci soliton, and in fact this soliton is given by the current  $\omega_1$ .*

It follows from [BN, Corollary 3.6] that the stabilizer of  $W$  in  $SL^G$  is reductive, and so we can find a  $\mathbf{C}^*$ -subgroup  $\lambda \in SL^G$  generated by a vector field  $w$  commuting with  $v$ , and an elements  $g \in SL^G$  such that

$$W = \lim_{t \rightarrow 0} \lambda(t)g \cdot M. \tag{75}$$

This is a special degeneration for  $M$ , whose central fiber is  $W$ . Since  $W$  admits a Kähler–Ricci soliton, the corresponding Futaki invariant  $\text{Fut}_v(W, w) = 0$ . By assumption  $(M, v)$  is K-stable, and so  $W$  must be biholomorphic to  $M$ . This means that  $\omega_1$  is a Kähler–Ricci soliton on  $M$ , which is what we wanted to obtain.

### 4 Some Applications

In this section, we look at some applications of Theorem 1 to existence of Kähler–Einstein metrics on Fano manifolds with large symmetry groups.

*Toric manifolds.* A compact Kahler manifold  $M$  of complex dimension  $n$  is *toric* if the compact torus  $T^n$  acts by isometries on  $M$  and the extension of the action to the complex torus  $(\mathbf{C}^*)^n$  acts holomorphically with a free, open, dense orbit. We can then recover the following theorem of Wang–Zhu [WZ04] as a consequence of Theorem 1.

**Theorem 17.** *There exists a Kähler–Ricci soliton, which is unique up to holomorphic automorphisms, on every toric Fano manifold. As a consequence, there exists a*

*Kähler–Einstein metric on a toric Fano manifold if and only if the Futaki invariant vanishes.*

*Proof.* Let  $M$  be a toric manifold with  $\dim_{\mathbf{C}} M = n$ . We wish to apply Theorem 1 with  $G = T^n$  with a fixed identification as a subgroup of  $GL(N + 1, \mathbf{C})$ . The key observation is that if  $v$  is a toric vector field, then any  $(\mathbf{C}^*)^n$ -equivariant special degeneration of  $(M, v)$  is necessarily trivial. Indeed, if  $\lambda : \mathbf{C}^* \rightarrow GL(N + 1, \mathbf{C})^G$  is a test configuration and if  $M_0 = \lim_{t \rightarrow 0} \lambda(t) \cdot M$  is not in the  $GL(N + 1, \mathbf{C})$ -orbit of  $M$ , then the stabilizer of  $M_0$  must contain a  $(\mathbf{C}^*)^{n+1}$ . On the other hand, since  $M_0$  is irreducible and not contained in any hyperplane, the action of this stabilizer on  $M_0$  must also be effective. This is a contradiction since any torus acting on an  $n$ -dimensional normal variety cannot have a dimension greater than  $n$ . The upshot is that  $M_0$  must be bi-holomorphic to  $M$  and the test configuration is induced by a toric vector field  $w$  on  $M$ . To verify K-stability of  $(M, v)$ , it then suffices to check that the modified Futaki invariant vanishes:  $\text{Fut}_v(M, w) = 0$ , for all toric vector fields  $w$  on  $M$ .

Next, recall that any toric manifold  $M$  with an ample line bundle corresponds to a unique (up to translations) polytope  $P \subset \mathbf{R}^n$  defined by a finite collection of affine linear inequalities  $l_j(\mathbf{x}) \geq 0$ . This polytope is in fact the image of the free  $(\mathbf{C}^*)^n$  orbit in  $M$  under the moment map. Since  $M$  is Fano, one can normalize the polytope so that  $l_j(0) = 1$  for all  $j$ . Any toric vector field can be written as  $w = \sum_{j=1}^n c_j z_j \frac{\partial}{\partial z_j}$  for some  $\mathbf{c} \in \mathbf{R}^n$  where  $(z_1, \dots, z_n)$  are the usual complex coordinates on  $(\mathbf{C}^*)^n$ . In terms of the polytope data, for a vector field  $v = \sum_{j=1}^n a_j z_j \frac{\partial}{\partial z_j}$ , equation (16) then reduces to

$$\text{Fut}_v(M, w) = \mathbf{c} \cdot \frac{\int_P \mathbf{x} e^{\mathbf{a} \cdot \mathbf{x}} d\mathbf{x}}{V},$$

where  $V = \text{Vol}(P)$  is the volume of  $M$ . But then, as in Tian–Zhu [TZ00], by minimizing the functional  $F(\mathbf{a}) = \int_P e^{\mathbf{a} \cdot \mathbf{x}} d\mathbf{x}$ , one can find a vector  $\mathbf{a}$  such that the integral on the right vanishes, and hence  $\text{Fut}_v(M, w)$  vanishes identically for the corresponding toric vector field  $v$ . □

If  $M$  does not admit a Kähler–Einstein metric and  $\alpha \in c_1(M)$  is a Kähler form, then

$$R(M) = \sup\{t \mid \exists \omega \in c_1(M) \text{ such that } \text{Ric}(\omega) = t\omega + (1 - t)\alpha\},$$

provides a natural obstruction. It follows from the work of the second author [Sze11] that  $R(M)$  is in fact independent of the choice of  $\alpha$ . We can then recover the following result of Li [Li11], expressing  $R(M)$  in terms of the corresponding polytope.

**Theorem 18.** *Let  $M$  be toric, Fano, and  $P$  be the canonical polytope as above with barycenter  $P_C$ . Let  $Q$  be the point of intersection of the ray  $-sP_C$ ,  $s \geq 0$  with  $\partial P$ . If  $O$  denotes the origin,*

$$R(M) = \frac{|QO|}{|QP_C|}.$$

*Proof.* By the above discussion and Proposition 10 it is enough to find the maximum  $t$  such that  $\text{Fut}_{(1-t)\psi}(M, w) \geq 0$  for all toric holomorphic vector fields  $w$  where  $\alpha = \sqrt{-1}\partial\bar{\partial}\psi$ . We once again write  $w = \sum_{j=1}^n c_j z_j \frac{\partial}{\partial z_j}$  for some  $\mathbf{c} \in \mathbb{R}^n$ . Then the twisted modified Futaki invariant (Equation (16)) takes the form

$$\text{Fut}_{(1-t)\psi}(M, w) = t\mathbf{c} \cdot P_C + (1-t) \max_{\mathbf{x} \in P} \mathbf{c} \cdot \mathbf{x}.$$

Now let the face of the polytope containing  $Q$  be given by the vanishing of the affine linear functional  $l(\mathbf{x}) := \mathbf{u} \cdot \mathbf{x} + 1$ . Note that since  $l(0) = 1$ , it follows from elementary arguments that  $|QO|/|QP_C| = 1/l(P_C)$ . We also remark that  $l(P_C) \geq 1$ .

CLAIM. For any  $\mathbf{c} \in \mathbb{R}^n$ ,

$$\frac{\mathbf{c} \cdot P_C}{\max_{\mathbf{x} \in P} \mathbf{c} \cdot \mathbf{x}} \geq 1 - l(P_C).$$

Assuming this, for  $t \leq 1/l(P_C)$  and any holomorphic toric vector field  $w$ , it is easily seen that  $\text{Fut}_{(1-t)w}(M, w) \geq 0$ , and hence  $R(M) \geq 1/l(P_C)$ . On the other hand, if  $w$  is a special holomorphic vector field corresponding to  $-\mathbf{u} \in \mathbb{R}^n$ , then  $\max_{\mathbf{x} \in P} (-\mathbf{u}) \cdot \mathbf{x} = 1$ , and hence

$$\text{Fut}_{(1-t)\psi}(M, w) = 1 - t \cdot l(P_C).$$

This is negative when  $t > 1/l(P_C)$ , which implies that  $R(M) = 1/l(P_C)$ , completing the proof of the theorem. To prove the claim, we first normalize  $\mathbf{c}$  so that  $\max_{\mathbf{x} \in P} \mathbf{c} \cdot \mathbf{x} = 1$ . If we now let  $\tilde{l}(\mathbf{x}) = -\mathbf{c} \cdot \mathbf{x} + 1$ , then  $\tilde{l}(\mathbf{x}) \geq 0$  for all  $\mathbf{x} \in P$ . Moreover, since  $\mathbf{c} \cdot P_C = 1 - \tilde{l}(P_C)$  it is enough to show that  $l(P_C) \geq \tilde{l}(P_C)$ . Once again consider the ray  $-sP_C$  with  $s \geq 0$ . If this does not intersect the hyperplane  $\{\tilde{l} = 0\}$ , then clearly  $\mathbf{c} \cdot P_C \geq 0$ , and hence  $\tilde{l}(P_C) \leq 1 \leq l(P_C)$ . On the other hand, suppose the ray does intersect the hyperplane, at say a point  $Q'$ . Since the polytope  $P$  lies entirely on one side of the hyperplane, we have  $|QP_C| < |Q'P_C|$ . In fact, since  $\tilde{l}(0) = l(0) = 1$ ,

$$\tilde{l}(P_C) = \frac{|Q'P_C|}{|Q'O|} = \frac{|QQ'| + |QP_C|}{|QQ'| + |QO|} \leq \frac{|QP_C|}{|QO|} = l(P_C),$$

and the claim is proved. □

**T-varieties.** Relaxing the toric condition, we consider Fano manifolds  $M$  with an effective action of the torus  $T^m$  for some  $m < n = \dim M$ . The simplest case is that of a complexity-one action, where  $m = n - 1$ . Kähler–Einstein metrics on such manifolds, in particular Fano 3-folds with 2-torus actions, was studied by Süß [Sus13, Sus14]. In particular in [Sus14, Theorem 1.1] a list of 9 such manifolds is given with vanishing Futaki invariant, for 5 of which it was not known whether they admit a Kähler–Einstein metric or not. Using Theorem 1 one only needs to check  $T$ -equivariant special degenerations, and such degenerations can be classified using combinatorial data. [Sus14, Section 5] lists all such degenerations to canonical toric

Fano varieties, while the more general degenerations to log-terminal toric Fanos are classified by Ilten–Süss [IS]. The conclusion is that all 9 Fano threefolds with vanishing Futaki invariant in [Sus14, Theorem 1.1] admit a Kähler–Einstein metric. *Other manifolds with large symmetry group* We expect that Theorem 1 can be used to show the existence of Kähler–Einstein metrics on many other classes of Fano manifolds with large symmetry group. One interesting class is that of reductive varieties, studied by Alexeev–Brion [AB04a, AB04b]. Let  $G$  be a connected compact group,  $T \subset G$  a maximal torus, and  $W$  the corresponding Weyl group. Denote by  $\Lambda$  the character group of  $T$ , which is a lattice in the real vector space  $\Lambda_{\mathbf{R}}$ . To every  $W$ -invariant maximal dimensional convex lattice polytope  $P \subset \Lambda_{\mathbf{R}}$  one can associate a variety  $V_P$ , which is a  $G^c \times G^c$ -equivariant compactification of  $G^c$ , the action being left and right multiplication. As shown in [AB04b] (see also Alexeev–Katzarkov [AK05]), the equivariant degenerations of  $V_P$  correspond to convex, rational,  $W$ -invariant, piecewise linear functions  $f$  on  $P$ , in analogy to the toric case studied in Alexeev [Ale02], Donaldson [Don02]. If we have an equivariant special degeneration, then in particular the central fiber is irreducible, and this will only happen when  $f$  is linear on  $P \cap \Lambda_{\mathbf{R}}^+$ , where  $\Lambda_{\mathbf{R}}^+ \subset \Lambda_{\mathbf{R}}$  is a positive Weyl chamber corresponding to a Borel subgroup of  $G^c$ , containing  $T^c$ . It follows that there are only a finite number of degenerations that need to be checked in order to apply Theorem 1.

In the case when  $P \cap \Lambda_{\mathbf{R}}^+$  is a maximal set on which  $f$  is linear, then the central fiber of the corresponding special degeneration is a horospherical variety. These are the homogeneous toric bundles studied by Podesta–Spiro [PS10], who showed that all such Fano manifolds admit a Kähler–Ricci soliton. This also follows from the above discussion together with our main result, since the polytope  $P$  can not be subdivided further, and so a horospherical variety has no non-trivial equivariant special degenerations, just as the toric manifolds discussed above.

### 5 The Partial $C^0$ -Estimate for Solitons

In this section we briefly outline the changes that have to be made to the arguments in [Sze16], using also techniques in Zhang [Zha10], Tian–Zhang [TZ12] and Phong–Song–Sturm [PSS], to prove the partial  $C^0$ -estimate for the family of metrics  $\omega_t \in c_1(M)$  solving

$$\text{Ric}(\omega_t) - L_v \omega_t = t\omega_t + (1 - t)\alpha, \tag{76}$$

where  $t \in [0, T)$  with  $T < 1$ . The case when  $T = 1$  has been established by Jiang–Wang–Zhu [JWZ]. Here  $v$  is a holomorphic vector field, such that  $\text{Im } v$  generates a compact torus of isometries of the metric  $\alpha$ . In particular  $\omega_t$  will also be invariant under this torus. To simplify notation, we will drop the subscript  $t$ , and so in what follows,  $\omega$  denotes a solution of (76) for some  $t \in [0, T)$ .

Recall that we have the Hamiltonian function  $\theta_v$  of  $v$ , with respect to the metric  $\omega$ , defined by

$$\iota_v \omega = \sqrt{-1} \partial \bar{\partial} \theta_v, \tag{77}$$



with the normalization

$$\int_M e^{\theta_v} \omega^n = \int_M \omega^n. \tag{78}$$

From Zhu [Zhu00], and Wang–Zhu [WZ, Lemma 6.1] we know that we have estimates

$$|\theta_v| + |\nabla\theta_v|_\omega + |\Delta_\omega\theta_v| < C. \tag{79}$$

The Equation (76) implies that

$$\text{Ric}(\omega) - L_v\omega \geq 0. \tag{80}$$

In addition as soon as  $t$  is bounded away from 0, the volume comparison and Myers type theorem in Wei–Wylie [WW09] implies that the diameter of  $(M, \omega)$  is bounded, and we have the non-collapsing property

$$\text{Vol}(B(p, 1), \omega) \geq c > 0. \tag{81}$$

There are two basic approaches to studying metrics satisfying this lower bound for the Bakry–Émery Ricci curvature, generalizing the theory of Cheeger–Colding [CC97] in the case when  $v = 0$ . One approach is to study the conformally related metrics  $\tilde{g}_{j\bar{k}} = e^{-\frac{1}{n-1}\theta_v} g_{j\bar{k}}$ , where  $g_{j\bar{k}}$  is the metric with Kähler form  $\omega$ . This approach, similar to that used in Zhang [Zha10] and Tian–Zhang [TZ12] (who used the Ricci potential instead of  $\theta_v$ ), effectively reduces the problem to studying non-collapsed metrics with a lower Ricci curvature bound so that the theory of Cheeger–Colding can be applied. Indeed, in real coordinates the Ricci tensor of  $\tilde{g}$  satisfies

$$\tilde{R}_{ij} = R_{ij} + \nabla_i \nabla_j \theta_v + \frac{1}{2(n-1)} \nabla_i \theta_v \nabla_j \theta_v - \frac{1}{2(n-1)} [|\nabla\theta_v|_g^2 - \Delta_g\theta_v] g_{ij}, \tag{82}$$

and so (80), (79) together with the fact that  $v$  is holomorphic, and so  $\nabla_i \nabla_j \theta_v$  is of type  $(1, 1)$ , imply that  $\tilde{g}$  has a Ricci lower bound. In addition it is clear that  $\tilde{g}$  is uniformly equivalent to  $g$ . The other approach is to build up the Cheeger–Colding theory using the bound (80) on the Bakry–Émery Ricci curvature. This approach is executed by Wang–Zhu [WZ]. We summarize the main conclusions from these works that we need.

If we have a sequence  $(M, \omega_i)$ , satisfying (79), (80) and (81), then up to choosing a subsequence, the Riemannian manifolds  $(M, g_i)$  converge in the Gromov–Hausdorff sense to a length space  $(Z, d)$ . At each point  $p \in Z$  there exists a tangent cone  $C(Y)$  which is a metric cone. We can stratify the space  $Z$  as

$$S_n \subset S_{n-1} \subset \dots \subset S_1 = S \subset Z, \tag{83}$$

where  $S_k$  consists of those points, where no tangent cone is of the form  $\mathbf{C}^{n-k+1} \times C(Y)$ .

The regular part of  $Z$  is defined to be  $\mathcal{R} = Z \setminus S$ , and at  $p \in \mathcal{R}$  every tangent cone is  $\mathbf{C}^n$ . We also write  $\mathcal{D} = S \setminus S_2$ . The following is analogous to Anderson’s regularity result [And90], showing that we have good control of the metrics on the regular set if we also have an upper bound of the Bakry–Émery Ricci curvature.

PROPOSITION 19. *Suppose that  $B(p, 1)$  is a unit ball in Kähler manifold  $(M, \omega)$ , together with a holomorphic vector field  $v$  with Hamiltonian  $\theta$ , satisfying bounds of the form*

1.  $\sup_M |\theta| + |\nabla\theta| + |\Delta\theta| < K$
2.  $0 \leq \text{Ric}(\omega) - L_v\omega \leq K\omega$ .

*There are constants  $\delta, \kappa > 0$  depending on  $K$  such that if  $d_{GH}(B(p, 1), B^{2n}) < \delta$ , then for each  $q \in B(p, \frac{1}{2})$ , the ball  $B(q, \kappa)$  is the domain of a holomorphic coordinate system in which the components of  $\omega$  satisfy*

$$\begin{aligned} \frac{1}{2}\delta_{jk} < \omega_{j\bar{k}} < 2\delta_{jk}, \\ \|\omega_{j\bar{k}}\|_{L^{2,p}} < 2, \quad \text{for all } p. \end{aligned} \tag{84}$$

*Proof.* We use the conformal scaling  $\tilde{g} = e^{-\frac{1}{n-1}\theta}g$ , so that by (82)  $\tilde{g}$  satisfies two-sided Ricci curvature bounds. Suppose that  $d_{GH}((B(p, 1), g), B^{2n}) < \delta$ . The bound on  $\nabla\theta$  implies that if  $q \in B(p, \frac{1}{2})$  and  $r$  is sufficiently small, then

$$d_{GH}((B(q, r), \tilde{g}), r\lambda B^{2n}) < 2\delta, \tag{85}$$

for a suitable scaling factor  $\lambda$  (depending on the value  $\theta(q)$ ).

If  $\delta$  is sufficiently small, then Colding’s volume convergence result [Col97] combined with Anderson’s gap theorem implies that there is a harmonic coordinate system on the ball  $B(q, r\theta\lambda, \tilde{g})$  in which the metric  $\tilde{g}$  is controlled in  $L^{2,p}$  for any  $p$ . The metrics  $\tilde{g}$  and  $g$  are  $C^1$ -equivalent, so we also control the components of  $g$  in  $C^1$ . The Laplacian bound on  $\theta$  then implies that we have  $L^{2,p}$  estimates on  $\theta$  so in fact  $g$  and  $\tilde{g}$  are equivalent in  $L^{2,p}$ . In particular in our harmonic coordinates (harmonic for  $\tilde{g}$ ) we control the coefficients of  $g$  in  $L^{2,p}$ . Using that the complex structure is covariant constant, this allows us to find holomorphic coordinates on a possibly smaller ball, in which the coefficients of  $g$  are controlled in  $L^{2,p}$ .  $\square$

Following Chen–Donaldson–Sun, define

$$I(\Omega) = \inf_{B(x,r) \subset \Omega} VR(x, r), \tag{86}$$

where  $\Omega$  is any domain in a Kähler manifold, and  $VR(x, r)$  is the ratio of volumes of the ball  $B(x, r)$  in  $\Omega$  and the Euclidean ball  $rB^{2n}$ . If the Ricci curvature is non-negative, the Bishop-Gromov comparison theorem and Colding’s volume convergence implies that if  $B$  is a unit ball in  $\Omega$ , then  $1 - I(B)$  controls  $d_{GH}(B, B^{2n})$ , and conversely  $d_{GH}(B, B^{2n})$  controls  $1 - I(B)$ . In our setting, with the bound (80), a similar statement will only hold once the metrics are scaled up by a sufficient amount. We have the following.

PROPOSITION 20. *Suppose that  $B$  is a unit ball in a Kähler manifold  $(M, \omega)$  satisfying*

$$\text{Ric}(\omega) - L_v\omega \geq 0, \tag{87}$$

as well as

$$\sup_B |\nabla\theta| + |\Delta\theta| \leq \delta, \tag{88}$$

where  $\theta$  is a Hamiltonian of  $X$ . Then

$$d_{GH}(B, B^{2n}) = \Psi(\delta, 1 - I(B)), \tag{89}$$

and for any  $\lambda < 1$ ,

$$1 - I(\lambda B) = \Psi(\delta, d_{GH}(B, B^{2n}), 1 - \lambda), \tag{90}$$

where  $\Psi(\epsilon_1, \dots, \epsilon_k)$  denotes a function converging to zero as  $\epsilon_i \rightarrow 0$ . We have suppressed the dependence of  $\Psi$  on the dimension  $n$ .

*Proof.* We can assume that  $\theta(0) = 0$ . Use the conformal metric  $\tilde{g} = e^{-\frac{1}{n-1}\theta}g$ . Then under our assumptions we have  $Ric(\tilde{g}) > -C'\delta\tilde{g}$  and the metric  $\tilde{g}$  is very close in  $C^0$  to the metric  $g$ . We can then apply the volume convergence under lower Ricci curvature bounds to the metric  $\tilde{g}$ .  $\square$

We now return to our original setup, of a metric  $\omega$  on  $M$  satisfying

$$Ric(\omega) - L_v\omega = t\omega + (1 - t)\alpha, \tag{91}$$

for some  $t \in [0, T)$ , and  $T < 1$ . The vector field  $v$  and background metric  $\alpha$  is fixed. As before we can assume that the metrics are non-collapsed, and in addition the Hamiltonian  $\theta_v$  of  $v$  satisfies

$$\sup_M (|\nabla\theta_v|^2 + |\Delta\theta_v|) \leq K, \tag{92}$$

for some fixed constant  $K$ . The square is inserted for scaling reasons. Note that for any point  $p \in M$  we can choose the  $\theta_v$  so that  $\theta_v(p) = 0$ . We will exploit the fact that  $\alpha$  is a fixed metric. In particular we can assume that  $K$  is chosen such that on any ball of radius at most  $K^{-1}$  with respect to  $\alpha$  we can find holomorphic coordinates in which the coefficients of  $\alpha$  are controlled in  $C^2$ .

To understand the tangent cones of the Gromov–Hausdorff limit of a sequence of metrics satisfying these conditions, we need to study very small balls in  $(M, \omega)$ , scaled up to unit size. Let  $(B, \eta)$  be a small ball in  $(M, \omega)$  scaled to unit size, so that  $\eta = \Lambda\omega$  for some large  $\Lambda$ . Let  $w = \Lambda^{-1}v$ . Then  $\eta$  satisfies

$$Ric(\eta) - L_w\eta = \lambda\eta + (1 - t)\alpha, \tag{93}$$

for some  $\lambda \in (0, 1]$  and  $t \in (0, T)$ . In addition we can choose the Hamiltonian  $\theta_w$  for  $w$  relative to  $\eta$  such that  $\theta_w(0) = 0$ , and

$$\sup_M (|\nabla\theta_w|_\eta^2 + |\Delta_\eta\theta_w|) \leq \Lambda^{-1}K. \tag{94}$$

The following is the generalization of Proposition 8 in [Sze16], showing that on the regular set the Gromov–Hausdorff limit behaves as if we had a two-sided Ricci curvature bound. Note that as in Proposition 20 we need an extra assumption ensuring that we have scaled our metrics up by a sufficient amount.

PROPOSITION 21. *There is a  $\delta > 0$  depending on  $K$  above, such that if  $1 - I(B) < \delta$ , and the scaling factor  $\Lambda > \delta^{-1}$  then*

$$\alpha < 4\eta \quad \text{in } \frac{1}{2}B. \tag{95}$$

*Proof.* The method of proof is the same as in [Sze16]. Suppose that

$$\sup_B d_x^2 |\alpha(x)|_\eta = M, \tag{96}$$

where  $d_x$  is the distance of  $x$  to the boundary of  $B$  with respect to  $\eta$ , and suppose that the supremum is achieved at  $q \in B$ . If  $M > 1$  then we can consider the ball

$$B\left(q, \frac{1}{2}d_q M^{-1/2}\right), \tag{97}$$

scaled to unit size  $\tilde{B}$ , with scaled metric  $\tilde{\eta} = 4Md_q^{-2}\eta$ . Note that  $\tilde{\eta}$  satisfies the same estimates as  $\eta$ , but in addition  $|\alpha|_{\tilde{\eta}} \leq 1$  on  $\tilde{B}$ . If  $\delta$  is sufficiently small, then we can apply Propositions 19 and 20 to find holomorphic coordinates  $z_i$  on a small ball  $\tau\tilde{B}$ , in which the components of  $\tilde{\eta}$  are controlled in  $C^{1,\alpha}$ .

The metric  $\tilde{\eta}$  satisfies

$$\begin{aligned} \text{Ric}(\tilde{\eta}) &= L_w\eta + \lambda\eta + (1 - t)\alpha \\ &\geq (4Md_q^{-2})^{-1}L_w\tilde{\eta} + (1 - t)\alpha, \end{aligned} \tag{98}$$

and for any  $\epsilon > 0$  we can choose the scaling factor  $\Lambda$  large enough, so that the Hamiltonian of  $w$  satisfies  $|\nabla\theta_w|_\eta^2 < \epsilon$ , which implies  $|w|_\eta^2 < 4Md_q^{-2}\epsilon$ . Since  $w$  is a holomorphic vector field, we obtain that in the coordinates  $z_i$ , on the half ball  $\frac{\tau}{2}\tilde{B}$ , the components of  $w$ , along with their derivatives are bounded by  $(4Md_q^{-2}\epsilon)^{1/2}$ . It follows that on this ball we have

$$|L_w\eta|_{\tilde{\eta}} < C\epsilon^{1/2}(4Md_q^{-2})^{-1/2}, \tag{99}$$

for some fixed constant  $C$ . In particular if  $\delta$  is chosen sufficiently small, then we will have  $L_w\eta < \epsilon\tilde{\eta}$  and so

$$\text{Ric}(\tilde{\eta}) \geq -\epsilon\tilde{\eta} + (1 - t)\alpha. \tag{100}$$

Using this, the rest of the proof is essentially identical to that in [Sze16]. □

Together with Proposition 19 it follows from this that in the Gromov–Hausdorff limit of a sequence of metrics  $\omega$  satisfying (91), with  $t < T < 1$ , the regular set is open and smooth, and the convergence of the metrics is  $C^{1,\alpha}$  on the regular set. In addition the same holds for iterated tangent cones.

What remains is to study tangent cones of the form  $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$ , i.e. the points in the set  $\mathcal{D}$  in the Gromov–Hausdorff limit. The arguments in [Sze16, Proposition 11, 12, 13] can be followed closely with a couple of remarks. First of all the results

of Chen–Donaldson–Sun [CDS15b] on good tangent cones can be applied. The main difference here is that a variant of the  $L^2$ -estimates in [DS14, Proposition 2.1] needs to be used, following [PSS, Proposition 4.1], with the Hamiltonian  $\theta_v$  replacing the Ricci potential  $u$ . This implies that if a scaled up ball  $(B, \eta)$  as above is sufficiently close to the unit ball in  $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$ , then on a smaller ball we have holomorphic coordinates, in which the metric  $\eta$  satisfies the conditions (1), (2), (3) in the proof of Proposition 13.

An additional important fact used several times is that by Cheeger–Colding–Tian [CCT02], no tangent cone of the form  $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$  can form in the Gromov–Hausdorff limit of a sequence of Kähler metrics with bounded Ricci curvature. The analogous result with the bound on Ricci curvature replaced by a bound on  $\text{Ric}(\omega) - L_v\omega$  was shown by Tian–Zhang [TZ12]. It also follows from the more recent work of Cheeger–Naber [CN] in the general Riemannian case using a conformal change as in (82), together with the bounds (1) in Proposition 19. With these observations the proof of the partial  $C^0$ -estimate for solutions of (76) follows the argument in [Sze16] closely.

### 6 Reductivity of the Automorphism Group and Vanishing of the Futaki Invariant

In this section we briefly outline the proofs of Propositions 5 and 7 following [Ber15, CDS15c, BN]. As before, let  $W$  be the normal  $\mathbb{Q}$ -Fano variety obtained as the Gromov–Hausdorff limit along the continuity method, and  $v \in H^0(W, TW)$  such that  $\text{Im}(v)$  generates the action of a torus  $T$  on  $W$ . We let  $\mathcal{H}_v$  denote the space of continuous  $T$ -invariant metrics  $h_\phi = e^{-\phi}$  on  $-K_W$  with non-negative curvature. Then the twisted Ding functional is defined as

$$\mathcal{D}_{(1-t)\psi, v}(\phi) = -tE_v(\phi) - \log \left( \int_W e^{-t\phi - (1-t)\psi} \right), \tag{101}$$

where  $E_v$  is defined by its variation at  $\phi$  in the direction  $\dot{\phi}$  by

$$\frac{d}{ds} E_v(\phi) = \frac{1}{V} \int_W \dot{\phi} e^{\theta_v} \omega_\phi^n, \tag{102}$$

as in Berman–Witt–Nyström [BN]. Next, we recall the definition of a geodesic in the path of Kähler metrics. We let  $\mathcal{R} = \{s \in \mathbb{C} \mid \text{Re}(s) \in [0, 1]\}$ . Recall that a path  $\phi_s \in \mathcal{H}_v$  is called a geodesic if  $\Phi : W \times \mathcal{R} \rightarrow \mathbb{R}$  defined by  $\Phi(x, s) = \phi_{\text{Re}(s)}(x)$  satisfies  $\sqrt{-1}\partial\bar{\partial}_{s, W}(\Phi) \geq 0$  and

$$(\sqrt{-1}\partial\bar{\partial}_{s, W}\Phi)^{n+1} = 0,$$

where the  $\partial\bar{\partial}$  is taken in both  $W$  and  $\mathcal{R}$  directions. Then the following is proved in [Ber15].

LEMMA 22. For any  $\phi_0, \phi_1 \in \mathcal{H}_v$ , there exists a geodesic  $\phi_s \in \mathcal{H}_v$  connecting them such that

$$\|\phi_{s'} - \phi_s\|_{L^\infty(W)} < C|s' - s|$$

The key point is that the Ding functional is convex along these geodesics. It is proved in Berman–Witt–Nyström [BN, Proposition 2.17] that the functional  $E_v(\phi)$  is affine along geodesics and continuous up to the boundary. So the convexity of the Ding functional is a consequence of the following result of Berndtsson [Ber15].

PROPOSITION 23. Let  $\phi_s$  be a geodesic as above. Then the functional

$$\mathcal{F}(s) = -\log \left( \int_W e^{-t\phi_s - (1-t)\psi} \right) \tag{103}$$

is convex. Moreover, if  $\mathcal{F}(s)$  is affine, then there exists a holomorphic vector fields  $w_s$  on  $W$  with  $i_{w_s} \sqrt{-1} \partial \bar{\partial} \psi = 0$ , and such that the flow  $F_s$  satisfies

$$F_s^*(\sqrt{-1} \partial \bar{\partial} \phi_s) = \sqrt{-1} \partial \bar{\partial} \phi_0.$$

This was proved on compact Kähler manifolds by Berndtsson [Ber15] and extended to normal varieties by Chen–Donaldson–Sun [CDS15c] when  $\sqrt{-1} \partial \bar{\partial} \psi$  is the current of integration along a divisor (see also [BBEGZ]). Though the above statement does not seem to follow directly from either of the works, the arguments can be easily adapted, and we briefly provide an outline of the proof.

*Proof.* For ease of notation, we let  $\tau_s = t\phi_s + (1-t)\psi$ . Let  $p : W' \rightarrow W$  be a log-resolution and  $\omega'$  be a fixed Kähler metric on  $W'$ . Since  $W$  has only log terminal singularities, one has the following adjunction formula

$$-K_{W'} = -p^*K_W - E + \Delta, \tag{104}$$

where  $E$  and  $\Delta$  are effective divisors, and  $\Delta = \sum a_j E_j$  with  $a_j \in (0, 1)$ . Suppose first that  $e^{-\tau_s}$  is a smooth family of metrics on  $-K_W$ , inducing a smooth family of pull-back metrics on  $-p^*K_W$  with curvature  $\omega_{\tau'_s} = \sqrt{-1} \partial \bar{\partial} \tau'_s$ . We write  $L = K_{W'}^{-1} \otimes E$ . Then from (104) it is clear that

$$\tau'_s = p^* \tau_s + \sum a_j \log |s_j|^2,$$

where  $s_j$  is the defining function of  $E_j$ , induces a family of singular metrics  $e^{-\tau'_s}$  on  $L$ . Moreover, if  $u$  is a holomorphic  $L$ -valued  $(n, 0)$  form with zero divisor  $E$  (which is unique up to multiplication by a constant) it can be easily checked that up to scaling  $u$  by a constant,

$$\mathcal{F}(s) = -\log \int_{W'} u \wedge \bar{u} e^{-\tau'_s}.$$

Let us pretend for the moment that the metrics  $e^{-\tau'_s}$  are smooth. Consider the equation

$$\nabla_s \nu_s = P_s \left( \frac{d\tau'_s}{ds} u \right), \tag{105}$$

where  $\nabla_s = \partial - \partial\tau'_s \wedge \cdot$  is the Chern connection of  $e^{-\tau'_s}$  and  $P_s$  is the projection onto the orthogonal complement of  $L$ -valued holomorphic  $(n, 0)$  forms. As argued in [Ber15], it can be shown that there always exists a smooth solution  $\nu_s$  to (105) satisfying  $\bar{\partial}\nu_s \wedge \omega' = 0$ . Next, the Hessian of  $\mathcal{F}$  is given by ([Ber15, Theorem 3.1], [CDS15c, Lemma 14])

$$\|u\|_{\tau'_s}^2 \sqrt{-1} \partial \bar{\partial} \mathcal{F}(s) = \int_{W'} \omega'_s \wedge \tilde{u} \wedge \bar{\tilde{u}} e^{-\tau'_s} + \|\bar{\partial}\nu_s\|_{\tau'_s}^2 \sqrt{-1} ds \wedge d\bar{s}, \tag{106}$$

where  $\tilde{u} = u - ds \wedge \nu_s$  and  $\omega'_s = \sqrt{-1} \partial \bar{\partial}_{s, W'}(\tau'_s)$ . This is in fact a special case of the general positivity of direct image sheaves discovered by Bendtsson [Ber09]. For smooth geodesics, the convexity follows directly from this formula.

In our case the metrics  $\tau'_s$  are not smooth, and hence we first need to use a regularization. First, if we let  $\eta = \omega' + \sqrt{-1} ds \wedge d\bar{s}$ , then by the approximation theorem of Demailly [Dem92] (see also Blocki–Kolodziej [BK07]) there exists a decreasing sequence of smooth metrics  $\rho_{s,\epsilon} \searrow p^* \tau_s$  such that  $\sqrt{-1} \partial \bar{\partial}_{s, W'}(\rho_{s,\epsilon}) \geq -C\eta$ . By averaging we can also suppose that  $\rho_{s,\epsilon}$  are independent of  $Re(s)$  and  $T$ -invariant. To approximate  $\tau'_s$  we then let  $\tau'_{s,\epsilon} = \rho_{s,\epsilon} + \log h_\epsilon$  where

$$\log h_\epsilon = \sum a_j (\log(|s_j|_{h_j}^2 + \epsilon) - \log h_j) \tag{107}$$

and  $h_j$  is a metric on the line bundle generated by  $E_j$ . Clearly  $e^{-\tau'_{s,\epsilon}}$  are metrics on  $L$  with  $\tau'_{s,\epsilon} \searrow \tau'_s$  and  $\sqrt{-1} \partial \bar{\partial}_{s, W'}(\tau'_{s,\epsilon}) > -C\eta$  for some  $C > 0$ . Moreover, for any neighborhood  $U$  of  $\Delta$  there exists a constant  $C_U$  such that

$$\sqrt{-1} \partial \bar{\partial}_{s, W'}(\tau'_{s,\epsilon}) > -\epsilon C_U \eta, \quad \text{on } W' \setminus U.$$

We then let  $\nu_{s,\epsilon}$  be the solutions to (105) corresponding to  $\tau_{s,\epsilon}$ . The key point now is the following lemma of Berndtsson which guarantees uniform estimates for these solutions independent of  $s$  and  $\epsilon$ .

LEMMA 24. [Ber15, Lemmas 6.3, 6.5], [CDS15c, Lemmas 17, 19]

- There exists a constant  $C$  (independent of  $s, \epsilon$ ) such that

$$\|\nu_{s,\epsilon}\|_{L^2(\tau'_{s,\epsilon})} \leq C \left\| \frac{d\tau'_{s,\epsilon}}{ds} u \right\|_{L^2(\tau'_{s,\epsilon})}.$$

- For every  $\delta$ -neighborhood  $U_\delta$  of  $\Delta$ , there exists a constant  $c_\delta$  such that  $c_\delta \rightarrow 0$  as  $\delta \rightarrow 0$  and

$$\int_{U_\delta} |\nu_{s,\epsilon}|_{\tau'_{s,\epsilon}}^2 \leq c_\delta \left( \int_{W'} |\nu_{s,\epsilon}|_{\tau'_{s,\epsilon}}^2 + |\bar{\partial}\nu_{s,\epsilon}|_{\tau'_{s,\epsilon}}^2 \right).$$



Note that the norms of  $\nu_{s,\epsilon}$  also involve a Kähler metric on  $W'$  which we take to be the fixed metric  $\omega'$ . We also remark that this was proved by Berndtsson for metrics  $e^{-\xi}$  where  $\xi$  is only upper bounded, and hence is applicable in our situation since  $\tau'_{s,\epsilon}$  are easily seen to be upper bounded. Once we have this uniform  $L^2$  estimate, the rest of the argument in [CDS15c] can be followed almost verbatim. That is, if we write for  $\mathcal{F}_\epsilon(s)$  for the functional corresponding to  $\tau'_{s,\epsilon}$ , then  $\mathcal{F}_\epsilon \searrow \mathcal{F}$ . Moreover, using the Hessian formula above one can show that for any  $r \in (0, 1)$  on  $[r, 1 - r]$  we have

$$\frac{d^2 \mathcal{F}_\epsilon}{ds^2} > -c_\epsilon \rightarrow 0.$$

This shows that  $\mathcal{F}$  is indeed convex.

Suppose now that  $\mathcal{F}$  is affine linear. Observe that since  $\tau'_{s,\epsilon}$  decrease to  $\tau'_s$  and  $\tau'_{s,\epsilon}$  are uniformly Lipschitz in  $s$ ,  $\|\nu_{s,\epsilon}\|_{L^2(\tau'_{s,\epsilon})}$  are uniformly bounded. Hence  $\nu_{s,\epsilon}$  converges weakly in  $L^2(\tau'_s)$  to an  $L$ -valued  $(n-1, 0)$  form  $\nu_s$  with  $\bar{\partial}\nu_s = 0$ . Integrating by parts, it can be shown that  $\nu_s$  solves (105) weakly on  $W' \setminus \{\psi = -\infty\}$  or equivalently,  $\nabla_s \nu_s - u \, d\tau'_s/ds$  is holomorphic on  $\{\psi \neq \infty\}$ , and it is in  $L^2$ . But since pluripolar sets are removable for  $L^2$  holomorphic forms,  $\nabla_s \nu_s - u \, d\tau'_s/ds$  is also holomorphic globally. Using the formula  $\bar{\partial}\nabla_s \nu_s + \nabla_s \bar{\partial}\nu_s = \omega_{\tau'_s} \wedge \nu_s$  it follows that

$$\omega_{\tau'_s} \wedge \nu_s = \sqrt{-1} \bar{\partial} \left( \frac{d\tau'_s}{ds} \right) \wedge u. \tag{108}$$

A family of holomorphic vector fields  $w'_s$  can now be defined on  $W' \setminus E$  by

$$\iota_{w'_s} u = \nu_s,$$

so that away from  $E$  we have  $\iota_{w'_s} \omega_{\tau'_s} = -\sqrt{-1} \bar{\partial} \tau'_s$ . Then  $w_s = p_* w'_s$  is a holomorphic vector field on  $W_0$  which by normality of  $W$  extends to a global time-dependent holomorphic vector field on  $W$ . Next, note that  $p^{-1}$  is a biholomorphism when restricted to  $W_0$ , and  $\omega_{\tau_s} = (p^{-1})^* \omega_{\tau'_s}$ . It then follows that on  $W_0$ ,  $\iota_{w_s} \omega_{\tau_s} = -\sqrt{-1} \bar{\partial} \tau_s$  and hence,

$$\mathcal{L}_{w_s} \omega_{\tau_s} = -\frac{\partial}{\partial s} \omega_{\tau_s}, \tag{109}$$

as currents. Moreover, it can be shown that  $\partial w_s / \partial \bar{s} = 0$ , and hence  $w_s$  generates a holomorphic flow  $F_s$  (see [BBEGZ, Lemma 5.2]). Also, note that  $w'_s$  has uniform  $L^2$  bound (independent of  $s$ ) away from  $E$ , and hence the flow  $F_s$  extends continuously to  $s = 0, 1$  such that  $F_0$  is the identity. From (109) it follows that on  $W_0$ ,

$$\frac{\partial}{\partial s} F_s^* \omega_{\tau_s} = F_s^* \left( \frac{\partial}{\partial s} \omega_{\tau_s} + \mathcal{L}_{w_s} \omega_{\tau_s} \right) = 0.$$

In particular  $F_s^* \omega_{\tau_s} = \omega_{\tau_0}$  on  $W_0$ , and hence globally on  $W$  by unique extension of closed positive  $(1, 1)$  currents over sets of Hausdorff co-dimensions greater than two.

Now, if we define a holomorphic vector field  $\mathcal{W}_s = \partial/\partial s - w_s$  on  $W \times \mathcal{R}$ , following the same line of argument as in [Ber15, Lemma 4.3] we can show that

$$\iota_{\mathcal{W}_s} \sqrt{-1} \partial \bar{\partial}_{s,W}(\tau_s) = 0.$$

Again following [Ber15]

$$0 = \iota_{\overline{\mathcal{W}}_s} \iota_{\mathcal{W}_s} \sqrt{-1} \partial \bar{\partial}_{s,W}(\tau_s) = t \iota_{\overline{\mathcal{W}}_s} \iota_{\mathcal{W}_s} \sqrt{-1} \partial \bar{\partial}_{s,W}(\phi_s) + (1-t) \iota_{\overline{w}_s} \iota_{w_s} \sqrt{-1} \partial \bar{\partial} \psi.$$

Since both the  $(1, 1)$  currents on the right are non-negative, each has to be zero. Again, since  $\sqrt{-1} \partial \bar{\partial} \psi \geq 0$ , by Cauchy’s inequality for any  $(1, 0)$  vector field  $\xi$ ,  $\iota_{\xi} \iota_{w_s} \sqrt{-1} \partial \bar{\partial} \psi = 0$ , and hence  $\iota_{w_s} \sqrt{-1} \partial \bar{\partial} \psi = 0$ . In particular,  $\mathcal{L}_{w_s} \sqrt{-1} \partial \bar{\partial} \psi = 0$ , and hence  $F_s^* \sqrt{-1} \partial \bar{\partial} \phi_s = \phi_0$ , which completes the proof of the proposition.  $\square$

*Proof of Proposition 5.* Let  $e^{-\phi_0}$  and  $e^{-\phi_1}$  be two soliton metrics on  $(W, (1-t)\psi, v)$  and  $\phi_s \in \mathcal{H}_v$  be a bounded geodesic connecting  $\phi_0$  and  $\phi_1$ . Since solitons are the stationary points of  $\mathcal{D}_{(1-t)\psi, v}$ , the one sided derivatives at  $s = 0$  and  $s = 1$  (which exist by convexity of the Ding functional) are zero. As a consequence  $\mathcal{D}_{(1-t)\psi, v}(\phi_s)$ , and hence  $\mathcal{F}(s)$ , is affine, and by Proposition 23 there exists a family of holomorphic vector fields  $w_s$  with flow  $F_s$  such that  $F_s^* \omega_{\phi_s} = \omega_{\phi_0}$ . Next, note that  $\phi_j$  for  $j = 0, 1$  satisfies

$$Ric(\omega_{\phi_j}) = t \omega_{\phi_j} + (1-t) \sqrt{-1} \partial \bar{\partial} \psi + \mathcal{L}_v \omega_{\phi_j} \tag{110}$$

on  $W_0$ . So on the one hand, since  $\phi_s$  are stationary points of  $\mathcal{D}_{(1-t)\psi, v}$ ,  $\omega_{\phi_s}$  also satisfies (110), while on the other hand  $\omega_{\phi_s}$  satisfies (110) with  $v$  replaced by  $(F_s)_* v$ . Hence if we set  $\xi_s = (F_s)_* v - v$ , then  $\mathcal{L}_{\xi_s} \omega_{\phi_s} = 0$ . This implies that if  $h_s$  is the hamiltonian of  $\xi_s$  with respect to  $\omega_{\phi_s}$ , then  $\sqrt{-1} \partial \bar{\partial} h_s = 0$  and consequently  $v = (F_s)_* v$ . To show the time-independence of the vector fields, arguing as in the proof of [Ber15, Proposition 4,5], we can show that

$$\iota_{(F_s^{-1})_* w_s - w_0} \omega_{\phi_0} = 0.$$

Since  $\phi_0$  is bounded, and hence in particular  $e^{-\phi_0}$  is integrable, by Berndtsson [Ber15, Proposition 8.2] the above equation forces  $(F_s^{-1})_* w_s = w_0$ . This shows that the vector fields are independent of time, and in fact  $F_s$  is just the flow generated by  $w_0$ . Finally since  $\iota_{w_0} \omega_{\phi_0} = -\sqrt{-1} \partial \bar{\partial} \phi_0$  and  $\phi_0$  is real valued,  $Im(w_0)$  is also a Killing field for  $\omega_{\phi_0}$ . This completes the proof of the proposition with  $w = w_0$ .  $\square$

*Proof of Proposition 7.* As shown in [CDS15c] reductivity follows from uniqueness, and we reproduce their arguments. Suppose  $\omega$  is the twisted Kähler–Ricci soliton on the triple  $(W, (1-t)\psi, v)$ , and let  $H$  be the connected group with Lie algebra  $\mathfrak{g}_{W, \psi, v}$  naturally identified as a subgroup of  $SL(N+1, \mathbf{C})$ . Let  $K \subset H$  be the subgroup of isometries of  $\omega$  with the corresponding Lie sub-algebra of  $\mathfrak{g}_{W, \psi, v}$  given by

$$\mathfrak{k}_{W, \psi, v} = \{w \in H^0(W, T^{1,0}W) : \mathcal{L}_{Re(w)} \omega = 0, \iota_w \omega_{\psi} = 0, [w, v] = 0\},$$

which can naturally be identified as a sub-algebra of  $\mathfrak{su}(N + 1, \mathbf{C})$ . Moreover, since the trace form on  $\mathfrak{su}(N + 1, \mathbf{C})$  given by  $B(x, y) = \text{tr}(xy)$  is negative definite, its restriction to  $\mathfrak{k}_{W,\psi,v}$  is a non-degenerate bilinear form, and hence  $\mathfrak{k}_{W,\psi,v}$  is a reductive Lie algebra. Next, if  $K^c \subset SL(N + 1, \mathbf{C})$  is the connected complexification of  $K$ , then clearly  $K^c \subset H$ . Conversely, for any  $h \in H$ , it can be checked that  $h^*\omega$  is also a twisted Kähler–Ricci soliton for the triple  $(W, (1 - t)\psi, v)$ , and hence by Proposition 4 there exists an element  $F \in K^c$  such that  $h^*\omega = F^*\omega$ . But then  $h \circ F^{-1} \in K$ , and hence  $H = K^c$ . As a consequence  $\mathfrak{g}_{W,\psi,v} = \mathfrak{k}_{W,\psi,v} \otimes_{\mathbf{R}} \mathbf{C}$ , and is reductive. The same proof suitably modified shows that the centralizer  $\mathfrak{g}_{W,\psi,v}^G$  is also reductive.  $\square$

*Proof of Proposition 8.* Suppose that  $e^{-\phi}$  is a smooth metric on  $K_W^{-1}$ , and  $f_t \in \text{Aut}(W)$  is a one-parameter group of biholomorphisms, generated by  $w \in \mathfrak{g}_{W,\psi,v}$ . In particular since  $f_t^*\omega_\psi = \omega_\psi$ , we must have  $f_t^*(e^{-\psi}) = c_t e^{-\psi}$  for some constants  $c_t$ . Similarly to [CDS15c, Lemma 12], we consider the quantity

$$\begin{aligned} I(e^{-\phi}) &= \frac{1}{V} \int_W \log \frac{(\int_W e^{-\phi})^{-1} e^{-\phi}}{(\int_W e^{-t\phi-(1-t)\psi})^{-1} e^{-t\phi-(1-t)\psi}} \omega_\phi^n \\ &= \log \frac{\int_W e^{-t\phi-(1-t)\psi}}{\int_W e^{-\phi}} - \frac{1-t}{V} \int_W (\phi - \psi) \omega_\phi^n, \end{aligned} \tag{111}$$

where we note that  $\phi - \psi$  is a globally defined integrable function. We have  $I(f_t^*(e^{-\phi})) = I(e^{-\phi})$ , and differentiating this at  $t = 0$  we obtain (using  $\dot{\phi} = \theta_w$ ), that

$$\begin{aligned} &\frac{\int_W \theta_w e^{-\phi}}{\int_W e^{-\phi}} - \frac{\int_W t \theta_w e^{-t\phi-(1-t)\psi}}{\int_W e^{-t\phi-(1-t)\psi}} - \frac{1-t}{V} \int_W \theta_w \omega_\phi^n \\ &\quad - n \frac{1-t}{V} \int_W (\phi - \psi) \sqrt{-1} \partial \bar{\partial} \theta_w \wedge \omega_\phi^{n-1} = 0. \end{aligned} \tag{112}$$

Integrating by parts in the last integral, and using the definition (15) of the twisted Futaki invariant, we obtain

$$\text{Fut}_{(1-t)\psi,v}(W, w) = \frac{t}{V} \int_W \theta_w e^{\theta_v} \omega_\phi^n - t \frac{\int_W \theta_w e^{-t\phi-(1-t)\psi}}{\int_W e^{-t\phi-(1-t)\psi}}. \tag{113}$$

Note that this formula is not well defined if  $e^{-(1-t)\psi}$  is not integrable, but we only need it in that case, since by assumption  $(W, (1 - t)\psi, v)$  admits a twisted Kähler–Ricci soliton.

By the convexity of  $\mathcal{D}_{(1-t)\psi,v}$ , twisted Kähler–Ricci solitons minimize the twisted Ding functional, we know that  $\mathcal{D}_{(1-t)\psi,v}$  is bounded below. At the same time (113) implies that

$$\frac{d}{dt} \mathcal{D}_{(1-t)\psi,v}(f_t^* \phi) = -\text{Fut}_{(1-t)\psi,v}(W, w), \tag{114}$$

and as a result the twisted Futaki invariant must vanish.  $\square$

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