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AN OBSTRUCTION TO THE SMOOTHABILITY OF SINGULAR NONPOSITIVELY CURVED METRICS ON 4-MANIFOLDS BY PATTERNS OF INCOMPRESSIBLE TORI

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Abstract. We give new examples of closed smooth 4-manifolds which support singular metrics of nonpositive curvature, but no smooth ones, thereby answering affirmatively a question of Gromov. The obstruction comes from patterns of incompressible 2-tori sufficiently complicated to force branching of geodesics for nonpositively curved metrics.

Contents

1 Introduction

The goal of this note is to exhibit new examples of closed smooth 4-manifolds which support singular metrics of nonpositive (sectional) curvature, but no smooth ones. Such manifolds had first been found by Davis, Januszkiewicz and Lafont [DJL12]. The approaches are different, but they both rely on a basic rigidity phenomenon in nonpositive curvature, namely that free abelian subgroups in the fundamental group of a closed nonpositively curved manifold are carried by totally-geodesically immersed flat tori. The fundamental groups Γ of the singular locally CAT(0) 4-manifolds M studied in [DJL12] contain few ("isolated") copies of $\mathbb{Z}^2$. The obstruction to the existence of a smooth nonpositively curved metric on M is that they, respectively, the corresponding invariant flats in the universal covering $\widetilde{M}$, are *knotted at infinity*. This is impossible for 2-flats in smooth Hadamard 4-manifolds. We consider fundamental groups Γ which contain *plenty* of copies of $\mathbb{Z}^2$ and exploit the fact that this rigidifies the geometry of nonpositively curved metrics on M, singular or smooth, since it enforces a complicated pattern of immersed flat 2-tori. Extreme cases occur in "higher rank": When Γ splits as a product $\Gamma_1 \times \Gamma_2$ of subgroups, then the universal cover splits as a metric product. Or (in dimensions ≥ 5) when Γ is the fundamental group of an irreducible higher rank locally symmetric space of noncompact type, then the geometry of nonpositively curved metrics is completely rigidified by Mostow rigidity, i.e. it is essentially unique up to rescaling. We consider "rank one" situations where there are still plenty of subgroups isomorphic to $\mathbb{Z}^2$ which however only partially rigidify the geometry. Heuristically, singular nonpositively curved metrics allow more complicated patterns of tori than smooth ones because, due to possible branching, the tori can be packed "more densely". It is therefore conceivable that sufficiently complicated patterns which occur for singular nonpositively curved metrics cannot occur in the smooth case because they enforce the branching of geodesics. Indeed, natural candidates to which this line of reasoning could apply had been pointed out by Gromov in (the first exercise of) [BGS85], namely the fundamental groups of branched coverings $M \to \Sigma \times \Sigma$ of products of higher genus surfaces Σ with themselves with branching locus the diagonal. The purpose of this note is to do this exercise. More precisely, we prove

Theorem 1 (Exercise 1 in [BGS85]). *Let V be a closed 4-dimensional manifold which admits a non-trivial finite branched covering $\beta : V \to \Sigma \times \Sigma$ over the product of a hyperbolic surface Σ with itself such that the branching locus equals the diagonal $\Delta_\Sigma \subset \Sigma \times \Sigma$. Then V admits no smooth Riemannian metric of nonpositive sectional curvature.*

The above theorem is an application of a more general result (Theorem 2), which provides an obstruction for a discrete group Γ to act geometrically on a Hadamard manifold. The obstruction comes from the existence of a CAT(0) model space X_{model} which contains a specific "singular configuration" and admits a geometric action $\Gamma \curvearrowright X_{model}$.

In short, the singular configuration in X_{model} consists of two rigid convex subsets. Their rigidity ensures that one finds corresponding convex subsets in every CAT(0) space which allows for a geometric action by Γ. The singular nature of Γ is reflected in the way these two sets interact. On the one hand, their correlation forces the presence of branching geodesics. On the other hand, they are inseparable, in the sense that their interaction persists when passing to corresponding sets in other CAT(0) spaces with geometric actions by Γ. Consequently, *any* CAT(0) space which permits a geometric action by Γ has to be singular.

2 Preliminaries

For notations and basics on CAT(0) spaces we refer the reader to the first two chapters of [B95] and Section 2 of [KL97].

2.1 Quasi-isometry invariance of flats. A *flat* F in a CAT(0) space X is a convex subset isometric to a Euclidean space. If a flat has dimension k, then we will also call it a *k-flat*.

If $\Gamma \curvearrowright X$ is an isometric action, then a flat $F \subset X$ is called Γ-*periodic* if its stabilizer $\mathrm{Stab}_\Gamma(F)$ acts cocompactly on it. In case of a discrete action, a finite index subgroup of $\mathrm{Stab}_\Gamma(F)$ acts on F by translations, and hence $\mathrm{Stab}_\Gamma(F)$ is virtually free abelian of rank equal to the dimension of F.

An isometric action $\Gamma \curvearrowright X$ of a discrete group Γ on a locally compact CAT(0) space X is called *geometric* if it is properly discontinuous and cocompact. Then every abelian subgroup $A \subset \Gamma$ preserves a flat in X on which it acts cocompactly.

Suppose that $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright X'$ are geometric actions of the same group on two locally compact CAT(0) spaces. Then there exists a Γ-equivariant quasi-isometry $\Phi : X \to X'$.

The following properties of CAT(0) spaces will imply Proposition 1 which will be used further in the proof.

If $F \subset X$ is a Γ-periodic flat, then its stabilizer $\mathrm{Stab}_\Gamma(F)$ is virtually abelian and preserves a flat $F' \subset X'$. Hence, a Γ-equivariant quasi-isometry $\Phi : X \to X'$ carries Γ-periodic flats in X Hausdorff close to Γ-periodic flats in X'.

One can also say something regarding the quasi-isometry invariance of *non*-periodic flats: Recall that, for locally compact CAT(0) spaces with cocompact isometry group, the maximal dimension of flats equals the maximal dimension of quasi-flats and is in particular a quasi-isometry invariant [K99, Thm. C]. By Theorem B in [LS97], quasi-flats of maximal dimension which are within finite Hausdorff distance from (maximal) flats are actually within uniformly bounded Hausdorff distance from these flats. Combining these results, one obtains the following useful

Proposition 1. *There exists a constant $D = D(L, A, X, X')$ such that a Γ-equivariant (L, A)-quasi-isometry $\Phi : X \to X'$, between CAT(0) spaces X and X', maps Γ-periodic flats of maximal dimension in X D-Hausdorff close to such flats in X'. As a consequence, also pointed Hausdorff limits of Γ-periodic flats of maximal dimension in X are carried D-Hausdorff close to such flats in X'.*

2.2 Product rigidity. Recall that an isometry of a CAT(0) space is called *axial*, if it preserves a complete geodesic on which it acts as a nontrivial translation. Such a geodesic is then called an *axis*. We will need the following product splitting result which is a special case of Corollary 10 in [M06]. See also Proposition 2.2 in [L00] and Theorem 1 in [S85].

Proposition 2. *Let X be a locally compact CAT(0) space and let $\Gamma \cong \Gamma_1 \times \Gamma_2$ be a product of non-abelian free groups Γ_i. Suppose that Γ acts on X discretely by axial isometries. Then there exists a minimal non-empty Γ-invariant closed convex subset $C \subset X$ which splits metrically as a product, $C \cong C_1 \times C_2$, such that Γ preserves the product splitting and Γ_i acts trivially on C_{3-i}.*

REMARK 1. If X is 4-dimensional, then the set C is unique. Indeed, the factors C_i would have to be 2-dimensional. Since two minimal non-empty Γ-invariant closed convex subsets are parallel and X is 4-dimensional, C is unique.

2.3 Coarse intersection of flats and quasi-isometry invariance. For a subset A of a metric space X we denote its closure by $\overline{A}$ and its tubular r-neighborhood by $N_r(A)$.

Let $F_1, F_2 \subset X$ be flats. We say that they *diverge* if $\partial_\infty F_1 \cap \partial_\infty F_2 = \emptyset$. Equivalently, the distance function $d(\cdot, F_2)|_{F_1}$ is proper and grows (at least) linearly.

DEFINITION 1. *Let $F_1, F_2 \subset X$ be diverging flats. We say that F_1* coarsely intersects *F_2 if there exists $R \geq 0$ such that for every $r \geq R$ holds: If $B_1 \subset F_1$ is a round ball such that $F_1 \cap \overline{N_r(F_2)} \subset \mathrm{int}(B_1)$, then its boundary sphere ∂B_1 is not contractible inside $X \backslash \overline{N_r(F_2)}$.*

REMARK 2. (i) This is independent of the choice of the ball $B_1 \subset F_1$.

(ii) The notion is asymptotic in the sense that it only depends on the ideal boundaries of the flats, i.e. passing to parallel flats does not affect coarse intersection.

(iii) Coarse intersection is not a symmetric relation.

(iv) In general, disjoint flats can coarsely intersect. However, this cannot occur in geodesically complete smooth spaces, i.e. in Hadamard manifolds.

We need a criterion to recognize whether flats coarsely intersect. In the smooth case "coarse intersection" simply becomes "nontrivial transversal intersection", i.e. two flats in a Hadamard manifold intersect coarsely if and only if they intersect transversely in one point. This is clear, because for a flat F in a Hadamard manifold X there is a deformation retraction of $X \backslash F$ onto $X \backslash \overline{N_r(F)}$ using the gradient flow of $d(\cdot, F)$.

More generally, we have:

LEMMA 1. *Let F_1 and F_2 be flats in a CAT(0) space X. Suppose that F_2 is contained in an open convex subset $C \subset X$ which is Riemannian, i.e. the metric on C is induced by a smooth Riemannian metric. If F_1 and F_2 intersect transversely in one point, then F_1 coarsely intersects F_2.*

Proof. Otherwise spheres in $F_1 \backslash F_2$ around the intersection point $F_1 \cap F_2$ could be contracted in $X \backslash F_2$. But this would be absurd since $X \backslash F_2$ retracts to $\overline{C} \backslash F_2$ along normal geodesics. □

It will be crucial for us that coarse intersection is quasi-isometry invariant.

LEMMA 2. *Let $\Phi : X \to X'$ be a quasi-isometry of CAT(0) spaces with a quasi-inverse $\Phi' : X' \to X$. Let $F_1, F_2 \subset X$ and ${F'}_1, {F'}_2 \subset X'$ be flats such that $\Phi(F_i)$ is Hausdorff close to ${F'}_i$. Then F_1 coarsely intersects F_2 if and only if ${F'}_1$ coarsely intersects ${F'}_2$.*

Proof. First note that F_1 and F_2 diverge if and only if F'_1 and F'_2 do. The quasi-flat $\Phi|_{F_1}$ may not be continuous, but since X is CAT(0), it is uniformly (in terms of the quasi-isometry constants) Hausdorff close to a continuous quasi-flat $q : F_1 \to X'$. Suppose that $q(F_1)$ is D-Hausdorff close to F'_1. If F'_1 does not coarsely intersect F'_2, then for every tubular neighborhood $N_r(F'_2)$ of F'_2 all large spheres in F'_1 are contractible in the complement of $\overline{N_r(F'_2)}$. Because the q-image of a sphere in F_1 can be homotoped to F'_1 by a D-short homotopy, we obtain that q-images of large spheres in F_1 are contractible in $X \backslash \overline{N_r(F'_2)}$. The Φ'-image of a (contracting) homotopy is again uniformly Hausdorff close to a continuous map. Since $\Phi' \circ \Phi$ is at finite distance from id_X, it follows that we can for every radius $r > 0$ contract sufficiently large spheres in F_1 in the complement of the tubular r-neighborhood of F_2. Consequently, F_1 does not coarsely intersect F_2. □

3 Configurations of convex product subsets in dimension 4

3.1 Flat half-strips in CAT(0) surfaces with symmetries. By a *flat strip*, respectively, *half-strip* of *width* $w \geq 0$ in a CAT(0) space we mean a convex subset isometric to $\mathbb{R} \times [0, w]$, respectively, to $[0, +\infty) \times [0, w]$.

The following observation restricts the possible positions of flat half-strips in a CAT(0) surface relative to the action of its isometry group.

LEMMA 3. *Let Y be a smooth CAT(0) surface, and let $h \subset Y$ be a flat half-strip. Suppose that h is asymptotic to a periodic geodesic $c \subset Y$, i.e. to an axis c of an axial isometry γ of Y.*

Then either $w = 0$, or h extends to a (periodic) flat strip in Y parallel to c.

Proof. We may assume that γ translates towards the ideal endpoint of h and preserves the orientation transversal to c. If $w > 0$ and $r(t)$ is a ray in $\mathrm{int}(h)$, then the ray $\gamma^{-1}r$ is strongly asymptotic to r, i.e. $d(\gamma^{-1}r(t), r) \to 0$ as $t \to +\infty$. Therefore, $\gamma^{-1}r$ must enter $\mathrm{int}(h)$, because $\mathrm{int}(h)$ is open in Y. Consequently, $\gamma^{-1}r$ extends r, and $\gamma^{-1}h$ extends h. It follows by induction that h is contained in a γ-invariant flat strip. □

3.2 Configurations not occuring in smooth spaces. Let X be a CAT(0) space.

We describe a configuration of convex product subsets which can occur if X is singular, but not if it is smooth.

We assume that X contains two closed convex subsets, namely a product

$$Y_1 \times Y_2$$

of smooth CAT(0) surfaces Y_1 and Y_2 with boundary such that $\mathrm{int}(Y_1) \times \mathrm{int}(Y_2)$ is open in X; and a product

$$Z \times \mathbb{R}$$

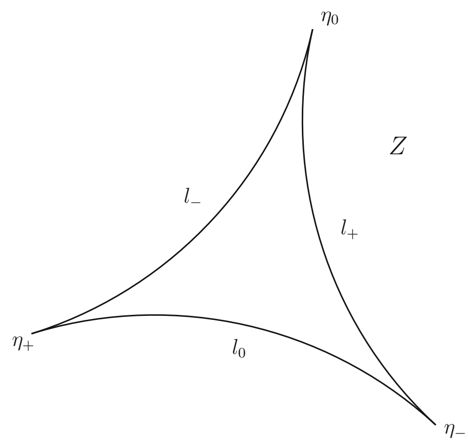

Figure 1: The ideal triangle contained in Z

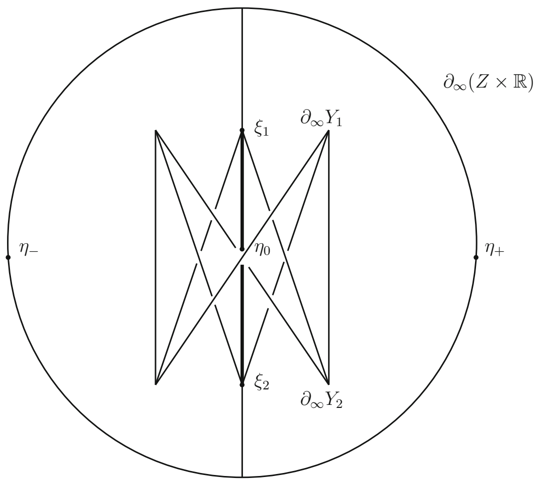

Figure 2: The configuration in $\partial_\infty X$

whose (not necessarily smooth) cross section Z contains an ideal triangle with three ideal vertices η_0, η_+, η_-. We denote the sides asymptotic to η_0 and $\eta_\pm$ by $l_\mp$ and the side asymptotic to η_+ and η_- by l_0 (Fig. 1).

We assume furthermore, that these product subsets interact as follows:[1]

(i) The intersection of the flat $F_\pm = l_\pm \times \mathbb{R} \subset Z \times \mathbb{R}$ with $Y_1 \times Y_2$ contains a quadrant $r_1^\pm \times r_2^\pm$, where $r_i^\pm$ are asymptotic rays in Y_i. We denote their common ideal endpoint by $\xi_i \in \partial_\infty Y_i$.

(ii) η_0 is an *interior* point of the Tits arc $\xi_1\xi_2$ of length $\frac{\pi}{2}$ in $\partial_\infty X$.

Then the intersection $Y_1 \times Y_2 \cap Z \times \mathbb{R}$ is nonempty and, by condition (ii), the product structures (i.e. the directions of the factors) do not match on it. The latter

[1] See Figure 2.

implies that the convex subset $Y_1 \times Y_2 \cap Z \times \mathbb{R}$ is *flat*.[2] As a consequence, subrays of the rays $r_i^{\pm}$ bound a *flat half-strip* $h_i \subset Y_i$.

In addition, we impose a *symmetry* condition:

(iii) The rays $r_i^{\pm}$ are asymptotic to a periodic geodesic $c_i \subset Y_i$.

Using Lemma 3 above, we conclude: Either subrays of the rays $r_i^{\pm}$ coincide, or subrays extend to geodesics $c_i^{\pm} \subset Y_i$ parallel to c_i.

Claim 1. *If conditions* (*i*)–(*iii*) *hold, then* X *cannot be smooth.*

Proof. Suppose that X is smooth. Then our discussion implies that the flats $F_{\pm}$ either have a quadrant in common and therefore coincide, or contain parallel half-planes and their intersection of ideal boundaries $\partial_\infty F_+ \cap \partial_\infty F_-$ contains an arc of length π of the form $\xi_1\xi_2\hat{\xi}_1$ or $\xi_2\xi_1\hat{\xi}_2$ with an antipode $\hat{\xi}_i \in \partial_\infty Y_i$ for $i = 1$ or 2. It follows that $\angle_{Tits}(\eta_{\pm}, \hat{\xi}_i) < \frac{\pi}{2}$ and hence $\angle_{Tits}(\eta_+, \eta_-) < \pi$, a contradiction. □

3.3 Not equivariantly smoothable configurations. Now, we restrict to *symmetric* situations and consider *geometric* actions

$$\Gamma \curvearrowright X$$

by discrete groups on locally compact CAT(0) spaces, i.e. actions which are isometric, properly discontinuous and cocompact.

We will tie the configuration considered above sufficiently closely to the action so that it will carry over to other geometric actions $\Gamma \curvearrowright X'$ on CAT(0) spaces. This will then be used to rule out such actions on *smooth* CAT(0) spaces, i.e. on Hadamard 4-manifolds.

In addition to the conditions (i)–(iii) above, we assume:

(iv) X contains no 3-flats.

(v) $Y_1 \times Y_2$ is preserved by a subgroup $\Gamma_1 \times \Gamma_2 \subset \Gamma$ with non-abelian free factors Γ_i, and the restricted action $\Gamma_1 \times \Gamma_2 \curvearrowright Y_1 \times Y_2$ is a product action (not necessarily cocompact).

(vi) The flats $F_{\pm}$ and the flat $F_0 = l_0 \times \mathbb{R}$ in $Z \times \mathbb{R}$ are Γ-*periodically approximable* (i.e. pointed Hausdorff limits of Γ-periodic flats).

(vii) The geodesics $c_i \subset Y_i$ are Γ_i-periodic. Moreover, there exist Γ_i-periodic geodesics $d_i \subset \mathrm{int}(Y_i)$ which intersect the rays $r_i^{\pm} \subset Y_i$ transversally in points.

Under the assumptions (i)–(vii), we look for a corresponding configuration in X'. Let $\Phi : X \to X'$ denote a Γ-equivariant quasi-isometry.

By (v) and Proposition 2, there exists a $\Gamma_1 \times \Gamma_2$-invariant closed convex product subset (in general singular)

$$Y_1' \times Y_2' \subset X'$$

on which $\Gamma_1 \times \Gamma_2$ acts by a product action. The Γ_i-periodic image quasigeodesics $\Phi(c_i)$ are Hausdorff close to Γ_i-periodic geodesics $c'_i \subset Y'_i$.

[2] This follows from the fact that a geodesic triangle in a product of CAT(0) spaces is flat if and only if its projections to the factors are flat.

By (iv+vi) and Proposition 1, the quasi-flats $\Phi(F_\pm)$ and $\Phi(F_0)$ are Hausdorff close to flats $F'_\pm$ and F'_0. We have that any one of these flats is contained in a tubular neighborhood of the union of the other two. Hence its ideal boudary circle is contained in the union of the ideal boudary circles of the other two. This leaves only the possibility that the union of their ideal boudaries is a spherical suspension of three points. It follows that the flats are contained in a closed convex product subset

$$Z' \times \mathbb{R} \subset X'$$

whose cross section Z' contains an ideal triangle with corresponding ideal vertices $\eta'_0, \eta'_+, \eta'_-$ and sides l'_+, l'_-, l'_0, such that $F'_\pm = l'_\pm \times \mathbb{R}$ and $F'_0 = l'_0 \times \mathbb{R}$. Furthermore, if $\rho \subset Z$ is a ray asymptotic to one of the ideal vertices η_0, η_+ or η_-, then Φ carries the vertical half-plane $\rho \times \mathbb{R} \subset Z \times \mathbb{R}$ Hausdorff close to a vertical half-plane $\rho' \times \mathbb{R} \subset Z' \times \mathbb{R}$ where $\rho' \subset Z'$ is a ray with corresponding ideal endpoint η'_0, η'_+ or η'_-. This follows from the fact that the half-plane $\rho \times \mathbb{R}$ is Hausdorff close to the intersection of sufficiently large tubular neighborhoods of two of the flats $F_\pm$ and F_0.

Since the c_i are periodic, Φ carries the quadrants $r_1^\pm \times r_2^\pm$ Hausdorff close to a quadrant $r'_1 \times r'_2$ for rays $r'_i \subset c'_i$. The quadrants $r_1^\pm \times r_2^\pm$ are contained in vertical half-planes with ideal boundary semicircle $\partial_\infty F_+ \cap \partial_\infty F_-$ and, by condition (ii), their ideal boundary arc $\xi_1\xi_2$ of length $\frac{\pi}{2}$ is contained in the interior of this semicircle. Denoting the ideal endpoints of the rays r'_i by $\xi'_i = \partial_\infty r'_i$ it follows that the arc $\xi'_1\xi'_2$ of length $\frac{\pi}{2}$ is contained in the *interior* of the semicircle $\partial_\infty F'_+ \cap \partial_\infty F'_-$, and η'_0 is an *interior* point of the arc $\xi'_1\xi'_2$ of length $\frac{\pi}{2}$.

In summary, the interaction of the product subsets $Y'_1 \times Y'_2$ and $Z' \times \mathbb{R}$ at infinity is as for the configuration in X. However, without further assumptions, the intersection $Y'_1 \times Y'_2 \cap Z' \times \mathbb{R}$ could be empty.

Claim 2. *X' cannot be smooth Riemannian.*

Proof. Suppose that X' is smooth. We show that then the intersection $Y'_1 \times Y'_2 \cap Z' \times \mathbb{R}$ must be nonempty.

Note that there exist Γ_i-periodic geodesics $d'_i \subset Y'_i$ with the same stabilizers as the geodesics d_i. By (vii), the periodic flat $d_1 \times d_2$ transversally intersects the flats $F_\pm$ in points inside the smooth region $\text{int}(Y_1) \times \text{int}(Y_2)$. Hence, by Lemma 1, $d_1 \times d_2$ *coarsely* intersects $F_\pm$. It follows from Lemma 2 that $d'_1 \times d'_2$ coarsely intersects $F'_\pm$. Now we use that X' is smooth to deduce that $d'_1 \times d'_2$ intersects $F'_\pm$ transversally in a point. In particular, $Y'_1 \times Y'_2 \cap Z' \times \mathbb{R} \neq \emptyset$.

It follows that conditions (i)–(iii) are satisfied by the product subsets $Y'_1 \times Y'_2$ and $Z' \times \mathbb{R}$ of X'. By Claim 1, this is a contradiction. □

We have proved:

Theorem 2 (Obstruction to smooth action). *If a discrete group Γ admits a geometric action $\Gamma \curvearrowright X$ on a locally compact CAT(0) space satisfying conditions (i)–(vii), then Γ does not act geometrically on any smooth Hadamard 4-manifold.*

Remark 3. The regularity assumptions can be relaxed. The argument works more generally and shows that Γ does not act geometrically on locally compact, geodesically complete CAT(0) spaces X' without branching geodesics, for instance $\mathcal{C}^2$-smooth Hadamard 4-manifolds [St13].

4 An example

In this section, we consider the geometric actions on 4-dimensional singular CAT(0) spaces suggested by Gromov in the first exercise of [BGS85] and verify that they contain configurations satisfying conditions (i)–(vii).

Let Σ be a closed surface of genus ≥ 2, and let

$$\beta : V \to \Sigma \times \Sigma$$

be a non-trivial finite branched covering with branching locus the diagonal $\Delta_\Sigma \subset \Sigma \times \Sigma$. Then the group

$$\Gamma := \pi_1(V)$$

admits geometric actions on 4-dimensional singular CAT(0) spaces: Let $\pi_V : X \to V$ denote the universal covering, and $\pi := \beta \circ \pi_V : X \to \Sigma \times \Sigma$. We equip Σ with a hyperbolic metric and pull back the corresponding product metric on $\Sigma \times \Sigma$ to singular metrics on V and X. In this way the 4-manifold X becomes a CAT(0) space, and the deck action

$$\Gamma \curvearrowright X$$

becomes a geometric action.

Regarding the geometry of X, note first that the *singular locus* $\pi^{-1}(\Delta_\Sigma) \subset X$ is a disjoint union of isometrically embedded hyperbolic planes. The restriction of π to any of them is a universal covering of the *branching locus* $\Delta_\Sigma \subset \Sigma \times \Sigma$.

We look for patterns of flats in X which obstruct the existence of geometric Γ-actions on Hadamard manifolds, as described in Sections 3.2 and 3.3.

The space X contains no 3-dimensional flats, but plenty of 2-dimensional ones. There are two kinds of them: flats disjoint from $\pi^{-1}(\Delta_\Sigma)$, and flats which intersect $\pi^{-1}(\Delta_\Sigma)$ orthogonally in one or several parallel geodesics.

Let $\mathcal{F}_0$ denote the set of flats disjoint from $\pi^{-1}(\Delta_\Sigma)$. There are obvious subfamilies of $\mathcal{F}_0$ which occur in convex product subsets of X. Namely, let

$$\Sigma = \Sigma^+ \cup \Sigma^- \tag{1}$$

be a decomposition of Σ into two subsurfaces $\Sigma^\pm$ along a finite family of disjoint closed geodesics. Then the open product block $\mathrm{int}(\Sigma^+ \times \Sigma^-) \subset \Sigma \times \Sigma$ is disjoint from Δ_Σ, and hence the connected components of its inverse image $\pi^{-1}(\mathrm{int}(\Sigma^+ \times \Sigma^-))$ in X are convex subsets isometric to $\mathrm{int}(\widetilde{\Sigma}^+ \times \widetilde{\Sigma}^-)$ on which π restricts to a universal covering of $\mathrm{int}(\Sigma^+ \times \Sigma^-)$.

The other flats in X important for our argument are, somewhat unexpectedly, the flats which intersect $\pi^{-1}(\Delta_\Sigma)$ in precisely *one* geodesic; let us denote the set of these flats by $\mathcal{F}_1$. Understanding them leads us to considering flat half-planes.

We define $\mathcal{H}$ as the set of *injectively* immersed flat half-planes $H \subset \Sigma \times \Sigma$ which intersect the branching locus precisely along their boundary line, $H \cap \Delta_\Sigma = \partial H$, and are orthogonal to it, $H \perp \Delta_\Sigma$. Furthermore, we define $\widetilde{\mathcal{H}}$ as the set of isometrically embedded flat half-planes $\widetilde{H} \subset X$ such that $\widetilde{H} \cap \pi^{-1}(\Delta_\Sigma) = \partial\widetilde{H}$ and $\widetilde{H} \perp \pi^{-1}(\Delta_\Sigma)$. We say that a half-plane $\widetilde{H} \in \widetilde{\mathcal{H}}$ *covers* or is a *lift* of a half-plane $H \in \mathcal{H}$ if $\pi|_{\widetilde{H}}$ is a local isometry onto H. A flat in $\mathcal{F}_1$ is the union of two half-planes in $\widetilde{\mathcal{H}}$ with common boundary line.

We collect some facts about $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ needed for our argument.

If $H \in \mathcal{H}$, then ∂H is an injectively immersed line in Δ_Σ and therefore of the form $\partial H = \Delta_c$ for a nonperiodic simple geodesic $c \subset \Sigma$. It follows that $H \subset c \times c$ because H is flat. We also see that half-planes in $\mathcal{H}$ occur in pairs of opposite half-planes with common boundary line.

A half-plane $H \in \mathcal{H}$ lifts to a half-plane $\widetilde{H} \in \widetilde{\mathcal{H}}$ because it is simply-connected and the branched covering β is a true covering over $\Sigma \times \Sigma - \Delta_\Sigma$. More precisely, for a point $p \in H - \partial H$ and a lift $\widetilde{p}$ of p there exists a unique lift $\widetilde{H}$ of H with $\widetilde{p} \in \widetilde{H}$. A lift $\widetilde{l} \subset \pi^{-1}(\Delta_\Sigma)$ of the boundary line ∂H extends in several ways to a lift $\widetilde{H}$ of H, because points close to ∂H can be lifted in several ways to points close to $\widetilde{l}$. The number of lifts is given by the local branching order of π at $\widetilde{l}$.

If $\widetilde{H} \in \widetilde{\mathcal{H}}$, then its boundary line $\partial\widetilde{H}$ projects to an immersed line Δ_c in Δ_Σ. The geodesic $c \subset \Sigma$ must be nonperiodic simple, because otherwise $(\widetilde{H} - \partial\widetilde{H}) \cap \pi^{-1}(\Delta_\Sigma) \neq \emptyset$. Thus, all half-planes in $\widetilde{\mathcal{H}}$ are lifts of half-planes in $\mathcal{H}$.

If $\widetilde{H}_1, \widetilde{H}_2 \in \widetilde{\mathcal{H}}$ are distinct half-planes with the same boundary line, $\partial\widetilde{H}_1 = \partial\widetilde{H}_2$, then their projections $H_1, H_2 \in \mathcal{H}$ either coincide or are a pair of opposite half-planes. The local geometry of branched coverings implies, that $\widetilde{H}_1, \widetilde{H}_2$ have angle π along their common boundary line and their union $\widetilde{H}_1 \cup \widetilde{H}_2$ is a flat in $\mathcal{F}_1$.

We will use the following consequence of this discussion: Let $c \times c \subset \Sigma \times \Sigma$ be an injectively immersed plane, and let $H_\pm$ be the half-planes into which it is divided by Δ_c. Then for every lift $\widetilde{H}_+$ of H_+ there exist at least two distinct lifts $\widetilde{H}_-^1, \widetilde{H}_-^2$ of H_- with the same boundary line $\partial\widetilde{H}_-^i = \partial\widetilde{H}_+$, and the union of any two of the three half-planes $\widetilde{H}_+, \widetilde{H}_-^1, \widetilde{H}_-^2$ is a flat in $\mathcal{F}_1$.

The flats in $\mathcal{F}_1$ are nonperiodic. Nevertheless, they are useful for investigating geometric Γ-actions on other CAT(0) spaces. This is due to the following fact:

Lemma 4. *Let $F \in \mathcal{F}_1$. Suppose that the nonperiodic simple geodesic $\pi(F \cap \pi^{-1}(\Delta_\Sigma))$ in Δ_Σ is the pointed Hausdorff limit of periodic simple geodesics in Δ_Σ. Then F is the pointed Hausdorff limit of Γ-periodic flats in X.*

Proof. We denote $\widetilde{l} = F \cap \pi^{-1}(\Delta_\Sigma)$. Let $(c_n, p_n) \to (c, p)$ be a sequence of pointed periodic simple geodesics in Σ converging to the nonperiodic simple geodesic $c \subset \Sigma$ with $\pi(\widetilde{l}) = \Delta_c$. There exist geodesics $\widetilde{l}_n \subset \pi^{-1}(\Delta_\Sigma)$ lifting the c_n and lifts $\widetilde{p}_n, \widetilde{p}$ of the base points p_n, p such that $(\widetilde{l}_n, \widetilde{p}_n) \to (\widetilde{l}, \widetilde{p})$. We choose embedded subsegments $s_n \subset c_n$ of increasing lengths centered at the base points p_n such that also $(s_n, p_n) \to (c, p)$ and lifted segments $\widetilde{s}_n \subset \widetilde{l}_n$ centered at the $\widetilde{p}_n$ such that $(\widetilde{s}_n, \widetilde{p}_n) \to (\widetilde{l}, \widetilde{p})$.

The main step of the argument is to approximate F by isometrically embedded flat squares $\widetilde{Q}_n \subset \pi^{-1}(s_n \times s_n)$ with diagonals $\widetilde{s}_n$, $(\widetilde{Q}_n, \widetilde{p}_n) \to (F, \widetilde{p})$. This will imply the assertion because isometrically embedded flat squares in $\pi^{-1}(c_n \times c_n)$ are contained in Γ-periodic flats. Indeed, the subsets $\pi^{-1}(c_n \times c_n) \subset X$ have cocompact stabilizers in Γ, and their connected components are convex subsets which split as metric products of the line with discrete metric trees. All flats contained in them are limits of Γ-periodic ones.

To find the squares $\widetilde{Q}^n$, we proceed as follows. The flat F is divided by $\widetilde{l}$ into two half-planes $\widetilde{H}_\pm \in \widetilde{\mathcal{H}}$. We will approximate these simultaneously by isometrically embedded right-angled isosceles triangles $\widetilde{T}^n_\pm \subset \pi^{-1}(s_n \times s_n)$ with sides $\widetilde{s}_n$.

Let $\widetilde{q}_\pm \in \widetilde{H}_\pm - \partial\widetilde{H}_\pm$ be base points close to $\widetilde{p}$, and let $\bar{q}_\pm = \pi(\widetilde{q}_\pm) \in c \times c - \Delta_c$ denote their projections. There exist sequences of points $\bar{q}^n_\pm \in s_n \times s_n - \Delta_{s_n}$ approximating them, $\bar{q}^n_\pm \to \bar{q}_\pm$. More precisely, we choose them such that they are close to $\Delta_{p_n} \in \Delta_{s_n}$ intrinsically in $s_n \times s_n$, i.e. such that the segments $\Delta_{p_n}\bar{q}^n_\pm \subset s_n \times s_n$. Furthermore, there exists a sequence of lifts $\widetilde{q}^n_\pm \in \pi^{-1}(\bar{q}^n_\pm)$ close to $\widetilde{p}_n$ such that $\widetilde{q}^n_\pm \to \widetilde{q}_\pm$.

The injectively immersed square $s_n \times s_n \subset \Sigma \times \Sigma$ is divided by Δ_{s_n} into two triangles. Let $T^n_\pm$ be the subtriangle containing $\bar{q}^n_\pm$. (Possibly $T^n_+ = T^n_-$.) Since the injectively immersed flat triangles $T^n_\pm$ meet Δ_Σ only along their hypotenuses Δ_{s_n}, we can lift them to isometrically embedded flat triangles $\widetilde{T}^n_\pm$ in X with hypotenuses $\widetilde{s}_n$, as we could lift the half-planes in $\mathcal{H}$ to half-planes in $\widetilde{\mathcal{H}}$. The lifts are again uniquely determined by the lift of one off-hypotenuse point. Thus we can choose them such that $\widetilde{q}^n_\pm \in \widetilde{T}^n_\pm \subset \pi^{-1}(c_n \times c_n)$. Then the pointed triangles $(\widetilde{T}^n_\pm, \widetilde{q}^n_\pm)$ Hausdorff converge to a flat half-plane in $\widetilde{\mathcal{H}}$ with base point $\widetilde{q}_\pm$ and boundary line $\widetilde{l}$. The only such half-plane is $\widetilde{H}_\pm$, i.e. $(\widetilde{T}^n_\pm, \widetilde{q}^n_\pm) \to (\widetilde{H}_\pm, \widetilde{q}_\pm)$.

The two triangles $T^n_\pm$ either coincide or have angle π along their common side Δ_{s_n}. The local geometry of branched coverings implies that the lifted triangles $\widetilde{T}^n_\pm$ have angle π along their common side $\widetilde{s}_n$. (They are distinct for large n, $\widetilde{T}^n_+ \cap \widetilde{T}^n_- = \widetilde{s}_n$.) Hence their union $\widetilde{Q}^n = \widetilde{T}^n_+ \cup \widetilde{T}^n_-$ is an embedded flat square in X. These are the squares we were looking for. As desired, they satisfy $(\widetilde{Q}^n, \widetilde{p}_n) \to (F, \widetilde{p})$. This finishes the proof. □

Now we describe a configuration in X which satisfies conditions (i)–(vii) formulated in Sections 3.2 and 3.3.

We consider a decomposition (1) of Σ and choose an injectively immersed geodesic line $c \subset \Sigma$ which intersects $\Sigma^+ \cap \Sigma^-$ transversally in precisely one point p. The geodesic c is divided by p into the injectively immersed rays $r^\pm = c \cap \Sigma^\pm$. We can arrange our choices (of Σ, $\Sigma^\pm$ and c) so that

(a) $r^\pm$ is asymptotic to a simple closed geodesic $c^\pm \subset \mathrm{int}(\Sigma^\pm)$, and

(b) c is a pointed Hausdorff limit of simple closed geodesics $c_n \subset \Sigma$.

Indeed, if $\Sigma^\pm$ and $c^\pm$ are chosen appropriately then there exists a simple closed curve a, which intersects c^+ and c^- transversally in one point each and $\Sigma^+ \cap \Sigma^-$ transversally in two points. It is divided by its intersection points with $c^\pm$ into two arcs a_{+-} and a_{-+}. The concatenations $a_{+-} * nc^- * a_{-+} * nc^+$ are freely homotopic to simple closed geodesics c_n which, when equipped with suitable base points, Hausdorff converge to an injectively immersed line c with the desired properties.

Let $H \in \mathcal{H}$ be the half-plane $H \subset c \times c$ with boundary line $\partial H = \Delta_c$ and containing the quadrant $r^+ \times r^-$. There exist two distinct flats $F_1, F_2 \in \mathcal{F}_1$ which contain the same lift $\widetilde{H} \in \widetilde{\mathcal{H}}$ of H (and branch along its boundary line $\partial\widetilde{H}$). Their union $F_1 \cup F_2$ splits metrically as $Z \times \mathbb{R}$, and the cross section Z is a degenerate ideal triangle (a tripod). By Lemma 4, the three flats contained in $Z \times \mathbb{R}$, i.e. F_1, F_2 and $(F_1 \cup F_2) - \mathrm{int}(\widetilde{H})$, are Γ-periodically approximable.

Let $\widetilde{r}^+ \times \widetilde{r}^- \subset \widetilde{H}$ be the quadrant lifting $r^+ \times r^-$. There exists a closed convex product subset $P = Y^+ \times Y^- \subset X$ such that $\pi|_P$ is a universal covering of $\Sigma^+ \times \Sigma^-$ and $F_j \cap P = \widetilde{r}^+ \times \widetilde{r}^-$ for $j = 1, 2$.

The product subsets $Y^+ \times Y^-$ and $Z \times \mathbb{R}$ satisfy conditions (i)–(vii). Applying Theorem 2, we therefore obtain:

Theorem 1 (Exercise 1 in [BGS85]). *Let V be a closed 4-dimensional manifold which admits a non-trivial finite branched covering $\beta : V \to \Sigma \times \Sigma$ over the product of a hyperbolic surface Σ with itself such that the branching locus equals the diagonal $\Delta_\Sigma \subset \Sigma \times \Sigma$. Then V admits no smooth Riemannian metric of nonpositive sectional curvature.*

Remark 4. As in Theorem 2, one can relax the regularity assumptions and rule out the existence of $\mathcal{C}^2$-smooth Riemannian metrics on V [St13].

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