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LIOUVILLE THEOREM FOR BELTRAMI FLOW

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Abstract. We prove that the Beltrami flow of ideal fluid in $\mathbb{R}^3$ of a finite energy is zero.

1 Introduction

Let $v(x)$, $x \in \mathbb{R}^n$, $n = 2, 3$ be a velocity of a steady flow of an ideal fluid. Then v is a solution of the system of Euler equations:

$$\begin{cases} v\nabla v + \nabla p = 0, & \text{in } \mathbb{R}^n \\ div\ v = 0, & \text{in } \mathbb{R}^n. \end{cases} \quad (1)$$

We assume that the vector field v is smooth.

The system of Euler equations (1) has equivalent forms. It can be written as the Helmholtz equation, see, e.g., [AK98].

$$[v, \omega] = 0,$$

where $\omega =$curl$v$ is the vorticity of v and $[\cdot]$ are the Lie brackets of vector fields.

In dimension 3 the Euler equation can be also written in Bernoulli form:

$$v \times \text{curl} v = \nabla b, \quad (2)$$

where

$$b = p + \tfrac{1}{2}||v||^2 \quad (3)$$

is the Bernoulli's function.

A stationary solution v of the system (1) is called the Beltrami flow if $b \equiv$const and hence v satisfies the equation

$$v \times \text{curl} v = 0. \quad (4)$$

The Beltrami flows are an important class of stationary solutions of the Euler equation. For basic properties of the Beltrami flows see [AK98], some recent results are in [EP1].

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In this paper we are concerned with vanishing at infinity solutions of (1). On the plane the ready example of compactly supported solution of the Euler equation comes from rotationally symmetric flows. Non-symmetric flow one can obtain pasting together finite or countable collection of rotationally symmetric flows with disjoint supports.

In dimension 3 the existence of compactly supported stationary solutions of the Euler equation is not known. However, there exists a Beltrami flow $v \in C^\infty(\mathbb{R}^3)$ such that $|v(x)| < C/|x|$, [EP1].

Notice that nonzero solutions of (1) in $\mathbb{R}^3$ can vanish on an open set, for instance, the cylinders of solutions of (1) in $\mathbb{R}^2$ with compact support. Explicit examples of a solution of the Euler equation which vanishes in the interior or exterior part of a given hyperboloid are constructed in [SV85]. In the contrast for the Beltrami flows the unique continuation property holds, [EP].

In this paper we show that the Beltrami flow of an ideal fluid in $\mathbb{R}^3$ of a finite energy is zero.

Theorem. *Let $v \in C^1(\mathbb{R}^3)$ be a Beltrami flow. Assume that either $v \in L_p(\mathbb{R}^3)$, $2 \le p \le 3$, or $v(x) = o(1/|x|)$ as $x \to \infty$. Then $v \equiv 0$.*

Notice, that Enciso and Peralta-Salas example of the Beltrami flow, [EP1], shows that the assumptions of Theorem are sharp.

If we consider the Navier-Stokes equations instead of the Euler equations then stronger Liouville type theorems hold. Any bounded in $\mathbb{R}^2$ solution u of the Navier-Stokes equations is a constant, see [KNSS09], and any solution of the Navier-Stokes equations in $\mathbb{R}^3$ with a sufficiently small L_3-norm is zero, see [G98].

To prove Theorem 1.2 we rewrite equations (1) as linear equations for a suitable tensor form.

2 Tensor Equations from the Euler Equation

First we introduce some tensor notations and then derive from (1) equations for corresponding tensor fields.

Denote by T^m the space of covariant tensors on $\mathbb{R}^n$ of the rang m; let $S^m \subset T^m$ be the symmetric subspace of T^m. The map $\sigma : T^m \to S^m$

$$\sigma f(x_1, \ldots, x_m) = \frac{1}{m!} \sum f(x_{i_1}, \ldots, x_{i_m})$$

where the summation is taken over all permutations of the indices 1,$\ldots$, m, is called the symmetrization of tensor f. For smooth tensor fields $C^\infty(T^m, \mathbb{R}^n)$ is defined covariant differentiation $\nabla : C^\infty(T^m, \mathbb{R}^n) \to C^\infty(T^{m+1}, \mathbb{R}^n)$,

$$\nabla f = f_{i_1,\ldots,i_m;j}$$

The operator d of inner differentiation is the symmetrization of ∇, $d = \sigma\nabla : C^\infty(S^m, \mathbb{R}^n) \to C^\infty(S^{m+1}, \mathbb{R}^n)$. The divergence operator δ, $\delta : C^\infty(S^m, \mathbb{R}^n) \to C^\infty(S^{m-1}, \mathbb{R}^n)$,

$$(\delta f)_{i_1,\dots,i_{m-1}} = \sum f_{i_1,\dots,i_m;i_m}$$

is an operator formally adjoint to $-d$.

Let $v \in C^\infty(R^3)$ be a solution of (1). We define the tensor $F \in C^\infty(S^2, R^3)$ of the flow v as

$$F = p(dx)^2 + \tilde{v}^2,$$

where $\tilde{v}$ is a convector dual to the vector v: $\tilde{v}(\cdot) = (v, \cdot)$ and

$$\tilde{v}^2 = \sum v^i v^j dx_i dx_j.$$

As a consequence of the system (1) one has the equations

$$p_i + \sum_j (v^i v^j)_j = 0.$$

Directly from the last equations we get the following linear equation for F:

$$\delta F = 0. \tag{5}$$

3 Proof of the Theorem

For a Beltrami flow v it follows from (4), (2), (3) that $p = -|v|^2/2+\text{const}$. Subtracting from p a constant we may assume that

$$p = -|v|^2/2. \tag{6}$$

Let F be the flow's tensor of v. Then from (6) it follows

$$F = -\frac{|v|^2}{2}(dx)^2 + \tilde{v}^2,$$

where $(dx)^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2$.

Let A (B) be the the spherical average of F ($\tilde{v}$), i.e.,

$$A = \int_{s\in O_3} F_s d\chi,$$

$$B = \int_{s\in O_3} \tilde{v}_s^2 d\chi,$$

where F_s ($\tilde{v}_s$) are the rotations of F ($\tilde{v}$) on $s \in O_3$ and $d\chi$ is the Haar measure on the group O_3. Then

$$tr\, A(x) = -\frac{1}{2} tr\, B(x)$$

and hence

$$A(x) = B(x) - \frac{1}{2}(tr\, B(x))(dx)^2.$$

Let $r \in \mathbb{R}$, $\theta \in S^2$ be the polar coordinates in $\mathbb{R}^3$, $r^2 dr d^2\theta$ is a standard element of volume in $\mathbb{R}^3$: $r^2 dr d^2\theta = (dx)^3$, where $d^2\theta$ is the area form on the unit sphere. Let

$$B = \alpha(x)(dr)^2 + \beta(x) r^2 (d\theta)^2,$$

where $(d\theta)^2$ is the metric tensor of the unit sphere and$\alpha(x) = \alpha(|x|)$, $\beta(x) = \beta(|x|)$. Since $\tilde{v}^2$ and hence B are nonnegative tensors then $\alpha, \beta \geq 0$. Therefore

$$A(x) = B(x) - \frac{1}{2}(\alpha + 2\beta)(dx)^2 = (\frac{1}{2}\alpha - \beta)(dr)^2 - \frac{1}{2}\alpha(d\theta)^2.$$

Since (5) is a linear equation it holds after the averaging of F,

$$\delta A = 0. \tag{7}$$

Denote by G_r the half ball $\{\{|x| < r\} \cap \{x_1 < 0\}\}$. Set $H_r = \{\{|x| < r\} \cap \{x_1 = 0\}\}$, $n = (1, 0, 0)$. Integrating equality (7) against the vector n we get

$$-\int_{H_r} (An, n) ds = \int_{\partial G_r \setminus H_r} (An, x/r) ds.$$

Since $(An, n)_{|\{x_1 = 0\}} = -\alpha/2 \leq 0$ we have

$$\int_0^r t\alpha(t) dt = -\frac{1}{2} r^2 (\alpha(r) - 2\beta(r)). \tag{8}$$

Hence

$$-\int_{H_r} (An, n) ds \leq \int_{\partial G_r \setminus H_r} |A| ds. \tag{9}$$

By our assumption either $v \in L_p(\mathbb{R}^3)$, $2 \leq p \leq 3$, and hence

$$\int_0^\infty \int_{\partial G_r \setminus H_r} |A|^{p/2} ds dr < \infty \tag{10}$$

or $v(x) = o(1/|x|)$ as $x \to \infty$, therefore $|A| = o(1/|x|^2)$ and

$$\int_{\partial G_r \setminus H_r} |A| ds = o(1/r). \tag{11}$$

In the first case by Hölder's inequality

$$\int_{\partial G_r \setminus H_r} |A| ds \leq \left(\int_{\partial G_r \setminus H_r} |A|^{p/2} ds \right)^{2/p} \left(\int_{\partial G_r \setminus H_r} ds \right)^{(p-2)/p}$$

or

$$\left(\int_{\partial G_r \setminus H_r} |A| ds \right)^{p/2} \leq 2\pi r^{p-2} \int_{\partial G_r \setminus H_r} |A|^{p/2} ds. \tag{12}$$

From inequality (10) follows the existence of the sequence $r_n \to \infty$ such that

$$r_n \int_{\partial G_{r_n} \setminus H_{r_n}} |A|^{p/2} ds \to 0$$

as $n \to \infty$. Thus from the inequality (12) follows that

$$\int_{\partial G_{r_n} \setminus H_{r_n}} |A| ds \to 0$$

as $n \to \infty$.

Since α is nonnegative taking $n \to \infty$ we get from the inequality (9)

$$\alpha \equiv 0$$

in case of inequality (10). In case (11) the last identity immediately follows from (9) and (11). Then from the equality (9) we conclude

$$\beta \equiv 0.$$

Thus $A = 0$ and hence $v(0) = 0$. Since the last equality holds for the any choice of origin in $\mathbb{R}^3$ it follows that $v \equiv 0$. The theorem is proved.

References

[AK98] V.I. Arnold, B.A. Khesin *Topological Methods in Hydrodynamics*, Springer, (1998).

[G98] G.P. Galdi An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Vol. 2. Springer (1998).

[EP1] A. Enciso, D. Peralta-Salas *Existence of knotted vortex tubes in steady Euler flows*. arXiv:1210.6271v1.

[EP] A. Enciso, D. Peralta-Salas *Beltrami fields with a non constant proportionalty factor are rare* . preprint.

[KNSS09] G. Koch, N. Nadirashvili, G. Seregin, V. Sverak Liouville theorems for the Navier-Stokes equations and applications. *Acta Math*, (1)203 (2009), 83–105.

[SV85] B. Shapiro, A. Vainstein Multididimensional analogues of the Newton and Ivory theorems. *Functional Analysis and its Applications*, 19 (1985), 17–20.

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