

# CLASSIFICATION OF INVARIANT FATOU COMPONENTS FOR DISSIPATIVE HÉNON MAPS

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**Abstract.** Fatou components for rational endomorphisms of the Riemann sphere are fully classified and play an important role in our view of one-dimensional dynamics. In higher dimensions, the situation is less satisfactory. In this work we give a nearly complete classification of invariant Fatou components for moderately dissipative Hénon maps. Namely, we prove that any such a component is either an attracting or parabolic basin, or the basin of a rotation domain. More specifically, recurrent Fatou components were classified about 20 years ago (modulo the problem of existence of Herman ring basins), while in this paper we prove that non-recurrent invariant Fatou components are semi-parabolic basins. Most of our methods apply in a more general setting.

## 1 Main Result

The basic dynamical dichotomy for a holomorphic dynamical system is given by partitioning the space into the *Fatou set* and the *Julia set*. The former is usually an open dense subset supporting the regular dynamics, while the latter is a smaller set supporting the chaotic dynamics. This dichotomy is classical for rational endomorphisms of the Riemann sphere (as the names suggest), and in the past 25 years, it has been extended to some classes of higher dimensional dynamical systems, including polynomial automorphisms of  $\mathbb{C}^2$ .

The first fundamental problem to address is to describe the dynamics on the Fatou set. There are some types of behavior on this set, convergence to attracting or parabolic cycles, that can be easily observed. There are more subtle types, irrational rotations in certain domains, which naturally appear in the theory but it is much harder to construct examples. And then the question arises, whether any other type of behavior can occur.

For rational endomorphisms of the Riemann sphere, this problem is now fully resolved, but it took 60 years to reach this satisfactory state. Dynamics on the periodic Fatou components was thoroughly analyzed in the classical work by Fatou and Julia. However, they were not able to decide whether rotation domains exist. Much later existence of such domains was demonstrated in the work of Siegel, Arnold, and Herman on small denominators. After that, the only pending issue was the

possible existence of wandering domains. Non-existence of such domains was proved by Sullivan by methods of quasiconformal deformations.

Interest to the dynamics of polynomial automorphisms of  $\mathbb{C}^2$  (with the Hénon maps as main examples) rose in late 1980's, due to the pioneering work of Friedland and Milnor, Hubbard and Obertse-Vorth, Bedford and Smillie, Fornæss and Sibony, followed by many other people. First results on the dynamics on periodic Fatou components were obtained by Bedford and Smillie in 1991 [BS91b], followed by Fornæss and Sibony in 1995 [FS95], with a later complement by Ueda [Ued08]. The case of *recurrent* Fatou components was analyzed in those papers. (Here “recurrence” of a component  $\Omega$  means that there is a point  $z \in \Omega$  whose limit set  $\omega(z)$  contains a point in  $\Omega$ .) These domains were classified as attracting basins or basins of rotation attractors, or rotation domains. (One question that remained unsolved was whether Herman rings can appear as attractors.)

After this work, the problem was reduced (modulo existence of Herman rings) to classification of *non-recurrent* Fatou components, where all orbits escape to the boundary of the domain. The main new phenomenon that can appear in this situation is existence of limit sets of rank 1. (In fact, such a phenomenon can appear indeed for holomorphic endomorphisms of the projective space  $\mathbb{P}^2$ .) For 20 years, no essential progress has been made in this problem.

In this paper, we solve the problem for *moderately dissipative* automorphisms:

**Theorem 1.** *Let  $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  be a non-elementary polynomial automorphism of degree  $d \geq 2$ , and let  $\delta = \det Df$  be its Jacobian. Assume that  $f$  is moderately dissipative, i.e.*

$$|\delta| < \frac{1}{d^2}. \quad (1)$$

*Let  $\Omega$  be an invariant non-recurrent Fatou component of  $f$  with bounded forward orbits. Then all the orbits in  $\Omega$  converge to a parabolic point  $\alpha \in \partial\Omega$  with multiplier 1.*

Thus, for moderately dissipative automorphisms, the answer turns out to be the same as in the one-dimensional case.

Note that Hénon maps

$$(z, w) \mapsto (p(z) - \delta w, z),$$

where  $p(z)$  is a polynomial of degree  $d \geq 2$ , with small Jacobian  $\delta$ , can be viewed as perturbations of one-dimensional polynomials. However, it is well known that various new phenomena and substantial technical problems can arise after such a perturbation. Newhouse phenomenon [New74] on possible co-existence of infinitely many attracting cycles, or non-rigidity of infinitely renormalizable maps [CLM05] are two examples of this kind. And though some conjectures are proposed for all dissipative Hénon maps, often progress has been made for a tiny Jacobian. (Most notable work of this kind is by Benedicks and Carleson [BC91].) Our assumption,

$|\text{Jac}f| < 1/4$  for quadratic Hénon maps, looks quite moderate from this point of view. Of course, we can make no conjecture of whether our result can be extended to all scales of the Jacobian.

Putting together the above results, we obtain a nearly complete classification of the dynamics on invariant Fatou components:

**Theorem 2.** *Let  $\Omega$  be an invariant Fatou component of a moderately dissipative non-elementary polynomial automorphism  $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  of degree  $d \geq 2$ . Then one of the following three cases is satisfied.*

- (1) *All orbits in  $\Omega$  converge to an attracting fixed point  $p \in \Omega$ . The component  $\Omega$  is biholomorphically equivalent to  $\mathbb{C}^2$ .*
- (2) *All orbits in  $\Omega$  converge to a properly embedded submanifold  $\Sigma \subset \Omega$ , and  $\Sigma$  is biholomorphically equivalent to either the unit disk or an annulus. The manifold  $\Sigma$  is invariant under  $f$  and  $f$  acts on  $\Sigma$  as an irrational rotation.*
- (3) *All orbits in  $\Omega$  converge to a fixed point  $p \in \partial\Omega$ . The eigenvalues  $\lambda_1$  and  $\lambda_2$  of  $Df(p)$  satisfy  $|\lambda_1| < 1$  and  $\lambda_2 = 1$ , and  $\Omega$  is biholomorphically equivalent to  $\mathbb{C}^2$ .*

The only uncertainty in this classification (that belongs to the recurrent case) is whether the submanifold in Case (2) can actually be biholomorphically equivalent to an annulus.

Our argument comprises several steps that we will now outline. We let  $\Gamma_f$  be the set of all limit functions  $h : \Omega \rightarrow \bar{\Omega}$  for the family of iterates  $f^n : \Omega \rightarrow \Omega$ . In the non-recurrent case the image of  $h$  is either a single point (rank 0 case) or a holomorphic curve (rank 1 case). Our main concern is to analyze the latter case. In this case, the image  $h(\Omega)$  is an analytic curve residing in the boundary of  $\Omega$ .

*Step 1.* First, we show that the curve  $h(\Omega)$  is non-singular (see Section 5 and Appendix).

*Step 2.* We then analyze the natural action of  $f$  on  $\Gamma_f$  and prove that  $\Gamma_f$  always contains a rank 0 map (Section 6). The image of such a map must be a fixed point  $p$ .

*Step 3.* We then prove that if there exists a rank one limit map, then there also exists one whose image lies in the strong stable manifold of some fixed point  $p$  constructed on the previous step (Section 8).

*Step 4.* We then apply classical Denjoy–Carleman–Ahlfors and Wiman Theorems to rule out the last situation (Section 8). This is the only time when the stronger assumption on the Jacobian of  $f$  is needed. At this stage we can conclude that all orbits in  $\Omega$  converge to a unique fixed point  $p$ .

*Step 5.* Finally, we prove, adapting the classical Fatou’s argument, the *Snail Lemma* asserting that the fixed point  $p$  is a semi-parabolic point with one multiplier equal to 1 (Section 7).

**REMARK 3.** An important supplement to Theorem 2 is that in cases (1) and (3) the domain  $\Omega$  contains a “critical point” of  $f$  (as long as the multipliers at  $p$  are different), i.e., a point of tangency between the strong stable foliation in  $\Omega$  and the unstable manifold of any saddle. This result had been obtained in [DL12].

Many of our intermediate results are valid in greater generality (for endomorphisms of projective spaces, for local dynamics, and in higher dimensions). To set the stage, we make some of these statements explicit.

## 2 Background

In the 1960's the astronomer Michel Hénon suggested that complicated behavior observed in the Poincaré section of the Lorenz model would already occur for the much simpler maps given by

$$(x, y) \rightarrow (x^2 + c - \delta y, x), \quad (2)$$

at least for specific parameters  $c, \delta$ . These maps are now called Hénon maps and have since become one of the most extensively studied dynamical systems, both in the real and in the complex setting.

We will work with a more general definition of Hénon maps. It was proved by Friedland and Milnor [FM89] that every polynomial automorphisms of  $\mathbb{C}^2$  is affinely conjugate to either an affine map, an elementary map, or a finite compositions of *generalized* Hénon maps. Here  $f$  is a generalized Hénon map if  $f$  is of the form

$$(z, w) \mapsto (p(z) - \delta w, z), \quad (3)$$

where  $p$  is a polynomial and  $\delta \in \mathbb{C} \setminus \{0\}$ . The dynamical behavior of affine and elementary maps is easy to describe. Therefore we will only look at finite compositions of Hénon maps, and for simplicity we will refer to these maps just as Hénon maps.

Complex Hénon maps have been studied extensively by Hubbard and Oberste-Vorth [Hub86, HO94, HO95], Bedford-Smithie [BS91a, BS91b], Fornæss-Sibony [FS92] and many other authors. A basic property of Hénon maps that will be useful to us is the existence of the following *filtration*. For  $R > 0$  sufficiently large we define

$$W = \{(z, w) \mid \max(|z|, |w|) \leq R\}, \quad (4)$$

$$V^+ = \{(z, w) \mid |z| \geq \max(|w|, R)\}, \quad \text{and}, \quad (5)$$

$$V^- = \{(z, w) \mid |w| \geq \max(|z|, R)\}. \quad (6)$$

One easily checks that for  $R > 0$  large enough one has  $f(V^+) \subset V^+$  and  $f^{-1}(V^-) \subset V^-$ . Moreover, the orbit of any point in  $V^+$  will converge to the attracting fixed point  $[1 : 0 : 0]$  on the line at infinity.

It follows that the escaping set

$$I_\infty = \bigcup_{n \in \mathbb{N}} f^{-n}(V^+) = \{z \in \mathbb{C}^2 \mid \|f^n(z)\| \rightarrow \infty\} \quad (7)$$

is one Fatou component, and that for any other Fatou component the forward orbits are bounded. In this article we will, in the Hénon setting, only consider Fatou components with bounded forward orbits.

Let us introduce Fatou components in a more general setting. Let  $X$  be a complex manifold, and let  $f : X \rightarrow X$  be a holomorphic map. We say that  $z \in X$  lies in the Fatou set  $\mathcal{F}$  of  $f$  if the family of iterates  $\{f^n\}$  is normal in a neighborhood of  $z$ . A connected component of the Fatou set is called a *Fatou component*.

When  $X$  is the Riemann sphere and  $f$  is a rational function the possible Fatou components have been precisely described. Sullivan [Sul85] proved in 1982 that every Fatou component is (pre-)periodic, and periodic Fatou components had already been classified in the works of Fatou and Julia Siegel and Arnol'd and Herman: an invariant Fatou component is either the basin of an attracting fixed point, a parabolic basin, a Siegel disk or a Herman ring (see Milnor [Mil99]).

Fatou components in two complex variables have been studied by a number of authors. In general there is no reason to believe that all Fatou components are (pre-)periodic, but there has been some progress in describing *periodic* Fatou components. Bedford-Smillie [BS91b] have introduced the notion of a recurrent Fatou component, which we will adopt here.

**DEFINITION 4.** An invariant Fatou component  $\Omega$  is called *recurrent* if there exists a point  $z \in \Omega$  whose orbit accumulates at a point in  $\Omega$ .

Normality implies that a Fatou component is recurrent precisely when it contains a recurrent orbit. If  $\Omega$  is not recurrent then all orbits in  $\Omega$  converge to  $\partial\Omega$ . For rational self-maps of the Riemann sphere the recurrent Fatou components are basins of attracting fixed points, Siegel disks and Herman rings, while the only non-recurrent components are basins of parabolic fixed points. Recurrent Fatou components in two complex dimensions have been studied by Bedford-Smillie [BS91b, BS99], Fornæss-Sibony [FS95] and Ueda [Ued08]. The following result was proved by Bedford and Smillie.

**Theorem 5.** *Let  $f$  be a Hénon map and suppose that  $\Omega$  is a recurrent invariant Fatou component. Then either:*

- (1)  $\Omega$  is an attracting basin of some fixed point in  $\Omega$ , and  $\Omega$  is biholomorphic to  $\mathbb{C}^2$ .
- (2) there exists a one-dimensional closed complex submanifold  $\Sigma$  of  $\Omega$  and  $f^n(K) \mapsto \Sigma$  for any compact set  $K$  in  $\Omega$ . The Riemann surface  $\Sigma$  is biholomorphic to a disk or an annulus and  $f|_{\Sigma}$  is conjugate to an irrational rotation, or
- (3) the domain  $\Omega$  is a Siegel domain.

Recall that a Fatou component  $\Omega$  is called a Siegel domain if there exists a sequence of iterates  $f^{n_j}$  that converges on  $\Omega$  to the identity map.

While there are still a number of open questions regarding Theorem 5, for example whether the Riemann surface  $\Sigma$  in Case (2) can really be biholomorphic to an annulus, the recurrent Fatou components are relatively well understood.

The situation was quite different for non-recurrent Fatou components. These components have been studied by Weickert [WeiJ03] and Jupiter-Lilov [JL03], but were far less well understood.

### 3 Outline of the Proof

Let us first describe the main difficulty for dealing with non-recurrent Fatou components. If  $\Omega$  is a non-recurrent Fatou component then all orbits converge to the boundary  $\partial\Omega$ . By normality there exists a sequence  $\{f^{n_j}\}$  that converges, uniformly on compact subsets of  $\Omega$ , to a limit map  $h : \Omega \rightarrow \partial\Omega$ . In general the map  $h$  is not unique and depends on the sequence  $(n_j)$ , we will see some examples of this in Section 4. The main difficulty lies in the fact that a priori it is not even clear whether the *limit set*  $h(\Omega)$  is always unique.

Theorem 2 follows from several intermediate results, most of which hold in a more general setting. If we do not assume that the Hénon map is moderately dissipative then we still have the following.

**Theorem 6.** *Let  $f$  be a Hénon map and suppose that  $\Omega$  is a non-recurrent invariant Fatou component. Then there exists a sequence  $\{f^{n_j}\}$  that converges uniformly on compact subsets of  $\Omega$  to a fixed point  $p \in \partial\Omega$ . If the entire sequence  $\{f^n\}$  converges to  $p$  then the eigenvalues  $\lambda_1$  and  $\lambda_2$  of  $Df(p)$  satisfy  $|\lambda_1| < 1$  and  $\lambda_2 = 1$ , and  $\Omega$  is biholomorphically equivalent to  $\mathbb{C}^2$ .*

We also obtain results that lie outside the class of Hénon maps, for example the following.

**Theorem 7.** *Let  $f$  be a holomorphic endomorphism of  $\mathbb{P}^2$  and let  $\Omega$  be a non-recurrent, invariant Fatou component. Suppose that the limit set  $h(\Omega)$  is unique. Then  $h(\Omega)$  either consists of one point, or  $h(\Omega)$  is an injectively immersed Riemann surface, conformally equivalent to either the unit disk, the punctured unit disk or an annulus, and  $f$  acts on  $h(\Omega)$  as an irrational rotation.*

Notice that both Theorem 6 and Theorem 7 give a precise classification of non-recurrent Fatou components under the assumption that  $h(\Omega)$  is unique. It is exactly the uniqueness of  $h(\Omega)$  that condition (1) is used for in Theorem 2

We note that in Theorem 7 all Fatou components are known to occur, except for the punctured unit disk, whose possible existence is still an open question. We will see that when the limit map is a single point, the condition on the eigenvalues that holds for Hénon maps does not hold for holomorphic endomorphisms of  $\mathbb{P}^2$ . Fatou components for holomorphic self-maps of projective space will be discussed in Section 9

Let us now discuss the intermediate results from which Theorems 2, 6 and 7 will follow. The first lemma that we will prove is the following:

**LEMMA 13.** *Let  $f$  be a holomorphic endomorphism of a complex manifold  $X$  of dimension 2, and let  $\Omega$  be a non-recurrent Fatou component. Let  $h = \lim f^{n_j}$  be a limit map on  $\Omega$  of generic rank 1. Then  $h(\Omega) \subset \partial\Omega$  is an injectively immersed Riemann surface.*

Next we consider the set of all limit maps  $h : \Omega \rightarrow \partial\Omega$ , which we will denote by  $\Gamma_f$ . We introduce in Definition 22 a (not necessarily anti-symmetric) partial ordering on  $\Gamma_f$  (roughly, by divisibility), and prove the following.

**LEMMA 25.** *The invariant minimal subsets of  $\Gamma_f$  are exactly equal to the maximal equivalence classes. It follows that maximal elements exist: there exists an  $h \in \Gamma_f$  so that if  $k \geq h$  then  $h \geq k$ .*

The next result is the first that uses properties of Hénon maps.

**LEMMA 27.** *Let  $f$  be a Hénon map, let  $\Omega \subset \mathbb{C}^2$  be a non-recurrent Fatou component and let  $h \in \Gamma_f$  be maximal. Then  $h(\Omega)$  is a fixed point.*

Lemmas 25 and 27 together give that there must always be a limit map of rank 0, and if the limit set is unique then all orbits converge to this fixed point  $p$ . The fact that the eigenvalues  $\lambda_1$  and  $\lambda_2$  of  $Df(p)$  must satisfy  $|\lambda_1| < 1$  and  $\lambda_2 = 1$  is a local property that will be proved in this generalization of Fatou's Snail Lemma:

**Theorem 28.** *Let  $f : (\mathbb{C}^k, 0) \rightarrow (\mathbb{C}^k, 0)$  be a germ of a holomorphic map that fixes the origin. Suppose that there exists an open set  $W$  with  $f(W) \cap W \neq \emptyset$  and such that on  $W$  the iterates  $(f^n)$  converge uniformly to the origin. Further suppose that  $Df(0)$  has eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$  with  $|\lambda_1| = 1$  and  $|\lambda_i| < 1$  for  $i \geq 2$ . Then  $\lambda_1 = 1$ .*

In the proof of Lemma 27 we will work in a normal hyperbolic setting, comparable to Theorem (4.1) in Hirsch–Pugh–Shub [HPS70]. In this setting the normal submanifold is not necessarily compact, but we will prove uniform estimates on the geometry of the normal submanifold, which will give us a thickening of the strong stable manifolds to obtain a stable lamination that fills up a neighborhood of the normal submanifold. The existence of strong stable manifolds of definite size near points whose orbits remain in a small neighborhood of a weakly hyperbolic fixed point will also be used in the proofs of Theorem 28 and Lemma 32.

The final step in the proof of Theorem 6 follows from the following result.

**Theorem 8.** *Let  $F$  be a holomorphic automorphism of  $\mathbb{C}^2$  with a fixed point  $p$ , such that  $DF(p)$  has eigenvalues  $\{1, \lambda\}$ , with  $|\lambda| < 1$ , and that  $F - \text{Id}$  has multiplicity  $k + 1$  in  $p$ . The attracting basin of  $p$  has then  $k$  components and each component is biholomorphic to  $\mathbb{C}^2$ .*

This result was first proved by Ueda in the case where  $k = 1$  in [Ued86], and generalized to higher multiplicities by Hakim [Hak94]. Note that if the multiplicity  $k + 1$  is not finite then there must be a curve of fixed points, which cannot occur for Hénon maps, so Theorem 8 does imply the last statement of Theorem 6.

Once Theorem 6 is proved, the proof of Theorem 2 is completed by using the bound on the Jacobian derivative to obtain a growth estimate for the Green's function on stable manifolds. A classical result of Wiman [Wim05] is then used to prove that the limit set must be unique. We note that this is the only point in the proof of Theorem 2 where we use that  $f$  is an at least moderately dissipative polynomial automorphism. Theorem 6 holds under the following weaker conditions.

**Theorem 9.** *Let  $f$  be a holomorphic automorphism of a 2-dimensional complex manifold  $X$ . Let  $\Omega \subset X$  be a non-recurrent Fatou component. If there exists a compact  $K \subset X$  so that all orbits in  $\Omega$  converge to  $K$ , and so that  $f$  is volume contracting at all points in  $K$ , then the conclusions in Theorem 6 hold.*

We also note that here the invertibility of  $f$  is only necessary in order to conclude that  $\Omega$  is biholomorphic to  $\mathbb{C}^2$ .

The organization of the rest of this paper is the following. In the next section we will describe all known examples of non-recurrent Fatou components in two complex dimensions. In Section 5 we prove that the image of rank 1 limit maps is smooth. In Section 6 we introduce the ordering on the set of limit maps and prove that the maximal limit maps for Hénon maps are fixed points. In Section 7 we prove a higher dimensional version of Fatou's Snail Lemma, which completes the proof of Theorem 6. In Section 8 we discuss the possibility of non-unique limit sets, and prove that for moderately dissipative Hénon maps the limit set is unique, completing the proof of Theorem 2. Finally in Section 9 we consider non-recurrent Fatou components in projective space and prove Theorem 7.

#### 4 Examples of Non-recurrent Fatou Components

While the only known examples of non-recurrent Fatou components of Hénon maps are parabolic basins, for holomorphic endomorphisms several other examples are known to occur.

For a description of the dynamics of holomorphic endomorphisms of  $\mathbb{P}^2$ , see for example the work of Hubbard–Papadopol [HP94] and Fornæss–Sibony [FS94, FS95b]. Fatou components for holomorphic endomorphisms of  $\mathbb{P}^2$  are known to be pseudoconvex and Kobayashi hyperbolic [Ued94, FS95].

Recurrent Fatou components have been classified for holomorphic endomorphisms and the classification is almost identical to the classification for Hénon maps.

**Theorem 10** (Fornaess-Sibony, Ueda). *Suppose that  $f$  is a holomorphic self-map of  $\mathbb{P}^2$  of degree  $d \geq 2$ . Suppose that  $\Omega$  is an invariant recurrent Fatou component. Then either:*

- (1)  $\Omega$  is an attracting basin of some fixed point in  $\Omega$ ,
- (2) there exists a one-dimensional closed complex submanifold  $\Sigma$  of  $\Omega$  and  $f^n(K) \mapsto \Sigma$  for any compact set  $K$  in  $\Omega$ . The Riemann surface  $\Sigma$  is biholomorphic to a disk or an annulus and  $f|_{\Sigma}$  is conjugate to an irrational rotation, or
- (3) the domain  $\Omega$  is a Siegel domain.

Examples of non-recurrent Fatou components in  $\mathbb{P}^2$  can easily be constructed by taking a cross product of two polynomials. For example, if  $p$  is a polynomial with a parabolic petal, and  $q$  is a polynomial of the same degree with a Siegel disk, then the map  $p \times q : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  extends to a holomorphic endomorphism  $f$  of  $\mathbb{P}^2$ . The



map  $f$  has a Fatou component  $\Omega$  where all orbits converge to a holomorphic disk, properly embedded in  $\partial\Omega$ , on which  $f$  acts as a rotation.

This idea cannot be applied to products of rational functions, so the product of a parabolic petal and a Herman ring cannot be obtained as easily. This can however be done in the following way.

**EXAMPLE 11.** Let  $g$  be a rational function that has two distinct Fatou components  $\Omega_1$  and  $\Omega_2$ , where the former is a Herman ring and the latter a parabolic petal. Then  $g \times g$  is a holomorphic endomorphism of  $\mathbb{P}^1 \times \mathbb{P}^1$  with a Fatou component  $\Omega_1 \times \Omega_2$ . Now we use a construction due to Ueda which was also used by Fornæss and Sibony to construct examples of recurrent Fatou components in [FS95].

Let  $\rho : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2$  be defined by  $\rho([z : t], [w : s]) = [zw : ts : zs + wt]$ . Then  $\rho$  exactly identifies pairs  $(a, b)$  and  $(b, a)$ , and we see that  $\rho$  pushes the action of  $g \times g$  down to  $\mathbb{P}^2$ . The new map  $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  has a Fatou component  $\Omega$  biholomorphic to  $\Omega_1 \times \Omega_2$ . We see that all orbits converge to an annulus  $A \subset \partial\Omega$ , and  $f$  acts on  $A$  as an irrational rotation.

Suppose that the sequence  $f^{n_j}$  converges on  $\Omega$  to a rank 1 limit map  $h : \Omega \rightarrow \partial\Omega$  as before, and assume that  $h(\Omega)$  does not depend on the sequence  $(n_j)$ . In [WeiJ03] Weickert showed that the action of  $f$  on the limit set  $h(\Omega)$  can be lifted to a holomorphic self-map  $F$  of the unit disk  $\Delta$ , and that  $F$  is either conjugate to an irrational rotation or  $F^n \rightarrow \partial\Delta$ , locally uniformly on  $\Delta$ . Weickert gave examples of the former, but stated that he has no examples of the latter nor a proof that it cannot occur. Our Example 11 shows that it can indeed occur. We can think of the universal cover of the annulus  $A$  to be an infinite horizontal strip, and it is clear that the lift of a rotation to this strip can be given as a horizontal translation. But that means that for the lift  $F : \Delta \rightarrow \Delta$  all orbits converge to a point on the boundary  $\partial\Delta$ . However, note from Theorem 7 that if  $h(\Omega)$  is biholomorphic to a disk itself then the action of  $f$  is indeed conjugate to a rotation.

To summarize we now have seen three distinct examples of non-recurrent Fatou components:

- (1) All orbits converge to a single fixed point,
- (2) All orbits converge to a holomorphic disk, on which  $f$  acts as an irrational rotation, and
- (3) All orbits converge to an annulus, on which  $f$  acts as an irrational rotation.

We note that these are the three components that arise immediately by combining 1-dimensional Fatou components. Is this everything, or can there be 2-dimensional Fatou components that are fundamentally different from combinations of 1-dimensional Fatou components?

### 5 Smoothness of Rank 1 Limit Sets

The main step towards the proof of Lemma 13 is made in Proposition 12 below. We give an elementary analytic proof here, and we will give two more sophisticated but shorter proofs in the appendix.

PROPOSITION 12. *Let  $f : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$  be the germ of a holomorphic map whose image is singular at the origin in  $\mathbb{C}^2$ . Let  $U$  be a neighborhood of 0 where  $f$  is defined. Then there exists an  $\epsilon > 0$  so that for every  $g : U \rightarrow \mathbb{C}^2$  with  $\|f - g\|_U \leq \epsilon$  we have that  $g(U) \cap f(U) \neq \emptyset$ .*

*Proof.* Without loss of generalization we may assume that  $U$  contains a neighborhood of the unit disk  $\Delta$ . After changing coordinates, both on the domain and the target, we can further assume that  $f$  is of the form

$$f(x) = (f_1(x), f_2(x)) = (x^p, x^q + h.o.t.), \tag{8}$$

where  $q > p \geq 2$  (the Puiseux expansion). After a further change of coordinates on the target of the form  $(z, w) \mapsto (z, w - \alpha(z))$  we may assume that none of the exponents occuring in  $f_2(x)$  are divisible by  $p$ .

Let  $k$  be the greatest common divisor of all the exponents occuring in  $f$ , and write

$$f(x) = \tilde{f}(y) = (y^m, y^n + h.o.t.), \tag{9}$$

where  $y = x^k$ ,  $m = p/k$  and  $n = q/k$ . Then near  $(0, 0) \in \mathbb{C}^2$  the image of  $f$  is given by the equation

$$\prod_{j=1}^m (w - \tilde{f}_2((z^{\frac{1}{m}})_j)) = w^m + \alpha_{m-2}(z)w^{m-2} + \dots + \alpha_0(z) = 0, \tag{10}$$

as in the Weierstrass Preperation Theorem. The fact that

$$\alpha_{m-1}(z) = \sum \tilde{f}_2((z^{\frac{1}{m}})_j) \tag{11}$$

vanishes follows from our earlier assumption that none of the exponents in  $f_2$  were divisible by  $p$ . Note that for every  $j = 0 \dots m - 2$  the coefficient  $\alpha_j(z)$  is divisible by  $z^l$ , where  $l \geq \frac{2q}{p}$ . Hence we can rewrite Equation (10) to obtain that a point  $(z, w)$  near  $(0, 0)$  lies in the image of  $f$  if and only if

$$\frac{w^m}{z^l r(z, w)} = 1, \tag{12}$$

where we write  $r(z, w)$  for the remainder. Now let  $g$  is a holomorphic function defined on  $U$ . In order to show that  $g(\Delta) \cap f(\Delta) \neq \emptyset$  it is sufficient to show that there exists an  $x \in \Delta$  such that

$$\phi(x) = \frac{g_2(x)^m}{g_1(x)^l \tilde{r}(x)} = 1. \tag{13}$$

We suppose that  $\|f - g\|_\Delta$  is sufficiently small so that by Rouché's Theorem both the numerator and denominator of  $\phi$  have exactly  $\frac{pq}{k}$  zeroes, counting multiplicity. Moreover, by decreasing  $\|f - g\|_\Delta$  if necessary, the value of  $\phi(x)$  will be close to 1 on  $\partial\Delta$ . Let us for the purpose of contradiction assume that  $f(U)$  and  $g(U)$  do not intersect. Then it follows that  $f(\Delta)$  and  $g(\Delta)$  do not intersect for sufficiently small perturbations of  $g$ . Hence we may assume that the meromorphic function  $\phi$  has zeroes and poles distinct from each other, exactly  $q$  zeroes, each of multiplicity  $m$ , and at least  $p$  poles of multiplicity at least  $l$ .

Let us change coordinates on the target by post-composing  $\phi$  with the map

$$\theta = x \rightarrow \frac{x}{x-1}, \quad (14)$$

effectively switching the roles of 1 and  $\infty$  while keeping 0 fixed. Then by our assumption  $\tilde{\phi} = \theta \circ \phi$  is bounded and close to infinity on the boundary  $\Delta$ , still has  $q$  zeroes, each of multiplicity  $m$ , and takes on the value 1 in at least  $p$  points of multiplicity at least  $l$ . Let

$$R = \min\{|\tilde{\phi}(x)| \mid x \in \partial\Delta\}. \quad (15)$$

Then by the maximum principle  $V = \tilde{\phi}^{-1}(\Delta_R)$  is a disjoint union of finitely many simply connected domains. Since  $\tilde{\phi}$  is a branched covering of degree  $\frac{pq}{k}$  from  $V$  to  $\Delta_R$ , the Riemann Hurwitz Theorem gives that the Euler characteristic of  $V$  satisfies

$$\chi(V) \leq \frac{pq}{k}\chi(\Delta_R) - q(m-1) - p\left(\frac{2q}{p} - 1\right) = \frac{pq}{k} - \frac{pq}{k} + q - 2q + p = p - q < 0. \quad (16)$$

But this is a contradiction since the Euler Characteristic of a finite union of disks is positive.  $\square$

We stress that in Proposition 12 we consider perturbations of the map  $f$ , not of the defining equation for  $f(\Delta)$ . For example, if  $f = (x^2, x^3)$  then the image is given by  $\{w^2 - z^3 = 0\}$ , which can easily be perturbed to  $\{w^2 - z^3 = \epsilon\}$ , which does not intersect the original cusp. Note that in this example the topology has changed: the intersection of  $\{w^2 - z^3 = \epsilon\}$  with a ball centered at the origin is no longer simply connected.

The following is a direct consequence of Proposition 12.

**LEMMA 13.** *Let  $f$  be a holomorphic endomorphism of a complex manifold  $X$  of dimension 2, and let  $\Omega$  be a non-recurrent Fatou component. Let  $h = \lim f^{n_j}$  be a limit map on  $\Omega$  of generic rank 1. Then  $h(\Omega) \subset \partial\Omega$  is an injectively immersed Riemann surface.*

*Proof.* Let  $z \in \Sigma = h(\Omega)$  and let  $x \in \Omega$  be such that  $h(x) = z$ . Since  $h$  is rank 1 there exist a small disk  $D$  through  $x$  such that  $h$  is non-constant on  $D$  and such that for some small neighborhood  $U$  of  $x$ , the image  $h(U)$  equals  $h(D)$ . We first show that  $h(D)$  is smooth for  $D$  small enough.

By reparametrizing we may assume that  $x = 0$ ,  $D$  is the unit disk, and  $h$  maps  $0 \in D$  to  $(0, 0) \in \mathbb{C}^2$ . Suppose for the purpose of a contradiction that  $h(D)$  is singular at  $(0, 0)$ . After a change of coordinates we can write  $h(\zeta) = (\zeta^p, \zeta^q + h.o.t.)$  with  $q > p \geq 2$  and  $q$  not divisible by  $p$ , as in the proof of Proposition 12. But then it follows from Proposition 12 that for  $j$  large enough, the set  $f^{n_j}(D)$  intersects  $h(D)$ . This is a contradiction since  $f^{n_j}(D) \subset \Omega$  and  $h(D) \subset \partial\Omega$ . Therefore the image  $h(D)$  is a smooth Riemann surface.

Now let  $y \in \Omega$  be such that  $h(y) = h(x)$ . Let  $V$  be a small neighborhood of  $y$ . Suppose for the purpose of contradiction that the images  $h(V)$  and  $h(U)$  do not agree as germs. Then we have that  $h(U)$  and  $h(V)$  are both holomorphic graphs over a straight disk through  $z$ , and intersect only in  $z$ . It follows that for  $j$  large enough  $f^{n_j}(U)$  must intersect  $h(V)$ , and again we have a contradiction. It follows that  $h(U)$  and  $h(V)$  agree as germs, and  $h(\Omega)$  is therefore smooth. □

### 6 Ordering of Limit Maps

Let us begin with a useful observation due to Weickert:

LEMMA 14. [WeiJ03] *If  $\Omega$  is an invariant Fatou component with a limit map  $h = \lim f^{n_j}$ , then  $h(\Omega)$  is invariant under  $f$ . In particular, if  $h(\Omega)$  is a point then it is fixed under  $f$ .*

*Proof.* Let  $z \in h(\Omega)$  and write  $z = h(x)$  for some  $x \in \Omega$ . Then  $f(x) \in \Omega$ , and since  $f$  and  $h$  commute we have  $f(z) = f(h(x)) = h(f(x)) \in h(\Omega)$ , and the conclusion follows. □

REMARK 15. Since we are dealing with invertible maps, the same argument shows that the limit set  $h(\Omega)$  is completely invariant under  $f$ .

Let us now recall the definition of a Fatou map.

DEFINITION 16. Let  $f$  be a holomorphic endomorphism of a complex manifold  $X$ . A holomorphic map  $\phi$  from a complex analytic space  $R$  into  $X$  is called a *Fatou map* for  $f$  if  $\{f^j \circ \phi\}$  is a normal family.

LEMMA 17. *Let  $X$  be a 2-dimensional complex manifold and  $f$  be a holomorphic endomorphism of  $X$ . Let  $\Omega$  be a non-recurrent Fatou component, and suppose that  $\{f^{n_j}\}$  converges uniformly on compact subsets of  $\Omega$  to a rank 1 limit map  $h : \Omega \rightarrow \partial\Omega$ . Then the inclusion map from  $\Sigma_h = h(\Omega)$  into  $X$  is a Fatou map.*

*Proof.* Let us denote the inclusion map by  $\phi$ . Let  $m_1, m_2 \dots$  be an increasing sequence of integers and let  $K_1 \subset K_2 \subset \dots$  be a compact exhaustion of  $\Omega$ . For each  $l$  we can choose  $j(l)$  large enough so that

$$\|f^{n_{j(l)}+m_l}(z) - f^{m_l} \circ h(z)\| < \frac{1}{2^l}, \tag{17}$$

for  $z \in K_l$ . By the definition of the Fatou component  $\Omega$  we obtain that a subsequence of  $f^{n_{j(l)}+m_l}$  converges uniformly on compact subsets of  $\Omega$ . But then it follows that there is a subsequence of  $f^{m_l} \circ h(z)$  that converges uniformly on compact subsets of  $\Omega$ , which implies that the corresponding subsequence of  $f^{m_l} \circ \phi$  converges uniformly on compact subsets of  $\Sigma$ .  $\square$

The normality of the family  $(f^n)$  restricted to  $\Sigma_h$  allows us to introduce a natural ordering on the family of limit maps.

DEFINITION 18. Given a non-recurrent Fatou component  $\Omega$  for  $f$ , we define

$$\Gamma_f := \{h : \Omega \rightarrow \partial\Omega \mid \exists n_1, n_2, \dots : f^{n_j}|_\Omega \rightarrow h\}. \quad (18)$$

REMARK 19. The set  $\Gamma_f$ , equipped with the topology of uniform convergence on compact subsets of  $\Omega$ , is compact. The map  $f$  acts on  $\Gamma_f$  both by pre- and post-composition, which by commutativity of the iterates of  $f$  induce the same action on  $\Gamma_f$ . It is well known that for any continuous group action on a compact set there exists a minimal invariant subset. These subsets are exactly the maximal equivalence classes with respect to the ordering that we introduce in this section.

REMARK 20. The set  $\Gamma_f$  is reminiscent of the *Sushkevich kernel* for almost periodic semigroup actions, compare [LL85].

Now let  $h \in \Gamma_f$  be a rank 1 limit map with image  $\Sigma_h \subset \partial\Omega$ . We have seen in Lemma 17 that the inclusion map  $\Sigma \rightarrow X$  is a Fatou map. Let  $f^{m_l}$  be a convergent subsequence on  $\Sigma$ , converging to a map  $\phi : \Sigma \rightarrow \phi(\Sigma)$ .

LEMMA 21. *The map  $\phi \circ h$  is an element of  $\Gamma_f$ .*

*Proof.* Write  $h = \lim f^{n_j}$ . Then there exist  $j(l)$  large enough such that the iterates  $f^{m_l+n_{j(l)}}$  converge to  $\phi \circ h$ , uniformly on compact subsets of  $\Omega$ .  $\square$

We would like to say that in the setting of Lemma 21 that  $k = \phi \circ h$  is larger than  $h$ . Instead we will use the following equivalent definition.

DEFINITION 22. Let  $h = \lim f^{n_j} \in \Gamma_f$  and let  $k \in \Gamma_f$ . If there exists a sequence  $(m_l)$  in  $\mathbb{Z}^+$  such that for any sufficiently large  $\{j(l)\}$  we have that

$$k = \lim f^{m_l+n_{j(l)}}, \quad (19)$$

uniformly on compact subsets of  $\Omega$ , then we say that  $k \geq h$ . This ordering is reflexive and transitive, but not necessarily anti-symmetric. For example, if  $f$  acts as a rotation on  $h(\Omega)$ , then  $\Gamma_f$  is equivalent to  $S^1$ , and  $k \leq h$  for any  $h, k \in \Gamma_f$ . By considering  $h$  and  $k$  equivalent if  $h \geq k$  and  $k \geq h$  we do obtain a partial ordering on the equivalence classes.

LEMMA 23. *Let  $h, k \in \Gamma_f$ , and write  $\Sigma_h = h(\Omega)$ . Then  $k \geq h$  if and only if there exists an sequence  $\{m_l\}$  in  $\mathbb{Z}^+$  such that  $f^{m_l}$  converges on  $\Sigma_h$  to  $\phi : \Sigma_h \rightarrow \partial\Omega$  and  $k = \phi \circ h$ .*

*Proof.* Let us write  $h = \lim f^{n_j}$ . If  $k = \phi \circ h$ , with  $\phi = \lim f^{m_l}$ , then it follows immediately that for  $j(l)$  large enough we have that  $k = \lim f^{m_l+n_j(l)}$ , so  $k \geq h$ .

To prove the other direction, suppose that  $k \geq h$ , which by our definition means that there exist a sequence  $\{m_l\}$  in  $\mathbb{Z}^+$  so that for  $j(l)$  large enough  $k = \lim f^{m_l+n_j(l)}$ . Since the inclusion map from  $\Sigma_h$  into  $X$  is a Fatou map, we can restrict to a subsequence of  $\{m_l\}$  if necessary so that  $f^{m_l}$  converges on  $\Sigma$  to a map  $\phi : \Sigma_h \rightarrow \partial\Omega$ . It follows that  $\phi \circ h = k$ . □

LEMMA 24. *If  $h, k \in \Gamma_f$  and  $k \geq h$ , then  $k(\Omega) \subset \overline{h(\Omega)}$ . Moreover, if  $x, y \in \Omega$  are such that  $h(x) = h(y)$  then  $k(x) = k(y)$ .*

We note that Lemma 24 follows immediately from Lemma 23.

LEMMA 25. *The invariant minimal subsets of  $\Gamma_f$  are exactly equal to the maximal equivalence classes. In particular, there exists an  $h \in \Gamma_f$  so that if  $k \geq h$  then  $h \geq k$ .*

*Proof.* Suppose that  $\omega \subset \Gamma_f$  is a maximal equivalence class. It follows immediately from the definition of the ordering that the closure of the orbit under  $f$  of any  $h \in \omega$  is equal to  $\omega$ , so  $\omega$  is a minimal invariant subset.

On the other hand, suppose that  $V \subset \Gamma_f$  is a minimal invariant subset, let  $h \in V$  and suppose that there exists an  $k \in \Gamma_f$  with  $k \geq h$ . By the minimality of  $V$  the map  $h$  must lie in the closure of the orbit of  $k$  under  $f$ , so by Lemma 23 we have that  $h \geq k$ . Hence  $h$  lies in a maximal equivalence class  $\omega$ , which as we have seen is also a minimal invariant subset. But then  $\omega$  must equal  $V$ , which completes the proof. □

We prove the following.

LEMMA 26. *Suppose that  $h_{max}$  lies in a maximal equivalence class in  $\Gamma_f$ . Suppose that  $h$  has rank 1 and write  $\Sigma_h = h_{max}(\Omega)$ . Then the tangential derivatives along  $\Sigma_h$  of the maps  $(f^n)$  are uniformly bounded away from 0 and  $\infty$  on compact subsets of  $\Sigma_h$ .*

*Proof.* By Lemmas 13 and 14,  $\Sigma_h$  is an invariant injectively immersed Riemann surface, so we can indeed talk about tangential derivatives. Let  $K$  be a compact subset of  $\Sigma_h$ . Since the inclusion map is a Fatou map, we immediately obtain that on  $K$  the modulus of the tangential derivatives is uniformly bounded from above. Suppose for the purpose of contradiction that moduli of the tangential derivatives is not bounded away from 0. Then we can find a sequence of iterates  $(f^{n_j})$  and a sequence of points  $a_j$  so that  $a_j \rightarrow a \in K$  and such that the tangential derivative of  $f^{n_j}$  at  $a_j$  goes to 0. By restricting to a convergent subsequence we obtain a limit map

$$\phi = \lim f^{n_j}|_{\Sigma_h}, \tag{20}$$

with  $\phi'(a) = 0$ . By Lemma 23 we have that  $\phi \circ h_{max} = k \in \Gamma_f$ , and  $k \geq h_{max}$ . Hence by Lemma 13  $\phi(\Sigma_h) = k(\Omega)$  is smooth, which implies that  $\phi : \Sigma_h \rightarrow \sigma_k$  is  $d : 1$  near the point  $a$ , for some  $d > 1$ . But then it follows from Lemma 24 that  $k$  is strictly larger than  $h$ , which contradicts the maximality of  $h$ . □

We will now restate and prove Lemma 27, formulated in Section 3, which will conclude the proof of the first part of Theorem 6.

LEMMA 27. *Let  $f$  be a Hénon map, let  $\Omega \subset \mathbb{C}^2$  be a non-recurrent Fatou component and let  $h \in \Gamma_f$  be a maximal limit map. Then  $h(\Omega)$  consists of a single fixed point.*

*Proof.* If  $h$  has rank 0 then the desired follows from Lemma 14. So assume for the purpose of a contradiction that  $h$  has rank 1. Recall from Lemma 13 that  $\Sigma_h = h(\Omega)$  is an injectively immersed Riemann surface, and from Lemma 26 that on any compact subset of  $\Sigma_h$  the tangential derivatives of the family  $\{f^n\}$  are uniformly bounded away from 0 and  $\infty$ .

Recall from [BS91a] that  $\Sigma_h$  is a bounded set, contained in the polydisk

$$W = \{(z, w) \in \mathbb{C}^2 \mid |z|, |w| \leq R\}, \quad (21)$$

for some sufficiently large  $R$  depending on  $f$ .

Since  $\Omega$  is non-recurrent, the Jacobian derivative  $\delta$  of  $f$  necessarily satisfies  $|\delta| < 1$ . Let  $z \in \Sigma_h$  and let  $D$  be a relatively compact holomorphic disk in  $\Sigma_h$ , centered at  $z$ . For  $x \in \Sigma_h$  and  $m > 0$  we define the tangent cone

$$C_x^m = \{v = v_1 + v_2 \in T_x(\Sigma_h) \oplus N_x(\Sigma_h) \mid |v_2| \leq m|v_1|\}. \quad (22)$$

Since  $\Omega$  lies in the compact set  $W$ , the map  $f$  is volume contracting and the tangential derivatives of the iterates  $f^j$  along  $\Sigma_h$  are uniformly bounded from above and away from 0, we can find  $N \in \mathbb{N}$  and  $0 < m' < m$  so that

$$df^N(C_x^m) \subset C_{f^N(x)}^{m'}, \quad (23)$$

for every integer  $n$  and every  $x \in f^n(D)$ .

Note that since the sequence of iterates  $(f^n)$  restricted to  $\Sigma_h$  is a normal family, the second derivatives of  $f^n$  along  $\Sigma_h$  are bounded from above on  $D$ , uniformly over  $n$ . Hence we can make  $D$  smaller if necessary so that for every  $n$  the holomorphic disk  $f^n(D)$  is a graph over the complex line through  $f^n(z)$  tangent to  $\Sigma_h$ . It follows that, by again decreasing  $D$  if necessary, we can extend the invariant cone fields  $C^m$  and  $C^{m'}$  on each  $f^n(D)$  to a bidisk centered at  $f^n(z)$  whose radii are independent of  $n$ . Here the axes of these bidisks are the tangent and normal directions of  $\Sigma_h$  at  $f^n(z)$ .

By standard construction (see for example [HPS70]) there exists through every  $x \in D$  a strong stable manifold  $W^s(x)$ . Moreover, the uniform size of the bidisks guarantees that the stable manifolds through  $x \in D$  extend almost vertically through the bidisk containing  $D$ . Hence we obtain a stable foliation, filling a neighborhood of  $D$ . But since the sequence of iterates is normal when restricted to  $D$ , the iterates also form a normal family in the union of these stable manifolds, which implies that  $\Sigma_h$  does not lie on the boundary of the Fatou set but in the interior. This contradicts our hypothesis and completes the proof.  $\square$

Of course Lemma 27 does not rule out the existence of rank 1 limit maps. It does follow that given any rank 1 limit map  $h \in \Gamma_f$ , there exists a sequence  $f^{n_j}$  that converges uniformly on compact subsets of  $\Sigma_h = h(\Omega)$  to a fixed point in  $\bar{\Sigma}_h$ .

## 7 Parabolic Basins

Let  $f$  be a Hénon map with a non-recurrent Fatou component  $\Omega$ , and suppose that  $\{f^n\}$  converges uniformly on compact subsets of  $\Omega$  to a point  $p \in \Gamma_f$ . Since the Jacobian derivative of  $f$  must be strictly  $<1$  in absolute value, at least one of the eigenvalues must have modulus strictly  $<1$ . As  $p$  lies in the Julia set, it cannot be an attractive fixed point, and since there is uniform convergence to  $p$  on an open subset,  $p$  cannot be a hyperbolic fixed point. It follows that the other eigenvalue must have modulus exactly equal to 1. In fact, we now prove that the other eigenvalue must be equal to 1.

**Theorem 28.** *Let  $f : (\mathbb{C}^k, 0) \rightarrow (\mathbb{C}^k, 0)$  be a germ of a holomorphic map that fixes the origin. Suppose that there exists an open set  $W$  with  $f(W) \cap W \neq \emptyset$  and such that on  $W$  the iterates  $(f^n)$  converges uniformly to the origin. Further suppose that  $Df(0)$  has eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$  with  $|\lambda_1| = 1$  and  $|\lambda_i| < 1$  for  $i \geq 2$ . Then  $\lambda_1 = 1$ .*

*Proof.* After changing coordinates we may assume that the eigenvector corresponding to  $\lambda_1$  is  $(1, 0, \dots, 0)$  and all other eigenvectors of  $Df(0)$  are orthogonal to  $(1, 0, \dots, 0)$ .

Recall that we have a strong stable manifold through 0 of complex dimension  $k - 1$ . Since this strong stable manifold is locally a graph over the hyperplane  $\{z_1 = 0\}$ , we can locally change coordinates such that the local stable manifold of 0 is equal to the hyperplane  $\{z_1 = 0\}$ . We denote by  $C$  a (not necessarily unique)  $\mathcal{C}^\infty$  center manifold through 0. Since there may not be a holomorphic center manifold we cannot assume that there is a holomorphic change of coordinates that maps the center manifold to the  $z_1$ -plane, but we do have that  $C$  is tangent to the  $z_1$ -plane at 0.

Let  $V$  be a small ball centered at 0 so that  $f|_V$  is invertible and acts weakly hyperbolically. That is, we choose  $V$  small enough such that  $\{(v, v') \in T(V) \mid |v| \geq \|v'\|\}$  is an invariant cone field. We denote by  $K$  the subset of points in  $V$  whose orbits stay in  $V$ , so in particular for every  $z \in W$  we have that  $f^n(z) \in K$  for large enough  $n \in \mathbb{N}$ . By choosing  $V$  small enough we have that for every point  $z \in K$  we locally have a strong stable manifold through  $z$  that extends to the boundary of  $V$ . It follows from the invariant cone field that the strong stable manifolds are *close to vertical*, that is, for every pair  $(x, x'), (y, y')$  in a local stable manifold we have that  $\|x' - y'\| < |x - y|$ .

Let  $w \in W$  be such that  $f(w)$  also lies in  $W$ . Then for some  $N \in \mathbb{N}$  we have that  $f^n(W) \subset V$  for all  $n \geq N$ . We write  $W' = f^N(W)$ .

Then through every point in  $W'$  we have a local stable manifold as noted before. Since the strong stable manifolds are close to vertical they must intersect the  $z_1$ -plane in a unique point. We denote by  $h : K \rightarrow \mathbb{C}_{z_1}$  the holonomy map induced by the strong stable manifolds:  $h$  maps a point  $z \in K$  to the intersection of the strong stable manifold through  $z$  with the  $z_1$ -plane. Then  $U = h(W')$  is a connected subset of  $W \cap \mathbb{C}_{z_1}$  that contains the points  $u = h(f^N(w))$  and  $v = h(f^{N+1}(w))$ . Note that  $v$  is not necessarily equal to  $f(u)$  since the  $z_1$ -plane is not invariant, but  $v$  lies in the strong



stable manifold through  $f(u)$ . Also note that  $U$  cannot contain 0: if  $U$  did contain 0 then  $W$  would contain a neighborhood of 0 which would contradict our assumptions.

Let  $\psi$  be a conformal map from the unit disk  $\Delta$  to  $U$  that maps 0 to  $u$ , and define

$$\phi_n(\zeta) = \frac{\pi_1 \circ f^n \circ \psi(\zeta)}{\pi_1 \circ f^n(u)}, \quad (24)$$

where  $\pi_1$  is the (straight) projection onto the  $z_1$ -plane.

Note that since  $f^n(U)$  avoids the strong stable manifold  $W^s(0)$ , the maps  $\phi_n$  are well-defined and  $\phi_n(\zeta) \neq 0$  for any  $\zeta \in \Delta$ . It is clear that  $\phi_n(0) = 1$  for every  $n$ , and it follows from the invariant cone field for  $f$  in  $V$  that  $f^n(W)$  is "almost horizontal" and the map  $\phi_n$  is univalent. It follows that the family  $\{\phi_n\}$  is normal. Moreover, we note that  $\phi_n(\psi^{-1}(v)) \rightarrow \lambda_1$ , and by Hurwitz Theorem it follows that any limit map of the family must be univalent. Then we must have a constant  $\epsilon > 0$  such that  $|\phi'_n(0)| > \epsilon$  for any  $n \in \mathbb{N}$ . Hence by the Koebe  $\frac{1}{4}$ -Theorem we have that

$$d(\phi_n(0), \partial(\phi_n(\Delta))) > \frac{1}{4}\epsilon, \quad (25)$$

and hence there is a constant  $\tilde{\epsilon} > 0$  so that

$$d(\pi_1(f^n(u)), \partial(\pi_1 f^n(W))) > \tilde{\epsilon} \|f^n(u)\|. \quad (26)$$

Assume for the purpose of a contradiction that  $\lambda_1 \neq 1$ , and let  $j \in \mathbb{N}$  be such that  $d(\lambda_1^j, 1) < \frac{\tilde{\epsilon}}{2}$ .

Let us write  $f^n(u) = u_n = (x_n, y_n)$ , with  $x_n \in \mathbb{C}_{z_1}$  and  $y_n \in \mathbb{C}^{k-1}$ . For large  $n$  we claim that  $\|y_n\| < |x_n|^2$ . Indeed,  $w$  lies in the stable manifold of a point in the invariant center manifold  $C$ , and the distance  $d(u_n, C)$  decreases exponentially. The claim follows from the fact that  $C$  is tangent to the  $z_1$ -plane at 0. Hence it follows from the fact that the local stable manifolds in  $V$  are vertical and the uniform convergence of  $f^n$  on  $U$  that for large  $n$  we have

$$d(\pi_1(x_n), h(x_n)) < \frac{1}{8}\tilde{\epsilon}\|u_n\|. \quad (27)$$

It follows from the fact that the local stable manifolds through  $u_n$  are almost vertical and  $u_n \rightarrow 0$  that for large  $n$  we have that

$$d(h(u_n), h(u_{n+j})) < \frac{2}{3}\tilde{\epsilon}\|u_n\|, \quad (28)$$

while

$$d(h(u_n), \partial(hf^n(U))) > \frac{3}{4}\tilde{\epsilon}\|u_n\|. \quad (29)$$

It follows from estimates (28) and (29) that  $h(u_{n+j})$  lies in  $h(f^n(U))$  and hence that  $V \cap \Omega \cap \mathbb{C}_{z_1}$  contains a Jordan curve that winds around the origin. The vertical foliation through this Jordan curve plus the maximum principle now give us that  $\Omega$  contains a neighborhood of 0, which is in contradiction with the assumption that  $0 \in \partial\Omega$ .  $\square$

Lemma 27 and Theorem 28 combined with Theorem 8 together imply Theorem 6. Let us note that the dynamical behavior of a germ  $f : (\mathbb{C}^k, 0) \rightarrow (\mathbb{C}^k, 0)$  with eigenvalues  $\lambda_1, \dots, \lambda_k$  satisfying  $\lambda_1^p = 1$  and  $|\lambda_i| < 1$  for  $i \geq 2$  is very well understood due to the following result of Di Giuseppe [Giu10].

**Theorem 29** (Di Giuseppe). *Let  $f = (f_1, \dots, f_k)$  be a semi-hyperbolic germ as above. Then up to holomorphic conjugacy one of the following is satisfied:*

- (i)  $f_1(z) = \lambda_1 z_1$ ;
- (ii)  $f_1(z) = \lambda_1 z_1 + a_k z_1^{kq+1} + o(\|z\|^{kq+1})$ , with  $a_k \neq 0$  and  $k \geq 1$ .

Moreover, in case (i),  $f$  is locally topologically conjugate at the origin with  $g := df_0$ ; and in case (ii),  $f$  is locally topologically conjugate at the origin with  $g(z) := (\lambda_1 + z_1^{kq+1}, \lambda_2 z_2, \dots, \lambda_k z_k)$ .

Di Giuseppe’s result actually applies in greater generality:  $|\lambda_i| > 1$  is allowed for  $i \geq 2$  as long as there is *quasi-absence of resonance*. In our setting quasi-absence of resonance is automatically satisfied.

### 8 Uniqueness of the Limit Set

Let  $f$  be a holomorphic endomorphism of a complex manifold  $X$ , and let  $\Omega$  be a non-recurrent Fatou component for  $f$ .

QUESTION 30. Given a holomorphic endomorphism of a complex manifold  $X$  with a non-recurrent Fatou component. Is it possible for the limit set  $h(\Omega)$  to depend on the map  $h \in \Gamma_f$ ?

Although non-uniqueness may be hard to imagine for Hénon maps or holomorphic endomorphisms of  $\mathbb{P}^2$ , the following construction gives an affirmative answer to the above question. The complex manifold will be a domain in  $\mathbb{C}^2$ .

**Theorem 31.** *There exists an open connected set  $D \subset \mathbb{C}^2$  and a map  $f : D \rightarrow D$  such that  $\{f^n\}$  is normal on  $D$ , all orbits  $f^n(z)$  converge to  $\partial D$  and  $h(D)$  depends on  $h \in \Gamma_f$ .*

*Proof.* We let

$$f(z, w) = (p(z), q_z(w)), \tag{30}$$

where

$$p(z) = z - z^3 + h.o.t. \tag{31}$$

is such that

$$p\left(\frac{1}{\sqrt{u}}\right) = \frac{1}{\sqrt{u+1}}, \tag{32}$$

for  $u$  in the strip  $S = \{0 < \text{Im}(u) < 2\pi\}$ . We define the map  $q_z(w)$  by

$$q_z(w) = \frac{\phi(p(z))}{\phi(z)} w, \tag{33}$$

where

$$\phi(z) = 1 - e^{i\frac{1}{z}}. \quad (34)$$

The domain  $D$  is given by  $\Omega \times \mathbb{C}$ , where  $\Omega$  contains the  $z$ -values that correspond to  $u$ -values in the half-strip  $S$ . Note that  $\phi(z) \neq 0$  for any  $z \in \Omega$ , so the map  $f$  is well defined.

Let  $K$  be a compact subset of  $\Omega$ . Then  $p^n(z)$  converges to the origin along the positive real axis, uniformly for  $z \in K$ . Moreover, we have that

$$\max_{z, z' \in K} \operatorname{dist} \left( \frac{1}{p^n(z)}, \frac{1}{p^n(z')} \right) \rightarrow 0. \quad (35)$$

Hence given any increasing sequence in  $\mathbb{N}$  we can find a subsequence  $\{n_j\}$  for which

$$e^{i\frac{1}{p^{n_j}(z)}} \rightarrow \theta, \quad (36)$$

with  $|\theta| = 1$ , uniformly for  $z \in K$ . Given that

$$w_n = \frac{\phi(f^{n+1}(z))}{\phi(f^n(z))} \cdots \frac{\phi(f(z))}{\phi(z)} w_0 = \frac{\phi(f^{n+1}(z))}{\phi(z)} w_0, \quad (37)$$

it follows that  $\{f^n\}$  is normal on  $\Omega \times \mathbb{C}$ . The first coordinate  $z_n$  will always converge to the origin, so all the orbits  $(z_n, w_n)$  converge to  $\partial D$ .

To see that  $h(D)$  depends on the sequence  $\{n_j\}$ , note that by choosing the  $n_j \in \mathbb{N}$  appropriately we can make sure that  $e^{i\frac{1}{z_{n_j}}}$  converges to 1 for some initial  $(z_0, w_0)$ . It follows that  $(z_{n_j}, w_{n_j})$  converges to  $(0, 0)$  for any starting point  $(z, w)$  in a compact subset of  $D$ .

On the other hand, we can make sure that  $e^{i\frac{1}{z_{n_j}}}$  converges to  $-1$  for some point  $z_0$ . In this case the map  $h$  will satisfy  $h(z, w) = (0, \frac{2}{z}w)$  and the image of  $h$  is the complex line  $\{0\} \times \mathbb{C}$ , showing that the image  $h(D)$  depends on the subsequence  $\{f^{n_j}\}$ .  $\square$

We note that the above construction is quite different from a Fatou component of a Hénon map or a holomorphic map of projective space. The map  $f$  is not defined on the boundary of  $D$ , and  $f$  cannot be extended holomorphically to any neighborhood of 0.

Let us return to our Hénon map  $f$  with a non-recurrent Fatou component  $\Omega$ . If  $h(\Omega)$  depends on  $h \in \Gamma_f$ , then by Lemma 27 there must exist both an  $k \in \Gamma$  of rank 0 (for example any maximal  $h$ ), and an  $h \in \Gamma$  of rank 1. The latter follows from the following classical argument, already used in higher dimensions by Jupiter and Lilov in [JL03]. The union of the limit sets must be connected, the image of a rank 0 limit map is a fixed point, and a Hénon map has only finitely many fixed points. Therefore if all limit maps have rank 0, the limit map is unique. The same argument holds for holomorphic endomorphisms, the fact that these maps have only finitely many fixed points was proved by Fornæss and Sibony in [FS94].

So let us suppose that there exist limit maps in  $\Gamma_f$  both with rank 0 and with rank 1. Let us call  $k$  a limit map of rank 0. Then  $p = k(\Omega)$  is a fixed point, and since  $f$  is dissipative,  $p$  is either a hyperbolic fixed point or a semi-attracting fixed point. In either case there exists a strong stable manifold  $W^s(p)$ , which is biholomorphic to the complex plane.

LEMMA 32. *Let  $f$  be a Hénon map and  $\Omega$  be an invariant non-recurrent Fatou component with bounded forward orbits. Suppose that a subsequence  $f^{n_j}$  converges uniformly on compact subsets of  $\Omega$  to the fixed point  $p$ , which necessarily has a strong stable manifold  $W^s(p)$ . Further suppose that the limit set  $\{p\}$  is not unique. Then there exists a sequence of iterates  $f^{m_i}$  that converges uniformly on compact subsets of  $\Omega$  to a rank 1 limit map  $h : \Omega \rightarrow W^s(p)$ .*

*Proof.* Since  $f$  has only finitely many fixed points, we can find an  $\epsilon > 0$  so that  $\overline{B_{2\epsilon}(p)}$  contains no fixed points besides  $p$ . Further decrease  $\epsilon$  if necessary so that (1) there exists a dominated splitting on  $B_{2\epsilon}(p)$  and (2) there exists a  $z \in \Omega$  whose orbit enters and leaves  $B_{2\epsilon}(p)$  infinitely often. The dominated splitting guarantees the existence of a strong stable manifold in  $B_\epsilon$  through every point whose orbits remain in  $B_\epsilon$ . These strong stable manifolds lie *properly* in  $B_\epsilon$ .

Let  $0 < \epsilon' < \epsilon$  be such that  $f^{-1}(B_{\epsilon'}(p)) \subset B_\epsilon(p)$ . Then there exist increasing integers  $m_1, m_2, \dots$  and  $k_1, k_2, \dots$  such that for every integer  $l$

- (1)  $f^{m_i}(z) \in B_\epsilon(p) \setminus B_{\epsilon'}(p)$ ,
- (2)  $f^{m_i+j}(z) \in B_\epsilon(p)$  for  $j = \{1, \dots, k_l\}$ , and
- (3)  $f^{m_i+k_l}(z) \in B_{\frac{1}{2l}}(p)$ .

By restricting to a subsequence if necessary we may assume that  $f^{m_i}$  converges uniformly on compact subsets of  $\Omega$  to a limit map  $h : \Omega \rightarrow \partial\Omega$ . Denote  $x = h(z)$ . Note that  $x$  must lie in  $\overline{B_\epsilon(p)} \setminus B_{\epsilon'}(p)$ , and hence cannot be a fixed point. Therefore  $h$  must have rank 1.

Note that the orbit of  $x$  must remain in  $\overline{B_\epsilon}$ . If not, there could not be points  $f^{m_i}(z)$  arbitrarily close to  $x$  whose orbits approach  $p$  arbitrarily nearly before leaving  $\overline{B_\epsilon}$ . Since the orbit of  $x$  remains in  $\overline{B_\epsilon}$ , the dominated splitting guarantees the existence of a unique strong stable manifold  $W^s(x)$ .

The limit set  $h(\Omega)$  either intersects  $W^s(x)$  locally only in  $x$  (where by locally we refer to the topology on the Riemann surfaces  $h(\Omega)$  and  $W^s(x)$ ), or locally coincides with  $W^s(x)$ . By normality of the family  $f^n$  restricted to  $h(\Omega)$ , the tangential derivatives of the family  $f^n$  must be uniformly bounded, locally in  $h(\Omega)$ . Therefore it follows that there is a neighborhood of  $x$  in  $h(\Omega)$  for which the orbits stay in  $B_{2\epsilon}(p)$ , which implies the existence of strong stable manifolds through any point in  $h(\Omega)$ . Hence if  $W^s(x)$  and  $h(\Omega)$  intersect locally only in  $x$ , the strong stable manifolds fill some neighborhood of  $x$ . It follows that the orbits of all points in a small neighborhood of  $x$  remain in  $B_{2\epsilon}(p)$ , which contradicts the fact that  $x$  is the limit of the sequence  $f^{m_i}(z)$ . Hence  $h(\Omega)$  and  $W^s(x)$  must locally coincide.

By Lemma 14,  $h(\Omega)$  is invariant under  $f$ , and by Lemma 27 there must be a sequence of iterates on  $h(\Omega)$  that converges to a fixed point, which in this case can only be  $p$ . This is not possible unless  $W^s(x) = W^s(p)$ . Hence  $h(\Omega)$  must be contained in  $W^s(p)$ .  $\square$

Recall that  $K^+, K^- \subset \mathbb{C}^2$  are the sets of points with bounded forward resp. backward orbits, and that  $J^+ = \partial K^+$  and  $J^- = \partial K^-$  are the forward and backward Julia sets.

**LEMMA 33.** *Let  $h \in \Gamma_f$  be of rank 1 so that  $h(\Omega) \subset W^s(p)$ . Then  $h(\Omega)$  lies in  $J = J^+ \cap J^-$ .*

*Proof.* Since  $h(\Omega)$  lies in  $W^s(p)$ , it is clear that  $h(\Omega)$  lies in  $K^+$ . It is equally clear that no point of  $h(\Omega)$  can lie in the interior of  $K^+$ , as it would imply normality of the sequence of iterates in a neighborhood of the point. Therefore  $h(\Omega)$  lies in  $J^+$ .

Let  $x \in h(\Omega)$ , and let  $z \in \Omega$  be such that  $h(z) = x$ . Suppose for the purpose of contradiction that  $x$  does not lie in  $K^-$ . Then there is an  $N \in \mathbb{N}$  such that  $f^{-N}(x) \in V^-$ . Write  $h = f^{n_j}$ . Then we see that for  $j$  large enough  $f^{-N}(f^{n_j}(z)) = f^{n_j-N}(z)$  also lies in  $V^-$ . But this is a contradiction, as  $n_j - N \rightarrow \infty$  as  $j \rightarrow \infty$ . Therefore  $h(\Omega) \subset K^-$ . Since  $f$  is dissipative,  $J^- = K^-$  and we are done.  $\square$

The following Lemma completes the proof of Theorem 2.

**LEMMA 34.** *Let  $f$  be a Hénon map of degree  $d$ , whose Jacobian determinant  $\delta$  satisfies*

$$|\delta| < \frac{1}{d^2}, \quad (38)$$

*and let  $p$  be a fixed point that is not attracting. Suppose that  $\Omega$  is an invariant non-recurrent Fatou component with a limit map  $h = \lim f^{n_j}$  that maps  $\Omega$  into  $W^s(p)$ . Then  $h(\Omega) = \{p\}$ .*

Finally, we will make use of a subharmonic version of the classical result by Wiman [Wim05] (see also [GO08], Ch. V, Thm 1.3, and [Hor94], Thm. 3.3.26) which is also a particular case of a (subharmonic version) of the Denjoy–Carleman–Ahlfors Theorem, see [Sod00] and [HK76], Section 4.6. It had been recently used in the same context by Dujardin and the first author in [DL12], see Remark 3 in the Introduction. (In the context of one-dimensional complex dynamics, it had been earlier used by Eremenko and Levin [EL89].)

**Theorem 35** (Wiman). *Let  $g$  be a non-constant subharmonic function on  $\mathbb{C}$  whose order of growth is  $< 1/2$ , i.e.,  $g(z) = O(|z|^\rho)$  for some  $\rho < \frac{1}{2}$ . Then all components of  $\{g = 0\}$  are bounded.*

**PROOF OF LEMMA 34.** Suppose for the purpose of a contradiction that  $h(\Omega)$  is not equal to  $\{p\}$ , in which case  $h$  must have rank 1. By the classical Poincaré Theorem,

the restriction  $f|_{W^s(p)}$  is globally linearizable, i.e., there exists a biholomorphism  $\Psi : \mathbb{C} \rightarrow W^s(p)$  that satisfies the functional equation

$$\Psi(\lambda^{-1}\zeta) = f^{-1}(\Psi(\zeta)), \tag{39}$$

where  $\lambda$  is the stable multiplier of  $p$ . Note that by our assumption,

$$\lambda \leq \delta < \frac{1}{d^2} \tag{40}$$

Let us now consider the *backward Green function*  $G_- : \mathbb{C}^2 \rightarrow \mathbb{R}$

$$G_-(z) = \lim_{n \rightarrow +\infty} \frac{1}{d^n} \log \|f^{-n}z\|.$$

It is a non-negative plurisubharmonic function vanishing on  $K^-$  and satisfying the functional equation

$$G_-(f^{-1}z) = dG_-(z). \tag{41}$$

Let us restrict this function to  $W^s(p)$  and pull it back to  $\mathbb{C}$ , i.e., let  $g = G_- \circ \Psi$ . This is a non-negative subharmonic function on  $\mathbb{C}$  vanishing on

$$\Psi^{-1}(K_-) \supset \Psi^{-1}(h(\Omega)) =: \Lambda.$$

Note that  $\Lambda$  is connected and by (39) is invariant under the scaling  $z \mapsto \lambda^{-1}z$ . Hence it is an unbounded continuum in  $\mathbb{C}$ .

Moreover, by (39) and (41), it satisfies the functional equation

$$g(\lambda^{-1}z) = dg(z).$$

It follows that

$$g(z) = O(|z|^\rho), \quad \text{with } \rho = -\frac{\log d}{\log \lambda} < 1/2,$$

where the last estimate follows from (40).

By the Wiman Theorem, all components of  $\{g = 0\}$  are bounded, contradicting unboundedness of  $\Lambda$ . □

REMARK 36. As we have alluded earlier, the original Wiman Theorem was concerned with entire (rather than subharmonic) functions. In fact, it can be directly used in our context as well, using the coordinate function  $w$  instead of the Green function  $G_-$ .

## 9 Holomorphic Endomorphisms of Projective Space

In this section we give a description of Fatou components under the assumption that the image of the limit map  $h = \lim f^{n_j}$  is independent of the sequence  $(n_j)$ .

Before we prove Theorem 7, let us note that for holomorphic endomorphisms of  $\mathbb{P}^2$  we cannot expect the same description as for Hénon maps in Theorem 6. First of all, we have already seen in Example 11 a Fatou component  $\Omega$  for which all orbits converge to an invariant disk or annulus lying in the boundary of  $\Omega$ . Also, even in the case where all orbits converge to a point  $p \in \partial\Omega$ , we cannot expect the eigenvalues  $\lambda_1$  and  $\lambda_2$  of  $Df(p)$  to satisfy  $|\lambda_1| < 1$  and  $\lambda_2 = 1$ . Of course, if one of the eigenvalues  $\lambda_1$  satisfies  $|\lambda_1| < 1$ , then  $|\lambda_2| = 1$  and Theorem 28 implies that  $\lambda_2 = 1$ . However, by taking the cross product of two polynomials of the same degree that both have a parabolic fixed point, we can obtain an example of a Fatou components where all orbits converge to a point  $p \in \partial\Omega$  with  $Df(p) = \text{Id}$ .

Naively one might then expect that at least one of the eigenvalues has to equal 1, but the examples below show that this does not hold either. The first map has a *quasi-parabolic* fixed point, as studied by Bracci and Molino in [BM04].

EXAMPLE 37. Let us first construct an example where both eigenvalues have modulus 1 but one of the eigenvalues is not equal to 1. Let  $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  be given by

$$f(z, w) = (z(1 + z), \lambda w(1 + z)), \quad (42)$$

where  $|\lambda| = 1$ . Let us write  $(z_n, w_n) = f^n(z, w)$ . It is clear that  $z_n$  only depends on  $z_0$ , and

$$z_n = z_0 \prod_{i=0}^{n-1} (1 - z_i). \quad (43)$$

Similarly,

$$w_n = w_0 \prod_{i=0}^{n-1} (1 - z_i). \quad (44)$$

So we see that if  $z_n$  converges to 0 then so will  $w_n$ , and the basin of the origin is given by  $\Omega \times \mathbb{C}$ , where  $\Omega \subset \mathbb{C}$  is the parabolic basin of the map  $z \mapsto z(1 + z)$ .

This map does not extend holomorphically to  $\mathbb{P}^2$ , but we can consider the following modification:

$$f(z, w) = (z(1 + z) + z^d, \lambda w(1 + z) + w^d), \quad (45)$$

where  $d \geq 3$ . It is easy to see that the dynamical behavior near the origin is similar, yet this map extends holomorphically to  $\mathbb{P}^2$ . Hence we see that for any  $|\lambda| = 1$  there exist a holomorphic endomorphism of  $\mathbb{P}^2$  with a parabolic basin and corresponding eigenvalues equal to 1 and  $\lambda$ .

With a little more effort we can construct a parabolic basin in  $\mathbb{P}^2$  where neither eigenvalue equals 1.

EXAMPLE 38. Again we first consider a selfmap of  $\mathbb{C}^2$ .

$$f(z, w) = (e^{i\theta}z(1 + zw), e^{-i\theta}w(1 + zw)). \quad (46)$$

We claim that for any  $\theta \in \mathbb{R}$  this map has a parabolic basin at the origin. Notice first that in a neighborhood of the origin, the axes are completely invariant. We also have that  $z_{n+1}w_{n+1} = z_nw_n(1 + z_nw_n)$ . Writing  $y$  for  $zw$  we obtain

$$y_n = y_0 \prod_{i=0}^{n-1} (1 + y_i)^2. \quad (47)$$

We also have that

$$z_n = z_0 \prod_{i=0}^{n-1} (1 + y_i), \quad (48)$$

and

$$w_n = w_0 \prod_{i=0}^{n-1} (1 + y_i). \quad (49)$$

We therefore see that if  $y_n \rightarrow 0$  then  $z_n, w_n \rightarrow 0$  as well, and we conclude that the origin has a parabolic basin.

Again the map  $f$  does not extend to a holomorphic endomorphism of  $\mathbb{P}^2$ , but we can consider a similar modification

$$F(z, w) = f(z, w) + (z^d, w^d), \quad (50)$$

where  $d \geq 7$  and the map does extend to  $\mathbb{P}^2$ .

LEMMA 39. *The holomorphic map  $F$  has an open set of orbits converging uniformly to the origin.*

*Proof.* Let  $(z_0, w_0) \neq 0$  be such that the following induction hypotheses are satisfied for some  $\epsilon > 0$ :

- (i)  $|z_0| < \epsilon, \quad |w_0| < \epsilon,$
- (ii)  $|\frac{z_0^2}{w_0}| < \epsilon, \quad |\frac{w_0^2}{z_0}| < \epsilon,$
- (iii)  $|\operatorname{Re}(z_0w_0)| < 0, \quad |\operatorname{Im}(z_0w_0)| < \epsilon|\operatorname{Re}(z_0w_0)|.$

A straightforward calculation shows that, as long as  $\epsilon$  is small enough, the hypotheses are also satisfied for  $(z_n, w_n)$ , and (again assuming that  $\epsilon$  is small enough) that  $z_n$  and  $w_n$  both converge to 0.  $\square$

REMARK 40. Examples of this kind have been independently constructed by Bracci and Zaitsev in [BZ13] (see also [BRZ13]). In fact, in those papers, a broad class of germs exhibiting a similar phenomenon is introduced.

We now restate and prove Theorem 7.



**Theorem 7.** *Let  $f$  be a holomorphic endomorphism of  $\mathbb{P}^2$  and let  $\Omega$  be a non-recurrent, invariant Fatou component. Suppose that the limit set  $h(\Omega)$  is unique. Then  $h(\Omega)$  either consist of one point  $p$ , or  $h(\Omega)$  is a injectively immersed Riemann surface, conformally equivalent to either the unit disk, the punctured unit disk or an annulus, and  $f$  acts on  $h(\Omega)$  as an irrational rotation.*

*Proof.* If  $h(\Omega)$  is a point then we are done, so we may assume that  $h$  has rank 1. By Lemma 13 the image  $\Sigma_h = h(\Omega)$  is a injectively immersed Riemann surface, invariant under  $f$ . By the work of Weickert [WeiJ03],  $\Sigma_h$  is hyperbolic. We claim that every orbit in  $\Sigma_h$  is recurrent in the topology of  $\Sigma_h$ . If not there would be an orbit in  $\Sigma_h$  accumulating on a point in  $\partial\Omega \setminus \Sigma_h$ , which by Lemma 17 would imply that  $h(\Omega)$  is not unique.

A hyperbolic Riemann surface either has a discrete automorphism group, or is biholomorphic to the unit disk, the punctured unit disk or an annulus. By Lemma 25 we may assume that  $h$  is minimal. Hence  $f : \Sigma_h \rightarrow \Sigma_h$  is an automorphism, and as we noted above its action on  $\Sigma_h$  is recurrent. As was shown in [FS94], a holomorphic endomorphism of  $\mathbb{P}^2$  can only have finitely many fixed points. Therefore the automorphism group of  $\Sigma_h$  cannot be discrete. Hence  $\Omega$  must be the unit disk, the punctured unit disk or an annulus, and  $f$  acts on  $\Sigma_h$  as an irrational rotation.  $\square$

We note that it is unknown whether  $h(\Omega)$  can be equivalent to a punctured disk.

## 10 Appendix: Perturbations of Singular Riemann Surfaces

We give two alternative proofs of Proposition 12, which we first restate.

**PROPOSITION 12.** *Let  $f : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$  be the germ of a holomorphic map whose image is singular at the origin in  $\mathbb{C}^2$ . Let  $U$  be a neighborhood of 0 where  $f$  is defined. Then there exists an  $\epsilon > 0$  so that for every  $g : U \rightarrow \mathbb{C}^2$  with  $\|f - g\|_U \leq \epsilon$  we have that  $g(U) \cap f(U) \neq \emptyset$ .*

*Geometric proof.* The following argument is a more geometric presentation of the argument given in the main body of the paper. To fix the idea, let us assume that the image of  $f$  is the standard cusp  $\{w^2 = z^3\}$ . Let us include into holomorphic foliation  $\mathcal{F}$  with leaves

$$L_\lambda = \{w^2 = \lambda z^3\}, \quad \lambda \in \hat{\mathbb{C}}.$$

Let us puncture out 0, and consider the space  $\mathcal{O}$  of leaves in the punctured neighborhood of the origin. This space has a natural Riemann orbifold structure (supported on the sphere) whose local charts are obtained by taking local transversals to  $\mathcal{F}$  and slicing the leaves to it. There are two orbifold points on  $\mathcal{O}$ : the leaf  $w = 0$  is an orbifold point of order 3 and the leaf  $z = 0$  is an orbifold point of order 2. So, the Euler characteristic of  $\mathcal{O}$  is equal to  $1/2 + 1/3 < 1$ .

Let  $[\lambda]$  be the point of  $\mathcal{O}$  corresponding to the leaf  $L_\lambda$ . The function  $g$  naturally induces a holomorphic orbifold map  $g : U \rightarrow \mathcal{O}$  that does not assume value  $[\lambda = 1]$

but whose boundary values are close to this point. It follows that  $g$  is proper over  $\mathcal{O} \setminus B$ , where  $B$  is a small neighborhood of  $[1]$ . Hence  $g$  is an orbifold branched covering over  $\mathcal{O} \setminus B$  of some degree  $d$ . By the orbifold Riemann-Hurwitz Theorem, the Euler characteristic of  $g^{-1}(\mathcal{O} \setminus B)$  is at most  $d \cdot \chi(\mathcal{O} \setminus B) < 0$ . On the other hand, it follows from the Maximum Principle that  $g^{-1}(\mathcal{O} \setminus B)$  is the union of disks, which has a positive Euler characteristic.

In general, consider the foliation with leaves  $\{w^m = \lambda z^l r(z, w)\}$  [using notation of (12)]. The space of its leaves in the punctured neighborhood of 0 is an orbifold with Euler characteristic  $< 1$  (as it has at least two orbifold points, and one of them has order at least 3). The rest of the argument is the same.  $\square$

*Topological proof* The following argument was proposed to us by Gabriellov and Milnor.

We recall a few facts that can be found in [Mil68]. Let us consider a small closed ball  $B = \mathbb{B}_\epsilon$  in  $\mathbb{C}^2$  bounded by a 3-sphere  $S = \partial B$ . We may assume that  $V := f(U) \cap B$  lies properly in  $B$ , and we denote its intersection with  $S$  by  $\gamma$ . If  $\epsilon$  is chosen sufficiently small then  $\gamma$  is a non-trivial knot in  $S$ . Moreover, there exists a retraction  $\pi : B \setminus V \rightarrow S \setminus \gamma$ .

Let us consider the component  $U_0$  of  $g^{-1}(B)$  containing 0. Then  $g : U_0 \rightarrow B$  is a singular 2-cell bounded by a knot  $\eta$  in  $S$ . Since by assumption it is disjoint from  $V$ , we can retract it by  $\pi$  to  $S \setminus \gamma$ . We obtain a singular 2-cell  $\Delta$  in  $S \setminus \gamma$  bounded by  $\eta$ .

Since  $g$  is close to  $f$ , the knots  $\gamma$  and  $\eta$  are parallel. [Two disjoint knots are called *parallel* if they bound an embedded annulus  $S^1 \times [0, 1] \rightarrow S$ . It is easy to see that if both knots are smooth and  $\eta$  is a small  $C^1$ -perturbation of  $\gamma$ , then the knots are parallel.] Let  $A$  be the embedded annulus with boundary  $\gamma - \eta$ . Then the sum  $D := \Delta + A$  is a singular 2-cell bounded by  $\gamma$ . We can apply to it the following classical result, proved by Papakyriakopoulos in [Pap57]:

LEMMA 41. (Dehn's Lemma). *Let  $D$  be a singular 2-cell in  $S^3$  bounded by a knot  $\gamma$  that has an annular neighborhood in  $D$ . Then there exists an embedded 2-cell  $D'$  bounded by  $\gamma$ .*

It follows that the knot  $\gamma$  is trivial—contradiction.  $\square$

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