

# POLAR FOLIATIONS OF SYMMETRIC SPACES

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**Abstract.** We prove that a polar foliation of codimension at least three in an irreducible compact symmetric space is hyperpolar, unless the symmetric space has rank one. For reducible symmetric spaces of compact type, we derive decomposition results for polar foliations.

## 1 Introduction

The following is the most important special case of our results:

**Theorem 1.1.** *Let  $M$  be a simply connected, irreducible, non-negatively curved symmetric space, and let  $\mathcal{F}$  be a polar foliation on  $M$  of codimension at least 3. Then either all leaves of  $\mathcal{F}$  are points, or  $\mathcal{F}$  is hyperpolar, or the symmetric space has rank one. Moreover, in the last case,  $M$  is not the Cayley plane and the foliation lifts via the Hopf fibration to a polar foliation of the round sphere.*

The result confirms a folklore conjecture in the field of polar foliations and actions. We explain the origin, ambience and generalizations of this result below. But first, we would like to emphasize that the main idea of the proof may be more interesting than the result itself. Namely, the main step of the proof is an application of the famous theorem of Tits which classifies spherical Tits buildings and shows that spherical buildings of dimension at least 2 are homogeneous and of algebraic origin. This combinatorial-algebraic theorem, seemingly very far remote from the world of Riemannian geometry, was already used as the main tool in two very important papers [Tho91] and [BS87]. Here we provide another application of this classification result along with another theorem of Tits on chamber systems slightly more general than buildings.

We hope that this approach via the combinatorial theory of buildings might be fruitful in the field of polar foliations, providing insights barely attainable by direct

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differential-geometric means. Our hope is supported by the fact that independently the same idea was applied by Fang et al. in [FGT12], to obtain a classification of polar actions on positively curved manifolds. Moreover, this approach gives more weight and interest to the exotic two-dimensional combinatorial objects which has appeared in the work of Tits on chamber complexes and buildings. It seems to come as a surprise in the building community, that such two-dimensional local buildings not coming as quotients of global buildings, which were discovered in the 80ies ([Neu84]) and considered as something extremely bizarre and exotic, appear naturally as problems and examples in Riemannian geometry. We refer the reader to [KL12] for more on these combinatorial objects, finish the advertisement part and come to the introduction.

A *polar foliation*  $\mathcal{F}$  of a complete  $m$ -dimensional Riemannian manifold  $M$  is a singular Riemannian foliation (cf. Section 2) with regular leaves of dimension  $(m-k)$ , such that each point  $x \in M$  is contained in a complete, totally geodesic, immersed submanifold  $\Sigma$  of dimension  $k$ , by definition, the *codimension of the foliation*, that intersects all leaves of  $\mathcal{F}$  orthogonally. Such a submanifold is called a *section* of  $\mathcal{F}$ . The polar foliation  $\mathcal{F}$  is called *hyperpolar* if one and hence all sections are flat. If the foliation is given by the orbit foliation of an isometric action it is called *homogeneous* and the action is called a *polar action*.

In space forms, the investigation of polar foliations of codimension one has been initiated by Segre and Cartan and in higher codimensions by Terng ([Ter85]) under the name of *isoparametric foliations*. We refer to the excellent surveys [Tho00], [Tho10] and the huge list of references therein. It turns out that polar foliations in Euclidean spaces come from polar foliations on spheres. Polar foliations of codimension at least two in round spheres have been shown by Thorbergsson to be homogeneous (if they are “irreducible and full”) and related to *non-positively curved* symmetric spaces and their buildings at infinity ([Tho91]). On the other hand, in codimension one, there are series of inhomogeneous examples and the classification is still not complete, despite great recent progress in the area ([FKM81], [Sto99], [CCJ07], [Imm08], and [Tho10] for more references).

The investigation of polar foliations in (from now on always) non-negatively curved symmetric spaces  $M$  has been initiated in [TT95]. It has been shown that, using a Riemannian submersion  $\mathcal{H} \rightarrow M$  from a Hilbert space of paths to  $M$ , one can lift hyperpolar (!) foliations from  $M$  to  $\mathcal{H}$ . From this observation one could “understand” all hyperpolar (“full, irreducible”) foliations of codimension at least 2, by showing that they are homogeneous ([HL99], [Ewe98], [Chr02]). In irreducible symmetric spaces, such hyperpolar actions have been classified in [Kol02].

On the other hand, in symmetric spaces of rank one, there are lots of polar foliations (cf. [PT99], [Dom12]) which are never hyperpolar if the codimension is at least two. Motivated by the known examples and confirmed by the partial classification of polar *actions* on irreducible symmetric spaces obtained in [Kol09], it has been conjectured that polar foliations on irreducible symmetric spaces of higher rank are hyperpolar. Our Theorem 1.1 confirms this conjecture if the codimension is not equal to two.

In our approach, the irreducibility of  $M$  does not play a major role. More important is the irreducibility of the sections, more precisely of the quotient spaces  $M/\mathcal{F}$ . Without the assumption of irreducibility we prove:

**Theorem 1.2.** *Let  $\mathcal{F}$  be a polar foliation on a simply connected non-negatively curved symmetric space  $M$ . Then we have a splitting  $M = M_{-1} \times M_0 \times M_1 \times \dots \times M_l$ , such that  $\mathcal{F}$  is a direct product of polar foliations  $\mathcal{F}_i$  on  $M_i$ . The foliation  $\mathcal{F}_{-1}$  on  $M_{-1}$  is given by the fibers of the projection of  $M_{-1}$  onto a direct factor of  $M_{-1}$ . The foliation  $\mathcal{F}_0$  is hyperpolar. For  $i \geq 1$ , the sections of the foliation  $\mathcal{F}_i$  on  $M_i$  have constant positive sectional curvature. For  $i \geq 1$ , if the codimension of  $\mathcal{F}_i$  on  $M_i$  is at least 3 then  $M_i$  is irreducible and of rank one; moreover, in this case, the foliation  $\mathcal{F}_i$  lifts to a polar foliation of the round sphere.*

In the special case of *polar action* it is possible to understand the arising additional difficulties in cohomogeneity two. Based on this work, in [KL12] (and, previously, in [KL13], in the irreducible case), it is proved that the additional assumption on the codimension being at least 3 is redundant. Thus any polar action on a non-negatively curved symmetric space is (up to orbit-equivalence) a direct product of hyperpolar actions and of polar actions on spaces of rank 1.

The method of proving our main result is inspired by the proof of the homogeneity result of polar foliation in Euclidean spaces due to Thorbergsson [Tho91]. We reduce the statement to the case in which the sections have constant curvature 1. We investigate the horizontal object of our foliation, that is a length metric space defined by measuring the lengths of broken horizontal geodesics with respect to the foliation. We use Wilking's results about the dual foliation to see that (in the irreducible case) this new metric space is connected. Since the local structure of this metric space is given by polar foliations on the Euclidean space, this new metric space is locally isometric to some spherical building (possibly up to one special case that can be handled directly). Now we use a theorem of [CL01], stating that, if the codimension  $k$  of the foliation (i.e., the dimension of our horizontal object) is at least 3, this horizontal space is covered by a spherical building. Moreover, we use our coarser manifold topology, to find a coarser compact topology on our building. If this building is reducible, one can use direct methods to detect the structure of our symmetric space. In the “main” irreducible case, we use the theorems of Burns–Spatzier and Tits ([BS87], [Tit74]) saying that our building is the building of a simple non-compact real Lie group. In particular, its coarser topology is that of a sphere. Then our manifold turns out to be the base of a principal fibration of a sphere. Therefore it is homeomorphic to a projective space. We conclude that our symmetric space has rank 1.

Finally, we would like to mention that the case of cohomogeneity 2 is different not only for technical reasons. The main point is that the universal covering of our horizontal space need not be a building (i.e. the local–global result from [CL01] may fail). We are aware of only one example in which this problem arises, namely for the polar action of  $SU(3) \cdot SU(3)$  on the Cayley projective plane  $\text{CaP}^2$ . In [KL12], it is

shown that this is the only such example in the case of polar *actions*, i.e., in the case, when the horizontal simplicial object is homogeneous. Unfortunately, in the general case, nothing is known about the combinatorial structure of arising objects.

In Section 2 we shortly recollect all notions and results needed later in the proof. In Section 3 and Section 4 we study dual foliations and derive the product decomposition of Theorem 1.2, reducing Theorem 1.2 and Theorem 1.1 to the case where sections have constant positive curvature and the dual foliation has only one leaf. In Section 6 we introduce our main tool: the horizontal singular metric  $d^{hor}$  on our manifold  $M$  and study its basic properties. It turns out that there are two essentially different cases to be investigated, depending on whether the spherical Coxeter group in question is reducible or not. In Section 7 we study the reducible case and apply some basic results of [Nag92] about special totally geodesic subspaces, called *polars* and *meridians*, to prove that our symmetric space  $M$  has rank 1. In Section 8, together with Section 6, the heart of the paper, we use the fact the universal covering of our singular metric space  $(M, d^{hor})$  is a spherical building. We define a coarser topology on this space and use the main theorem of [BS87] to prove that this coarser topology is the topology of a sphere. Then we deduce that  $M$  has rank 1. In the final section, we use a simple argument inspired by [PT99], to describe polar foliations on symmetric spaces of rank 1, thus finishing the proof of our main theorems.

## 2 Preliminaries

**2.1 Foliations.** We refer to [Wil07], [LT10], [Lyt10] for more on singular Riemannian foliations. Here we just recall the basic notions. Let  $M$  be a Riemannian manifold. A singular Riemannian foliation  $\mathcal{F}$  on  $M$  is a decomposition of  $M$  into smooth, injectively immersed submanifolds  $L(x)$ , called *leaves*, such that it is a singular foliation and such that any geodesic starting orthogonally to a leaf remains orthogonal to all leaves it intersects. Such a geodesic is called a *horizontal geodesic*. For all  $x \in M$ , we denote by  $H_x$  the orthogonal complement to the tangent space  $T_x(L(x))$ , and call it the *horizontal space at  $x$* . A leaf and all of its points are called *regular* if it has maximal dimension. On the set of regular points, the foliation is locally given by a Riemannian submersion. The dimension of the regular leaves is called the dimension of the foliation, and their codimension in  $M$  is called the codimension of the foliation.

The foliation is called *polar* if through any point  $x \in M$  one finds a totally geodesic submanifold whose dimension equals the codimension of  $\mathcal{F}$  and which intersects all leaves orthogonally. This happens if and only if the horizontal distribution in the regular part is integrable. If  $M$  is complete, then the totally geodesic submanifolds can be chosen to be complete. They are called section of the polar foliation  $\mathcal{F}$ . We refer to [Ale04], [AT06], [Lyt10] for more on polar foliations.

If the foliation  $\mathcal{F}$  is polar and  $M$  is simply connected then all leaves are closed. The quotient space (the space of all leaves) will be denoted by  $\Delta$ . It comes along

with the canonical projection  $p : M \rightarrow \Delta$  which is a *submetry*. The quotient  $\Delta$  is a *good Riemannian Coxeter orbifold* (*reflectofold*, in terms of [Dav10]). Moreover, the restriction  $p : \Sigma \rightarrow \Delta$  to any section  $\Sigma$  is a Riemannian branched covering. Thus  $\Delta$  is isometric to  $\tilde{\Sigma}/\Gamma$ , where  $\tilde{\Sigma}$  is the universal covering of  $\Sigma$  and  $\Gamma$  is a *reflection group*, i.e., a discrete group of isometries of  $\tilde{\Sigma}$  generated by reflections at totally geodesic hypersurfaces.

For any point  $x \in M$ , the singular Riemannian foliation defines an infinitesimal singular Riemannian foliation  $T_x\mathcal{F}$  on  $T_xM$ , that factors as a projection of  $T_xM$  to  $H_x$  and the restriction of  $T_x\mathcal{F}$  on  $H_x$ . If  $\mathcal{F}$  is polar then  $T_x\mathcal{F}$  is polar and sections of  $\mathcal{F}$  through  $x$  are in one-to-one correspondence with sections of  $T_x\mathcal{F}$  through the origin. Any horizontal geodesic is contained in a section of  $\mathcal{F}$ . Moreover, either the foliation is regular or there are two sections  $\Sigma_{1,2}$  whose intersection  $\Sigma_1 \cap \Sigma_2$  is a hypersurface in both sections  $\Sigma_{1,2}$ .

**2.2 Dual foliation.** The dual foliation  $\mathcal{F}^\#$  of a singular Riemannian foliation  $\mathcal{F}$  is defined by letting the leaf  $L^\#(x)$  be the set of all points in  $M$  that can be connected with  $x$  by a broken horizontal geodesic. In [Wil07] it is shown that  $\mathcal{F}^\#$  is indeed a singular foliation. The following important results has been shown in [Wil07] (we use slightly weaker formulations, suitable for our aims):

**PROPOSITION 2.1.** *Let  $M$  be a complete non-negatively curved manifold with a singular Riemannian foliation  $\mathcal{F}$ . Let  $\gamma$  be an  $\mathcal{F}$ -horizontal geodesic starting at a point  $x \in M$ . Let  $W(t) := \nu_{\gamma(t)}L^\#(x)$  denote the normal space to the dual leaf  $L^\#(x) = L^\#(\gamma(t))$  at the point  $\gamma(t)$ . Then  $W(t)$  is parallel along  $\gamma$ . Moreover, for all  $w \in W(t)$ , the sectional curvature  $\text{sec}(w \wedge \gamma'(t))$  of the plane spanned by  $w$  and  $\gamma'(t)$  is 0.*

**PROPOSITION 2.2.** *Under the assumptions of Proposition 2.1, if all dual leaves are complete in their induced metric then the dual foliation is a singular Riemannian foliation.*

We provide an easy application of these results:

**LEMMA 2.3.** *Let  $M$  be a simply connected, complete, non-negatively curved manifold. If  $\mathcal{F}$  is a polar regular foliation of  $M$  then  $M$  splits isometrically as a product  $M = M_1 \times M_2$  and  $\mathcal{F}$  is given by the projection  $p_1 : M \rightarrow M_1$ .*

*Proof.* By definition, the leaves of the dual foliation  $\mathcal{F}^\#$  are exactly the sections of  $\mathcal{F}$ . In particular, they are complete. Due to Proposition 2.2,  $\mathcal{F}^\#$  is a Riemannian foliation as well. Moreover, the horizontal distribution of  $\mathcal{F}^\#$  coincides with  $\mathcal{F}$ , hence it is integrable. Thus the leaves of  $\mathcal{F}$  are the sections of  $\mathcal{F}^\#$ . Thus they are totally geodesic. A polar foliation with totally geodesic leaves is locally given by a projection onto a section, which is locally a direct factor of  $M$ . Since  $M$  is simply connected, we get a global decomposition  $M = M/\mathcal{F} \times M/\mathcal{F}^\#$ .  $\square$

**2.3 Spherical Coxeter groups.** A *spherical Coxeter group* is a reflection group  $\Gamma$  on a round sphere  $\mathbb{S}^k$ . We will call it reducible if the corresponding action on  $\mathbb{R}^{k+1}$  is reducible. There is a unique decomposition  $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_l$  and a  $\Gamma$ -invariant orthogonal decomposition  $\mathbb{R}^{k+1} = V_0 \oplus V_1 \oplus \cdots \oplus V_l$ , such that  $\Gamma_i, i = 1, \dots, l$  acts as an irreducible reflection group on  $V_i$  and trivial on all  $V_j, j \neq i$ .

The quotient  $\Delta = \mathbb{S}^k/\Gamma$  is the spherical join  $\Delta = \Delta_0 * \Delta_1 \dots * \Delta_l$  of the round sphere  $\Delta_0$  and irreducible Coxeter simplices  $\Delta_i = S_i/\Gamma_i$ , where  $S_i$  is the unit sphere of  $V_i$ .

The group  $\Gamma$  is called *crystallographic*, if all dihedral angles of the spherical polyhedron  $\Delta$  are given by  $\pi/m$ , where  $m$  can only take the values 1, 2, 3, 4, 6. If none of the direct factors  $\Gamma_i$  is one-dimensional, then none of the dihedral angles of  $\Delta$  is equal to  $\pi/6$ .

Assume now that  $\Delta = \mathbb{S}^k/\Gamma$  is the quotient  $\Delta = M/\mathcal{F}$  of a polar foliation  $\mathcal{F}$  on a simply connected manifold  $M$ . Take a point  $y$  in a face of  $\Delta$  of codimension 2 in  $\Delta$ . Take a point  $x$  in the leaf over  $y$ . Then the tangent space  $T_y\Delta$  is the quotient space of the polar foliation  $T_x\mathcal{F}$  on  $T_xM$  (cf. [Lyt10]). The famous theorem of Münzner ([Mue80], [Mue81]) implies that the dihedral angle at  $y$  can be only given by  $\pi/m$ , with  $m = 1, 2, 3, 4, 6$ . We deduce that  $\Gamma$  is crystallographic.

**2.4 Spherical buildings.** We define buildings as metric spaces in contrast to their original simplicial definition of Tits. We refer to [CL01] and [KL98] for more on buildings considered from our point of view. Let  $\Gamma$  be a spherical Coxeter group acting on  $\mathbb{S}^n$ . A *spherical building* of type  $\Gamma$  is a metric space  $X$  with a set of isometric embeddings  $\phi : \mathbb{S}^n \rightarrow X$ , called *apartments*, such that the following two conditions hold true: Any pair of points of  $X$  is contained in some apartment and the transition maps between different apartments are given by restrictions of elements of  $\Gamma$ .

Consider the natural decomposition of  $\mathbb{S}^n$  into polyhedra isometric to  $\mathbb{S}^n/\Gamma$ . This polyhedral structure is preserved by  $\Gamma$ , hence we obtain a natural polyhedral structure on  $X$ . The building  $X$  is called *thick* if all walls of codimension 1 bound at least 3 polyhedra.

A spherical join of spherical buildings is a spherical building, in particular so is the suspension of a spherical building (cf. [BH99], for spherical joins and suspensions). A spherical building  $X$  is called *irreducible* if it is indecomposable as a spherical join. For a thick building of dimension at least 1 this is equivalent to the irreducibility of the Coxeter group  $\Gamma$ .

**2.5 Obtaining new foliations.** Let again  $p : M \rightarrow \Delta$  be the projection whose fibers are leaves of a polar foliation  $\mathcal{F}$  on  $M$ . Write again  $\Delta = N/\Gamma$ , where  $N$  is the universal covering of any section. Assume that there is a  $\Gamma$ -invariant polar foliation  $\mathcal{G}$  on  $N$ . Then  $\Gamma$  acts on the quotient orbifold  $N/\mathcal{G}$  by isometries. Assume that this action has closed (i.e. discrete) orbits and let  $\Delta'$  be the quotient orbifold  $\Delta' = (N/\mathcal{G})/\Gamma$ . The projection  $N \rightarrow \Delta'$  factors by definition through  $\Delta$ .

Then the composition  $p' = q \circ p : M \rightarrow \Delta'$  is the quotient map of a new polar foliation  $\mathcal{F}'$  on  $M$ .

Namely,  $p$  is a submetry (i.e., its fibers are equidistant) and so is  $q$ , hence the fibers of  $p'$  are equidistant as well. Around the preimage of a regular point of  $\Delta'$ ,  $p'$  is the composition of two Riemannian submersions with sections, hence it is itself a Riemannian submersion with sections. It only remains to prove that  $\mathcal{F}'$  is a singular foliation. This can be done directly. A slightly more elegant and sophisticated proof is obtained as follows. It is a direct consequence of our construction and the main definition of [Toe06], that the regular fiber has *parallel focal structure*. The main result of [Toe06] now implies that  $\mathcal{F}'$  is a singular Riemannian foliation.

By construction, each dual leaf of  $\mathcal{F}'$  is contained in a dual leaf of  $\mathcal{F}$ . On the other hand, if the polar foliation  $\mathcal{G}$  on  $N$  has only one dual leaf, then the dual leaves of  $\mathcal{F}$  and of  $\mathcal{F}'$  coincide.

We are going to use this construction only in two simple cases. First assume that  $\Delta$  is a direct metric product  $\Delta = \Delta' \times \Delta''$ . Then the composition  $p'$  of  $p : M \rightarrow \Delta$  and the projection  $q : \Delta \rightarrow \Delta'$  defines a polar foliation on  $M$ .

We will consider only one other case. Assume that  $\Delta$  is given as the quotient  $\mathbb{S}^k/\Gamma$ , where  $k \geq 2$  and  $\Gamma$  is a spherical Coxeter group. Assume that  $\Gamma$  is reducible. Consider the  $\Gamma$ -invariant orthogonal decomposition  $\mathbb{R}^{k+1} = V_1 \oplus V_2$ . Then  $\Delta$  is a spherical join  $\Delta = \Delta_1 * \Delta_2$ . Collapsing  $\Delta_i$  to points, we obtain a projection  $\Delta \rightarrow [0, \pi/2]$ , which corresponds to the reducible, polar, codimension one foliation on  $\mathbb{S}^k$  which is given by the distance function  $p' : \mathbb{S}^k \rightarrow [0, \pi/2]$  to the sphere  $\mathbb{S}^k \cap V_1$ .

Note, that any non-trivial singular Riemannian foliation on the round sphere has only one dual leaf, due to [Wil07]. Collecting the previous observations we arrive at:

**LEMMA 2.4.** *Let  $\mathcal{F}$  be a polar foliation on a complete Riemannian manifold  $M$ . Assume that the quotient  $\Delta$  is isometric to  $\mathbb{S}^k/\Gamma$ , with a reducible Coxeter group  $\Gamma$ . Then there is a coarser polar foliation  $\mathcal{F}'$  on  $M$ , which has the same dual leaves as  $\mathcal{F}$  and whose quotient space  $\Delta'$  is the interval  $[0, \pi/2]$ .*

### 3 Dual Foliations on Symmetric Spaces

We will use a general observation about dual leaves in symmetric spaces. Marco Radeschi has pointed out that a variant of the two subsequent results appeared in [MT11].

**PROPOSITION 3.1.** *Let  $M$  be a non-negatively curved symmetric space. Let  $\mathcal{F}$  be a singular Riemannian foliation on  $M$  and let  $\mathcal{F}^\#$  be the dual foliation. Then any leaf  $L^\#$  of the dual foliation is contained in a totally geodesic submanifold  $Z$  of the same dimension as  $L^\#$ . Moreover,  $Z$  is a direct factor of  $M$ . In particular, if the dual leaf  $L^\#$  is complete, it is a direct factor of  $M$ .*

*Proof.* Take a point  $x \in L^\#$ . Let  $W_x$  denote the normal space  $W_x = \nu_x(L^\#)$  to the dual leaf. We let  $W'_x$  be the set of all vectors  $w'$  in  $T_x M$  such that, for all  $w \in W_x$ , the sectional curvature  $\text{sec}(w \wedge w')$  is 0. Identifying  $x$  with the origin of the symmetric

space  $M = G/K$  and writing  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ , with the usual identification of  $\mathfrak{p}$  and  $T_x M$ , we have

$$W'_x := \{w \in T_x M \mid [W_x, w] = 0\}$$

Due to the Jacobi identity,  $[[W'_x, W'_x], W'_x] \subset W'_x$ . Thus, by definition,  $W'_x$  is a *Lie triple system*. The subspace  $W'_x \cap W_x$  commutes with  $W'_x$ . Hence the orthogonal complement  $W''_x$  of  $(W_x \cap W'_x)$  in  $W'_x$  is a Lie triple system as well. Exponentiating the Lie triple system  $W''_x$ , we obtain a totally geodesic submanifold  $Z = \exp(W''_x)$ . By definition,  $W''_x \cap W_x = \{0\}$ . Hence  $\dim(Z) \leq \dim(M) - \dim(W_x) = \dim(L^\#(x))$ .

We are going to prove that  $Z$  contains  $L^\#(x)$ . Take an  $\mathcal{F}$ -horizontal broken geodesic  $\gamma$  that starts in  $x = x_1$  and consists of a finite concatenation of  $\mathcal{F}$ -horizontal geodesics  $\gamma_i$  connecting  $x_i$  and  $x_{i+1}$ . Due to Proposition 2.1, the starting direction of  $\gamma_i$  is contained in  $W''_{x_i}$ . Moreover, the parallel translation along  $\gamma_i$  sends  $W_{x_i}$  to  $W_{x_{i+1}}$ .

Since the curvature tensor is invariant under parallel translation in the symmetric space  $M$ , the parallel translation along  $\gamma_i$  sends  $W''_{x_i}$  to  $W''_{x_{i+1}}$ . By induction on the number of concatenations, we deduce that  $\gamma$  is contained in  $Z$ . Since any point of  $L^\#$  can be reached from  $x$  by a broken  $\mathcal{F}$ -horizontal geodesic, we deduce that  $L^\#$  is contained in  $Z$ .

Thus we must have  $\dim(Z) = \dim(L^\#(x))$ . Then  $W_x$  and  $W''_x$  are complementary commuting subspaces of  $\mathfrak{p}$ . Therefore,  $W_x$  is a Lie triple system as well, and  $M$  splits as the product of  $Z$  and its orthogonal complement.  $\square$

In particular, we deduce:

**COROLLARY 3.2.** *If  $\mathcal{F}$  is a singular Riemannian foliation on a compact irreducible symmetric space then the dual foliation has only one leaf, unless  $\mathcal{F}$  has only one leaf.*

Another consequence, we will use is:

**COROLLARY 3.3.** *Let  $\mathcal{F}$  be a singular Riemannian foliation on a simply connected symmetric space  $M$ . If the dual leaves of  $\mathcal{F}$  are complete then  $M$  splits as  $M = M_1 \times M_2$  such that the dual leaves of  $\mathcal{F}$  are exactly the  $M_2$ -factors, i.e., all dual leaves have the form  $\{x_1\} \times M_2$ .*

*Proof.* Due to Proposition 2.2, the dual foliation is a singular Riemannian foliation. Due to Proposition 3.1, all leaves must be factors of  $M$ . Since these factors are equidistant they must be  $M_2$ -factors of the same product decomposition  $M = M_1 \times M_2$ .  $\square$

## 4 Product Decomposition

Here and in the sequel, let  $M$  be a simply connected non-negatively curved symmetric space and let  $\mathcal{F}$  be a polar foliation on  $M$ .



**4.1 Decomposition of the factor.** We recall that our foliation has closed leaves and that the quotient space  $\Delta = M/\mathcal{F}$  is a Coxeter orbifold. Moreover,  $\Delta$  is a discrete quotient of a section, the last being a totally geodesic submanifold of  $M$ , hence a symmetric space itself. Thus  $\Delta$  is given as  $\Delta = N/\Gamma$  with a symmetric non-negatively curved simply connected manifold  $N$  (the universal covering of a section  $\Sigma$ ), on which  $\Gamma$ , the orbifold fundamental group of  $\Delta$ , acts as a reflection group.

Let  $N = N_0 \times N_1 \times \cdots \times N_l$  be the direct product decomposition, where  $N_0$  is the Euclidean space and where  $N_i$  are irreducible of dimension at least 2. Any reflection (always at a wall of codimension 1!) on  $N$  respects this product decomposition. Hence it induces a reflection on one factor and identity on all other factors. Therefore,  $\Gamma$  is a direct product  $\Gamma = \Gamma_0 \times \Gamma_1 \times \cdots \times \Gamma_l$ , where  $\Gamma_i$  is the subgroup of  $\Gamma$  generated by all reflections fixing all factors  $N_j, j \neq i$ . Moreover, the quotient  $\Delta$  splits isometrically as the direct product  $\Delta = \Delta_0 \times \cdots \times \Delta_l$ , with  $\Delta_i = N_i/\Gamma_i$ .

Finally, the only simply connected irreducible symmetric spaces of compact type which admit a totally geodesic hypersurface are round spheres. Thus, for all  $i \geq 1$ , either  $N_i$  is a round sphere or  $\Gamma_i$  is trivial. Therefore, in the above product decomposition all  $\Delta_i, i \geq 1$  either have constant positive curvature or they coincide with the Riemannian manifolds  $N_i$  (this fact has been observed in [Kol09]).

**4.2 Decomposition of the space.** We call a polar foliation  $\mathcal{F}$  on a symmetric space  $M$  *decomposable* if  $M$  can be decomposed non-trivially as  $M = M_1 \times M_2$  such that  $\mathcal{F}$  splits as  $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ , a product of polar foliations on the factors. Otherwise we call  $\mathcal{F}$  *indecomposable*.

The proof of the following observation is postponed to Section 6:

LEMMA 4.1. *Assume that the sections of  $\mathcal{F}$  have constant positive curvature. Then the dual leaves are compact. In particular, they are factors of  $M$ .*

Now we can prove:

PROPOSITION 4.2. *Let  $\mathcal{F}$  be indecomposable. Then either  $\mathcal{F}$  is trivial, or hyperpolar, or  $\Delta$  has constant positive curvature and the dual foliation  $\mathcal{F}^\#$  has only one leaf.*

*Proof.* Assume that  $\Delta$  is non-trivially decomposed as  $\Delta_1 \times \Delta_2$ , with  $\Delta_1$  either a manifold or of constant positive curvature. Consider the induced submetry  $p_1 : M \rightarrow \Delta_1$  that is given by a polar foliation  $\mathcal{F}'_1$ . Due to the preceding lemmas (Lemma 2.3, Lemma 4.1, Corollary 3.3), the leaves of the dual foliation of  $\mathcal{F}'_1$  are  $M_1$ -factors in a decomposition  $M = M_1 \times M_2$ .

Any  $\mathcal{F}'_1$ -horizontal geodesic is mapped by the projection to  $\Delta_1 \times \Delta_2$  into a  $\Delta_1$  factor, hence by the projection  $p_2 : M \rightarrow \Delta_2$  to a point. This shows that any dual leaf of  $\mathcal{F}'_1$  is contained in a leaf of the foliation  $\mathcal{F}'_2$  defined by the projection  $p_2 : M \rightarrow \Delta_2$ . Thus the foliation  $\mathcal{F}'_2$  is coarser than the foliation defined by the  $M_1$ -factors. Hence,  $p_2$  factors through the projection  $q_2 : M \rightarrow M_2$ .

Taking  $\mathcal{F}_1$  to be the restriction of  $\mathcal{F}$  to  $M_1$  (any  $M_1$ -factor) and  $\mathcal{F}_2$  the restriction of  $\mathcal{F}$  to  $M_2$  (any  $M_2$ -factor) we get  $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ . This contradicts to the assumption that  $\mathcal{F}$  is indecomposable.

If a decomposition of  $\Delta$  as above does not exist, then  $\Delta$  must be either flat, or a manifold, or of constant positive curvature. If it is flat then the foliation is hyperpolar. If  $\Delta$  is a manifold then  $\mathcal{F}$  must be given by a projection to a factor. Since  $\mathcal{F}$  is indecomposable, this factor and, therefore,  $\mathcal{F}$  must be trivial. In the remaining case,  $\Delta$  must have constant positive curvature. Then by Lemma 4.1 and Corollary 3.3, the dual foliation has only one leaf.  $\square$

REMARK 4.1. Note, that the hyperpolar factor may be decomposed further until the quotient  $\Delta$  is irreducible ([Ewe98]).

Given a polar foliation  $\mathcal{F}$  on  $M$ , we now decompose it in indecomposable pieces. Taking trivial pieces together we obtain a foliation given by a projection to a direct factor. Collecting hyperpolar pieces together we get a hyperpolar foliation. Thus we arrive at:

PROPOSITION 4.3. *Let  $\mathcal{F}$  be a polar foliation on a non-negatively curved simply connected symmetric space  $M$ . Then we have a splitting  $M = M_{-1} \times M_0 \times M_1 \times \cdots \times M_l$ , such that  $\mathcal{F}$  is a direct product of polar foliations  $\mathcal{F}_i$  on  $M_i$ . The foliation  $\mathcal{F}_{-1}$  on  $M_{-1}$  is given by the fibers of the projection of  $M_{-1}$  onto a direct factor of  $M_{-1}$ . The foliation  $\mathcal{F}_0$  is hyperpolar. For  $i \geq 1$ , the sections of the foliation  $\mathcal{F}_i$  on  $M_i$  have constant positive sectional curvature; moreover, for  $i \geq 1$ , there is only one dual leaf of  $\mathcal{F}_i$ .*

## 5 New Setting

Due to Proposition 4.3, in order to prove Theorem 1.2 and Theorem 1.1, we only need to study the case in which the sections of  $\mathcal{F}$  have constant positive curvature.

From now on, we will assume the sections of  $\mathcal{F}$  to have constant positive curvature. Hence the sections are either spheres or projective spaces. We normalize the space such that the sections and the quotient have constant curvature 1. Thus for any horizontal vector  $v$  the geodesic in direction  $v$  is closed of period  $\pi$  or of period  $2\pi$ . Since in a symmetric space, for a continuous variation of closed geodesics the period of the geodesics cannot change, we deduce that all horizontal geodesics have the same period. This period is equal to  $2\pi$  if all sections are spheres, and it is equal to  $\pi$  if all sections are projective spaces.

The quotient  $\Delta$  is equal to  $\Delta = \mathbb{S}^k/\Gamma$  for a spherical Coxeter group  $\Gamma$ , that must be crystallographic. By assumption,  $k \geq 2$ .

## 6 Horizontal Metric

**6.1 Definition.** We now define a new metric  $d^{hor}$  on our manifold  $M$  by declaring  $d^{hor}(x, y)$  to be the infimum over all lengths of broken horizontal geodesics that connect  $x$  and  $y$ . By definition  $d^{hor} \geq d$ . The dual leaf  $L^\#(x)$  is exactly the set of

points that have a finite distance to the point  $x$ . We denote by  $X$  the set  $M$  with the horizontal metric  $d^{hor}$ .

By construction, the identity  $i : X \rightarrow M$  is 1-Lipschitz and the projection  $p : X \rightarrow \Delta$  is still a submetry. Since any horizontal geodesic is contained in a section, we see that the metric space  $X$  is defined by gluing together spherical polyhedra (each one isometric to the quotient  $\Delta$ ). A pair of polyhedra may be glued only along some union of faces. By definition,  $X$  is a length space and since it is a polyhedral complex with only one type of polyhedra, it is a geodesic space, i.e., any pair of points at a finite distance are connected by a geodesic with respect to  $d^{hor}$  (cf. [BH99, p. 105]).

Given a point  $x \in X$ , a small ball  $U_x$  around  $x$  in  $X$  is given by the image of a small ball in the horizontal space  $H_x$  under the exponential map. (Note, however, that the exponential map, considered as a map from  $H_x$  to  $X$  is not continuous). Consider the induced infinitesimal polar foliation  $\mathcal{F}_x$  on the Euclidean space  $H_x$ . The sections of  $\mathcal{F}$  through  $x$  are in one-to-one correspondence with the sections of  $\mathcal{F}_x$ . Hence a small neighborhood of  $x$  in  $X$  is isometric to a small ball in the spherical suspension over the "horizontal metric space"  $Y = (\mathbb{S}^r, d^{hor})$  that is defined by the polar foliation  $\mathcal{F}_x$  on the unit sphere  $\mathbb{S}^r = H_x^1$  in  $H_x$ .

Thus,  $X$  is a  $k$ -dimensional locally spherical space in the sense of [CL01]. Moreover, the space of directions  $S_x X$  at each point  $x \in X$  is isometric to the horizontal space defined by the infinitesimal polar foliation on the unit sphere  $H_x^1$ .

**6.2 Classical case and the irreducible case.** We are going to use the following result due to Immervoll and Thorbergsson ([Imm03], [Tho91]):

**PROPOSITION 6.1.** *Let  $\mathcal{G}$  be a polar foliation on the round sphere  $\mathbb{S}^r$ . Let  $C$  be the quotient  $C = \mathbb{S}^r / \mathcal{G}$ . If the Coxeter polyhedron  $C$  does not have dihedral angles equal to  $\pi/6$  then the horizontal space  $Y = (\mathbb{S}^r, d^{hor})$  defined by the foliation is a spherical building.*

**REMARK 6.1.** The conclusion of the previous lemma is true without any assumptions on the angles, if the foliation comes from a group action, in which case Proposition 6.4 below is a direct consequence of Proposition 6.1.

Let again  $M$  be a symmetric space with our polar foliation  $\mathcal{F}$  and quotient  $\Delta$  of dimension  $k \geq 2$ . We say that  $\Delta$  is *irreducible*, if the corresponding spherical Coxeter group  $\Gamma$  is irreducible. Otherwise, we say that  $\Delta$  is *reducible* and find a non-trivial decomposition  $\Delta = \Delta_1 * \Delta_2$  of  $\Delta$  as a spherical join.

If  $\Delta$  is irreducible, then  $\Delta$  does not have faces meeting at the dihedral angle  $\pi/6$ . Therefore we conclude:

**LEMMA 6.2.** *If  $\Delta$  is irreducible then for any point  $x$  in the horizontal space  $X$ , a small neighborhood of  $x$  is isometric to an open subset of a spherical building (possibly depending on the point).*

REMARK 6.2. Just to avoid confusion, we remark that in our convention the suspension over a building is again a building of one dimension larger.

Applying [CL01] we deduce:

COROLLARY 6.3. *Assume that  $\Delta$  is irreducible. Let  $X'$  be any dual leaf of  $\mathcal{F}$  with the metric induced from  $X$ . Then  $X'$  has diameter at most  $\pi$ . If  $k = \dim(\Delta) \geq 3$  then the universal covering of  $X'$  is a spherical building.*

**6.3 Reducible case.** We are going to prove that in the reducible case, our manifold  $M$  contains two submanifolds that behave like a projective subspace and its cut locus in a projective space.

PROPOSITION 6.4. *Let  $\Delta$  be reducible  $\Delta = \Delta_1 * \Delta_2$ . Let  $A_i = p^{-1}(\Delta_i)$  and let  $\mathcal{F}'$  be the polar foliation given by the submetry  $p' : M \rightarrow [0, \pi/2]$ , with  $p'(x) = d(A_1, x)$ . Then for any pair of points  $x_i \in A_i$  that are contained in the same dual leaf  $L^\#$  of  $\mathcal{F}$ , we have  $d(x_1, x_2) = \pi/2$ .*

*Proof.* Since  $k = \dim(\Delta) \geq 2$ , at least one  $\Delta_i$  is not a point. Without loss of generality, let  $\Delta_1$  have positive dimension. Due to Lemma 2.4, the dual leaves of  $\mathcal{F}$  and  $\mathcal{F}'$  coincide. The horizontal metric  $d'_{hor}$  induced by  $\mathcal{F}'$  on  $M$  is the metric on a graph with each edge being a horizontal geodesic from  $A_1$  to  $A_2$  of length  $\pi/2$ .

Thus, if the claim of the proposition is wrong, we find  $x_1 \in A_1$  and  $x_2 \in A_2$  such that there is a shortest geodesic  $\gamma$  with respect to  $d'_{hor}$  from  $x_1$  to  $x_2$  that has length  $3\pi/2$ . Let  $x_+ = \gamma(\pi/2) \in A_2$  and let  $x_- = \gamma(\pi) \in A_1$ .

Consider the polar foliation  $T_{x_+}\mathcal{F}$  on the Euclidean space  $H_{x_+}$ . The quotient is given by the tangent cone to  $\Delta$  at a point of  $\Delta_2$ . Hence it splits as a product of the tangent space to  $\Delta_2$  (that may be trivial) and the orthogonal complement  $Q$ . This implies a corresponding splitting of  $H_{x_+}$ , into a part tangent to  $A_2$  and the part  $H'$  of all  $\mathcal{F}'$ -horizontal vectors. Moreover, we have  $H'/T_{x_+}\mathcal{F} = Q$ . Since the restriction of  $\mathcal{F}'$  to the unit sphere of  $H'$  is non-trivial, it has only one dual leaf. Thus we find a broken horizontal geodesic in this sphere that connects the incoming and outgoing direction of  $\gamma$  at  $x_+$ . Exponentiated to the length  $\pi/2$ , we obtain a broken  $\mathcal{F}$ -horizontal geodesic  $\eta : [s, t] \rightarrow A_1$  that connects  $x_1$  with  $x_-$ .

But, at any point  $y \in A_1$ , any pair of  $\mathcal{F}$ -horizontal vectors  $h, v \in T_y M$  are tangent to a section of  $\mathcal{F}$ , whenever  $v$  is tangent to  $A_1$  and  $h$  orthogonal to  $A_1$  (i.e.,  $h$  is  $\mathcal{F}'$ -horizontal). Therefore, if  $d(x_2, \eta(r)) = \pi/2$ , for some  $r \in (s, t]$ , then  $x_2, \eta(r)$  and  $\eta(r - \epsilon)$  are contained in some section of  $\mathcal{F}$ . Thus  $d(x_2, \eta(r - \epsilon)) = \pi/2$  as well. Running  $\eta$  backwards from  $x_-$  to  $x_1$  we deduce  $d(x_1, x_2) = \pi/2$ .  $\square$

**6.4 The dual foliation.** We are going to prove Lemma 4.1 now.

First, let us assume that  $\Delta$  is irreducible. Take a dual leaf  $X' = L^\#(x)$  of  $\mathcal{F}$ . We have seen in Corollary 6.3 that  $X'$  has diameter at most  $\pi$  with respect to  $d^{hor}$ . Since  $X$  consists of simplices of the same size, any point of the dual leaf  $L^\#(x)$  can be connected with  $x$  by a broken horizontal geodesic with at most  $n$  breaks (for

some  $n$  depending only on  $\Delta$ ), of total length at most  $\pi$ . Since a limit of a sequence of such broken horizontal geodesic is again a broken horizontal geodesic, we deduce that  $L^\#$  is compact.

If the quotient  $\Delta$  is reducible as spherical join, the conclusion follows in the same way, using Proposition 6.4.

**6.5 More conclusions.** After having closed the gap Lemma 4.1 the proof of Proposition 4.3 is complete. We deduce:

**COROLLARY 6.5.** *If the foliation  $\mathcal{F}$  is indecomposable, there is only one dual leaf.*

From now on, in addition to our assumptions from Section 5,  $\mathcal{F}$  will be indecomposable as a product.

## 7 Polars and Meridians

We assume here that  $\Delta$  is reducible as a spherical join  $\Delta = \Delta_1 * \Delta_2$ , and are going to prove that  $M$  has rank 1.

Set  $A_i = p^{-1}(\Delta_i)$ . Using Proposition 6.4 we know that  $A_i$  are smooth manifolds, and we have  $d(x_1, x_2) = \pi/2$ , for all  $x_i \in A_i$ . Since there is only one dual leaf of  $\mathcal{F}$ , any point  $x$  in  $M$  lies on a unique shortest geodesic from  $A_1$  to  $A_2$ . Finally, any geodesic that starts horizontally on  $A_1$  is closed of period  $\pi$  or of period  $2\pi$ .

We are going to use a few easy facts about *polars* and *meridians* ([Nag92], [CN78]). Recall that in a symmetric space  $M$ , a *polar* of a point  $o$  is a connected component of the fixed point set of the geodesic symmetry  $s_o$  at the point  $o$ . The *meridian*  $M^-(p)$  through a point  $p$  in a polar  $M^+$  of  $o$  is the component through  $p$  of the fixed point set of the isometry  $s_o \circ s_p$ . We recall that the tangent spaces at  $p$  of  $M^+$  and  $M^-$  are complementary orthogonal subspaces. Moreover, the rank of the meridian  $M^-$  is equal to the rank of  $M$ .

Assume first that all sections of  $\mathcal{F}$  are projective spaces, i.e., all horizontal geodesics are closed of period  $\pi$ . Then, for any  $o \in A_1$ , the reflection  $s_o$  at  $o$  must leave  $A_2$  pointwise fixed. Choose any point  $p \in A_2$ . Then  $p$  is contained in a polar  $M^+$  of  $o$ , that contains  $A_2$ . Thus the tangent space to the meridian  $M^-(p)$  through  $p$  is contained in the orthogonal space to  $A_2$ . Thus, in the symmetric space  $M^-(p)$ , all geodesics starting at  $p$  are closed. Hence  $M^-(p)$  has rank one. Therefore the rank of  $M$  must be 1 as well.

Assume now that all sections are spheres. For any  $o \in A_1$ , the geodesic symmetry  $s_o$  leaves  $A_2$  invariant, but no point in  $A_2$  is fixed by  $s_o$ . Therefore,  $s_o(A_1) = A_1$  as well. Moreover, all polars of  $o$  must be contained in  $A_1$ . Let  $p \neq o$  be fixed by  $s_o$  and let  $M^-(p)$  be the meridian through  $p$ . Since the polar  $M^+(p)$  is contained in  $A_1$ , the normal space to  $A_1$  is tangent to the meridian. Hence, the meridian contains  $A_2$ . In the meridian  $M^- := M^-(p)$ , the point  $p$  is a *pole* of  $o$ , meaning that in the symmetric space  $M^-$ , the point  $p$  is a one-point polar of  $o$ . In such a case there is a two-to-one covering  $c : M^- \rightarrow M_1$ , such that  $M_1$  is symmetric and  $c$  sends  $o$

and  $p$  to the same point  $\bar{o}$  ([Nag92]). In  $M_1$ , the projections of horizontal geodesics starting in  $\bar{o}$  have period  $\pi$ . Hence a polar of  $\bar{o}$  inside  $M_1$  contains the image  $\bar{A}_2$  of  $A_2$ . Therefore, the meridian in  $M_1$  through any point of  $\bar{A}_2$  is orthogonal to  $\bar{A}_2$ . Thus all geodesics in this meridian are closed and it must have rank 1. Due to [Nag92],  $M_1$  and hence  $M$  must have rank 1 as well.

## 8 Topological Buildings

We assume here that  $\Delta = \mathbb{S}^k/\Gamma$  is an irreducible Coxeter simplex. Moreover, we assume that the universal covering  $\tilde{X}$  is a building. Due to Corollary 6.3, the last assumption is always fulfilled if  $k \geq 3$ . Again we are going to prove that  $M$  has rank 1.

We denote by  $K$  the fundamental group of  $X$  acting on  $\tilde{X}$  by deck transformations. By  $\pi : \tilde{X} \rightarrow X$  we denote the projection. We are going to define a compact  $K$ -invariant topology  $\mathcal{T}$  on  $\tilde{X}$ .

In order to do so, we will use the following construction several times. Let  $N$  be a compact, geodesic, simply connected metric space (for us an interval or a disc). Let  $h_i : N \rightarrow \tilde{X}$  be a sequence of uniformly Lipschitz maps. Consider the projections  $\bar{h}_i = \pi \circ h_i : N \rightarrow \tilde{X} \rightarrow X \rightarrow M$ . Then as Lipschitz maps to the compact manifold  $M$  the sequence is equicontinuous and we find a subsequence converging to a Lipschitz map  $\bar{h} : N \rightarrow M$ . Since all  $\bar{h}_i$  map a small neighborhood of any point into the union of sections through the image, the same is true for  $\bar{h}$ . Thus  $\bar{h}$  is in fact a Lipschitz map to  $X$ . Assuming that all  $h_i$  send a base point of  $N$  to the same point  $q \in \tilde{X}$  we have a unique lift of  $\bar{h}$  to a Lipschitz map  $h : N \rightarrow \tilde{X}$  sending the base point of  $N$  to  $q$ . We will say that the sequence  $h_i$  *weakly subconverges* to  $h$ .

To define the topology  $\mathcal{T}$ , we first fix a point  $q \in \tilde{X}$ . We will say that a point  $p \in \tilde{X}$  is contained in the  $\mathcal{T}$ -closure of a subset  $C \subset \tilde{X}$  if and only if for some sequence  $p_n \in C$  there is a curve  $\gamma$  from  $q$  to  $p$  and a sequence of shortest geodesics  $\gamma_n$  from  $q$  to  $p_n$ , such that  $\gamma_n$  weakly converges to  $\gamma$ .

From the Theorem of Arzela-Ascoli we see that the topology  $\mathcal{T}$  is sequentially compact. Moreover,  $(\tilde{X}, \mathcal{T})$  has a dense countable subset. We are going to prove that  $\mathcal{T}$  is Hausdorff and does not depend on the base point  $q$ .

**LEMMA 8.1.** *Let  $p_n$  converge to  $p$  in  $\mathcal{T}$ . Let  $\gamma'_n : [0, 1] \rightarrow \tilde{X}$  be a curve of length  $\leq L < \infty$  from  $q$  to  $p_n$ , parameterized proportionally to arc length. Let  $\gamma' : [0, 1] \rightarrow \tilde{X}$  be a weak limit of  $\gamma'_n$ . Then  $\gamma'$  ends in  $p$ .*

*Proof.* Let  $\gamma_n, \gamma$  be as in the definition of  $\mathcal{T}$  above. Let  $r_n : S^1 \rightarrow \tilde{X}$  be the concatenation of  $\gamma_n$  and of the reversed  $\gamma'_n$ . Since  $\tilde{X}$  is a spherical building of dimension at least 2,  $r_n$  can be retracted to a point uniformly, i.e.,  $r_n$  can be extended to some  $L' = L'(L)$ -Lipschitz map  $r_n : D^2 \rightarrow \tilde{X}$  (Straighten  $\gamma'_n$  to be a broken geodesic with uniformly many geodesic parts, using that the injectivity radius is  $\pi$ . Then subdivide  $S^1$  into uniformly finitely many intervals, such that  $q$  and the image of any of

these intervals are contained together in an apartment). Consider now a weak limit  $r : D^2 \rightarrow \tilde{X}$  of the sequence  $r_n$ . By construction, the left half-circle in  $r(S^1)$  is  $\gamma$  and the right half-circle in  $r(S^1)$  is  $\gamma'$ .  $\square$

The lemma implies that a sequence cannot converge in  $\mathcal{T}$  to two different points. Thus  $\mathcal{T}$  is Hausdorff. Since  $\mathcal{T}$  is separable and sequentially compact it is a compact metrizable topology. Taking another point  $q' \in \tilde{X}$  and considering concatenations with a fixed geodesic from  $q$  to  $q'$ , the lemma implies that the topology does not depend on the base point  $q$ . Thus it is defined only in terms of the projection  $\pi : \tilde{X} \rightarrow M$ . Therefore, it is invariant under the action of  $K$ .

By construction, a small metric ball around any point  $x \in \tilde{X}$  is sent by  $\pi$  bijectively onto a small ball in (the exponential image of) the normal space to the leaf through  $\pi(x)$ . The topology  $\mathcal{T}$  we have defined, restricts to this ball as the usual Euclidean topology in the normal space. Thus the intersections of a preimage of a regular leaf and a small ball around any point of  $\tilde{X}$  is connected.

Thus we have a compact irreducible building  $(\tilde{X}, \mathcal{T})$ , in the sense of [BS87]. Since the preimages of the leaves of  $\mathcal{F}$  (i.e., points of the same type in  $\tilde{X}$ , in other words, the set of chambers of the building) contain non-trivial connected subsets, the set of chambers is connected ([GvMKW12]). From [BS87] it follows, that the space  $(\tilde{X}, \mathcal{T})$  must be homeomorphic to a sphere. Moreover, the building is the spherical building of a simple non-compact real Lie group and can be identified with the boundary at infinity of a non-compact irreducible symmetric space. In particular, the group of automorphisms of the building acts on the sphere in a linear way.

Consider now the action of  $K$  on  $\tilde{X}$ . The orbit of any point is the preimage of a point under the continuous projection  $\pi : \tilde{X} \rightarrow M$ . Thus it is a compact set. The group of topological automorphisms  $G$  of the compact building  $\tilde{X}$  is locally compact with respect to the compact-open topology ([BS87]). We claim that  $K$  is a compact subgroup of  $G$ . In fact, take a sequence  $g_n \in K$ . Choose a point  $p \in \tilde{X}$ . Then there is some  $g \in K$ , such that  $g_n \cdot p$  converges to  $g \cdot p$ . We call  $h_n = g^{-1}g_n$  and have  $h_n \cdot p \rightarrow p$ . We claim that  $h_n$  converges to the identity in  $G$ .

Choose a shortest geodesic  $\gamma_n$  from  $p_n = h_n(p)$  to  $p$ . Choose now a point  $q \in \tilde{X}$  and a shortest geodesic  $\eta$  from  $p$  to  $q$ . Then  $h_n(q)$  is given by the lift starting at  $p$  of the projection of the concatenation of  $\gamma_n$  and  $\eta$ . These projections converge to a curve which lifts to a curve ending at  $q$ . Hence,  $q_n$  converges to  $q$ . Therefore,  $g_n$  converges to  $g$ . Thus  $K$  is compact.

Thus our compact group  $K$  of automorphisms acts freely and linearly on the sphere  $\tilde{X}$ . The projection map  $\pi : \tilde{X} \rightarrow M$  has as fibers the orbits of  $K$ , hence  $M$  is the quotient space  $M = \tilde{X}/K$ . By assumption,  $M$  is simply connected, hence  $K$  is connected. The only connected groups that act without fixed points on a sphere are the trivial group,  $U(1)$  and  $SU(2)$ . Then the quotient space  $M$  is homeomorphic to a sphere, or projective space over the complex or over the quaternions.

But only symmetric spaces of rank 1 have the topology of a sphere or of a projective space (cf. [Zil77]).

## 9 Polar Foliations on Symmetric Spaces of Rank One

Polar *actions* on symmetric spaces of rank one have been studied and classified in [PT99]. The geometric structure of polar foliations on such spaces is not more complicated. The following result is folklore (cf. [Dom12]):

**PROPOSITION 9.1.** *Let  $M$  be a projective space  $F\mathbb{P}^m$ , where  $F$  denotes the field of complex or quaternionic numbers. Let  $h : \mathbb{S}^n \rightarrow M$  be the Hopf fibration from the round sphere. If  $\mathcal{F}$  is a polar foliation on  $M$  then its lift  $\hat{\mathcal{F}} := h^{-1}(\mathcal{F})$  is a polar foliation on  $\mathbb{S}^n$ .*

*Proof.* We normalize our space, such that the round sphere  $\mathbb{S}^n$  has curvature 1. The Hopf fibration  $h$  is a Riemannian submersion, hence  $\hat{\mathcal{F}}$  is a singular Riemannian foliation on  $\mathbb{S}^n$ , with the same quotient space  $\Delta = M/\mathcal{F} = \mathbb{S}^n/\hat{\mathcal{F}}$ . If the dimension  $k$  of  $\Delta$  is 1, then  $\hat{\mathcal{F}}$  is of codimension 1, hence polar. If  $k \geq 2$ , then  $\hat{\mathcal{F}}$  is polar if and only if the orbifold  $\Delta$  has constant curvature 1. Thus  $\hat{\mathcal{F}}$  is polar if and only if the sections of  $\mathcal{F}$  have constant curvature 1.

The sections of  $\mathcal{F}$  are either spheres or projective spaces of constant curvature. Maximal totally geodesic spheres in  $M$  are given by the projective lines  $F\mathbb{P}^1$  ([PT99]). However, any pair of such projective lines intersect in at most one point and never in a one-dimensional or three-dimensional subset. Thus if all sections are projective lines, the foliation must be regular. This contradicts Lemma 2.3.

Hence all sections of  $\mathcal{F}$  are real projective spaces  $\mathbb{R}\mathbb{P}^k$ . But such projective spaces have curvature 1 (cf. [PT99]).  $\square$

The same proof as above shows:

**PROPOSITION 9.2.** *Let  $\mathcal{F}$  be a polar foliations on the Cayley projective plane. Then either  $\mathcal{F}$  has codimension 1 or the sections of  $\mathcal{F}$  are real projective subspaces  $\mathbb{R}\mathbb{P}^2$  and  $\mathcal{F}$  has codimension 2.*

Combining the above propositions with the results of Section 7 and Section 8, we finish the proof of Theorem 1.2 and Theorem 1.1.

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