Geom. Funct. Anal. Vol. 22 (2012) 240–265
DOI 10.1007/s00039-012-0151-1
Published online January 26, 2012

GAFA Geometric And Functional Analysis

METRIC FLIPS WITH CALABI ANSATZ

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Abstract. We study the limiting behavior of the Kähler–Ricci flow on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)})$ for $m, n \geq 1$, assuming the initial metric satisfies the Calabi symmetry. We show that the flow either shrinks to a point, collapses to $\mathbb{P}^n$ or contracts a subvariety of codimension $m+1$ in the Gromov–Hausdorff sense. We also show that the Kähler–Ricci flow resolves a certain type of cone singularities in the Gromov–Hausdorff sense.

1 Introduction

The formation of singularities of the Kähler–Ricci flow on a compact Kähler manifold M reveals the analytic and algebraic structures of M. It is well known that the Kähler–Ricci flow converges to a Kähler–Einstein metric if M admits negative or vanishing first Chern class for any initial Kähler metric [Y1], [Ca1], [Ts].

When the canonical bundle K_M is not nef, the Kähler–Ricci flow will develop a finite time singularity and one expects the Ricci flow to carry out surgeries through the singularities in some natural and unique way. The flow $(M, g(t))$ should converge in some suitable sense to a 'limit manifold' $(\overline{M}, g_T)$ as t tends to the singular time T and continue on the new manifold starting at g_T. This is referred to as *canonical surgery by the Ricci flow*. If the Kähler manifold M is projective, then one hopes that the canonical surgeries correspond to algebraic transformations such as divisorial contractions or flips.

An analytic analogue of Mori's minimal model program is laid out in [SoT3] for how the Kähler–Ricci flow will behave on a general projective variety. More precisely, it is conjectured that the Kähler–Ricci flow will either deform a projective variety M to its minimal model after finitely many divisorial contractions and flips in the Gromov–Hausdorff sense, or collapse in finite time. The existence and uniqueness is proved in [SoT3] for the weak solution of the Kähler–Ricci flow through divisorial contractions and flips. However, the Gromov–Hausdorff convergence at the singular time is largely open. The program is established for Kähler surfaces in [SoW2,3]. More precisely, for the Kähler–Ricci flow on a Kähler surface M with an initial Kähler metric g_0, either the flow deforms M to a minimal surface or the volume

Keywords and phrases: Kähler-Ricci flow, Gromov-Hausdorff convergence,, small contraction, flip

2010 *Mathematics Subject Classification*: 53c55, 53c44

The first named author is supported in part by National Science Foundation grant DMS-0847524 and a Sloan Foundation Fellowship.

tends to 0 in finite time, after finitely many contractions of (-1)-curves in the Gromov–Hausdorff sense. In the work of [LT] the conjectural behavior of the flow through a flip is discussed in relation to their V-soliton equation.

The goal of the current paper is to construct examples of small contractions and resolution of singularities by the Kähler–Ricci flow. Let (X, g_0) be a compact Kähler manifold of complex dimension $n \geq 2$. We write $\omega_0 = \frac{\sqrt{-1}}{2\pi}(g_0)_{i\bar{j}}dz^i \wedge d\overline{z^j}$ for the Kähler form associated to g_0. We consider the following Kähler–Ricci flow $\omega = \omega(t)$ given by

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega)\,, \quad \omega|_{t=0} = \omega_0\,, \tag{1.1}$$

for $\operatorname{Ric}(\omega) = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\omega^n$, and $g = g(t)$ is the metric associated to ω. The flow admits a smooth solution on $[0, t+\epsilon)$ for some $\epsilon > 0$, if and only if the cohomology class of $\omega(t)$ given by $[\omega(t)] = [\omega_0] + t[K_X]$ is Kähler. The first singular time T is characterized by

$$T = \sup\big\{t \in \mathbb{R} \mid [\omega_0] + t[K_X] > 0\big\}\,. \tag{1.2}$$

Clearly T depends only on X and the Kähler class $[\omega_0]$, and satisfies $0 < T \leq \infty$.

The manifold

$$X_{m,n} = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)}\big)$$

is a projective toric manifold for $m \geq 0$ and $n \geq 1$. $X_{0,n}$ is exactly $\mathbb{P}^{n+1}$ blown up at one point. $X_{m,n}$ does not admit a definite or vanishing first Chern class when $n \leq m$ and $X_{m,n}$ is Fano if and only if $n > m$. $X_{m,n}$ has a special subvariety P_0 of codimension $m+1$, defined as the zero section of projection $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)}) \to \mathbb{P}^n$. There exists a morphism

$$\Phi_{m,n} : X_{m,n} \to \mathbb{P}^{(m+1)(n+1)} \tag{1.3}$$

which is an immersion on $X_{m,n} \setminus P_0$ and contracts P_0 to a point. $Y_{m,n}$, the image of $X_{m,n}$ via $\Phi_{m,n}$ is smooth if and only if $m = 0$ and then $Y_{0,n}$ is simply $\mathbb{P}^{n+1}$. When $m \geq 1$, $Y_{m,n}$ has a cone singularity where P_0 is contracted. In particular, $Y_{m,n} = Y_{n,m}$ is the projective cone in $\mathbb{P}^{(m+1)(n+1)}$ over $\mathbb{P}^m \times \mathbb{P}^n$ via the Segre map for $m \geq 1$. It is also well known that $X_{m,n}$ and $X_{n,m}$ are birationally equivalent for $m \geq 1$, and differ by a flip for $m \neq n$.

In this paper, we always consider the Kähler metrics on $X_{m,n}$ satisfying the Calabi symmetric condition defined in [C1]. The precise definition is given in section 2.2.

Our first main result characterizes the limiting behavior of the Kähler–Ricci flow (1.1) on $X_{m,n}$ as $t \to T$.

Theorem 1.1. *Let $g(t)$ be the solution of the Kähler–Ricci flow* (1.1) *on $X_{m,n}$ with the initial Kähler metric $\omega_0 \in a_0[D_H] + b_0[D_\infty]$ satisfying the Calabi symmetry. Let $T > 0$ be the first singular time of the flow.*

(1) *If $m < n$ and $b_0/(m+2) > a_0/(n-m)$, then $T = a_0/(n-m)$ and on $X_{m,n} \setminus P_0$, $g(t)$ converges smoothly to a Kähler metric g_T. Let (X_T, d_T) be the metric completion of $(X_{m,n} \setminus P_0, g_T)$. Then (X_T, d_T) has finite diameter and is homeomorphic to $Y_{m,n}$ as the projective cone in $\mathbb{P}^{(m+1)(n+1)}$ over $\mathbb{P}^m \times \mathbb{P}^n$*

via the Segre map. Furthermore, $(X_{m,n}, g(t))$ converges to (X_T, d_T) in the Gromov–Hausdorff sense as $t \to T$.

(2) *If $m < n$ and $b_0/(m+2) = a_0/(n-m)$, then $T = a_0/(n-m)$ and $(X_{m,n}, g(t))$ converges to a point in the Gromov–Hausdorff sense as $t \to T$.*

(3) *If $m < n$ and $b_0/(m+2) < a_0/(n-m)$, then $T = b_0/(m+2)$ and $(X_{m,n}, g(t))$ converges to $\big(\mathbb{P}^n, \big(a_0 - \frac{n-m}{m+2} b_0\big)\omega_{FS}\big)$ in the Gromov–Hausdorff sense as $t \to T$, where ω_{FS} is the Fubini–Study metric on $\mathbb{P}^n$.*

(4) *If $m \geq n$, then $T = b_0/(m+2)$ and $(X_{m,n}, g(t))$ converges to $\big(\mathbb{P}^n, \big(a_0 - \frac{n-m}{m+2} b_0\big)\omega_{FS}\big)$ in the Gromov–Hausdorff sense as $t \to T$.*

In the cases (3) and (4), the Kähler–Ricci flow can be continued on $\mathbb{P}^n$ starting with $\big(\mathbb{P}^n, \big(a_0 - \frac{n-m}{m+2} b_0\big)\omega_{FS}\big)$ and the flow will eventually become extinct in finite time. The case (2) is related to the result in [So] that the Kähler–Ricci flow shrinks to a point if and only if X is Fano, and the initial Kähler class is proportional to $c_1(X)$, establishing the smooth case of a conjecture in [T2]. If this occurs, it is natural to renormalize the flow so that the volume is constant. The problem of how this normalized flow behaves is related to various notions of stability [Y2], [T2], [Do], and is still open in general. Assuming the existence of a Kähler–Einstein metric [P2], [TZh] or soliton [TZh], the flow is shown to converge to a Kähler–Einstein metric or soliton respectively (see also [ST], [Zh]). The connection between stability conditions and the behavior of the Kähler–Ricci flow has been studied in [PhS], [PhSSW1,2], [R], [Sz], [To], [MuS], [ChW] for example.

Theorem 1.1 can also be viewed as an analogue of Theorem 1.1 in [SoW1]. We apply ideas and techniques from [SoW1] to obtain many estimates in the proof of Theorem 1.1. In fact, Theorem 1.1 can be generalized to

$$X_{m,n,k} = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-k)^{\oplus(m+1)}\big), \quad k = 1, 2, \ldots$$

In particular, $X_{m,n,1} = X_{m,n}$ and $X_{0,1,k}$ are exactly the rational ruled surfaces considered in [SoW1].

We would also like to mention some known results about Kähler–Ricci solitons on these manifolds. $X_{n,0}$ admits a Kähler–Ricci soliton [Ko], [Ca2]. Complete Kähler–Ricci solitons are also constructed on vector bundles $\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)}$ by [FIK] when $m = 0$ and by [Li] when $m \geq 1$.

The general conjecture in [SoT3] predicts that the flow can also be continued in the first case in Theorem 1.1 and the contracted variety should jump to its minimal resolution by the flip. Our next result shows how the Kähler–Ricci flow can resolve a certain type of projective cone singularities and confirms the weaker statement of the general conjecture.

Theorem 1.2. *Let $Y_{m,n} = Y_{n,m}$ be a projective cone in $\mathbb{P}^{(m+1)(n+1)}$ over $\mathbb{P}^m \times \mathbb{P}^n$ via the Segre map and let $g_0 \in [\mathcal{O}_{\mathbb{P}^{(m+1)(n+1)}}(1)]$ be the restriction of the Fubini–Study metric of $\mathbb{P}^{(m+1)(n+1)}$ on $Y_{m,n}$.*

(1) *If $m > n \geq 1$, there exists a smooth solution $g(t) \in \Phi^*_{m,n}[\mathcal{O}_{\mathbb{P}^{(m+1)(n+1)}}(1)] + t[K_{X_{m,n}}]$ of the Kähler–Ricci flow on $(0, T = 1/(m+2)) \times X_{m,n}$ such that on $X_{m,n} \setminus P_0 \simeq Y_{m,n} \setminus \{O\}$, $g(t)$ converges smoothly to g_0 as $t \to 0$, and $(X_{m,n}, g(t))$ converges*

to $(Y_{m,n}, g_0)$ in the Gromov–Hausdorff sense as $t \to 0$. Furthermore, $g(t)$ has uniformly bounded local potential in L^∞ for $t \in [0, T)$. If there exists another solution $\hat{g}(t)$ satisfying the above conditions, then $g(t) = \hat{g}(t)$.

(2) *If $m = n \geq 1$, there exists a smooth solution $g(t) \in (1 - (m+2)t)[g_0]$ of the Kähler–Ricci flow on $Y_{n,n} \setminus \{O\}$ for $t \in (0, T = 1/(m+2))$ such that*

- *(Y_t, d_t), the metric completion of $(Y_{n,n} \setminus \{O\}, g(t))$ is homeomorphic to $(Y_{n,n}, g_0)$.*
- *$g(t)$ converges to g_0 smoothly on $Y_{n,n} \setminus \{O\}$ as $t \to 0$, and $g(t)$ has uniformly bounded local potential in L^∞ for $t \in [0, T)$.*
- *$(Y_{n,n}, d_t)$ converges to $(Y_{n,n}, g_0)$ in the Gromov–Hausdorff sense as $t \to 0$ and converges to a point in the Gromov–Hausdorff sense as $t \to T$.*

If there exists another solution $\hat{g}(t)$ satisfying the above conditions, then $g(t) = \hat{g}(t)$.

The above theorem shows that the Kähler–Ricci flow resolves the cone singularity of $Y_{m,n}$ for $m > n \geq 1$ in the Gromov–Hausdorff sense. It suggests that the Kähler–Ricci flow smooths out not only the initial singular metric, but also the initial underlying variety. Combining (1) in Theorem 1.1 and (1) in Theorem 1.2, the Kähler–Ricci flow replaces $X_{m,n}$ by $X_{n,m}$ as an analytic flip in the Gromov–Hausdorff sense, if we are allowed to continue the Kähler–Ricci flow at $t = T$ with the Fubini–Study metric restricted on $Y_{m,n}$. We believe that the Kähler–Ricci flow should perform the flip for $X_{m,n}$ without replacing the singular metric g_T by the Fubini–Study metric as the initial metric at the singular time.

On the other hand, the Kähler–Ricci flow does not change the underlying manifold $Y_{n,n}$ for $n \geq 1$ even though $K_{Y_{n,n}}$ is not a Cartier divisor. This is because $X_{n,n}$ is the resolution of $Y_{n,n}$ and the canonical divisor $K_{X_{n,n}}$ vanishes along the exceptional locus over the singularity O.

The organization of the paper is as follows. In section 2, we describe flips, the Calabi ansatz and the Kähler–Ricci flow on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)})$. In section 3, we prove the small contraction by the Kähler–Ricci flow if the volume does not tend to zero when approaching the singular time. In section 4, we show that the Kähler–Ricci flow collapses if the volume tends to zero when approaching the singular time. In section 5, we describe how the Kähler–Ricci flow resolves singularities of $Y_{m,n}$.

2 Background

2.1 An example of flips. We will describe a family of projective bundles over $\mathbb{P}^n$ so that one can construct a flip. The detailed algebraic construction can be found in section 1.9 of [D].

Let E be the vector bundle over a projective space $\mathbb{P}^n$ defined by $\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)}$. We let

$$X_{m,n} = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)}\big)$$

be the projectivization of E and it is a $\mathbb{P}^{m+1}$ bundle over $\mathbb{P}^n$. In particular,

$X_{0,n}$ is $\mathbb{P}^{n+1}$ blown up at one point. Let D_∞ be the divisor in $X_{m,n}$ given by $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)})$, the quotient of $\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)}$. We also let D_0 be the divisor in $X_{m,n}$ given by $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m})$, the quotient of $\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m}$. In fact, $N^1(X_{m,n})$ is spanned by $[D_0]$ and $[D_\infty]$. We also define the divisor D_H on $X_{m,n}$ by the pullback of the divisor on $\mathbb{P}^n$ associated to $\mathcal{O}_{\mathbb{P}^n}(1)$. Then

$$[D_\infty] = [D_0] + [D_H]$$

and

$$[K_{X_{m,n}}] = -(m+2)[D_\infty] - (n-m)[D_H] = -(n+2)[D_\infty] + (n-m)[D_0]\,. \quad (2.1)$$

The above formulas can be easily obtained by induction on m and the adjunction formula. In particular, D_∞ is a big and semi-ample divisor and any divisor $a[D_H] + b[D_\infty]$ is ample if and only if $a > 0$ and $b > 0$. Hence $X_{m,n}$ is Fano if and only if $n > m$.

Let P_0 be the zero section of $\pi_{m,n} : X_{m,n} \to \mathbb{P}^n$, which is the intersection of the $m+1$ effective divisors as the quotient of $\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m}$. In fact, the linear system $|[D_\infty]|$ is base-point-free and it induces a morphism

$$\Phi_{m,n} : X_{m,n} \to \mathbb{P}^{(m+1)(n+1)}. \quad (2.2)$$

$\Phi_{m,n}$ is an immersion on $X_{m,n} \setminus P_0$ and it contracts P_0 to a point. $Y_{m,n}$, the image of $\Phi_{m,n}$ in $\mathbb{P}^{(m+1)(n+1)}$, is a projective cone over $\mathbb{P}^m \times \mathbb{P}^n$ in $\mathbb{P}^{(m+1)(n+1)}$ by the Segre embedding

$$[Z_0, \ldots, Z_m] \times [W_0, \ldots, W_n] \to [Z_0W_0, \ldots, Z_iW_j, \ldots, Z_mW_n] \in \mathbb{P}^{(m+1)(n+1)-1}.$$

Note that $Y_{m,n} = Y_{n,m}$.

The following diagram gives a flip from $X_{m,n}$ to $X_{n,m}$ for $0 < m < n$:

$$\begin{array}{ccc} X_{m,n} & \overset{\tilde{\Phi}}{\dashrightarrow} & X_{n,m} \\ & \searrow^{\Phi_{m,n}} \quad \swarrow^{\Phi_{n,m}} & \\ & Y_{m,n} & \end{array} \quad (2.3)$$

Furthermore, $X_{m,n}$ and $X_{n,m}$ are birational to each other.

2.2 Calabi ansatz. In this section, we will define the Calabi ansatz constructed by Calabi [C1] (also see [Li]). To apply the Calabi symmetry, we instead consider the vector bundle

$$E = \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)}.$$

Let ω_{FS} be the Fubini–Study metric on $\mathbb{P}^n$. Let h be the hermitian metric on $\mathcal{O}_{\mathbb{P}^n}(-1)$ such that $\mathrm{Ric}(h) = -\omega_{FS}$. The induced hermitian metric h_E on E is given by $h_E = h^{\oplus(m+1)}$. Under local trivialization of E, we write

$$e^\rho = h_\xi(z)|\xi|^2, \quad \xi = (\xi^1, \xi^2, \ldots, \xi^{m+1})\,,$$

where $h_\xi(z)$ is a local representation for h (note that h_E has the same eigenvalues as $h(z)$ does). In particular, if we choose the inhomogeneous coordinates $z = (z_1, z_2, \ldots, z_n)$ on $\mathbb{P}^n$, we have

$$h_\xi(z) = \left(1 + |z|^2\right).$$

Therefore,

$$\rho = \log\big((1+|z|^2)|\xi|^2\big)\,. \tag{2.4}$$

In the future calculations, we will always compute in terms of ρ.

We would like to find appropriate conditions for $a \in \mathbb{R}$ and a smooth real valued function $u = u(\rho)$ such that

$$\omega = a\omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u(\rho) \tag{2.5}$$

defines a Kähler metric on $X_{m,n}$. In fact,

$$\omega = \big(a + u'(\rho)\big)\omega_{FS} + \frac{\sqrt{-1}}{2\pi}h_\xi e^{-\rho}\big(u'\delta_{\alpha\beta} + h_\xi e^{-\rho}(u''-u')\xi^{\bar{\alpha}}\xi^{\beta}\big)\nabla\xi^\alpha \wedge \overline{\nabla\xi^\beta}. \tag{2.6}$$

Here,

$$\nabla\xi^\alpha = d\xi^\alpha + h_\xi^{-1}\partial h_\xi \xi^\alpha$$

and $\{dz^i, \nabla\xi^\alpha\}$ is dual to the basis

$$\nabla_{z^i} = \frac{\partial}{\partial z^i} - h_\xi^{-1}\frac{\partial h_\xi}{\partial z^i}\sum_\alpha \xi^\alpha \frac{\partial}{\partial \xi^\alpha}, \quad \frac{\partial}{\partial \xi^\alpha}\,.$$

The following criterion is due to Calabi [C1].

Proposition 2.1. *ω as defined above is a Kähler metric if and only if*

(1) $a > 0$.
(2) $u' > 0$ *and* $u'' > 0$ *for* $\rho \in (-\infty, \infty)$.
(3) $U_0(e^\rho) = u(\rho)$ *is smooth on* $(-\infty, 0]$ *and* $U_0'(0) > 0$.
(4) $U_\infty(e^{-\rho}) = u(\rho) - b\rho$ *is smooth on* $[0, \infty)$ *for some* $b > 0$ *and* $U_\infty'(0) > 0$.

We remark that given $a, b > 0$, the Kähler metric constructed above lies in the Kähler class

$$\omega = a\omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u(\rho) \in a[D_H] + b[D_\infty] \tag{2.7}$$

and

$$0 < u'(\rho) \le b\,. \tag{2.8}$$

2.3 The Kähler–Ricci flow on $X_{m,n}$. Straightforward calculations show that the induced volume form of ω is given by

$$\omega^{m+n+1} = (a+u')^n h_\xi^{m+1} e^{-(m+1)\rho}(u')^m u'' \left(\omega_{FS}^n \wedge \prod_{\alpha=1}^{m+1}\frac{\sqrt{-1}}{2\pi}d\xi^\alpha \wedge d\xi^{\bar{\alpha}}\right). \tag{2.9}$$

Therefore,

$$-\operatorname{Ric}(\omega) = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\big(\log[(a+u')^n(u')^m u''] - (m+1)\rho\big) + (m-n)\omega_{FS}\,.$$

It is straightforward to check that the Calabi ansatz is preserved by the Ricci flow. Indeed, the Kähler–Ricci flow

$$\frac{\partial\omega}{\partial t} = -\operatorname{Ric}(\omega)\,, \quad \omega|_{t=0} = \omega_0 = a_0\omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u_0 \in a_0[D_H] + b_0[D_\infty] \tag{2.10}$$

is equivalent to the following parabolic equation:

$$a'(t)\omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\frac{\partial u}{\partial t} = (m-n)\omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\big(\log[(a+u')^n(u')^m u''] - (m+1)\rho\big)\,.$$

Separating the variables, we have that

$$a = a(t) = a_0 - (n-m)t \tag{2.11}$$

and

$$\frac{\partial u}{\partial t} = \log\big[(a+u')^n(u')^m u''\big] - (m+1)\rho + c_t\,, \tag{2.12}$$

where

$$c_t = -\log u''(0,t) - m\log u'(0,t) - n\log\big(a(t)+u'(0,t)\big)\,. \tag{2.13}$$

The constant c_t is chosen such that $\frac{\partial u}{\partial t}(0,t) = 0$. From the formula (2.1) and the Kähler class evolves by $[\omega] = (a_0 - (n-m)t)[D_H] + (b_0 - (m+2)t)[D_\infty]$, and so

$$b = b(t) = b_0 - (m+2)t\,. \tag{2.14}$$

It is straightforward to show that equation (2.12) admits a smooth solution u satisfying the Calabi ansatz as long as the Kähler–Ricci flow admits a smooth solution, by comparing u to the solution of the Monge–Ampère flow associated to the Kähler–Ricci flow.

Next, the evolution equations for u' and u'' are given by

$$\frac{\partial u'}{\partial t} = \frac{u'''}{u''} + \frac{mu''}{u'} + \frac{nu''}{a+u'} - (m+1)\,, \tag{2.15}$$

$$\frac{\partial u''}{\partial t} = \frac{u^{(4)}}{u''} - \frac{(u''')^2}{(u'')^2} + \frac{mu'''}{u'} - \frac{m(u'')^2}{(u')^2} + \frac{nu'''}{a+u'} - \frac{n(u'')^2}{(a+u')^2}\,, \tag{2.16}$$

as can be seen from differentiating (2.12).

3 Small Contractions by the Kähler–Ricci Flow

The first singular time of the Kähler–Ricci flow on $X_{m,n}$ is given by

$$T = \sup\big\{t > 0 \mid [\omega_0] + t[K_{X_{m,n}}] > 0\big\}\,. \tag{3.1}$$

Since $X_{m,n}$ is not a minimal model, $T < \infty$.

In the section, we assume that at the singular time T, $a(T) = 0$ and $b(T) > 0$, i.e. the Kähler–Ricci flow does not collapse. This is equivalent to

$$n > m\,, \quad \frac{b_0}{m+2} > \frac{a_0}{n-m}\,.$$

In this case, the first singular time of the flow is given by $T = \frac{a_0}{n-m}$.

Let us first explicitly write down the contraction map $\Phi_{m,n}$. In local trivialization for $X_{m,n}$, $\{1, \xi^\alpha, z_i\xi^\alpha\}_{i=1,\dots,n,\alpha=1,\dots,m+1}$ extend to global holomorphic sections in $[D_\infty]$, furthermore, they span $H^0(X_{m,n}, \mathcal{O}([D_\infty]))$. Then the free linear system of $|[D_\infty]|$ induces the following morphism

$$\Phi_{m,n} : (z_i, \xi^\alpha) \in X_{m,n} \to [1, \xi^\alpha, z_i\xi^\alpha] \in \mathbb{P}^{(m+1)(n+1)}.$$

The pullback of the Fubini–Study metric is given by

$$\begin{aligned}\hat{\omega} &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Big(1+\sum_{\alpha=1}^{m+1}|\xi^\alpha|^2+\sum_{1\le i\le n, 1\le\alpha\le m+1}|z_i\xi^\alpha|^2\Big)\\ &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Big(1+\Big(1+\sum_{i=1}^{n}|z_i|^2\Big)\Big(\sum_{\alpha=1}^{m+1}|\xi^\alpha|^2\Big)\Big)\\ &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(1+e^\rho)\,.\end{aligned}$$

Let

$$\hat{u}(\rho)=\log(1+e^\rho)\,.$$

Then $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\hat{u}$ extends to the pullback of the Fubini–Study metric $\hat{\omega}$ given by $\Phi_{m,n}$. In particular, $Y_{m,n}$ has an isolated cone singularity and $\hat{\omega}$ is a asymptotically cone metric on $Y_{m,n}$ near the cone singularity.

Now we list some well-known results for some useful uniform estimates. We begin by rewriting the Kähler–Ricci flow as a parabolic flow of Monge–Ampere type. We let ω_0 be the initial Kähler metric and Ω a smooth volume form on $X_{m,n}$. Let $\chi=\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Omega$ and $\omega_t=\omega_0+t\chi\in[\omega_0]+t[K_{X_{m,n}}]$ be the reference form. Then the Kähler–Ricci flow is equivalent to

$$\frac{\partial\varphi}{\partial t}=\log\frac{\big(\omega_t+\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi\big)^{m+n+1}}{\Omega}\,,\quad \varphi|_{t=0}=0\,. \tag{3.2}$$

Then there exists a unique solution $\varphi\in C^\infty([0,T)\times X_{m,n})$. Furthermore, we have the following well-known estimates due to [TZ]:

1. There exists $C>0$ such that on $[0,T)\times X_{m,n}$,
$$|\varphi|\le C\,. \tag{3.3}$$
2. There exists $C>0$ such that on $[0,T)\times X_{m,n}$,
$$\Big(\omega_t+\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi\Big)^{m+n+1}\le C\Omega\,. \tag{3.4}$$
3. For any $K\subset\subset X_{m,n}\setminus P_0$, there exists for each $k=0,1,2,\ldots$, a constant $C_{k,K}>0$ such that on $[0,T)\times X_{m,n}$,
$$\|\varphi\|_{C^k([0,T)\times K)}\le C_{k,K}\,. \tag{3.5}$$

We also have the following estimates as the parabolic Schwarz lemma.

LEMMA 3.1. *There exists $C>0$ such that on $[0,T)\times X_{m,n}$,*

$$\omega\ge C\hat{\omega}\,.$$

Proof. The proof is given in [SoW1], [So] and makes use of the L^∞-estimate of φ. □

By comparing $\omega(t)$ and $\hat{\omega}$, we immediately have the following estimate.

COROLLARY 3.1. *There exists $C>0$ such that on $[0,T)\times X_{m,n}$,*

$$a+u'\ge C\hat{u}'=\frac{Ce^\rho}{1+e^\rho}\,.$$

LEMMA 3.2. *There exists $C > 0$ such that on $[0,T) \times X_{m,n}$,*

$$\big((u')^{m+n+1}\big)' \le \frac{Ce^{(m+1)\rho}}{(1+e^\rho)^{m+2}}. \tag{3.6}$$

Proof. The inequality (3.6) follows immediately from the volume estimate (3.4) by the following observation:

$$\big((u')^{m+n+1}\big)' \le (m+n+1)(a+u')^n(u')^m u'' \le C_1(1+\hat{u}')^n(\hat{u}')^m\hat{u}'' \le \frac{C_2e^{(m+1)\rho}}{(1+e^\rho)^{m+2}}.$$

The last inequality follows from the definition of $\hat{u}$. □

COROLLARY 3.2. *There exists $C > 0$ such that on $[0,T) \times X_{m,n}$,*

$$u'(\rho) \le Ce^{\frac{m+1}{m+n+1}\rho}. \tag{3.7}$$

Proof. Using Proposition 2.1 and integrating (3.6) from $-\infty$ to ρ, we have

$$\big(u'(\rho)\big)^{m+n+1} \le C\int_{-\infty}^{\rho} e^{(m+1)\rho}d\rho + \lim_{\rho\to-\infty}\big(u'(\rho)\big)^{m+n+1} = \frac{C}{m+1}e^{(m+1)\rho}.$$

□

We also notice that $0 < u'(\rho) < b(t)$ for $\rho \in (-\infty,\infty)$ by (2.7) because $\omega \in a(t)[D_H] + b(t)[D_\infty]$. Therefore, u' is uniformly bounded above for $t \in [0,T)$.

PROPOSITION 3.1. *There exists $C > 0$ such that on $[0,T) \times X_{m,n}$,*

$$u'' \le Cu'. \tag{3.8}$$

Proof. Let $H = \log u'' - \log u'$. Notice that by Proposition 2.1, for fixed $t \in [0,T)$ and near $\rho = -\infty$, $u(\rho) = U_0(e^\rho)$ for some smooth function U_0, and near $\rho = \infty$, $u(\rho) = U_\infty(e^{-\rho}) + b\rho$ for some smooth function U_∞ and $b > 0$.

$$\lim_{\rho\to-\infty}\frac{u''}{u'} = \lim_{\rho\to-\infty}\frac{(U_0(e^\rho))''}{(U_0(e^\rho))'} = \lim_{\rho\to-\infty}\left(\frac{U_0' + e^\rho U_0''}{U_0'}\right) = 1 + \lim_{\rho\to-\infty}e^\rho\frac{U_0''}{U_0'} = 1,$$

$$\lim_{\rho\to\infty}\frac{u''}{u'} = \lim_{\rho\to\infty}\frac{(U_\infty(e^{-\rho})+b\rho)''}{(U_\infty(e^{-\rho})+b\rho)'} = \lim_{\rho\to\infty}\frac{e^{-\rho}U_\infty' + e^{-2\rho}U_\infty''}{-e^{-\rho}U_\infty' + b} = 0.$$

And so we can apply maximum principle for H in $[0,T) \times (-\infty,\infty)$.

$$\frac{\partial H}{\partial t} = \frac{1}{u''}\left\{\frac{u^{(4)}}{u''} - \frac{(u''')^2}{(u'')^2} + \frac{mu'''}{u'} - \frac{m(u'')^2}{(u')^2} + \frac{nu'''}{a+u'} - \frac{n(u'')^2}{(a+u')^2}\right\}$$
$$- \frac{1}{u'}\left\{\frac{u'''}{u''} + \frac{mu''}{u'} + \frac{nu''}{a+u'} - (m+1)\right\}.$$

Suppose that $H(t_0,\rho_0) = \sup_{[0,t_0]\times(-\infty,\infty)} H(t,\rho)$ is achieved for some $t_0 \in (0,T)$, $\rho_0 \in (-\infty,\infty)$. At (t_0,ρ_0), we have

$$H' = \frac{u'''}{u''} - \frac{u''}{u'} = 0$$

and

$$H'' = \frac{u^{(4)}}{u''} - \frac{(u''')^2}{(u'')^2} - \frac{u'''}{u'} + \frac{(u'')^2}{(u')^2} = \frac{u^{(4)}}{u''} - \frac{u'''}{u'} \leq 0\,.$$

Then at (t_0, ρ_0),

$$\begin{aligned}
0 &\leq \frac{\partial H}{\partial t} \\
&= \frac{1}{u''}\left\{\frac{u^{(4)}}{u''} - \frac{(u''')^2}{(u'')^2} + \frac{mu'''}{u'} - \frac{m(u'')^2}{(u')^2}\right\} - \frac{1}{u'}\left\{\frac{u'''}{u''} + \frac{mu''}{u'} - (m+1)\right\} \\
&\quad + \frac{n}{a+u'}\left\{\frac{u'''}{u''} - \frac{u''}{a+u'} - \frac{u''}{u'}\right\} \\
&\leq -\frac{(m+1)u''}{(u')^2} + \frac{m+1}{u'} - \frac{nu''}{(a+u')^2} \\
&\leq \frac{m+1}{u'}(1-e^H)\,.
\end{aligned}$$

Therefore, by the maximum principle, $H(t_0, x_0) \leq 0$ and so

$$\sup_{[0,T)\times(-\infty,\infty)} H(t,\rho) \leq \sup_{(-\infty,\infty)} H(0,\rho) < \infty\,.$$

The proposition then follows. □

We have the following immediate corollary by combining Proposition 3.1 and Corollary 3.2.

Corollary 3.3. *There exists $C > 0$ such that on $[0,T) \times X_{m,n}$,*

$$u''(\rho) \leq Ce^{\frac{m+1}{m+n+1}\rho}\,.$$

Corollary 3.4. *There exists $C > 0$ such that on $[0,T) \times X_{m,n}$,*

$$\omega \leq C\big(\omega_{FS} + \hat{\omega} + e^{-\frac{n}{m+n+1}\rho}\hat{\omega}\big)\,. \tag{3.9}$$

Proof. The corollary holds for $t \in [0,T)$ and $\rho \in [0,\infty)$ from the estimates in (3.5) away from P_0 since $\omega_{FS} + \hat{\omega}$ is a smooth Kähler metric on $X_{m,n}$. It suffices to prove the corollary for $\rho \leq 0$.

Applying Corollary 3.2 and Corollary 3.3, for $(t,\rho) \in [0,T) \times (-\infty, 0]$, we have

$$\begin{aligned}
\omega &\leq C_1\big(1 + e^{\frac{m+1}{m+n+1}\rho}\big)\omega_{FS} + C_1 h_\xi e^{-\frac{n}{m+n+1}\rho}(\delta_{\alpha\beta} + h_\xi e^{-\rho}\xi^{\bar{\alpha}}\xi^\beta)\frac{\sqrt{-1}}{2\pi}\nabla\xi^\alpha \wedge \overline{\nabla\xi^\beta} \\
&\leq C_2\omega_{FS} + C_2 h_\xi e^{-\frac{n}{m+n+1}\rho}\sum_{\alpha=1}^{m+1}\frac{\sqrt{-1}}{2\pi}\nabla\xi^\alpha \wedge \overline{\nabla\xi^\alpha}\,.
\end{aligned}$$

The corollary follows by comparing the above estimates to

$$\begin{aligned}
\hat{\omega} &= \frac{e^\rho}{1+e^\rho}\omega_{FS} + \frac{h_\xi}{1+e^\rho}\left(\delta_{\alpha\beta} - \frac{h_\xi}{1+e^\rho}\xi^{\bar{\alpha}}\xi^\beta\right)\frac{\sqrt{-1}}{2\pi}\nabla\xi^\alpha \wedge \overline{\nabla\xi^\beta} \\
&\geq C_3 e^\rho\omega_{FS} + C_3 h_\xi\frac{\sqrt{-1}}{2\pi}\nabla\xi^\alpha \wedge \overline{\nabla\xi^\alpha}
\end{aligned}$$

for $\rho \leq 0$ and some $C_3 > 0$. □

Let $\omega(T)=\lim_{t\to T^-}\omega(t)$ be the closed positive $(1,1)$-form with bounded local potentials. Then by the estimates of $\omega(t)$ away from P_0 as in (3.5), $\omega(T)$ is a smooth Kähler metric on $X_{m,n}\setminus P_0$ and $\omega(t)$ converges in $C^\infty(X_{m,n}\setminus P_0)$ to $\omega(T)$ as $t\to T$.

Theorem 3.1. *Let (X_T,d_T) be the metric completion of $(X_{m,n}\setminus P_0,\omega(T))$. Then (X_T,d_T) has finite diameter and is homeomorphic to $Y_{m,n}$ as the projective cone in $\mathbb{P}^{(m+1)(n+1)}$ over $\mathbb{P}^m\times\mathbb{P}^n$ via the Segre map. Furthermore, $(X_{m,n},g(t))$ converges to (X_T,d_T) in the Gromov–Hausdorff sense as $t\to T^-$, and there exists $C>0$ such that, for $t\in[0,T)$,*

$$\operatorname{diam}\big(X_{m,n},g(t)\big)\le C\,.$$

Proof. Let $U_\kappa=\{e^\rho\le\kappa\}$ be a κ-tubular neighborhood of the zero section P_0. We will use local coordinates (z_i,ξ_α) for $i=1,\dots,n$ and $\alpha=1,\dots,m+1$. For any fixed fibre $X_z=(\pi_{m,n})^{-1}(z)$ for $z=(z_1,\dots,z_n)\in\mathbb{C}^n$, there exists $C_1>0$, such that the restriction of the evolving metric is bounded by

$$\begin{aligned}\omega|_{X_z}&=\frac{\sqrt{-1}}{2\pi}u''e^{-2\rho}\partial e^\rho\wedge\bar\partial e^\rho+u'e^{-\rho}\frac{\sqrt{-1}}{2\pi}\partial\bar\partial e^\rho-\frac{\sqrt{-1}}{2\pi}u'e^{-2\rho}\partial e^\rho\wedge\bar\partial e^\rho\\&\le\frac{\sqrt{-1}}{2\pi}u''e^{-2\rho}\partial e^\rho\wedge\bar\partial e^\rho+u'e^{-\rho}\frac{\sqrt{-1}}{2\pi}\partial\bar\partial e^\rho\\&\le C_1h_\xi e^{-\frac{n}{m+n+1}\rho}\sum_\alpha\frac{\sqrt{-1}}{2\pi}d\xi^\alpha\wedge d\xi^{\bar\alpha}.\end{aligned}$$

We first show that for any $\epsilon>0$, there exists $\kappa_\epsilon>0$ such that for any $z\in\mathbb{C}^n$, $\kappa<\kappa_\epsilon$ and $t\in[0,T)$,

$$\operatorname{diam}\big(X_z\cap U_\kappa,g(t)\big)<\epsilon\,.$$

- We begin with estimates in the radial direction. We can always assume $\rho\le0$. For any point $\xi\in X_z$, we consider the radial line segment $\gamma(r)=r\xi$ joining 0 and ξ in $\mathbb{C}^{m+1}$ for $0\le r\le1$. Note that $e^\rho=(1+|z|^2)|\xi|^2$, then the arc length of γ is given by

$$\begin{aligned}|\gamma|_{g(t)}&\le C_2\int_0^{|\xi|}e^{-\frac{n}{2(m+n+1)}\rho}\big(1+|z|^2\big)^{1/2}dr\\&=C_2\int_0^{|\xi|}\big(1+|z|^2\big)^{\frac{m+1}{2(m+n+1)}}r^{-\frac{n}{m+n+1}}dr\\&\le C_3\big\{(1+|z|^2)|\xi|^2\big\}^{\frac{m+1}{2(m+n+1)}}\\&\le C_3\ \kappa^{\frac{m+1}{2(m+n+1)}}\end{aligned}$$

  for some fixed constant $C_i>0$, $i=2,3$.
- We now consider the behavior of $g(t)$ on $S_{|\xi|}$, the sphere centered at $\xi=0$ with radius $r_\xi=|\xi|$ in $\mathbb{C}^{m+1}$ with respect to the Euclidean metric. Let $g_{S_{2m+1}}$ be the standard metric on the unit sphere S_{2m+1} in $\mathbb{C}^{m+1}$. If $S_{|\xi|}\subset X_z\cap U_\kappa$, then there exist $C_4>0$ such that

$$\begin{aligned}g(t)|_{S_{|\xi|}}&\le\sqrt{-1}C_1e^{-\frac{n}{m+n+1}\rho}\big(1+|z|^2\big)d\xi\wedge d\bar\xi|_{S_{|\xi|}}\\&\le C_4e^{-\frac{n}{m+n+1}\rho}\big(1+|z|^2\big)|\xi|^2g_{S_{2m+1}}\end{aligned}$$

$$= C_4 e^{\frac{m+1}{m+n+1}\rho} g_{S_{2m+1}}$$
$$= C_4 \kappa^{\frac{m+1}{m+n+1}} g_{S_{2m+1}} .$$

Combining the above estimates, any two points in $X_z \cap U_\kappa$ can be connected with by a piecewise smooth curve in $X_z \cap U_\kappa$ with arbitrarily small arc length if κ is chosen sufficiently small.

Now we consider points $(0,\xi)$ and $(w,\xi) \in U_\kappa$ and we can assume $\xi = (\xi^1, 0, \ldots, 0)$ after the unitary transformation. We will then consider a straight line segment

$$\gamma(s) = \big\{(z,\xi) \mid z = sw,\ \xi = (\xi^1, 0, \ldots, 0)\big\} .$$

There exists $C_5 > 0$, such that the restriction of $g(t)$ on the submanifold $V = \{\xi = (\xi^1, 0, 0, \ldots, 0)\}$ is bounded by

$$\begin{aligned}
g(t)|_{V\cap U_\kappa} &= (a+u')\omega_{FS} + \frac{\sqrt{-1}}{2\pi} u'' \frac{\bar{z}_i z_j}{(1+|z|^2)^2} \sum_{i,j} dz_i \wedge dz_{\bar{j}} \\
&\le C_5\big(a + e^{\frac{m+1}{m+n+1}\rho}\big)\omega_{FS} + \sqrt{-1} C_5 e^{\frac{m+1}{m+n+1}\rho} \frac{\bar{z}_i z_j}{(1+|z|^2)^2} \sum_{i,j} dz_i \wedge dz_{\bar{j}} \\
&\le C_5\big(a_0 - (n-m)t + \kappa^{\frac{m+1}{m+n+1}}\big)\omega_{FS} + \sqrt{-1} C_5 e^{\frac{m+1}{m+n+1}\rho} \frac{\bar{z}_i z_j}{(1+|z|^2)^2} \sum_{i,j} dz_i \wedge dz_{\bar{j}} .
\end{aligned}$$

Therefore, there exist $C_6, C_7 > 0$ such that the arc length of $\gamma(s)$ for $0 \le s \le 1$ is bounded by

$$\begin{aligned}
|\gamma|_{g(t)} &\le C_6 \int_0^1 e^{\frac{m+1}{2(m+n+1)}\rho} \frac{s|w|^2}{1+|z|^2} ds + C_6\big(a_0 - (n-m)t + \kappa^{\frac{m+1}{m+n+1}}\big)^{1/2} \\
&\le C_6 \int_0^1 \big(1+|z|^2\big)^{\frac{m+1}{2(m+n+1)}-1} |\xi|^{\frac{m+1}{m+n+1}} s|w|^2 ds + C_6\big(a_0 - (n-m)t + \kappa^{\frac{m+1}{m+n+1}}\big)^{1/2} \\
&\le C_7 \big\{(1+|w|^2)|\xi|^2\big\}^{\frac{m+1}{2(m+n+1)}} + C_7\big(a_0 - (n-m)t + \kappa^{\frac{m+1}{m+n+1}}\big)^{1/2} \\
&\le C_7 \kappa^{\frac{m+1}{2(m+n+1)}} + C_7\big(a_0 - (n-m)t + \kappa^{\frac{m+1}{m+n+1}}\big)^{1/2}.
\end{aligned}$$

In general, given two points (z,ξ) and $(z',\xi') \in U_\kappa$, we can assume $|\xi| \le |\xi'|$ without loss of generality. Let $\hat{\xi} = (\xi^1, 0, \ldots, 0)$ such that $|\hat{\xi}| = |\xi|$.

$$\begin{aligned}
&\mathrm{dist}_{g(t)}\big((z,\xi),(z',\xi')\big) \\
&\quad\le \mathrm{dist}_{g(t)}\big((z,\xi),(z,\hat{\xi})\big) + \mathrm{dist}_{g(t)}\big((z',\xi'),(z',\hat{\xi})\big) \\
&\qquad + \mathrm{dist}_{g(t)}\big((0,\hat{\xi}),(z',\hat{\xi})\big) + \mathrm{dist}_{g(t)}\big((z,\hat{\xi}),(0,\hat{\xi})\big) \\
&\quad\le C_8 \kappa^{\frac{n+1}{2(m+n+1)}} + C_8\big(a_0 - (n-m)t\big)^{1/2}
\end{aligned}$$

for $C_8 > 0$. Hence for any $\epsilon > 0$, there exist $\kappa_\epsilon > 0$ and $T_\epsilon \in (0,T)$ such that for any $0 < \kappa < \kappa_\epsilon$ and $t \in (T_\epsilon, T)$,

$$\mathrm{diam}(U_\kappa \setminus P_0, g(t)) < \epsilon .$$

This shows that $\mathrm{diam}(X_{m,n}, g(t))$ is uniformly bounded above for $t \in [0,T)$. Similar arguments show that the metric completion (X_T, d_T) of $(X_{m,n} \setminus P_0, \omega(T))$ is compact and is homeomorphic to $(Y_{m,n}, \hat{g})$ as a metric space after replacing

$a = a_0 - (m-n)t$ by 0. Standard argument shows that $(X_{m,n}, g(t))$ converges to (X_T, d_T) in the Gromov–Hausdorff sense as $t \to T$ (cf. [SoW1]). □

4 Finite Time Collapsing

4.1 The case $m \geq n$, or $m < n$ and $\frac{b_0}{m+2} < \frac{a_0}{n-m}$. In this section, we consider the Kähler–Ricci flow on $X_{m,n}$ with the initial class $a_0[D_H] + b_0[D_\infty]$ such that

$$m \geq n$$

or

$$n > m\,, \quad \frac{b_0}{m+2} < \frac{a_0}{n-m}\,.$$

The first singular time of the flow is given by

$$T = \frac{b_0}{m+2}\,. \tag{4.1}$$

The following lemma is an immediate consequence of the observation (2.8).

LEMMA 4.1. *For $t \in [0,T)$ and $\rho \in (-\infty, \infty)$, we have*

$$0 < u' < b = b_0 - (m+2)t = (m+2)(T-t)\,. \tag{4.2}$$

The general volume estimate (3.4) gives us the upper bound for the volume form.

LEMMA 4.2. *There exists $C > 0$ such that on $[0,T) \times X_{m,n}$,*

$$\omega(t)^{m+n+1} \leq C\Omega\,.$$

COROLLARY 4.1. *There exists $C > 0$ such that for $\rho \in (-\infty, \infty)$ and $t \in [0,T)$,*

$$0 < u' \leq C \min(T-t, e^\rho) \tag{4.3}$$

and

$$0 < b - u' \leq C e^{-\frac{1}{m+1}\rho}. \tag{4.4}$$

Proof. We apply similar arguments as in Corollary 3.2.

- By Lemma 4.2,
$$\left[(u')^{m+1}\right]' \leq C_1 (a+u')^n (u')^m u'' \leq C_2 (1+\hat{u}')^n (\hat{u}')^m \hat{u}''.$$
For $\rho \in (-\infty, \infty)$, integrating the above inequality from $-\infty$ to ρ, we have
$$(u')^{m+1}(\rho) \leq C_3 \int_{-\infty}^{\rho} e^{(m+1)\rho} d\rho \leq C_4 e^{(m+1)\rho}.$$
The estimate (4.3) follows by combining Lemma 4.1.
- We also have
$$\left[(u')^{m+1}\right]' \leq C_5 (a+u')^n (u')^m u'' \leq C_6 (1+\hat{u}')^n (\hat{u}')^m \hat{u}'' \leq C_7 e^{-\rho}.$$
Then after integrating the above inequality from ρ to ∞, we have
$$b^{m+1} - (u')^{m+1}(\rho) \leq C_8 \int_{\rho}^{\infty} e^{-\rho} d\rho \leq C_9 e^{-\rho}.$$

The estimate (4.4) follows immediately from the elementary inequality

$$(A-B)^p \le A^p - B^p$$

for $A \ge B \ge 0$ and $p \in \mathbb{Z}^+$. □

PROPOSITION 4.1. *There exists $C>0$ such that on $[0,T)\times(-\infty,\infty)$,*

$$u'' \le C\min\{u', b_t - u'\}\,. \tag{4.5}$$

Proof. The same argument as in Proposition 3.1 can be applied to show that u''/u' is uniformly bounded above on $[0,T)\times(-\infty,\infty)$.

Let $H = \log\{u''/(b-u')\}$. Then $\lim_{\rho\to-\infty} H = -\infty$ and $\lim_{\rho\to\infty} H = 0$. The evolution equation for H is given as follows.

$$\frac{\partial H}{\partial t} = \frac{1}{u''}\left\{\frac{u^{(4)}}{u''} - \frac{(u''')^2}{(u'')^2} + \frac{mu'''}{u'} - \frac{m(u'')^2}{(u')^2} + \frac{nu'''}{a+u'} - \frac{n(u'')^2}{(a+u')^2}\right\} + \frac{1}{b-u'}\left\{\frac{u'''}{u''} + \frac{mu''}{u'} + \frac{nu''}{a+u'} - (m+1)\right\}.$$

We also have

$$H' = \frac{u'''}{u''} + \frac{u''}{b-u'}$$

and

$$H'' = \frac{u^{(4)}}{u''} - \frac{(u''')^2}{(u'')^2} + \frac{u''}{b-u'}H'.$$

Suppose $\sup_{[0,t_0)\times(-\infty,\infty)} H = H(t_0,\rho)$. Then at (t_0,ρ_0), straightforward calculations show that

$$0 \le \frac{\partial H}{\partial t} < 0\,,$$

which is a contradiction. Thus

$$\sup_{[0,T)\times(-\infty,\infty)} H \le \sup_{\{0\}\times(-\infty,\infty)} H \le C\,.$$

□

We then have the following immediate corollary.

PROPOSITION 4.2. *There exists $C>0$ such that on $[0,T)\times(-\infty,\infty)$,*

$$u'' \le C\min(T-t, e^{\rho}, e^{-\rho})\,. \tag{4.6}$$

Proof. It suffices to prove that $e^{\rho}u''$ is bounded above by the previous lemma. Let $H_\gamma = e^{-t}e^{\gamma\rho}u''$ for $\gamma\in(0,1)$. Then the evolution of H_γ is given by

$$\frac{\partial}{\partial t}H_\gamma = e^{-t}e^{\gamma\rho}\left(\frac{u^{(4)}}{u''} - \frac{(u''')^2}{(u'')^2} + \frac{mu'''}{u'} - \frac{m(u'')^2}{(u')^2} + \frac{nu'''}{a+u'} - \frac{n(u'')^2}{(a+u')^2}\right) - H_\gamma\,.$$

Also for any $t\in[0,T)$,

$$\lim_{|\rho|\to\infty} H_\gamma(\rho) = 0\,.$$

Suppose $H_\gamma(t_0,\rho_0) = \sup_{[0,t_0]\times(-\infty,\infty)} H_\gamma(t,\rho)$. Then at (t_0,ρ_0), we have

$$u^{(4)} \le -\gamma u''', \quad u''' = -\gamma u'',$$

by the maximum principle, and then

$$0 \le \frac{\partial}{\partial t} H_\gamma \le -H_\gamma \,.$$

Hence $H_\gamma \le \sup_{\rho\in(-\infty,\infty)} H_\gamma(0,\rho)$ and there exists $C>0$ such that for $t\in[0,T)$ and $\gamma\in(0,1)$, $H_\gamma \le C$. By letting $\gamma\to 1$, we can uniformly bound $e^{-t}e^{\rho}u''$ on $[0,T)\times(-\infty,\infty)$ from above and the lemma follows. □

We then obtain uniform bounds for the evolving metrics from the upper bound on u' and u''.

Corollary 4.2. *There exists $C>0$ such that on $[0,T)\times X_{m,n}$,*

$$a(t)\omega_{FS} \le \omega(t) \le \big(a(t)+C(T-t)\big)\omega_{FS} + C\min\big\{(T-t)(e^{-\rho}+e^{\rho})\hat\omega, \hat\omega\big\}\,. \quad (4.7)$$

Proof. It suffices to compare u', u'' with $\hat u' = \frac{e^\rho}{1+e^\rho}$, $\hat u'' = \frac{e^\rho}{(1+e^\rho)^2}$. Notice that

$$T-t \le C_1(T-t)(e^\rho+e^{-\rho})\frac{e^\rho}{1+e^\rho} = C_1(T-t)(e^\rho+e^{-\rho})\hat u',$$

$$T-t \le C_2(T-t)(e^\rho+e^{-\rho})\frac{e^\rho}{(1+e^\rho)^2} = C_2(T-t)(e^\rho+e^{-\rho})\hat u'',$$

$$e^\rho \le C_3\frac{e^\rho}{1+e^\rho} = C_3\hat u' \quad \text{when} \quad \rho\to-\infty\,,$$

$$e^\rho \le C_4\frac{e^\rho}{(1+e^\rho)^2} = C_4\hat u'' \quad \text{when} \quad \rho\to-\infty\,,$$

and

$$e^{-\rho} \le C_5\frac{e^\rho}{(1+e^\rho)^2} = C_5\hat u'' \;\; \text{when} \;\; \rho\to\infty\,.$$

The corollary follows from Corollary 4.1 and Proposition 4.2. □

Proposition 4.3. *For any $\epsilon>0$, there exists $T_\epsilon\in(0,T)$ such that for $t\in(T_\epsilon,T)$ and any fibre X_z with $z\in\mathbb{P}^n$,*

$$\operatorname{diam}\big(X_z, g(t)|_{X_z}\big) < \epsilon\,.$$

Proof. We consider the following open set $V_\kappa\subset X_{m,n}$ for $\kappa>0$ defined by

$$V_\kappa = \{\kappa^{-1} < e^\rho < \kappa\}\,.$$

Since $\omega|_{X_z} \le C\hat\omega|_{X_z}$, for any $\epsilon>0$, there exists $\kappa_\epsilon>0$ such that for all $t\in[0,T)$ and $\kappa>\kappa_\epsilon$,

$$\operatorname{diam}\big(X_z\cap(X\setminus V_\kappa), g(t)\big) < \epsilon/2$$

by a similar argument in the proof of Theorem 3.1. On the other hand, in $V_{2\kappa_\epsilon}$, $\omega|_{X_z} \le C(T-t)\hat\omega|_{X_z}$. Then there exists $T_\epsilon<T$ such that

$$\operatorname{diam}\big(X_z\cap V_{2\kappa_\epsilon}, g(t)\big) < \epsilon/2\,.$$

The proposition then follows easily. □

Theorem 4.1. *There exists $C > 0$ such that for $t \in [0, T)$,*

$$\operatorname{diam}\big(X_{m,n}, g(t)\big) \le C\,.$$

Furthermore, $(X_{m,n}, g(t))$ converges to $\big(\mathbb{P}^n, \big(a_0 - \frac{n-m}{m+2} b_0\big)\omega_{FS}\big)$ in the Gromov–Hausdorff sense as $t \to T$.

Proof. Let $V_\kappa = \{\kappa^{-1} \le e^\rho \le \kappa\}$ for $\kappa > 0$. From the calculation above, there exists $C > 0$ such that on V_κ,

$$a\omega_{FS} \le \omega(t) \le a\omega_{FS} + C(T-t)\omega_{FS} + C_\kappa(T-t)\hat{\omega}\,,$$

and so $\omega(t)$ converges to ω_{FS} uniformly in $C^0(V_{\kappa_1})$ as $t \to T$. On the other hand, the diameter of any fibre X_z for $z \in \mathbb{P}^n$ tends to 0 uniformly as $t \to T$.

We now choose a smooth map $\sigma : \mathbb{P}^m \to X_{m,n}$ such that the image of σ sits in the interior of V_1. Then the theorem follows by a similar argument in the proof of Theorem 5.1 in [SoW1]. □

4.2 The case $m < n$ and $\frac{b_0}{m+2} = \frac{a_0}{n-m}$. In this case, $X_{m,n}$ is Fano and the initial Kähler class is proportional to $c_1(X_{m,n})$. The first singular time of the Kähler–Ricci flow is $T = \frac{b_0}{m+2} = \frac{a_0}{n-m}$. By Perelman's diameter estimates, we have

$$\operatorname{diam}\big(X_{m,n}, g(t)\big) \le C(T-t)$$

for a constant $C > 0$ and so the flow becomes extinct at $t = T$.

5 Resolution of Singularities by the Kähler–Ricci Flow

5.1 Resolution by the Fubini–Study metric and its Ricci curvature. Consider the morphism $\Phi_{m,n} : X_{m,n} \to \mathbb{P}^{(m+1)(n+1)}$ as defined in (2.2). We assume that $m, n \ge 1$. The restriction of the Fubini–Study metric on $Y_{m,n}$, the image of $\Phi_{m,n}$, is given by

$$\begin{aligned}\hat{\omega} &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(1+e^\rho)\\ &= \frac{e^\rho}{1+e^\rho}\omega_{FS} + h_\xi\left(\frac{1}{1+e^\rho}\delta_{\alpha\beta} - \frac{1}{(1+e^\rho)^2}h_\xi\xi^{\bar{\alpha}}\xi^\beta\right)\frac{\sqrt{-1}}{2\pi}\nabla\xi^\alpha \wedge \overline{\nabla\xi^\beta}.\end{aligned}$$

Its induced volume from on $Y_{m,n}$ is given by

$$\hat{\omega}^{m+n+1} = h_\xi^{m+1}\frac{e^{n\rho}}{(1+e^\rho)^{m+n+2}}\left(\omega_{FS}^n \wedge \prod_{\alpha=1}^{m+1}\frac{\sqrt{-1}}{2\pi}d\xi^\alpha \wedge d\xi^{\bar{\alpha}}\right).$$

We can now calculate the Ricci form.

$$\begin{aligned}-\operatorname{Ric}(\hat{\omega}) &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\big(n\rho - (m+n+2)\log(1+e^\rho)\big) + (m-n)\omega_{FS}\\ &= (m-n)\omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\hat{u}_{\mathrm{Ric}}\end{aligned}$$

$$= \left(m - \frac{(m+n+2)e^\rho}{1+e^\rho}\right)\omega_{FS} + h_\xi e^{-\rho}\Bigg\{\left(n - \frac{(m+n+2)e^\rho}{1+e^\rho}\right)\delta_{\alpha\beta}$$
$$+ e^{-\rho}\left(\frac{(m+n+2)e^{2\rho}}{(1+e^\rho)^2} - n\right) h_\xi \xi^{\bar\alpha}\xi^\beta)\Bigg\}\frac{\sqrt{-1}}{2\pi}\nabla\xi^\alpha \wedge \overline{\nabla\xi^\beta},$$

where $\hat u_{\mathrm{Ric}} = n\rho - (m+n+2)\log(1+e^\rho)$.

Let O be the vertex of $Y_{m,n}$ as the projective cone over $\mathbb{P}^m \times \mathbb{P}^n$. We define $(Y_{m,n})_{\mathrm{reg}} = Y_{m,n} \setminus \{O\}$ as the nonsingular part of $Y_{m,n}$.

Lemma 5.1. *We define $\hat\omega_\epsilon = \hat\omega - \epsilon Ric(\hat\omega)$. Then there exists $\epsilon_0 > 0$, such that $\hat\omega_\epsilon > 0$ on $(Y_{m,n})_{\mathrm{reg}}$, the nonsingular part of of $Y_{m,n}$ for $\epsilon \in (0, \epsilon_0)$.*

Proof. It suffices to check for $\rho \le 0$ because $\hat\omega$ is Kähler on $Y_{m,n} \setminus \{O\}$ and $Ric(\hat\omega)$ is smooth away from O. The calculation is straightforward by assuming $\xi = (|\xi|, 0, \dots, 0)$ after certain $U(m+1)$ transformation. □

Let $\tilde X_{m,n}$ be the blow-up of $X_{m,n}$ along the zero section P_0. Then we have the following commutative diagram from section 1.9 in [D].

$$\begin{array}{ccccc} X_{m,n} & \xleftarrow{\vartheta_1} & \tilde X_{m,n} & \xrightarrow{\vartheta_2} & X_{n,m} \\ \downarrow{\scriptstyle \pi_{m,n}} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \pi_{n,m}} \\ \mathbb{P}^n & \xleftarrow{p_1} & \mathbb{P}^m \times \mathbb{P}^n & \xrightarrow{p_2} & \mathbb{P}^m \end{array} \tag{5.1}$$

Proposition 5.1. *The metric completion of $((Y_{m,n})_{\mathrm{reg}}, \hat\omega_\epsilon)$ is a compact metric space isomorphic to $\tilde X_{m,n}$ for sufficiently small $\epsilon > 0$. In particular, $\hat\omega_\epsilon$ extends to a smooth Kähler metric on $\tilde X_{m,n}$.*

Proof. The potential $\hat u_\epsilon$ of $\hat\omega_\epsilon$ is given by

$$\hat u_\epsilon = \hat u + \epsilon \hat u_{\mathrm{Ric}} = n\epsilon\rho + \big(1 - (m+n+2)\epsilon\big)\log(1+e^\rho)\,.$$

It suffices to compare $\hat\omega_\epsilon$ to a smooth Kähler metric on $\tilde X_{m,n}$. We let

$$\tilde u = a\rho + b\log(1+e^\rho)$$

with $a, b > 0$. In particular, $\tilde u = \hat u_\epsilon$ when $a = n\epsilon$ and $b = 1 - (m+n+2)\epsilon$. Then

$$\tilde\omega = \frac{\sqrt{-1}}{2\pi}\partial\overline\partial\tilde u$$
$$= \left(a + b\frac{e^\rho}{1+e^\rho}\right)\omega_{FS} + \frac{\sqrt{-1}}{2\pi}e^{-\rho}h_\xi\big(\tilde u'\delta_{\alpha\beta} + h_\xi e^{-\rho}(\tilde u'' - \tilde u')\xi^{\bar\alpha}\xi^\beta\big)\nabla\xi^\alpha \wedge \overline{\nabla\xi^\beta}.$$

$\tilde\omega$ restricted on each fibre $\mathbb{P}^{m+1} \cap (X_{m,n} \setminus P_0)$ is give by

$$h_\xi e^{-\rho}\big(\tilde u'\delta_{\alpha\beta} + e^{-\rho}(\tilde u'' - \tilde u')h_\xi\xi^{\bar\alpha}\xi^\beta\big)\frac{\sqrt{-1}}{2\pi}d\xi^\alpha \wedge d\xi^{\bar\beta}$$

whose metric completion is exactly $\mathbb{P}^{m+1}$ blown up at one point. Note that

$$\frac{\sqrt{-1}}{2\pi}\partial\overline\partial\rho = \frac{\sqrt{-1}}{2\pi}\partial\overline\partial\log\big((1+|z|^2)|\xi|^2\big)$$

is exactly the pullback of the product of Fubini–Study metrics on $\mathbb{P}^m \times \mathbb{P}^n$ by Ψ. Therefore, $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\tilde{u}$ blows up along the zero section P_0 and replaces P_0 by the exceptional divisor $E = \mathbb{P}^m \times \mathbb{P}^n$.

Furthermore, $\tilde{\omega}$ lies in a Kähler class $a\Psi^*\big[\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\rho\big] + b(\Phi_{m,n}\circ\vartheta_1)^*[\hat{\omega}]$ on $\tilde{X}_{m,n}$. Also $\hat{\omega}$ is positive on $\tilde{X}_{m,n}\setminus(\mathbb{P}^m\times\mathbb{P}^n)$. Therefore, $a\Psi^*\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\rho + b(\Phi_{m,n}\circ\vartheta_1)^*\hat{\omega}$ defines a smooth Kähler metric on $\tilde{X}_{m,n}$ for a, $b>0$ and the proposition follows by choosing $\epsilon > 0$ sufficiently small. □

For a given projective embedding $X \hookrightarrow \mathbb{P}^N$ of a normal variety X, the Ricci curvature is well defined on X_{reg}, the nonsingular part of X, for the restriction of the Fubini–Study metric ω_{FS}. We consider $\omega_\epsilon = \omega_{FS} - \epsilon\,\mathrm{Ric}(\omega_{FS})$ for sufficiently small $\epsilon > 0$. Let $(\tilde{X}, d_\epsilon)$ be the metric completion of $(X_{\mathrm{reg}}, \omega_\epsilon)$. Then Proposition 5.1 suggests that $\tilde{X}$ is possibly a resolution of singularities for X. However, such a resolution is not necessarily minimal as shown in the example above. This leads us to consider the Kähler–Ricci flow as a certain smoothing process to resolve the singularity of a general normal variety. The goal of the section is to show that indeed the Kähler–Ricci flow gives an optimal resolution of singularities for $Y_{m,n}$.

5.2 The case $m \neq n$. In the section, we will consider the Kähler–Ricci flow on $Y_{m,n}$ with the initial metric $\omega_0 = b_0\hat{\omega}$ for some $b_0 > 0$. Since $Y_{m,n} = Y_{n,m}$, we can assume that $m > n$.

We choose the potential for $\hat{\omega}$ to be

$$\hat{u} = \log(1+e^\rho)\,.$$

Then, the calculation in section 2.3 suggests that the Kähler–Ricci flow should be equivalent to a parabolic PDE for u as below,

$$\frac{\partial u}{\partial t} = \log\big[(a+u')^m(u')^n u''\big] - (n+1)\rho\,,\quad u|_{t=0} = b_0\hat{u}\,, \tag{5.2}$$

with $a(t) = (n-m)t$, or

$$\frac{\partial u}{\partial t} = \log\big[(a+u')^n(u')^m u''\big] - (m+1)\rho\,,\quad u|_{t=0} = b_0\hat{u}\,, \tag{5.3}$$

with $a(t) = (m-n)t$, since $a_0 = 0$.

We have to choose (5.3) because $m > n$ and $a(t)$ should be nonnegative for $t > 0$. This can be seen by the class evolution of the Kähler–Ricci flow because $X_{m,n}$ is the only resolution of $Y_{m,n}$ such that $K_{X_{m,n}}$ is $\mathbb{Q}$-Cartier and the class $[\hat{\omega}]+\epsilon[K_{X_{m,n}}] > 0$ for sufficiently small $\epsilon > 0$. Hence, now we can lift the Kähler–Ricci flow on $Y_{m,n}$ to the one on $X_{m,n}$ starting with $\hat{\omega}$.

Note that $\hat{\omega}$ has bounded local potential and for any smooth Kähler metric ω_0 on $X_{m,n}$, there exists $C > 0$ such that

$$\hat{\omega} \le C\omega_0\,.$$

By [SoT3], $\hat{\omega} \in \mathcal{K}_{[\hat{\omega}],\infty}(X_{m,n})$ (cf. [SoW2]) and there exists a unique weak Kähler–Ricci flow on $X_{m,n}$ starting with $\hat{\omega}$. Furthermore, the solution becomes a smooth Kähler metric on $X_{m,n}$ once $t > 0$. Therefore, it suffices to study the behavior of the solution as $t \to 0^+$.

We first write down the equivalent parabolic flow of Monge–Ampere type for the Kähler–Ricci flow. Since $[\hat\omega] + t[K_{X_{m,n}}] > 0$ for sufficiently small $t > 0$, there exists a smooth volume form Ω with $\chi = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Omega$, such that

$$\omega_t = \hat\omega + t\chi > 0\,,$$

for $t \in (0, T)$, where $T = \sup\{t > 0 \mid [\hat\omega] + t[K_{X_{m,n}}] > 0\}$. Let the solution of the Kähler–Ricci flow be given as $\omega(t) = \omega_t + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi$. Then

$$\frac{\partial\varphi}{\partial t} = \log\frac{\left(\omega_t + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi\right)^{m+n+1}}{\Omega}\,, \quad \varphi|_{t=0} = 0\,. \tag{5.4}$$

It is proved in [SoT3], that $\varphi \in C^\infty((0,T)\times X_{m,n}) \cap C^\infty([0,T)\times(X_{m,n}\setminus P_0))$ and

$$\|\varphi\|_{L^\infty([0,T/2]\times X_{m,n})} < \infty\,.$$

Lemma 5.2. *Then there exists $C > 0$, such that on $[0, T/2]\times X_{m,n}$,*

$$\omega^{m+n+1} \le C\max\left\{1, e^{-(m-n)\rho}\right\}\Omega\,, \tag{5.5}$$

and on $[T/2, T)\times X_{m,n}$,

$$\omega^{m+n+1} \le C\Omega\,.$$

Proof. It suffices to show that the lemma holds on $[0, T/2]\times(-\infty, 0]$, as one can easily obtain the estimate $\omega^{m+n+1} \le C_1\Omega$ away from the zero section P_0 (see [SoT3]), as well as for $t \ge T/2$ (see [SoW1]), for $C_1 > 0$.

Let $v = \frac{\partial u}{\partial t}$. Then the evolution of v is given by

$$\frac{\partial v}{\partial t} = \frac{nv'}{a+u'} + \frac{mv'}{u'} + \frac{v''}{u''} + \frac{n(m-n)}{a+u'}\,. \tag{5.6}$$

Let $H = e^{-t}(v + (m-n)\rho)$. Then by (5.3), $H(0) \le C_2 + m\rho \le C_2$ on $\rho \in (-\infty, 0]$ for $C_2 > 0$. One can calculate the evolution for H,

$$\frac{\partial H}{\partial t} = \frac{nH'}{a+u'} + \frac{mH'}{u'} + \frac{H''}{u''} - H - \frac{m(m-n)e^{-t}}{u'} \le \frac{nH'}{a+u'} + \frac{mH'}{u'} + \frac{H''}{u''} - H. \tag{5.7}$$

One notices that $\lim_{\rho\to-\infty} H(t,\rho) = -\infty$ for any t by Proposition 2.1. Hence the maximum of $H(t,\cdot)$ is achieved away from P_0 for $t \ge 0$. Since the Käher–Ricci flow is smoothly defined away from P_0, it follows from the maximum principle that $H \le C_3$ on $[0, T/2]\times(-\infty, 0]$ for $C_3 > 0$. Therefore, by (5.3) again, there exists $C_4 > 0$, such that

$$(a+u')^n(u')^m u'' = e^{v+(m+1)\rho} = e^{e^t H + (n+1)\rho} \le C_4 e^{(n+1)\rho}.$$

On the other hand, there exists $C_5 > 0$ such that, on $(-\infty, 0]$,

$$(1+\hat u')^n(\hat u')^m\hat u'' \ge C_5 e^{(m+1)\rho}.$$

Combining them, there exists $C_6 > 0$, such that on $(-\infty, 0]$,

$$\omega^{m+n+1} \le Ce^{-(m-n)\rho}\Omega\,.$$

□

Lemma 5.3. *There exists $C > 0$ such that on $[0, T/2] \times X_{m,n}$,*

$$\omega \geq C\hat{\omega}\,. \tag{5.8}$$

Proof. Let θ be a smooth Kähler metric on $X_{m,n}$. We consider the Kähler–Ricci flow on $X_{m,n}$ with initial metric

$$\omega_{\epsilon,0}|_{t=0} = b_0\hat{\omega} + \epsilon\theta\,.$$

Then by the same argument as in [SoW1], [So], we can show that there exists $C > 0$ such that for any $\epsilon \in (0, 1)$, the solution $\omega_\epsilon(t)$ of the Kähler–Ricci flow is bounded below by

$$\omega \geq C\hat{\omega}$$

on $t \in (0, T/2] \times X_{m,n}$. □

Proposition 5.2. *There exist A_1 and $A_2 > 0$ such that for $t \in [0, T/2]$ and $\rho \leq 0$,*

$$u'' \leq A_1 u' \leq A_2\, e^{\frac{n+1}{m+n+1}\rho}\,,$$

Proof. By the volume comparison in Lemma 5.2, there exist $C_1, C_2 > 0$ such that for $t \in [0, T/2)$ and $\rho \leq 0$,

$$(a + u')^m (u')^n u'' \leq C_1 e^{-(m-n)\rho}(1 + \hat{u}')^n (\hat{u}')^m \hat{u}'' \leq C_2 e^{(n+1)\rho}.$$

Applying the same argument in Corollary 3.2, there exists $C_3 > 0$ such that for $t \in [0, T)$ and $\rho \leq 0$,

$$u' \leq C_3 e^{\frac{n+1}{m+n+1}\rho}.$$

Let $H = \log u'' - \log u'$. To prove H is uniformly bounded from above, one just imitates the argument as in Proposition 3.1 and in addition checks at $t = 0$ when $u = b_0\hat{u}$,

$$H(0) = \log \hat{u}'' - \log \hat{u}' = -\log(1 + e^\rho) \leq 0\,.$$

□

Corollary 5.1. *There exists $C > 0$ such that on $[0, T/2] \times X_{m,n}$,*

$$C^{-1}\hat{\omega} \leq \omega \leq C\big\{a\omega_{FS} + \hat{\omega} + e^{-\frac{m}{m+n+1}\rho}\hat{\omega}\big\}\,, \tag{5.9}$$

where $a = (m - n)t$.

Theorem 5.1. *$(X_{m,n}, g(t))$ converges to $(Y_{m,n}, \hat{g})$ in the Gromov–Hausdorff sense as $t \to 0^+$.*

Proof. It is proved in [SoT3] that $g(t)$ converges to $\hat{g}$ in C^∞ topology of $X_{m,n} \setminus P_0$. Let $U_\kappa = \{e^\rho \leq \kappa\}$ be the κ-tubular neighborhood of P_0. Then it suffices to show that for any $\epsilon > 0$, there exist $\kappa_\epsilon > 0$ and $T_\epsilon \in (0, T/2]$ such that for any $\kappa < \kappa_\epsilon$ and $t \in (0, T_\epsilon)$,

$$\operatorname{diam}\big(U_\kappa \setminus P_0, g(t)\big) < \epsilon\,.$$

This can be proved by a similar argument as in the proof of Theorem 3.1. □

Theorem 5.2. *$(X_{m,n}, g(t))$ converges to $\left(\mathbb{P}^n, \frac{(m-n)b_0}{m+2}\omega_{FS}\right)$ in the Gromov–Hausdorff sense as $t \to T^-$.*

Proof. It follows directly from the equation (2.14) that $b(T) = 0$ and so the singular time is given by $T = \frac{b_0}{m+2}$. Immediately, we have $a(T) = \frac{(m-n)b_0}{m+2}$ from (5.3). The theorem thus follows from Theorem 4.1 as the Kähler–Ricci flow on $Y_{m,n}$ becomes the Kähler–Ricci flow on $X_{m,n}$ after arbitrary short time $t > 0$. In particular, the limiting metric is equal to $a(T)\omega_{FS}$ on $\mathbb{P}^n$. □

5.3 The case $m = n$. We now consider the Kähler–Ricci flow on $Y_{n,n}$ starting with $b_0\hat{\omega} = b_0\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\hat{u}$, where $\hat{u} = \log(1 + e^\rho)$. We would like to lift the flow to the one on $X_{n,n}$. If we let

$$T = \sup\left\{t > 0 \mid [\hat{\omega}] + t[K_{X_{n,n}}] \text{ is big and semi-ample}\right\},$$

then

$$T = \frac{b_0}{n+2} > 0\,.$$

By [SoT3], the Kähler–Ricci flow can always be lifted to the one on $X_{n,n}$ for $t \in [0, T)$. However the solution is in general not smooth since $b_0[\hat{\omega}] + t[K_{X_{n,n}}] = (b_0 - (n+2)t)[\hat{\omega}]$ vanishes on P_0 of $X_{n,n}$ for any $t \geq 0$.

We apply the same method in [SoT3] by approximating the flow (5.4) by smooth data. We consider the family of flows for $\delta \in (0, 1)$,

$$\frac{\partial \varphi_\delta}{\partial t} = \log \frac{(\omega_t + \delta\omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_\delta)^{m+n+1}}{\Omega}\,, \quad \varphi_\delta|_{t=0} = 0\,, \tag{5.10}$$

where ω_{FS} is the pullback of the Fubini–Study metric on $\mathbb{P}^n$, $\omega_t = (b_0 - (n+2))\hat{\omega}$ and Ω is a smooth volume form on $X_{n,n}$ with $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Omega = -(n+2)\hat{\omega}$.

The above perturbed flow is equivalent to the following family of parabolic flows,

$$\frac{\partial u_\delta}{\partial t} = \log\left[(\delta + u_\delta')^n(u_\delta')^n u_\delta''\right] - (n+1)\rho\,, \quad u_\delta|_{t=0} = b_0\hat{u}\,. \tag{5.11}$$

Lemma 5.4. *There exists $C > 0$ such that for $t \in [0, T)$ and $\delta \in (0, 1)$,*

$$(\delta + u_\delta')^n(u_\delta')^n u_\delta'' \leq C(1 + \hat{u}')^n(\hat{u}')^n\hat{u}''.$$

Proof. It suffices to prove the lemma for $\rho \leq 0$ as the volume estimate holds true away from the zero section P_0(see [SoT3]). Let $v_\delta = \frac{\partial u_\delta}{\partial t}$. Then

$$\frac{\partial v_\delta}{\partial t} = \frac{nv_\delta'}{\delta + u_\delta'} + \frac{nv_\delta'}{u_\delta'} + \frac{v_\delta''}{u_\delta''}\,.$$

Let $H_{\delta,\epsilon} = e^{-t}(v_\delta + \epsilon\rho)$ for $\epsilon \in (0, 1)$. Then there exists $C_1 > 0$ such that $\lim_{\rho\to-\infty} H_{\delta,\epsilon} \leq C_1$ for fixed δ, ϵ and $t \in [0, T)$.

$$\frac{\partial H_{\delta,\epsilon}}{\partial t} = \frac{nH_{\delta,\epsilon}'}{\delta + u_\delta'} + \frac{nH_{\delta,\epsilon}'}{u_\delta'} + \frac{H_{\delta,\epsilon}''}{u_\delta''} - \frac{n\epsilon e^{-t}}{\delta + u_\delta'} - \frac{n\epsilon e^{-t}}{u_\delta'} - H_{\delta,\epsilon}\,.$$

The maximum principle implies that there exists $C_2 > 0$ such that for $t \in [0, T)$, $\delta \in (0, 1)$, and $\epsilon \in (0, 1)$,

$$\sup_{t\in[0,T),\rho\in(-\infty,0]} H_{\delta,\epsilon} \leq \sup_{t=0,\rho\in(-\infty,0]} H_{\delta,\epsilon} + C_2\,.$$

The lemma is then proved by checking $H_{\delta,\epsilon}(0,\cdot)$ is bounded from above and letting $\epsilon \to 0$. □

The following proposition can be proved in the same way as in Proposition 3.1.

PROPOSITION 5.3. *There exist $C_1, C_2 > 0$ such that for $t \in [0,T)$, $\rho \in (-\infty,\infty)$ and $\delta \in (0,1)$,*

$$u_\delta'' \le C_1 u_\delta' \le C_2 \min\bigl(1, e^{\frac{n+1}{2n+1}\rho}\bigr)\,. \tag{5.12}$$

COROLLARY 5.2. *There exists $C > 0$ such that for $t \in [0,T)$, $\delta \in (0,1)$ and $\rho \le 0$,*

$$\omega(t) \le \delta\omega_{FS} + Ce^{-\frac{n}{2n+1}\rho}\hat{\omega}\,. \tag{5.13}$$

Furthermore, for any $t \in [0,T)$, there exists $C_t > 0$ such that

$$\omega(t) \ge C_t\hat{\omega}\,. \tag{5.14}$$

Proof. Letting $\delta \to 0$ in equation (5.12), we have $u'' \le C_1 u' \le C_2 e^{\frac{n+1}{2n+1}\rho}$, for constants $C_1, C_2 > 0$. Equation (5.13) then follows easily.

For any $T' \in (0,T)$, we can apply the argument in Lemma 5.3 to show that there exists $C_{T'} > 0$ such that for $\delta \in (0,1)$ and on $[0,T') \times X_{n,n}$,

$$\omega_\delta = \delta\omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u_\delta \ge C_{T'}\hat{\omega}\,.$$

Inequality (5.14) follows by letting $\delta \to 0$. □

Let (X_t, d_t) be the metric completion of $(X_{n,n} \setminus P_0, \omega(t))$ for $t \in (0,T)$.

Theorem 5.3. *For any $t \in (0,T)$, (X_t, d_t) has finite diameter and (X_t, d_t) is homeomorphic to the projective cone $Y_{n,n}$ over $\mathbb{P}^n \times \mathbb{P}^n$ via the Segre map. Furthermore, the Gromov–Hausdorff distance $D(t) = d_{GH}((X_t, d_t), (Y_{n,n}, \hat{g}))$ is a continuous function in $t \in [0,T)$ and*

$$\lim_{t\to 0} D(t) = 0\,. \tag{5.15}$$

Proof. We will consider the approximating Kähler–Ricci flow defined by (5.10). The solution $\omega_\delta(t) = \omega_t + \delta\theta + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_\delta$ is a smooth Kähler metric on $(0,T) \times X_{n,n}$. Let $U_\kappa = \{e^\rho \le \kappa\}$ be the κ-tubular neighborhood of P_0. By a similar argument in the proof of Theorem 3.1, we can show that for any fixed $t \in (0,T)$ and $\epsilon > 0$, there exist $\kappa_\epsilon > 0$ and $\delta_\epsilon > 0$, such that for any $\kappa \in (0,\kappa_\epsilon)$ and $\delta \in (0,\delta_\epsilon]$,

$$\operatorname{diam}\bigl(U_\kappa, g_\delta(t)\bigr) < \epsilon\,.$$

Since $g_\delta(t)$ converges to $g(t)$ in C^∞-topology on $(0,T) \times X_{n,n} \setminus P_0$, we can show by a similar argument in [SoW1] that (X_t, d_T) has finite diameter and is homeomorphic to $(Y_{n,n}, \hat{g})$.

Now it suffices to show that $D(t)$ is continuous and the rest of the theorem can be proved by a similar argument in the proofs of Theorem 3.1 and Theorem 5.1. Fix $t_0 \in (0,T)$, we consider

$$\hat{D}(t) = d_{GH}\bigl((X_t, d_t), (X_{t_0}, d_{t_0})\bigr)$$

for $t \in [0, T)$. We claim that

$$\lim_{t \to t_0} \hat{D}(t) = 0 .$$

First note that $\omega(t)$ converges to $\omega(t_0)$ in C^∞ topology of $X_{n,n} \setminus P_0$ as $t \to t_0$. On the other hand, we can apply the same argument as before that, for any $\epsilon > 0$, there exists $\eta > 0$ and $\kappa_\epsilon > 0$ such that for any $\kappa < \kappa_\epsilon$ and $t \in (t_0 - \eta, t_0 + \eta) \cap (0, T/2]$,

$$\operatorname{diam}\big(U_\kappa \setminus P_0, g(t)\big) < \epsilon .$$

Also for any $\epsilon > 0$, any $\kappa > 0$ and $k > 0$, there exists $\eta > 0$ such that for $t \in [t_0 - \eta, t_0 + \eta) \cap (0, T/2]$,

$$\big\|\omega(t) - \omega(t_0)\big\|_{C^k(X_{n,n} \setminus U_\kappa)} < \epsilon .$$

Here the C^k norm is taken with respect to a fixed Kähler metric θ on $X_{n,n}$. Then the claim follows by a similar argument in [SoW1]. □

Finally, we consider the limiting behavior of the Kähler–Ricci flow on $Y_{n,n}$ starting with $b_0\hat{\omega}$ for some $b_0 > 0$. We have shown the existence of the solution $g(t)$ as in section 4.

Theorem 5.4. *Let $(Y_{n,n}, d_t)$ be the metric completion of $(Y_{n,n} \setminus \{O\}, g(t))$. Then*

$$\lim_{t \to T} \operatorname{diam}(Y_{n,n}, d_t) = 0 .$$

Proof. We again consider the perturbed flow (5.10). Notice that the Kähler class along the flow is given by $\delta[\omega_{FS}] + (b_0 - (n+2)t)[\hat{\omega}]$, hence for all $\delta \in (0, 1)$, we have

$$0 < u'_\delta \le \big(b_0 - (n+2)\big)t = (n+2)(T - t) .$$

By a similar argument as in the section 4.1, there exist C_1 and $C_2 > 0$ such that on $[0, T) \times (-\infty, \infty)$

$$u''_\delta \le C_1 u'_\delta \le C_2 \min\big(T - t, e^{\frac{n+1}{2n+1}\rho}\big) , \quad u''_\delta \le C_2 e^{-\rho},$$

for $\delta \in (0, 1)$. By letting $\delta \to 0$, we have

$$u'' \le C_1 u' \le C_2 \min\big(T - t, e^{\frac{n+1}{2n+1}\rho}\big) , \quad u'' \le C_2 e^{-\rho}.$$

We thus have the estimates on $Y_{n,n} \setminus \{O\}$,

$$\omega(t) \le C \min\Big((T - t)(e^{-\rho} + e^{\rho})\hat{\omega}, \big(1 + e^{-\frac{n}{2n+1}\rho}\big)\hat{\omega}\Big) .$$

Then by a similar argument in [SoW1], we can show that

$$\lim_{t \to T} \operatorname{diam}\big(Y_{n,n} \setminus \{O\}, \omega(t)\big) = 0 .$$

The theorem follows since $(Y_{n,n}, d_t)$ is the metric completion of $(Y_{n,n} \setminus \{O\}, g(t))$. □

Acknowledgements. The authors are grateful to Professor D.H. Phong for his support and encouragement. They also thank Professor Gang Tian, Professor Ben Weinkove, Chenyang Xu and Chi Li for many helpful discussions. In addition, the authors would like to thank V. Tosatti for a number of useful suggestions. The second named author is also grateful to Professor X. Huang for his constant support.

Part of the work was carried out during the first-named author's visit at Columbia University, and he thanks the department for its kind hospitality. Part of the work was also carried out during the second-named author's graduate study at Rutgers University. Finally, the authors would like to thank the referees for the valuable comments.

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Received: January 19, 2011
Revision: June 23, 2011
Accepted: June 29, 2011