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# ALGEBRAIC TORSION IN CONTACT MANIFOLDS

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(with appendix by Michael Hutchings)

**Abstract.** We extract an invariant taking values in $\mathbb{N}\cup\{\infty\}$, which we call the *order of algebraic torsion*, from the Symplectic Field Theory of a closed contact manifold, and show that its finiteness gives obstructions to the existence of symplectic fillings and exact symplectic cobordisms. A contact manifold has algebraic torsion of order 0 if and only if it is algebraically overtwisted (i.e. has trivial contact homology), and any contact 3-manifold with positive Giroux torsion has algebraic torsion of order 1 (though the converse is not true). We also construct examples for each $k \in \mathbb{N}$ of contact 3-manifolds that have algebraic torsion of order $k$ but not $k-1$, and derive consequences for contact surgeries on such manifolds.

The appendix by Michael Hutchings gives an alternative proof of our cobordism obstructions in dimension three using a refinement of the contact invariant in Embedded Contact Homology.

## 1 Introduction

**1.1 Main results.** Symplectic field theory (SFT) is a very general theory of holomorphic curves in symplectic manifolds which was outlined by Eliashberg, Givental and Hofer [EGH], and whose analytical foundations are currently under development by Hofer, Wysocki and Zehnder, cf. [H2]. It contains as special cases several theories that have been shown to have powerful consequences in contact topology – notably contact homology and Gromov–Witten theory – but the more elaborate structure of "full" SFT has yet to find application, as it is usually far too complicated to compute. Our goal here is to introduce a numerical invariant, which we call *algebraic torsion*, that is extracted from the full SFT algebra and whose finiteness gives obstructions to the existence of symplectic fillings and exact symplectic cobordisms. Algebraic torsion is defined in all dimensions, and we illustrate its effectiveness by proving explicit nonexistence results for exact symplectic cobordisms whose ends are certain prescribed nonfillable contact 3-manifolds, see Corollary 1 below. To the best of our knowledge, results of this type are new and seem to be beyond the present reach of more topologically oriented methods such as Heegaard Floer homology.

From the point of view taken in this paper, which is adapted from [CL] and described in more detail in section 2, the SFT of a contact manifold $(M,\xi)$ is the

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homology $H_*^{\mathrm{SFT}}(M,\xi)$ of a $\mathbb{Z}_2$-graded $BV_\infty$-algebra $(\mathcal{A}[[\hbar]],\mathbf{D}_{\mathrm{SFT}})$, where $\mathcal{A}$ has generators $q_\gamma$ for each good closed Reeb orbit $\gamma$ with respect to some nondegenerate contact form for $\xi$, $\hbar$ is an even variable, and the operator

$$\mathbf{D}_{\mathrm{SFT}}:\mathcal{A}[[\hbar]]\to\mathcal{A}[[\hbar]]$$

is defined by counting rigid solutions to a suitable abstract perturbation of a $J$-holomorphic curve equation in the symplectization of $(M,\xi)$. The domains for these solutions are punctured closed Riemann surfaces, and near the punctures the solutions have so-called positive or negative cylindrical ends. It follows from the exactness of the symplectic form in the symplectization that all such curves must have at least one positive end. Algebraically, this translates into the fact that the ground ring $\mathbb{R}[[\hbar]]$ of $\mathcal{A}$ consists of closed elements with respect to $\mathbf{D}_{\mathrm{SFT}}$. This motivates the following:

Definition 1.1. Let $(M,\xi)$ be a closed manifold of dimension $2n-1$ with a positive, co-oriented contact structure. For any integer $k\geq 0$, we say that $(M,\xi)$ has *algebraic torsion of order $k$* (or simply *algebraic $k$-torsion*) if $[\hbar^k]=0$ in $H_*^{\mathrm{SFT}}(M,\xi)$.

Note that although the version of SFT described in [EGH] has coefficients in the group ring of $H_2(M)$, the homology $H_*^{\mathrm{SFT}}(M,\xi)$ above is defined without group ring coefficients – one can always do this at the cost of reducing the usual $\mathbb{Z}$-grading to a $\mathbb{Z}_2$-grading (see section 2 for details). We will introduce group ring coefficients later to obtain a more refined invariant, cf. Definition 1.8.

In order to state our first main result, we need a few standard concepts. Recall that a strong symplectic filling of a contact manifold $(M,\xi)$ is a compact symplectic manifold $(W,\omega)$ with $\partial W=M$ for which there exists a vector field $Y$, defined near the boundary and pointing transversely outward there, with $\mathcal{L}_Y\omega=\omega$ (i.e. $Y$ is a *Liouville* vector field) and such that $\iota_Y\omega|_M$ is a contact form for $\xi$ giving the correct co-orientation. More generally, a symplectic cobordism with positive end $(M^+,\xi^+)$ and negative end $(M^-,\xi^-)$ is a compact symplectic manifold $(W,\omega)$ with boundary $M^+\sqcup(-M^-)$ and a vector field as above with $\xi^\pm=\ker(\iota_Y\omega|_{M^\pm})$, with the difference that $Y$ is required to point outward only along $M^+$ and inward along $M^-$. Note that since $\mathcal{L}_Y\omega=d(\iota_Y\omega)=\omega$, the symplectic form is always exact near the boundary of a symplectic cobordism, though it need not be exact globally. The flow of $Y$ can be used to identify a neighborhood of $\partial W$ with

$$\left([0,\epsilon)\times M^-,d(e^s(\iota_Y\omega)|_{M^-})\right)\sqcup\left((-\epsilon,0]\times M^+,d(e^s(\iota_Y\omega)|_{M^+})\right),$$

and so any symplectic cobordism in the above sense can be *completed* by gluing a positive half of the symplectization of $(M^+,\xi^+)$ and a negative half of the symplectization of $(M^-,\xi^-)$ to the respective boundaries. Holomorphic curves in completed symplectic cobordisms are the main object of study in SFT, with the symplectization $\mathbb{R}\times M$ being an important special case of a completed symplectic cobordism.

A symplectic cobordism $(W,\omega)$ is called *exact* if the vector field $Y$ as described above extends globally over $W$; equivalently, this means $\omega=d\lambda$ for a 1-form $\lambda$ on $W$ whose restrictions to $M^\pm$ define contact forms for $\xi^\pm$. From the above definition of algebraic torsion and the general formalism of SFT, we draw the following consequence, which is our first main result and is proven in section 2.

**Theorem 1.** *If $(M, \xi)$ has algebraic torsion then it is not strongly fillable. Moreover, suppose there is an exact symplectic cobordism having contact manifolds $(M^+, \xi^+)$ and $(M^-, \xi^-)$ as positive and negative ends respectively: then if $(M^+, \xi^+)$ has algebraic $k$-torsion, so does $(M^-, \xi^-)$.*

Remark 1.2. It is time for a more or less standard disclaimer: All the theorems regarding SFT that we shall state in this introduction depend on the analytical foundations of SFT, which remains a large project in progress by Hofer, Wysocki and Zehnder (see e.g. [H2]). In particular, the main technical difficulty which is the subject of their work is to establish a sufficiently well-behaved abstract perturbation scheme so that $H_*^{\mathrm{SFT}}(M, \xi)$ is well defined and the natural maps induced by counting solutions to a perturbed holomorphic curve equation in symplectic cobordisms exist. We shall take it for granted throughout the following that such a perturbation scheme exists and has the properties that its architects claim (cf. Remark 3.7) – the further details of this scheme will be irrelevant to our arguments. Note however that our main applications, Corollaries 1 and 3, can also be proved using the embedded contact homology techniques described in the appendix (cf. Theorem 7), and thus do not depend on any unpublished work in progress.

Remark 1.3. Algebraic torsion has some obvious applications beyond those that we will consider in this paper, e.g. it is immediate from the formalism of SFT discussed in section 2 that any contact manifold with algebraic torsion satisfies the Weinstein conjecture.

The simplest example of algebraic torsion is the case $k = 0$: we will show in section 2 (Proposition 2.9) that this is equivalent to $(M, \xi)$ having trivial contact homology, in which case it is called *algebraically overtwisted*, cf. [BN1]. This is the case, for instance, whenever $(M, \xi)$ is an overtwisted contact 3-manifold, and in higher dimensions it has been shown to hold whenever $(M, \xi)$ contains a *plastikstufe* [BN2], or when $(M, \xi)$ is a connected sum with a certain exotic contact sphere [BK].

In dimension three, there are also many known examples of contact manifolds that are tight but not fillable. An important class of examples is the following: $(M, \xi)$ is said to have *Giroux torsion* if it admits a contact embedding of $(T^2 \times [0, 1], \xi_T)$ where

$$\xi_T = \ker\big[\cos(2\pi t)\ d\theta + \sin(2\pi t)\ d\phi\big]$$

in coordinates $(\phi, \theta, t) \in T^2 \times [0, 1] = S^1 \times S^1 \times [0, 1]$. It was shown by D. Gay [G] that contact 3-manifolds with Giroux torsion are never strongly fillable, and a computation of the twisted Ozsváth–Szabó contact invariant due to Ghiggini and Honda [GhH] shows that Giroux torsion is also an obstruction to weak fillings whenever the submanifold $T^2 \times [0, 1] \subset M$ separates $M$. There are obvious examples of manifolds with these properties that are also tight. On $T^3 = S^1 \times S^1 \times S^1$ for example with coordinates $(\phi, \theta, t)$, the contact form

$$\cos(2\pi Nt)\ d\theta + \sin(2\pi Nt)\ d\phi$$

has Giroux torsion for any integer $N \geq 2$, but it also has no contractible Reeb orbits, which implies that its contact homology cannot vanish. The original motivation for this project was to find an algebraic interpretation of Giroux torsion that implies nonfillability. The solution to this problem is the following result, which is implied by the more general Theorem 6 below:

**Theorem 2.** *If $(M, \xi)$ is a contact 3-manifold with Giroux torsion, then it has algebraic 1-torsion.*

While it is possible that "overtwisted" and "algebraically overtwisted" could be equivalent notions in dimension three, it turns out that the converse of Theorem 2 is not true. We will show this using a special class of contact manifolds constructed as follows: assume $\Sigma_+$ and $\Sigma_-$ are compact (not necessarily connected) oriented surfaces with nonempty diffeomorphic boundaries, and denote by

$$\Sigma = \Sigma_+ \cup \Sigma_-$$

the closed oriented surface obtained by gluing them along some orientation reversing diffeomorphism $\partial\Sigma_+ \to \partial\Sigma_-$. We shall assume $\Sigma$ to be connected. The common boundary of $\Sigma_\pm$ forms a multicurve $\Gamma \subset \Sigma$. Then by a construction originally due to Lutz [L], the product $S^1 \times \Sigma$ admits a unique (up to isotopy) $S^1$-invariant contact structure $\xi_\Gamma$ for which the loops $S^1 \times \{z\}$ are positively/negatively transverse for $z$ in the interior of $\Sigma_\pm$, and Legendrian for $z \in \Gamma$. (We will give a more explicit construction of this contact structure in section 4.) By an argument due to Giroux (see [M]), $(S^1 \times \Sigma, \xi_\Gamma)$ has no Giroux torsion whenever it has the following two properties:

- No connected component of $\Gamma$ is contractible in $\Sigma$;
- No two connected components of $\Gamma$ are isotopic in $\Sigma$.

It is easy to find examples (see Figure 1) for which both these conditions are satisfied, as well as the assumption in the following result:

**Theorem 3.** *If either of $\Sigma_+$ or $\Sigma_-$ is disconnected, then the $S^1$-invariant contact manifold $(S^1 \times \Sigma, \xi_\Gamma)$ described above has algebraic 1-torsion. In particular, there exist contact 3-manifolds that have algebraic 1-torsion but no Giroux torsion.*

Remark 1.4. Theorem 1 implies that the examples in Theorem 3 are not strongly fillable. The latter has been established previously via vanishing results for the Ozsváth–Szabó contact invariant in sutured Floer homology, see [HoKM], [M], [Ma].

Examples showing that algebraic torsion is interesting for all orders can be constructed in almost the same way. In the construction of $S^1$-invariant contact manifolds $(S^1 \times \Sigma, \xi_\Gamma)$ above, assume that $\Sigma_\pm$ are both connected with $k \geq 1$ boundary components, and that $\Sigma_-$ has genus 0 and $\Sigma_+$ has genus $g' > 0$. The surface $\Sigma$ obtained by gluing will have genus $g = g' + k - 1$. We denote the resulting contact manifold by $(V_g, \xi_k) := (S^1 \times \Sigma, \xi_\Gamma)$. We then obtain

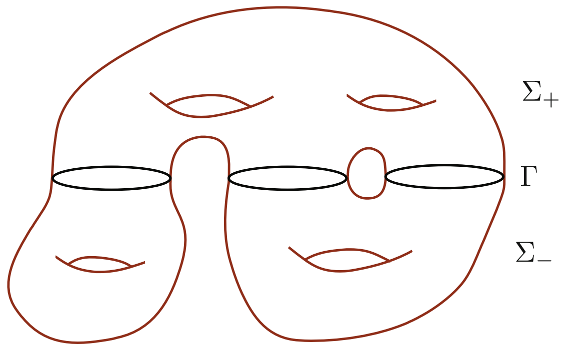


Figure 1: A surface $\Sigma = \Sigma_+ \cup_\Gamma \Sigma_-$ such that $(S^1 \times \Sigma, \xi_\Gamma)$ has algebraic 1-torsion but no Giroux torsion.

**Theorem 4.** *$(V_g, \xi_k)$ has algebraic torsion of order $k-1$, but not $k-2$.*

The proof that $(V_g, \xi_k)$ has algebraic torsion of order $k-1$ will be a consequence of Theorem 6 below, which relates algebraic torsion in dimension 3 to the geometric notion of *planar torsion* recently introduced by the second author [We4]. This is discussed in detail in section 3. The proof that there is no algebraic torsion of lower order occupies a large part of section 4. It is based on a combination of algebraic properties of SFT and a construction of certain explicit contact forms for the contact structures $\xi_k$, for which the Reeb dynamics and the holomorphic curves can be understood sufficiently well.

Combining Theorems 1 and 4 yields the following consequence.

Corollary 1. *Suppose $g \ge k \ge 2$. Then for any exact symplectic cobordism with negative end $(V_g, \xi_k)$, the positive end does not have algebraic $(k-2)$-torsion. In particular, there exists no exact symplectic cobordism with positive end $(V_{g_+}, \xi_{k_+})$ and negative end $(V_{g_-}, \xi_{k_-})$ if $k_+ < k_-$ (Figure 2).*

Remark 1.5. The inclusion of the word "exact" in the above corollary is crucial, as a recent construction due to the second author [We5] shows that non-exact symplectic cobordisms exist between *any* two contact 3-manifolds with planar torsion.

Remark 1.6. Sometimes exact cobordisms are known to exist when the negative end has a smaller order of algebraic torsion than the positive end, e.g. Etnyre and Honda [EtH] have shown that *any* positive end is allowed if the negative end is overtwisted (meaning 0-torsion, in the present context). Similarly, Jeremy Van Horn-Morris has explained to us that a Stein cobordism with negative end $(V_g, \xi_k)$ and positive end $(V_{g+1}, \xi_{k+1})$ does always exist; cf. Remark 4.18 in section 4 for an outline of the construction. Together with Corollary 1, this gives infinite sequences of contact 3-manifolds such that each is exactly cobordant to its successor, but not vice versa.

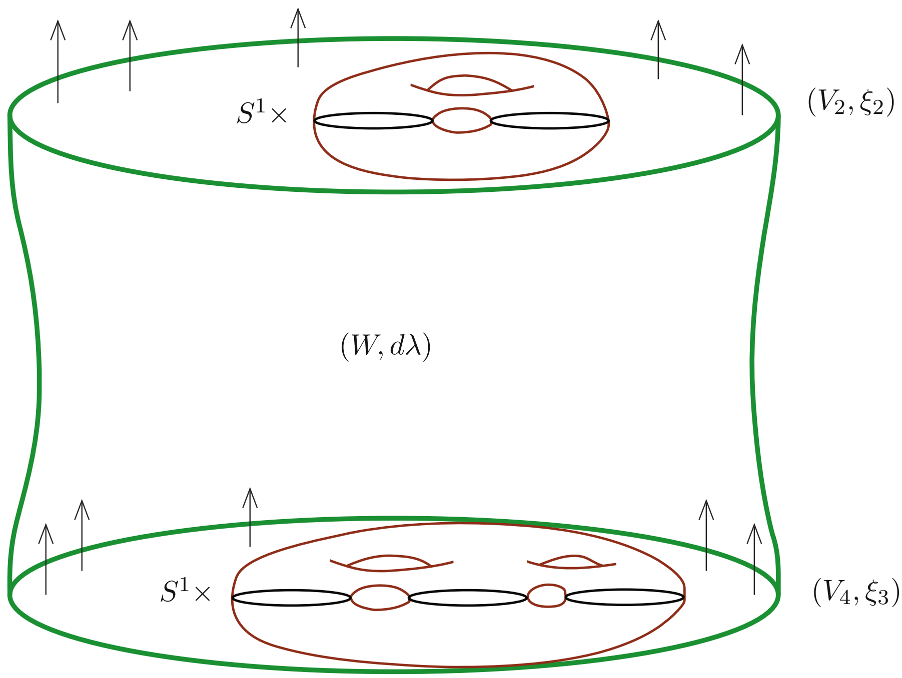


Figure 2: An example of an exact symplectic cobordism that cannot exist according to Corollary 1.

Remark 1.7. The case $k_+ = 1$ of Corollary 1 can be deduced already from the argument used by Hofer [H1] to prove the Weinstein conjecture for overtwisted contact structures. Indeed, $(V_{g_+}, \xi_{k_+})$ is always overtwisted if $k_+ = 1$, and transplanting Hofer's argument from the symplectization to an exact symplectic cobordism shows that $(V_{g_-}, \xi_{k_-})$ must then have a contractible Reeb orbit for all nondegenerate contact forms, which is easily shown to be false if $k_- \geq 2$. In this sense, the obstructions coming from algebraic torsion may be seen as a "higher order" generalization of Hofer's argument, which incidentally was the starting point for the development of SFT.

To obtain a more sensitive invariant, we now introduce a more general notion of algebraic torsion using SFT with group ring coefficients. Namely, for any linear subspace $\mathcal{R} \subset H_2(M;\mathbb{R})$, one can define the algebra of SFT with coefficients in the group ring $\mathbb{R}[H_2(M;\mathbb{R})/\mathcal{R}]$, which means keeping track of the classes in $H_2(M;\mathbb{R})/\mathcal{R}$ represented by the holomorphic curves that are counted. We shall denote the SFT with corresponding coefficients by $H_*^{\mathrm{SFT}}(M,\xi;\mathcal{R})$. The most important special cases are $\mathcal{R} = H_2(M;\mathbb{R})$ and $\mathcal{R} = \{0\}$, called the *untwisted* and *fully twisted* cases respectively, and $\mathcal{R} = \ker\Omega$ with $\Omega$ a closed 2-form on $M$. We shall abbreviate the untwisted case by $H_*^{\mathrm{SFT}}(M,\xi) = H_*^{\mathrm{SFT}}(M,\xi;H_2(M;\mathbb{R}))$, and often write the case

$\mathcal{R} = \ker \Omega$ as

$$H_*^{\mathrm{SFT}}(M, \xi, \Omega) := H_*^{\mathrm{SFT}}(M, \xi; \ker \Omega)\,.$$

DEFINITION 1.8. If $(M, \xi)$ is a closed contact manifold, for any integer $k \geq 0$ and closed 2-form $\Omega$ on $M$ we say that $(M, \xi)$ has $\Omega$-twisted algebraic $k$-torsion if $[\hbar^k] = 0$ in $H_*^{\mathrm{SFT}}(M, \xi, \Omega)$. If this is true for all $\Omega$, or equivalently, if $[\hbar^k] = 0$ in $H_*^{\mathrm{SFT}}(M, \xi; \{0\})$, then we say that $(M, \xi)$ has *fully twisted* algebraic $k$-torsion.

To see the significance of algebraic torsion with more general coefficients, we consider a more general notion of symplectic fillings, for which the symplectic form need not be exact near the boundary.

DEFINITION 1.9. Suppose $(W, \omega)$ is a compact symplectic manifold with boundary $\partial W = M$, and $\xi$ is a positive (with respect to the boundary orientation) co-oriented contact structure on $M$. We call $(W, \omega)$ a *stable symplectic filling* of $(M, \xi)$ if the following conditions are satisfied:

(1) $\omega|_\xi$ is nondegenerate and the induced orientation on $\xi$ is compatible with its co-orientation;
(2) $\xi$ admits a nondegenerate contact form $\lambda$ such that the Reeb vector field $X_\lambda$ generates the characteristic line field on $\partial W$;
(3) $\xi$ admits a complex bundle structure $J$ which is tamed by both $d\lambda|_\xi$ and $\omega|_\xi$.

Note that the compactness results in [BEHWZ] are stated for compatible $J$, but they hold without change for tamed $J$ as well. A strong filling with Liouville vector field $Y$ is also a stable filling whenever the contact form $\iota_Y \omega|_M$ is nondegenerate, which can always be assumed after a small perturbation. In general, the boundary of a stable filling is a *stable hypersurface* as defined in [HZ], meaning it belongs to a 1-parameter family of hypersurfaces in $(W, \omega)$ whose Hamiltonian dynamics are all conjugate. In particular, the pair $(\lambda, \omega|_M)$ defines a *stable Hamiltonian structure* on $M$ (cf. [CV]).

**Theorem 5.** *If $(M, \xi)$ is a closed contact manifold with $\Omega$-twisted algebraic torsion for some closed 2-form $\Omega$ on $M$, then it does not admit any stable filling $(W, \omega)$ for which $\omega|_M$ is cohomologous to $\Omega$. In particular, if $(M, \xi)$ has fully twisted algebraic torsion, then it is not stably fillable.*

Recall that for $\dim M = 3$, $(W, \omega)$ with $\partial W = M$ is said to be a *weak symplectic filling* of $(M, \xi)$ if $\omega|_\xi > 0$. Thus a stable filling is also a weak filling. What's far less obvious is that the converse is true up to deformation: by [NW, Th. 2.8], every weak filling can be deformed near its boundary to a stable filling of the same contact manifold, hence weak and stable fillability are completely equivalent notions in dimension three. Theorem 5 thus implies

COROLLARY 2. *Contact 3-manifolds with fully twisted algebraic torsion are not weakly fillable.*

Figure 3 in section 3 below shows some examples to which this result applies, including one that has no Giroux torsion; see also Theorem 6 below, and [NW].

In higher dimensions, it is not hard to find examples of stable fillings for which the symplectic form is not exact near the boundary, though it's less obvious whether there are also examples which are not strongly fillable. Such examples are found in the work in progress by Massot, Niederkrüger and the second author [MNW], which defines a suitable generalization of weak fillings to arbitrary dimensions: in a nutshell, $(W, \omega)$ with $\partial W = M$ is a weak filling of $(M, \xi)$ if $\omega$ tames an almost complex structure $J$ that preserves $\xi$ and is also tamed by the natural conformal symplectic structure on $\xi$. Under this definition, one can use an existence result of Cieliebak–Volkov [CV] to show that weak and stable fillability are equivalent, see [MNW] for details. Thus SFT also gives obstructions to weak filling in all dimensions, where the distinction between "strong" and "weak" is detected algebraically via the choice of coefficients.

As already mentioned, the second author [We4] recently introduced a new class of filling obstructions in dimension three called *planar torsion*, which also has a non-negative integer-valued *order*. A contact 3-manifold is then overtwisted if and only if it has planar 0-torsion, and Giroux torsion implies planar 1-torsion. We will recall the definition of planar torsion and $\Omega$-separating planar torsion in section 3, and prove the following generalization of Theorem 2.

**Theorem 6.** *Suppose $(M, \xi)$ is a closed contact 3-manifold, $\Omega$ is a closed 2-form on $M$ and $k \geq 0$ is an integer.*

(1) *If $(M, \xi)$ has planar $k$-torsion then it also has algebraic $k$-torsion.*
(2) *If $(M, \xi)$ has $\Omega$-separating planar $k$-torsion then it also has $\Omega$-twisted algebraic $k$-torsion.*

REMARK 1.10. Together with Theorem 1 and Corollary 2, this yields new proofs that contact 3-manifolds with planar torsion are not strongly fillable, and also not weakly fillable if the planar torsion is fully separating. These two results were first proved in [We4] and [NW] respectively. The former also proves a vanishing result for the ECH contact invariant which is closely analogous to Theorem 6 and has thus far been inaccessible from the direction of Heegaard Floer homology. Our argument in fact implies a refinement of this vanishing result in terms of the relative filtration on ECH introduced in the appendix; see Theorem 7 below.

We can now state a more geometric analogue of Corollary 1. The notion of planar torsion gives rise to a contact invariant $\mathrm{PT}(M, \xi) \in \mathbb{N} \cup \{0, \infty\}$, the *minimal order* of planar torsion, defined by

$$\mathrm{PT}(M, \xi) := \sup \left\{ k \geq 0 \;\middle|\; (M, \xi) \text{ has no planar } \ell\text{-torsion for any } \ell < k \right\}.$$

This number is infinite whenever $(M, \xi)$ is strongly fillable, and is positive if and only if $(M, \xi)$ is tight. Recall that contact connected sums and $(-1)$-surgeries always yield Stein cobordisms between contact 3-manifolds (see e.g. [Ge2]). The following can then be thought of as demonstrating a higher-order variant of the well-known conjecture that such surgeries always *preserve tightness*.

Corollary 3. *For any $g \geq k \geq 1$, $\mathrm{PT}(V_g, \xi_k) = k - 1$. Moreover, suppose $(M, \xi)$ is any contact 3-manifold that can be obtained from $(V_g, \xi_k)$ by a sequence of*

- *contact connected sums with itself or exactly fillable contact manifolds, and/or*
- *contact $(-1)$-surgeries.*

*Then $\mathrm{PT}(M, \xi) \geq k - 1$.*

At present, we do not know any example for which the minimal order of algebraic torsion is strictly smaller than the minimal order of planar torsion, but Theorem 3 seems to suggest that such examples are likely to exist.

Here is a summary of the remainder of the paper. In section 2 we review the algebraic formalism of SFT as a $BV_\infty$-algebra, in particular proving Theorems 1 and 5. In section 3 we review the definition of planar torsion and prove Theorem 6, as an easy application of some results on holomorphic curves from [We4]. The $S^1$-invariant examples $(S^1 \times \Sigma, \xi_\Gamma)$ are then treated at length in section 4, leading to the proofs of Theorems 3 and 4. We close with a brief discussion of open questions and related issues in section 5.

In Michael Hutchings' appendix to this paper, it is shown that the applications to 3-dimensional contact topology described above can also be proved using methods from embedded contact homology. Indeed, as remarked above, all of our examples of contact 3-manifolds with algebraic torsion can also be shown to have vanishing ECH contact invariant, suggesting that a refinement of the latter should exist which could detect the order of torsion. The appendix carries out enough of this program to suffice for our applications. In particular, Hutchings associates to any closed contact 3-manifold $(M, \xi)$ with generic contact form $\lambda$, compatible complex structure $J$ and positive number $T \in (0, \infty]$, two non-negative (possibly infinite) integers $f^T(M, \lambda, J)$ and $f^T_{\mathrm{simp}}(M, \lambda, J)$. These can be finite only if the ECH contact invariant vanishes, and they have the property that

$$f^{T_+}_{\mathrm{simp}}(M^+, \lambda^+, J^+) \geq f^{T_-}(M^-, \lambda^-, J^-)$$

whenever there is an exact cobordism $(X, d\lambda)$ with $\lambda = e^s \lambda^\pm$ at the positive/negative end and $T_- \geq T_+$ (cf. Theorem A.9). Since $f^T$ and $f^T_{\mathrm{simp}}$ are defined by counting embedded holomorphic curves in symplectizations, our SFT computations can be reinterpreted as estimates of these integers, leading to the following.

**Theorem 7.** (1) *If $(M, \xi)$ has planar $k$-torsion, then $\xi$ admits a nondegenerate contact form $\lambda$ and generic complex structure $J$ such that $f^\infty_{\mathrm{simp}}(M, \lambda, J) \leq k$.*

(2) *For any $g \geq k \geq 1$, $(V_g, \xi_k)$ admits a sequence of generic contact forms and complex structures $(\lambda_i, J_i)$ such that*

(a) *$f^{T_i}(V_g, \lambda_i, J_i) \geq k - 1$ for some sequence of real numbers $T_i \to +\infty$;*
(b) *For $i < j$, there is an exact symplectic cobordism $(X, d\lambda)$ such that $\lambda$ matches $e^s \lambda_i$ at the positive end and $e^s \lambda_j$ at the negative end.*

As mentioned in Remark 1.2 above, this immediately implies an alternative proof of Corollaries 1 and 3, cf. Corollary A.10 in the appendix.

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## 2 Review of SFT as a $BV_\infty$-Algebra

The general framework of SFT, in particular its algebraic structure, was laid out in [EGH] (see also [E2] for a more recent point of view), whereas the analytic foundations are the subject of ongoing work by Hofer–Wysocki–Zehnder (see [H2]). In this section, we will take the existence of SFT as described in [EGH] for granted and review a version of the theory which is readily derived from this description (cf. [CL] for some details of this translation). To keep the discussion reasonably brief, we will frequently refer to these sources for details. Theorems 1 and 5 will be simple consequences of the algebraic properties of SFT.

**2.1 Review of the basic setup of SFT.** Let $(M,\xi)$ be a closed manifold of dimension $2n-1$ with a co-oriented contact structure. To describe SFT, one needs to fix a nondegenerate contact form $\lambda$, as well as some additional choices, which we denote by a single letter $\mathfrak{f}$ (for framing). The most important of these are a cylindrical almost complex structure $J$ on the symplectization of $M$, coherent orientations for the moduli space of finite energy $J$-holomorphic curves, an abstract perturbation scheme for the $J$-holomorphic curve equation and suitable spanning surfaces for Reeb orbits.

Given a linear subspace $\mathcal{R} \subset H_2(M;\mathbb{R})$, let $R_{\mathcal{R}} := \mathbb{R}[H_2(M;\mathbb{R})/\mathcal{R}]$ denote the group ring over $\mathbb{R}$ of $H_2(M;\mathbb{R})/\mathcal{R}$, whose elements we write as $\sum a_i z^{d_i}$ with $a_i \in \mathbb{R}$ and $d_i \in H_2(M;\mathbb{R})/\mathcal{R}$. Define $\mathcal{A} = \mathcal{A}(\lambda)$ to be the $\mathbb{Z}_2$-graded algebra with unit over the group ring $R_{\mathcal{R}}$, generated by variables $q_\gamma$, where $\gamma$ ranges over the collection of good closed Reeb orbits for $\lambda$ (cf. [EGH, fnote p. 566; Rems. 1.9.2, 1.9.6]), and the degree of $q_\gamma$ is defined as

$$|q_\gamma| := n - 3 + \mu_{\mathrm{CZ}}(\gamma) \mod 2\,.$$

Here $\mu_{\mathrm{CZ}}(\gamma)$ denotes the mod 2 Conley–Zehnder index of the closed orbit $\gamma$, which is defined in terms of the linearized Poincare return map for $\gamma$ (cf. [EGH, p. 567]). We also introduce an extra variable $\hbar$ of even degree and consider the algebra of formal power series $\mathcal{A}[[\hbar]]$.

To construct the differential, one chooses a cylindrical almost complex structure $J$ on the symplectization $(\mathbb{R} \times M, \omega = d(e^s\lambda))$. To be precise, we say that an almost complex structure $J$ on $\mathbb{R} \times M$ is *adapted to* $\lambda$ if it is $\mathbb{R}$-invariant, maps the unit vector $\partial_s$ in the $\mathbb{R}$-direction to the Reeb vector field $X_\lambda$ of $\lambda$, and restricts to a tamed complex structure on the symplectic vector bundle $(\xi, d\lambda)$. After a choice of spanning surfaces as in [EGH, p. 566, see also p. 651], the projection to $M$ of each finite energy holomorphic curve $u$ can be capped off to a 2-cycle in $M$, and so it gives rise to a homology class in $H_2(M)$, which we project to define $[u] \in H_2(M;\mathbb{R})/\mathcal{R}$.

As explained in [CL, §6], the count of suitably perturbed $J$-holomorphic curves in $\mathbb{R} \times M$ with finite Hofer energy gives rise to a differential operator

$$\mathbf{D}_{\mathrm{SFT}} : \mathcal{A}[[\hbar]] \to \mathcal{A}[[\hbar]]$$

such that

- $\mathbf{D}_{\mathrm{SFT}}$ is odd and squares to zero;
- $\mathbf{D}_{\mathrm{SFT}}(1) = 0$; and
- $\mathbf{D}_{\mathrm{SFT}} = \sum_{k \geq 1} D_k \hbar^{k-1}$, where $D_k : \mathcal{A} \to \mathcal{A}$ is a differential operator of order $\leq k$.

More precisely,

$$D_k = \sum_{\substack{\Gamma_+, \Gamma_-, g, d \\ |\Gamma_+| + g = k}} n_g(\Gamma_-, \Gamma_+, d) \frac{1}{C(\Gamma_-, \Gamma_+)} q_{\gamma_1^-} \cdots q_{\gamma_{s_-}^-} z^d \frac{\partial}{\partial q_{\gamma_1^+}} \cdots \frac{\partial}{\partial q_{\gamma_{s_+}^+}} ,$$

where the sum ranges over all non-negative integers $g \geq 0$, homology classes $d \in H_2(M;\mathbb{R})/\mathcal{R}$ and ordered (possibly empty) collections of good closed Reeb orbits $\Gamma_\pm = (\gamma_1^\pm, \ldots, \gamma_{s_\pm}^\pm)$ such that $s_+ + g = k$. The number $n_g(\Gamma_-, \Gamma_+, d) \in \mathbb{Q}$ denotes the count of (suitably perturbed) holomorphic curves of genus $g$ with positive asymptotics $\Gamma_+$ and negative asymptotics $\Gamma_-$ in the homology class $d$, including asymptotic markers as explained in [EGH, p. 622f]. Finally, $C(\Gamma_-, \Gamma_+) \in \mathbb{N}$ is a combinatorial factor defined as

$$C(\Gamma_-, \Gamma_+) = s_-! s_+! \kappa_{\gamma_1^-} \cdots \kappa_{\gamma_{s_-}^-} \kappa_{\gamma_1^+} \cdots \kappa_{\gamma_{s_+}^+} ,$$

where $\kappa_\gamma$ denotes the covering multiplicity of the Reeb orbit $\gamma$.

Observe in particular that for $Q = q_{\gamma_1} \cdots q_{\gamma_r}$, the constant coefficient (i.e. the element of the ground ring) in $D_k(Q)$ for $k \geq r$ corresponds to the count of holomorphic curves of genus $k - r$ with positive asymptotics $\Gamma = \{\gamma_1, \ldots, \gamma_r\}$ and no negative ends.

The homology of $(\mathcal{A}[[\hbar]], \mathbf{D}_{\mathrm{SFT}})$ is denoted by $H_*^{\mathrm{SFT}}(M, \lambda, \mathfrak{f}; \mathcal{R})$. Note that by definition the operator $\mathbf{D}_{\mathrm{SFT}}$ commutes with $\hbar$ and with elements of $R_{\mathcal{R}}$. As $\mathbf{D}_{\mathrm{SFT}}$ is not a derivation, the homology is not an algebra, but only an $R_{\mathcal{R}}[[\hbar]]$-module. However, the element $1 \in \mathcal{A}$ and all its $R_{\mathcal{R}}[[\hbar]]$-multiples are always closed by the second property above, and so they define preferred homology classes. The special case $\mathcal{R} = H_2(M;\mathbb{R})$ is of particular importance: then $R_{\mathcal{R}}$ reduces to the trivial group ring $\mathbb{R}$ and we abbreviate

$$H_*^{\mathrm{SFT}}(M, \lambda, \mathfrak{f}) := H_*^{\mathrm{SFT}}\big(M, \lambda, \mathfrak{f}; H_2(M;\mathbb{R})\big) ,$$

which we refer to as the SFT with *untwisted* coefficients. Similarly, for any closed 2-form $\Omega$ on $M$, we abbreviate the special case $\mathcal{R} = \ker \Omega \subset H_2(M;\mathbb{R})$ by

$$H_*^{\mathrm{SFT}}(M, \lambda, \mathfrak{f}, \Omega) := H_*^{\mathrm{SFT}}(M, \lambda, \mathfrak{f}; \ker \Omega)$$

and call this the SFT with $\Omega$-*twisted* coefficients. The *fully* twisted SFT is

$$H_*^{\mathrm{SFT}}\big(M,\lambda,\mathfrak{f};\{0\}\big)\,,$$

defined by taking $\mathcal{R}$ to be the trivial subspace. Observe that the inclusions $\{0\}\hookrightarrow\ker\Omega\hookrightarrow H_2(M;\mathbb{R})$ induce natural $\mathbb{R}[[\hbar]]$-module morphisms

$$H_*^{\mathrm{SFT}}\big(M,\lambda,\mathfrak{f};\{0\}\big)\to H_*^{\mathrm{SFT}}(M,\lambda,\mathfrak{f},\Omega)\to H_*^{\mathrm{SFT}}(M,\lambda,\mathfrak{f})\,.$$

A framed cobordism $(X,\omega,\mathfrak{f}_X)$ with positive end $(M^+,\lambda^+,\mathfrak{f}^+)$ and negative end $(M^-,\lambda^-,\mathfrak{f}^-)$ is a symplectic cobordism $(X,\omega)$ with oriented boundary $M^+\sqcup(-M^-)$, together with the following additional data:

- a Liouville vector field $Y$, defined near the boundary, pointing outward at $M^+$ and inward at $M^-$, such that $\iota_Y\omega|_{M^\pm}=\lambda^\pm$;
- a tamed almost complex structure $J$ interpolating between the given cylindrical structures $J^\pm$ at the ends;
- coherent orientations for the moduli spaces of finite energy $J$-holomorphic curves in the completion of $X$;
- an abstract perturbation scheme compatible with $\mathfrak{f}^+$ and $\mathfrak{f}^-$; and
- spanning surfaces for the cobordism as described in [EGH, p. 571f].

As explained in [CL, §8], such a cobordism gives rise to a morphism from $H_*^{\mathrm{SFT}}(M^+,\lambda^+,\mathfrak{f}^+)$ to $H_*^{\mathrm{SFT}}(M^-,\lambda^-,\mathfrak{f}^-)$ after suitably twisting the differential as follows.

Suppose $\mathcal{R}^\pm\subset H_2(M^\pm;\mathbb{R})$ and $\mathcal{R}(X)\subset\ker\omega\subset H_2(X;\mathbb{R})$ are linear subspaces such that the maps $H_2(M^\pm;\mathbb{R})\to H_2(X;\mathbb{R})$ induced by the inclusions $M^\pm\hookrightarrow X$ map $\mathcal{R}^\pm$ into $\mathcal{R}(X)$. Define the group rings $R_{\mathcal{R}^\pm}=\mathbb{R}[H_2(M;\mathbb{R})/\mathcal{R}^\pm]$ and $R_{\mathcal{R}(X)}=\mathbb{R}[H_2(X;\mathbb{R})/\mathcal{R}(X)]$, and let $(\mathcal{A}^\pm[[\hbar]],\mathbf{D}^\pm_{\mathrm{SFT}})$ denote the $BV_\infty$-algebras as defined above for $(M^\pm,\lambda^\pm,\mathfrak{f}^\pm)$ with coefficients in $R_{\mathcal{R}^\pm}$. We also denote by $\mathcal{A}_X^-$ the algebra generated by the $q_\gamma^-$ with coefficients in $R_{\mathcal{R}(X)}$ instead of $R_{\mathcal{R}^-}$, Novikov completed as described in [EGH, p. 624] (note that integration of $\omega$ gives a well defined homomorphism $H_2(X;\mathbb{R})/\mathcal{R}(X)\to\mathbb{R}$). The inclusions $M^\pm\hookrightarrow X$ give rise to morphisms $H_2(M^\pm;\mathbb{R})/\mathcal{R}^\pm\to H_2(X;\mathbb{R})/\mathcal{R}(X)$ and $R_{\mathcal{R}^\pm}\to R_{\mathcal{R}(X)}$, which in particular determine a morphism of algebras $\mathcal{A}^-\to\mathcal{A}_X^-$.

Now $(X,\omega,\mathfrak{f}_X)$ gives rise to several structures, the first of which is an element $A\in\hbar^{-1}\mathcal{A}_X^-[[\hbar]]$ satisfying $\mathbf{D}^-_{\mathrm{SFT}}(e^A)=0$, which is obtained from counting holomorphic curves in $X$ with no positive punctures (these may exist only if $X$ is not exact). Using this, one can define a twisted differential $\mathbf{D}_X^-:\mathcal{A}_X^-[[\hbar]]\to\mathcal{A}_X^-[[\hbar]]$ by the formula

$$\mathbf{D}_X^-(Q)=e^{-A}\mathbf{D}^-_{\mathrm{SFT}}(e^A\cdot Q)\,.$$

In this way, we get a twisted version of SFT for $(M^-,\lambda^-,\mathfrak{f}^-)$, which depends on $(X,\omega,\mathfrak{f}_X)$.

Remark 2.1. Above we have defined two kinds of twisted versions of SFT, namely SFT twisted with respect to a closed two-form, and the twisted SFT of the negative end of a (non-exact) symplectic cobordism. We hope that it is always clear from the context which kind of twisting is meant.

The other structure one obtains is a chain map $\Phi = e^{\phi} : (\mathcal{A}^+[[\hbar]], \mathbf{D}^+_{\mathrm{SFT}}) \to (\mathcal{A}^-_X[[\hbar]], \mathbf{D}^-_X)$ determined by a map $\phi = \phi_X : \mathcal{A}^+ \to \mathcal{A}^-_X[[\hbar]]$ satisfying

- $\phi$ is even and $\phi(1) = 0$;
- $e^{\phi}\mathbf{D}^+_{\mathrm{SFT}} = \mathbf{D}^-_X e^{\phi}$; and
- $\phi = \sum_{k\geq 1} \phi_k \hbar^{k-1}$, where each $\phi_k : \mathcal{A}^+ \to \mathcal{A}^-_X$ is a differential operator of order $\leq k$ over the zero morphism.

Here we remind the reader that, given a morphism $\rho : A_1 \to A_2$ between graded commutative algebras, a homogeneous linear map $D : A_1 \to A_2$ is a differential operator of order $\leq k$ over $\rho$ if for each homogeneous element $a \in A_1$ the map $x \mapsto D(ax) - (-1)^{|D||a|}\rho(a)D(x)$ is a differential operator of order $\leq k-1$, with the convention that the zero map has order $\leq -1$.

The map $\phi$ counts holomorphic curves in $X$ with at least one positive puncture. The first condition above translates to the fact that $\Phi(1) = 1$. Again $\Phi$ is $\hbar$-linear, so it induces a morphism of $\mathbb{R}[[\hbar]]$-modules $H_*(\mathcal{A}^+, \mathbf{D}^+_{\mathrm{SFT}}) \to H_*(\mathcal{A}^-_X, \mathbf{D}^-_X)$, which maps the preferred class $[1] \in H_*(\mathcal{A}^+, \mathbf{D}^+_{\mathrm{SFT}})$ and its $R_{M^+}[[\hbar]]$-multiples to the corresponding classes in $H_*(\mathcal{A}^-_X, \mathbf{D}^-_X)$.

To discuss the invariance properties of SFT, one studies holomorphic curves in topologically trivial cobordisms $\mathbb{R} \times M$. More precisely, given two contact forms $\lambda^{\pm}$ for the same contact structure $\xi$, there is a constant $c > 0$ and an exact symplectic form $\omega = d(e^s\lambda_s)$ on $\mathbb{R} \times M$ such that the primitive $\lambda_s$ agrees with $c\lambda^-$ at the negative end and with $\lambda^+$ at the positive end of the cobordism. Similarly, one finds a framing $\mathfrak{f}_{\mathbb{R}\times M}$ compatible with given framings $\mathfrak{f}^{\pm}$ at the ends. Note that in this case $\ker\omega = H_2(X) = H_2(M)$, so we can choose $\mathcal{R}^{\pm} = \mathcal{R} = \mathcal{R}(X)$ and observe that the completion process in the definition of $\mathcal{A}^-_X$ is trivial since $\omega$ is exact, giving rise to a natural identification of $\mathcal{A}^-_X$ with $\mathcal{A}^-$. Likewise, $A \in \hbar^{-1}\mathcal{A}^-$ vanishes as the cobordism is exact. Since rescaling of $\lambda$ does not influence the count of holomorphic curves, we obtain a chain map $(\mathcal{A}^+[[\hbar]], \mathbf{D}^+_{\mathrm{SFT}}) \to (\mathcal{A}^-[[\hbar]], \mathbf{D}^-_{\mathrm{SFT}})$.

Reversing the roles of $\lambda^+$ and $\lambda^-$, one obtains a similar chain map in the other direction, and a deformation argument implies that both compositions are chain homotopic to the identity maps on $(\mathcal{A}^{\pm}, \mathbf{D}^{\pm}_{\mathrm{SFT}})$, respectively. In particular, they induce $R_{\mathcal{R}}[[\hbar]]$-module isomorphisms on homology, so that the contact invariant

$$H^{\mathrm{SFT}}_*(M, \xi; \mathcal{R}) := H^{\mathrm{SFT}}_*(M, \lambda, \mathfrak{f}; \mathcal{R})$$

is well defined up to natural isomorphisms. It is important for us to observe that, by construction, these morphisms are the identity on $R_{\mathcal{R}}[[\hbar]] \subset \mathcal{A}^{\pm}$, thus $H^{\mathrm{SFT}}_*(M, \xi; \mathcal{R})$ comes with preferred homology classes associated to the elements of $R_{\mathcal{R}}[[\hbar]]$. Considering the special cases where $\mathcal{R}$ is $\{0\}$, $\ker\Omega$ or $H_2(M;\mathbb{R})$ again gives rise to the fully twisted, $\Omega$-twisted and untwisted versions respectively, with natural $\mathbb{R}[[\hbar]]$-module morphisms

$$H^{\mathrm{SFT}}_*\big(M, \xi; \{0\}\big) \to H^{\mathrm{SFT}}_*(M, \xi, \Omega) \to H^{\mathrm{SFT}}_*(M, \xi)\,. \tag{2.1}$$

Remark 2.2. The above discussion of morphisms can be refined slightly as follows. Given a nondegenerate contact form $\lambda$ and a constant $T > 0$, we can consider the linear subspace $\mathcal{A}(\lambda, T) \subset \mathcal{A}(\lambda)$ in the corresponding chain level algebra generated by all the monomials of the form $q_{\gamma_1} \cdots q_{\gamma_r}$ for which the total action is bounded by $T$, i.e.

$$\sum_{j=1}^{r} \int_{\gamma_j} \lambda < T .$$

Since the energy of holomorphic curves contributing to $\mathbf{D}_{\mathrm{SFT}}$ is non-negative and given by the action difference of the asymptotics, the operator $\mathbf{D}_{\mathrm{SFT}}$ restricts to define a differential

$$\mathbf{D}_{\mathrm{SFT}} : \mathcal{A}(\lambda, T)[[\hbar]] \to \mathcal{A}(\lambda, T)[[\hbar]] .$$

Moreover, if $\omega = d(e^s \lambda_s)$ is a symplectic form on $\mathbb{R} \times M$ such that $\lambda$ agrees with $\lambda^+$ at the positive end and $c\lambda^-$ at the negative end, then the resulting morphism respects the truncation with suitable rescaling, i.e. it gives rise to a chain map

$$\Phi_T : \left(\mathcal{A}(\lambda^+, T)[[\hbar]], \mathbf{D}^+_{\mathrm{SFT}}\right) \to \left(\mathcal{A}(c\lambda^-, T)[[\hbar]], \mathbf{D}^-_{\mathrm{SFT}}\right) = \left(\mathcal{A}(\lambda^-, T/c)[[\hbar]], \mathbf{D}^-_{\mathrm{SFT}}\right) .$$

Beware however that, due to the rescaling of forms for the cylindrical cobordisms, there is no meaningful filtration on $H_*^{\mathrm{SFT}}(M, \xi; \mathcal{R})$.

In the proof of Theorem 4 we will use this refinement in the situation where $\lambda^-$ has only its periodic orbits of action at most $T$ nondegenerate, in which case the truncated complex $(\mathcal{A}(\lambda^-, T)[[\hbar]], \mathbf{D}^-_{\mathrm{SFT}})$ can still be constructed with all the required properties.

It is useful to consider how the chain map $\Phi : (\mathcal{A}^+[[\hbar]], \mathbf{D}^+_{\mathrm{SFT}}) \to (\mathcal{A}^-_X[[\hbar]], \mathbf{D}^-_X)$ induced by a symplectic cobordism $(X, \omega)$ simplifies whenever certain natural extra assumptions are placed on $X$. First, suppose that $(X, \omega)$ is an *exact* cobordism. As we already observed above, in this case $X$ contains no holomorphic curves without positive ends, hence the "twisting" term $A \in \hbar^{-1} \mathcal{A}^-_X[[\hbar]]$ vanishes. Moreover, since $\ker \omega = H_2(X; \mathbb{R})$, we can set $\mathcal{R}(X) = H_2(X; \mathbb{R})$ and reduce $R_{\mathcal{R}(X)}$ to the untwisted coefficient ring $\mathbb{R}$. Making corresponding choices $\mathcal{R}^\pm = H_2(M^\pm; \mathbb{R})$ so that $R_{\mathcal{R}^\pm} = \mathbb{R}$ for the positive and negative ends, we then have a natural identification of the two chain complexes $(\mathcal{A}^-_X[[\hbar]], \mathbf{D}^-_X)$ and $(\mathcal{A}^-[[\hbar]], \mathbf{D}^-_{\mathrm{SFT}})$, hence the aforementioned chain map yields the following:

Proposition 2.3. *Any exact symplectic cobordism $(X, \omega)$ with positive end $(M^+, \xi^+)$ and negative end $(M^-, \xi^-)$ gives rise to a natural $\mathbb{R}[[\hbar]]$-module morphism on the untwisted SFT,*

$$\Phi_X : H_*^{SFT}(M^+, \xi^+) \to H_*^{SFT}(M^-, \xi^-) .$$

Now suppose $(X, \omega)$ is a strong filling of $(M^+, \xi^+)$, which we may view as a symplectic cobordism whose negative end $(M^-, \xi^-)$ is the empty set. For any given subspace $\mathcal{R}(X) \subset \ker \omega$, the Novikov completion $\overline{R_{\mathcal{R}(X)}}$ of $R_{\mathcal{R}(X)}$ need not be trivial, but the chain complex $(\mathcal{A}^-_X[[\hbar]], \mathbf{D}^-_X)$ has no generators other than the unit, and its differential vanishes, hence its homology is simply $\overline{R_{\mathcal{R}(X)}}[[\hbar]]$. Choosing $\mathcal{R} \subset H_2(M; \mathbb{R})$ so that the natural map $H_2(M; \mathbb{R}) \to H_2(X; \mathbb{R})$ induced by the inclusion $M \hookrightarrow X$ takes $\mathcal{R}$ into $\mathcal{R}(X)$, we also obtain a natural $\mathbb{R}[[\hbar]]$-module morphism $R_{\mathcal{R}}[[\hbar]] \to R_{\mathcal{R}(X)}[[\hbar]]$. Note that since $\omega$ is necessarily exact near $\partial X$, we can always choose $\mathcal{R}(X) = \ker \omega$ and $\mathcal{R} = H_2(M; \mathbb{R})$. We obtain

Proposition 2.4. *Suppose* $(X,\omega)$ *is a strong filling of* $(M,\xi)$, *and* $\mathcal{R}(X) \subset \ker\omega \subset H_2(X;\mathbb{R})$ *and* $\mathcal{R} \subset H_2(M;\mathbb{R})$ *are linear subspaces for which the natural map from* $H_2(M;\mathbb{R})$ *to* $H_2(X;\mathbb{R})$ *takes* $\mathcal{R}$ *into* $\mathcal{R}(X)$. *Then there is a natural* $\mathbb{R}[[\hbar]]$*-module morphism*

$$\Phi_X : H_*^{SFT}(M,\xi;\mathcal{R}) \to \overline{R_{\mathcal{R}(X)}}[[\hbar]]\,,$$

*which acts on* $R_{\mathcal{R}}[[\hbar]] \subset H_*^{SFT}(M,\xi;\mathcal{R})$ *as the natural map to* $R_{\mathcal{R}(X)}[[\hbar]]$ *induced by the inclusion* $M \hookrightarrow X$. *In particular, the untwisted SFT of* $(M,\xi)$ *admits an* $\mathbb{R}[[\hbar]]$*-module morphism*

$$\Phi_X : H_*^{SFT}(M,\xi) \to \overline{R_{\ker\omega}}[[\hbar]]\,.$$

Finally, we generalize the above to allow for stable symplectic fillings as defined in the introduction. Recall that if $(X,\omega)$ is a stable filling of $(M,\xi)$ and we write $\Omega := \omega|_M$, then $\xi$ admits a nondegenerate contact form $\lambda$ and complex structure $J_\xi$ such that $\omega|_\xi$ and $d\lambda|_\xi$ both define symplectic bundle structures taming $J_\xi$, and the Reeb vector field $X_\lambda$ generates $\ker\Omega$. In particular, the pair $(\lambda,\Omega)$ is then a stable Hamiltonian structure, meaning it satisfies

1. $\lambda \wedge \Omega^{n-1} > 0$;
2. $d\Omega = 0$;
3. $\ker\Omega \subset \ker d\lambda$.

A routine Moser deformation argument shows that a neighborhood of $\partial X$ in $(X,\omega)$ can then be identified symplectically with the collar

$$\big((-\epsilon,0]\times M, d(t\lambda)+\Omega\big)$$

for $\epsilon > 0$ sufficiently small. Choose a small number $\epsilon_0 > 0$ and define

$$\mathcal{T} := \big\{\varphi \in C^\infty([0,\infty)\to[0,\epsilon_0)) \mid \varphi' > 0 \text{ everywhere and } \varphi(t) = t \text{ near } t = 0\big\}\,.$$

Then, if $\epsilon_0$ is small enough, every $\varphi \in \mathcal{T}$ gives rise to a symplectic form $\omega_\varphi$ on the completion $\widehat{X} := X \cup_M ([0,\infty)\times M)$, defined by

$$\omega_\varphi = \begin{cases} \omega & \text{on } X,\\ d\big(\varphi(t)\lambda\big) + \Omega & \text{on } [0,\infty)\times M\,. \end{cases}$$

Define a cylindrical almost complex structure on $[0,\infty)\times M$ which maps $\partial_s$ to $X_\lambda$ and restricts to $J_\xi$ on $\xi$; due to the compatibility assumptions on $J_\xi$, this is $\omega_\varphi$-tamed for all possible choices of $\varphi \in \mathcal{T}$. We can thus extend it to a generic $\omega_\varphi$-tamed almost complex structure $J$ on $\widehat{X}$. Then one can generalize the previous discussion by considering punctured $J$-holomorphic curves $u : \dot{S} \to \widehat{X}$ that satisfy the finite energy condition

$$E(u) := \sup_{\varphi\in\mathcal{T}} \int_{\dot{S}} u^*\omega_\varphi\,.$$

This definition of energy is equivalent to the one given in [BEHWZ] in the sense that bounds on either imply bounds on the other; it follows that the compactness theorems of [BEHWZ] apply to sequences $u_k$ of punctured $J$-holomorphic curves for which $E(u_k)$ is uniformly bounded. Such a bound exists for any sequence of curves with fixed genus, asymptotics and homology class. Note also that the restriction of $J$ to the cylindrical end is also adapted to $\lambda$ in the usual sense, thus the upper level curves that appear in holomorphic buildings arising from the compactness theorem are precisely the curves that are counted in the definition of $H_*^{\mathrm{SFT}}(M,\lambda,\mathfrak{f};\mathcal{R})$.

The above observations yield the following generalization of Proposition 2.4.

Proposition 2.5. *Suppose $(X,\omega)$ is a stable symplectic filling of $(M,\xi)$, and $\mathcal{R}(X) \subset \ker\omega \subset H_2(X;\mathbb{R})$ and $\mathcal{R} \subset H_2(M;\mathbb{R})$ are linear subspaces such that the natural map $H_2(M;\mathbb{R}) \to H_2(X;\mathbb{R})$ takes $\mathcal{R}$ into $\mathcal{R}(X)$. Then there exists a natural $\mathbb{R}[[\hbar]]$-module morphism*

$$\Phi_X : H_*^{SFT}(M,\xi;\mathcal{R}) \to \overline{R_{\mathcal{R}(X)}}[[\hbar]]\,,$$

*which acts on $R_{\mathcal{R}}[[\hbar]]$ as the natural map to $R_{\mathcal{R}(X)}[[\hbar]]$ induced by the inclusion $M \hookrightarrow X$. In particular, defining a 2-form on $M$ by $\Omega = \omega|_M$, the $\Omega$-twisted SFT of $(M,\xi)$ admits an $\mathbb{R}[[\hbar]]$-module morphism*

$$\Phi_X : H_*^{SFT}(M,\xi,\Omega) \to \overline{R_{\ker\omega}}[[\hbar]]\,.$$

Example 2.6. The following shows that aside from defining filling obstructions, SFT can also provide information as to the classification of symplectic fillings. Consider for instance the tight contact structure $\xi_0$ on $S^1 \times S^2$, which it acquires as the boundary of the Stein domain $S^1 \times B^3 \subset T^*S^1 \times \mathbb{R}^2$. Presenting $(S^1 \times S^2, \xi_0)$ via a symmetric summed open book with disk-like pages (see Definition 3.1), one can find a Reeb orbit that is uniquely spanned by two rigid holomorphic planes whose homology classes differ by the generator $[S^2] := [\{*\} \times S^2] \in H_2(S^1 \times S^2;\mathbb{R})$. Hence, in the notation established at the beginning of this section, the fully twisted SFT satisfies a relation of the form

$$\left[1 - z^{[S^2]}\right] = 0 \in H_*^{\mathrm{SFT}}\left(S^1 \times S^2, \xi_0; \{0\}\right).$$

Then, if $(X,\omega)$ is any weak filling of $(S^1 \times S^2, \xi_0)$, Proposition 2.5 gives a map from $H_*^{\mathrm{SFT}}(S^1 \times S^2, \xi_0; \{0\})$ to the Novikov completion of $\mathbb{R}[H_2(X;\mathbb{R})]$ whose action on $\mathbb{R}[H_2(M;\mathbb{R})][[\hbar]]$ is determined by the inclusion $S^1 \times S^2 \hookrightarrow X$. In light of the above relation, this implies that the natural map $H_2(S^1 \times S^2;\mathbb{R}) \to H_2(X;\mathbb{R})$ takes $[S^2]$ to zero. In fact, this is known to be true: it follows from the disk filling argument of Eliashberg [E1], which implies that every weak filling of $S^1 \times S^2$ is diffeomorphic to a blow-up of $S^1 \times B^3$.

Another example is provided by the standard 3-torus $(T^3, \xi_0)$, which is the boundary of the Stein domain $T^2 \times \mathbb{D} \subset T^*T^2$ and can also be presented by a symmetric summed open book, but with cylindrical pages. One can then choose a 1-dimensional subspace $\mathcal{R} \subset H_2(T^3;\mathbb{R})$ with generator $d_0$ represented by a pre-Lagrangian torus, so that counting holomorphic cylinders yields relations of the form

$$\left[(1 - z^{d_1})\hbar\right] = \left[(1 - z^{d_2})\hbar\right] = 0 \in H_*^{\mathrm{SFT}}(T^3, \xi_0; \mathcal{R})$$

for both of the other canonical generators $d_1, d_2 \in H_2(T^3;\mathbb{R})$. Applying Proposition 2.5 again, one can use this to show that for any weak filling $(X,\omega)$ of $(T^3,\xi_0)$ such that $\int_{d_0} \omega = 0$, and in particular for any strong filling, the natural map

$H_2(T^3, \mathbb{R}) \to H_2(X; \mathbb{R})$ has its image in a space of dimension at most one. This is also known to be true: by a combination of arguments in [We3] and [NW], $(X, \omega)$ must in this case be a symplectic blow-up of the standard Stein filling $T^2 \times \mathbb{D}$.

**2.2 Algebraic torsion and its consequences.** As above, we write $\mathcal{R}$ for some given linear subspace in $H_2(M; \mathbb{R})$, and use the notation $R_{\mathcal{R}} = \mathbb{R}[H_2(M; \mathbb{R})/\mathcal{R}]$ for the corresponding group ring. Recall the following definition from the Introduction.

Definition 2.7. For any integer $k \geq 0$, we say that $(M, \xi)$ has algebraic torsion of order $k$ with coefficients in $R_{\mathcal{R}}$ if $[\hbar^k] = 0$ in $H_*^{\mathrm{SFT}}(M, \xi; \mathcal{R})$. We single out the following special cases:

- $(M, \xi)$ has (untwisted) algebraic $k$-torsion if $[\hbar^k] = 0 \in H_*^{\mathrm{SFT}}(M, \xi)$.
- For a closed 2-form $\Omega$ on $M$, $(M, \xi)$ has $\Omega$-*twisted* algebraic $k$-torsion if $[\hbar^k] = 0 \in H_*^{\mathrm{SFT}}(M, \xi, \Omega)$.
- $(M, \xi)$ has *fully twisted* algebraic $k$-torsion if $[\hbar^k] = 0 \in H_*^{\mathrm{SFT}}(M, \xi; \{0\})$.

By default, when we speak of algebraic torsion without specifying the coefficients, we will always mean the untwisted version. Observe that due to the morphisms (2.1), fully twisted torsion implies $\Omega$-twisted torsion for all closed 2-forms $\Omega$, and it is not hard to show that the converse is also true. Likewise, $\Omega$-twisted torsion for *any one* closed 2-form $\Omega$ implies untwisted torsion, and $k$-torsion for any choice of coefficients implies $(k+1)$-torsion for the same coefficients since $\mathbf{D}_{\mathrm{SFT}}(Q) = \hbar^k$ implies $\mathbf{D}_{\mathrm{SFT}}(\hbar Q) = \hbar^{k+1}$.

Remark 2.8. Since all power series in $\mathbb{R}[[\hbar]]$ are naturally closed elements of the SFT chain complex, one can define a seemingly more general notion than algebraic torsion via the condition

$$[f(\hbar)] = 0 \in H_*^{\mathrm{SFT}}(M, \xi)$$

for any nonzero power series $f \in \mathbb{R}[[\hbar]]$. In fact, this is not more general: all elements of the form $1 + \mathcal{O}(\hbar)$ can be inverted in $\mathbb{R}[[\hbar]]$ via alternating series, thus $[f(\hbar)] = 0$ implies untwisted algebraic $k$-torsion where $k \geq 0$ is the largest integer with $f(\hbar) = \hbar^k g(\hbar)$ for some $g \in \mathbb{R}[[\hbar]]$. The situation changes when one considers the vanishing of nonzero elements of $R_{\mathcal{R}}[[\hbar]]$ in $H_*^{\mathrm{SFT}}(M, \xi; \mathcal{R})$: as shown by Example 2.6 above, this does not always imply nonfillability, but it can yield topological restrictions on the symplectic fillings that exist.

The special case $k = 0$ is not a new concept; the following result is stated for the untwisted theory but has obvious analogues for any choice of coefficients $R_{\mathcal{R}}$.

Proposition 2.9. *The following statements are equivalent.*

(i) $(M, \xi)$ *has algebraic* 0-*torsion.*
(ii) $H_*^{SFT}(M, \xi) = 0$.
(iii) $(M, \xi)$ *is* algebraically overtwisted *in the sense of [BN1], i.e. its contact homology is trivial.*

*Proof.* The only claim not immediate from the definitions is that (i) implies (ii), for which we use a variation on the main argument in [BN1]. For $Q_1, Q_2 \in \mathcal{A}[[\hbar]]$, define

$$[Q_1, Q_2] := \mathbf{D}_{\mathrm{SFT}}(Q_1 Q_2) - \mathbf{D}_{\mathrm{SFT}}(Q_1) Q_2 - (-1)^{|Q_1|} Q_1 \mathbf{D}_{\mathrm{SFT}}(Q_2)$$

to be the deviation of $\mathbf{D}_{\mathrm{SFT}}$ from being a derivation. Note that since the first term $D_1$ in the expansion of $\mathbf{D}_{\mathrm{SFT}}$ is a derivation, we always have $[Q_1, Q_2] = \mathcal{O}(\hbar)$. One also easily checks that $\mathbf{D}_{\mathrm{SFT}}$ is a derivation of this bracket, in the sense that

$$\mathbf{D}_{\mathrm{SFT}}[Q_1, Q_2] = -[\mathbf{D}_{\mathrm{SFT}} Q_1, Q_2] - (-1)^{|Q_1|}[Q_1, \mathbf{D}_{\mathrm{SFT}} Q_2] .$$

These signs are correct because the bracket has odd degree.

Now suppose $\mathbf{D}_{\mathrm{SFT}}(P) = 1$, and define a map $B : \mathcal{A}[[\hbar]] \to \mathcal{A}[[\hbar]]$ as an alternating sum of iterated brackets with $P$, i.e. as

$$B(Q) := Q - [P, Q] + [P, [P, Q]] - \dots$$

Clearly $[P, B(Q)] = Q - B(Q)$ and $\mathbf{D}_{\mathrm{SFT}}(B(Q)) = B(\mathbf{D}_{\mathrm{SFT}}(Q))$, and so, if $\mathbf{D}_{\mathrm{SFT}}(Q) = 0$, then

$$\mathbf{D}_{\mathrm{SFT}}\big(P \cdot B(Q)\big) = \big[P, B(Q)\big] + \mathbf{D}_{\mathrm{SFT}}(P) \cdot B(Q) = Q - B(Q) + B(Q) = Q ,$$

proving that every closed element in $\mathcal{A}[[\hbar]]$ is exact. □

With the algebraic formalism in place, the proofs of Theorems 1 and 5 are now immediate.

*Proofs of Theorems 1 and 5.* Suppose $(X, \omega)$ is an exact symplectic cobordism with positive end $(M^+, \xi^+)$ and negative end $(M^-, \xi^-)$. Then, if $[\hbar^k] = 0 \in H_*^{\mathrm{SFT}}(M^+, \xi^+)$, the same must be true in $H_*^{\mathrm{SFT}}(M^-, \xi^-)$ due to Proposition 2.3.

Likewise, if $(X, \omega)$ is a strong filling of $(M, \xi)$, then Proposition 2.4 gives an $\mathbb{R}[[\hbar]]$-module morphism from $H_*^{\mathrm{SFT}}(M, \xi)$ to $\overline{R_{\mathcal{R}(X)}}[[\hbar]]$, where $\overline{R_{\mathcal{R}(X)}}$ is the Novikov completion of $\mathbb{R}[H_2(X; \mathbb{R}) / \ker \omega]$. Since no power of $\hbar$ vanishes in $\overline{R_{\mathcal{R}(X)}}[[\hbar]]$, the same must be true in $H_*^{\mathrm{SFT}}(M, \xi)$, completing the proof of Theorem 1. Theorem 5 follows by exactly the same argument, using Proposition 2.5 and observing that $H_*^{\mathrm{SFT}}(M, \xi, \Omega)$ depends only on $(M, \xi)$ and the cohomology class of $\Omega$. □

## 3 Relation to Planar Torsion in Dimension 3

This section describes the relation of algebraic torsion to planar torsion, and in particular provides the proof of Theorem 6.

**3.1 Review of planar torsion.** We begin by reviewing briefly the notion of planar torsion, which is defined in more detail in [We4]. A *planar torsion domain* is a special type of contact manifold with boundary which generalizes the thickened torus $(T^2 \times [0, 1], \xi_T)$ in the definition of Giroux torsion. We can define it in terms of open book decompositions as follows.

Recall first that if $\check{M}$ is a closed oriented (not necessarily connected) 3-manifold with an open book decomposition $\check{\pi} : \check{M} \setminus \check{B} \to S^1$, then the open book can be "blown up" along part of its binding to produce a manifold with boundary: for any given binding component $\gamma \subset \check{B}$, this means replacing $\gamma$ with its unit normal bundle. The latter is then a 2-torus $T$ in the boundary of the blown-up manifold $M$, and it comes with a canonical homology basis $\{\mu, \lambda\} \subset H_1(T)$, where $\mu$ is the meridian around the boundary of a neighborhood of $\gamma$ and $\lambda$ is a boundary component of a page. Given any two binding components $\gamma_1, \gamma_2 \subset \check{B}$, one can then produce a new manifold via a so-called *binding sum*, which consists of the following two steps:

1. Blow up at $\gamma_1$ and $\gamma_2$ to produce boundary tori $T_1$ and $T_2$ with canonical homology bases $\{\mu_1, \lambda_1\}$ and $\{\mu_2, \lambda_2\}$ respectively.
2. Attach $T_1$ to $T_2$ via an orientation reversing diffeomorphism $T_1 \to T_2$ that maps $\lambda_1$ to $\lambda_2$ and $\mu_1$ to $-\mu_2$.

Combining both the blow-up and binding sum operations for a given closed manifold with an open book $\check{\pi} : \check{M} \setminus \check{B} \to S^1$, one obtains a compact manifold $M$, possibly with boundary, carrying a fibration

$$\pi : M \setminus (B \cup \mathcal{I}) \to S^1,$$

where $B$ is an oriented (possibly empty) link consisting of all components of $\check{B}$ that have not been blown up, and $\mathcal{I}$ is a special (also possibly empty) collection of 2-tori which are each the result of identifying two blown-up binding components in a binding sum. The tori $T \subset \mathcal{I} \cup \partial M$ each carry canonical homology bases $\{\mu, \lambda\} \subset H_1(T)$, where for $T \in \mathcal{I}$, $\mu$ is defined only up to a sign. These homology bases together with the fibration $\pi$ determine a so-called *blown-up summed open book* $\boldsymbol{\pi}$ on $M$, with *binding* $B$ and *interface* $\mathcal{I}$. Its *pages* are the connected components of the fibers $\pi^{-1}(\text{const})$. We call a blown-up summed open book *irreducible* if the fibers $\pi^{-1}(\text{const})$ are connected, which means it contains only a single $S^1$-family of pages. In general, every manifold $M$ with a blown-up summed open book $\boldsymbol{\pi}$ can be written as a union of *irreducible subdomains*,

$$M = M_1 \cup \ldots \cup M_n \,,$$

where $M_i$ are manifolds with boundary that each carry irreducible blown-up summed open books $\boldsymbol{\pi}_i$, whose pages are pages of $\boldsymbol{\pi}$, and they are attached to each other along tori in the interface of $\boldsymbol{\pi}$.

Just as an open book on $M$ determines a special class of contact forms, we define a *Giroux form* on a manifold $M$ with a blown-up summed open book to be any contact form $\lambda$ with the following properties:

1. The Reeb vector field $X_\lambda$ is everywhere positively transverse to the pages and positively tangent to the oriented boundaries of their closures.
2. The characteristic foliation cut out by $\xi = \ker \lambda$ on each boundary or interface torus $T \subset \mathcal{I} \cup \partial M$ has closed leaves in the homology class of the meridian.

Note that whenever $\lambda$ is a Giroux form, the binding consists of periodic orbits of $X_\lambda$, and each torus in $\mathcal{I} \cup \partial M$ is foliated by periodic orbits. A Giroux form can be defined for any blown-up summed open book that contains no closed pages, and it is then unique up to deformation. We say that a contact structure $\xi$ on $M$ is *supported* by a given blown-up summed open book if and only if it can be written as the kernel of a Giroux form. The effect of a binding sum on supported contact structures is then equivalent to a special case of the *contact fiber sum* defined by Gromov [Gr] and Geiges [Ge1].

Definition 3.1. A blown-up summed open book is called *symmetric* if it has no boundary and contains exactly two irreducible subdomains, each with pages of the same topological type, and each with empty binding and (interior) interface.

Symmetric examples are constructed in general by taking any two open books with diffeomorphic pages, choosing an oriented diffeomorphism from the binding of one to the binding of the other and constructing the corresponding binding sum on their disjoint union. Supported contact manifolds that arise in this way include the tight $S^1 \times S^2$ (with disk-like pages) and the standard $T^3$ (cylindrical pages).

We call an irreducible blown-up summed open book *planar* if its pages have genus 0, and a general blown-up summed open book is then *partially planar* if it contains a planar irreducible subdomain in its interior.

Definition 3.2. For any integer $k \geq 0$, a *planar torsion domain of order $k$* (or simply *planar $k$-torsion domain*) is a connected contact 3-manifold $(M, \xi)$, possibly with boundary, with a supporting blown-up summed open book $\boldsymbol{\pi}$, such that

(1) $M$ contains a planar irreducible subdomain $M^P \subset M$ in its interior, whose pages have $k+1$ boundary components;
(2) $M \setminus M^P$ is not empty; and
(3) $\boldsymbol{\pi}$ is not symmetric.

We then call the subdomains $M^P$ and $\overline{M \setminus M^P}$ the *planar piece* and the *padding* respectively.

A contact 3-manifold is said to have *planar $k$-torsion* whenever it admits a contact embedding of a planar $k$-torsion domain.

Definition 3.3. Suppose $(M, \xi)$ is a contact 3-manifold containing a planar $k$-torsion domain $M_0 \subset M$ with planar piece $M_0^P$ for some $k \geq 0$, and $\Omega$ is a closed 2-form on $M$. If every interface torus $T \subset M_0$ lying in $M_0^P$ satisfies $\int_T \Omega = 0$, then we say that $(M, \xi)$ has *$\Omega$-separating* planar $k$-torsion. We say that $(M, \xi)$ has *fully separating* planar $k$-torsion if this is true for every closed 2-form on $M$, or equivalently, each of the relevant interface tori separates $M$.

Example 3.4. The simplest examples of planar torsion domains have the form $S^1 \times \Sigma$, where $\Sigma$ is an orientable surface (possibly with boundary), the contact structure is $S^1$-invariant and the resulting dividing set $\Gamma \subset \Sigma$ contains the boundary. This may be viewed as a blown up summed open book whose pages are the connected components of $\Sigma \setminus \Gamma$, so the binding is empty, and the interface and boundary together are $S^1 \times \Gamma$. Some special cases are shown in Figure 3.

Remark 3.5. Another phenomenon that is allowed by the definition but not seen in the cases $S^1 \times \Sigma$ of Example 3.4 is for an irreducible subdomain to have interface tori in its interior, due to summing of a single connected open book to itself at

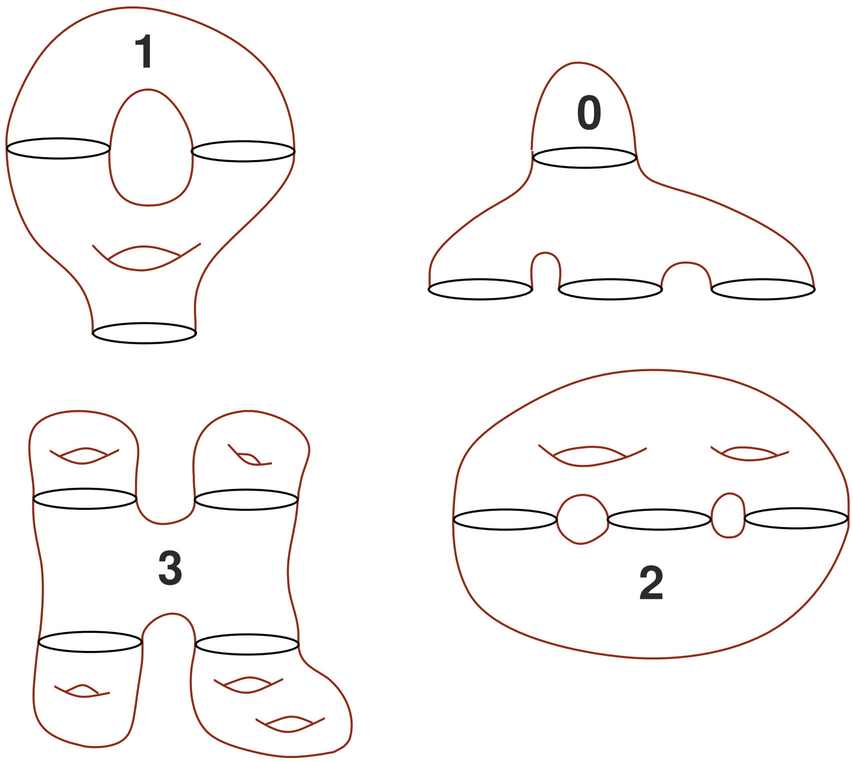


Figure 3: Some examples of convex surfaces and dividing sets that determine $S^1$-invariant planar torsion domains, of orders 1, 0, 3 and 2 respectively. The examples at the top right and bottom left are both fully separating. The bottom right example defines a closed manifold contactomorphic to the example $(V_4, \xi_3)$ from Theorem 4. Note that in this case, it's important that the two surfaces on either side of the dividing set are not diffeomorphic (so that the summed open book is not symmetric).

different binding components. Examples of this are shown in Figure 4, which also illustrates the fact that the choice of planar piece (and consequently the order of planar torsion) is not always unique, even for a fixed planar torsion domain.

It is shown in [We4] that a contact manifold has planar 0-torsion if and only if it is overtwisted, and every contact manifold with Giroux torsion also has planar 1-torsion. The latter is the reason why Theorem 6 implies Theorem 2.

**3.2 Proof of Theorem 6.** With these definitions in place, Theorem 6 follows easily from an existence and uniqueness result proved in [We4] for $J$-holomorphic curves in blown-up summed open books. Namely, suppose $(M, \xi)$ is a closed contact 3-manifold containing a compact and connected 3-dimensional submanifold $M_0$, possibly with boundary, on which $\xi$ is supported by a blown-up summed open book $\boldsymbol{\pi}$ with binding $B$, interface $\mathcal{I}$ and induced fibration $\pi : M_0 \setminus (B \cup \mathcal{I}) \to S^1$. Assume

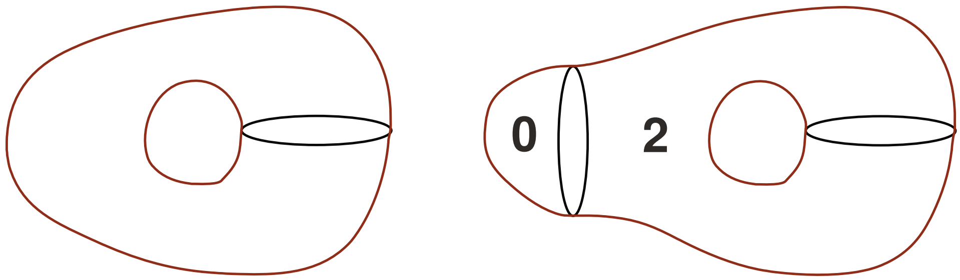


Figure 4: Schematic representations of two summed open books that include "self summing", i.e. interface tori in the interior of an irreducible subdomain. Assuming trivial monodromy, the example at the left is obtained from the tight $S^1 \times S^2$ with its obvious cylindrical open book by summing one binding component to the other: the result is a Stein fillable contact structure on the torus bundle over $S^1$ with monodromy $-1$. At the right, the additional subdomain with disk-like pages turns it into a planar torsion domain: the 3-manifold is the same, but the contact structure is changed by a half Lutz twist and is thus overtwisted. Note that in this example either irreducible subdomain can be taken as the planar piece, so it is both a 0-torsion domain and a 2-torsion domain.

there are $N \geq 2$ irreducible subdomains

$$M_0 = M_1 \cup \ldots \cup M_N ,$$

of which $M_1$ lies fully in the interior of $M_0$, and denote the corresponding restrictions of $\pi$ by

$$\pi_i : M_i \setminus (B_i \cup \mathcal{I}_i) \to S^1$$

for $i = 1, \ldots, N$, with $B_i := B \cap M_i$ and $\mathcal{I}_i := \mathcal{I} \cap \operatorname{int} M_i$. Note that while $\pi$ itself is not necessarily well defined at $\partial M_i$, $\pi_i$ always has a continuous extension to $\partial M_i$. Assume the pages in $M_i$ have genus $g_i \geq 0$, where $g_1 = 0$. In particular, $M_0$ is a planar torsion domain with planar piece $M_1$.

Proposition 3.6 [We4]. *For any number $\tau_0 > 0$, $(M, \xi)$ admits a Morse–Bott contact form $\lambda$ and compatible Fredholm regular almost complex structure $J$ with the following properties.*

(1) *On $M_0$, $\lambda$ is a Giroux form for $\boldsymbol{\pi}$.*
(2) *The Reeb orbits in $B$ are nondegenerate and elliptic, and the components of $\mathcal{I} \cup \partial M_0$ are all Morse–Bott submanifolds.*
(3) *All Reeb orbits in $B_1 \cup \mathcal{I}_1 \cup \partial M_1$ have minimal period at most $\tau_0$, while every other closed orbit of the Reeb vector field $X_\lambda$ in $M$ has minimal period at least 1.*
(4) *For each irreducible subdomain $M_i$ with $g_i = 0$, the fibration $\pi_i : M_i \setminus (B_i \cup \mathcal{I}_i) \to S^1$ admits a $C^\infty$-small perturbation $\hat{\pi}_i : M_i \setminus (B_i \cup \mathcal{I}_i) \to S^1$ such that the*

*interior of each fiber* $\hat{\pi}_i^{-1}(\tau)$ *for* $\tau \in S^1$ *lifts uniquely to an* $\mathbb{R}$*-invariant family of properly embedded surfaces*

$$S^{(i)}_{\sigma,\tau} \subset \mathbb{R} \times M_i\,, \quad (\sigma, \tau) \in \mathbb{R} \times S^1,$$

*which are the images of embedded finite energy* $J$*-holomorphic curves*

$$u^{(i)}_{\sigma,\tau} = \left(a^{(i)}_\tau + \sigma, F^{(i)}_\tau\right) : \dot{S}_i \to \mathbb{R} \times M_i\,,$$

*all of them Fredholm regular with index* 2*, and with only positive ends.*

(5) *Suppose* $u : \dot{S} \to \mathbb{R} \times M$ *is a finite energy punctured* $J$*-holomorphic curve which is not a cover of a trivial cylinder, and such that all its positive asymptotic orbits are simply covered and contained in* $B_1 \cup \mathcal{I}_1 \cup \partial M_1$*, with at most one positive end approaching each connected component of* $B_1 \cup \partial M_1$ *and at most two approaching each connected component of* $\mathcal{I}_1$*. Then* $u$ *has genus zero and parametrizes one of the surfaces* $S^{(i)}_{\sigma,\tau}$ *described above.*

Recall that a $J$-holomorphic curve is called *Fredholm regular* if it corresponds to a transversal intersection of the appropriate section of a Banach space bundle with the zero-section, see for example [We2]. We also say that $J$ is Fredholm regular if every somewhere injective $J$-holomorphic curve is Fredholm regular; this is a generic condition due to [D]. If $u$ is a rigid curve that is Fredholm regular, this implies in particular that $u$ can be perturbed uniquely to a solution of any sufficiently small perturbation of the nonlinear Cauchy–Riemann equation.

*Proof of Theorem 6.* The following is an adaptation of the argument used in [We4] to show that planar torsion kills the ECH contact invariant, and it can similarly be used to compute an upper bound on the integer $f^T_{\text{simp}}(M, \lambda, J)$ defined via ECH in the appendix. Given a closed 2-form $\Omega$ on $M$, let $k_0 \leq k$ be the smallest order of $\Omega$-separating planar torsion that $(M, \xi)$ admits. We will prove that $(M, \xi)$ then has $\Omega$-twisted algebraic $k_0$-torsion, which as previously observed, implies algebraic $k$-torsion. The statement for untwisted algebraic torsion is then the special case where $\Omega = 0$. Throughout the proof, for any $d \in H_2(M;\mathbb{R})$, we denote by

$$\bar{d} \in H_2(M;\mathbb{R})/\ker\Omega$$

the corresponding equivalence class.

Suppose $M_0 \subset (M, \xi)$ is a planar $k_0$-torsion domain with planar piece $M_0^P \subset M_0$, such that $[T] \subset \ker\Omega \subset H_2(M;\mathbb{R})$ for every interface torus $T$ lying in $M_0^P$. Denote by

$$\pi^P : M_0^P \setminus (B^P \cup \mathcal{I}^P) \to S^1$$

the corresponding fibration in the planar piece. Write the connected components of the binding, interface and boundary respectively as

$$B^P = \gamma_1 \cup \ldots \cup \gamma_m\,,$$
$$\partial M_0^P = T_1 \cup \ldots \cup T_n\,,$$
$$\mathcal{I}^P = T_{n+1} \cup \ldots \cup T_{n+r}\,,$$

where by definition we have

$$m + n + 2r = k_0 + 1 \quad \text{and} \quad n \geq 1\,.$$

Now, given the special Morse–Bott contact form $\lambda_0$ and compatible almost complex structure $J_0$ provided by Proposition 3.6, we consider the moduli space

$$\mathcal{M}(J_0) := \mathcal{M}_0(\gamma_1, \ldots, \gamma_m, T_1, \ldots, T_n, T_{n+1}, T_{n+1}, \ldots, T_{n+r}, T_{n+r}; J_0)$$

of unparametrized $J_0$-holomorphic curves $u : \dot{S} \to \mathbb{R} \times M$ such that

1. $\dot{S}$ has genus 0, no negative punctures and $m + n + 2r$ positive punctures
$$z_1, \ldots, z_m, \zeta_1, \ldots, \zeta_n, w_1^+, w_1^-, \ldots, w_r^+, w_r^- .$$
2. For the punctures listed above, $u$ approaches the simply covered orbit $\gamma_i$ at $z_i$, any simply covered orbit in $T_i$ at $\zeta_i$ and any simply covered orbit in $T_{n+i}$ at both $w_i^+$ and $w_i^-$.

By Proposition 3.6, $\mathcal{M}$ is a connected 2-dimensional manifold consisting of an $\mathbb{R}$-invariant family of embedded Fredholm regular curves that project to the pages in $M_0^P$. Note here we are using the fact that the blown-up summed open book on $M_0$ is not symmetric, so in particular the padding $M_0 \setminus M_0^P$ cannot contain additional genus 0 curves with the asymptotic behavior that defines $\mathcal{M}(J_0)$. It also cannot contain any genus 0 curves asymptotic to a proper subset of the same orbits, as this would mean the existence of an $\Omega$-separating planar torsion domain with order less than $k_0$.

We next perturb the Morse–Bott data $(\lambda_0, J_0)$ to generic nondegenerate data $(\lambda, J)$ by the scheme described in [B], extend $J$ to a suitable framing $\mathfrak{f}$ and assume that $H_*^{\mathrm{SFT}}(M, \lambda, \mathfrak{f}, \Omega)$ is well defined (see Remark 3.7 below). Recall that the perturbation to nondegenerate data is achieved by choosing a Morse function on each of the relevant Morse–Bott families of orbits and using it to alter the contact form in small neighborhoods of these families. In our case, each Morse-Bott family is parametrized by a circle, so we may assume without loss of generality that our Morse function on $S^1$ has exactly two critical points, which correspond to the two orbits in the family that survive as nondegenerate orbits after the perturbation. Moreover, $J$-holomorphic curves are obtained as perturbations of $J_0$-holomorphic "cascades", i.e. multi-level buildings composed of a mixture of holomorphic curves with gradient flow lines along the Morse–Bott manifolds. We may therefore assume after the perturbation that each of the tori $T_i$ for $i = 1, \ldots, n + r$ contains two nondegenerate simple Reeb orbits $\gamma_i^e$ and $\gamma_i^h$, elliptic and hyperbolic respectively. These orbits come with preferred framings determined by the tangent spaces to $T_i$, and in these framings their Conley–Zehnder indices are

$$\mu_{\mathrm{CZ}}(\gamma_i^e) = 1 \quad \text{and} \quad \mu_{\mathrm{CZ}}(\gamma_i^h) = 0 .$$

There are also two embedded $J$-holomorphic index 1 cylinders (corresponding to gradient flow lines along the Morse–Bott family)

$$v_i^{\pm} : \mathbb{R} \times S^1 \to \mathbb{R} \times M$$

whose projections to $M$ are disjoint and fill the two regions in $T_i$ separated by $\gamma_i^e$ and $\gamma_i^h$, so the homology classes they represent are related to each other by

$$[v_i^+] - [v_i^-] = [T_i] \in H_2(M; \mathbb{R}) ,$$

and for a suitable choice of coherent orientation, these two together contribute terms of the form

$$\left(z^{\overline{[T_i]}}-1\right)q_{\gamma_i^h}\frac{\partial}{\partial q_{\gamma_i^e}}$$

to the operator $\mathbf{D}_{\mathrm{SFT}}$. The curves in $\mathcal{M}(J_0)$ likewise give rise to a unique $J$-holomorphic punctured sphere in the space

$$\mathcal{M}(J) := \mathcal{M}_0(\gamma_1,\dots,\gamma_m,\gamma_1^h,\gamma_2^e,\dots,\gamma_n^e,\gamma_{n+1}^e,\gamma_{n+1}^e,\dots,\gamma_{n+r}^e,\gamma_{n+r}^e;J)$$

with puncture $\zeta_1$ asymptotic to $\gamma_1^h$ and all other punctures asymptotic to elliptic orbits. This curve is embedded and has index 1, thus if $d \in H_2(M;\mathbb{R})$ denotes the homology class defined by the pages in $M_0^P$ with attached capping surfaces, then this curve produces a term

$$z^{\bar{d}}\hbar^{m+n+2r-1}\frac{\partial}{\partial q_{\gamma_1^h}}\prod_{i=1}^{m}\frac{\partial}{\partial q_{\gamma_i}}\prod_{i=2}^{n}\frac{\partial}{\partial q_{\gamma_i^e}}\prod_{i=1}^{r}\frac{1}{2}\frac{\partial}{\partial q_{\gamma_{n+i}^e}}\frac{\partial}{\partial q_{\gamma_{n+i}^e}}$$

in $\mathbf{D}_{\mathrm{SFT}}$. We thus define the monomial

$$F = q_{\gamma_1}\dots q_{\gamma_m}q_{\gamma_1^h}q_{\gamma_2^e}\dots q_{\gamma_n^e}q_{\gamma_{n+1}^e}q_{\gamma_{n+1}^e}\dots q_{\gamma_{n+r}^e}q_{\gamma_{n+r}^e}$$

and compute,

$$\mathbf{D}_{\mathrm{SFT}}F = z^{\bar{d}}\hbar^{k_0} + \sum_{i=2}^{n+r}\left(z^{\overline{[T_i]}}-1\right)q_{\gamma_i^h}\frac{\partial F}{\partial q_{\gamma_i^e}}\,.$$

Every term in the summation now vanishes since $[T_i] \subset \ker\Omega$, implying that $\hbar^{k_0}$ is exact. □

Remark 3.7. To make the above computation fully rigorous, one must show that the relevant count of curves doesn't change under a suitable abstract perturbation, e.g. as provided by [H2]. The curves that were counted in the above argument are Fredholm regular and will thus survive any such perturbation, but we also need to check that no additional curves appear. If any such curves exist, then in the unperturbed limit they must give rise to nontrivial holomorphic *cascades* in the natural compactification of $\mathcal{M}(J_0)$, see [BEHWZ]. It suffices therefore to observe that in the above setup, all possible cascades are accounted for by the $J_0$-holomorphic pages in $M_0^P$, due to the uniqueness statement in Proposition 3.6.

## 4 $S^1$-Invariant Examples in Dimension 3

In this section we consider the special examples $(S^1\times\Sigma,\xi_\Gamma)$ described in the introduction, and prove in particular Theorems 3 and 4. Note that the examples $(V_g,\xi_k)$ of Theorem 4 can be constructed via a summed open book as follows. Fix $g \ge k \ge 1$, and let $(M_-,\xi_-)$ denote the closed contact 3-manifold supported by a planar open book $\pi_- : M_- \setminus B_- \to S^1$ with $k$ binding components and trivial monodromy. Similarly, let $(M_+,\xi_+)$ be the contact manifold supported by an open book $\pi_+ : M_+ \setminus B_+ \to S^1$ with pages of genus $g-k+1>0$, $k$ binding components and trivial monodromy. Choosing any one-to-one correspondence between the connected components of $B_+$ and $B_-$, we produce a new closed contact manifold $(M, \xi)$ by taking the binding sum of $(M_+, \xi_+) \sqcup (M_-, \xi_-)$ along corresponding binding components as described in section 3; this produces a closed planar $(k-1)$-torsion domain which is contactomorphic to $(V_g, \xi_k)$.

To complete the proof of Theorem 4, we will have to show that certain types of holomorphic curves in $\mathbb{R} \times V_g$ do *not* exist (at least algebraically), which would need to exist if $\hbar^{k-2}$ were exact (see Lemma 4.15 below). To do this, we will construct a precise model for contact manifolds of the form $(S^1 \times \Sigma, \xi_\Gamma)$, in which all the relevant holomorphic curves can be classified. The proof of Theorem 3 will also follow immediately from this classification.

**4.1 Holomorphic curves in $(S^1 \times \Sigma, \xi_\Gamma)$.** The basic idea of our model for $(S^1 \times \Sigma, \xi_\Gamma)$ will be to choose data so that the singular foliation of $\Sigma$ defined by the gradient flow lines of a suitable Morse function gives rise to a foliation of the symplectization by holomorphic cylinders, which can be counted by Morse homology. We will then be able to exclude all the other relevant curves by a combination of intersection arguments and index estimates.

For the constructions carried out below, the following lemma turns out to be convenient.

LEMMA 4.1. *Suppose $\Sigma$ is a compact connected oriented surface with nonempty boundary, and $\tilde{h} : \Sigma \to \mathbb{R}$ is a smooth Morse function with all critical points in the interior and none of index 2, and with $\partial\Sigma = \tilde{h}^{-1}(1)$. Then there exists a conformal structure $j$ on $\Sigma$, compatible with the orientation, and a smooth, strictly increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $h := \varphi \circ \tilde{h} : \Sigma \to \mathbb{R}$ satisfies*

$$-d(dh \circ j) > 0\,,$$

*and each boundary component has a collar neighborhood biholomorphically identified with $(-\delta, 0] \times S^1$ for some small $\delta > 0$, so that in these holomorphic coordinates $(s, t) \in (-\delta, 0] \times S^1$ we have*

$$h(s,t) = e^s.$$

*Proof.* To construct $j$ with the required properties, we start by choosing oriented coordinates $(s, t) \in (-2\delta, 0] \times S^1$ on a collar neighborhood of each boundary component such that $\tilde{h}(s,t) = e^s$ in these coordinates. In this collar neighborhood, we simply define $j$ by requiring $j(\partial_s) = \partial_t$ and $j(\partial_t) = -\partial_s$. Note that

$$-d(d\tilde{h} \circ j) = e^s\, ds \wedge dt > 0$$

on these collars.

Next we choose oriented Morse coordinates near the critical points, such that locally

$$\tilde{h}(x,y) = x^2 \pm y^2 + \tilde{h}(0)\,.$$

In such coordinates, we can define $j$ such that $j(\partial_x) = \lambda \partial_y$ and $j(\partial_y) = -\frac{1}{\lambda}\partial_x$ for some $\lambda > 0$. A computation then yields

$$-d(d\tilde{h} \circ j) = \left(\tfrac{2}{\lambda} \pm 2\lambda\right) dx \wedge dy\,,$$

which is positive whenever $0 < \lambda < 1$.

Now extend $j$ arbitrarily to all of $\Sigma$ and consider the function $h = \varphi \circ \tilde{h}$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth function with $\varphi' > 0$ and $\varphi'' \geq 0$. Observe that the 2-form

$$\mu := -d\tilde{h} \wedge (d\tilde{h} \circ j)$$

is everywhere non-negative, and vanishes precisely at the critical points of $\tilde{h}$. We then compute,

$$-d(dh \circ j) = -(\varphi' \circ \tilde{h})\, d(d\tilde{h} \circ j) + (\varphi'' \circ \tilde{h})\mu\,. \tag{4.1}$$

This is already positive whenever $-d(d\tilde{h} \circ j)$ is positive, which is true on a neighborhood of the critical points and the boundary. Outside of this neighborhood, we have $\mu > 0$ and can thus arrange $-d(dh \circ j) > 0$ by choosing $\varphi$ so that

$$\frac{\varphi''}{\varphi'} \geq K$$

for a sufficiently large constant $K > 0$. Since $-d(d\tilde{h} \circ j) > 0$ on the collar neighborhoods $(-2\delta, 0] \times S^1$ of $\partial\Sigma$, we are free to set $\varphi'' = 0$ in $[-\delta, 0] \times S^1$. Now since $-d(dh \circ j) > 0$ everywhere, (4.1) implies that this property will survive a further post-composition with an increasing affine function, hence through such a composition we can arrange without loss of generality that $\varphi(s) = s$ on the collar neighborhoods $[-\delta, 0] \times S^1$. □

Let $\Sigma_-$ and $\Sigma_+$ denote compact oriented and possibly disconnected surfaces, such that each connected component has non-empty boundary and the total number of boundary components of $\Sigma_-$ and $\Sigma_+$ agrees. On each of the surfaces $\Sigma_\pm$, we choose a function $h_\pm$ and conformal structure $j_\pm$ as provided by the lemma and define a 1-form by

$$\beta_\pm = -dh_\pm \circ j_\pm\,.$$

This induces a symplectic form $\sigma_\pm$ and Riemannian metric $g_\pm$ on $\Sigma_\pm$, defined by

$$\sigma_\pm = d\beta_\pm, \quad g_\pm = \sigma_\pm(\,\cdot\,, j_\pm \cdot\,)\,.$$

Since $dh_\pm = e^s\, ds$ in holomorphic coordinates $(s,t) \in (-\delta, 0] \times S^1$ near each component of the boundary, we find

$$\sigma_\pm = e^s\, ds \wedge dt\,, \quad \nabla h_\pm = \partial_s\,.$$

Denote the union of all these collar neighborhoods of $\partial\Sigma_\pm$ by

$$\mathcal{U}_\pm \subset \Sigma_\pm\,.$$

The gradient $\nabla h_\pm$ is a Liouville vector field pointing orthogonally outward at $\partial\Sigma_\pm$.

Remark 4.2. Since the subharmonicity condition on the pair $(h_\pm, j_\pm)$ is open, there is some freedom in the construction. In particular, by perturbing the conformal structure if necessary we can achieve that the flow of $\nabla h_\pm$ is Morse–Smale.

We now glue $\Sigma_+$ and $\Sigma_-$ together along an orientation preserving diffeomorphism $\partial\Sigma_+ \to \partial\Sigma_-$ to create a closed oriented surface

$$\Sigma = \Sigma_+ \cup (-\Sigma_-)\,,$$

divided into two halves by a special set of circles $\Gamma := \partial\Sigma_+ \subset \Sigma$. We will always assume $\Sigma$ is connected, and as the above notation suggests we assign it the same orientation as $\Sigma_+$, which is opposite the given orientation on $\Sigma_-$. On each connected component of $\mathcal{U}_+$ and $\mathcal{U}_-$, one can define new coordinates

$$S^1 \times [0,\delta) \ni (\theta,\rho) := (t,-s) \text{ for } (s,t) \in \mathcal{U}_+ ,$$
$$S^1 \times (-\delta,0] \ni (\theta,\rho) := (t,s) \text{ for } (s,t) \in \mathcal{U}_- ,$$

and then define the gluing map and the smooth structure on $\Sigma$ so that each component of $\mathcal{U} := \mathcal{U}_+ \cup \mathcal{U}_- \subset \Sigma$ inherits smooth positively oriented coordinates $(\theta,\rho) \in S^1 \times (-\delta,\delta)$.

Choose a function $g_0 : [-\delta,\delta] \to \mathbb{R}$ with $g_0(\rho) = \pm 1$ for $\rho$ near $\pm\delta$, $g_0(0) = 0$, $g_0' \geq 0$ and $g_0' > 0$ near $\rho = 0$ and a function $\gamma : [-\delta,\delta] \to \mathbb{R}$ with $\gamma(\rho) = \mp e^{\mp\rho}$ for $\rho$ near $\pm\delta$, $\gamma' > 0$ wherever $g_0' = 0$, $\gamma(\rho) > 0$ for $\rho < 0$ and $\gamma(\rho) < 0$ for $\rho > 0$. For $\epsilon \in (0,1)$, we then set

$$g_\epsilon(\rho) = g_0(\rho) + \epsilon^2\gamma(\rho) ,$$

which satisfies

- $g_\epsilon' > 0$ for sufficiently small $\epsilon > 0$;
- $g_\epsilon(\rho) = \pm(1 - \epsilon^2 e^{\mp\rho})$ for $\rho$ near $\pm\delta$;
- $g_\epsilon(0) = 0$.

Now define a smooth family of functions $h_\epsilon : \Sigma \to \mathbb{R}$ by

$$h_\epsilon = \begin{cases} 1 - \epsilon^2 h_+ & \text{on } \Sigma_+ \setminus \mathcal{U}_+ , \\ g_\epsilon(\rho) & \text{for } (\theta,\rho) \in \mathcal{U} , \\ -1 + \epsilon^2 h_- & \text{on } \Sigma_- \setminus \mathcal{U}_- . \end{cases}$$

For each fixed $\epsilon > 0$, $h_\epsilon$ is a Morse function with all its critical points in $\Sigma \setminus \mathcal{U}$, and they are precisely the critical points of $h_\pm$.

Next choose a function $f_0 : [-\delta,\delta] \to \mathbb{R}$ such that $f_0(\rho) = 0$ for $\rho$ near $\pm\delta$, $f_0 \geq 0$ everywhere and $\rho \cdot f_0'(\rho) \leq 0$ for $\rho \neq 0$ and $f_0''(0) < 0$, and a function $\psi : [-\delta,\delta] \to \mathbb{R}$ with $\psi(\rho) = e^{\pm\rho}$ for $\rho$ near $\mp\delta$, $\psi \geq 0$ everywhere and $\rho \cdot \psi'(\rho) < 0$ for $\rho \neq 0$. Then we define

$$f_\epsilon(\rho) = f_0(\rho) + \epsilon\psi(\rho) .$$

With these choices in place, we denote the coordinate in $S^1$ by $\phi$ and define a smooth family of 1-forms $\lambda_\epsilon$ on $S^1 \times \Sigma$ by

$$\lambda_\epsilon = \begin{cases} \epsilon\beta_+ + h_\epsilon \, d\phi & \text{on } S^1 \times (\Sigma_+ \setminus \mathcal{U}_+) , \\ f_\epsilon(\rho)\, d\theta + g_\epsilon(\rho)\, d\phi & \text{on } S^1 \times \mathcal{U} , \\ \epsilon\beta_- + h_\epsilon \, d\phi & \text{on } S^1 \times (\Sigma_- \setminus \mathcal{U}_-) . \end{cases} \tag{4.2}$$

Observe that $S^1 \times \Sigma$ admits a natural summed open book with empty binding, interface $\mathcal{I} = S^1 \times \Gamma$, fibration

$$\pi : S^1 \times (\Sigma \setminus \Gamma) \to S^1 : (\phi, z) \mapsto \begin{cases} \phi & \text{if } z \in \Sigma_+ , \\ -\phi & \text{if } z \in \Sigma_- , \end{cases}$$

and the meridians on $S^1 \times \Gamma$ generated by the circles $S^1 \times \{\text{const}\}$.

Proposition 4.3. *There exists $\epsilon_0 > 0$ with the following properties:*

(i) *For any $\epsilon \in (0, \epsilon_0]$, $\lambda_\epsilon$ is a positive contact form on $S^1 \times \Sigma$ and is a Giroux form for the summed open book described above. Moreover, for all these contact forms each component of the interface $S^1 \times \Gamma$ is a Morse–Bott submanifold of Reeb orbits pointing in the $\partial_\theta$-direction.*

(ii) *For any $\epsilon \in (0, \epsilon_0]$ and for each $\phi \in S^1$, the leaves of the characteristic foliation on $\{\phi\} \times \Sigma$ are precisely the gradient flow lines of $h_\epsilon$.*

(iii) *The 2-form $\omega = d(e^s \lambda_s)$ is symplectic on $(0, \epsilon_0] \times S^1 \times \Sigma$, where $s$ denotes the coordinate on the first factor.*

*Proof.* To prove (i), note that the natural co-orientation induced by the summed open book on its pages is compatible with the orientations defined on $\Sigma_\pm$ by $j_\pm$, for which $\sigma_\pm$ are positive volume forms. To prove the contact condition on $S^1 \times (\Sigma_\pm \setminus \mathcal{U}_\pm)$, observe that $\lambda_\epsilon \to \pm d\phi$ on this region as $\epsilon \to 0$, so the contact planes are almost tangent to the pages. Thus it suffices to observe that $d\lambda_\epsilon$ is positive on $\Sigma_\pm \setminus \mathcal{U}_\pm$, which is clear since $d\lambda_\epsilon = \epsilon\sigma_\pm$ when restricted to the pages.

On $S^1 \times \mathcal{U}$, a routine computation shows that the contact condition follows from $f_\epsilon g_\epsilon' - f_\epsilon' g_\epsilon > 0$. But this is easily computed to equal

$$f_\epsilon g_\epsilon' - f_\epsilon' g_\epsilon = f_0 g_0' - f_0' g_0 + \epsilon(\psi g_0' - \psi' g_0) + \mathcal{O}(\epsilon^2)\,.$$

Our conditions on the various functions ensure that all four summands are non-negative, with the first one strictly positive for $\rho$ near 0 and the last one strictly positive for $\rho$ away from zero. So for $\epsilon_0 > 0$ sufficiently small, the contact condition holds for all $\epsilon \in (0, \epsilon_0]$ on $S^1 \times \mathcal{U}$ as well. Here it is also easy to compute the Reeb vector field $X_{\lambda_\epsilon}$: writing $D_\epsilon = f_\epsilon g_\epsilon' - f_\epsilon' g_\epsilon$, we have

$$X_{\lambda_\epsilon}(\phi, \rho, \theta) = \frac{1}{D_\epsilon(\rho)} \left[ g_\epsilon'(\rho)\, \frac{\partial}{\partial\theta} - f_\epsilon'(\rho)\, \frac{\partial}{\partial\phi} \right]. \tag{4.3}$$

Our assumptions on $f_\epsilon'(\rho)$ then imply that $X_{\lambda_\epsilon}$ always has a component in the negative $\partial_\phi$-direction for $\rho \in (-\delta, 0)$, and in the positive $\partial_\phi$-direction for $\rho \in (0, \delta)$, while at $\rho = 0$ it points in the $\partial_\theta$-direction. Moreover, the condition $g_\epsilon(0) = 0$ implies that the contact planes at $\rho = 0$ are tangent to the circles $S^1 \times \{\text{const}\}$, thus $\lambda_\epsilon$ is a Giroux form. The Morse–Bott condition at $S^1 \times \Gamma$ follows from $f_\epsilon''(0) < 0$, which for small $\epsilon > 0$ follows from $f_0''(0) < 0$. This concludes the proof of (i).

Next we verify that the characteristic foliation on $\{\phi\} \times \Sigma$ matches the gradient flow of $h_\epsilon$. This is obvious in $\mathcal{U}$, where both characteristic leaves and gradient flow lines are simply straight lines in the $\partial_\rho$-direction. On $\Sigma_\pm \setminus \mathcal{U}_\pm$, a vector $v \in T\Sigma_\pm$ is tangent to the characteristic foliation if and only if $\beta_\pm(v) = 0$, implying $dh_\pm(j_\pm v) = 0$ and thus $v$ is orthogonal to the level sets of $h_\pm$, which makes it proportional to $\nabla h_\pm$ as claimed, and establishes (ii).

Finally, consider the two-form $\omega = d(e^s \lambda_s)$. On $\mathbb{R} \times S^1 \times \mathcal{U}$, we have $\lambda_s = f_s\, d\theta + g_s\, d\phi$ and so

$$\omega = e^s (ds \wedge \lambda_s + df_s \wedge d\theta + dg_s \wedge d\phi)\,,$$

with

$$df_s = f_s'\, d\rho + \psi\, ds\,,$$

$$dg_s = g_s'\, d\rho + 2s\gamma\, ds\,.$$

One then computes

$$\omega \wedge \omega = e^s (f_s g_s' - f_s' g_s + \psi g_s' - 2\gamma s f_s')\, ds \wedge d\theta \wedge d\rho \wedge d\phi$$

here, and observe that all four terms are non-negative, with the first one strictly positive for small $s > 0$, so $\omega$ is symplectic here.

On $\mathbb{R} \times S^1 \times (\Sigma_+ \setminus \mathcal{U})$, we have $\lambda_s = s\beta_+ + (1 - s^2 h_+)\, d\phi$, and so another computation shows

$$\omega \wedge \omega = e^{2s}\big(s\, \sigma_+ \wedge ds \wedge d\phi + \mathcal{O}(s^2)\big)$$

here, which is also a positive volume form for small enough $s > 0$. A similar computation on $\mathbb{R} \times S^1 \times (\Sigma_- \setminus \mathcal{U})$ finishes the proof of part (iii). □

From now on, denote the contact structure on $S^1 \times \Sigma$ for $\epsilon \in (0, \epsilon_0]$ by

$$\xi_\epsilon = \ker \lambda_\epsilon\,.$$

Due to Gray's stability theorem, $\xi_\epsilon$ is independent of $\epsilon$ up to isotopy, and it is isomorphic to $\xi_\Gamma$.

Remark 4.4. From the discussion above it is clear that for every $\phi \in S^1$, $\{\phi\} \times \Sigma$ is a convex surface for $\xi_\epsilon$ with dividing set $\Gamma$, positive part $\Sigma_+$ and negative part $\Sigma_-$. In particular, the Euler class $e(\xi_\epsilon) \in H^2(S^1 \times \Sigma)$ satisfies $\langle e(\xi_\epsilon), [\{*\} \times \Sigma]\rangle = \chi(\Sigma_+) - \chi(\Sigma_-)$. It follows from the $S^1$-invariance of $\xi_\epsilon$ that the Euler class vanishes on all cycles of the form $S^1 \times \gamma$ for closed curves $\gamma \subset \Sigma$. Thus

$$e(\xi_\epsilon) = \big[\chi(\Sigma_+) - \chi(\Sigma_-)\big]\, \mathrm{PD}\big[S^1 \times \{*\}\big]\,.$$

The following assertion can be checked by a routine computation.

Lemma 4.5. *The Reeb vector field $X_{\lambda_\epsilon}$ on $S^1 \times (\Sigma_\pm \setminus \mathcal{U}_\pm)$ is given by*

$$X_{\lambda_\epsilon} = \frac{1}{1 + \epsilon^2(|\nabla h_\pm|^2_{g_\pm} - h_\pm)} \left(\pm\frac{\partial}{\partial\phi} + \epsilon j_\pm \nabla h_\pm\right). \tag{4.4}$$

□

In particular, this shows that every critical point $z \in \mathrm{Crit}(h_\epsilon)$ gives rise to a periodic orbit

$$\gamma_z := S^1 \times \{z\}$$

of $X_{\lambda_\epsilon}$. We shall denote by $\gamma_z^n$ the $n$-fold cover of $\gamma_z$ for any $n \in \mathbb{N}$ and $z \in \mathrm{Crit}(h_\epsilon)$. Observe that there is always a natural trivialization of the contact bundle along $\gamma_z^n$, defined by choosing any frame at a point and transporting by the $S^1$-action.

We next define a compatible complex structure $J_\epsilon$ on $\xi_\epsilon$ as follows. On $S^1 \times (\Sigma_\pm \setminus \Gamma)$, the projection $S^1 \times \Sigma \to \Sigma$ defines a bundle isomorphism

$$\pi_\Sigma : \xi_\epsilon|_{S^1 \times (\Sigma \setminus \Gamma)} \to T\Sigma|_{S^1 \times (\Sigma \setminus \Gamma)}\,,$$

which we can use to define $J_\epsilon : \xi_\epsilon \to \xi_\epsilon$ on $S^1 \times (\Sigma_\pm \setminus \mathcal{U}_\pm)$ by

$$J_\epsilon = \pi_\Sigma^* j_\pm\,. \tag{4.5}$$

Since $\partial_\rho \in \xi_\epsilon$ on $S^1 \times \mathcal{U}$, we can now extend $J_\epsilon$ to this region by setting

$$J_\epsilon \partial_\rho = \alpha_\epsilon(\rho)\big[f_\epsilon(\rho)\partial_\phi - g_\epsilon(\rho)\partial_\theta\big]\,,$$

for any smooth family of functions $\alpha_\epsilon : (-\delta, \delta) \to (0, \infty)$ which equals $\pm 1/g_\epsilon$ near $\rho = \pm\delta$, so in particular for $\epsilon > 0$, $J_\epsilon$ satisfies

$$d\rho(J_\epsilon \partial_\rho) = 0 \quad \text{and} \quad d\lambda_\epsilon(\partial_\rho, J_\epsilon \partial_\rho) > 0\,.$$

Extend $J_\epsilon$ to an $\mathbb{R}$-invariant almost complex structure

$$J_\epsilon : T\left(\mathbb{R} \times (S^1 \times \Sigma)\right) \to T\left(\mathbb{R} \times (S^1 \times \Sigma)\right)$$

in the standard way, i.e. by setting $J_\epsilon \partial_s = X_{\lambda_\epsilon}$ where $s$ is the $\mathbb{R}$-coordinate. Then for each $z \in \operatorname{Crit}(h_\epsilon)$, there is a *trivial cylinder*

$$\mathbb{R} \times S^1 \to \mathbb{R} \times (S^1 \times \Sigma) : (s,t) \mapsto (s,t,z)\,,$$

which can be reparametrized to define an embedded $J_\epsilon$-holomorphic curve of Fredholm index 0. We shall abbreviate this curve by $\mathbb{R} \times \gamma_z$, and similarly write $\mathbb{R} \times \gamma_z^n$ for the obvious $J_\epsilon$-holomorphic $n$-fold cover of $\mathbb{R} \times \gamma_z$.

PROPOSITION 4.6. *For $\epsilon \in (0, \epsilon_0]$, suppose $x : \mathbb{R} \to \Sigma$ is a solution to the gradient flow equation $\dot{x} = \nabla h_\epsilon(x)$ approaching $z_\pm \in \operatorname{Crit}(h_\epsilon)$ at $\pm\infty$. Then there exists a proper function $a : \mathbb{R} \to \mathbb{R}$, unique up to a constant, such that the embedding*

$$u_x : \mathbb{R} \times S^1 \to \mathbb{R} \times (S^1 \times \Sigma) : (s,t) \mapsto \big(a(s), t, x(s)\big)$$

*is a $J_\epsilon$-complex curve. Both ends of $u$ are positive if and only if the two critical points $z_+$ and $z_-$ lie on opposite sides of the interface.*

*Proof.* For any $z \in \Sigma$, regard $\nabla h_\epsilon(z)$ as a vector in $T_{(\phi,z)}(S^1 \times \Sigma)$ for some fixed $\phi \in S^1$, and observe that $\nabla h_\epsilon(z) \in (\xi_\epsilon)_z$ due to Proposition 4.3. Thus we can define an $S^1$-invariant vector field

$$v(\phi, z) = J_\epsilon \nabla h_\epsilon(z)\,,$$

which takes values in $\xi_\epsilon$ and vanishes only at $S^1 \times \operatorname{Crit}(h_\epsilon)$. For $z \in \Sigma_\pm \setminus \mathcal{U}_\pm$, (4.5) implies that $v(\phi, z)$ is a linear combination of $j_\pm \nabla h_\epsilon(z)$ and $\partial_\phi$, and the same is true for $z \in \mathcal{U}$ due to the condition $d\rho(J_\epsilon \partial_\rho) = 0$. By (4.3) and (4.4), the Reeb vector field $X_{\lambda_\epsilon}$ is also a linear combination of the same two vector fields everywhere, and is of course linearly independent of $v$ except when the latter vanishes, from which we conclude

$$\partial_\phi \in \mathbb{R}X_{\lambda_\epsilon} \oplus \mathbb{R}v$$

everywhere on $S^1 \times \Sigma$. It follows that $J_\epsilon \partial_\phi$ is everywhere a linear combination of $\partial_s$ and $\nabla h_\epsilon$, so the desired complex curves are obtained by integrating the distribution

$$\mathbb{R}\partial_\phi \oplus \mathbb{R}J_\epsilon \partial_\phi\,.$$

In particular, this generates a foliation whose leaves include an $\mathbb{R}$-invariant family of cylinders of the form $u_x$ described above for each nontrivial gradient flow line $x : \mathbb{R} \to \Sigma$, and the trivial cylinders $\mathbb{R} \times \gamma_z$ defined above for each $z \in \operatorname{Crit}(h_\epsilon)$. The signs of the cylindrical ends can now be deduced from the orientations of the Reeb orbits, using the fact that the orientations of $\gamma_z$ and $\gamma_\zeta$ in the $S^1$-direction match if and only if $z$ and $\zeta$ lie on the same side of the dividing set $\Gamma$. □

From the proposition it follows that each of the embeddings $u_x$ is a (not necessarily $J_\epsilon$-holomorphic) parametrization of a finite energy $J_\epsilon$-holomorphic curve, whose Fredholm index $\operatorname{ind}(u_x)$ is the sum of the Conley–Zehnder indices at its ends if both are positive, or the difference if one end is negative. We shall abuse notation by identifying the map $u_x : \mathbb{R} \times S^1 \to \mathbb{R} \times (S^1 \times \Sigma)$ with the unique unparametrized $J_\epsilon$-holomorphic curve it determines, and do the same with the obvious unbranched multiple cover

$$u_x^n(s,t) := u_x(s,nt)$$

for each $n \in \mathbb{N}$.

Proposition 4.7. *Assume $h_+$ and $h_-$ are chosen so that their gradient flows are Morse–Smale (see Remark 4.2). Then after possibly adjusting the gluing map $\partial\Sigma_+ \to \partial\Sigma_-$, there exist functions*

$$(0,\epsilon_0] \to (0,\infty) : \epsilon \mapsto T_\epsilon$$
$$(0,\epsilon_0] \to \mathbb{N} : \epsilon \mapsto N_\epsilon$$

*with $\lim_{\epsilon\to 0} T_\epsilon = \lim_{\epsilon\to 0} N_\epsilon = +\infty$ such that the following conditions hold for all $\epsilon > 0$:*

1. *$\nabla h_\epsilon$ is Morse–Smale.*
2. *Every closed orbit of $X_{\lambda_\epsilon}$ with period less than $T_\epsilon$ is either in $S^1 \times \mathcal{U}$ or is $\gamma_z^n$ for some $z \in \operatorname{Crit}(h_\epsilon)$ and $n \le N_\epsilon$.*
3. *For all $n \le N_\epsilon$, $\gamma_z^n$ is nondegenerate as an orbit of $X_{\lambda_\epsilon}$ and has Conley–Zehnder index*

$$\mu_{\mathrm{CZ}}(\gamma_z^n) = \begin{cases} 1 & \text{if } \operatorname{ind}(z) = 0 \text{ or } 2\,, \\ 0 & \text{if } \operatorname{ind}(z) = 1\,, \end{cases} \tag{4.6}$$

*with respect to the $S^1$-invariant trivialization of $\xi_\epsilon$ along $\gamma_z^n$, where $\operatorname{ind}(z)$ denotes the Morse index of $z$.*

*Proof.* Up to parametrization, the flow of $\nabla h_\epsilon$ matches that of $\nabla h_\pm$ on $\Sigma_\pm \setminus \mathcal{U}_\pm$ and $\partial_\rho$ on $\mathcal{U}$. Thus if $\nabla h_\pm$ are both Morse–Smale, any flow lines of $\nabla h_\epsilon$ connecting two index 1 critical points must pass through $\Gamma$, and can thus be eliminated by a small rotation of the gluing map $\partial\Sigma_+ \to \partial\Sigma_-$. The existence of the function $T_\epsilon$ with $\lim_{\epsilon\to 0} T_\epsilon = \infty$ follows from (4.4), as all orbits outside of $S^1 \times \mathcal{U}$ other than the $\gamma_z^n$ for $z \in \operatorname{Crit}(h_\epsilon)$ correspond to closed orbits of $j_\pm \nabla h_\pm$ in level sets of $h_\pm$, with periods that become infinitely large as $\epsilon \to 0$. We can then define

$$N_\epsilon := \max\{n \in \mathbb{N} \mid \text{All } \gamma_z^n \text{ have periods} < T_\epsilon \text{ as orbits of } X_{\lambda_\epsilon}\}\,,$$

and observe that $N_\epsilon \to \infty$ as $\epsilon \to 0$ since the periods of $\gamma_z$ converge to 1. The formula for $\mu_{\mathrm{CZ}}(\gamma_z^n)$ is a standard computation from Floer theory relating Conley–Zehnder indices to Morse indices, see for example [SZ]. □

We will assume from now on that the conditions of Proposition 4.7 are satisfied. Then $\nabla h_\epsilon$ is Morse–Smale for all $\epsilon \in (0,\epsilon_0]$, and it will follow that each of the $J_\epsilon$-holomorphic cylinders $u_x$ corresponding to gradient flow lines $x : \mathbb{R} \to \Sigma$ between critical points $z_-, z_+ \in \operatorname{Crit}(h_\epsilon)$ has positive Fredholm index. Indeed, these cylinders come in five types:

1. $z_- \in \Sigma_-$ with index 0 and $z_+ \in \Sigma_+$ with index 2: then $\mathrm{ind}(u_x) = 2$ and both ends are positive.
2. $z_-, z_+ \in \Sigma_+$ with indices 1 and 2: then $\mathrm{ind}(u_x) = 1$ and one end is negative.
3. $z_-, z_+ \in \Sigma_-$ with indices 0 and 1: then $\mathrm{ind}(u_x) = 1$ and one end is negative.
4. $z_- \in \Sigma_-$ with index 0 and $z_+ \in \Sigma_+$ with index 1: then $\mathrm{ind}(u_x) = 1$ and both ends are positive.
5. $z_- \in \Sigma_-$ with index 1 and $z_+ \in \Sigma_+$ with index 2: then $\mathrm{ind}(u_x) = 1$ and both ends are positive.

This classification is exactly the same for the multiply covered cylinders $u_x^n(s,t)$ for all $n \le N_\epsilon$.

PROPOSITION 4.8. *For every gradient flow line $x : \mathbb{R} \to \Sigma$, the corresponding $J_\epsilon$-holomorphic cylinders $u_x^n$ for $n \le N_\epsilon$ are all Fredholm regular.*

*Proof.* By the criterion in [We2, Th. 1] an immersed, connected finite energy $J_\epsilon$-holomorphic curve $u$ with genus $g$ asymptotic to nondegenerate Reeb orbits is Fredholm regular whenever

$$\mathrm{ind}(u) > 2g - 2 + \#\Gamma_0 \,,$$

where the integer $\#\Gamma_0 \ge 0$ denotes the number of ends at which $u$ approaches orbits with even Conley–Zehnder index. In the case at hand, we always have $g = 0$ and either $\mathrm{ind}(u) = 2$ with $\#\Gamma_0 = 0$ or $\mathrm{ind}(u) = 1$ with $\#\Gamma_0 = 1$, so the criterion is satisfied in all cases. □

It follows that the embedded cylinders $u_x$ for all gradient flow lines $x$ on $\Sigma$, together with the trivial cylinders $\mathbb{R} \times \gamma_z$ for $z \in \mathrm{Crit}(h_\epsilon)$, form a stable finite energy foliation in the sense of [HWZ2], [We1].

In the following, we will make use of the intersection theory for punctured holomorphic curves, defined by Siefring [Si]. This theory defines an intersection number

$$u * v \in \mathbb{Z}$$

for any two asymptotically cylindrical maps $u, v$ from punctured Riemann surfaces into the symplectization of a contact 3-manifold, with the following properties:

- $u * v$ is invariant under homotopies of $u$ and $v$ through asymptotically cylindrical maps.
- $u * v \ge 0$ whenever both are finite energy pseudoholomorphic curves that are not covers of the same somewhere injective curve, and the inequality is strict if they have nonempty intersection.

LEMMA 4.9. *Suppose $u$ and $v$ are finite energy pseudoholomorphic curves in the symplectization $\mathbb{R} \times M$ of a contact manifold $(M, \xi)$, such that $u$ has no negative ends, and the positive punctures $\zeta \in \Gamma_v^+$ of $v$ are asymptotic to Reeb orbits denoted by $\gamma_\zeta$. Then*

$$u * v = \sum_{\zeta \in \Gamma_v^+} u * (\mathbb{R} \times \gamma_\zeta)\,.$$

*Proof.* By $\mathbb{R}$-translation we can assume the image of $u$ is contained in $[0,\infty) \times M$, and can then homotop $v$ through a family of asymptotically cylindrical maps so that its intersection with $[0,\infty) \times M$ consists only of the trivial half-cylinders $[0,\infty) \times \gamma_\zeta$ for $\zeta \in \Gamma_v^+$. The lemma thus follows from the homotopy invariance of $u * v$. □

It is possible in general to have $u * v > 0$ even if $u$ and $v$ are disjoint holomorphic curves: in this case intersections can "emerge from infinity" under generic perturbations, and excluding this typically requires the computation of certain winding numbers. We will only need to worry about this in one special case:

LEMMA 4.10. *For any $z \in \mathrm{Crit}(h_\epsilon)$, a gradient flow line $x : \mathbb{R} \to \Sigma$ that begins and ends on opposite sides of the interface, and $n \le N_\epsilon$, $(\mathbb{R} \times \gamma_z^n) * u_x = 0$.*

*Proof.* The curves $\mathbb{R} \times \gamma_z^n$ and $u_x$ obviously do not intersect since $x$ does not pass through any critical points, so we only have to check that there are no asymptotic contributions to $(\mathbb{R} \times \gamma_z^n) * u_x$. This is trivially true unless $z$ is one of the end points of $x$, so assume the latter. Then the definition of the intersection number in [Si] implies that $(\mathbb{R} \times \gamma_z^n) * u_x = 0$ if and only if the asymptotic end of $u_x^n$ approaching $\gamma_z^n$ has the largest possible asymptotic winding about the orbit. This bound on the winding is an integer $\alpha_-(\gamma_z^n)$, which is the winding of a particular eigenfunction of the Hessian of the contact action functional, and was shown in [HWZ1] to be related to the Conley–Zehnder index by

$$\mu_{\mathrm{CZ}}(\gamma_z^n) = 2\alpha_-(\gamma_z^n) + p(\gamma_z^n)\,,$$

where $p(\gamma_z^n) \in \{0,1\}$. Since $\mu_{\mathrm{CZ}}(\gamma_z^n)$ is either 0 or 1 by Proposition 4.7, we conclude $\alpha_-(\gamma_z^n) = 0$, which is obviously the same as the winding of $u_x^n$ about $\gamma_z^n$ as it approaches asymptotically. □

PROPOSITION 4.11. *Suppose $u : \dot{S} \to \mathbb{R} \times (S^1 \times \Sigma)$ is a finite energy $J_\epsilon$-holomorphic curve which is not a cover of a trivial cylinder and has all its positive ends asymptotic to Reeb orbits of the form $\gamma_z^n$ for $z \in \mathrm{Crit}(h_\epsilon)$ and $n \le N_\epsilon$. Then $u$ is a cover of $u_x$ for some gradient flow line $x : \mathbb{R} \to \Sigma$.*

*Proof.* If $u$ is neither a cover of any $u_x$ nor of a trivial cylinder over $\gamma_z$ for some $z \in \mathrm{Crit}(h_\epsilon)$, then it must have a nontrivial intersection with one of the curves $u_x$, implying $u * u_x > 0$. By a small perturbation using positivity of intersections, we can assume also that $x$ is a *generic* flow line, connecting an index 0 critical point $z_- \in \Sigma_-$ to an index 2 critical point $z_+ \in \Sigma_+$. Then $u_x$ has no negative ends, so $u * u_x$ is the sum of the intersection numbers of $u_x$ with all the positive asymptotic orbits of $u$ by Lemma 4.9. But these are all zero by Lemma 4.10, giving a contradiction. □

PROPOSITION 4.12. *Suppose $x : \mathbb{R} \to \Sigma$ is a gradient flow line of $h_\epsilon$ and $u : \dot{S} \to \mathbb{R} \times (S^1 \times \Sigma)$ is a $J_\epsilon$-holomorphic multiple cover of $u_x$ with covering multiplicity at most $N_\epsilon$. Then $\mathrm{ind}(u) \ge 1$, and the inequality is strict unless the cover is unbranched, i.e. $u = u_x^n$ for some $n \le N_\epsilon$.*

*Proof.* The index formula for $u$ is

$$\operatorname{ind}(u) = -\chi(\dot{S}) + 2c_1(u^*\xi) + \mu_{\mathrm{CZ}}(u)\,,$$

where $\mu_{\mathrm{CZ}}(u)$ is the sum of the Conley–Zehnder indices of its positive asymptotic orbits minus those of its negative asymptotic orbits, and $c_1(u^*\xi)$ is the relative first Chern number of the bundle $u^*\xi \to \dot{S}$ with respect to the natural trivialization of each orbit $\gamma_z^n$. The latter vanishes due to the $S^1$-invariance (cf. Remark 4.4). For the Conley–Zehnder indices, we use Proposition 4.7, distinguishing between two cases:

- If $x$ passes through $\Gamma$, then both ends of $u_x$ are positive and thus all ends of $u$ are positive. Moreover, the Morse–Smale condition guarantees that $u_x$ cannot have both its ends at hyperbolic critical points with Conley–Zehnder index 0, hence $\mu_{\mathrm{CZ}}(u) \geq 1$.
- Otherwise, $u_x$ has a positive end at an elliptic critical point $z_+$ with $\mu_{\mathrm{CZ}}(\gamma_{z_+}) = 1$ and a negative end at a hyperbolic critical point $z_-$ with $\mu_{\mathrm{CZ}}(\gamma_{z_-}) = 0$, so again $\mu_{\mathrm{CZ}}(u) \geq 1$.

As a result, $\operatorname{ind}(u) \geq -\chi(\dot{S}) + 1$, which is strictly greater than 1 unless $\dot{S}$ is a cylinder, in which case there are no branch points. □

PROPOSITION 4.13. *Suppose $z \in \operatorname{Crit}(h_\epsilon)$ and $u : \dot{S} \to \mathbb{R} \times (S^1 \times \Sigma)$ is a $J_\epsilon$-holomorphic multiple cover of $\mathbb{R} \times \gamma_z$ with covering multiplicity at most $N_\epsilon$. Then $\operatorname{ind}(u) \geq 0$, and the inequality is strict unless $u$ has exactly one positive end.*

*Proof.* If $\operatorname{ind}(z) = 1$, then Proposition 4.7 implies that all asymptotic orbits of $u$ have Conley–Zehnder index 0 in the natural trivialization, hence $\operatorname{ind}(u) = -\chi(\dot{S}) \geq 0$, with equality if and only if $\dot{S}$ is a cylinder, implying it has one positive and one negative end. Otherwise, the asymptotic orbits of $u$ all have Conley–Zehnder index 1, so if $g \geq 0$ is the genus of $u$ and its sets of positive and negative punctures are denoted by $\Gamma^+$ and $\Gamma^-$ respectively, we have

$$\begin{aligned}\operatorname{ind}(u) &= -\chi(\dot{S}) + \#\Gamma^+ - \#\Gamma^- = -(2 - 2g - \#\Gamma^+ - \#\Gamma^-) + \#\Gamma^+ - \#\Gamma^- \\ &= 2g - 2 + 2\#\Gamma^+ = 2g + 2\left(\#\Gamma^+ - 1\right) \geq 0\,.\end{aligned}$$

□

REMARK 4.14. The moduli spaces of $J_\epsilon$-holomorphic curves in $\mathbb{R} \times (S^1 \times \Sigma)$ can be oriented coherently whenever all asymptotic orbits are nondegenerate and "good", see [EGH], [BM]. In particular, the spaces of cylinders $u_x^n$ covering gradient flow lines $x$ can be given orientations that match a corresponding set of coherent orientations for the spaces of Morse gradient flow lines.

**4.2 Proofs of Theorems 3 and 4.** The results of the previous subsection give enough information on $J_\epsilon$-holomorphic curves in $\mathbb{R} \times (S^1 \times \Sigma)$ to prove the main theorems. Recall that the natural compactification of the moduli space of finite energy punctured holomorphic curves consists of holomorphic *buildings*, which in general may have multiple levels and nodes, see [BEHWZ].

*Proof of Theorem 3.* Assume $\Sigma_-$ is disconnected and let $\Sigma_-^1$ and $\Sigma_-^2$ denote two of its connected components. Then we can choose the Morse functions $h_\pm$ so that $h_-$ has exactly one index 0 critical point in each of $\Sigma_-^1$ and $\Sigma_-^2$, denoted by $z_-^1$ and $z_-^2$ respectively, and $h_+$ has an index 1 critical point $z_+ \in \Sigma_+$ such that the two negative gradient flow lines of $h_\epsilon$ flowing out of $z_+$ end at $z_-^1$ and $z_-^2$ respectively. In particular, there is a *unique* gradient flow line $x_1$ connecting $z_-^1$ to $z_+$. By Proposition 4.11, the set of all $J_\epsilon$-holomorphic buildings with no negative ends and positive ends approaching any subset of the two simply covered orbits $\gamma_{z_+}$ and $\gamma_{z_-^1}$ consists of the following:

1. The cylinder $u_{x_1}$ with two positive ends at $\gamma_{z_+}$ and $\gamma_{z_-^1}$.
2. All cylinders $u_x$ corresponding to gradient flow lines $x$ connecting $z_-^1$ to index 1 critical points in $\Sigma_-^1$. Each of these cylinders has one positive and one negative end, with the positive end approaching $\gamma_{z_-^1}$.

Since both of these orbits are nondegenerate and all of the holomorphic curves in question are Fredholm regular by Proposition 4.8, they all survive any sufficiently small perturbation to make $\lambda_\epsilon$ nondegenerate and $J_\epsilon$ generic, as well as the introduction of an abstract perturbation for the holomorphic curve equation. The chain complex for SFT can therefore be defined so as to contain two special generators $q_{\gamma_{z_-^1}}$ and $q_{\gamma_{z_+}}$ such that $\mathbf{D}_{\mathrm{SFT}}(q_{\gamma_{z_-^1}} q_{\gamma_{z_+}})$ is computed by counting the $J_\epsilon$-holomorphic curves listed above (cf. Remark 3.7). We claim now that for a suitable choice of coherent orientations, the algebraic count of cylinders of the second type is zero. Indeed, the orientations can be chosen compatibly with a choice of coherent orientations for the space of gradient flow lines (cf. Remark 4.14), thus the count of these cylinders matches the count of all gradient flow lines connecting $z_-^1$ to index 1 critical points in $\Sigma_-^1$. The latter computes a part of the term $d\langle z_-^1\rangle$ in the Morse cohomology of $\Sigma$, but since $z_-^1$ is the only index 0 critical point in $\Sigma_-^1$, $\langle z_-^1\rangle$ is a closed generator of the Morse cohomology, and the claim follows. We conclude that only the cylinder $u_{x_1}$ with two positive ends gives a nontrivial count, and thus

$$\mathbf{D}_{\mathrm{SFT}}(q_{\gamma_{z_-^1}} q_{\gamma_{z_+}}) = \hbar\,.$$

□

Recall from Remark 2.2 that if all the Reeb orbits below some given action $T > 0$ are nondegenerate, then one can define a truncated complex $(\mathcal{A}(\lambda, T)[[\hbar]], \mathbf{D}_{\mathrm{SFT}})$. The proof that $(V_g, \xi_k)$ has no algebraic $(k-2)$-torsion for $k \ge 2$ depends on establishing the following criterion.

LEMMA 4.15. *Suppose $K$ is a non-negative integer and $(M, \xi)$ is a closed contact manifold admitting a contact form $\lambda$, compatible almost complex structure $J$ and constant $T > 0$ with the following properties:*

(1) *All Reeb orbits of $\lambda$ with period less than $T$ are nondegenerate.*
(2) *For every pair of integers $g \ge 0$ and $r \ge 1$ with $g + r \le K + 1$, let $\overline{\mathcal{M}}^1_{g,r}(J; T)$ denote the space of all index 1 connected $J$-holomorphic buildings in $\mathbb{R} \times M$ with arithmetic genus $g$, no negative ends, and $r$ positive ends approaching*

*orbits whose periods add up to less than $T$. Then $\overline{\mathcal{M}}^1_{g,r}(J;T)$ consists of finitely many smooth curves (i.e. buildings with only one level and no nodes), which are all Fredholm regular.*

(3) *There is a choice of coherent orientations for which the algebraic count of curves in $\overline{\mathcal{M}}^1_{g,r}(J;T)$ is zero whenever $g + r \leq K + 1$.*

*Then if $\mathbf{D}_{SFT} : \mathcal{A}(\lambda, T)[[\hbar]] \to \mathcal{A}(\lambda, T)[[\hbar]]$ is defined by counting solutions to a sufficiently small abstract perturbation of the $J$-holomorphic curve equation, there is no $Q \in \mathcal{A}(\lambda, T)[[\hbar]]$ such that*

$$\mathbf{D}_{SFT}(Q) = \hbar^K + \mathcal{O}(\hbar^{K+1})\,.$$

*Proof.* We begin by observing that since all the buildings in $\overline{\mathcal{M}}^1_{g,r}(J;T)$ are smooth Fredholm regular curves, the count of the corresponding moduli space of solutions under any suitable abstract perturbation will remain 0 (cf. Remark 3.7).

Recall now that $\mathbf{D}_{\mathrm{SFT}}$ has an expansion $\mathbf{D}_{\mathrm{SFT}} = \sum D_\ell \hbar^\ell$ in powers of $\hbar$, where $D_\ell$ counts (perturbed) holomorphic curves whose genus and number of positive punctures add up to $\ell$. The assumption (3) now guarantees that, for every $Q \in \mathcal{A}(\lambda, T)$ each term of $D_\ell(Q)$ with $\ell \leq K$ contains at least one $q$-variable. So if $Q \in \mathcal{A}(\lambda, T)[[\hbar]]$ is arbitrary, we can write its differential uniquely as

$$\mathbf{D}_{\mathrm{SFT}}(Q) = P + \mathcal{O}(\hbar^{K+1})\,,$$

with $P$ a polynomial of degree at most $K$ in $\hbar$ whose nontrivial terms each contain at least one $q$-variable. This establishes the claim. □

We now fix one of our specific examples $(V_g, \xi_k)$. The two sides $\Sigma_+$ and $\Sigma_-$ of $\Sigma$ are then both connected, so we can choose each of the functions $h_\pm : \Sigma_\pm \to \mathbb{R}$ to have a unique local minimum; in this case $h_\epsilon : \Sigma \to \mathbb{R}$ for $\epsilon > 0$ has a unique index 0 critical point in $\Sigma_-$ and a unique index 2 critical point in $\Sigma_+$. Recall that for any $\epsilon \in (0, \epsilon_0]$, Proposition 4.3 gives an exact symplectic cobordism

$$\big([\epsilon, \epsilon_0] \times (S^1 \times \Sigma), d(e^s \lambda_s)\big)$$

relating the contact forms $e^\epsilon \lambda_\epsilon$ and $e^{\epsilon_0} \lambda_{\epsilon_0}$. Then for any sufficiently $C^\infty$-small function $F_\epsilon : S^1 \times \Sigma \to \mathbb{R}$, the subdomain

$$X_\epsilon := \big\{(s, m) \in \mathbb{R} \times (S^1 \times \Sigma) \mid \epsilon + F_\epsilon(m) \leq s \leq \epsilon_0\big\}$$

gives an exact symplectic cobordism between $e^{\epsilon_0} \lambda_{\epsilon_0}$ and $e^\epsilon \lambda'_\epsilon$, where $\lambda'_\epsilon$ is the perturbed contact form

$$\lambda'_\epsilon := e^{F_\epsilon} \lambda_\epsilon\,.$$

By Proposition 4.7, $\lambda_\epsilon$ has nondegenerate orbits up to period $T_\epsilon$ except in $S^1 \times \mathcal{U}$, thus one can choose a generic $C^\infty$-small function $F_\epsilon$ with compact support in $S^1 \times \mathcal{U}$ so that $\lambda'_\epsilon$ has *only* nondegenerate orbits up to period $T_\epsilon$ (the fact that generic perturbations in an open subset suffice follows from the appendix of [ABW]). Choose a corresponding complex structure $J'_\epsilon$ on the perturbed contact structure $\xi'_\epsilon := \ker \lambda'_\epsilon$ such that $J'_\epsilon$ is $C^\infty$-close to $J_\epsilon$. The proof of Theorem 4 now rests on the following observation.

Lemma 4.16. *The assumptions of Lemma 4.15 are satisfied with $\lambda = \lambda'_\epsilon$, $J = J'_\epsilon$, $T = T_\epsilon$ and $K = k - 2$.*

*Proof.* It will turn out that it suffices to count holomorphic buildings for the unperturbed structure $J_\epsilon$, so to start with, suppose $u$ is an index 1 $J_\epsilon$-holomorphic building in $\mathbb{R} \times (S^1 \times \Sigma)$ with no negative ends and at most $k-1$ positive ends, asymptotic to orbits whose periods add up to less than $T_\epsilon$. We claim that $u$ is then a smooth curve (with only one level and no nodes), and is a cylinder of the form $u_x^n$ for some gradient flow line $x : \mathbb{R} \to \Sigma$ of $h_\epsilon$ and $n \leq N_\epsilon$. Indeed, we start by arguing that none of the asymptotic orbits of $u$ can lie in the region $S^1 \times \mathcal{U}$. By Proposition 4.7, all asymptotic orbits of $u$ outside this region are of the form $\gamma_z^n$ for $z \in \mathrm{Crit}(h_\epsilon)$, and thus have trivial projections to $\Sigma$. Moreover, all closed Reeb orbits in $S^1 \times \mathcal{U}$ project to $\mathcal{U}$ as closed curves homologous to some positive multiple of a component of $\Gamma$, oriented as boundary of $\Sigma_+$. It follows that the projection of $u$ to $\Sigma$ provides a homology from the sum of these curves to zero. Since there are $k$ components of $\Gamma$, but only at most $k-1$ ends of $u$, there is at least one component of $S^1 \times \mathcal{U}$ which does not contain any asymptotics of $u$. Using this interface component, it is easy to construct a closed curve on $\Sigma$ which has nonzero intersection number with the projected asymptotics of $u$ in $\mathcal{U} \subset \Sigma$, proving that the sum cannot be homologous to zero. This contradiction proves our claim that none of the asymptotics can lie in $S^1 \times \mathcal{U}$.

Now Proposition 4.7 implies that all the asymptotic orbits of $u$ are of the form $\gamma_z^n$ for $z \in \mathrm{Crit}(h_\epsilon)$ and $n \leq N_\epsilon$. Proposition 4.11 then implies that every component curve in the levels of $u$ is one of the following:

1. a cover of a trivial cylinder $\mathbb{R} \times \gamma_z$ for some $z \in \mathrm{Crit}(h_\epsilon)$;
2. a cover of the cylinder $u_x$ for some gradient flow line $x : \mathbb{R} \to \Sigma$ of $h_\epsilon$, connecting critical points of $h_\epsilon$ on opposite sides of $\Gamma$.

By Proposition 4.13, all curves of the first type have non-negative index. Proposition 4.12 implies in turn that all curves of the second type have index at least 1, and there must be at least one such curve since $u$ has no negative ends. Since $\mathrm{ind}(u) = 1$, it follows that $u$ contains exactly one curve of the second type, which is an unbranched cover $u_x^n$ for some gradient flow line $x$ and $n \leq N_\epsilon$, and all components of $u$ that are covers of trivial cylinders have exactly one positive end. Combinatorially, this is only possible if $u$ has precisely one nontrivial connected component, which is of the form $u_x^n$.

By Proposition 4.8, the curves $u_x^n$ are all Fredholm regular, thus they will all survive the small perturbation of $J_\epsilon$ to $J'_\epsilon$; in fact the lack of nontrivial $J_\epsilon$-holomorphic buildings means that no additional $J'_\epsilon$-holomorphic buildings can appear under this perturbation. Thus it will suffice to show that the algebraic count of the $J_\epsilon$-holomorphic cylinders $u_x^n$ for $n \leq N_\epsilon$ is zero. For this, choose a system of coherent orientations for the gradient flow lines of $h_\epsilon$, and a corresponding system of orientations for the moduli spaces of $J_\epsilon$-holomorphic curves (see Remark 4.14). The relevant count of holomorphic curves is then the same as a certain count of gradient flow lines: we are interested namely in all index 1 holomorphic cylinders $u_x^n$ for which both ends are positive, and these correspond to the gradient flow lines $x$ that pass through $\Gamma$ and connect an index 1 critical point on one side to an index 0 or 2 critical point on the other. Consider in particular the set of all gradient flow lines that connect the unique index 2 critical point $z_+ \in \Sigma_+$ to any index 1 critical point in $\Sigma_-$. The count of these flow lines calculates part of the differential $\partial\langle z_+\rangle$ in the Morse homology of $\Sigma$, but since there is no other critical point of index 2, $\langle z_+\rangle$ is necessarily closed in Morse homology, implying that the relevant algebraic count of flow lines is zero. Applying the same argument to the unique index 0 critical point in $\Sigma_-$ using Morse cohomology, we find indeed that the algebraic count of cylinders $u_x^n$ with two positive ends for any $n \le N_\epsilon$ vanishes. □

Remark 4.17. The preceding result also establishes the conditions of Proposition A.6 in the appendix, thus implying the lower bound stated in Theorem 7.

*Proof of Theorem 4.* In light of Theorem 6, it remains to show that $[\hbar^{k-2}]$ does not vanish in $H_*^{\mathrm{SFT}}(V_g, \xi_k)$.

We will argue by contradiction and suppose $\hbar^{k-2}$ vanishes in $H_*^{\mathrm{SFT}}(V_g, \xi_k)$. Choose a nondegenerate contact form $\lambda$ such that there is a topologically trivial cobordism $X$ with positive end $(V_g, \lambda)$ and negative end $(V_g, e^{\epsilon_0}\lambda_{\epsilon_0})$. Choose all necessary data to define $\mathbf{D}_{\mathrm{SFT}}$ on $\mathcal{A}(\lambda)[[\hbar]]$ such that it computes $H_*^{\mathrm{SFT}}(V_g, \xi_k)$. In particular, there exists $Q \in \mathcal{A}(\lambda)[[\hbar]]$ such that

$$\mathbf{D}_{\mathrm{SFT}}(Q) = \hbar^{k-2}.$$

Writing $Q = Q_1 + \mathcal{O}(\hbar^{k-1})$, we find a polynomial $Q_1$ of degree at most $k-2$ in $\hbar$ with the property that

$$\mathbf{D}_{\mathrm{SFT}}(Q_1) = \hbar^{k-2} + \mathcal{O}(\hbar^{k-1})\,.$$

Note that since $Q_1$ is a polynomial in the $q$-variables, there exists some $T > 0$ such that in fact $Q_1 \in \mathcal{A}(\lambda, T)[[\hbar]]$.

Now choose $\epsilon > 0$ so small that $e^{\epsilon}T_\epsilon > T$. Gluing the cobordism $X_\epsilon$ constructed above to $X$, we obtain an exact cobordism with positive end $(V_g, \lambda)$ and negative end $(V_g, e^{\epsilon}\lambda'_\epsilon)$ which according to Remark 2.2 gives rise to a chain map,

$$\Phi_T : \big(\mathcal{A}(\lambda, T)[[\hbar]], \mathbf{D}_{\mathrm{SFT}}\big) \to \big(\mathcal{A}(\lambda'_\epsilon, e^{-\epsilon}T)[[\hbar]], \mathbf{D}_{\mathrm{SFT}}\big)\,,$$

where the right-hand side admits the obvious inclusion into $(\mathcal{A}(\lambda'_\epsilon, T_\epsilon)[[\hbar]], \mathbf{D}_{\mathrm{SFT}})$. But then $\mathbf{D}_{\mathrm{SFT}}\Phi_T(Q_1) = \Phi_T\mathbf{D}_{\mathrm{SFT}}(Q_1) = \hbar^{k-2} + \mathcal{O}(\hbar^{k-1})$, which contradicts Lemmas 4.15 and 4.16. This contradiction shows that $\hbar^{k-2}$ cannot vanish in $H_*^{\mathrm{SFT}}(V_g, \xi_k)$, completing the proof of the theorem. □

Remark 4.18. We conclude this section by giving the rough idea of how to construct the exact cobordisms with positive end $(V_{g+1}, \xi_{k+1})$ and negative end $(V_g, \xi_k)$ alluded to in Remark 1.6; this was explained to us by J. Van Horn-Morris. First observe that if $V_g = S^1 \times \Sigma$ with $\Sigma = \Sigma_+ \cup_\Gamma \Sigma_-$ and $V_{g+1} = S^1 \times \Sigma'$ with $\Sigma' = \Sigma'_+ \cup_{\Gamma'} \Sigma'_-$, then one can transform the former to the latter by picking two distinct points $p_-, p_+$ in the same connected component of $\Gamma$ and attaching

2-dimensional 1-handles $\mathcal{H} := \mathbb{D}^1 \times \mathbb{D}^1$ along the corresponding points in both $\partial\Sigma_+$ and $\partial\Sigma_-$, producing $\Sigma'_+$ and $\Sigma'_-$ respectively with a preferred orientation reversing diffeomorphism $\partial\Sigma'_+ \to \partial\Sigma'_-$. A Stein cobordism between $(V_g, \xi_k)$ and $(V_{g+1}, \xi_{k+1})$ is then constructed by "multiplying the handle attachment by an annulus". More precisely, we define the two Legendrian loops $\ell_\pm = S^1 \times \{p_\pm\} \subset V_g$, and attach to these a 4-dimensional *round* 1*-handle*

$$\widehat{\mathcal{H}} := \mathcal{H} \times [-1,1] \times S^1 \cong \mathbb{D}^1 \times \left(\mathbb{D}^2 \times S^1\right)$$

with boundary

$$\partial\widehat{\mathcal{H}} = \partial_-\widehat{\mathcal{H}} \cup \partial_+\widehat{\mathcal{H}} := \left(\partial\mathbb{D}^1 \times (\mathbb{D}^2 \times S^1)\right) \cup \left(\mathbb{D}^1 \times \partial(\mathbb{D}^2 \times S^1)\right).$$

This produces a smooth cobordism from $V_g$ to $V_{g+1}$, and one can make it into a Stein cobordism by regarding $\widehat{\mathcal{H}}$ as an "$S^1$-invariant Weinstein handle", with a Morse–Bott plurisubharmonic function with critical set $\{(0,0)\} \times S^1$, isotropic unstable manifold $\mathbb{D}^1 \times \{0\} \times S^1$ and coisotropic stable manifold $\{0\} \times \mathbb{D}^2 \times S^1$. Perturbing the Morse–Bott function to a Morse function with critical points of index 1 and 2 along $\{(0,0)\} \times S^1$, one sees that the same cobordism can be obtained by attaching a combination of standard Stein 1-handles and 2-handles. One can then use open book decompositions [V] to show that the resulting contact structure on $V_{g+1}$ is the one determined by the dividing curves $\Gamma' \subset \Sigma'$.

## 5 Outlook

We close by mentioning a few questions that arise from the results of this paper.

As shown in the appendix, algebraic torsion in dimension three seems to be closely related to the ECH contact invariant; indeed, all of our examples are contact manifolds for which the latter vanishes, and they exhibit a correspondence between the minimal order of algebraic torsion and the integers $f$ and $f_{\text{simp}}$ defined by Hutchings. It is unclear however whether a precise relationship between these invariants exists in general, as SFT counts a much larger class of holomorphic curves than ECH.

It is presumably also possible to define a corresponding invariant in Heegaard Floer homology, but the latter is apparently still unknown.

Question 1. *Is there a Heegaard Floer theoretic contact invariant that implies obstructions to Stein cobordisms between pairs of contact* 3*-manifolds whose Ozsváth–Szabó invariants vanish?*

Remark 5.1. There is an obvious Stein cobordism obstruction in Heegaard Floer homology, defined in terms of the largest integer $k \geq 1$ for which the contact invariant is in the image of the $k$th power of the so-called $U$-map. (Note that one could define an exact cobordism obstruction in ECH in precisely the same way.) Nontrivial examples of this obstruction have been computed by Karakurt [K]. Interestingly, since this invariant is only really interesting in cases where the contact invariant is *nonvanishing*, Karakurt's results are completely disjoint from ours.

In contrast to ECH or Heegaard Floer homology, SFT is also well defined in higher dimensions, and it remains to find interesting examples beyond the 0-torsion examples that are known from [BN2], [BK]. Some candidates arise in [MNW]: in particular, the authors define a higher-dimensional generalization of Giroux torsion which obstructs strong fillability and conjecturally implies algebraic 1-torsion. They also find examples of contact forms in all dimensions that have this form of torsion but don't admit any contractible Reeb orbits, implying there is no algebraic 0-torsion, and in some cases the examples are also known to be weakly (and hence stably) fillable, implying that they do not have any fully twisted algebraic torsion.

Conjecture. *For all integers $k \geq 1$ and $n \geq 2$, there exist infinitely many closed $(2n-1)$-dimensional contact manifolds that have algebraic torsion of order $k$ but not $k-1$. There also exist $(2n-1)$-dimensional contact manifolds that have (untwisted) algebraic $k$-torsion but admit stable symplectic fillings.*

Finally, one wonders to what extent algebraic torsion might also give obstructions to *non-exact* cobordisms. Results in [We5] show that Corollary 1 for instance is false without the exactness assumption, and the reason is that a non-exact cobordism between $(M^+, \xi^+)$ and $(M^-, \xi^-)$ does not in general imply a morphism

$$H_*^{\mathrm{SFT}}(M^+, \xi^+) \to H_*^{\mathrm{SFT}}(M^-, \xi^-)\,.$$

On the other hand, if $(M^+, \xi^+)$ has algebraic torsion, then $(M^-, \xi^-)$ clearly cannot be fillable, and as was explained in section 2, a non-exact cobordism does give a map from $H_*^{\mathrm{SFT}}(M^+, \xi^+)$ to a suitably twisted version of $H_*^{\mathrm{SFT}}(M^-, \xi^-)$, where the twisting is defined by a count of holomorphic curves without positive ends in the cobordism. It is however unclear whether the vanishing of $[\hbar^k]$ in this twisted SFT also implies a result for the untwisted theory. A promising class of test examples is provided by the so-called *capping* and *decoupling* cobordisms constructed in [We5], for which the holomorphic curves without positive ends can be enumerated precisely.

Question 2. *If $(M^+, \xi^+)$ and $(M^-, \xi^-)$ are related by a non-exact symplectic cobordism and $(M^+, \xi^+)$ has algebraic torsion of some finite order, must $(M^-, \xi^-)$ also have algebraic torsion of some (possibly higher) finite order? Is there a precise relation between these orders for the capping/decoupling cobordisms constructed in [We5]?*

## Appendix

## ECH Analogue of Algebraic *k*-Torsion

by Michael Hutchings

The purpose of this appendix is to define an analogue of algebraic $k$-torsion in embedded contact homology (ECH). Specifically, given a closed oriented 3-manifold $Y$, a nondegenerate contact form $\lambda$ on $Y$, and an almost complex structure $J$ on $\mathbb{R} \times Y$ as needed to define the ECH chain complex, we define a number $f(Y, \lambda, J) \in \mathbb{N} \cup \{\infty\}$, which is similar to the order of algebraic torsion. It is not known whether this number is an invariant of the contact manifold $(Y, \xi = \operatorname{Ker} \lambda)$. Nonetheless, this number, together with some variants thereof, can be used to reprove some of the results on nonexistence of exact symplectic cobordisms between contact manifolds that are proved in the main paper using algebraic torsion. In addition, the results in this appendix do not depend on any unpublished work: in particular we do not use any symplectic field theory or Seiberg–Witten theory here.

**A.1 Basic recollections about ECH.** We begin by recalling what we will need to know about the definition of ECH.

Let $Y$ be a closed oriented 3-manifold with a nondegenerate contact form $\lambda$. Let $R$ denote the Reeb vector field determined by $\lambda$, and let $\xi = \operatorname{Ker}(\lambda)$ denote the corresponding contact structure. Choose a generic almost complex structure $J$ on $\mathbb{R} \times Y$ such that $J$ is $\mathbb{R}$-invariant, $J(\partial_s) = R$ where $s$ denotes the $\mathbb{R}$ coordinate, and $J(\xi) = \xi$, with $d\lambda(v, Jv) \geq 0$ for $v \in \xi$. To save verbiage below, we refer to the pair $(\lambda, J)$ as *ECH data* for $(Y, \xi)$. From these data one defines the ECH chain complex $\operatorname{ECC}(Y, \lambda, J)$ as follows.

An *orbit set* is a finite set of pairs $\alpha = \{(\alpha_i, m_i)\}$ where the $\alpha_i$'s are distinct embedded Reeb orbits and the $m_i$'s are positive integers. The homology class of the orbit set $\alpha$ is defined by $[\alpha] := \sum_i m_i[\alpha_i] \in H_1(Y)$. The orbit set $\alpha$ is called *admissible* if $m_i = 1$ whenever $\alpha_i$ is hyperbolic (i.e. its linearized return map has real eigenvalues). The ECH chain complex is freely generated over $\mathbb{Z}$ by admissible orbit sets.

Now let $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$ be two orbit sets with $[\alpha] = [\beta] \in H_1(Y)$.

Definition A.1. Define $\mathcal{M}_J(\alpha, \beta)$ to be the moduli space of holomorphic curves $u : (\Sigma, j) \to (\mathbb{R} \times Y, J)$, where the domain $\Sigma$ is a (possibly disconnected) punctured compact Riemann surface, and $u$ has positive ends at covers of $\alpha_i$ with total covering multiplicity $m_i$, negative ends at covers of $\beta_j$ with total covering multiplicity $n_j$, and no other ends. We consider two such holomorphic curves to be equivalent if they represent the same 2-dimensional current in $\mathbb{R} \times Y$.

Let $H_2(Y, \alpha, \beta)$ denote the set of relative homology classes of 2-chains in $Y$ with $\partial Y = \sum_i m_i \alpha_i - \sum_j n_j \beta_j$; this is an affine space over $H_2(Y)$. Any holomorphic curve $u \in \mathcal{M}_J(\alpha, \beta)$ determines a class $[u] \in H_2(Y, \alpha, \beta)$. If $Z \in H_2(Y, \alpha, \beta)$, define

$$\mathcal{M}_J(\alpha, \beta, Z) = \left\{ u \in \mathcal{M}_J(\alpha, \beta) \mid [u] = Z \right\}.$$

Also the *ECH index* is defined by

$$I(\alpha, \beta, Z) := c_\tau(Z) + Q_\tau(Z) + \sum_i \sum_{k=1}^{m_i} \operatorname{CZ}_\tau(\alpha_i^k) - \sum_j \sum_{k=1}^{n_j} \operatorname{CZ}_\tau(\beta_j^k). \tag{A.1}$$

Here $\tau$ is a trivialization of $\xi$ over the Reeb orbits $\alpha_i$ and $\beta_j$; $c_\tau(Z)$ denotes the relative first Chern class of $\xi$ over $Z$ with respect to the boundary trivializations $\tau$; $Q_\tau(Z)$ denotes the relative self-intersection pairing; and $\operatorname{CZ}_\tau(\gamma^k)$ denotes the Conley–Zehnder index with respect to $\tau$ of the $k^{th}$ iterate of $\gamma$. These notions are explained in detail in [Hu1,2]. The ECH index of a holomorphic curve $u \in \mathcal{M}_J(\alpha, \beta)$ is defined by $I(u) := I(\alpha, \beta, [u])$.

We will need the following facts, which are proved in [Hu2, Th. 4.15] and [HuS, Cor. 11.5]:

PROPOSITION A.2. (a) *If* $u \in \mathcal{M}_J(\alpha,\beta)$ *does not multiply cover any component of its image, then* $\operatorname{ind}(u) \le I(u)$*, where* ind *denotes the Fredholm index.*

(b) *If* $J$ *is generic and* $u \in \mathcal{M}_J(\alpha,\beta)$*, then:*

- $I(u) \ge 0$*, with equality if and only if* $u$ *is* $\mathbb{R}$*-invariant (as a current).*
- *If* $I(u) = 1$*, then* $u = u_0 \sqcup u_1$ *where* $u_1$ *is embedded and connected,* $\operatorname{ind}(u_1) = I(u_1) = 1$*, and* $u_0$ *is* $\mathbb{R}$*-invariant (as a current).*

The differential $\partial$ on the ECH chain complex is now defined as follows: If $\alpha$ is an admissible orbit set, then

$$\partial\alpha := \sum_\beta \sum_{\{u\in\mathcal{M}_J(\alpha,\beta)/\mathbb{R} \mid I(u)=1\}} \varepsilon(u)\cdot\beta\,.$$

Here the sum is over admissible orbit sets $\beta$ with $[\alpha] = [\beta]$, and $\varepsilon(u) \in \{\pm 1\}$ is a sign explained in [HuT2, §9]. The signs depend on some orientation choices, but the chain complexes for different sign choices are canonically isomorphic to each other. It is shown in [HuT1,2] that $\partial$ is well-defined and (what is much harder) $\partial^2 = 0$. The homology of the chain complex is the *embedded contact homology* $\operatorname{ECH}(Y,\lambda,J)$. Note that the empty set $\emptyset$ is a legitimate generator of the ECH chain complex, and $\partial\emptyset = 0$. The homology class $[\emptyset] \in \operatorname{ECH}(Y,\lambda,J)$ is called the *ECH contact invariant*.

Taubes has shown that $\operatorname{ECH}(Y,\lambda,J)$ is canonically isomorphic to a version of Seiberg–Witten Floer cohomology [T], and in particular depends only on $Y$. In addition, under this isomorphism the ECH contact invariant depends only on $\xi$ and agrees with an analogous contact invariant in Seiberg–Witten Floer cohomology. However, we will not need these facts here.

There is also a filtered version of ECH which is important in applications. If $\alpha = \{(\alpha_i, m_i)\}$ is an orbit set, define the *symplectic action*

$$\mathcal{A}(\alpha) := \sum_i m_i \int_{\alpha_i} \lambda\,.$$

It follows from the conditions on $J$ that the ECH differential decreases symplectic action, i.e. if $\langle\partial\alpha,\beta\rangle \neq 0$ then $\mathcal{A}(\alpha) > \mathcal{A}(\beta)$. Hence for each $L \in (0,\infty]$, the submodule $\operatorname{ECC}^L(Y,\lambda,J)$ of $\operatorname{ECC}(Y,\lambda,J)$ generated by admissible orbit sets of action less than $L$ is a subcomplex. The homology of this subcomplex is denoted by $\operatorname{ECH}^L(Y,\lambda,J)$, and called *filtered ECH*. Of course, taking $L = \infty$ recovers the usual ECH.

It is shown in [HuT3] that filtered ECH does not depend on $J$ (we will not use this fact here). However, filtered ECH does depend on the contact form $\lambda$. In particular, if $c$ is a positive constant, then an almost complex structure $J$ as needed to define the ECH of $\lambda$ determines an almost complex structure (which we also denote by $J$) as needed to define the ECH of $c\lambda$, with the same holomorphic curves. There is then a canonical isomorphism of chain complexes

$$\operatorname{ECC}^L(Y,\lambda,J) = \operatorname{ECC}^{cL}(Y,c\lambda,J)\,, \tag{A.2}$$

induced by the obvious bijection on generators.

**A.2 The relative filtration $J_+$.** We now recall from [Hu2, §6] how to define a relative filtration on ECH which is similar to the exponent of $\hbar$ in SFT.

Let $\alpha$ and $\beta$ be admissible orbit sets with $[\alpha]{=}[\beta]{\in}H_1(Y)$, and let $Z{\in}H_2(Y,\alpha,\beta)$. Similarly to (A.1), define

$$J_+(\alpha,\beta,Z) := -c_\tau(Z) + Q_\tau(Z) + \sum_i \sum_{k=1}^{m_i-1} \mathrm{CZ}_\tau(\alpha_i^k) - \sum_j \sum_{k=1}^{n_j-1} \mathrm{CZ}_\tau(\beta_j^k) + |\alpha| - |\beta| \,. \tag{A.3}$$

Here $|\alpha|$ denotes the cardinality of the admissible orbit set $\alpha$. (There is also a more general definition of $J_+$ when the orbit sets are not necessarily admissible, but we will not need this here.) If $u \in \mathcal{M}_J(\alpha,\beta)$, define $J_+(u) := J_+(\alpha,\beta,[u])$. There is now the following analogue of Proposition A.2, proved in [Hu2, Prop. 6.9 & Th. 6.6].

Proposition A.3. *Let $\alpha$ and $\beta$ be admissible orbit sets with $[\alpha] = [\beta]$.*

(a) *If $u \in \mathcal{M}_J(\alpha,\beta)$ is irreducible and not multiply covered and has genus $g$, then*

$$J_+(u) \geq 2\Big(g - 1 + |\alpha| + \sum_i (N_i^+ - 1) + \sum_j (N_j^- - 1)\Big) \,. \tag{A.4}$$

*Here $N_i^+$ denotes the number of positive ends of $u$ at covers of $\alpha_i$, and $N_j^-$ denotes the number of negative ends of $u$ at covers of $\beta_j$. Moreover, equality holds in* (A.4) *when* $\mathrm{ind}(u) = I(u)$.

(b) *If $J$ is generic, and if $u \in \mathcal{M}_J(\alpha,\beta)$, then $J_+(u) \geq 0$.*

Note that if $u$ contributes to the ECH differential, then $J_+(u)$ is even. (Comparing (A.1) and (A.3) shows that the parity of $J_+(u) - I(u)$ is the parity of the number of Reeb orbits $\alpha_i$ or $\beta_j$ that are positive hyperbolic, which is the parity of $\mathrm{ind}(u)$.) Thus we can decompose the ECH differential $\partial$ as

$$\partial = \partial_0 + \partial_1 + \partial_2 + \cdots \tag{A.5}$$

where $\partial_k$ denotes the contribution from holomorphic curves $u$ with $J_+(u) = 2k$.

Since $J_+$ is additive under gluing [Hu2, Prop. 6.5(a)], it follows that $\partial_0^2 = 0$, $\partial_0\partial_1 + \partial_1\partial_0 = 0$, etc. Thus we obtain a spectral sequence $E^*(Y,\lambda,J)$, where $E^1$ is the homology of $\partial_0$, and $E^2$ is the homology of $\partial_1$ acting on $E^1$. Unfortunately, this spectral sequence is not invariant under deformation of the contact form. The reason is that although an exact symplectic cobordism induces a map on ECH which (up to a given symplectic action) is induced by a chain map that somehow counts (possibly broken) holomorphic curves [HuT3], Proposition A.3(b) does not generalize to exact symplectic cobordisms. That is, the chain map induced by a cobordism can include contributions from multiply covered holomorphic curves with $J_+$ negative. However, we can still use this spectral sequence to define a useful analogue of the order of algebraic $k$-torsion.

**A.3 The analogue of order of algebraic torsion.** Let $Y$ be a closed oriented 3-manifold, and let $(\lambda, J)$ be ECH data on $Y$.

Definition A.4. Define $f(Y,\lambda,J)$ to be the smallest non-negative integer $k$ such that there exists $x \in \mathrm{ECC}(Y,\lambda,J)$ with

$$(\partial_0 + \cdots + \partial_k)x = \emptyset\,.$$

Equivalently, $f(Y,\lambda,J)$ is the smallest $k$ such that $\emptyset$ does not survive to the $E^{k+1}$ page of the spectral sequence $E^*(Y,\lambda,J)$. If no such $k$ exists, define $f(Y,\lambda,J) := \infty$.

Of course, $f(Y,\lambda,J) < \infty$ if and only if the ECH contact invariant vanishes. One can use the cobordism maps on ECH defined in [HuT3] (using Seiberg–Witten theory) to show that $f(Y,\lambda,J)$ does not depend on $J$. However, we will not need this fact here.

There are now two difficulties in using $f$ to obstruct exact symplectic cobordisms. First, we would like to show that if there is an exact symplectic cobordism from $(Y_+,\lambda_+)$ to $(Y_-,\lambda_-)$ then

$$f(Y_+,\lambda_+,J_+) \geq f(Y_-,\lambda_-,J_-)\,. \tag{A.6}$$

This would imply that $f$ depends only on the contact structure and is monotone with respect to exact symplectic cobordisms. Unfortunately, we cannot prove (A.6) or these consequences (and we do not know whether these are true), due to the aforementioned lack of invariance of the spectral sequence. Second, $f(Y,\lambda,J)$ is difficult to compute in practice, because often one only understands the ECH chain complex up to a given symplectic action.

To deal with the latter difficulty, we can define a filtered version of $f$.

Definition A.5. Given $L \in (0,\infty]$, define $f^L(Y,\lambda,J)$ to be the smallest non-negative integer $k$ such that there exists $x \in \mathrm{ECC}^L(Y,\lambda,J)$ with

$$(\partial_0 + \cdots + \partial_k)x = \emptyset\,.$$

The following proposition can be used in calculations to give lower bounds on $f^L$.

Proposition A.6. *Let $(\lambda,J)$ be ECH data on $Y$, and fix $L \in (0,\infty]$. Let $k$ be a positive integer. Suppose that the algebraic count*

$$\sum_{\{u\in\mathcal{M}_J(\alpha,\emptyset,Z)/\mathbb{R}\}} \varepsilon(u) = 0\,,$$

*whenever*

- *$\alpha$ is an admissible orbit set with $\mathcal{A}(\alpha) < L$; and*
- *$Z \in H_2(Y,\alpha,\emptyset)$ is such that $I(\alpha,\emptyset,Z) = 1$; and*
- *curves in $\mathcal{M}_J(\alpha,\emptyset,Z)$ have genus $g$ and $N_+$ positive ends with $g + N_+ \leq k$.*

*Then $f^L(Y,\lambda,J) \geq k$.*

In the third bullet point above, note that curves in $\mathcal{M}_J(\alpha,\emptyset,Z)$ are embedded and connected by Proposition A.2(b), and then $g$ and $N_+$ are uniquely determined by $\alpha$ and $Z$. Here $N_+$ is determined by [Hu2, Th. 4.15], while $g$ is determined by Proposition A.3(a).

*Proof.* Let $\alpha$ be an admissible orbit set with $\mathcal{A}(\alpha) < L$ and let $Z \in H_2(Y,\alpha,\emptyset)$ such that $I(\alpha,\emptyset,Z) = 1$ and $J_+(\alpha,\emptyset,Z) < 2k$. Then by Proposition A.2(b), curves in

$\mathcal{M}_J(\alpha, \emptyset, Z)$ are embedded and connected, so by Proposition A.3(a), such curves have $g + N_+ \leq k$. Then by hypothesis, the algebraic count of such curves is zero. This means that $\langle \partial_i \alpha, \emptyset \rangle = 0$ whenever $i < k$. □

We now prove a weaker version of (A.6), which will still allow us to obstruct exact symplectic cobordisms. This requires the following additional definitions.

Definition A.7. An orbit set $\alpha = \{(\alpha_i, m_i)\}$ is *simple* (with respect to $J$) if the following hold:

- $m_i = 1$ for each $i$.
- If $\beta = \{(\beta_j, n_j)\}$ is another orbit set (not necessarily admissible), and if there is a (possibly broken) $J$-holomorphic curve from $\alpha$ to $\beta$, then $n_j = 1$ for each $j$.

Given $L \in (0, \infty]$, let $\mathrm{ECC}^L_{\mathrm{simp}}(Y, \lambda, J)$ denote the subcomplex of $\mathrm{ECC}(Y, \lambda, J)$ generated by simple admissible orbit sets $\alpha$ with $\mathcal{A}(\alpha) < L$.

Note that even when $L = \infty$, the homology of the subcomplex $\mathrm{ECC}^L_{\mathrm{simp}}$ is not invariant under deformation of $\lambda$, as shown by the ellipsoid example in [Hu3].

Definition A.8. Define $f^L_{\mathrm{simp}}(Y, \lambda, J)$ to be the smallest non-negative integer $j$ such that there exists $x \in \mathrm{ECC}^L_{\mathrm{simp}}(Y, \lambda, J)$ with

$$(\partial_0 + \cdots + \partial_k)x = \emptyset\,.$$

If no such $x$ exists, define $f^L_{\mathrm{simp}}(Y, \lambda, J) := \infty$.

Of course we always have $f^L_{\mathrm{simp}}(Y, \lambda, J) \geq f^L(Y, \lambda, J)$. The main result of this appendix is now the following theorem.

**Theorem A.9**. *Let $(\lambda_\pm, J_\pm)$ be ECH data on $Y_\pm$. Suppose there is an exact symplectic cobordism from $(Y_+, \lambda_+)$ to $(Y_-, \lambda_-)$. Then*

$$f^L_{\mathrm{simp}}(Y_+, \lambda_+, J_+) \geq f^L(Y_-, \lambda_-, J_-)$$

*for each $L \in (0, \infty]$.*

Here is how Theorem A.9 can be used in practice to obstruct symplectic cobordisms. Below, write $f_{\mathrm{simp}} := f^\infty_{\mathrm{simp}}$.

Corollary A.10. *Suppose there exists an exact symplectic cobordism from $(Y_+, \xi_+)$ to $(Y_-, \xi_-)$. Fix ECH data $(\lambda_+, J_+)$ for $(Y_+, \xi_+)$ and a contact form $\lambda'_-$ with $\mathrm{Ker}(\lambda'_-) = \xi_-$. Fix a positive integer $k$. Suppose that for each $L > 0$ there exist ECH data $(\lambda_-, J_-)$ for $(Y_-, \xi_-)$ with $f^L(Y_-, \lambda_-, J_-) \geq k$ and an exact symplectic cobordism from $(Y_-, \lambda'_-)$ to $(Y_-, \lambda_-)$. Then $f_{\mathrm{simp}}(Y_+, \lambda_+, J_+) \geq k$.*

*Proof.* The first hypothesis implies that there exist a positive constant $c$ and an exact symplectic cobordism from $(Y_+, c\lambda_+)$ to $(Y_-, \lambda'_-)$. The second hypothesis then implies that for each $L > 0$ there exist ECH data $(\lambda_-, J_-)$ for $(Y_-, \xi_-)$ with $f^L(Y_-, \lambda_-, J_-) \geq k$ and an exact symplectic cobordism from $(Y_+, c\lambda_+)$ to $(Y_-, \lambda_-)$. By the scaling isomorphism (A.2) and Theorem A.9 we have

$$f^{c^{-1}L}_{\mathrm{simp}}(Y_+, \lambda_+, J_+) = f^L_{\mathrm{simp}}(Y_+, c\lambda_+, J_+) \geq k\,.$$

Since $L$ was arbitrary, we conclude that $f_{\mathrm{simp}}(Y_+, \lambda_+, J_+) \geq k$. □

Here is another corollary of Theorem A.9 which tells us a bit more about the meaning of $f$.

Corollary A.11. *Suppose $(Y,\xi)$ is overtwisted. Then $f(Y,\lambda,J)=0$ whenever $(\lambda,J)$ is ECH data for $(Y,\xi)$.*

*Proof.* The argument in the appendix to [Y] shows that one can find ECH data $(\lambda_+,J_+)$ for $(Y,\xi)$ such that there is an embedded Reeb orbit $\gamma$ with the following properties:

- $\gamma$ has smaller symplectic action than any other Reeb orbit.
- There is a unique Fredholm index 1 holomorphic plane $u$ in $\mathbb{R}\times Y$ with positive end at $\gamma$.

The holomorphic plane $u$ is embedded in $\mathbb{R}\times Y$, so $I(u)=1$ also, and $J_+(u)=0$. This means that $\partial_0\{(\gamma,1)\}=\pm\emptyset$. Since $\gamma$ has minimal symplectic action, $\{(\gamma,1)\}$ is simple. Thus $f_{\mathrm{simp}}(Y,\lambda_+,J_+)=0$. We can also assume, by multiplying $\lambda_+$ by a large positive constant, that there is an exact (product) symplectic cobordism from $(Y,\lambda_+)$ to $(Y,\lambda)$. Theorem A.9 with $L=\infty$ then implies that $f(Y,\lambda,J)=0$. □

One might conjecture that the converse of Corollary A.11 holds:

Conjecture A.12. *Given a closed contact 3-manifold $(Y,\xi)$, if $f(Y,\lambda,J)=0$ for all ECH data $(\lambda,J)$ for $(Y,\xi)$, then $(Y,\xi)$ is overtwisted.*

Remark A.13. Conjecture A.12 implies the well-known conjecture that if $(Y_-,\xi_-)$ is a closed tight contact 3-manifold, and if $(Y_+,\xi_+)$ is obtained from $(Y_-,\xi_-)$ by Legendrian surgery, then $(Y_+,\xi_+)$ is also tight.

*Proof.* Suppose $(Y_+,\xi_+)$ is obtained from $(Y_-,\xi_-)$ by Legendrian surgery. Recall from [W] that there is an exact symplectic cobordism from $(Y_+,\xi_+)$ to $(Y_-,\xi_-)$. If $(Y_+,\xi_+)$ is overtwisted, then as explained above one can find ECH data $(\lambda_+,J_+)$ for $(Y_+,\xi_+)$ such that $f_{\mathrm{simp}}(Y_+,\lambda_+,J_+)=0$. Theorem A.9 then implies that $f(Y_-,\lambda_-,J_-)=0$ for all ECH data $(\lambda_-,J_-)$ for $(Y_-,\xi_-)$. If we knew Conjecture A.12, then we could conclude that $(Y_-,\xi_-)$ is overtwisted. □

**A.4 A cobordism chain map.** We now state and prove the key lemma in the proof of Theorem A.9.

Lemma A.14. *Under the assumptions of Theorem A.9, there is a chain map*

$$\Phi:\mathrm{ECC}^L_{\mathrm{simp}}(Y_+,\lambda_+,J_+)\longrightarrow \mathrm{ECC}^L(Y_-,\lambda_-,J_-)$$

*with the following properties:*

(a) $\Phi(\emptyset)=\emptyset$.

(b) *There is a decomposition $\Phi=\Phi_0+\Phi_1+\cdots$ such that*

$$\sum_{i+j=k}(\partial_i\Phi_j-\Phi_i\partial_j)=0 \qquad \text{(A.7)}$$

*for each non-negative integer $k$.*

*Proof.* The proof has four steps.

*Step 1.* We begin with the definition of $\Phi$. Let $(X,\omega)$ be an exact symplectic cobordism from $(Y_+,\lambda_+)$ to $(Y_-,\lambda_-)$. Let $\lambda$ be a corresponding 1-form on $X$. There exists a neighborhood $N_+ \simeq (-\varepsilon,0]\times Y_+$ of $Y_+$ in $X$ in which $\lambda = e^s\lambda_+$ where $s$ denotes the $(-\varepsilon,0]$ coordinate. Likewise there exists a neighborhood $N_- \simeq [0,\varepsilon)\times Y_-$ of $Y_-$ in $X$ in which $\lambda = e^s\lambda_-$. We then define the "completion"

$$\overline{X} = \big((-\infty,0]\times Y_-\big)\cup_{Y_-} X \cup_{Y_+} \big([0,\infty)\times Y_+\big)\,,$$

with smooth structure defined using the above neighborhoods. Choose a generic almost complex structure $J$ on $\overline{X}$ which agrees with $J_+$ on $[0,\infty)\times Y_+$, which agrees with $J_-$ on $(-\infty,0]\times Y_-$, and which is $\omega$-tame on $X$. If $\alpha^+$ and $\alpha^-$ are orbit sets in $Y_+$ and $Y_-$ respectively, define $\mathcal{M}_J(\alpha^+,\alpha^-)$ to be the moduli space of $J$-holomorphic curves in $\overline{X}$ satisfying the obvious analogues of the conditions in Definition A.2.

The crucial point in all of what follows is this:

(∗) If the orbit set $\alpha^+$ is simple, then a holomorphic curve in $\mathcal{M}_J(\alpha^+,\alpha^-)$ cannot have any multiply covered component. Also, a broken holomorphic curve arising as a limit of a sequence of curves in $\mathcal{M}_J(\alpha^+,\alpha^-)$ cannot have any multiply covered component in the cobordism level.

Note that the proof of (∗) uses exactness of the cobordism to deduce that every component of a holomorphic curve in $\overline{X}$ has at least one positive end.

Another key point is that the definition of the ECH index $I$, and the index inequality in Proposition A.2(a), carry over directly to holomorphic curves in $\overline{X}$, see [Hu2, Th. 4.15]. In particular, if $\alpha^+$ is simple and if $u\in\mathcal{M}_J(\alpha^+,\alpha^-)$ has $I(u)=0$, then the index inequality applies to give $\operatorname{ind}(u)\le I(u)$, and since $J$ is generic we conclude that $I(u)=0$ and $u$ is an isolated point in the moduli space, cut out transversely. As a result, we can define the map $\Phi$ as follows: If $\alpha^+$ is a simple admissible orbit set in $Y_+$ with $\mathcal{A}(\alpha^+)<L$, then

$$\Phi(\alpha^+) := \sum_{\alpha^-}\ \sum_{\{u\in\mathcal{M}_J(\alpha^+,\alpha^-)\mid I(u)=0\}} \varepsilon(u)\,, \tag{A.8}$$

where the first sum is over admissible orbit sets $\alpha^-$ in $Y_-$, and $\varepsilon(u)\in\{\pm1\}$ is a sign defined as in [HuT2, §9].

*Step 2.* We now show that $\Phi$ is well-defined, i.e. that the sum on the right-hand side of (A.8) is finite, and we also prove that $\Phi$ satisfies property (a).

To start, note that if there exists $u\in\mathcal{M}_J(\alpha^+,\alpha^-)$, then exactness of the cobordism and Stokes's theorem imply that $\mathcal{A}(\alpha^+)\ge\mathcal{A}(\alpha^-)$, with equality only if $u$ is the empty holomorphic curve. This has three important consequences. First, $\Phi$ maps $\mathrm{ECC}^L_{\mathrm{simp}}$ to $\mathrm{ECC}^L$ as required. Second, $\Phi(\emptyset)=\emptyset$. (The sign here follows from the conventions in [HuT2, §9].) Third, for any simple admissible orbit set $\alpha^+$, only finitely many admissible orbit sets $\alpha^-$ can make a nonzero contribution to the right-hand side of (A.8). So to prove that $\Phi$ is well-defined, we need to show that if $\alpha^+$ is a simple admissible orbit set in $Y_+$ and if $\alpha^-$ is an admissible orbit set in $Y_-$, then there are only finitely many curves $u\in\mathcal{M}_J(\alpha^+,\alpha^-)$ with $I(u)=0$.

Suppose to obtain a contradiction that there are infinitely many such curves. By a Gromov compactness argument as in [Hu1, Lem. 9.8] we can then pass to a subsequence that converges to a broken holomorphic curve with total ECH index and total Fredholm index both equal to 0. By $(*)$, the level of the broken curve in $\overline{X}$ cannot contain any multiply covered component. Consequently the index inequality implies that this level has $I \geq 0$, and so by Proposition A.2(a) all levels have $I = 0$. The proof of [HuT1, Lem. 7.19] then shows that there is only one level (i.e. there cannot be symplectization levels containing branched covers of $\mathbb{R}$-invariant cylinders). Thus the limiting curve is also an element of $\mathcal{M}_J(\alpha^+, \alpha^-)$ with $I = 0$, and since this is an isolated point in the moduli space we have a contradiction.

*Step 3.* We now show that $\Phi$ is a chain map. If $\alpha^+$ is a simple admissible orbit set in $Y_+$, then to prove that $(\partial\Phi - \Phi\partial)\alpha^+ = 0$, we analyze ends of the $I = 1$ part of $\mathcal{M}_J(\alpha^+, \alpha^-)$ where $\alpha^-$ is an admissible orbit set in $Y_-$. Again, by $(*)$, a broken curve arising as a limit of such curves cannot contain a multiply covered component in the cobordism level. Thus the proof of [HuT1, Lem. 7.23] carries over to show that a broken curve arising as a limit of such curves consists of an $\operatorname{ind} = I = 0$ piece $u_0$ in the cobordism level, an $\operatorname{ind} = I = 1$ piece $u_1$ in a symplectization level, and (if $u_1$ is in $\mathbb{R} \times Y_-$) possibly additional levels in $\mathbb{R} \times Y_-$ between $u_0$ and $u_1$ consisting of branched covers of $\mathbb{R}$-invariant cylinders. The gluing analysis to prove that the ECH differential has square zero [HuT1, Th. 7.20] then carries over with minor modifications to prove that $\partial\Phi = \Phi\partial$.

*Step 4.* We now show that $\Phi$ satisfies property (b). To do so, note that if $u$ is a holomorphic curve counted by $\Phi$, then $J_+(u)$ is even by the same parity argument as before. Also, since $u$ contains no multiply covered component, and since every component of $u$ has a positive end, the proof of [Hu2, Th. 6.6] carries over to show that $J_+(u) \geq 0$. We now define $\Phi_k$ to be the contribution to $\Phi$ from curves $u$ with $J_+(u) = 2k$. Equation (A.7) then follows from the fact that $J_+$ is additive under gluing. □

### A.5 Conclusion.

*Proof of Theorem A.9.* The decomposition (A.5) of the differential for $(\lambda_+, J_+)$, restricted to the subcomplex $ECC^L_{\mathrm{simp}}(Y_+, \lambda_+, J_+)$, gives rise to a spectral sequence ${}^L E^*_{\mathrm{simp}}(Y_+, \lambda_+, J_+)$, whose $E^1$ term is the homology of $\partial_0$ acting on $ECC^L_{\mathrm{simp}}(Y_+, \lambda_+, J_+)$, and so forth. Likewise the decomposition (A.5) of the differential for $(\lambda_-, J_-)$, restricted to the subcomplex $ECC^L(Y_-, \lambda_-, J_-)$, gives rise to a spectral sequence ${}^L E^*(Y_-, \lambda_-, J_-)$. By Lemma A.14(b), $\Phi$ induces a morphism of spectral sequences

$$\Phi^* : {}^L E^*_{\mathrm{simp}}(Y_+, \lambda_+, J_+) \longrightarrow {}^L E^*(Y_-, \lambda_-, J_-)\,,$$

which by Lemma A.14(a) sends $\emptyset$ to $\emptyset$. If $f^L_{\mathrm{simp}}(Y_+, \lambda_+, J_+) = k < \infty$, then $\emptyset$ does not survive to ${}^L E^{k+1}_{\mathrm{simp}}$. Applying the morphism $\Phi^*$ then shows that $\emptyset$ does not survive to ${}^L E^{k+1}(Y_-, \lambda_-, J_-)$, so $f^L(Y_-, \lambda_-, J_-) \leq k$. □

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