

ON THE PERIODS OF AUTOMORPHIC FORMS ON SPECIAL ORTHOGONAL GROUPS AND THE GROSS–PRASAD CONJECTURE

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Dedicated to Professor Hiroyuki Yoshida on the occasion of his sixtieth birthday

Abstract. In this paper, we would like to formulate a conjecture on a relation between a certain period of automorphic forms on special orthogonal groups and some L -value. Our conjecture can be considered as a refinement of the global Gross–Prasad conjecture.

Introduction

In the early 90’s, Gross and Prasad [GrP1,2] gave a series of fascinating conjectures on the restriction of an automorphic representation of a special orthogonal group to a smaller special orthogonal subgroup. We now recall their global conjecture. Let k be a global field with $\text{char}(k) \neq 2$. Let $(V_0, Q_0) \subset (V_1, Q_1)$ be quadratic forms over k with rank n and $n + 1$, respectively. We assume that $n \geq 2$ and that (V_0, Q_0) is not isomorphic to the hyperbolic plane. We regard $G_0 = \text{SO}_{Q_0}$ as a subgroup of $G_1 = \text{SO}_{Q_1}$. Let $\pi_1 \simeq \otimes_v \pi_{1,v}$ and $\pi_0 \simeq \otimes_v \pi_{0,v}$ be irreducible tempered cuspidal automorphic representations of $G_1(\mathbb{A})$ and $G_0(\mathbb{A})$, respectively. Assume that $\text{Hom}_{G_0(k_v)}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ for any place v of k . Then the global Gross–Prasad conjecture [GrP1] asserts that

$$\langle \varphi_1|_{G_0}, \varphi_0 \rangle := \int_{G_0(k) \backslash G_0(\mathbb{A})} \varphi_1(g_0) \overline{\varphi_0(g_0)} dg_0 \neq 0$$

for some $\varphi_1 \in \pi_1$ and $\varphi_0 \in \pi_0$ if and only if $L(1/2, \pi_1 \boxtimes \pi_0) \neq 0$. Here, $L(s, \pi_1 \boxtimes \pi_0)$ is the “product” L -function of π_1 and π_0 .

In this paper, we would like to formulate a conjecture, which expresses the period $\langle \varphi_1|_{G_0}, \varphi_0 \rangle$ in terms of L -values. Put

$$\Delta_{G_1} = \begin{cases} \zeta(2)\zeta(4) \cdots \zeta(2l) & \text{if } \dim V_1 = 2l + 1, \\ \zeta(2)\zeta(4) \cdots \zeta(2l - 2) \cdot L(l, \chi_{Q_1}) & \text{if } \dim V_1 = 2l, \end{cases}$$

where χ_{Q_1} is the quadratic Hecke character associated with the discriminant of Q_1 . Let $\pi_1 \simeq \otimes_v \pi_{1,v}$ and $\pi_0 \simeq \otimes_v \pi_{0,v}$ be irreducible cuspidal automorphic representations of $G_1(\mathbb{A})$ and $G_0(\mathbb{A})$, respectively. We assume, for simplicity, π_1 and π_0 are

tempered. We put

$$\mathcal{P}_{\pi_1, \pi_0}(s) = \frac{L(s, \pi_1 \boxtimes \pi_0)}{L(s + (1/2), \pi_1, \text{Ad})L(s + (1/2), \pi_0, \text{Ad})},$$

where $L(s, \pi_1, \text{Ad})$ and $L(s, \pi_0, \text{Ad})$ are the adjoint L -function of π_1 and that of π_0 , respectively. We assume that the L -functions $L(s, \pi_1 \boxtimes \pi_0)$, $L(s, \pi_1, \text{Ad})$, and $L(s, \pi_0, \text{Ad})$ have meromorphic continuation. For a sufficiently large finite set of bad places S , we denote the partial Euler products for $\mathcal{P}_{\pi_1, \pi_0}(s)$ and Δ_{G_1} by $\mathcal{P}_{\pi_1, \pi_0}^S(s)$ and $\Delta_{G_1}^S$, respectively.

Let $\varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$ be cusp forms. We consider the matrix coefficients

$$\begin{aligned} \Phi_{\varphi_{1,v}, \varphi_{1,v}}(g_1) &= \langle \pi_{1,v}(g_1)\varphi_{1,v}, \varphi_{1,v} \rangle_v, & g_1 \in G_1(k_v), \\ \Phi_{\varphi_{0,v}, \varphi_{0,v}}(g_0) &= \langle \pi_{0,v}(g_0)\varphi_{0,v}, \varphi_{0,v} \rangle_v, & g_0 \in G_0(k_v). \end{aligned}$$

Put

$$I(\varphi_{1,v}, \varphi_{0,v}) = \int_{G_0(k_v)} \Phi_{\varphi_{1,v}, \varphi_{1,v}}(g_{0,v}) \overline{\Phi_{\varphi_{0,v}, \varphi_{0,v}}(g_{0,v})} dg_{0,v}.$$

It will be proved that this integral is convergent (Proposition 1.1).

Then we conjecture that there exists an integer β such that

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = 2^\beta C_0 \Delta_{G_1}^S \mathcal{P}_{\pi_1, \pi_0}^S(1/2) \prod_{v \in S} \frac{I(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}, \tag{★}$$

where C_0 is a constant determined by the choice of the local and global Haar measures of $G_0(\mathbb{A})$ (Conjecture 1.5). For more precise definitions, see section 1. When $n = 2$, our conjecture reduces to the theorem of Waldspurger [W].

One can give a possible interpretation of the factor 2^β in (★) in terms of the Arthur conjecture [Ar]. Let \mathcal{L}_k be the hypothetical Langlands group for k . Then, if we admit the Arthur conjecture, for an irreducible cuspidal tempered automorphic representation π_i of $G_i(\mathbb{A})$ ($i = 0, 1$), one can attach an L -homomorphism $\psi_i : \mathcal{L}_k \rightarrow {}^L G_i = \hat{G}_i \rtimes W_k$, where W_k is the Weil group [T] of k . It is generally believed that the structure of the L -packet for π_i is closely related to the finite group $\mathcal{S}_{\psi_i} = \text{Cent}_{\hat{G}_i}(\text{Im}(\psi_i))$. Then, we conjecture that

$$2^\beta = \frac{1}{|\mathcal{S}_{\psi_1}| \cdot |\mathcal{S}_{\psi_0}|}$$

(cf. Conjecture 2.1).

This paper consists of three parts. In Part I (sections 1–3), we formulate our conjecture in detail. We first formulate our conjecture in the tempered case. Then we discuss the relation with the Arthur conjecture. In particular, a possible interpretation of the factor 2^β in terms of Arthur parameter will be given. In section 3, we discuss the non-tempered case. In the non-tempered case, several difficulties will arise. One is that the factor $\mathcal{P}_{\pi_1, \pi_0}(s)$ may not be holomorphic at $s = 1/2$. Another difficulty is that the integral $I(\varphi_{1,v}, \varphi_{0,v})$ may not be convergent. Nevertheless, several examples suggest that an analogue of (★) holds in non-tempered case. We give a somewhat optimistic conjecture in section 3 for non-tempered case.

In Part II (sections 4–5), we develop some local theory to show that our conjecture (★) makes sense. In section 4, we prove that the local integral $I(\varphi_{1,v}, \varphi_{0,v})$ is

convergent if both $\pi_{1,v}$ and $\pi_{0,v}$ are tempered. In section 5, we show that

$$I(\varphi_{1,v}, \varphi_{0,v}) = \Delta_{G_1,v} \mathcal{P}_{\pi_{1,v}, \pi_{0,v}}(1/2)$$

for unramified case (Theorem 1.2). In particular, the right-hand side of (★) is independent of the choice of the set S of bad primes. In the course of the proof, we make use of the results of Ginzburg, Piatetski-Shapiro, Rallis [GiPR] and those of Kato, Murase, Sugano [KMS]. We emphasise the fact that the factor $\mathcal{P}_{\pi_1, \pi_0}(s)$ already appeared in [GiPR].

In Part III (sections 6–12), we give several examples over number fields. One can also give several examples over function fields, but we do not discuss such cases in this paper. In section 6, we show that our conjecture is compatible with the theorem of Waldspurger [W]. In section 7, we prove our conjecture for $n = 3$ by using the first named author's result [I2]. Then we show that our conjecture is compatible with the result of Watson [Wa] in some cases. We also discuss the relation with the conjecture of Deligne [D] and the conjecture of Shimura [S2,3]. In section 8, we consider the restriction of the Yoshida lift to the diagonal subgroup. We recall the result of Gan and the first named author [GI], which is compatible with our conjecture. In section 9, we consider the restriction of the Saito–Kurokawa lift to the diagonal subset. We show that the first named author's result [I1] is compatible with our conjecture. Note that this example is non-tempered. In section 10, we consider our result on the restriction of the hermitian Maass lift to the space of Saito–Kurokawa lifts [II]. This example is also non-tempered, and is compatible with our conjecture. In section 11, we consider the trivial representation. This example reduces to the mass formula for the quadratic forms. In section 12, we collect the calculation over the real place, which is necessary to get the result of section 7, section 9, and section 10.

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Part I. Global Theory

1 Formulation of the Conjecture

In this paper, we would like to formulate a conjecture on a relation between a certain period of automorphic forms on special orthogonal groups and some L -value. Our conjecture can be considered as a refinement of the global Gross–Prasad conjecture [GrP1].

Let k be a global field with $\text{char}(k) \neq 2$. Let (V_1, Q_1) and (V_0, Q_0) be quadratic forms over k with rank $n + 1$ and n , respectively. We assume $n \geq 2$. When $n = 2$, we also assume (V_0, Q_0) is not isomorphic to the hyperbolic plane over k . We denote the special orthogonal group of (V_i, Q_i) by G_i ($i = 0, 1$). From now on, the subscript i will indicate either 0 or 1, except for some obvious situations. We assume there is an embedding $\iota : V_0 \hookrightarrow V_1$ of quadratic spaces. Then we have an embedding of the corresponding special orthogonal groups $\iota : G_0 \hookrightarrow G_1$. We regard G_0 as a subgroup of G_1 by this embedding. The group $G_i(k_v)$ of k_v -valued points of G_i is denoted by $G_{i,v}$.

For an even-dimensional quadratic form (V, Q) , the discriminant field K_Q is defined by $K_Q = k(\sqrt{(-1)^{\dim V/2} \det Q})$. We put $K = K_{Q_0}$ (resp. $K = K_{Q_1}$), if $\dim V_0$ is even (resp. if $\dim V_1$ is even). We call K the discriminant field for the pair (V_1, V_0) . Let $\chi = \chi_{K/k}$ be the Hecke character associated to K/k by the class field theory.

Put

$$\Delta_{G_i, v} = \begin{cases} \zeta_v(2)\zeta_v(4) \cdots \zeta_v(2l) & \text{if } \dim V_i = 2l + 1, \\ \zeta_v(2)\zeta_v(4) \cdots \zeta_v(2l - 2) \cdot L_v(l, \chi) & \text{if } \dim V_i = 2l, \end{cases}$$

$$\Delta_{G_i} = \begin{cases} \zeta(2)\zeta(4) \cdots \zeta(2l) & \text{if } \dim V_i = 2l + 1, \\ \zeta(2)\zeta(4) \cdots \zeta(2l - 2) \cdot L(l, \chi) & \text{if } \dim V_i = 2l. \end{cases}$$

Note that $\Delta_{G_i} = L(M_i^\vee(1))$, where M_i^\vee is the dual motive of the motive M_i associated to G_i by Gross [Gr].

Let $\pi_i \simeq \otimes_v \pi_{i,v}$ be an irreducible square-integrable automorphic representation of $G_i(\mathbb{A})$. There is a canonical inner product $\langle \cdot, \cdot \rangle$ on forms on $G_i(k) \backslash G_i(\mathbb{A})$ defined by

$$\langle \varphi_i, \varphi'_i \rangle = \int_{G_i(k) \backslash G_i(\mathbb{A})} \varphi_i(g_i) \overline{\varphi'_i(g_i)} dg_i,$$

where dg_i is the Tamagawa measure on $G_i(\mathbb{A})$. We choose a Haar measure $dg_{i,v}$ on $G_{i,v}$ for each v . There exists a positive number C_i such that $dg_i = C_i \prod_v dg_{i,v}$, when the right-hand side is well-defined. In this paper, we call C_i the Haar measure constant. Since $\pi_{i,v}$ is a unitary representation, there is an inner product $\langle \cdot, \cdot \rangle_v$ on $\pi_{i,v}$ for any place v of k . We put $\|\varphi_{i,v}\| = \langle \varphi_{i,v}, \varphi_{i,v} \rangle_v^{1/2}$, as usual. There exists a positive constant C_{π_i} such that $\langle \varphi_i, \varphi'_i \rangle = C_{\pi_i} \prod_v \langle \varphi_{i,v}, \varphi'_{i,v} \rangle_v$ for any decomposable vectors $\varphi_i = \otimes_v \varphi_{i,v} \in \otimes_v \pi_{i,v}$ and $\varphi'_i = \otimes_v \varphi'_{i,v} \in \otimes_v \pi_{i,v}$.

We fix maximal compact subgroups $\mathcal{K}_1 = \prod_v \mathcal{K}_{1,v} \subset G_1(\mathbb{A})$ and $\mathcal{K}_0 = \prod_v \mathcal{K}_{0,v} \subset G_0(\mathbb{A})$ such that $[\mathcal{K}_0 : \mathcal{K}_1 \cap \mathcal{K}_0] < \infty$. We choose a \mathcal{K}_i -finite decomposable vector $\varphi_i = \otimes_v \varphi_{i,v} \in \otimes_v \pi_{i,v}$. We are interested in the period $\langle \varphi_1|_{G_0}, \varphi_0 \rangle$ where $\varphi_1|_{G_0}$ is the restriction of φ_1 to $G_0(\mathbb{A})$.

Let S be a finite set of bad places containing all archimedean places. We may and do assume the following conditions hold for $v \notin S$:

- (U1) G_i is unramified over k_v .
- (U2) $\mathcal{K}_{i,v}$ is a hyperspecial maximal compact subgroup of $G_{i,v}$.
- (U3) $\mathcal{K}_{0,v} \subset \mathcal{K}_{1,v}$.
- (U4) $\pi_{i,v}$ is an unramified representation of $G_{i,v}$.
- (U5) The vector $\varphi_{i,v}$ is fixed by $\mathcal{K}_{i,v}$ and $\|\varphi_{i,v}\| = 1$.
- (U6) $\int_{\mathcal{K}_{i,v}} dg_{i,v} = 1$.

When G_i is unramified over k_v , we shall say that a Haar measure on $G_{i,v}$ is the standard Haar measure if the volume of a hyperspecial maximal compact subgroup is 1. Thus the condition (U6) means that the measure $dg_{i,v}$ is the standard Haar measure.

The L -group ${}^L G_i$ of G_i is a semi-direct product $\hat{G}_i \rtimes W_k$. Here, W_k is the Weil group of k and

$$\hat{G}_i = \begin{cases} \mathrm{Sp}_l(\mathbb{C}) & \text{if } \dim V_i = 2l + 1, \\ \mathrm{SO}(2l, \mathbb{C}) & \text{if } \dim V_i = 2l. \end{cases}$$

We denote by st the standard representation of ${}^L G_i$. The completed standard L -function for π_i is denoted by $L(s, \pi_i, \mathrm{st})$ for an irreducible automorphic representation π_i of $G_i(\mathbb{A})$. For simplicity, we sometimes denote $L(s, \pi_i, \mathrm{st})$ by $L(s, \pi_i)$. For $v \notin S$, the Euler factor for $L(s, \pi_i)$ is given by $\det(1 - \mathrm{st}(A_{\pi_{i,v}}) \cdot q_v^{-s})^{-1}$, where $A_{\pi_{i,v}}$ is the Satake parameter of $\pi_{i,v}$. We consider the tensor product L -function $L(s, \pi_1 \boxtimes \pi_0)$. The Euler factor of $L(s, \pi_1 \boxtimes \pi_0)$ for $v \notin S$ is given by $\det(1 - \mathrm{st}(A_{\pi_{1,v}}) \otimes \mathrm{st}(A_{\pi_{0,v}}) \cdot q_v^{-s})^{-1}$.

Consider the adjoint representation $\mathrm{Ad} : {}^L G_i \rightarrow \mathrm{GL}(\mathrm{Lie}(\hat{G}_i))$. The associated L -function $L(s, \pi_i, \mathrm{Ad})$ is called the adjoint L -function. We assume that $L(s, \pi_1 \boxtimes \pi_0)$ and $L(s, \pi_i, \mathrm{Ad})$ can be analytically continued to the whole s -plane.

We put

$$\mathcal{P}_{\pi_1, \pi_0}(s) = \frac{L(s, \pi_1 \boxtimes \pi_0)}{L(s + (1/2), \pi_1, \mathrm{Ad})L(s + (1/2), \pi_0, \mathrm{Ad})}.$$

Let $\pi_{i,v}$ be an irreducible admissible representation of $G_{i,v}$. We denote the complex conjugate of $\pi_{i,v}$ by $\bar{\pi}_{i,v}$. It is believed that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \leq 1 \tag{MF}$$

for any place v of k . We do not assume (MF) in this paper. Note that an analogue of (MF) for orthogonal groups has been proved by Aizenbud, Gourevitch, Rallis, Schiffmann [AGRS] for the non-archimedean place and by Sun and Zhu [SunZ] for irreducible Harish-Chandra smooth representations for the archimedean place.

We consider the matrix coefficient

$$\Phi_{\varphi_{i,v}, \varphi'_{i,v}}(g_i) = \langle \pi_{i,v}(g_i) \varphi_{i,v}, \varphi'_{i,v} \rangle_v, \quad g_i \in G_{i,v},$$

for $\mathcal{K}_{1,v}$ -finite vectors $\varphi_{1,v}, \varphi'_{1,v} \in \pi_{1,v}$ and $\mathcal{K}_{0,v}$ -finite vectors $\varphi_{0,v}, \varphi'_{0,v} \in \pi_{0,v}$. Put

$$I(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}) = \int_{G_{0,v}} \Phi_{\varphi_{1,v}, \varphi'_{1,v}}(g_{0,v}) \overline{\Phi_{\varphi_{0,v}, \varphi'_{0,v}}(g_{0,v})} dg_{0,v},$$

$$\alpha_v(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}) = \Delta_{G_{1,v}}^{-1} \mathcal{P}_{\pi_{1,v}, \pi_{0,v}}(1/2)^{-1} I(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}).$$

When $\varphi_{1,v} = \varphi'_{1,v}$ and $\varphi_{0,v} = \varphi'_{0,v}$, we simply denote these objects by $I(\varphi_{1,v}, \varphi_{0,v})$ and $\alpha_v(\varphi_{1,v}, \varphi_{0,v})$, respectively.

PROPOSITION 1.1. *If both $\pi_{1,v}$ and $\pi_{0,v}$ are tempered, then the integral $I(\varphi_{1,v}, \varphi_{0,v})$ is absolutely convergent and $I(\varphi_{1,v}, \varphi_{0,v}) \geq 0$ for any $\mathcal{K}_{i,v}$ -finite vector $\varphi_{i,v} \in \pi_{i,v}$.*

Theorem 1.2. *Let v be a non-archimedean place. Assume that the conditions (U1), (U2), (U3), (U4), (U5), and (U6) hold. If the integral $I(\varphi_{1,v}, \varphi_{0,v})$ is absolutely convergent, then we have $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) = 1$.*

The proofs of Proposition 1.1 and Theorem 1.2 will be given in Part II.

CONJECTURE 1.3. Assume that both $\pi_{1,v}$ and $\pi_{0,v}$ are tempered. Then $\dim_{\mathbb{C}} \mathrm{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ if and only if $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) > 0$ for some $\mathcal{K}_{i,v}$ -finite vector $\varphi_{i,v} \in \pi_{i,v}$.

Now let $\pi_i \simeq \otimes_v \pi_{i,v}$ be an irreducible cuspidal automorphic representation of $G_i(\mathbb{A})$. We shall say that π_i is almost locally generic if π_i satisfies the following condition (ALG).

(ALG) For almost all v , the constituent $\pi_{i,v}$ is generic.

It is believed that π_i is almost locally generic if and only if $\pi_{i,v}$ is generic for some v . It is also believed that π_i is almost locally generic if and only if π_i is tempered (the generalized Ramanujan conjecture).

CONJECTURE 1.4. Let $\pi_i \simeq \otimes_v \pi_{i,v}$ be an irreducible cuspidal automorphic representation of $G_i(\mathbb{A})$. We assume both π_1 and π_0 are almost locally generic. Then

- (1) The integral $I(\varphi_{1,v}, \varphi_{0,v})$ should be absolutely convergent and $I(\varphi_{1,v}, \varphi_{0,v}) \geq 0$ for any $\mathcal{K}_{i,v}$ -finite vector $\varphi_{i,v} \in \pi_{i,v}$.
- (2) $\dim_{\mathbb{C}} \text{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ if and only if $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) > 0$ for some $\mathcal{K}_{i,v}$ -finite vector $\varphi_{i,v} \in \pi_{i,v}$.

Now we state our global conjecture.

CONJECTURE 1.5. Let $\pi_1 \simeq \otimes_v \pi_{1,v}$ and $\pi_0 \simeq \otimes_v \pi_{0,v}$ be irreducible cuspidal automorphic representations of $G_1(\mathbb{A})$ and $G_0(\mathbb{A})$, respectively. We assume π_1 and π_0 are almost locally generic. Then there should be an integer β such that

$$\frac{|\langle \varphi_1 |_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = 2^\beta C_0 \Delta_{G_1} \mathcal{P}_{\pi_1, \pi_0}(1/2) \prod_{v \in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}$$

for any non-zero vectors $\varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$.

We will discuss the nature of the integer β in the next section.

REMARK 1.6. When π_1 and π_0 are tempered, it is believed that the local L -factors $L(s, \pi_{1,v}, \text{Ad})$, $L(s, \pi_{0,v}, \text{Ad})$, and $L(s, \pi_{1,v} \boxtimes \pi_{0,v})$ are holomorphic for $\text{Re}(s) > 0$. Therefore in this case our conjecture is equivalent to

$$\frac{|\langle \varphi_1 |_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = 2^\beta C_0 \Delta_{G_1}^S \mathcal{P}_{\pi_1, \pi_0}^S(1/2) \prod_{v \in S} \frac{I(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2},$$

where $\Delta_{G_1}^S$ and $\mathcal{P}_{\pi_1, \pi_0}^S(s)$ are the partial Euler products. In particular, the definition of the L -factors for bad primes plays no role in this case. Note also that it is believed that $L(1, \pi_i, \text{Ad}) \neq 0$ if π_i is tempered.

REMARK 1.7. One can formulate Conjecture 1.5 in a different way as follows. Assume the local measure $dg_{i,v}$ and the local inner product $\langle \cdot, \cdot \rangle_v$ are normalised so that $C_i = C_{\pi_i} = 1$. Put

$$H_{\pi_1, \pi_0} = \text{Hom}_{G_0(\mathbb{A}) \times G_0(\mathbb{A})}((\pi_1 \boxtimes \tilde{\pi}_1) \otimes (\bar{\pi}_0 \boxtimes \tilde{\bar{\pi}}_0), \mathbb{C}).$$

We define two elements $L_{\pi_1, \pi_0}^{\text{global}}, L_{\pi_1, \pi_0}^{\text{local}} \in H_{\pi_1, \pi_0}$ by

$$\begin{aligned} L_{\pi_1, \pi_0}^{\text{global}}(\varphi_1, \varphi'_1; \varphi_0, \varphi'_0) &= \langle \varphi_1 |_{G_0}, \varphi_0 \rangle \overline{\langle \varphi'_1 |_{G_0}, \varphi'_0 \rangle}, \\ L_{\pi_1, \pi_0}^{\text{local}}(\varphi_1, \varphi'_1; \varphi_0, \varphi'_0) &= \prod_v \alpha_v(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}). \end{aligned}$$

Then Conjecture 1.5 can be reformulated as

$$L_{\pi_1, \pi_0}^{\text{global}} = 2^\beta \Delta_{G_1} \mathcal{P}_{\pi_1, \pi_0}(1/2) L_{\pi_1, \pi_0}^{\text{local}}.$$

2 Relation to the Arthur Conjecture

This section is devoted to a somewhat speculative argument based on the Arthur conjecture [Ar]. We recall the Arthur conjecture for an automorphic representation of reductive algebraic groups. We assume, for simplicity, G is a reductive algebraic group defined over k with anisotropic center. The local Langlands group \mathcal{L}_v is defined by

$$\mathcal{L}_v = \begin{cases} W_{k_v} \times \mathrm{SU}(2) & \text{if } v \text{ is non-archimedean,} \\ W_{k_v} & \text{if } v \text{ is archimedean,} \end{cases}$$

where W_{k_v} is the Weil group of k_v . A Langlands parameter is a homomorphism $\phi_v : \mathcal{L}_v \rightarrow {}^L G$ which satisfies certain additional conditions. Two Langlands parameters are equivalent if they are conjugate by an element of \hat{G} . Langlands conjectured that for each equivalence class of Langlands parameter, one can associate a finite set $\Pi_{\phi_v}(G)$ of irreducible admissible representations of G_v . The finite set $\Pi_{\phi_v}(G)$ is called the L -packet for ϕ_v . The set $\Pi(G_v)$ of all equivalence classes of irreducible admissible representations of G_v should be decomposed into a disjoint union

$$\Pi(G_v) = \coprod_{\phi_v} \Pi_{\phi_v}(G),$$

where ϕ_v extends over the equivalence classes of Langlands parameters. The L -packet $\Pi_{\phi_v}(G)$ should contain a tempered representation if and only if the Langlands parameter ϕ_v has a bounded image, in which case ϕ_v is called tempered. If ϕ_v is tempered, then all members of $\Pi_{\phi_v}(G)$ should be tempered.

A homomorphism $\psi_v : \mathcal{L}_v \times \mathrm{SL}_2(\mathbb{C}) \rightarrow {}^L G$ whose restriction to $\mathrm{SL}_2(\mathbb{C})$ is holomorphic is called a (local) Arthur parameter if $\psi_v|_{\mathcal{L}_v}$ is a tempered Langlands parameter. One can consider the equivalence of Arthur parameters as in the case of Langlands parameters. Arthur conjectured that for each equivalence class of Arthur parameters ψ_v , one can associate a finite set of unitary representations $\Pi_{\psi_v}(G)$. The set $\Pi_{\psi_v}(G)$ is called the A -packet of ψ_v . A -packets are not necessarily disjoint.

For each representation ρ_v of $\mathcal{L}_v \times \mathrm{SL}_2(\mathbb{C})$, we associate an L -factor as follows. We may assume ρ_v is irreducible. Then there exists an irreducible representation ϕ_v of \mathcal{L}_v and an integer $t \geq 0$ such that

$$\rho_v \simeq \phi_v \boxtimes \mathrm{Sym}^t,$$

where Sym^t is the unique irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of degree $t + 1$. We put

$$L(s, \rho_v) = \prod_{j=0}^t L(s - j + (t/2), \phi_v).$$

For each element $\pi_v \in \Pi_{\psi_v}(G)$ and a finite-dimensional representation r of ${}^L G$, we put $L(s, \pi_v, r) = L(s, r \circ \psi_v)$. Note that $L(s, \pi_v, r)$ depends not only on π_v , but also on ψ_v , since A -packets are not necessarily disjoint, although the symbol suggests it does not.

Langlands conjectured that there exists a locally compact group \mathcal{L}_k such that the equivalence classes of irreducible n -dimensional representation of \mathcal{L}_k is in one-to-one correspondence with the set of irreducible cuspidal automorphic representations of

$GL_n(\mathbb{A})$. There should be a homomorphism $\iota_v : \mathcal{L}_v \rightarrow \mathcal{L}_k$ for each v . A (global) Arthur parameter is a certain equivalence class of homomorphisms

$$\psi : \mathcal{L}_k \times SL_2(\mathbb{C}) \longrightarrow {}^L G$$

such that the image of \mathcal{L}_k is bounded. Let $\Pi_\psi(G)$ be the set of square-integrable automorphic representations $\pi \simeq \otimes_v \pi_v$ of $G(\mathbb{A})$ such that $\pi_v \in \Pi_{\psi \circ \iota_v}(G)$ for each v . The set $\Pi_\psi(G)$ is called the A -packet of ψ . Arthur conjectured that the set of square-integrable automorphic representations of $G(\mathbb{A})$ is a union

$$\bigcup_{\psi} \Pi_\psi(G).$$

If $\pi \in \Pi_\psi(G)$, then ψ is called the Arthur parameter of π . In general, ψ is not uniquely determined by the equivalence class of π , but for special orthogonal groups or unitary groups, ψ should be determined by π .

It is believed that the Arthur parameter $\psi : \mathcal{L}_k \times SL_2(\mathbb{C}) \rightarrow {}^L G$ associated with a square-integrable automorphic representation should be elliptic in the sense that $\text{Im}(\psi)$ is not contained in any proper Levi subgroup of ${}^L G$. This is the case if and only if $\text{Cent}_{\hat{G}}(\text{Im}(\psi))$ is finite. If ψ is an elliptic Arthur parameter such that $\Pi_\psi(G)$ is non-empty, the A -packet $\Pi_\psi(G)$ consists of only irreducible tempered cuspidal automorphic representations if and only if the restriction $\psi|_{SL_2(\mathbb{C})}$ is trivial. In this case, the Arthur parameter ψ is said to be tempered. For an elliptic Arthur parameter ψ , we put

$$\mathcal{S}_\psi = \text{Cent}_{\hat{G}}(\text{Im}(\psi)).$$

Now we go back to the situation that $G_1 = SO(n + 1)$ and $G_0 = SO(n)$. Let ψ_i be an elliptic Arthur parameter for the group G_i . In this case, the group \mathcal{S}_{ψ_i} can be calculated as follows. Let st be the standard representation of ${}^L G_i$. Then $\text{st} \circ \psi_i$ can be decomposed into a direct sum of irreducible representations of $\mathcal{L}_k \times SL_2(\mathbb{C})$:

$$\text{st} \circ \psi_i = \bigoplus_{j=1}^r \psi_i^{(j)}.$$

Here, the representations $\psi_i^{(1)}, \dots, \psi_i^{(r)}$ are mutually distinct orthogonal (resp. symplectic) representations of $\mathcal{L}_k \times SL_2(\mathbb{C})$ if $\dim V_i$ is even (resp. odd). Then

$$\mathcal{S}_{\psi_i} \simeq \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{r-1} & \text{if } \dim V_i \text{ is even and rank } \psi_i^{(j)} \text{ is odd for some } j, \\ (\mathbb{Z}/2\mathbb{Z})^r & \text{otherwise.} \end{cases}$$

In particular, \mathcal{S}_{ψ_i} is an elementary 2-abelian group.

Now we admit the Arthur conjecture. Let π_i be an irreducible cuspidal automorphic representation of $G_i(\mathbb{A})$, which satisfies the condition (ALG). Then corresponding Arthur parameter ψ_i must be tempered, since otherwise $\pi_{i,v}$ cannot be generic for any v .

CONJECTURE 2.1. Assume that π_i is an irreducible tempered cuspidal automorphic representation of $G_i(\mathbb{A})$ with Arthur parameter ψ_i . Then the constant 2^β in Conjecture 1.5 should be equal to $1/(|\mathcal{S}_{\psi_1}| \cdot |\mathcal{S}_{\psi_0}|)$. Equivalently, the equation

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = \frac{C_0 \Delta_{G_1}}{|\mathcal{S}_{\psi_1}| \cdot |\mathcal{S}_{\psi_0}|} \mathcal{P}_{\pi_1, \pi_0}(1/2) \prod_{v \in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}$$

holds.

3 The Non-tempered Case

Let $\pi_{i,v}$ be an irreducible representation of $G_{i,v}$, which we do not assume to be unitary for a moment. Note that if both $\pi_{1,v}$ and $\pi_{0,v}$ are tempered, then $\alpha_v(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v})$ gives an element of

$$\text{Hom}_{G_{0,v} \times G_{0,v}} \left((\pi_{1,v} \boxtimes \tilde{\pi}_{1,v}) \otimes (\bar{\pi}_{0,v} \boxtimes \tilde{\bar{\pi}}_{0,v}), \mathbb{C} \right),$$

where $\tilde{\pi}_{i,v}$ is the contragredient of $\pi_{i,v}$.

CONJECTURE 3.1. The quantity $\alpha_v(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v})$ should be somehow “analytically continued” for any $\pi_{1,v}$ and $\pi_{0,v}$. If $\text{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$, then the continuation $\alpha_v(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v})$ is unique and gives an element of

$$\text{Hom}_{G_{0,v} \times G_{0,v}} \left((\pi_{1,v} \boxtimes \tilde{\pi}_{1,v}) \otimes (\bar{\pi}_{0,v} \boxtimes \tilde{\bar{\pi}}_{0,v}), \mathbb{C} \right).$$

Now we consider the global situation. Let π_i be a square-integrable automorphic representation of $G_i(\mathbb{A})$, which may not be almost locally generic. We assume that $\text{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ for any v . For $v \notin S$, we may assume $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) = 1$ by Theorem 1.2, as long as it is meaningful.

CONJECTURE 3.2. Let π_i be as above. Then

- (1) The integral $\langle \varphi_1|_{G_0}, \varphi_0 \rangle$ should be convergent for any $\varphi_1 \in \pi_1$ and $\varphi_0 \in \pi_0$.
- (2) There should be an integer β such that

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = 2^\beta \Delta_{G_1} C_0 \mathcal{P}_{\pi_1, \pi_0}(1/2) \prod_{v \in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}$$

for any non-zero decomposable vectors $\varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$.

REMARK 3.3. Contrary to the almost locally generic case, the factor 2^β is not necessarily equal to $1/(|\mathcal{S}_{\psi_1}| \cdot |\mathcal{S}_{\psi_0}|)$, and depends not only on global data, but also on local data. See the examples in section 9, section 10, and section 11.

Part II. Local Theory

Until section 5, we consider only local objects and drop subscript v .

4 Convergence of the Integral: Proof of Proposition 1.1

In this section, we assume that k is a local field with $\text{char}(k) \neq 2$. Let (V, Q) be a non-degenerate quadratic space over k . We denote the anisotropic kernel of (V, Q) by $(V^{\text{an}}, Q^{\text{an}})$. Then there is a decomposition $V = X \oplus V^{\text{an}} \oplus Y$, where X and Y are totally isotropic subspaces. The Witt rank r of (V, Q) is, by definition, equal to the dimension of X or Y . We put $d = \dim V^{\text{an}}$. Choosing a basis of X , we get a minimal parabolic subgroup $P_{\min} = M_{\min} N_{\min}$ of G . The Levi factor M_{\min} is isomorphic to $(k^\times)^r \times \text{SO}_{Q^{\text{an}}}$. The split component A_{\min} of M_{\min} is isomorphic to $(k^\times)^r$, and the Weyl group $W(G, A_{\min})$ is of type B or D according as $d \neq 0$ or $d = 0$. We will denote an element of $A_{\min} \simeq (k^\times)^r$ by $x = (x_1, \dots, x_r)$. The simple roots of (P_{\min}, A_{\min}) are given by

$$\alpha_1(x) = x_1 x_2^{-1}, \dots, \alpha_{r-1}(x) = x_{r-1} x_r^{-1},$$

$$\alpha_r(x) = \begin{cases} x_r & \text{if } d \neq 0, \\ x_{r-1}x_r & \text{if } d = 0. \end{cases}$$

These roots are also regarded as a character of M_{\min} . Let $\delta_{P_{\min}}(x)$ be the modulus character of P_{\min} . Then

$$\delta_{P_{\min}}(x) = \prod_{i=1}^r |x_i|^{d+2r-2i}.$$

Fix a special maximal compact subgroup \mathcal{K} of G . Then we have a Cartan decomposition $G = \mathcal{K}M_{\min}^+ \mathcal{K}$, where

$$M_{\min}^+ = \{m \in M_{\min} \mid |\alpha_i(m)| \leq 1 \ (i = 1, \dots, r)\}.$$

Fix a suitable embedding $\eta : G \rightarrow \text{GL}_m$. Then the height function $\sigma(g)$ (with respect to the embedding η) is given by

$$\sigma(g) = \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}} (\log |\eta(g)_{ij}|, \log |\eta(g^{-1})_{ij}|).$$

When k is non-archimedean, the following integral formula holds

$$\int_G f(g) dg = \int_{M_{\min}^+} \mu(m) \int_{\mathcal{K} \times \mathcal{K}} f(k_1 m k_2) dk_1 dk_2 dm, \quad f \in L^1(G),$$

where $\mu(m) = \text{Vol}(\mathcal{K}m\mathcal{K})/\text{Vol}(\mathcal{K})$. Moreover, there exists a positive constant A such that $A^{-1}\delta_{P_{\min}}^{-1}(m) \leq \mu(m) \leq A\delta_{P_{\min}}^{-1}(m)$ for any $m \in M_{\min}^+$. (See Silberger [Si, p. 149].)

When k is archimedean, similar integral formula holds. (See, e.g. Helgason [Hel, Th. 5.8].) In particular, there exists a non-negative function $\mu(m)$ on M_{\min}^+ such that

$$\int_G f(g) dg = \int_{M_{\min}^+} \mu(m) \int_{\mathcal{K} \times \mathcal{K}} f(k_1 m k_2) dk_1 dk_2 dm, \quad f \in L^1(G).$$

Moreover, there exists a constant $A > 0$ such that $\mu(m) \leq A\delta_{P_{\min}}^{-1}(m)$ for $m \in M_{\min}^+$.

Harish-Chandra's spherical function $\Xi(g)$ of G is given by

$$\Xi(g) = \int_{\mathcal{K}} h_0(kg) dk$$

where $h_0 \in \text{Ind}_{P_{\min}}^G 1$ is a function whose restriction to \mathcal{K} is identically equal to 1. Note that Ξ is a matrix coefficient of a tempered representation $\text{Ind}_{P_{\min}}^G 1$. It is known that there exists positive constants A, B such that

$$A^{-1}\delta_{P_{\min}}^{1/2}(m) \leq \Xi(m) \leq A\delta_{P_{\min}}^{1/2}(m)(1 + \sigma(m))^B$$

for any $m \in M_{\min}^+$. (See Silberger [Si, p. 154, Th. 4.2.1] and Harish-Chandra [H, p. 129, Lem. 1 in §10].)

Recall that a function $f(g)$ on G satisfies the weak inequality if

$$|f(g)| \leq A\Xi(g)(1 + \sigma(g))^B$$

for some positive constant A, B . A matrix coefficient of a tempered representation satisfies the weak inequality.

Applying these results for $G_1 = \text{SO}(n + 1)$ and $G_0 = \text{SO}(n)$, we can now prove Proposition 1.1. As before, we define $P_{i,\min}, A_{i,\min}, r_i$, etc., for the group G_i .

Proof of Proposition 1.1. Let π_1 and π_0 be irreducible tempered representations of G_1 and G_0 , respectively. We may assume $A_{0,\min} \subset A_{1,\min}$. Then we have estimates

$$|\Phi_{\varphi_1, \varphi'_1}(m)| \leq A \delta_{P_{1,\min}}^{1/2}(m) (1 + \sigma(m))^B, \quad (m \in M_{1,\min}^+),$$

$$|\Phi_{\varphi_0, \varphi'_0}(m)| \leq A \delta_{P_{0,\min}}^{1/2}(m) (1 + \sigma(m))^B, \quad (m \in M_{0,\min}^+),$$

for some positive constants A, B . When $W(G_0, A_{0,\min})$ is of type B, it is enough to show the following integral

$$\int_{A_{0,\min}^+} \delta_{P_{0,\min}}^{-1/2}(m) \delta_{P_{1,\min}}^{1/2}(m) (1 + \sigma(m))^{2B} dm$$

is convergent. This is reduced to the convergence of

$$\int_{|x_1| \leq |x_2| \leq \dots \leq |x_{r_0}| \leq 1} |x_1 x_2 \dots x_{r_0}|^{1/2} \left(1 - \sum_{j=1}^{r_0} \log |x_j|\right)^{2B} d^\times x_1 d^\times x_2 \dots d^\times x_{r_0}.$$

One can easily prove the convergence of this integral. Note that when $W(G_0, A_{0,\min})$ is of type D, $A_{0,\min}^+$ is not contained in $A_{1,\min}^+$. In this case, one needs to consider the integral

$$\begin{aligned} & \int_{|x_1| \leq |x_2| \leq \dots \leq |x_{r_0}| \leq 1} |x_1 x_2 \dots x_{r_0}|^{1/2} \left(1 - \sum_{j=1}^{r_0} \log |x_j|\right)^{2B} d^\times x_1 d^\times x_2 \dots d^\times x_{r_0} \\ & + \int_{|x_1| \leq |x_2| \leq \dots \leq |x_{r_0-1}| \leq |x_{r_0}|^{-1} \leq 1} |x_1 x_2 \dots x_{r_0-1} x_{r_0}^{-1}|^{1/2} \\ & \quad \times \left(1 - \sum_{j=1}^{r_0-1} \log |x_j| + \log |x_{r_0}|\right)^{2B} d^\times x_1 d^\times x_2 \dots d^\times x_{r_0}. \end{aligned}$$

One can show the convergence of this integral similarly.

To prove the latter part of the proposition, we make use of the result of He [He]. Let Ξ_1 and Ξ_0 be Harish-Chandra’s spherical functions for G_1 and G_0 , respectively. Then the function $g_0 \mapsto \Xi_1(g_0)\Xi_0(g_0)$ belongs to $L^1(G_0)$ by the first part of the proposition. Note that Harish-Chandra’s spherical function is a matrix coefficient of a tempered representation.

Then the latter part of the proposition follows from Theorem 2.1 of He’s paper [He]. Note that He [He] used the estimates of almost L^2 matrix coefficients [CoHH], which is valid for p -adic groups as well. □

5 Calculation of the Unramified Integral: Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We assume the conditions (U1)–(U6) in section 1 holds. In particular, both G_1 and G_0 are quasi-split. We should consider the following two cases:

- (Case A) $G_1 = \mathrm{SO}(2l + 1)$ and $G_0 = \mathrm{SO}(2l)$,
- (Case B) $G_1 = \mathrm{SO}(2l + 2)$ and $G_0 = \mathrm{SO}(2l + 1)$.

Let K be the discriminant field. Note that K is equal to either k or the unramified quadratic extension of k . Let q be the number of elements of the residue field of k . The local zeta function $\zeta(s)$ is defined by $(1 - q^{-s})^{-1}$.

Let $B_i = T_i N_i$ be a Borel subgroup of G_i , where T_i and N_i are a maximal torus of G_i and the unipotent radical of B_i , respectively. Let $A_i \subset T_i$ be the maximal split subtorus. Without loss of generality, we may assume $N_0 \subset N_1$ and $A_0 \subset A_1$.

Let $\pi_1 = I(\Xi) = \text{Ind}_{B_1}^{G_1}(\Xi)$ and $\pi_0 = I(\xi) = \text{Ind}_{B_0}^{G_0}(\xi)$ be unramified principal series of G_1 and G_0 , respectively. Here, Ξ and ξ are unramified quasi-characters of T_1 and T_0 , respectively. Let Φ_Ξ and Φ_ξ be the class-one matrix coefficients of $I(\Xi)$ and $I(\xi)$ such that $\Phi_\Xi(1) = \Phi_\xi(1) = 1$, respectively. We consider the integral

$$I(g_1; \Phi_\Xi, \Phi_\xi) = \int_{G_0} \Phi_\Xi(g_1^{-1}g_0)\Phi_\xi(g_0)dg_0.$$

We assume that both Ξ and ξ are sufficiently close to the unitary axis. As shown in section 4, this condition implies that the integral $I(g_1; \Phi_\Xi, \Phi_\xi)$ is absolutely convergent. In this section, we calculate the value of $I(g_1; \Phi_\Xi, \Phi_\xi)$ at $g_1 = 1$.

Let $f_\Xi \in I(\Xi)$ and $f_\xi \in I(\xi)$ be the class-one vectors such that $f_\Xi(1) = f_\xi(1) = 1$. Then we have

$$\begin{aligned} \Phi_\Xi(g_1) &= \int_{\mathcal{K}_1} f_\Xi(k_1g_1)dk_1, \quad g_1 \in G_1, \\ \Phi_\xi(g_0) &= \int_{\mathcal{K}_0} f_\xi(k_0g_0)dk_0, \quad g_0 \in G_0. \end{aligned}$$

We recall the theory of Shintani functions [KMS]. We denote the Hecke algebra $\mathcal{H}(\mathcal{K}_i \backslash G_i / \mathcal{K}_i)$ by \mathcal{H}_i . By the Satake isomorphism, there are algebra homomorphisms

$$\omega_1 : \mathcal{H}_1 \longrightarrow \mathbb{C} \quad \text{and} \quad \omega_0 : \mathcal{H}_0 \longrightarrow \mathbb{C},$$

corresponding to the unramified principal series π_1 and π_0 , respectively. Recall that a smooth function S on G_1 is called a Shintani function for π_1 and π_0 , if the following conditions are satisfied:

- $\mathcal{L}(k_0)\mathcal{R}(k_1)S = S$ for any $k_1 \in \mathcal{K}_1$ and $k_0 \in \mathcal{K}_0$.
- $\mathcal{L}(\varphi_0)\mathcal{R}(\varphi_1)S = \omega_0(\varphi_0)\omega_1(\varphi_1)S$ for any $\varphi_0 \in \mathcal{H}_0$ and $\varphi_1 \in \mathcal{H}_1$.

Here, \mathcal{L} and \mathcal{R} are the left regular representation and the right regular representation, respectively. Note that $I(g_1; \Phi_\Xi, \Phi_\xi)$ is a Shintani function for $\tilde{\pi}_1$ and $\tilde{\pi}_0$. Kato, Murase, and Sugano [KMS] have proved that if both G_1 and G_0 are split, then a Shintani function exists and is unique up to scalar. In this paper, we do not use the uniqueness of Shintani functions.

Recall that the double coset $B_1 \backslash G_1 / B_0$ has a unique open orbit and the open orbit has a representative $\eta \in \mathcal{K}_1$ (cf. [GiPR, §7]). Note that $\eta^{-1}B_1\eta \cap B_0 = \{1\}$. Let $Y_{\Xi, \xi}$ be the function on G_1 determined by the following conditions:

- (1) $Y_{\Xi, \xi}(b_1g_1b_0) = (\Xi^{-1}\delta_1^{1/2})(b_1)(\xi\delta_0^{-1/2})(b_0)Y_{\Xi, \xi}(g_1)$ for any $b_1 \in B_1$ and $b_0 \in B_0$.
- (2) $Y_{\Xi, \xi}(\eta) = 1$.
- (3) $Y_{\Xi, \xi}(g_1) = 0$ if $g_1 \notin B_1\eta B_0$.

Here, δ_i is the modulus character of B_i . Note that a function satisfying (1) and (3) is unique up to scalar. We define $l_{\Xi, \xi} \in \text{Hom}_{G_0}(\pi_1, \tilde{\pi}_0) = \text{Hom}_{G_0}(I(\Xi), I(\xi^{-1}))$ by

$$l_{\Xi, \xi}(\text{pr}_1(f))(g_0) = \int_{G_1} f(g_1g_0)Y_{\Xi, \xi}(g_1)dg_1, \quad g_0 \in G_0.$$

Here, $\text{pr}_1 : C_c^\infty(G_1) \rightarrow \pi_1 = I(\Xi)$ is given by

$$\text{pr}_1(f)(g_1) = \int_{B_1} (\Xi^{-1}\delta_1^{1/2})(b_1)f(b_1g_1)db_1.$$

Let $\langle \cdot, \cdot \rangle$ be the natural pairing on $\pi_0 \times \tilde{\pi}_0$ defined by

$$\langle \varphi_0, \varphi'_0 \rangle = \int_{\mathcal{K}_0} \varphi_0(k_0)\varphi'_0(k_0) dk_0$$

for $\varphi_0 \in \pi_0$ and $\varphi'_0 \in \tilde{\pi}_0$. Put

$$S_{\Xi,\xi}(g_1) = \langle f_\xi, l_{\Xi,\xi}(\pi_1(g_1)f_\Xi) \rangle.$$

Then $S_{\Xi,\xi}$ is a Shintani function, and we have

$$\begin{aligned} S_{\Xi,\xi}(g_1) &= \int_{\mathcal{K}_0} f_\xi(k_0) \int_{G_1} \mathbf{1}_{\mathcal{K}_1}(g'_1k_0g_1)Y_{\Xi,\xi}(g'_1)dg'_1 dk_0 \\ &= \int_{\mathcal{K}_1 \times \mathcal{K}_0} Y_{\Xi,\xi}(k_1g_1^{-1}k_0)dk_1 dk_0. \end{aligned}$$

Here, $\mathbf{1}_{\mathcal{K}_1}$ is the characteristic function of \mathcal{K}_1 . Put

$$T_{\Xi,\xi}(g_1) = \begin{cases} \int_{G_0} f_\Xi(g_1g_0)f_\xi(g_0) dg_0 & \text{if } g_1 \in B_1\eta B_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $T_{\Xi,\xi}(g_1) = T_{\Xi,\xi}(\eta) \cdot Y_{\Xi^{-1},\xi^{-1}}(g_1)$, since $T_{\Xi,\xi}$ satisfies the conditions (1) and (3) for Ξ^{-1} and ξ^{-1} . Therefore we have

$$\begin{aligned} I(g_1; \Phi_\Xi, \Phi_\xi) &= \int_{G_0} \int_{\mathcal{K}_1} \int_{\mathcal{K}_0} f_\Xi(k_1g_1^{-1}g_0)f_\xi(k_0g_0) dk_0 dk_1 dg_0 \\ &= \int_{G_0} \int_{\mathcal{K}_1} \int_{\mathcal{K}_0} f_\Xi(k_1g_1^{-1}k_0g_0)f_\xi(g_0) dk_0 dk_1 dg_0 \\ &= \int_{\mathcal{K}_1 \times \mathcal{K}_0} T_{\Xi,\xi}(k_1g_1^{-1}k_0) dk_1 dk_0 \\ &= T_{\Xi,\xi}(\eta) \int_{\mathcal{K}_1 \times \mathcal{K}_0} Y_{\Xi^{-1},\xi^{-1}}(k_1g_1^{-1}k_0) dk_1 dk_0 \\ &= T_{\Xi,\xi}(\eta)S_{\Xi^{-1},\xi^{-1}}(g_1). \end{aligned}$$

In particular, $T_{\Xi,\xi}(\eta)$ and $S_{\Xi^{-1},\xi^{-1}}(g_1)$ are convergent if Ξ and ξ are sufficiently close to the unitary axis. Indeed, since the first part of Proposition 1.1 holds for $I(|\Xi|)$ and $I(|\xi|)$ if Ξ and ξ are sufficiently close to the unitary axis, $I(g_1; \Phi_{|\Xi|}, \Phi_{|\xi|})$ is convergent, and hence the above integral is absolutely convergent. It follows that, for each $g_1 \in G_1$, $T_{\Xi,\xi}(k_1g_1^{-1}k_0)$ is convergent for almost all $k_1 \in \mathcal{K}_1$ and $k_0 \in \mathcal{K}_0$ such that $k_1g_1^{-1}k_0 \in B_1\eta B_0$. By definition, $T_{\Xi,\xi}(g_1)$ is convergent for some $g_1 \in B_1\eta B_0$ if and only if $T_{\Xi,\xi}(g_1)$ is convergent for all $g_1 \in B_1\eta B_0$. Therefore $T_{\Xi,\xi}(\eta)$ is convergent, and the convergence of the above integral also implies that $S_{\Xi^{-1},\xi^{-1}}(g_1)$ is convergent.

We first assume that the residual characteristic of k is not 2. We consider the case when $K = k$. In this case, both $T_{\Xi,\xi}(\eta)$ and $S_{\Xi^{-1},\xi^{-1}}(1)$ are already calculated. Note that

$$T_1 = A_1 \simeq \begin{cases} (k^\times)^l & \text{if } G_1 = \text{SO}(2l + 1), \\ (k^\times)^{l+1} & \text{if } G_1 = \text{SO}(2l + 2), \end{cases}$$

$$T_0 = A_0 \simeq (k^\times)^l \quad \text{if } G_0 = \mathrm{SO}(2l) \text{ or } G_0 = \mathrm{SO}(2l+1).$$

We write

$$\Xi = \begin{cases} (\Xi_1, \dots, \Xi_l) & \text{if } G_1 = \mathrm{SO}(2l+1), \\ (\Xi_1, \dots, \Xi_{l+1}) & \text{if } G_1 = \mathrm{SO}(2l+2), \end{cases}$$

$$\xi = (\xi_1, \dots, \xi_l) \quad \text{if } G_0 = \mathrm{SO}(2l) \text{ or } G_0 = \mathrm{SO}(2l+1).$$

There exists a quadratic space $(\tilde{V}_1, \tilde{Q}_1) \subset (V_0, Q_0)$ such that V_1 is isomorphic to the direct sum of \tilde{V}_1 and the hyperbolic plane. Without loss of generality, we may assume that (V_0, \tilde{V}_1) satisfies the conditions (U1)–(U6). Put

$$\tilde{\Xi} = \begin{cases} (\Xi_2, \dots, \Xi_l) & \text{if } G_1 = \mathrm{SO}(2l+1), \\ (\Xi_2, \dots, \Xi_{l+1}) & \text{if } G_1 = \mathrm{SO}(2l+2). \end{cases}$$

Since $T_{\Xi, \xi}(\eta)$ is independent of the choice of η , we set $\zeta(\Xi, \xi) = T_{\Xi, \xi}(\eta)$. By Ginzburg, Piatetski-Shapiro, and Rallis [GiPR, p. 22, Cor. to Lem. 1.1 & p. 179, Cor. to Lem. 7.2], we have

$$\zeta(\Xi, \xi) = \zeta(\xi, \tilde{\Xi}) \frac{L(1/2, I(\xi), \Xi_1)}{L(1, I(\tilde{\Xi}), \Xi_1)} \times \begin{cases} L(1, \Xi_1^2)^{-1} & \text{(Case A)}, \\ 1 & \text{(Case B)}. \end{cases}$$

Here, $L(s, I(\xi), \Xi_1)$ is the standard L -factor of $I(\xi)$ twisted by the character Ξ_1 . By induction, we have

$$\begin{aligned} \zeta(\Xi, \xi) &= \prod_{i=1}^l L(1, \Xi_i^2)^{-1} \prod_{1 \leq i < j \leq l} L(1, \Xi_i \Xi_j)^{-1} L(1, \Xi_i \Xi_j^{-1})^{-1} \\ &\quad \times \prod_{1 \leq i < j \leq l} L(1, \xi_i \xi_j)^{-1} L(1, \xi_i \xi_j^{-1})^{-1} \\ &\quad \times \prod_{1 \leq i < j \leq l} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i \xi_j^{-1}) \\ &\quad \times \prod_{1 \leq j < i \leq l} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i^{-1} \xi_j) \end{aligned}$$

in Case A, and

$$\begin{aligned} \zeta(\Xi, \xi) &= \prod_{1 \leq i < j \leq l+1} L(1, \Xi_i \Xi_j)^{-1} L(1, \Xi_i \Xi_j^{-1})^{-1} \\ &\quad \times \prod_{i=1}^l L(1, \xi_i^2)^{-1} \prod_{1 \leq i < j \leq l} L(1, \xi_i \xi_j)^{-1} L(1, \xi_i \xi_j^{-1})^{-1} \\ &\quad \times \prod_{1 \leq i < j \leq l} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i \xi_j^{-1}) \\ &\quad \times \prod_{1 \leq j < i \leq l+1} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i^{-1} \xi_j) \end{aligned}$$

in Case B. On the other hand, Theorem 10.8 of [KMS] implies

$$\begin{aligned}
 S_{\Xi^{-1}, \xi^{-1}}(1) &= \Delta_{G_1} \zeta(1)^{-2l} \prod_{i=1}^l L(1, \Xi_i^{-2})^{-1} \prod_{1 \leq i < j \leq l} L(1, \Xi_i^{-1} \Xi_j^{-1})^{-1} L(1, \Xi_i^{-1} \Xi_j)^{-1} \\
 &\quad \times \prod_{1 \leq i < j \leq l} L(1, \xi_i^{-1} \xi_j^{-1})^{-1} L(1, \xi_i^{-1} \xi_j)^{-1} \\
 &\quad \times \prod_{1 \leq i < j \leq l} L(1/2, \Xi_i^{-1} \xi_j^{-1}) L(1/2, \Xi_i^{-1} \xi_j) \\
 &\quad \times \prod_{1 \leq j < i \leq l} L(1/2, \Xi_i^{-1} \xi_j^{-1}) L(1/2, \Xi_i \xi_j^{-1})
 \end{aligned}$$

in Case A, and

$$\begin{aligned}
 S_{\Xi^{-1}, \xi^{-1}}(1) &= \Delta_{G_1} \zeta(1)^{-2l-1} \prod_{1 \leq i < j \leq l+1} L(1, \Xi_i^{-1} \Xi_j^{-1})^{-1} L(1, \Xi_i^{-1} \Xi_j)^{-1} \\
 &\quad \times \prod_{i=1}^l L(1, \xi_i^{-2})^{-1} \prod_{1 \leq i < j \leq l} L(1, \xi_i^{-1} \xi_j^{-1})^{-1} L(1, \xi_i^{-1} \xi_j)^{-1} \\
 &\quad \times \prod_{1 \leq i < j \leq l} L(1/2, \Xi_i^{-1} \xi_j^{-1}) L(1/2, \Xi_i^{-1} \xi_j) \\
 &\quad \times \prod_{1 \leq j < i \leq l+1} L(1/2, \Xi_i^{-1} \xi_j^{-1}) L(1/2, \Xi_i \xi_j^{-1})
 \end{aligned}$$

in Case B. Combining these results, we have

$$I(1; \Phi_{\Xi}, \Phi_{\xi}) = \Delta_{G_1} \mathcal{P}_{\pi_1, \pi_0}(1/2),$$

when both G_1 and G_0 are split. Thus we have proved Theorem 1.2 in the case $2 \nmid q$ and both G_1 and G_0 are split.

Now we consider the case when the discriminant field K is equal to the unramified quadratic extension of k . Note that the character χ of k^\times associated to K/k by the class field theory is equal to the unique unramified quasi-character of order 2. As in the split case, we should consider the following two cases:

- (Case A) $G_1 = \text{SO}(2l + 1)$ and $G_0 = \text{SO}(2l)$.
- (Case B) $G_1 = \text{SO}(2l + 2)$ and $G_0 = \text{SO}(2l + 1)$.

Note that

$$\begin{cases} A_1 \simeq (k^\times)^l, A_0 \simeq (k^\times)^{l-1} & \text{(Case A),} \\ A_1 \simeq A_0 \simeq (k^\times)^l & \text{(Case B).} \end{cases}$$

The unramified characters Ξ and ξ are determined by their restriction to A_1 and A_0 , respectively. We write

$$\begin{aligned}
 \Xi &= (\Xi_1, \dots, \Xi_l), \\
 \xi &= \begin{cases} (\xi_1, \dots, \xi_{l-1}) & \text{(Case A),} \\ (\xi_1, \dots, \xi_l) & \text{(Case B).} \end{cases}
 \end{aligned}$$

Put $\tilde{\Xi} = (\Xi_2, \dots, \Xi_l)$. We set $\zeta(\Xi, \xi) = T_{\Xi, \xi}(\eta)$. As before, we have

$$\zeta(\Xi, \xi) = \zeta(\xi, \tilde{\Xi}) \frac{L(1/2, I(\xi), \Xi_1)}{L(1, I(\tilde{\Xi}), \Xi_1)} \times \begin{cases} L(1, \Xi_1^2)^{-1} & \text{(Case A),} \\ 1 & \text{(Case B),} \end{cases}$$

by [GiPR, p. 22, Cor. to Lem. 1.1 & p. 179, Cor. to Lem. 7.2]. By induction, we have

$$\begin{aligned} \zeta(\Xi, \xi) &= \prod_{i=1}^l L(1, \Xi_i^2)^{-1} \prod_{1 \leq i < j \leq l} L(1, \Xi_i \Xi_j)^{-1} L(1, \Xi_i \Xi_j^{-1})^{-1} \\ &\quad \times \prod_{i=1}^{l-1} L(1, \xi_i)^{-1} L(1, \chi \xi_i)^{-1} \prod_{1 \leq i < j \leq l-1} L(1, \xi_i \xi_j)^{-1} L(1, \xi_i \xi_j^{-1})^{-1} \\ &\quad \times \prod_{1 \leq i \leq j \leq l-1} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i \xi_j^{-1}) \prod_{i=1}^l L(1/2, \Xi_i) L(1/2, \chi \Xi_i) \\ &\quad \times \prod_{1 \leq j < i \leq l} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i^{-1} \xi_j) \end{aligned}$$

in Case A, and

$$\begin{aligned} \zeta(\Xi, \xi) &= \prod_{i=1}^l L(1, \Xi_i)^{-1} L(1, \chi \Xi_i)^{-1} \prod_{1 \leq i < j \leq l} L(1, \Xi_i \Xi_j)^{-1} L(1, \Xi_i \Xi_j^{-1})^{-1} \\ &\quad \times \prod_{i=1}^l L(1, \xi_i^2)^{-1} \prod_{1 \leq i < j \leq l} L(1, \xi_i \xi_j)^{-1} L(1, \xi_i \xi_j^{-1})^{-1} \\ &\quad \times \prod_{1 \leq i \leq j \leq l} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i \xi_j^{-1}) \prod_{i=1}^l L(1/2, \xi_i) L(1/2, \chi \xi_i) \\ &\quad \times \prod_{1 \leq j < i \leq l} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i^{-1} \xi_j) \end{aligned}$$

in Case B. As for $S_{\Xi, \xi}(1)$, we can prove the following lemma.

LEMMA 5.1. *We have*

$$S_{\Xi, \xi}(1) = \Delta_{G_1} \zeta(1)^{-\dim A_1 - \dim A_0} L(1, \chi)^{-1} \zeta(\Xi, \xi).$$

The proof of this lemma will be given in the appendix to this section. Note that

$$\mathcal{P}_{\pi_1, \pi_0}(1/2) = \zeta(1)^{-\dim A_1 - \dim A_0} L(1, \chi)^{-1} \zeta(\Xi, \xi) \zeta(\Xi^{-1}, \xi^{-1}).$$

We would like to emphasise that this relation has been already noted by Ginzburg, Piatetski-Shapiro, and Rallis [GiPR]. Combining these results, we have $I(1; \Phi_{\Xi}, \Phi_{\xi}) = \Delta_{G_1} \mathcal{P}_{\pi_1, \pi_0}(1/2)$. Thus we have proved Theorem 1.2 in the case $2 \nmid q$.

Now we consider the case $2 \mid q$. It is enough to prove that $I(1; \Phi_{\Xi}, \Phi_{\xi})$ is an element of $\mathbb{Q}(q^{1/2}, \Xi, \xi)$. More precisely, we will show that there exists a rational function $\mathcal{I}(t, X_1, \dots, x_1, \dots) \in \mathbb{Q}(t, X_1, \dots, x_1, \dots)$, where $t, X_1, \dots, x_1, \dots$ are indeterminates, such that if the order of residue field of k is q , then

$$I(1; \Phi_{\Xi}, \Phi_{\xi}) = \mathcal{I}(q^{1/2}, \Xi_1, \dots, \xi_1, \dots).$$

To prove this, we make use of Macdonald's formula for the spherical function. Recall that Macdonald's formula [C, p. 403, Th. 4.2] says that the spherical functions Φ_{Ξ} and Φ_{ξ} are of the form

$$\Phi_{\Xi}(m_1) = Q_1^{-1} \sum_{w_1 \in W_1} \gamma_1(w_1 \Xi) \cdot ((w_1 \Xi) \delta_1^{-1/2})(m_1), \quad m_1 \in A_1^+,$$

$$\Phi_\xi(m_0) = Q_0^{-1} \sum_{w_0 \in W_0} \gamma_0(w_0\xi) \cdot ((w_0\xi)\delta_0^{-1/2})(m_0), \quad m_0 \in A_0^+.$$

Here, $Q_1, Q_0, \gamma_1(\Xi), \gamma_0(\xi) \in \mathbb{Q}(q^{1/2}, \Xi, \xi)$ and δ_i is the modulus function of the Borel subgroup B_i . The integral $I(1; \Phi_\Xi, \Phi_\xi)$ is equal to

$$\int_{A_0^+} \Phi_\Xi(m_0)\Phi_\xi(m_0)\text{Vol}(\mathcal{K}_0 m_0 \mathcal{K}_0) dm_0.$$

Note that $\text{Vol}(\mathcal{K}_0 m_0 \mathcal{K}_0) = [\mathcal{K}_0 : \mathcal{K}_0 \cap m_0 \mathcal{K}_0 m_0^{-1}]$. One can show easily this integral gives an element of $\mathbb{Q}(q^{1/2}, \Xi, \xi)$. Therefore the proof of Theorem 1.2 is complete.

Appendix to Section 5: Proof of Lemma 5.1

In this appendix, we prove Lemma 5.1. The proof of Lemma 5.1 consists of three steps.

Step 1. The Weyl invariance. The Weyl group $W_1 \times W_0$ acts on the character group of $A_1 \times A_0$ by $(\Xi, \xi) \mapsto (w_1 \Xi, w_0 \xi)$.

LEMMA 5.2. *The quantity $S_{\Xi, \xi}(g_1)\zeta(\Xi, \xi)^{-1}$ is $W_1 \times W_0$ -invariant as a function of Ξ and ξ (cf. [KMS, Th. 10.8]).*

Proof. Note that both $\zeta(\Xi, \xi)\zeta(\Xi^{-1}, \xi^{-1})$ and

$$I(g_1; \Phi_\Xi, \Phi_\xi) = \zeta(\Xi, \xi)S_{\Xi^{-1}, \xi^{-1}}(g_1)$$

are $W_1 \times W_0$ -invariant. It follows that

$$\frac{I(g_1; \Phi_\Xi, \Phi_\xi)}{\zeta(\Xi, \xi)\zeta(\Xi^{-1}, \xi^{-1})} = \frac{S_{\Xi^{-1}, \xi^{-1}}(g_1)}{\zeta(\Xi^{-1}, \xi^{-1})}$$

is also $W_1 \times W_0$ -invariant. Hence the lemma. □

Step 2. An explicit formula for $S_{\Xi, \xi}(g_1)$. Now we closely follow the argument of [KMS]. Fix a hyperspecial maximal compact subgroup $\mathcal{K}_i \subset G_i$ and a maximal split torus $A_i \subset G_i$. Then the centralizer T_i of A_i is a maximally split maximal torus of G_i . We assume $\mathcal{K}_0 \subset \mathcal{K}_1$ and $A_0 \subset A_1$. Note that T_0 need not be a subgroup of T_1 . Choose a Borel subgroup $B_i = T_i N_i \subset G_i$. We also assume $N_0 \subset N_1$. The opposite Borel subgroup of $B_i = T_i N_i$ is denoted by $\bar{B}_i = T_i \bar{N}_i$. We put $T_i^{(0)} = T_i \cap \mathcal{K}_i$, $N_i^{(0)} = N_i \cap \mathcal{K}_i$, and $\bar{N}_i^{(0)} = \bar{N}_i \cap \mathcal{K}_i$. Choose a longest element $w_{i, \text{long}}$ of the Weyl group $W_i = W(G_i, A_i)$. We assume $w_{i, \text{long}} \in \mathcal{K}_i$. There exists an Iwahori subgroup $\mathcal{B}_i \subset \mathcal{K}_i$ such that $N_i^{(0)} \subset \mathcal{B}_i$. We put $\bar{N}_i^{(1)} = \bar{N}_i \cap \mathcal{B}_i$ and $N_i^{(1)} = w_{i, \text{long}}^{-1} \bar{N}_i^{(1)} w_{i, \text{long}}$. Then we have an Iwahori decomposition $\mathcal{B}_i = \bar{N}_i^{(1)} T_i^{(0)} N_i^{(0)}$.

Recall that the element $\eta \in G_1$ is a representative of the unique open orbit of $B_1 \backslash G_1 / B_0$ such that $\eta \in \mathcal{K}_1$. Let \mathfrak{o} and \mathfrak{o}_K be the ring of integers of k and K , respectively. The maximal ideal of \mathfrak{o} and \mathfrak{o}_K are denoted by \mathfrak{p} and \mathfrak{p}_K , respectively.

LEMMA 5.3. *One can choose the representative η of the open orbit of $B_1 \backslash G_1 / B_0$ such that the following conditions hold:*

- (1) $\eta \bar{N}_0^{(1)} \subset \mathcal{B}_1 \eta$,
- (2) $\bar{N}_1^{(1)} \eta \subset T_1^{(0)} N_1^{(0)} \eta T_0^{(0)} N_0^{(0)}$.

Proof. We first consider Case B. Note that in this case N_0 is a normal subgroup of N_1 . By [GiPR, p.171, Lem. 7.1] N_1/N_0 is isomorphic to $k^{l-1} \times (K/k)$ as a left module of $A_0 = A_1 \simeq (k^\times)^l$. We fix an isomorphism $N_1/N_0 \simeq k^{l-1} \times (K/k)$, which induces an isomorphism $N_1^{(0)}/N_0^{(0)} \simeq \mathfrak{o}^{l-1} \times (\mathfrak{o}_K/\mathfrak{o})$. Since K/k is unramified, $\mathfrak{o}_K/\mathfrak{o}$ is isomorphic to \mathfrak{o} , and so $N_1^{(0)}/N_0^{(0)} \simeq \mathfrak{o}^l$. There exists a cross section (i.e. “épinglage”) ι of the map $N_1^{(0)} \rightarrow N_1^{(0)}/N_0^{(0)} \simeq \mathfrak{o}^l$. Let η' be the image of the cross section of $(1, 1, \dots, 1) \in \mathfrak{o}^l$. We put $\eta = w_{1, \text{long}} \eta'$. Then η is a representative of the open orbit of $B_1 \backslash G_1/B_0$. Let \mathcal{U}_1 be the group generated by $N_1^{(1)}$ and $\bar{N}_1^{(1)}$. Then \mathcal{U}_1 is a normal subgroup of \mathcal{K}_1 . It follows that $\eta \bar{N}_0^{(1)} \subset \eta \mathcal{U}_1 = \mathcal{U}_1 \eta \subset \mathcal{B}_1 \eta$. As for (2), $\bar{N}_1^{(1)} \eta = w_{1, \text{long}} N_1^{(1)} \eta' \subset w_{1, \text{long}} \iota(\mathfrak{p}^l) \eta' N_0^{(1)}$. It suffices to prove that $\iota(\mathfrak{p}^l) \eta' \subset T_1^{(0)} \eta' T_0^{(0)}$. This is easily seen by the facts $1 + \mathfrak{p} \subset \mathfrak{o}^\times$.

Now we consider Case A. Let P_1 be the standard parabolic subgroup of G_1 with Levi factor $(k^\times)^{l-1} \times \text{SO}(3) \simeq (k^\times)^{l-1} \times \text{PGL}_2$. Let N_{P_1} be the unipotent radical of P_1 . Then as in Case B, N_{P_1}/N_0 is isomorphic to k^{l-1} as a left module of $A_0 \simeq (k^\times)^{l-1}$. We fix an isomorphism $N_{P_1}/N_0 \simeq k^{l-1}$, which induces an isomorphism $(N_{P_1} \cap N_1^{(0)})/N_0^{(0)} \simeq \mathfrak{o}^{l-1}$. Take a cross section ι of the map $(N_{P_1} \cap N_1^{(0)}) \rightarrow (N_{P_1} \cap N_1^{(0)})/N_0^{(0)} \simeq \mathfrak{o}^{l-1}$. Put $\eta = w_{1, \text{long}} \iota((1, 1, \dots, 1))$. Then η is a representative of the open orbit of $B_1 \backslash G_1/B_0$, since $\text{PGL}_2 = (\text{PGL}_2 \cap N_1) \cdot (\text{PGL}_2 \cap T_0)$ (cf. [GiPR, App. 1 to §7]). One can prove (1) in the same way as in Case B. As for (2), observe that $\bar{N}_1^{(1)} = (\bar{N}_1^{(1)} \cap \bar{N}_{P_1}) \cdot (\bar{N}_1^{(1)} \cap \text{PGL}_2)$, where \bar{N}_{P_1} is the unipotent radical of the opposite parabolic subgroup of P_1 with respect to the Levi subgroup $(k^\times)^{l-1} \times \text{PGL}_2$. One can prove that $(\bar{N}_1^{(1)} \cap \bar{N}_{P_1}) \eta \subset T_1^{(0)} \eta T_0^{(0)} N_0^{(0)}$ in the same way as in Case B. Now (2) follows from the fact $(T_1^{(0)} N_1^{(0)} \cap \text{PGL}_2) \cdot (T_0^{(0)} \cap \text{PGL}_2) = \mathcal{K}_1 \cap \text{PGL}_2$. \square

LEMMA 5.4. *We have*

$$\mathcal{B}_0 \eta^{-1} \mathcal{B}_1 \subset T_0^{(0)} N_0^{(0)} \eta^{-1} T_1^{(0)} N_1^{(0)}.$$

Proof. By Lemma 5.3, we have

$$\begin{aligned} \mathcal{B}_0 \eta^{-1} \mathcal{B}_1 &= T_0^{(0)} N_0^{(0)} \bar{N}_0^{(1)} \eta^{-1} \mathcal{B}_1 \\ &\subset T_0^{(0)} N_0^{(0)} \eta^{-1} \mathcal{B}_1 \\ &= T_0^{(0)} N_0^{(0)} \eta^{-1} \bar{N}_1^{(1)} T_1^{(0)} N_1^{(0)} \\ &\subset T_0^{(0)} N_0^{(0)} \eta^{-1} T_1^{(0)} N_1^{(0)}. \end{aligned} \quad \square$$

Put

$$\begin{aligned} A_1^+ &= \{t \in A_1 \mid |\alpha(t)| \leq 1 \text{ for any positive root } \alpha \text{ of } (G_1, A_1)\}, \\ A_0^+ &= \{t \in A_0 \mid |\alpha(t)| \leq 1 \text{ for any positive root } \alpha \text{ of } (G_0, A_0)\}. \end{aligned}$$

Then we have Cartan decompositions $G_1 = \mathcal{K}_1 A_1^+ \mathcal{K}_1$, $G_0 = \mathcal{K}_0 A_0^+ \mathcal{K}_0$.

For each positive root α of G_1 (resp. G_0), we denote Harish-Chandra's c -function (cf. e.g., Casselman [C]) by $c_\alpha(\Xi)$ (resp. $c_\alpha(\xi)$). We put

$$\bar{c}_{w_1}(\Xi) = \prod_{\substack{\alpha > 0 \\ w_1 \alpha > 0}} c_\alpha(\Xi) \quad \left(\text{resp. } \bar{c}_{w_0}(\xi) = \prod_{\substack{\alpha > 0 \\ w_0 \alpha > 0}} c_\alpha(\xi) \right).$$

When w_1 (resp. w_0) is the identity element, we set

$$\mathbf{c}_1(\Xi) = \prod_{\alpha>0} c_\alpha(\Xi) \quad \left(\text{resp. } \mathbf{c}_0(\xi) = \prod_{\alpha>0} c_\alpha(\xi) \right).$$

LEMMA 5.5. *There exists a basis $\{g_{1,w_1}\}_{w_1 \in W_1}$ of $I(\Xi)^{\mathcal{B}_1}$ with the following properties:*

- (1₁) $\mathcal{R}(\mathbf{1}_{\mathcal{B}_1 t^{-1} \mathcal{B}_1}) g_{1,w_1} = \text{Vol}(\mathcal{B}_1 t \mathcal{B}_1) \cdot (w_1 \Xi)^{-1} \delta_1^{1/2}(t) \cdot g_{1,w_1}$ for any $t \in A_1^+$;
- (2₁) *The restriction of $g_{1,1}$ to \mathcal{K}_1 is the characteristic function of \mathcal{B}_1 ;*
- (3₁) $f_\Xi = [N_1^{(0)} : N_1^{(1)}] \sum_{w_1 \in W_1} \bar{c}_{w_1}(\Xi) \cdot g_{1,w_1}$.

Similarly, there exists a basis $\{g_{0,w_0}\}_{w_0 \in W_0}$ of $I(\xi)^{\mathcal{B}_0}$ with the following properties.

- (1₀) $\mathcal{R}(\mathbf{1}_{\mathcal{B}_0 t^{-1} \mathcal{B}_0}) g_{0,w_0} = \text{Vol}(\mathcal{B}_0 t \mathcal{B}_0) \cdot (w_0 \xi)^{-1} \delta_0^{1/2}(t) \cdot g_{0,w_0}$ for any $t \in A_0^+$;
- (2₀) *The restriction of $g_{0,1}$ to \mathcal{K}_0 is the characteristic function of \mathcal{B}_0 .*
- (3₀) $f_\xi = [N_0^{(0)} : N_0^{(1)}] \sum_{w_0 \in W_0} \bar{c}_{w_0}(\xi) \cdot g_{0,w_0}$.

Proof. See [KMS, p. 8, Prop. 1.10]. □

LEMMA 5.6. *We have*

$$S_{\Xi,\xi}(t_0 \eta^{-1} t_1^{-1}) = \text{Vol}(\mathcal{B}_0 t_0^{-1} \mathcal{B}_0)^{-1} \text{Vol}(\mathcal{B}_1 t_1^{-1} \mathcal{B}_1)^{-1} \\ \times (\mathcal{L}(\mathbf{1}_{\mathcal{B}_0 t_0^{-1} \mathcal{B}_0}) \mathcal{R}(\mathbf{1}_{\mathcal{B}_1 t_1^{-1} \mathcal{B}_1}) S_{\Xi,\xi})(\eta^{-1})$$

for $t_0 \in A_0^+$, $t_1 \in A_1^+$.

Proof. It suffices to show that

$$(\mathcal{B}_0 t_0 \mathcal{B}_0) \eta^{-1} (\mathcal{B}_1 t_1^{-1} \mathcal{B}_1) \subset \mathcal{K}_0 t_0 \eta^{-1} t_1^{-1} \mathcal{K}_1$$

for $t_0 \in A_0^+$, $t_1 \in A_1^+$. By Lemma 5.4, we have

$$\mathcal{B}_0 t_0 \mathcal{B}_0 \eta^{-1} \mathcal{B}_1 t_1^{-1} \mathcal{B}_1 \subset \mathcal{B}_0 t_0 T_0^{(0)} N_0^{(0)} \eta^{-1} T_1^{(0)} N_1^{(0)} t_1^{-1} \mathcal{B}_1.$$

Since $t_i T_i^{(0)} N_i^{(0)} t_i^{-1} \subset T_i^{(0)} N_i^{(0)}$, the lemma follows. □

Recall that

$$S_{\Xi,\xi}(g_1) = \langle f_\xi, l_{\Xi,\xi}(\pi_1(g_1) f_\Xi) \rangle.$$

By (1₁), (3₁), (1₀), and (3₀) of Lemma 5.5, we have

$$S_{\Xi,\xi}(t_0 \eta^{-1} t_1^{-1}) = [N_1^{(0)} : N_1^{(1)}] [N_0^{(0)} : N_0^{(1)}] \\ \times \sum_{\substack{w_1 \in W_1 \\ w_0 \in W_0}} \bar{c}_{w_1}(\Xi) \bar{c}_{w_0}(\xi) (w_1 \Xi)^{-1} \delta_1^{1/2}(t_1) \cdot (w_0 \xi)^{-1} \delta_0^{1/2}(t_0) \\ \times \int_{\mathcal{K}_0 \times \mathcal{K}_1} g_{0,w_0}(k_0) g_{1,w_1}(k_1) Y_{\Xi,\xi}(k_0 \eta k_1) dk_0 dk_1.$$

By (2₁) and (2₀) of Lemma 5.5, we have

$$\int_{\mathcal{K}_0 \times \mathcal{K}_1} g_{0,1}(k_0) g_{1,1}(k_1) Y_{\Xi,\xi}(k_0 \eta k_1) dk_0 dk_1 \\ = \text{Vol}(\mathcal{B}_1) \text{Vol}(\mathcal{B}_0) \\ = \Delta_{G_1} \Delta_{G_0} \zeta(1)^{-\dim A_1 - \dim A_0} L(1, \chi)^{-1} / ([N_1^{(0)} : N_1^{(1)}] [N_0^{(0)} : N_0^{(1)}]).$$

Put $\mathbf{c}_{\text{WS}}(\Xi, \xi) = \mathbf{c}_1(\Xi)\mathbf{c}_0(\xi)\zeta(\Xi, \xi)^{-1} = \mathbf{b}(\Xi, \xi)\mathbf{d}_1(\Xi)^{-1}\mathbf{d}_0(\xi)^{-1}$, where

$$\begin{aligned} \mathbf{b}(\Xi, \xi)^{-1} &= \prod_{1 \leq i \leq j \leq l-1} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i \xi_j^{-1}) \prod_{i=1}^l L(1/2, \Xi_i) L(1/2, \chi \Xi_i) \\ &\quad \times \prod_{1 \leq j < i \leq l} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i^{-1} \xi_j) \\ \mathbf{d}_1(\Xi)^{-1} &= \prod_{i=1}^l L(0, \Xi_i^2) \prod_{1 \leq i < j \leq l} L(0, \Xi_i \Xi_j) L(0, \Xi_i \Xi_j^{-1}) \\ \mathbf{d}_0(\xi)^{-1} &= \prod_{i=1}^{l-1} L(0, \xi_i) L(0, \chi \xi_i) \prod_{1 \leq i < j \leq l-1} L(0, \xi_i \xi_j) L(0, \xi_i \xi_j^{-1}) \end{aligned}$$

in Case A, and

$$\begin{aligned} \mathbf{b}(\Xi, \xi)^{-1} &= \prod_{1 \leq i \leq j \leq l} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i \xi_j^{-1}) \prod_{i=1}^l L(1/2, \xi_i) L(1/2, \chi \xi_i) \\ &\quad \times \prod_{1 \leq j < i \leq l} L(1/2, \Xi_i \xi_j) L(1/2, \Xi_i^{-1} \xi_j) \\ \mathbf{d}_1(\Xi)^{-1} &= \prod_{i=1}^l L(0, \Xi_i) L(0, \chi \Xi_i) \prod_{1 \leq i < j \leq l} L(0, \Xi_i \Xi_j) L(0, \Xi_i \Xi_j^{-1}) \\ \mathbf{d}_0(\xi)^{-1} &= \prod_{i=1}^l L(0, \xi_i^2) \prod_{1 \leq i < j \leq l} L(0, \xi_i \xi_j) L(0, \xi_i \xi_j^{-1}) \end{aligned}$$

in Case B. By the Weyl-invariance, we have

$$\begin{aligned} \frac{S_{\Xi, \xi}(t_0 \eta^{-1} t_1^{-1})}{\zeta(\Xi, \xi)} &= \Delta_{G_1} \Delta_{G_0} \zeta(1)^{-\dim A_1 - \dim A_0} L(1, \chi)^{-1} \\ &\quad \times \sum_{\substack{w_1 \in W_1 \\ w_0 \in W_0}} \mathbf{c}_{\text{WS}}(w_1 \Xi, w_0 \xi) \cdot (w_1 \Xi)^{-1} \delta_1^{1/2}(t_1) \cdot (w_0 \xi)^{-1} \delta_0^{1/2}(t_0) \end{aligned}$$

(cf. [KMS, Th. 10.7]). Note that

$$\mathbf{b}(\Xi, \xi), \mathbf{d}_1(\Xi), \mathbf{d}_0(\xi) \in \mathbb{Z}[q^{\pm 1/2}, \Xi_1, \Xi_2, \dots, \xi_1, \xi_2, \dots].$$

Here and from now on, we identify an unramified quasi-character of k^\times with its value at a prime element.

Step 3. Calculation of $S_{\Xi, \xi}(1)/\zeta(\Xi, \xi)$. Our next task is to prove the following lemma.

LEMMA 5.7. *The sum*

$$\frac{S_{\Xi, \xi}(1)}{\zeta(\Xi, \xi)} = \Delta_{G_1} \Delta_{G_0} \zeta(1)^{-\dim A_1 - \dim A_0} L(1, \chi)^{-1} \sum_{\substack{w_1 \in W_1 \\ w_0 \in W_0}} \mathbf{c}_{\text{WS}}(w_1 \Xi, w_0 \xi)$$

is independent of Ξ and ξ .

Proof. We shall prove the lemma only in Case B. One can handle Case A in a similar way. Put

$$A_{\Xi, \xi} = \sum_{\substack{w_1 \in W_1 \\ w_0 \in W_0}} \mathbf{c}_{\text{WS}}(w_1 \Xi, w_0 \xi).$$

We are going to prove that $A_{\Xi, \xi}$ is independent of Ξ and ξ . Put

$$\mathcal{D}(\Xi) = \Xi^{-\rho_1} \mathbf{d}_1(\Xi) = \sum_{w_1 \in W_1} \text{sgn}(w_1) \cdot (w_1 \Xi)^{-\rho_1},$$

$$\mathcal{D}(\xi) = \xi^{-\rho_0} \mathbf{d}_0(\xi) = \sum_{w_0 \in W_0} \text{sgn}(w_0) \cdot (w_0 \xi)^{-\rho_0},$$

where

$$\rho_1 = \rho_0 = (l, l - 1, \dots, 1).$$

Then we have $\mathcal{D}(w_1 \Xi) = \text{sgn}(w_1) \mathcal{D}(\Xi)$ and $\mathcal{D}(w_0 \xi) = \text{sgn}(w_0) \mathcal{D}(\xi)$ for $w_1 \in W_1$ and $w_0 \in W_0$. Note that ρ_1 and ρ_0 are half the sum of the positive roots of type C. It follows that $A_{\Xi, \xi}$ is equal to

$$(\mathcal{D}(\Xi) \mathcal{D}(\xi))^{-1} \sum_{\substack{w_1 \in W_1 \\ w_0 \in W_0}} \text{sgn}(w_1) \text{sgn}(w_0) \cdot (w_1 \Xi)^{-\rho_1} (w_0 \xi)^{-\rho_0} \mathbf{b}(w_1 \Xi, w_0 \xi).$$

Put $B_{\Xi, \xi} = \Xi^{-\rho_1} \xi^{-\rho_0} \mathbf{b}(\Xi, \xi)$. Observe that $B_{\Xi, \xi}$ is equal to

$$\prod_{1 \leq j \leq l} (\xi_j^{-1} - q^{-1} \xi_j) \prod_{1 \leq i \leq j \leq l} (\Xi_i^{-1} - q^{-1/2} \xi_j^{-1}) \\ \times \prod_{1 \leq j < i \leq l} (\xi_j^{-1} - q^{-1/2} \Xi_i^{-1}) \prod_{\substack{1 \leq i \leq l \\ 1 \leq j \leq l}} (1 - q^{-1/2} \Xi_i \xi_j).$$

We express $B_{\Xi, \xi}$ as a sum of monomials

$$B_{\Xi, \xi} = \sum_{\lambda, \mu} c_{\lambda, \mu} \Xi^\lambda \xi^\mu, \quad \lambda, \mu \in \mathbb{Z}^l, \quad c_{\lambda, \mu} \in \mathbb{Z}[q^{\pm 1/2}].$$

We say that a monomial $\Xi^\lambda \xi^\mu$ is regular if $\Xi^{w_1 \lambda} \xi^{w_0 \mu} = \Xi^\lambda \xi^\mu$ implies $w_1 = w_0 = 1$. We also say that a monomial is singular if it is not regular. Here the action of the Weyl group on \mathbb{Z}^l is given by $(w_1 \Xi)^{w_1 \lambda} = \Xi^\lambda$, $(w_0 \xi)^{w_0 \mu} = \xi^\mu$, as usual.

We would like to show that if a regular monomial $\Xi^\lambda \xi^\mu$ appears in $B_{\Xi, \xi}$, then it is of the form $\Xi^{w_1 \rho_1} \xi^{w_0 \rho_0}$ with $w_1 \in W_1$, $w_0 \in W_0$. It is enough to show $|\lambda_i|, |\mu_j| \leq l$, since such a monomial is either singular or Weyl-equivalent to $\Xi^{\rho_1} \xi^{\rho_0}$. Choose $i_0, j_0 \in \{1, 2, \dots, l\}$. The positive contribution of Ξ_{i_0} comes from

$$\prod_{1 \leq j \leq l} (1 - q^{-1/2} \Xi_{i_0} \xi_j),$$

and the negative contribution of Ξ_{i_0} comes from

$$\prod_{i_0 \leq j \leq l} (\Xi_{i_0}^{-1} - q^{-1/2} \xi_j^{-1}) \prod_{1 \leq j < i_0} (\xi_j^{-1} - q^{-1/2} \Xi_{i_0}^{-1}).$$

Therefore $|\lambda_{i_0}| \leq l$. Similarly, the positive contribution of ξ_{j_0} comes from

$$(\xi_{j_0}^{-1} - q^{-1} \xi_{j_0}) \prod_{1 \leq i \leq l} (1 - q^{-1/2} \Xi_i \xi_{j_0})$$

and the negative contribution of ξ_{j_0} comes from

$$(\xi_{j_0}^{-1} - q^{-1}\xi_{j_0}) \prod_{1 \leq i \leq j_0} (\Xi_i^{-1} - q^{-1/2}\xi_{j_0}^{-1}) \prod_{j_0 < i \leq l} (\xi_{j_0}^{-1} - q^{-1/2}\Xi_i^{-1}).$$

Therefore $|\mu_{j_0}| \leq l + 1$. It follows that if a regular monomial $\Xi^\lambda \xi^\mu$ occurs in $B_{\Xi, \xi}$, then $l \leq |\mu_{j_0}| \leq l + 1$ for some j_0 . We will show that no regular monomial $\Xi^\lambda \xi^\mu$ such that $|\mu_{j_0}| > l$ occurs in $B_{\Xi, \xi}$. Assume that the monomial $\Xi^\lambda \xi^\mu$ occurs in $B_{\Xi, \xi}$ and $|\mu_{j_0}| > l$. We must show that such a monomial $\Xi^\lambda \xi^\mu$ is singular. Note that the monomial $\Xi^\lambda \xi^\mu$ occurs in

$$\begin{aligned} & q^{-1}\xi_{j_0} \cdot \prod_{i_0 \leq j \leq l} (\Xi_{i_0}^{-1} - q^{-1/2}\xi_j^{-1}) \\ & \times \prod_{1 \leq j < i_0} (\xi_j^{-1} - q^{-1/2}\Xi_{i_0}^{-1}) \cdot q^{-1/2}\Xi_{i_0}\xi_{j_0} \prod_{\substack{1 \leq j \leq l \\ j \neq j_0}} (1 - q^{-1/2}\Xi_{i_0}\xi_j) \\ & \times (\text{terms not containing } \Xi_{i_0} \text{ or } \xi_{j_0}). \end{aligned}$$

In particular, we have $\lambda_{i_0} \neq -l$. If $\lambda_{i_0} = l$, then the factor $\xi_{j_0}^{-1}$ must occur in the factor

$$\prod_{i_0 \leq j \leq l} (\Xi_{i_0}^{-1} - q^{-1/2}\xi_j^{-1}) \prod_{1 \leq j < i_0} (\xi_j^{-1} - q^{-1/2}\Xi_{i_0}^{-1}),$$

which would contradict the condition $\mu_{j_0} > l$. It follows that the condition $\mu_{j_0} > l$ implies $|\lambda_{i_0}| < l$. Therefore no regular monomial such that $\mu_{j_0} > l$ occurs in $B_{\Xi, \xi}$. Assume now $\mu_{j_0} < -l$. Then the monomial $\Xi^\lambda \xi^\mu$ occurs in

$$\begin{aligned} & \xi_{j_0}^{-1} \cdot (q^{-1/2}\xi_{j_0}^{-1})^{j_0} \prod_{\substack{i_0 \leq j \leq l \\ j \neq j_0}} (\Xi_{i_0}^{-1} - q^{-1/2}\xi_j^{-1}) \\ & \times \xi_{j_0}^{-l+j_0} \prod_{1 \leq j < i_0} (\xi_j^{-1} - q^{-1/2}\Xi_{i_0}^{-1}) \prod_{1 \leq j \leq l} (1 - q^{-1/2}\Xi_{i_0}\xi_j) \\ & \times (\text{terms not containing } \Xi_{i_0} \text{ or } \xi_{j_0}) \end{aligned}$$

if $i_0 \leq j_0$, and

$$\begin{aligned} & \xi_{j_0}^{-1} \cdot (q^{-1/2}\xi_{j_0}^{-1})^{j_0} \prod_{i_0 \leq j \leq l} (\Xi_{i_0}^{-1} - q^{-1/2}\xi_j^{-1}) \\ & \times \xi_{j_0}^{-l+j_0} \prod_{\substack{1 \leq j < i_0 \\ j \neq j_0}} (\xi_j^{-1} - q^{-1/2}\Xi_{i_0}^{-1}) \prod_{1 \leq j \leq l} (1 - q^{-1/2}\Xi_{i_0}\xi_j) \\ & \times (\text{terms not containing } \Xi_{i_0} \text{ or } \xi_{j_0}) \end{aligned}$$

if $i_0 > j_0$. In particular, $\lambda_{i_0} \neq -l$. If $\lambda_{i_0} = l$, then the factor ξ_{j_0} occurs, and so the condition $\mu_{j_0} < -l$ fails. It follows that the condition $\mu_{j_0} < -l$ implies $|\lambda_{i_0}| < l$. Therefore no regular monomial $\Xi^\lambda \xi^\mu$ such that $\mu_{j_0} < -l$ occurs in $B_{\Xi, \xi}$.

We have proved that the regular monomials $\Xi^\lambda \xi^\mu$ which occur in $B_{\Xi, \xi}$ are of the form $(w_1 \Xi)^{-\rho_1} (w_0 \xi)^{-\rho_0}$, for some $w_1 \in W_1$ and $w_0 \in W_0$. Therefore, up to a constant, $A_{\Xi, \xi}$ is equal to

$$(\mathcal{D}(\Xi)\mathcal{D}(\xi))^{-1} \sum_{\substack{w_1 \in W_1 \\ w_0 \in W_0}} \text{sgn}(w_1)\text{sgn}(w_0) \cdot (w_1 \Xi)^{-\rho_1} (w_0 \xi)^{-\rho_0} = 1.$$

Hence the lemma. \square

Recall that

$$A_{\Xi, \xi} = \sum_{\substack{w_1 \in W_1 \\ w_0 \in W_0}} \mathbf{c}_{\text{WS}}(w_1 \Xi, w_0 \xi).$$

LEMMA 5.8. *The constant $A_{\Xi, \xi}$ is equal to $\Delta_{G_0}^{-1}$.*

Proof. We shall prove the lemma only in Case B. One can handle Case A in a similar way. We put

$$\begin{aligned} \tilde{\Xi} &= (q^{-l}, q^{-l+1}, \dots, q^{-1}), \\ \tilde{\xi} &= (q^{-l+(1/2)}, q^{-l+(3/2)}, \dots, q^{-1/2}). \end{aligned}$$

As in the proof of [KMS, Lem. 11.9], we shall prove that $\mathbf{b}(w_1 \tilde{\Xi}, w_0 \tilde{\xi}) \neq 0$ implies $w_1 = w_0 = 1$. Note that $\mathbf{b}(\Xi, \xi)$ is equal to

$$\begin{aligned} \prod_{1 \leq i \leq j \leq l} (1 - q^{-1/2} \Xi_i \xi_j^{-1}) \prod_{1 \leq j < i \leq l} (1 - q^{-1/2} \Xi_i^{-1} \xi_j) \prod_{\substack{1 \leq i \leq l \\ 1 \leq j \leq l}} (1 - q^{-1/2} \Xi_i \xi_j) \\ \times \prod_{1 \leq j \leq l} (1 - q^{-1} \xi_j^2). \end{aligned}$$

Note that $W_1 \simeq W_0 \simeq \{\pm 1\}^l \rtimes \mathfrak{S}_l$, where \mathfrak{S}_l is the symmetric group. Therefore, for every $w_1 \in W_1, w_0 \in W_0$, one can find $\sigma, \tau \in \mathfrak{S}_l$ and $\varepsilon_i, \varepsilon'_j \in \{\pm 1\}$ such that

$$\begin{aligned} w_1 \Xi &= (\Xi_{\sigma(1)}^{\varepsilon_1}, \dots, \Xi_{\sigma(l)}^{\varepsilon_l}), \\ w_0 \xi &= (\xi_{\tau(1)}^{\varepsilon'_1}, \dots, \xi_{\tau(l)}^{\varepsilon'_l}). \end{aligned}$$

Put $i_s = \sigma^{-1}(l + 1 - s), j_t = \tau^{-1}(l + 1 - t)$. Then we have

$$\begin{aligned} (w_1 \tilde{\Xi})_{i_s} &= \tilde{\Xi}_{l+1-s}^{\varepsilon_{i_s}} = q^{-\varepsilon_{i_s} \cdot s}, \\ (w_0 \tilde{\xi})_{j_t} &= \tilde{\xi}_{l+1-t}^{\varepsilon'_{j_t}} = q^{-\varepsilon'_{j_t} (t - (1/2))}. \end{aligned}$$

Assume $\mathbf{b}(w_1 \tilde{\Xi}, w_0 \tilde{\xi}) \neq 0$. Firstly, $1 - q^{-1} (w_0 \tilde{\xi})_{j_1}^2 \neq 0$ implies $\varepsilon'_{j_1} = 1$. Secondly, $1 - q^{-1/2} (w_1 \tilde{\Xi})_{i_s} (w_0 \tilde{\xi})_{j_s} \neq 0$ and $1 - q^{-1/2} (w_1 \tilde{\Xi})_{i_{t+1}} (w_0 \tilde{\xi})_{j_t} \neq 0$ imply

$$\varepsilon'_{j_1} = \varepsilon_{i_1} = \varepsilon'_{j_2} = \varepsilon_{i_2} = \dots = \varepsilon'_{j_l} = \varepsilon_{i_l} = 1.$$

Now, if $j_s < i_s$, then the second factor would contain the factor $1 - q^{-1/2} (w_1 \tilde{\Xi})_{i_s}^{-1} (w_0 \tilde{\xi})_{j_s} = 0$, therefore we have $j_s \geq i_s$. Similarly, if $i_s \leq j_{s+1}$, the first factor would contain the factor $1 - q^{-1/2} (w_1 \tilde{\Xi})_{i_s} (w_0 \tilde{\xi})_{j_{s+1}}^{-1} = 0$, therefore we have $i_s > j_{s+1}$. It follows that

$$j_1 \geq i_1 > j_2 \geq i_2 > \dots > j_l \geq i_l,$$

and so $w_1 = w_0 = 1$. It follows that $A_{\Xi, \xi} = \mathbf{b}(\tilde{\Xi}, \tilde{\xi}) \mathbf{d}_1(\tilde{\Xi})^{-1} \mathbf{d}_0(\tilde{\xi})^{-1}$. By direct calculation, one can easily show that it is $\Delta_{G_0}^{-1}$. \square

Now Lemma 5.1 follows from Lemma 5.7 and Lemma 5.8.

Part III. Examples

In sections 6–12, k is an algebraic number field. The Dedekind zeta function of k is denoted by $\zeta_k(s)$. The Γ -factors of L -functions are normalized as in Tate [T]. In

particular, $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. The completed Dedekind zeta function of k is denoted by $\xi_k(s)$. When $k = \mathbb{Q}$, the subscript k is dropped. The symbol $L(s, \pi, r)$ is the Euler product $\prod_{v < \infty} L(s, \pi_v, r)$ and the completed L -function for $L(s, \pi, r)$ is denoted by $\Lambda(s, \pi, r)$.

6 Waldspurger's Theorem

The following example is due to Waldspurger [W]. Let D be a quaternion algebra over an algebraic number field k . Then $G_1 = D^\times/k^\times$ can be considered as a special orthogonal group associated to a 3-dimensional quadratic space over k . Note that $\Delta_{G_1} = \xi_k(2)$. Let $G_0 = T$ be an anisotropic torus of G_1 . Then T can be considered as a special orthogonal group associated to a 2-dimensional quadratic space over k . Let K be a splitting field of T over k . Then there exists an exact sequence

$$1 \longrightarrow k^\times \longrightarrow K^\times \longrightarrow T \longrightarrow 1.$$

By means of this exact sequence, a character ω of $T(\mathbb{A})/T(k)$ can be regarded as a character of $\mathbb{A}_K^\times/K^\times$ whose restriction to $\mathbb{A}_k^\times/k^\times$ is trivial. As in [W], we choose a Haar measure of $T(k_v)$ as follows. Fix a non-trivial additive character ψ of \mathbb{A}/k . Then we give the Haar measure $\zeta_v(1)^{-1}|t|_v^{-1} dt_v$ on k_v^\times , where dt_v is the self-dual Haar measure of k_v with respect to ψ_v . We give a Haar measure on K_v^\times in a similar way. Then the Haar measure on $T(k_v)$ is defined by the exact sequence

$$1 \longrightarrow k_v^\times \longrightarrow K_v^\times \longrightarrow T(k_v) \longrightarrow 1.$$

Let C_0 be the Haar measure constant. It is easily seen that $C_0 = \Lambda(1, \chi_{K/k})^{-1}$ for this choice of measure. Note that in [W], Waldspurger considered the measure on $T(\mathbb{A})$ such that $\text{Vol}(T(\mathbb{A})/T(k)) = 2\Lambda(1, \chi_{K/k})$.

An irreducible cuspidal automorphic representation π of $G_1(\mathbb{A})$ can be considered as a representation of $D^\times(\mathbb{A})$ with trivial central character. We assume π is almost locally generic. The base change of π to $\text{GL}_2(\mathbb{A}_K)$ is denoted by Π . Choose a non-zero cusp form $\varphi = \otimes_v \varphi_v \in \pi \simeq \otimes_v \pi_v$.

Then among other things, Waldspurger ([W, Prop. 7]) proved that the integral $I(\varphi_v, \omega_v)$ is convergent and that

$$\begin{aligned} \frac{|\langle \varphi|_{G_0}, \omega \rangle|^2}{\langle \varphi, \varphi \rangle \langle \omega, \omega \rangle} &= \frac{1}{4} \Delta_{G_1} C_0 \frac{\Lambda(1/2, \Pi \otimes \omega^{-1})}{\Lambda(1, \pi, \text{Ad}) \Lambda(1, \chi_{K/k})} \prod_{v \in S} \frac{\alpha_v(\varphi_v, \omega_v)}{\|\varphi_v\|^2} \\ &= \frac{1}{4} \Delta_{G_1} C_0 \mathcal{P}_{\pi_1, \pi_0}(1/2) \prod_{v \in S} \frac{\alpha_v(\varphi_v, \omega_v)}{\|\varphi_v\|^2 \cdot \|\omega_v\|^2}, \end{aligned}$$

where $\pi_1 = \pi$, $\pi_0 = \omega$. Thus Conjecture 1.5 is true for $n = 2$. Note that we have $|\mathcal{S}_{\psi_1}| = |\mathcal{S}_{\psi_0}| = 2$, if we admit the Arthur conjecture. Thus Waldspurger's result is compatible with Conjecture 2.1.

7 The Case $n = 3$

In this section, we prove Conjecture 1.5 for $n = 3$. Let D be a quaternion algebra over an algebraic number field k . Let k' be either $k \times k$ or a quadratic extension

of k . We put

$$\begin{aligned} \tilde{G}_1 &= (D \otimes_k k')^\times / k^\times, \\ G_1 &= \{g \in (D \otimes_k k')^\times \mid \nu(g) \in k^\times\} / k^\times, \\ G_0 &= D^\times / k^\times. \end{aligned}$$

Here ν is the reduced norm of D . Then G_1 (resp. G_0) can be considered as a special orthogonal group associated to a 4-dimensional (resp. 3-dimensional) quadratic space over k . We regard G_0 as a subgroup of G_1 . Note that

$$\Delta_{G_1} = \begin{cases} \xi_k(2)^2 & \text{if } k' = k \times k, \\ \xi_{k'}(2) & \text{otherwise.} \end{cases}$$

Let $Z_{\tilde{G}_1}$ be the identity component of the center of \tilde{G}_1 .

Let π_i be an irreducible cuspidal automorphic representation of $G_i(\mathbb{A})$ on the space \mathcal{V}_{π_i} . We assume π_i is almost locally generic. By the result of Hiraga and Saito [HiS, Th. 4.13], there exists an irreducible unitary cuspidal automorphic representation τ of $\tilde{G}_1(\mathbb{A})$ on the space \mathcal{V}_τ such that $\mathcal{V}_{\pi_1} \subset \mathcal{V}_\tau^1|_{G_1(\mathbb{A})}$. Here, \mathcal{V}_τ^1 is the subspace of \mathcal{V}_τ on which the group

$$\mathfrak{X}_\tau = \{ \omega \in \text{Hom}_{\text{cont}}(Z_{\tilde{G}_1}(\mathbb{A})G_1(\mathbb{A})\tilde{G}_1(k)\backslash\tilde{G}_1(\mathbb{A}), \mathbb{C}^\times) \mid \tau \otimes \omega \simeq \tau \}$$

acts trivially, and $\mathcal{V}_\tau^1|_{G_1(\mathbb{A})}$ is the restriction of \mathcal{V}_τ^1 to $G_1(\mathbb{A})$ as functions. Note that \mathfrak{X}_τ is an elementary 2-abelian group.

Let $\langle \cdot, \cdot \rangle$ be the canonical inner product on \mathcal{V}_τ and $\langle \cdot, \cdot \rangle_v$ an inner product on τ_v for any place v of k . Then Ichino's result [I2, Th. 1.1] says

$$\frac{|\langle \tilde{\varphi}|_{G_0}, \varphi_0 \rangle|^2}{\langle \tilde{\varphi}, \tilde{\varphi} \rangle \langle \varphi_0, \varphi_0 \rangle} = 2^{\tilde{\beta}} \Delta_{G_1} C_0 \mathcal{P}_{\pi_1, \pi_0}(1/2) \prod_{v \in S} \frac{\tilde{\alpha}_v(\tilde{\varphi}_v, \varphi_{0,v})}{\langle \tilde{\varphi}_v, \tilde{\varphi}_v \rangle_v \cdot \|\varphi_{0,v}\|^2}$$

for any non-zero vectors $\tilde{\varphi} = \otimes_v \tilde{\varphi}_v \in \tau$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$. Here,

$$\begin{aligned} \tilde{\alpha}_v(\tilde{\varphi}_v, \varphi_{0,v}) &= \Delta_{G_1,v}^{-1} \mathcal{P}_{\pi_1,v,\pi_0,v}(1/2)^{-1} \\ &\times \int_{G_{0,v}} \langle \tau_v(g_{0,v})\tilde{\varphi}_v, \tilde{\varphi}_v \rangle_v \overline{\langle \pi_{0,v}(g_{0,v})\varphi_{0,v}, \varphi_{0,v} \rangle_v} dg_{0,v} \end{aligned}$$

and

$$\tilde{\beta} = \begin{cases} -3 & \text{if } k' = k \times k, \\ -2 & \text{otherwise.} \end{cases}$$

Choose a non-zero cusp form $\varphi_1 = \otimes_v \varphi_{1,v} \in \mathcal{V}_{\pi_1}$. We choose $\tilde{\varphi} = \otimes_v \tilde{\varphi}_v \in \mathcal{V}_\tau^1$ such that $\tilde{\varphi}|_{G_1(\mathbb{A})} = \varphi_1$. Then $\tilde{\varphi}$ belongs to the $\otimes_v \pi_{1,v}$ -isotypic subspace of \mathcal{V}_τ^1 and we have

$$\frac{\tilde{\alpha}_v(\tilde{\varphi}_v, \varphi_{0,v})}{\langle \tilde{\varphi}_v, \tilde{\varphi}_v \rangle_v} = \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2}.$$

By Remark 4.20 of [HiS], we have

$$\langle \tilde{\varphi}, \tilde{\varphi} \rangle = \frac{1}{|\mathfrak{X}_\tau|} \langle \varphi_1, \varphi_1 \rangle \times \begin{cases} 2 & \text{if } k' = k \times k, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore we obtain the following theorem.

Theorem 7.1. *We have*

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = \frac{1}{4|\mathfrak{X}_\tau|} \Delta_{G_1} C_0 \mathcal{P}_{\pi_1, \pi_0}(1/2) \prod_{v \in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}$$

for any non-zero vectors $\varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$.

Thus Conjecture 1.5 is true for $n = 3$. Note that we have $|\mathcal{S}_{\psi_1}| = 2|\mathfrak{X}_\tau|$ and $|\mathcal{S}_{\psi_0}| = 2$, if we admit the Arthur conjecture.

We show that Theorem 7.1 is compatible with the result of Watson [Wa] in some cases. Put $G_1 = \text{SO}(2, 2)$ and $G_0 = \text{SO}(2, 1) = \text{PGL}_2$, defined over $k = \mathbb{Q}$. By definition, we have $\Delta_{G_1} = \xi(2)^2$. When v is non-archimedean, the local measure $dg_{0,v}$ of $G_{0,v}$ is the standard measure. In particular, the volume of the hyperspecial maximal compact subgroup $\mathcal{K}_v = \mathcal{K}_{0,v} = \text{PGL}_2(\mathbb{Z}_v)$ is 1. For the real place, we choose a Haar measure as follows. The topological identity component of $G_0(\mathbb{R})$ is denoted by $G_0(\mathbb{R})^0$. Let $\mathcal{K}_\infty = \mathcal{K}_{0,\infty} = \text{S}(\text{O}(2) \times \text{O}(1))$ be a maximal compact subgroup of $G_0(\mathbb{R})$. We put $\mathcal{K}_\infty^0 = G_0(\mathbb{R})^0 \cap \mathcal{K}_\infty$. Then $G_0(\mathbb{R})^0/\mathcal{K}_\infty^0$ can be identified with the upper-half plane \mathfrak{H}_1 . Let dk be the Haar measure on \mathcal{K}_∞^0 with total volume 1. Then the Haar measure $dg_{0,\infty}$ on $G_0(\mathbb{R})^0$ is such that $dg_{0,\infty}/dk$ induces the measure $y^{-2} dx dy$ on $G_0(\mathbb{R})^0/\mathcal{K}_\infty^0 \simeq \mathfrak{H}_1$. The Haar measure $dg_{0,\infty}$ can be naturally extended to $G_0(\mathbb{R})$. Let $G_0(\mathbb{R})^0 = AN\mathcal{K}_\infty^0$ be an Iwasawa decomposition, which induces a bijection $\mathfrak{H}_1 \simeq AN$. Let $X \subset AN$ be an image of a fundamental domain for $\text{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}_1$. Then there is a bijection

$$X \times \mathcal{K}_\infty^0 \times \prod_{v < \infty} \mathcal{K}_v \simeq G_0(\mathbb{Q}) \backslash G_0(\mathbb{A}).$$

It follows that

$$\int_{G_0(\mathbb{Q}) \backslash G_0(\mathbb{A})} \prod_{v \leq \infty} dg_{0,v} = \text{Vol}(\text{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}_1) = 2\xi(2).$$

Therefore we have $C_0 = \xi(2)^{-1} = 6\pi^{-1}$, where C_0 is the Haar measure constant.

Let $f_j \in S_{\kappa_j}(\text{SL}_2(\mathbb{Z}))$ ($j = 1, 2, 3$) be normalized Hecke eigenforms. We assume $\kappa_1 + \kappa_2 = \kappa_3$. We denote the automorphic form on $\text{GL}_2(\mathbb{A})$ corresponding to f_j by \mathbf{f}_j . Let τ_j be the irreducible automorphic representation of $\text{PGL}_2(\mathbb{A})$ generated by \mathbf{f}_j . Note that $\varphi_1 = \mathbf{f}_1 \times \mathbf{f}_2$ induces a cusp form on $\text{SO}(2, 2)(\mathbb{A})$ and its restriction to $\text{SO}(2, 1)$ is $\mathbf{f}_1 \mathbf{f}_2$. Put $\pi_1 = \tau_1 \boxtimes \tau_2$, $\pi_0 = \tau_3$ and $\varphi_0 = \mathbf{f}_3$. By the result of Watson [Wa] (see also Harris–Kudla [HaK]), we have

$$\Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3) = 2^{2\kappa_3+2} \langle f_1 f_2, f_3 \rangle^2.$$

It is well-known that $\Lambda(1, \tau_j, \text{Ad}) = 2^{\kappa_j} \langle f_j, f_j \rangle$. Here $\langle \cdot, \cdot \rangle$ is the usual Petersson inner product.

As both the Tamagawa number of $\text{SO}(2, 2)$ and that of $\text{SO}(2, 1)$ are equal to 2, we have

$$\begin{aligned} \frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} &= 2\xi(2) \frac{|\langle f_1 f_2, f_3 \rangle|^2}{\prod_{j=1}^3 \langle f_j, f_j \rangle} \\ &= \frac{1}{2} \xi(2) \frac{\Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3)}{\prod_{j=1}^3 \Lambda(1, \tau_j, \text{Ad})}. \end{aligned}$$

By easy calculation,

$$\mathcal{P}_{\pi_1, \pi_0}(s) = \frac{\Lambda(s, \tau_1 \times \tau_2 \times \tau_3)}{\prod_{j=1}^3 \Lambda(s + (1/2), \tau_j, \text{Ad})},$$

$$\mathcal{P}_{\pi_{1,\infty}, \pi_{0,\infty}}(1/2) = \frac{\Gamma_{\mathbb{C}}(1)\Gamma_{\mathbb{C}}(\kappa_1)\Gamma_{\mathbb{C}}(\kappa_2)\Gamma_{\mathbb{C}}(\kappa_3 - 1)}{\Gamma_{\mathbb{R}}(2)^3\Gamma_{\mathbb{C}}(\kappa_1)\Gamma_{\mathbb{C}}(\kappa_2)\Gamma_{\mathbb{C}}(\kappa_3)} = \frac{2\pi^3}{\kappa_3 - 1}.$$

PROPOSITION 7.2. *Let $\tau_{j,\infty}$ ($j = 1, 2, 3$) be the holomorphic discrete series of $\text{SO}(2, 1) \simeq \text{PGL}_2(\mathbb{R})$ with lowest weight $\pm\kappa_j$. Put $\pi_{1,\infty} = \tau_{1,\infty} \boxtimes \tau_{2,\infty}$ and $\pi_{0,\infty} = \tau_{3,\infty}$. Let $\varphi_{1,\infty} \in \pi_{1,\infty}$ be the vector with weight (κ_1, κ_2) . Let $\varphi_{0,\infty} \in \pi_{0,\infty}$ be the vector with weight κ_3 . We assume $\|\varphi_{1,\infty}\| = \|\varphi_{0,\infty}\| = 1$. Then we have*

$$I(\varphi_{1,\infty}, \varphi_{0,\infty}) = 4\pi(\kappa_3 - 1),$$

$$\alpha_{\infty}(\varphi_{1,\infty}, \varphi_{0,\infty}) = 2.$$

The proof of this proposition will be given in section 12. Putting together, we recover Theorem 7.1 in this case. Note that we have $|\mathfrak{X}_{\tau}| = 1$.

In fact, Watson [Wa] obtained a more general result. Let B be an indefinite quaternion algebra over \mathbb{Q} . The reduced discriminant d_B of B is, by definition, the product of primes which ramify in B . Let N be a square-free integer such that $(N, d_B) = 1$. Put S_f be the set of primes which divide $d_B N$. Let $\tau_j = \otimes_v \tau_{j,v}$ ($j = 1, 2, 3$) be an irreducible cuspidal automorphic representation of $\mathbb{A}^{\times} \backslash B^{\times}(\mathbb{A})$ with new vector $f_j = \otimes_v f_{j,v}$ which satisfies the following conditions:

- (1) When $v < \infty$ and $v \notin S_f$, the local components $\tau_{j,v}$ ($j = 1, 2, 3$) are unramified representations and $f_{j,v}$ are unramified vectors.
- (2) When $v \mid d_B$, the local component $\tau_{j,v}$ ($j = 1, 2, 3$) are one-dimensional representations of the form $\chi_j \circ \nu_{B_v}$, where χ_j are unramified quadratic characters and ν_{B_v} is the reduced norm. We also assume $\chi_1 \chi_2 \chi_3 = 1$.
- (3) When $v \mid N$, the local component $\tau_{j,v}$ ($j = 1, 2, 3$) are representations of the form $\chi_j \otimes (\text{Steinberg})$, where χ_j are unramified quadratic characters. We assume that $\chi_1 \chi_2 \chi_3$ is the unique unramified character of order 2 and that $f_{j,v}$ are Iwahori fixed vectors.
- (4) When $v = \infty$, we assume that $\tau_{j,v}$ ($j = 1, 2, 3$) are discrete series representations with minimal weight $\pm\kappa_j$. We assume $\kappa_3 = \kappa_1 + \kappa_2$ and $f_{j,v}$ have weight $\kappa_j > 0$.

Then Watson’s result [Wa, Th. 3] says

$$\frac{|\int_X f_1(z)f_2(z)f_3(z)\overline{\text{Im}(z)^{\kappa_3-2}} dz|^2}{\prod_{j=1}^3 \int_X |f_j(z)|^2 \overline{\text{Im}(z)^{\kappa_j-2}} dz} = \frac{2^{|S_f|-2} \Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3)}{(d_B N)^2 \prod_{j=1}^3 \Lambda(1, \tau_j, \text{Ad})}.$$

Here, $X = \mathcal{O}^{(1)}(d_B, N) \backslash \mathfrak{H}_1$, where $\mathcal{O}^{(1)}(d_B, N)$ is the arithmetic subgroup defined in Watson [Wa, Ch. 1]. Watson proved that

$$\text{Vol}(X) = 2\xi(2) \prod_{p \mid d_B} (p - 1) \prod_{p \mid N} (p + 1).$$

Watson also considered the cases when $\tau_{j,\infty}$ are not discrete series, but we do not discuss such cases.

Let V_1 be the vector space B equipped with the reduced norm form ν_B . The subspace $V_0 \subset V_1$ is defined by the space of elements of reduced trace 0. Then we have

$$\begin{aligned} G_1 &= \{(g_1, g_2) \in B^\times \times B^\times \mid \nu_B(g_1) = \nu_B(g_2)\} / \mathbb{Q}^\times, \\ G_0 &= B^\times / \mathbb{Q}^\times. \end{aligned}$$

As in the case of $\mathrm{SO}(2, 2)$, we regard $\pi_1 = \tau_1 \boxtimes \tau_2$ as a representation of $G_1(\mathbb{A})$, and $\pi_0 = \tau_3$ as a representation of $G_0(\mathbb{A})$. We put $\varphi_1 = f_1 \times f_2$, and $\varphi_0 = f_3$. We may assume $\|\varphi_{1,v}\| = \|\varphi_{0,v}\| = 1$ for any v . Note that Watson's result implies

$$\begin{aligned} \frac{|\langle \varphi_1 |_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} &= \mathrm{Vol}(X) \frac{|\int_X f_1(z) f_2(z) \overline{f_3(z)} \mathrm{Im}(z)^{\kappa_3-2} dz|^2}{\prod_{j=1}^3 \int_X |f_j(z)|^2 \mathrm{Im}(z)^{\kappa_j-2} dz} \\ &= 2^{-1} \xi(2) \mathcal{P}_{\pi_1, \pi_0}(1/2) \\ &\quad \times \prod_{p|d_B} (2p^{-1}(1-p^{-1})) \prod_{p|N} (2p^{-1}(1+p^{-1})). \end{aligned}$$

We describe local calculations below. Since G_0 is an inner form of PGL_2 , we can transfer the local measure of $\mathrm{PGL}_2(\mathbb{Q}_v)$ to $G_{0,v} = B^\times(\mathbb{Q}_v)/\mathbb{Q}_v^\times$. Note that $\Delta_{G_{1,v}} = \zeta_v(2)^2$ and $C_0 = 6\pi^{-1}$ are unchanged. When $p \mid d_B$, we have

$$\begin{aligned} \mathrm{Vol}(G_{0,p}) &= I(\varphi_{1,p}, \varphi_{0,p}) = 2p^{-1}(1-p^{-1})^{-1}, \\ \mathcal{P}_{\pi_1,p,\pi_0,p}(1/2) &= \zeta_p(1)^2 \zeta_p(2)^{-2}. \end{aligned}$$

It follows that $\alpha_p(\varphi_{1,p}, \varphi_{0,p}) = 2p^{-1}(1-p^{-1})$ for $p \mid d_B$. When $p \mid N$, let ε_p be the unique unramified character of \mathbb{Q}_p^\times of order 2. Then we have

$$\begin{aligned} \mathcal{P}_{\pi_1,p,\pi_0,p}(1/2) &= L(1, \varepsilon_p)^2 L(2, \varepsilon_p) \zeta_p(2)^{-3} \\ &= (1+p^{-1})^{-2} (1+p^{-2})^{-1} (1-p^{-2})^3. \end{aligned}$$

The integral $I(\varphi_{1,p}, \varphi_{0,p})$ can be calculated as follows (cf. Godement and Jacquet [GoJ, §7]). The image of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{PGL}_2(\mathbb{Q}_p)$ is denoted by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right]$. Let

$$I = \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right] \in \mathrm{PGL}_2(\mathbb{Q}_p) \mid a, b, d \in \mathbb{Z}_p, c \in p\mathbb{Z}_p \right\}$$

be an Iwahori subgroup of $G_{0,p} = \mathrm{PGL}_2(\mathbb{Q}_p)$. Let W_a be the affine Weyl group generated by $w_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 0 & p^{-1} \\ p & 0 \end{bmatrix}$. The extended affine Weyl group \tilde{W} is defined by $\tilde{W} = W_a \rtimes \Omega$, where Ω is the group of order 2 generated by $\omega = \begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix}$. Then we have a Bruhat decomposition $G_{0,p} = \coprod_{w \in \tilde{W}} IwI$. The extended Weyl group \tilde{W} has a length function $l(w)$ such that $l(w_1) = l(w_2) = 1$, $l(\omega) = 0$. The Poincaré series $\sum_{w \in W_a} t^{l(w)}$ is equal to $(1+t)(1-t)^{-1}$. Then the function

$$\Phi(b_1 \omega^j w b_2) = (-1)^j (-p^{-1})^{l(w)}, \quad b_1, b_2 \in I, j \in \{0, 1\}, w \in W_a,$$

is a bi- I -invariant matrix coefficient of the Steinberg representation of G_0 . From this, we have

$$\begin{aligned} I(\varphi_{1,p}, \varphi_{0,p}) &= \sum_{j=0}^1 (-1)^j \sum_{w \in W_a} \mathrm{Vol}(I\omega^j w I) \Phi(\omega^j w)^3 \\ &= 2(p+1)^{-1} \sum_{w \in W_a} (-p^{-2})^{l(w)} \end{aligned}$$

$$= 2p^{-1}(1 - p^{-1})(1 + p^{-2})^{-1}.$$

Note that

$$\text{Vol}(IwI) = (1 + p)^{-1}p^{l(w)}, \quad w \in \tilde{W}.$$

It follows that $\alpha_p(\varphi_{1,p}, \varphi_{0,p}) = 2p^{-1}(1 + p^{-1})$ for $p \mid N$.

Putting together, we recover Theorem 7.1 in this case. Note that we have $|\mathfrak{X}_\tau| = 1$, since the Steinberg representation does not come from a quadratic field.

We remark that Theorem 7.1 is compatible with algebraicity results for the triple product L -functions. For $j = 1, 2, 3$, let f_j be a primitive cusp form with weight κ_j , level N_j , and character ε_j . We assume that $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$ and $\kappa_1 \leq \kappa_2 \leq \kappa_3$. We denote by τ_j the automorphic representation of $\text{GL}_2(\mathbb{A})$ generated by f_j .

We use the symbol $a \sim b$ for $a, b \in \mathbb{C}$, which means that $b \neq 0$ and $a/b \in \bar{\mathbb{Q}}$. It is well-known that $\Lambda(1, \tau_j, \text{Ad}) \sim \langle f_j, f_j \rangle$. Then Harris–Kudla [HaK] proved that

$$\Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3) \sim p(f_1, f_2, f_3),$$

where

$$p(f_1, f_2, f_3) = \begin{cases} \langle f_1, f_1 \rangle \langle f_2, f_2 \rangle \langle f_3, f_3 \rangle & \text{if } \kappa_3 < \kappa_1 + \kappa_2, \\ \langle f_3, f_3 \rangle^2 & \text{if } \kappa_3 \geq \kappa_1 + \kappa_2. \end{cases}$$

We assume $\Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3) \neq 0$. They also proved the Jacquet conjecture which states that there exist a unique quaternion algebra D and some automorphic forms $F_j^D \in \tau_j^D$ such that

$$\int_{\mathbb{A}^\times D^\times (\mathbb{Q}) \backslash D^\times (\mathbb{A})} F_1^D(g) F_2^D(g) F_3^D(g) dg \neq 0.$$

Here τ_j^D is the Jacquet–Langlands–Shimizu correspondence of τ_j . Assume that $\varepsilon_1\varepsilon_2 = \varepsilon_3 = 1$ and $F_j^D \in \tau_j^D$. Then $\varphi_0 = F_3^D$ can be regarded as an automorphic form on $G_0 = D^\times / \mathbb{Q}^\times$ and $\varphi_1 = F_1^D \times F_2^D$ can be regarded as an automorphic form on

$$G_1 = \{(d_1, d_2) \in D^\times \times D^\times \mid \nu(d_1) = \nu(d_2)\} / \mathbb{Q}^\times.$$

Here ν is the reduced norm of D . As before, we transfer the Haar measure dg_v on $\text{GL}_2(\mathbb{Q}_v)$ to $G_0(\mathbb{Q}_v)$. In particular, $C_0 = 6/\pi$.

For each finite prime p , the component π_p has a $\bar{\mathbb{Q}}$ -structure. Note that for $\bar{\mathbb{Q}}$ -rational vectors $\varphi_{1,p}$ and $\varphi_{0,p}$, the quantity $\alpha_p(\varphi_{1,p}, \varphi_{0,p}) \in \bar{\mathbb{Q}}$.

In the balanced case $\kappa_3 < \kappa_1 + \kappa_2$, the quaternion algebra D is definite. We choose arithmetic automorphic forms $F_j^D \in \tau_j^D$. Then we have

$$\langle \varphi_1, \varphi_1 \rangle, \langle \varphi_0, \varphi_0 \rangle \in \bar{\mathbb{Q}}^\times, \quad \langle \varphi_1|_{G_0}, \varphi_0 \rangle \in \bar{\mathbb{Q}}.$$

Note that in this case we have

$$\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) \sim \Delta_{G_{1,\infty}}^{-1} \mathcal{P}_{\pi_{1,\infty}, \pi_{0,\infty}} (1/2)^{-1} \cdot \text{Vol}(G_0(\mathbb{R})) \sim \pi^{-1}.$$

Note that $\text{Vol}(G_0(\mathbb{R})) = \text{Vol}(\text{U}(2)/(\text{U}(1) \times \text{U}(1))) \sim \pi$. Therefore in this case Theorem 7.1 is compatible with the known result

$$\Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3) \sim \langle f_1, f_1 \rangle \langle f_2, f_2 \rangle \langle f_3, f_3 \rangle.$$

Now we consider the unbalanced case $\kappa_3 \geq \kappa_1 + \kappa_2$. We choose arithmetic holomorphic automorphic form $F_3^D \in \tau_3^D$ of weight κ_3 and arithmetic nearly anti-holomorphic forms $F_1^D \in \tau_1^D$ and $F_2^D \in \tau_2^D$ with some weight. Then we have

(see Shimura [S1])

$$\begin{aligned}\langle \varphi_0, \varphi_0 \rangle &\sim \xi(2)^{-1} \langle f_3, f_3 \rangle, \\ \langle \varphi_1, \varphi_1 \rangle &\sim \xi(2)^{-2} \langle f_1, f_1 \rangle \langle f_2, f_2 \rangle, \\ \langle \varphi_1|_{G_0}, \varphi_0 \rangle &\sim \xi(2)^{-1} \langle f_3, f_3 \rangle.\end{aligned}$$

Note that in this case, we have $\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) \sim 1$. Therefore in this case Theorem 7.1 is compatible with the known result

$$\Lambda(1/2, \tau_1 \times \tau_2 \times \tau_3) \sim \langle f_3, f_3 \rangle^2.$$

REMARK 7.3. More generally, Theorem 7.1 is compatible with Shimura's conjecture [S2,3] for Hilbert modular forms, which was proved by Harris [Ha1,2,3] and Yoshida [Y2] in most cases.

8 Restriction of the Yoshida Lift to the Diagonal Subgroup

In this section, we recall the result of Gan and Ichino [GI], in which a formula for the restriction of the Yoshida lift [Y1] to the diagonal subgroup by Böcherer, Furusawa, Schulze-Pillot [BFS] has been generalized. They have proved Conjecture 1.5 for $n = 4$ in some cases and given strong evidence for Conjecture 2.1.

Let k be a totally real algebraic number field. Let k' be either $k \times k$ or a totally real quadratic extension of k . We put

$$\begin{aligned}G_1 &= \text{PGSp}_2, \\ \tilde{G}_0 &= \text{GL}_2(k')/k^\times, \\ G_0 &= \{g \in \text{GL}_2(k') \mid \det g \in k^\times\}/k^\times.\end{aligned}$$

Then G_1 (resp. G_0) can be considered as a special orthogonal group associated to a 5-dimensional (resp. 4-dimensional) quadratic space over k . We regard G_0 as a subgroup of G_1 . Note that $\Delta_{G_1} = \xi_k(2)\xi_k(4)$.

Let (V, Q) be another 4-dimensional quadratic space over k with discriminant field K_Q . We put $H = \text{GO}_Q$ and

$$k'' = \begin{cases} k \times k & \text{if } K_Q = k, \\ K_Q & \text{if } [K_Q : k] = 2. \end{cases}$$

Then there exists a quaternion algebra D over k such that

$$1 \longrightarrow k''^\times \longrightarrow (D \otimes_k k'')^\times \times k^\times \longrightarrow H^0 \longrightarrow 1$$

(cf. e.g. Roberts [Ro, §2]). Here, H^0 is the identity component of H .

Let σ be an irreducible unitary cuspidal automorphic representation of $H(\mathbb{A})$ with trivial central character. We assume the following conditions:

- The Jacquet–Langlands lift of $\sigma|_{D^\times(\mathbb{A}_{k''})}$ to $\text{GL}_2(\mathbb{A}_{k''})$ is cuspidal.
- $\sigma_v \otimes \text{sgn} \simeq \sigma_v$ for some v .
- If $\sigma_v \otimes \text{sgn} \not\simeq \sigma_v$, then $\sigma_v \not\simeq \sigma_{0,v}^-$ for any distinguished representation $\sigma_{0,v}$ of H_v^0 (cf. [GI, Def. 5.4]).

Let π_1 be the theta lift of σ to $G_1(\mathbb{A})$. Note that π_1 is a non-zero irreducible cuspidal automorphic representation of $G_1(\mathbb{A})$. This theta lift was first considered by Yoshida

[Y1] in a certain case. Later, it was considered by Howe and Piatetski-Shapiro [HoP], Böcherer and Schulze-Pillot [BS], Harris, Soudry, and Taylor [HaST], Roberts [Ro] more generally. For this reason, we call π_1 the Yoshida lift of σ .

Let π_0 be an irreducible cuspidal automorphic representation of $G_0(\mathbb{A})$. As in section 7, we choose an irreducible unitary cuspidal automorphic representation τ of $\tilde{G}_0(\mathbb{A}) = \mathrm{GL}_2(\mathbb{A}_{k'})/\mathbb{A}^\times$ such that $\mathcal{V}_{\pi_0} \subset \mathcal{V}_\tau^1|_{G_0(\mathbb{A})}$. We assume the following conditions:

- The base change $\mathcal{BC}(\tau)$ of τ to $\tilde{G}_0(\mathbb{A}_{k''}) = \mathrm{GL}_2(\mathbb{A}_{k' \otimes_k k''})/\mathbb{A}_{k''}^\times$ is cuspidal.
- The Jacquet–Langlands lift of $\mathcal{BC}(\tau)$ to $D^\times(\mathbb{A}_{k' \otimes_k k''})/\mathbb{A}_{k''}^\times$ exists.

Then Theorem 1.1 of [GI] says

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = \frac{1}{2^{\beta'} |\mathfrak{X}_\tau|} \Delta_{G_1} C_0 \mathcal{P}_{\pi_1, \pi_0}(1/2) \prod_{v \in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}$$

for any non-zero vectors $\varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$. Here,

$$\beta' = \begin{cases} 3 & \text{if } K_Q = k, \\ 2 & \text{if } [K_Q : k] = 2, \end{cases}$$

and \mathfrak{X}_τ is the elementary 2-group as in section 7. Thus Conjecture 1.5 is true in this case. Note that we have

$$|\mathcal{S}_{\psi_1}| = \begin{cases} 4 & \text{if } K_Q = k, \\ 2 & \text{if } [K_Q : k] = 2, \end{cases}$$

and $|\mathcal{S}_{\psi_0}| = 2|\mathfrak{X}_\tau|$, if we admit the Arthur conjecture.

9 Restriction of the Saito–Kurokawa Lift to the Diagonal Subset $\mathfrak{H}_1 \times \mathfrak{H}_1$

Let $\kappa > 0$ be an odd integer. Let $f \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$ and $g \in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z}))$ be normalized Hecke eigenforms. We denote the Kohnen plus subspace by $S_{\kappa+(1/2)}^+(\Gamma_0(4)) \subset S_{\kappa+(1/2)}(\Gamma_0(4))$ (cf. Kohnen [Ko]). Let $h \in S_{\kappa+(1/2)}^+(\Gamma_0(4))$ be a Hecke eigenform associated to f by Shimura correspondence. Let $\mathcal{F} \in S_{\kappa+1}(\mathrm{Sp}_2(\mathbb{Z}))$ be the Saito–Kurokawa lift of h . Let τ and σ be the automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ generated by f and g , respectively. Then it is shown in Ichino [I1] that

$$\Lambda(1/2, \mathrm{Ad}(\sigma) \boxtimes \tau) = 2^{\kappa+1} \frac{\langle f, f \rangle |\langle \mathcal{F}|_{\mathfrak{H}_1 \times \mathfrak{H}_1}, g \times g \rangle^2}{\langle h, h \rangle \langle g, g \rangle^2}.$$

Here, $\langle \cdot, \cdot \rangle$ is the usual Petersson inner product on \mathfrak{H}_1 . We interpret this result in terms of automorphic representations. Let φ_1 be the automorphic form on $G_1(\mathbb{A}_\mathbb{Q}) = \mathrm{SO}(3, 2)(\mathbb{A}_\mathbb{Q})$ corresponding to \mathcal{F} . Similarly, let φ_0 be the automorphic form on $G_0(\mathbb{A}_\mathbb{Q}) = \mathrm{SO}(2, 2)(\mathbb{A}_\mathbb{Q})$ corresponding to $g \times g$. As in section 7, let $dg_{0,v}$ be the standard Haar measure of $G_0(\mathbb{Q}_v)$ for $v < \infty$. Let $G_0(\mathbb{R})^0$ be the topological identity component of $G_0(\mathbb{R})$. The maximal compact subgroup \mathcal{K}_∞^0 of $G_0(\mathbb{R})^0$ is defined by $\mathcal{K}_\infty^0 = \mathrm{SO}(2) \times \mathrm{SO}(2)$. Let $dg_{0,\infty}$ be the Haar measure of $G_0(\mathbb{R})^0$ such that $dg_{0,\infty}/dk$ is equal to the measure $(y_1 y_2)^{-2} dx_1 dx_2 dy_1 dy_2$ on $G_0(\mathbb{R})^0/\mathcal{K}_\infty^0 \simeq \mathfrak{H}_1 \times \mathfrak{H}_1$. Here, dk is the Haar measure on \mathcal{K}_∞^0 with total measure 1. The Haar measure $dg_{0,\infty}$ can

be naturally extended to $G_0(\mathbb{R})$. We calculate the Haar measure constant C_0 . Let $G_0(\mathbb{R})^0 = AN\mathcal{K}_\infty^0$ be an Iwasawa decomposition, and $X \subset AN$ be a set bijective to a fundamental domain for $(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}_1)^2$. Then each element of $G_0(\mathbb{Q}) \backslash G_0(\mathbb{A})$ has exactly two representatives in $X \times \mathcal{K}_\infty^0 \times \prod_{v < \infty} \mathcal{K}_{0,v}$. It follows that

$$\int_{G_0(\mathbb{Q}) \backslash G_0(\mathbb{A})} \prod_{v \leq \infty} dg_{0,v} = \frac{1}{2} \mathrm{Vol}(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}_1)^2 = 2\xi(2)^2.$$

Therefore, we have $C_0 = \xi(2)^{-2} = 36\pi^{-2}$. Note that $\Delta_{G_1} = \xi(2)\xi(4)$. Note also that the volume of $\mathrm{Sp}_2(\mathbb{Z}) \backslash \mathfrak{H}_2$ is $2\xi(2)\xi(4)$, where \mathfrak{H}_2 is the Siegel upper-half space of genus 2. It follows that

$$\langle \varphi_1, \varphi_1 \rangle = \frac{\langle \mathcal{F}, \mathcal{F} \rangle}{\xi(2)\xi(4)},$$

$$\langle \varphi_0, \varphi_0 \rangle = \frac{\langle g, g \rangle^2}{2\xi(2)^2},$$

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = \frac{\xi(4)}{2\xi(2)} \frac{|\langle \mathcal{F}|_{\mathfrak{H}_1 \times \mathfrak{H}_1}, g \times g \rangle|^2}{\langle \mathcal{F}, \mathcal{F} \rangle \langle g, g \rangle^2}.$$

As noticed in section 7, it is well-known that $\langle f, f \rangle = 2^{-2\kappa} \Lambda(1, \mathrm{Ad}(\tau))$. By Kohnen–Skoruppa [KoS], we have

$$\frac{\langle \mathcal{F}, \mathcal{F} \rangle}{\langle h, h \rangle} = 2^{\kappa-2} \pi^{-1} \xi(2) \Lambda(3/2, \tau).$$

(Note that there is a minor error in the unfolding argument of [KoS, p. 547]. Since the action of the center of $\mathrm{Sp}_2(\mathbb{Z})$ on \mathfrak{H}_2 is trivial, the right-hand side of the equation of [KoS, p. 547, l. 23] must be multiplied by 2.) It follows that

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = \pi \cdot \frac{\xi(4)}{\xi(2)} \cdot \frac{\Lambda(1/2, \mathrm{Ad}(\sigma) \boxtimes \tau)}{\xi(2) \Lambda(3/2, \tau) \Lambda(1, \mathrm{Ad}(\tau))}.$$

It is easy to check that

$$\Lambda(s, \pi_0) = \Lambda(s, \mathrm{Ad}(\sigma)) \xi(s),$$

$$\Lambda(s, \pi_1) = \Lambda(s, \tau) \xi(s + (1/2)) \xi(s - (1/2)),$$

$$\Lambda(s, \pi_0, \mathrm{Ad}) = \Lambda(s, \mathrm{Ad}(\sigma))^2,$$

$$\begin{aligned} \Lambda(s, \pi_1, \mathrm{Ad}) &= \Lambda(s, \mathrm{Ad}(\tau)) \Lambda(s + (1/2), \tau) \Lambda(s - (1/2), \tau) \\ &\quad \times \xi(s + 1) \xi(s) \xi(s - 1). \end{aligned}$$

From this, one can show that $\mathcal{P}_{\pi_1, \pi_0}(s)$ is equal to

$$\frac{\Lambda(s - (1/2), \mathrm{Ad}(\sigma)) \Lambda(s, \mathrm{Ad}(\sigma) \boxtimes \tau)}{\xi(s + (3/2)) \Lambda(s + 1, \tau) \Lambda(s + (1/2), \mathrm{Ad}(\sigma)) \Lambda(s + (1/2), \mathrm{Ad}(\tau))}.$$

It follows that

$$\begin{aligned} \mathcal{P}_{\pi_1, \pi_0}(1/2) &= \frac{\Lambda(0, \mathrm{Ad}(\sigma)) \Lambda(1/2, \mathrm{Ad}(\sigma) \boxtimes \tau)}{\xi(2) \Lambda(3/2, \tau) \Lambda(1, \mathrm{Ad}(\sigma)) \Lambda(1, \mathrm{Ad}(\tau))} \\ &= \frac{\Lambda(1/2, \mathrm{Ad}(\sigma) \boxtimes \tau)}{\xi(2) \Lambda(3/2, \tau) \Lambda(1, \mathrm{Ad}(\tau))}. \end{aligned}$$

Observe that

$$\begin{aligned} \mathcal{P}_{\pi_1, \infty, \pi_0, \infty}(1/2) &= \frac{\Gamma_{\mathbb{R}}(1) \Gamma_{\mathbb{C}}(\kappa) \cdot \Gamma_{\mathbb{C}}(\kappa) \Gamma_{\mathbb{C}}(2\kappa) \Gamma_{\mathbb{C}}(1)}{\Gamma_{\mathbb{R}}(2) \cdot \Gamma_{\mathbb{C}}(\kappa + 1) \cdot \Gamma_{\mathbb{R}}(2) \Gamma_{\mathbb{C}}(\kappa + 1) \cdot \Gamma_{\mathbb{R}}(2) \Gamma_{\mathbb{C}}(2\kappa)} \\ &= 4\kappa^{-2} \pi^4. \end{aligned}$$

PROPOSITION 9.1. *Let $\pi_{1,\infty}$ be the irreducible holomorphic discrete series representation of $\mathrm{SO}(3, 2)$ with lowest K -type $(\det)^{\pm(\kappa+1)}$. Let $\pi_{0,\infty}$ be the irreducible discrete series representation of $\mathrm{SO}(2, 2)$ with lowest K -type $\pm(\kappa + 1, \kappa + 1)$. Choose lowest weight vectors $\varphi_{1,\infty} \in \pi_{1,\infty}$ and $\varphi_{0,\infty} \in \pi_{0,\infty}$ such that $\|\varphi_{1,\infty}\| = \|\varphi_{0,\infty}\| = 1$. Then we have*

$$I(\varphi_{1,\infty}, \varphi_{0,\infty}) = 16\kappa^{-2}\pi^2,$$

$$\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) = 4\pi.$$

The proof of Proposition 9.1 will be given in section 12. Using Proposition 9.1, we have

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = \frac{1}{4} \Delta_{G_1} C_0 \mathcal{P}_{\pi_1, \pi_0}(1/2) \cdot \frac{\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty})}{\|\varphi_{1,\infty}\|^2 \cdot \|\varphi_{0,\infty}\|^2}.$$

Therefore in this case, it seems Conjecture 3.2 holds with $2^\beta = 1/4$. Note that we have $|\mathcal{S}_{\psi_1}| = 4$ and $|\mathcal{S}_{\psi_0}| = 2$, and hence $2^\beta \neq 1/(|\mathcal{S}_{\psi_1}| \cdot |\mathcal{S}_{\psi_0}|)$, if we admit the Arthur conjecture.

REMARK 9.2. Now choose another normalized Hecke eigenform $g' \in S_{\kappa+1}(\mathrm{SL}_2(\mathbb{Z}))$ such that $g \neq g'$. Let σ' be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by g' . Let φ_1 be as before and φ_0 the lifting of $g \times g'$ to $G_0(\mathbb{A})$. Then we have $\langle \varphi_1|_{G_0}, \varphi_0 \rangle = 0$. Note that $\mathrm{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) = \{0\}$ for some v (see, e.g. [Ik1, Prop. 3.1]). After a little calculation, one can show the numerator of $\mathcal{P}_{\pi_1, \pi_0}(s)$ is equal to

$$\Lambda(s, \tau \times \sigma \times \sigma') \Lambda(s + (1/2), \sigma \times \sigma') \Lambda(s - (1/2), \sigma \times \sigma'),$$

and the denominator is

$$\Lambda(s + (1/2), \mathrm{Ad}(\tau)) \Lambda(s + 1, \tau) \Lambda(s, \tau)$$

$$\times \xi(s + (3/2)) \xi(s + (1/2)) \xi(s - (1/2))$$

$$\times \Lambda(s + (1/2), \mathrm{Ad}(\sigma)) \Lambda(s + (1/2), \mathrm{Ad}(\sigma')).$$

Note that as far as we know, any relation between $\mathrm{ord}_{s=1/2} \Lambda(s, \tau \times \sigma \times \sigma')$ and $\mathrm{ord}_{s=1/2} \Lambda(s, \tau)$ is not known, and so $\mathcal{P}_{\pi_1, \pi_0}(s)$ might have a pole at $s = 1/2$. It seems this example suggests that there is no relation between the period $\langle \varphi_1|_{G_0}, \varphi_0 \rangle$ and the L -value $\mathcal{P}_{\pi_1, \pi_0}(1/2)$, when π_1 or π_0 is non-tempered and the condition $\mathrm{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ fails. Note that when both π_1 and π_0 are tempered, Conjecture 1.5 still makes sense even if the condition $\mathrm{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ fails, since it is believed that $\mathcal{P}_{\pi_1, \pi_0}(s)$ is holomorphic at $s = 1/2$.

10 Restriction of the Hermitian Maass Lift to \mathfrak{H}_2

Now we discuss the case $n = 5$ and $k = \mathbb{Q}$. We put $G_0 = \mathrm{SO}(3, 2) \simeq \mathrm{PGSp}_2$. Let $\kappa > 0$ be an odd integer and $f \in S_{2\kappa}(\mathrm{SL}_2(\mathbb{Z}))$, $h \in S_{\kappa+(1/2)}^+(\Gamma_0(4))$, $\mathcal{F} \in S_{\kappa+1}(\mathrm{Sp}_2(\mathbb{Z}))$, and τ be as in section 9. Let

$$h(\tau) = \sum_{\substack{n>0 \\ -n=0,1(4)}} c(n)q^n$$

be the Fourier expansion of $h(\tau)$.

Let K be an imaginary quadratic field with discriminant $-D$. We assume that $c(D) \neq 0$. We denote by χ and w_K the associated Dirichlet character for K/\mathbb{Q} and the number of units in K , respectively. We put $G_1 = \mathrm{SO}(4, 2)_{K/\mathbb{Q}} \simeq \mathrm{SU}(2, 2)_{K/\mathbb{Q}}/\{\pm 1\}$.

Now let $\Gamma_K = \mathrm{SU}(2, 2)(\mathbb{Q}) \cap \mathrm{GL}_4(\mathcal{O}_K)$ be the special hermitian modular group, where \mathcal{O}_K is the integer ring of K .

By using the fact that the Tamagawa number of $\mathrm{SU}(2, 2)$ is 1, one can show that the volume of the fundamental domain for Γ_K is equal to

$$\mathrm{Vol}(\Gamma_K \backslash \mathcal{H}_2) = 2^{-3} D^{5/2} (4, w_K) \xi(2) \Lambda(3, \chi) \xi(4),$$

where \mathcal{H}_2 is the hermitian upper-half space of degree 2. Here, we have given an invariant measure on \mathcal{H}_2 as follows. Put $X = (Z + {}^t\bar{Z})/2$, $Y = (Z - {}^t\bar{Z})/(2\sqrt{-1})$ for $Z \in \mathcal{H}_2$. The measure dX on the space of hermitian matrices is defined by $dX = \prod_{i \leq j} dX_{ij}^{(r)} \prod_{i < j} dX_{ij}^{(i)}$, where $X = X^{(r)} + \sqrt{-1}X^{(i)}$, $X_{ij}^{(r)}, X_{ij}^{(i)} \in \mathbb{R}$. Then the invariant measure is given by $(\det Z)^{-4} dX dY$. This calculation will be carried out in the appendix to this section.

Let $g \in S_\kappa(\Gamma_0(D), \chi)$ be a primitive form and $\mathcal{G} \in S_{\kappa+1}(\Gamma_K)$ the hermitian Maass lift of g (cf. Kojima [Koj], Krieg [Kr], Ikeda [Ik2]). We assume that $\mathcal{G} \neq 0$. Let ρ be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by g . By using Sugano [Su, Cor. 8.3] and Ikeda [Ik2, Prop. 17.4], we have

$$\langle \mathcal{G}, \mathcal{G} \rangle = 2^{-2\kappa-7} D^{\kappa+2} \pi^{-2} (4, w_K) \xi(2) \Lambda(2, \mathrm{Sym}^2(\rho)) \Lambda(1, \mathrm{Ad}(\rho)).$$

One can prove this formula using Raghavan–Sengupta [RS]. The main theorem of Ichino and Ikeda [II] says

$$|c(D)|^2 \frac{|\langle \mathcal{G}|_{\mathfrak{H}_2}, \mathcal{F} \rangle|^2}{\langle \mathcal{F}, \mathcal{F} \rangle^2} = 2^{-4\kappa-2} D^{2\kappa-1} \frac{\Lambda(1/2, \rho \times \rho \times \tau)}{\langle f, f \rangle^2}.$$

Combining these results and the Kohnen–Zagier formula [KoZ]

$$|c(D)|^2 \frac{\langle f, f \rangle}{\langle h, h \rangle} = 2^{\kappa-1} D^{\kappa-(1/2)} \Lambda(1/2, \tau \otimes \chi),$$

we have

$$\begin{aligned} \frac{|\langle \mathcal{G}|_{\mathfrak{H}_2}, \mathcal{F} \rangle|^2}{\langle \mathcal{G}, \mathcal{G} \rangle \langle \mathcal{F}, \mathcal{F} \rangle} &= 2\pi \cdot \mathrm{Vol}(\Gamma_K \backslash \mathcal{H}_2)^{-1} \xi(2) \Lambda(3, \chi) \xi(4) \\ &\quad \times \frac{\Lambda(1/2, \mathrm{Sym}^2(\rho) \boxtimes \tau) \Lambda(3/2, \tau)}{\Lambda(2, \mathrm{Sym}^2(\rho)) \Lambda(1, \mathrm{Ad}(\rho)) \Lambda(1, \mathrm{Ad}(\tau))}. \end{aligned}$$

We translate these results to adelic language. Let φ_1 (resp. φ_0) be the automorphic form on $G_1(\mathbb{A})$ (resp. $G_0(\mathbb{A})$) corresponding to \mathcal{G} (resp. \mathcal{F}). We put $S = S_f \cup \{\infty\}$, where S_f is the set of primes which divide D . When $v < \infty$, let $dg_{0,v}$ be the standard Haar measure of $G_0(\mathbb{Q}_v)$. The topological identity component of $G_0(\mathbb{R})$ is denoted by $G_0(\mathbb{R})^0$. Let $\mathcal{K}_\infty^0 = \mathrm{SO}(3) \times \mathrm{SO}(2)$ be a maximal compact subgroup of $G_0(\mathbb{R})^0$. Let dk be the Haar measure of \mathcal{K}_∞^0 with the total measure 1, and $dg_{0,\infty}$ the Haar measure of $G_0(\mathbb{R})^0$ such that $dg_{0,\infty}/dk$ is equal to the measure $(\det Y)^{-3} dX dY$ on $\mathfrak{H}_2 \simeq G_0(\mathbb{R})^0/\mathcal{K}_\infty^0$. Then we have $\mathrm{Vol}(\mathrm{PGSp}_2(\mathbb{Z}) \backslash G_0(\mathbb{R})) = \mathrm{Vol}(\mathrm{Sp}_2(\mathbb{Z}) \backslash \mathfrak{H}_2) = 2\xi(2)\xi(4)$. Let C_0 be the Haar measure constant. It follows that $C_0 = \xi(2)^{-1}\xi(4)^{-1} = 540\pi^{-3}$, since there is a

bijection $G_0(\mathbb{Q}) \backslash G_0(\mathbb{A}) \simeq (\text{PGSp}_2(\mathbb{Z}) \backslash G_0(\mathbb{R})) \times \prod_{p < \infty} \mathcal{K}_{0,p}$. Note also that $\Delta_{G_1} = \xi(2)\Lambda(3, \chi)\xi(4)$.

Let π_1 (resp. π_0) be the irreducible cuspidal automorphic representation of $G_1(\mathbb{A}_{\mathbb{Q}})$ (resp. $G_0(\mathbb{A}_{\mathbb{Q}})$) generated by φ_1 (resp. φ_0). Note that both π_1 and π_0 are non-tempered. It is easy to check that

$$\begin{aligned} \Lambda(s, \pi_1) &= \Lambda(s, \text{Sym}^2(\rho))\xi(s+1)\xi(s)\xi(s-1), \\ \Lambda(s, \pi_0) &= \Lambda(s, \tau)\xi(s+(1/2))\xi(s-(1/2)), \\ \Lambda(s, \pi_1, \text{Ad}) &= \Lambda(s+1, \text{Sym}^2(\rho))\Lambda(s, \text{Sym}^2(\rho))\Lambda(s-1, \text{Sym}^2(\rho)) \\ &\quad \times \Lambda(s, \text{Ad}(\rho))\xi(s+1)\xi(s)\xi(s-1), \\ \Lambda(s, \pi_0, \text{Ad}) &= \Lambda(s, \text{Ad}(\tau))\Lambda(s+(1/2), \tau)\Lambda(s-(1/2), \tau) \\ &\quad \times \xi(s+1)\xi(s)\xi(s-1). \end{aligned}$$

It follows that $\mathcal{P}_{\pi_1, \pi_0}(s) = R(s)/Q(s)$, where

$$\begin{aligned} R(s) &= \Lambda(s, \text{Sym}^2(\rho) \boxtimes \tau)\Lambda(s-1, \tau)\xi(s-(3/2)), \\ Q(s) &= \Lambda(s+(3/2), \text{Sym}^2(\rho))\Lambda(s+(1/2), \text{Ad}(\rho)) \\ &\quad \times \Lambda(s+(1/2), \text{Ad}(\tau))\xi(s+(3/2)). \end{aligned}$$

Observe that

$$\begin{aligned} \mathcal{P}_{\pi_1, \pi_0}(1/2) &= \frac{\Lambda(1/2, \text{Sym}^2(\rho) \boxtimes \tau)\Lambda(-1/2, \tau)\xi(-1)}{\Lambda(2, \text{Sym}^2(\rho))\Lambda(1, \text{Ad}(\rho))\Lambda(1, \text{Ad}(\tau))\xi(2)} \\ &= -\frac{\Lambda(1/2, \text{Sym}^2(\rho) \boxtimes \tau)\Lambda(3/2, \tau)}{\Lambda(2, \text{Sym}^2(\rho))\Lambda(1, \text{Ad}(\rho))\Lambda(1, \text{Ad}(\tau))} \end{aligned}$$

by the functional equations $\Lambda(1-s, \tau) = -\Lambda(s, \tau)$, $\xi(1-s) = \xi(s)$.

We consider the local factor $\alpha_v(\varphi_{1,v}, \varphi_{0,v})$. For $v \notin S$, we may consider $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) = 1$. For $v \in S_f$, the conditions (U1) and (U2) in section 1 fail. Instead of (U1) and (U2), we consider the following conditions:

- (U1') $G_{i,v}$ is quasi-split.
- (U2') $\mathcal{K}_{i,v}$ is a special maximal compact subgroup of $G_{i,v}$.

LEMMA 10.1. *Assume $n = 5$. Let v be a non-archimedean place such that the conditions (U1'), (U2'), (U3), (U4), (U5), and (U6) hold. Then we have $I(\varphi_{1,v}, \varphi_{0,v}) = \Delta_{G_{1,v}} \mathcal{P}_{\pi_{1,v}, \pi_{0,v}}(1/2)$, if it is convergent.*

The authors have verified this lemma by using computer calculation. By this lemma we may consider $\alpha_v(\varphi_{1,v}, \varphi_{0,v}) = 1$ by ‘‘analytic continuation’’.

For $v = \infty$, one can easily see that $\mathcal{P}_{\pi_{1,\infty}, \pi_{0,\infty}}(1/2)$ is equal to

$$\frac{\Gamma_{\mathbb{C}}(1)\Gamma_{\mathbb{C}}(\kappa)\Gamma_{\mathbb{C}}(2\kappa-1) \cdot \Gamma_{\mathbb{C}}(\kappa-1) \cdot \Gamma_{\mathbb{R}}(-1)}{\Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{C}}(\kappa+1) \cdot \Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{C}}(\kappa) \cdot \Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{C}}(2\kappa) \cdot \Gamma_{\mathbb{R}}(2)} = -\frac{16\pi^7}{\kappa(\kappa-1)(2\kappa-1)}.$$

Note that $\pi_{1,\infty}$ is a discrete series representation of $\text{SO}(4, 2)$, and the K -type of $\varphi_{1,\infty}$ is the lowest K -type. Similarly, $\pi_{0,\infty}$ is a discrete series representation of $\text{SO}(3, 2)$, and $\varphi_{0,\infty}$ is a lowest K -type vector. We may assume $\|\varphi_{1,\infty}\| = \|\varphi_{0,\infty}\| = 1$.

PROPOSITION 10.2. *We have*

$$I(\varphi_{1,\infty}, \varphi_{0,\infty}) = \frac{64\pi^3}{\kappa(\kappa - 1)(2\kappa - 1)},$$

$$\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) = -4\pi.$$

A proof of Proposition 10.2 will be given in section 12.

By Proposition 10.2, we have

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = \frac{1}{4} \Delta_{G_1} C_0 \mathcal{P}_{\pi_1, \pi_0}(1/2) \cdot \frac{\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty})}{\|\varphi_{1,\infty}\|^2 \cdot \|\varphi_{0,\infty}\|^2}$$

$$= \frac{1}{4} \Delta_{G_1} C_0 \mathcal{P}_{\pi_1, \pi_0}(1/2) \prod_{v \in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}$$

under the assumption $c(D) \neq 0$. Therefore in this case, Conjecture 3.2 seems to hold with $2^\beta = 1/4$. Note that we have $|\mathcal{S}_{\psi_1}| = 2$ and $|\mathcal{S}_{\psi_0}| = 4$, and hence $2^\beta \neq 1/(|\mathcal{S}_{\psi_1}| \cdot |\mathcal{S}_{\psi_0}|)$, if we admit the Arthur conjecture.

Appendix to Section 10: Calculation of the Volume of the Fundamental Domain for $\Gamma_K \backslash \mathcal{H}_2$

In this appendix, we calculate the volume of the fundamental domain for the hermitian modular group. Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D$. We put $K_p = K \otimes \mathbb{Q}_p$ and $\mathcal{O}_p = \mathcal{O}_K \otimes \mathbb{Z}_p$, where \mathcal{O}_K is the integer ring of K .

Let $\Gamma_K^{(n)} = \text{SU}(n, n)(\mathbb{Q}) \cap \text{GL}_{2n}(\mathcal{O}_K)$ be the special hermitian modular group. By using the fact that the Tamagawa number of $\text{SU}(n, n)$ is 1, we shall show that

$$\text{Vol}(\Gamma_K^{(n)} \backslash \mathcal{H}_n) = 2^{-n^2+1} D^{(2n^2-n-1)/2} (2n, w_K) \prod_{i=2}^{2n} \Lambda(i, \chi^i),$$

where \mathcal{H}_n is the hermitian upper half space of degree n .

Put $\mathfrak{G} = \text{SU}(n, n)$. Then

$$\text{Lie}(\mathfrak{G}) = \{X \in \text{M}_{2n}(K) \mid XJ + J^t \bar{X} = 0, \text{tr}(X) = 0\},$$

where $J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$. We choose a basis of the $\text{Lie}(\mathfrak{G})$ as follows. Let $E[i, j] \in \text{M}_n(\mathbb{Z})$ be the (i, j) -elementary matrix of size n . Set

$$S[i, j] = \begin{cases} E[i, i] & (i = j), \\ E[i, j] + E[j, i] & (i \neq j), \end{cases}$$

$$A[i, j] = E[i, j] - E[j, i].$$

Put

$$X_{ij} = \begin{pmatrix} E[i, j] & 0 \\ 0 & -E[j, i] \end{pmatrix},$$

$$Y_{ij} = \begin{pmatrix} 0 & S[i, j] \\ 0 & 0 \end{pmatrix},$$

$$Y'_{ij} = \begin{pmatrix} 0 & 0 \\ S[i, j] & 0 \end{pmatrix},$$

$$\begin{aligned}
 V_{ij} &= \sqrt{-D} \begin{pmatrix} 0 & A[i, j] \\ 0 & 0 \end{pmatrix}, \\
 V'_{ij} &= -\sqrt{-D} \begin{pmatrix} 0 & 0 \\ A[i, j] & 0 \end{pmatrix}, \\
 W_{ij} &= \sqrt{-D} \begin{pmatrix} E[i, j] & 0 \\ 0 & E[j, i] \end{pmatrix}, \\
 W'_i &= \sqrt{-D} \begin{pmatrix} E[i, i] - E[i + 1, i + 1] & 0 \\ 0 & E[i, i] - E[i + 1, i + 1] \end{pmatrix}.
 \end{aligned}$$

The following vectors make up a basis of $\text{Lie}(\mathfrak{G})$.

$$\begin{aligned}
 X_{ij} & \quad (1 \leq i, j \leq n), \\
 Y_{ij} & \quad (1 \leq i \leq j \leq n), \\
 Y'_{ij} & \quad (1 \leq i \leq j \leq n), \\
 V_{ij} & \quad (1 \leq i < j \leq n), \\
 V'_{ij} & \quad (1 \leq i < j \leq n), \\
 W_{ij} & \quad (1 \leq i, j \leq n, i \neq j), \\
 W'_i & \quad (1 \leq i < n).
 \end{aligned}$$

Let $\mathfrak{L} \subset \text{Lie}(\mathfrak{G})$ be the lattice generated by this basis. This basis determines a Haar measure dg_v on $\mathfrak{G}(\mathbb{Q}_v)$ for each place v , and the product measure $\prod_v dg_v$ is the Tamagawa measure on $\mathfrak{G}(\mathbb{A})$. For each prime p , we define a maximal compact subgroup $\mathcal{K}_{\mathfrak{G}_p}$ of $\mathfrak{G}(\mathbb{Q}_p)$ by $\mathcal{K}_{\mathfrak{G}_p} = \mathfrak{G}(\mathbb{Q}_p) \cap \text{GL}_{2n}(\mathcal{O}_p)$. Since $[\mathcal{O}_p : \mathbb{Z}_p + \sqrt{-D}\mathbb{Z}_p] = (2, p)$, we have

$$[\text{Lie}(\mathfrak{G})(\mathbb{Q}_p) \cap M_{2n}(\mathcal{O}_p) : \mathfrak{L} \otimes \mathbb{Z}_p] = (2, p)^{2n^2-n-1}.$$

It follows that the volume of $\mathcal{K}_{\mathfrak{G}_p}$ is equal to $(2, p)^{2n^2-n-1} \prod_{i=2}^{2n} L(i, \chi_p^i)^{-1}$.

For the real place, the vectors

$$\begin{aligned}
 X_{ij} - X_{ji} & \quad (1 \leq i < j \leq n), \\
 Y_{ij} - Y'_{ij} & \quad (1 \leq i \leq j \leq n), \\
 V_{ij} + V'_{ij} & \quad (1 \leq i < j \leq n), \\
 W_{ij} + W_{ji} & \quad (1 \leq i < j \leq n), \\
 W'_i & \quad (1 \leq i < n),
 \end{aligned}$$

generate the Lie algebra of a maximal compact subgroup $\mathcal{K}_{\mathfrak{G}_\infty}$ of $\mathfrak{G}(\mathbb{R})$. The maximal compact subgroup $\mathcal{K}_{\mathfrak{G}_\infty}$ is isomorphic to

$$\{(u_1, u_2) \in \text{U}(n) \times \text{U}(n) \mid \det u_1 \cdot \det u_2 = 1\}.$$

This isomorphism is explicitly given by $\text{Ad}(A) : \kappa \mapsto A\kappa A^{-1}$, where

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_n & -\sqrt{-1} \cdot \mathbf{1}_n \\ \mathbf{1}_n & \sqrt{-1} \cdot \mathbf{1}_n \end{pmatrix}.$$

Note that

$$\begin{aligned}
 \text{Ad}(A)(X_{ij} - X_{ji}) &= \begin{pmatrix} A[i, j] & 0 \\ 0 & A[i, j] \end{pmatrix}, \\
 \text{Ad}(A)(Y_{ij} - Y'_{ij}) &= \sqrt{-1} \begin{pmatrix} S[i, j] & 0 \\ 0 & -S[i, j] \end{pmatrix},
 \end{aligned}$$

$$\begin{aligned} \text{Ad}(A)(V_{ij} + V'_{ij}) &= \sqrt{D} \begin{pmatrix} -A[i, j] & 0 \\ 0 & A[i, j] \end{pmatrix}, \\ \text{Ad}(A)(W_{ij} + W_{ji}) &= \sqrt{-D} \begin{pmatrix} S[i, j] & 0 \\ 0 & S[i, j] \end{pmatrix}, \\ \text{Ad}(A)(W'_i) &= W'_i. \end{aligned}$$

Let dk_∞ be the Haar measure on $\mathcal{K}_{\mathfrak{G}_\infty}$ determined by these vectors. By Macdonald [M], the volume of $U(n)$ is equal to $(2\pi)^{n(n+1)/2} \prod_{i=1}^n \Gamma(i)^{-1}$, if the Haar measure is normalized by a Chevalley basis of $\text{Lie}(U(n)) \otimes \mathbb{C}$. Using this, we have

$$\text{Vol}(\mathcal{K}_{\mathfrak{G}_\infty}; dk_\infty) = D^{(-n^2+1)/2} 2^{-n^2+2n} \pi^{n^2+n-1} \prod_{i=1}^n \Gamma(i)^{-2}.$$

We now consider the invariant measure on the hermitian upper half space \mathcal{H}_n . We define an invariant measure on \mathcal{H}_n as follows. Let $\text{Her}_n(\mathbb{C}/\mathbb{R})$ be the space of hermitian matrices of size n . Then the Haar measures dX and dY on $\text{Her}_n(\mathbb{C}/\mathbb{R})$ are such that the covolume of the lattice $\text{Her}_n(\mathbb{C}/\mathbb{R}) \cap M_n(\mathbb{Z}[\sqrt{-1}])$ is 1. Then the measure $(\det Y)^{-2n} dX dY$ is invariant under the action of $\mathfrak{G}(\mathbb{R}) = \text{SU}(n, n)(\mathbb{R})$.

Note that $\mathfrak{G}(\mathbb{R})/\mathcal{K}_{\mathfrak{G}_\infty} \simeq \mathcal{H}_n$. We claim that dg_∞/dk_∞ is equal to $2^{-n} D^{-(n^2-n)/2} (\det Y)^{-2n} dX dY$. To prove this, we consider the Iwasawa decomposition $\mathfrak{G}(\mathbb{R}) = A_{\mathfrak{G}_\infty} N_{\mathfrak{G}_\infty} \mathcal{K}_{\mathfrak{G}_\infty}$, where $A_{\mathfrak{G}_\infty}$ and $N_{\mathfrak{G}_\infty}$ are Lie subgroup of $\mathfrak{G}(\mathbb{R})$ corresponding to the Lie algebras generated by

$$\{X_{ii} \mid 1 \leq i \leq n\}$$

and

$$\{X_{ij}, V_{ij}, W_{ij} \mid 1 \leq i < j \leq n\} \cup \{Y_{ij} \mid 1 \leq i \leq j \leq n\},$$

respectively. Then it is easy to check the left-invariant Haar measure determined by these basis induces $2^{-n} D^{-(n^2-n)/2} (\det Y)^{-2n} dX dY$ on \mathcal{H}_n , which implies the claim.

Now we consider the adèle space $\mathfrak{G}(\mathbb{A})$. Let \mathfrak{X} be a fundamental domain for $\Gamma_K^{(n)} \backslash \mathcal{H}_n$. We regard \mathfrak{X} as a subset of $A_{\mathfrak{G}_\infty} N_{\mathfrak{G}_\infty}$ by the bijection $A_{\mathfrak{G}_\infty} N_{\mathfrak{G}_\infty} \simeq \mathfrak{G}(\mathbb{R})/\mathcal{K}_{\mathfrak{G}_\infty} \simeq \mathcal{H}_n$. Then each fibre of the map

$$\left(\prod_p \mathcal{K}_{\mathfrak{G}_p} \right) \times \mathfrak{X} \times \mathcal{K}_{\mathfrak{G}_\infty} \rightarrow \mathfrak{G}(\mathbb{Q}) \backslash \mathfrak{G}(\mathbb{A})$$

has exactly $|Z(\Gamma_K^{(n)})|$ elements, where $Z(\Gamma_K^{(n)})$ is the center of $\Gamma_K^{(n)}$. Note that $|Z(\Gamma_K^{(n)})| = (2n, w_K)$. It follows that

$$\begin{aligned} (2n, w_K)^{-1} \cdot 2^{2n^2-n-1} \prod_{i=2}^{2n} L(i, \chi^i)^{-1} \cdot D^{(-n^2+1)/2} 2^{-n^2+2n} \pi^{n^2+n-1} \prod_{i=1}^n \Gamma(i)^{-2} \\ \times 2^{-n} D^{-(n^2-n)/2} \text{Vol}(\mathfrak{X}) = 1. \end{aligned}$$

It follows that

$$\text{Vol}(\Gamma_K^{(n)} \backslash \mathcal{H}_n) = 2^{-n^2+1} D^{(2n^2-n-1)/2} (2n, w_K) \prod_{i=2}^{2n} \Lambda(i, \chi^i),$$

as desired.

11 The Trivial Representation

Let k be a totally real algebraic number field and S the set of archimedean places of k . The discriminant of k is denoted by D_k . Recall that the completed Dedekind zeta function $\xi_k(s)$ satisfies the functional equation $\xi_k(1-s) = D_k^{s-(1/2)} \xi_k(s)$. Put $d = [k : \mathbb{Q}]$. We assume the following conditions:

- (a) Both G_1 and G_0 are unramified over k_v for each $v \notin S$.
- (b) $G_{0,v}$ is compact for each $v \in S$.

Note that such a pair $G_0 \subset G_1$ exists if and only if the following (i), (ii), and (iii) hold:

- (i) The discriminant field K is unramified over k .
- (ii) K is totally real if $n \equiv 0 \pmod{4}$, and totally imaginary if $n \equiv 2 \pmod{4}$.
- (iii) d is even if $n \equiv 3, 4, 5, 6 \pmod{8}$.

Let $\mathcal{K}_0 = \prod_v \mathcal{K}_{0,v}$ be a maximal compact subgroup of $G_0(\mathbb{A})$. We assume $\mathcal{K}_{0,v}$ is a hyperspecial maximal compact subgroup for $v \notin S$. For $v \in S$, we give the standard Haar measure $dg_{0,v}$ on $G_{0,v}$. For $v \in S$, we give the Haar measure $dg_{0,v}$ with total volume 1 on $\mathcal{K}_{0,v} = G_{0,v}$. The Haar measure constant C_0 can be calculated directly, but here we make use of the mass formula. There exists a finite subset $\mathfrak{B} \subset G_0(\mathbb{A})$ such that $G_0(\mathbb{A}) = \coprod_{x \in \mathfrak{B}} G_0(k)x\mathcal{K}_0$. For each $x \in \mathfrak{B}$, the group $\Gamma^x = x^{-1}G_0(k)x \cap \mathcal{K}_0$ is a finite group. The left coset $G_0(k)\backslash G_0(\mathbb{A})$ is decomposed into a disjoint union

$$G_0(k)\backslash G_0(\mathbb{A}) = \coprod_{x \in \mathfrak{B}} x \cdot (\Gamma^x \backslash \mathcal{K}_0).$$

Let e_x be the order of the group Γ^x . The mass M is defined by $M = \sum_{x \in \mathfrak{B}} e_x^{-1}$. Then Shimura’s exact mass formula (Shimura [S4, p.27, Th.5.8]) says that

$$M = 2D_k^{m^2-(m/2)} [(2\pi)^{-m}\Gamma(m)]^d L(m, \chi) \prod_{j=1}^{m-1} \{[(2\pi)^{-2j}\Gamma(2j)]^d \zeta_k(2j)\}$$

if $n = 2m$ is even, and that

$$M = 2^{1-md} D_k^{m^2+(m/2)} \prod_{j=1}^m \{[(2\pi)^{-2j}\Gamma(2j)]^d \zeta_k(2j)\}$$

if $n = 2m + 1$ is odd.

Then we have

$$\int_{G_0(k)\backslash G_0(\mathbb{A})} \prod_v dg_{0,v} = M.$$

Since the Tamagawa number of G_0 is 2, we have $C_0 = 2M^{-1}$.

By definition, we have

$$\Delta_{G_1} = \begin{cases} \prod_{j=1}^m \xi_k(2j) & \text{if } n = 2m \text{ is even,} \\ \Lambda(m+1, \chi) \prod_{j=1}^m \xi_k(2j) & \text{if } n = 2m + 1 \text{ is odd.} \end{cases}$$

We now put $\varphi_1 = 1$ and $\varphi_0 = 1$. Then π_i is the trivial representation of $G_i(\mathbb{A})$. Obviously, we have

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = 1.$$

The L -function of the trivial representation of G_0 is given by

$$\Lambda(s, \pi_0) = \begin{cases} \Lambda(s, \chi) \prod_{j=1}^{2m-1} \xi_k(s - m + j) & \text{if } n = 2m \text{ is even,} \\ \prod_{j=1}^{2m} \xi_k(s - m + j - (1/2)) & \text{if } n = 2m + 1 \text{ is odd.} \end{cases}$$

Similarly, we have

$$\Lambda(s, \pi_1) = \begin{cases} \prod_{j=1}^{2m} \xi_k(s - m + j - (1/2)) & \text{if } n = 2m \text{ is even,} \\ \Lambda(s, \chi) \prod_{j=1}^{2m+1} \xi_k(s - m + j - 1) & \text{if } n = 2m + 1 \text{ is odd.} \end{cases}$$

When $n = 2m$ is even, we have

$$\begin{aligned} \Lambda(s, \pi_1 \boxtimes \pi_0) &= \prod_{i=1}^{2m} \Lambda(s - m + i - (1/2), \chi) \\ &\quad \times \prod_{i=1}^{2m} \prod_{j=1}^{2m-1} \xi_k(s - 2m + i + j - (1/2)) \\ \Lambda(s, \pi_0, \text{Ad}) &= \prod_{i=1}^{2m-1} \Lambda(s - m + i, \chi) \\ &\quad \times \prod_{1 \leq i < j \leq 2m-1} \xi_k(s - 2m + i + j) \\ \Lambda(s, \pi_1, \text{Ad}) &= \prod_{1 \leq i \leq j \leq 2m} \xi_k(s - 2m + i + j - 1). \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{P}_{\pi_1, \pi_0}(s) &= \frac{\Lambda(s - m + (1/2), \chi)}{\xi_k(s + 2m - (1/2))} \prod_{j=1}^{m-1} \frac{\xi_k(s - 2j + (1/2))}{\xi_k(s + 2j - (1/2))}, \\ \mathcal{P}_{\pi_1, \pi_0}(1/2) &= \frac{\Lambda(1 - m, \chi)}{\xi_k(2m)} \prod_{j=1}^{m-1} \frac{\xi_k(-2j + 1)}{\xi_k(2j)} \\ &= D_k^{m^2 - (m/2)} \frac{\Lambda(m, \chi)}{\xi_k(2m)}, \end{aligned}$$

if $n = 2m$ is even. A similar calculation shows that

$$\begin{aligned} \mathcal{P}_{\pi_1, \pi_0}(s) &= \Lambda(s + m + (1/2), \chi)^{-1} \prod_{j=1}^m \frac{\xi_k(s - 2j + (1/2))}{\xi_k(s + 2j - (1/2))}, \\ \mathcal{P}_{\pi_1, \pi_0}(1/2) &= \Lambda(m + 1, \chi)^{-1} \prod_{j=1}^m \frac{\xi_k(-2j + 1)}{\xi_k(2j)} \\ &= D_k^{m^2 + (m/2)} \Lambda(m + 1, \chi)^{-1}, \end{aligned}$$

if $n = 2m + 1$ is odd. When $v \in S$, the integral $I(\varphi_{1,v}, \varphi_{0,v})$ is clearly equal to 1. It follows that

$$\begin{aligned} \alpha_v(\varphi_{1,v}, \varphi_{0,v}) &= \Delta_{G_{1,v}}^{-1} \mathcal{P}_{\pi_{1,v}, \pi_{0,v}}(1/2)^{-1} \\ &= \begin{cases} \Gamma_{\mathbb{R}}(1 - m)^{-1} \prod_{j=1}^{m-1} \Gamma_{\mathbb{R}}(-2j + 1)^{-1} & \text{if } n = 2m \equiv 0 \pmod{4}, \\ \Gamma_{\mathbb{R}}(2 - m)^{-1} \prod_{j=1}^{m-1} \Gamma_{\mathbb{R}}(-2j + 1)^{-1} & \text{if } n = 2m \equiv 2 \pmod{4}, \\ \prod_{j=1}^m \Gamma_{\mathbb{R}}(-2j + 1)^{-1} & \text{if } n = 2m + 1 \text{ is odd.} \end{cases} \end{aligned}$$

Therefore we have

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = 2^\beta \Delta_{G_1} C_0 \mathcal{P}_{\pi_1, \pi_0}(1/2) \prod_{v \in S} \alpha_v(\varphi_{1,v}, \varphi_{0,v}),$$

where

$$\beta = \begin{cases} -md & \text{if } n = 2m \text{ is even,} \\ -2md & \text{if } n = 2m + 1 \text{ is odd.} \end{cases}$$

Note that the integer β depends on the number of bad places.

12 Calculation for the Real Place

In this section, we carry out the calculation of the archimedean local integrals which appeared in sections 7, 9, and 10. Every algebraic group is defined over \mathbb{R} in this section.

We first consider the case $G_0 = \text{SO}(2, 1) \simeq \text{PGL}_2(\mathbb{R})$. The (topological) identity component of G_0 is denoted by $G_0(\mathbb{R})^0$. Note that $G_0(\mathbb{R})^0 \simeq \text{SL}_2(\mathbb{R})/\{\pm 1\}$. The image of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\text{PGL}_2(\mathbb{R})$ is denoted by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right]$. The maximal compact subgroup $\text{O}(2)/\{\pm 1\} \subset \text{PGL}_2(\mathbb{R})$ is denoted by \mathcal{K} . Put $\mathcal{K}^0 = \text{SO}(2)/\{\pm 1\} \subset \mathcal{K}$. The Haar measure dk on \mathcal{K}^0 is such that the total measure is 1. By Iwasawa decomposition, an element $g \in G_0(\mathbb{R})^0$ can be uniquely written as

$$g = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} k,$$

$t, n \in \mathbb{R}, k \in \mathcal{K}^0$. We choose a Haar measure dg on $G_0(\mathbb{R})^0$ such that dg/dk induces the measure $y^{-2} dx dy$ on the upper half plane $\mathfrak{H}_1 \simeq G_0(\mathbb{R})^0/\mathcal{K}^0$. Note that $dg = 2 dt dn dk$. The Haar measure dg can be naturally extended to $G_0(\mathbb{R})$. We put

$$A^+ = \left\{ \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \mid t \geq 0 \right\}.$$

We consider the map

$$\begin{aligned} \mathcal{K}^0 \times A^+ \times \mathcal{K}^0 &\longrightarrow G_0(\mathbb{R})^0 \\ \left(k, \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, k' \right) &\longmapsto k \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} k'. \end{aligned}$$

By Cartan decomposition, this map is bijective outside the boundary of A^+ . It is well-known (e.g. [Hel, Th. 5.8]) that

$$dg = C \cdot \sinh(2t) dk dt dk'$$

for some constant $C > 0$. Let $A(T)$ be the area of the small disc with radius T and center $\sqrt{-1} \in \mathfrak{H}_1$. Then we have $A(T) \sim C \int_0^{T/2} \sinh(2t) dt$ when $T \rightarrow 0$, and so we have $C = 4\pi$.

Let τ_j be the (limit of) discrete series representation of $\text{PGL}_2(\mathbb{R})$ with minimal weight $\pm \kappa_j$. Let Φ_j be the matrix coefficient of $\tau_{j,\infty}$ with respect to the lowest weight vector with norm 1. Then the support of Φ_j is contained in $G_0(\mathbb{R})^0$ and

$$\Phi_j \left(\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \right) = \cosh(t)^{-\kappa_j}.$$

(See Knapp [Kn, p. 89].)

Proof of Proposition 7.2. Let $\varphi_{1,\infty}$ and $\varphi_{0,\infty}$ be as in Proposition 7.2. Then we have

$$\begin{aligned} I(\varphi_{1,\infty}, \varphi_{0,\infty}) &= 4\pi \int_0^\infty \cosh(t)^{-2\kappa_3} \sinh(2t) dt \\ &= 4\pi(\kappa_3 - 1)^{-1}. \end{aligned}$$

For the latter part of the proposition,

$$\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) = \Delta_{G_{1,\infty}}^{-1} \mathcal{P}_{\pi_{1,\infty}, \pi_{0,\infty}}(1/2)^{-1} I(\varphi_{1,\infty}, \varphi_{0,\infty}) = 2. \quad \square$$

Next, we consider the case $G_0 = \text{SO}(2, 2)$. Put

$$\text{GL}_2^{(2)} = \{(h_1, h_2) \in \text{GL}_2 \times \text{GL}_2 \mid \det h_1 = \det h_2\}.$$

Then, we have $\text{SO}(2, 2) \simeq \text{GL}_2^{(2)}(\mathbb{R})/\mathbb{R}^\times$. We denote the image of $(h_1, h_2) \in \text{GL}_2^{(2)}(\mathbb{R})$ in $\text{SO}(2, 2)$ by $[h_1, h_2]$. Put

$$\begin{aligned} A &= \left\{ \left[\begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{-t_1} \end{pmatrix}, \begin{pmatrix} e^{t_2} & 0 \\ 0 & e^{-t_2} \end{pmatrix} \right] \mid t_1, t_2 \in \mathbb{R} \right\}, \\ N &= \left\{ \left[\begin{pmatrix} 1 & n_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & n_2 \\ 0 & 1 \end{pmatrix} \right] \mid n_1, n_2 \in \mathbb{R} \right\}, \\ \mathcal{K} &= \{[k_1, k_2] \mid k_1, k_2 \in \text{O}(2), \det k_1 = \det k_2\}. \end{aligned}$$

For each $(t_1, t_2) \in \mathbb{R}^2$, we put

$$m(t_1, t_2) = \left[\begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{-t_1} \end{pmatrix}, \begin{pmatrix} e^{t_2} & 0 \\ 0 & e^{-t_2} \end{pmatrix} \right].$$

The connected component $\text{SO}(2, 2)^0$ is equal to the image of $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$. Put $\mathcal{K}^0 = \mathcal{K} \cap \text{SO}(2, 2)^0$. Then we have an Iwasawa decomposition $\text{SO}(2, 2)^0 = AN\mathcal{K}^0$. Then $\text{SO}(2, 2)^0/\mathcal{K}^0$ can be identified with $\mathfrak{H}_1 \times \mathfrak{H}_1$. The Haar measure dk on \mathcal{K}^0 is the Haar measure such that the total volume is 1. We choose a Haar measure dg on $\text{SO}(2, 2)^0$ such that the induced measure dg/dk on $\mathfrak{H}_1 \times \mathfrak{H}_1$ is equal to $y_1^{-2}y_2^{-2} dx_1 dx_2 dy_1 dy_2$. Then $dg = 4 dt_1 dt_2 dn_1 dn_2 dk$. The Haar measure dg can be naturally extended to $G_0(\mathbb{R}) = \text{SO}(2, 2)$. Put $A^+ = \{m(t_1, t_2) \mid t_1, t_2 \geq 0\}$. Consider the map

$$\begin{aligned} \lambda : \mathcal{K}^0 \times A^+ \times \mathcal{K}^0 &\longrightarrow \text{SO}(2, 2)^0 \\ (k, m(t_1, t_2), k') &\longmapsto k \cdot m(t_1, t_2) \cdot k'. \end{aligned}$$

Let ∂A^+ be the boundary of A^+ . If $g \in G_0(\mathbb{R})^0$ is not in the image of ∂A^+ , then $\lambda^{-1}(g)$ consists of two elements. In terms of the map λ , we have

$$\begin{aligned} \int_{G_0(\mathbb{R})^0} f(g) dg &= 16\pi^2 \int_{\mathcal{K}^0 \times A^+ \times \mathcal{K}^0} f(\lambda(k, m(t_1, t_2), k')) \sinh(2t_1) \sinh(2t_2) dk dt_1 dt_2 dk' \end{aligned}$$

for any integrable function f on $G_0(\mathbb{R})^0$.

Proof of Proposition 9.1. We need to calculate the matrix coefficient of $\varphi_{1,\infty} \in \pi_{1,\infty}$. In fact, it is enough to consider the pullback of the matrix coefficient by the map $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \rightarrow \text{SO}(2, 2) \subset \text{SO}(3, 2)$, since A^+ is contained in the image of this map. Note that the image of $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$ is contained in the identity

component $\mathrm{SO}(3, 2)^0 = \mathrm{Sp}_2(\mathbb{R})/\{\pm 1\}$. The restriction of $\pi_{1, \infty}$ is a direct sum of a holomorphic discrete series and an anti-holomorphic discrete series. Since the holomorphic discrete series is a lowest-weight representation, its pullback to $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ is a direct sum of lowest-weight representations. We denote τ_λ the holomorphic discrete series of $\mathrm{SL}_2(\mathbb{R})$ with lowest weight λ . Since the lowest weight $(\kappa + 1, \kappa + 1)$ occurs with multiplicity one, the summand contains $\tau_{\kappa+1} \boxtimes \tau_{\kappa+1}$ exactly once, and the other summands are of the form $\tau_{\lambda_1} \boxtimes \tau_{\lambda_2}$, where $\lambda_1, \lambda_2 \geq \kappa + 1$ and $(\lambda_1, \lambda_2) \neq (\kappa + 1, \kappa + 1)$. (In fact, the precise decomposition of the restriction is known in this case.) Therefore the value of the matrix coefficient at $m(t_1, t_2) \in A^+$ is equal to $\cosh(t_1)^{-\kappa-1} \cosh(t_2)^{-\kappa-1}$.

It follow that

$$\begin{aligned} I(\varphi_{1, \infty}, \varphi_{0, \infty}) &= 16\pi^2 \left(\int_0^\infty \cosh(t)^{-2\kappa-2} \sinh(2t) dt \right)^2 \\ &= 16\pi^2 / \kappa^2, \\ \alpha_\infty(\varphi_{1, \infty}, \varphi_{0, \infty}) &= \Delta_{G_{1, \infty}}^{-1} \mathcal{P}_{\pi_{1, \infty}, \pi_{0, \infty}} (1/2)^{-1} I(\varphi_{1, \infty}, \varphi_{0, \infty}) \\ &= 4\pi. \end{aligned} \quad \square$$

Now, we consider the case $G_0 = \mathrm{SO}(3, 2) = \mathrm{GSp}_2(\mathbb{R})/\mathbb{R}^\times$. We denote the image of $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_2(\mathbb{R})$ in $G_0(\mathbb{R})$ by $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$. Put

$$\begin{aligned} A &= \left\{ \left[\begin{array}{cc|cc} e^{t_1} & 0 & & 0 \\ 0 & e^{t_2} & & 0 \\ \hline 0 & & e^{-t_1} & 0 \\ & & 0 & e^{-t_2} \end{array} \right] \middle| t_1, t_2 \in \mathbb{R} \right\}, \\ N' &= \left\{ \left[\begin{array}{cc|cc} 1 & n'_1 & & 0 \\ 0 & 1 & & 0 \\ \hline 0 & & 1 & 0 \\ & & -n'_1 & 1 \end{array} \right] \middle| n'_1 \in \mathbb{R} \right\}, \\ N'' &= \left\{ \left[\begin{array}{cc|cc} & & n''_{11} & n''_{12} \\ & & n''_{12} & n''_{22} \\ \hline \mathbf{1}_2 & & & \\ 0 & & & \mathbf{1}_2 \end{array} \right] \middle| n''_{11}, n''_{12}, n''_{22} \in \mathbb{R} \right\}, \\ \mathcal{K}^0 &= \left\{ \left[\begin{array}{cc|c} A & B & \\ \hline -B & A & A + \sqrt{-1}B \in \mathrm{U}(2) \end{array} \right] \right\}. \end{aligned}$$

Then the topological identity component $G_0(\mathbb{R})^0 = \mathrm{SO}(3, 2)^0$ has an Iwasawa decomposition $G_0(\mathbb{R})^0 = AN\mathcal{K}^0$, where $N = N'N''$. Note that $G_0(\mathbb{R})^0/\mathcal{K}^0$ can be identified with \mathfrak{H}_2 . We take the Haar measures dk on \mathcal{K}^0 with the total volume 1. We choose the Haar measure dg of $G_0(\mathbb{R})^0$ such that the induced measure dg/dk is equal to $(\det Y)^{-3} dX dY$. Then we have

$$dg = 4 dt_1 dt_2 dn'_1 dn''_{11} dn''_{12} dn''_{22} dk.$$

The Haar measure dg can be naturally extended to $G_0(\mathbb{R})$. Put $\mathfrak{a} = \mathrm{Lie}(A)$. Then

\mathfrak{a} can be identified with \mathbb{R}^2 and we put

$$m(t_1, t_2) = \left[\begin{array}{cc|cc} e^{t_1} & 0 & & 0 \\ 0 & e^{t_2} & & \\ \hline & & e^{-t_1} & 0 \\ 0 & & 0 & e^{-t_2} \end{array} \right]$$

for each $(t_1, t_2) \in \mathbb{R}^2 \simeq \mathfrak{a}$. The positive chamber A^+ is defined by $A^+ = \{m(t_1, t_2) \in A \mid t_1 \geq t_2 \geq 0\}$. Then the map

$$\begin{aligned} \lambda : \mathcal{K}^0 \times A^+ \times \mathcal{K}^0 &\longrightarrow \mathrm{SO}(3, 2)^0 \\ (k, m(t_1, t_2), k') &\longmapsto k \cdot m(t_1, t_2) \cdot k' \end{aligned}$$

is a double covering outside the boundary of A^+ . In terms of this map, we have (cf. [Hel, Th. 5.8])

$$dg = C \sinh(2t_1) \sinh(2t_2) \sinh(t_1 - t_2) \sinh(t_1 + t_2) dk dt_1 dt_2 dk'$$

for some positive constant $C > 0$.

The constant C can be calculated as follows. We recall the argument of [Hel, Ch. I, Th. 5.8]. We shall calculate the Jacobian of the induced map

$$\bar{\lambda} : \mathcal{K}^0 \times A^+ \longrightarrow G_0(\mathbb{R})^0 / \mathcal{K}^0 \simeq AN$$

at $(k, m(t_1, t_2)) \in \mathcal{K}^0 \times A^+$. Let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be the Cartan decomposition of $\mathfrak{g} = \mathrm{Lie}(\mathrm{SO}(3, 2)^0)$. Then the tangent space of $\mathcal{K}^0 \times A^+$ at $(k, m(t_1, t_2))$ can be identified with $\mathfrak{k} + \mathfrak{a}$ by left translation. Let Σ^+ be the set of positive roots for $(G_0(\mathbb{R})^0, A)$. Then for each $\alpha \in \Sigma^+$, we put

$$\mathfrak{k}_\alpha = \{T \in \mathfrak{k} \mid \mathrm{ad}((x_1, x_2))^2 T = \alpha((x_1, x_2))^2 T \text{ for all } (x_1, x_2) \in \mathfrak{a}\}.$$

Then $\dim \mathfrak{a}_\alpha = 1$ for any $\alpha \in \Sigma^+$. Choose a non-zero vector $T_\alpha \in \mathfrak{k}_\alpha$ for each $\alpha \in \Sigma^+$. For example, we can choose

$$\begin{aligned} T_{\varepsilon_1 - \varepsilon_2} &= \left(\begin{array}{cc|cc} 0 & 1 & & 0 \\ -1 & 0 & & \\ \hline & & 0 & 1 \\ 0 & & -1 & 0 \end{array} \right), & T_{2\varepsilon_1} &= \left(\begin{array}{cc|cc} & & 1 & 0 \\ & & 0 & 0 \\ \hline -1 & 0 & & \\ 0 & 0 & & 0 \end{array} \right), \\ T_{\varepsilon_1 + \varepsilon_2} &= \left(\begin{array}{cc|cc} & & 0 & 1 \\ & & 1 & 0 \\ \hline 0 & -1 & & \\ -1 & 0 & & 0 \end{array} \right), & T_{2\varepsilon_2} &= \left(\begin{array}{cc|cc} & & 0 & 0 \\ & & 0 & 1 \\ \hline 0 & 0 & & \\ 0 & -1 & & 0 \end{array} \right). \end{aligned}$$

For each $\alpha \in \Sigma^+$,

$$U_\alpha = \alpha((t_1, t_2))^{-1} \mathrm{ad}((t_1, t_2))(T_\alpha)$$

belongs to \mathfrak{p} , and does not depend on $(t_1, t_2) \in \mathfrak{a}$. Note that

$$\begin{aligned} U_{\varepsilon_1 - \varepsilon_2} &= \left(\begin{array}{cc|cc} 0 & 1 & & 0 \\ 1 & 0 & & \\ \hline & & 0 & 1 \\ 0 & & 1 & 0 \end{array} \right), & U_{2\varepsilon_1} &= \left(\begin{array}{cc|cc} & & 1 & 0 \\ & & 0 & 0 \\ \hline 1 & 0 & & \\ 0 & 0 & & 0 \end{array} \right), \\ U_{\varepsilon_1 + \varepsilon_2} &= \left(\begin{array}{cc|cc} & & 0 & 1 \\ & & 1 & 0 \\ \hline 0 & 1 & & \\ 1 & 0 & & 0 \end{array} \right), & U_{2\varepsilon_2} &= \left(\begin{array}{cc|cc} & & 0 & 0 \\ & & 0 & 1 \\ \hline 0 & 0 & & \\ 0 & 1 & & 0 \end{array} \right). \end{aligned}$$

Then

$$T_\alpha(\alpha \in \Sigma^+), \quad (1, 0), (0, 1) \in \mathfrak{a},$$

make up a basis of $\mathfrak{k} + \mathfrak{a}$, and

$$U_\alpha(\alpha \in \Sigma^+), \quad (1, 0), (0, 1) \in \mathfrak{a},$$

make up a basis of \mathfrak{p} . By the proof of [Hel, Ch. I, Th. 5.8],

$$|\det(d\bar{\lambda}_{(k,m(t_1,t_2))})| = \prod_{\alpha \in \Sigma^+} \sinh(\alpha(t_1, t_2))$$

with respect to these basis.

Let ω_α ($\alpha \in \Sigma^+$) be the basis of the space of left-invariant 1-forms on \mathcal{K}^0 dual to T_α ($\alpha \in \Sigma^+$). Then it is easy to check that

$$\int_{\mathcal{K}^0} \left| \bigwedge_{\alpha \in \Sigma^+} \omega_\alpha \right| = 2\pi^3.$$

On the other hand, the dual basis of

$$(1, 0), (0, 1) \in \mathfrak{a}, \quad U_\alpha(\alpha \in \Sigma^+)$$

induces

$$\frac{1}{16} dt_1 dt_2 dn'_1 dn''_{11} dn''_{12} dn''_{22}$$

on $AN \simeq G_0(\mathbb{R})^0/\mathcal{K}^0$. It follows that $C = 64\pi^3$.

Proof of Proposition 10.2. As in the proof of Proposition 9.1, the value of the matrix coefficient $\langle \pi_{1,\infty}(g_0)\varphi_{1,\infty}, \varphi_{1,\infty} \rangle$ at $g_0=m(t_1, t_2)$ is equal to $\cosh(t_1)^{-\kappa-1} \cosh(t_2)^{-\kappa-1}$. It follows that

$$\begin{aligned} I(\varphi_{1,\infty}, \varphi_{0,\infty}) &= 64\pi^3 \int_{t_1 \geq t_2 \geq 0} \cosh(t_1)^{-2\kappa-2} \cosh(t_2)^{-2\kappa-2} \\ &\quad \times \sinh(2t_1) \sinh(2t_2) \sinh(t_1 + t_2) \sinh(t_1 - t_2) dt_1 dt_2 \\ &= 64\pi^3 \int_0^\infty \int_0^\infty \cosh(x + y)^{-2\kappa-2} \cosh(y)^{-2\kappa-2} \\ &\quad \times \sinh(2x + 2y) \sinh(2y) \sinh(x + 2y) \sinh(x) dx dy. \end{aligned}$$

By using the formulas

$$\sinh(2a) = 2 \sinh(a) \cosh(a),$$

$$\sinh(a + b) \sinh(a - b) = \cosh^2(a) - \cosh^2(b),$$

one can show that the integral $I(\varphi_{1,\infty}, \varphi_{0,\infty})$ is equal to

$$\begin{aligned} &256\pi^3 \int_0^\infty \cosh(y)^{-2\kappa-1} \sinh(y) \\ &\quad \times \int_0^\infty \cosh(x + y)^{-2\kappa-1} \sinh(x + y) [\cosh^2(x + y) - \cosh^2(y)] dx dy \\ &= 256\pi^3 \int_0^\infty \cosh(y)^{-2\kappa-1} \sinh(y) \\ &\quad \times \left\{ \left[-\frac{u^{-2\kappa+2}}{2\kappa - 2} \right]_{u=\cosh(y)}^\infty - \cosh^2(y) \left[-\frac{u^{-2\kappa}}{2\kappa} \right]_{u=\cosh(y)}^\infty \right\} dy \end{aligned}$$

$$\begin{aligned}
&= \frac{128\pi^3}{\kappa(\kappa-1)} \int_0^\infty \cosh(y)^{-4\kappa+1} \sinh(y) dy \\
&= \frac{64\pi^3}{\kappa(\kappa-1)(2\kappa-1)}.
\end{aligned}$$

Since $\Delta_{G_{1,\infty}} = \Gamma_{\mathbb{R}}(2)\Gamma_{\mathbb{R}}(4)^2 = \pi^{-5}$, we have $\alpha_\infty(\varphi_{1,\infty}, \varphi_{0,\infty}) = -4\pi$. \square

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