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SINGULARITIES, EXPANDERS AND TOPOLOGY OF MAPS. PART 1: HOMOLOGY VERSUS VOLUME IN THE SPACES OF CYCLES

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Abstract. We find lower bounds on the topological complexity of the critical (values) sets $\Sigma(F) \subset Y$ of generic smooth maps $F : X \to Y$, as well as on the complexity of the fibers $F^{-1}(y) \subset X$ in terms of the topology of X and Y, where the relevant topological invariants of X are often encoded in the geometry of some Riemannian metric supported by X.

Contents

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1 Corollaries and Background

The Morse theory tells you that the cardinality $N_{\text{crit}} = N_{\text{crit}}(F)$ of the critical value set $\Sigma = \Sigma(F) \subset \mathbb{R}$ of a generic smooth function on a closed n-dimensional manifold, $F : X \to \mathbb{R}$, is bounded from below by the sum of the Betti numbers of X,

$$N_{\text{crit}} \geq \big|H_*(X)\big|_{\mathbb{F}} =_{\text{def}} \sum_{i=0,1,\dots,n} \text{rank}\big(H_i(X;\mathbb{F})\big)\,,$$

where, recall, a point $x \in X$ is called *singular* and the value (point) $F(x) \in \mathbb{R}$ is called *critical* for F if the differential of F vanishes at x, and where the homology groups $H_i(X) = H^i(X;\mathbb{F})$ may be taken with any coefficient field $\mathbb{F}$.

If F ranges in $\mathbb{R}^m$ with $m > 1$, then the critical set (see 1.1) is a singular subvariety $\Sigma = \Sigma(F) \subset \mathbb{R}^m$ of codimension one and we introduce the following two numerical invariants of Σ similar to N_{crit}.

• *The self-crossing number* $N_m = N_m(\Sigma) = N_m(F)$ of the points in Σ of multiplicity m (see 1.1). For example, $N_1 = N_{\text{crit}}$ for Morse functions; more generally, if Σ equals the image of a *smoothly immersed* hypersurface $\hat{\Sigma}$ this is the number of the m-multiple self-intersection points of $\hat{\Sigma}$;

•• *The depth of* $\Sigma \subset \mathbb{R}^m$ (see 1.2). This is the minimal number N such that every point $y \in \mathbb{R}^m$ can be moved to infinity by a path which intersect Σ at most N times. For example if $\Sigma \subset \mathbb{R}$ is a finite set (e.g. of critical values of a Morse function) of cardinality $2N$ then $\text{dep}(\Sigma) = N$.

Below are examples of our "higher codimensional Morse inequalities", where the (mostly standard) definitions are detailed in 1.1 and 1.2 after the statements.

Let $F : X \to \mathbb{R}^2$ be a generic (C^∞-stable in the present case) smooth map, such that the restriction of F to the (1-dimensional) singularity $\hat{\Sigma} \subset X$ of F is an immersion. Then the number $N_2 = N_2(\Sigma)$ of the *self-crossings (double points) of the critical set* $\Sigma = \Sigma(F) = F(\hat{\Sigma}) \subset \mathbb{R}^2$, *satisfies,*

(A) $$N_2(\Sigma) \geq \tfrac{1}{2}\big|H_*(X)\big|_{\mathbb{F}} - N_{\mathrm{comp}}(\Sigma)$$

where N_{comp} *denotes the number of the connected components of* $\Sigma \subset \mathbb{R}^2$.

This is derived in 2.1 from the classical Morse inequality by composing F with a suitable function $f : \mathbb{R}^2 \to \mathbb{R}$, where f is constructed with the aid of the *Poincaré–Benedicson theorem*.

If X *is the Cartesian product of two connected manifolds,* $X = X_1 \times X_2$, *and* $F : X \to \mathbb{R}^2$ *is a generic smooth map, then the depth of the critical set* $\Sigma \subset \mathbb{R}^2$ *of* F *satisfies*

(B) $$\mathrm{dep}(\Sigma) \geq \tfrac{1}{6}\min\big(|H_*(X_1)|_{\mathbb{F}}, |H_*(X_2)|_{\mathbb{F}}]\big)\,.$$

This is proved in 2.2 by applying a *Lyusternik–Schnirelmann* type lower bound to the homologies of the levels (fibers) of continuous maps of X to certain (sub)trees in $\mathbb{R}^2$.

Let X *be a closed* n*-dimensional manifold with a metric of constant negative curvature. If* $n = 3$, *then there exists a sequence of integers* $s \to \infty$ *and a sequence of* s*-sheeted coverings* $X_s \to X$ *such that the critical set* Σ *of every generic smooth map of each* X_s *to the plane,* $F : X_s \to \mathbb{R}^2$, *satisfies*

(C) $$\mathrm{dep}(\Sigma) \geq \varepsilon s$$

and

(D) $$N_2(\Sigma) \geq \varepsilon s^2$$

for some $\varepsilon = \varepsilon(X) > 0$.

These inequalities are proved in 6.3 by studying certain families of surfaces $G(y)$ in *expander coverings* X_s of X, such that $\max_y \mathrm{area}(G(y)) \geq \mathrm{const}\cdot s$. Then these surfaces are simultaneously minimized in some way which makes

$$\begin{aligned}\max_y \mathrm{genus}(G(y)) &\geq \max_y \mathrm{genus}\big(G_{\min}(y)\big)\\ &\geq \mathrm{const}'\cdot\max_y \mathrm{area}\big(G_{\min}(y)\big) \geq \max_y \mathrm{area}(G(y))\,.\end{aligned}$$

This, applied to levels of maps of X_s to graphs Y, yields **(C)**, while the proof of **(D)** also relies on the *separator theorem* for planar graphs (see 6.6).

Our argument also delivers

lower bounds on the (Heegard) genera of 3*-manifolds in terms of the volumes and the spectra of (the Laplacians on) their hyperbolic Thurston components*,

and similarly

lower bounds on the crossing numbers of (diagrams of) links and knots in S^3 (see 6.3, 6.4).

If $n > 3$, we prove similar but much weaker inequalities for maps $X \to \mathbb{R}^m$, $m = n-1$,

(F) $\quad \operatorname{dep}(\Sigma) \geq \varepsilon s^\alpha \quad$ for $1 > \alpha = \alpha(n) > 0$ and $N_m(\Sigma) \geq \varepsilon s$,

where the first inequality is proved for coverings X_s with $\operatorname{Inj}\operatorname{Rad}(X_s) \geq \text{const} \cdot \log(s)$ similarly to **(C)** while the second one is obtained in 3.3 with the bound $\|X_s\|_\Delta \leq \text{const} \cdot N_m$ on the *simplicial volume* $\|X_s\|_\Delta$, where one knows that $\|X_s\|_\Delta = s\|X\||_\Delta$.

In the course of the proof of **(C)**–**(F)**, we shall introduce certain topological invariants similar to the simplicial volume which allow, for instance, new examples of closed hyperbolic n-manifolds, $n = 3, 4, \ldots$, say X_0^n and X_i^n, $i = 1, 2, \ldots$, where $\operatorname{vol}(X_i^n) \to \infty$ *for* $i \to \infty$ *and where no* X_i^n *admits a map to* X_0^n *of non-zero degree.*

Acknowledgments. I want to thank Larry Guth who explained his results to me, some of them contradicting (incorrect) statements in the first draft of this paper. I am also thankful to the anonymous referees who pointed out many errors and inconsistencies in the paper and suggested improvements of the exposition, and to Miriam Hercberg for her help in proofreading the paper.

1.1 Genericity, singularities, critical sets, folds and their self-crossing numbers N_m.

DEFINITION OF $\hat{\Sigma}$ AND Σ. The *singularity* $\hat{\Sigma}(F) \subset X$ of a smooth map $F : X \to Y$, for $\dim Y \leq \dim X + 1$, is the set of the points in X where the rank of the differential of F is $< m \dim Y$, while the image of $\hat{\Sigma}$, called the *critical set*, is denoted by $F(hat\Sigma(F)) = \Sigma(F) \subset Y$.

Self-crossing of $\Sigma = \Sigma(F)$. If F is *generic*, i.e. is a member of some open dense subset in the space of smooth maps $X \to Y$, then both $\hat{\Sigma}$ and Σ have dimensions equal to $\dim(Y) - 1$, where the map F sends $\hat{\Sigma}$ onto Σ finite-to-one.

Both sets may be rather singular but Σ is more complicated than $\hat{\Sigma}$ due to (*self-*)*crossing*, also called *self-intersection*, of the singularity in Y where the map F, which is finite-to-one on but *not* necessarily one-to-one on $\hat{\Sigma}$: the cardinality of the pullback in $\hat{\Sigma}$ of each $y \in \Sigma$ can be anything between 1 and $m \dim(Y)$ for generic maps F.

The number $N_m = N_m(F)$ of points of the maximal multiplicity (of self-intersections of $\hat{\Sigma}$ mapped by F to Y) i.e. of the points with m pullbacks in $\hat{\Sigma}$ is finite for generic maps of compact manifolds X. If $m = 1$ this is just the number of critical points of F.

> *Our basic problem is finding lower bounds on N_m in terms of suitable topological invariants of X.*

Also define the *local multiplicity* $\mu(\Sigma, y)$, $y \in \Sigma$ as the multiplicity of a generic smooth map $Y \to R^{m-1}$ on Σ near y and observe that the m-multiple points of the previous definition have local multiplicity m for generic maps F.

The local topology of F is constant on X away from $\hat{\Sigma}$: it is equivalent to the projection $\mathbb{R}^n = \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^m$ for $n \dim X \geq m \dim Y$ (if $m = n+1$ then $\Sigma = X$) and, if X is a compact manifold without boundary (more generally, if F is a proper map), then the global topology of the *fibers* (*levels*) $F^{-1}(y) \subset X$, $y \in Y$, can change only at the critical points $y \in \Sigma \subset Y$, where the rank of $H_*(F^{-1}(y))$ changes by ± 2 as you transversally cross Σ at a simple (i.e. multiplicity 1) point in the image of the *folding locus* (*stratum*) $\hat{\Sigma}_1 \subset \hat{\Sigma}$ (see below) of F.

The singularity $\hat{\Sigma}$ of a generic map F, contains the *principal stratum* $\hat{\Sigma}_1 \subset \hat{\Sigma}$, such that the complement $\hat{\Sigma} \setminus \hat{\Sigma}_1 \subset \hat{\Sigma}$ is a closed subset of codimension (at least) 1 in $\hat{\Sigma}$ and such that F *folds* along $\hat{\Sigma}_1 \subset X$ according to the following definitions.

Equidimensional folded maps. An equidimensional map $X \to Y$ is called *(purely) folded* if it is locally diffeomorphic away from some smooth hypersurface $\hat{\Sigma}_1 \subset X$, where the restriction of F to $\hat{\Sigma}_1$ is a *smooth immersion* (i.e. it has non-vanishing differential) and where the map is locally two-to-one at the points in X that are close to $\hat{\Sigma}_1$ but not lying in $\hat{\Sigma}_1$: every small ball in X centered in $\hat{\Sigma}_1$ is folded by F along $\hat{\Sigma}_1$ to a small half-ball in Y. Finally, one adds an extra technical condition (that ensures the smooth stability of foldings): *the second differential of F does not vanish on the kernel of the first differential.* (This condition does not affect the topology of the map; also it can always be achieved by a small perturbation.)

Every folded map between n-dimensional manifolds is locally equivalent (by local diffeomorphisms of both manifolds) to $(x_1, x_2, x_3, \ldots, x_n) \mapsto (x_1^2, x_2, x_3, \ldots, x_n)$.

The set of folded maps is open but not dense in the space of all smooth maps: a small smooth perturbation of a purely folded map remains purely folded due to the above technical condition; but there are other classes of generic smooth maps that have more drastic singularities, such as the *cuspidal submanifold* $\hat{\Sigma}_1' \subset \hat{\Sigma}$, where the map of $\hat{\Sigma} \to Y$ fails to be a *smooth* immersion (but remains a topological immersion, i.e. it is locally one-to-one on $\hat{\Sigma}$).

Self-transversal foldings. Every folded map can be smoothly perturbed, such that all self-intersections of it become *transversal* and the fold becomes a smooth immersed hypersurface with transversal self-crossing in Y. The multiplicity of the self-transversal fold intersection at y equals the local multiplicity $\mu(\Sigma, y)$.

Foldings with a drop in dimension. The notion of a folded map is defined whenever $\dim(Y) = m \leq n = \dim X$ as follows.

If $Y = \mathbb{R}$, then "folding" is equivalent to "Morse": the function (map) F is locally equivalent, by a local diffeomorphism of X, to a non-singular quadratic function on $\mathbb{R}^n$; in general, a folded map is locally equivalent, by local diffeomorphisms of X and Y, to the Cartesian sum of the identity map on $\mathbb{R}^{m-1}$ and a Morse function on $\mathbb{R}^{n-m+1}$,

$$\mathbb{R}^n = \mathbb{R}^{m-1} \times \mathbb{R}^{n-m+1} \to \mathbb{R}^{m-1} \times \mathbb{R} = \mathbb{R}^m.$$

Thus the singularity $\hat{\Sigma} = \hat{\Sigma}_1 \subset X$ of a purely folded map is a smooth submanifold of dimension $m - 1 \dim(Y) - 1$, the restriction of F to $\hat{\Sigma}$ is a smooth immersion while the *fold* itself, i.e. the critical set of a folded map is an immersed hypersurface in Y. Generically the fold has transversal self-crossing. For instance, if $Y = \mathbb{R}$, then self-transversal foldings are just Morse functions where the transversality amounts to the simplicity of their critical values.

Product Example: Folds associated to Morse functions. Let $X_0 \subset Y$ be a smooth (co)oriented hypersurface and $F' : X' \to \mathbb{R}$ a Morse function. Then the obvious map $F : X_0 \times X' \to X_0 \times \mathbb{R} = Y_0 \subset Y$, for some split neighborhood $Y_0 \supset X_0$, folds along several smooth disjoint hypersurfaces $X_i = X_0 \times r_i \subset X_0 \times \mathbb{R} = Y_0 \subset Y$ that correspond to the critical values $r_i \in \mathbb{R}$ of F'. Observe that the fold of this F is embedded to Y, and thus, has no multiple points at all.

Maps to surfaces. Let $\dim(Y) = 2$, e.g. $Y = \mathbb{R}^2$, where X may have any dimension $n \geq 2$. One knows that $\hat{\Sigma}(F) \subset X$ of a *generic* smooth map F is a smooth curve in X, a union of several smooth circles, while its F-image $\Sigma(F) \subset Y$ is a curve that is smooth away from finitely many transversal *self-crossing points* that are the double points of the map F on $\hat{\Sigma}(F)$; also this curve may have, for generic F (generic as maps of X not of $\hat{\Sigma}$) finitely many *cuspidal points* corresponding to the vanishing points of the differential of the map F restricted to $\hat{\Sigma}(F) \subset X$. The part of this curve away from the cusps is called the *folding locus* (*folding line for* $m = 2$) of F.

Examples. (a) Take a connected 2-manifold X_0 with the boundary consisting of $g+1$ smooth circles. Then the double X of X_0 is a closed surface of genus $\geq g$ which is naturally two-to-one mapped to X_0 with the fold made of the boundary circles. Thus every surface of genus g, represented as the double of a plane domain, admits a map to the plane (i.e. to $Y = \mathbb{R}^2$) where the critical set consists of $g + 1$ disjoint (non-singular) folding circles in the plane.

(b) Represent the 2-torus minus a disc, denoted X_0, as a narrow band around the figure ∞ made of a meridian and a parallel in the torus. This X_0 admits an immersion F_0 into the plane narrowly following that of the figure ∞ that is sent to the plane with a single self-intersection point (where the meridian meets the parallel) away from the native singular point. The boundary circle of this immersion has four self-crossing points; thus the double of F_0 gives us a map F of the surface X of genus two to the plane with a single non-cuspidal folding line having four double points.

(c) The above folded map F of the closed surface X of genus 2 to $\mathbb{R}^2$ naturally extends to a smooth *regular* map of the 3-ball with two handles (that is homeomorphic to $X_0 \times [0, 1]$), denoted X_1, with the boundary identified with X, where "regular" means the absence of the singular set. Finally the double X_2 of X_1, now a *closed* 3-manifold – the 3-sphere with two handles, goes to the plane where it has the same critical set as F that is the folding line with four self-crossings.

1.2 Depth of the critical set $\Sigma(F)$ and Leray bound on homology. Assume that Y is an open connected manifold, $\Sigma \subset Y$ is compact and consider smooth paths in Y which start at the points in the complement $Y \setminus \Sigma$ and that cross Σ transversally at non-singular points, i.e. at simple points of the folding part of Σ. Denote by $\mathrm{dep}(\Sigma)$ the minimal number of crossings needed to go from every point to infinity in Y by such a path. Clearly, this depth does not exceed the number of bounded components in $Y \setminus \Sigma$, that, in turn, is bounded by $\mathrm{rank}(H^{m-1}(\Sigma))$.

This depth may serve as an alternative to N_m generalization of the number of critical points of a Morse function F, since $\mathrm{dep}(\Sigma(F))$ equals one half of the number N of the critical points of F if this number is even and $\frac{1}{2}(N+1)$ if N is odd.

The depth of Σ is easier to bound from below than N_m but $\mathrm{rank}\, H_*(X)$ is not sufficient for this purpose as is seen from the

Product Example. Let $F : X = X_0 \times X' \to X_0 \times \mathbb{R} = Y_0 \subset Y$, where Y_0 is open in Y, be associated to a Morse function $X' \to \mathbb{R}$ (see above) with N critical points. If Y is an open manifold connected at infinity and X_0 is a closed hypersurface that bounds in Y (e.g. $Y = \mathbb{R}^m$ for $m \geq 2$), then $\mathrm{dep}(\Sigma(F)) = N$ (regardless of the "topological size" of the X_0-factor of X).

On the other hand, we obtain lower bounds on this depth with several kinds of invariants that are "stronger" than $\mathrm{rank}\, H_*(X)$ and that provide lower bounds on the maximum of $\mathrm{rank}\, H_*$ of the *fibers* $X_y = F^{-1}(y)$ of F over $y \in Y$, see 2.2, 2.4.

Let us look at these more closely at more comprehensive invariants of Σ that may serve better than N_m and/or $\mathrm{dep}(\Sigma)$ for bounding the topology of X.

Strata adjacency complex $\Upsilon(F)$. The set of the connected components S of the strata of $\Sigma(F)$ make a partially ordered set, where $S_1 \prec S_2$ if S_1 is contained in the closure of S_2 and its simplicial realization, called the *(strata) adjacency complex*, $\Upsilon(F)$ carries the essential combinatorics of the singularity. Observe that $\dim(\Upsilon(F)) \leq \dim Y$ and that there is a natural map $Y \to \Upsilon(F)$, that sends each S to the star of the corresponding vertex in $\Upsilon(F)$. This map is unique up to homotopy.

The above depth of Σ is, essentially, the diameter of the 1-skeleton of $\Upsilon(F)$.

Example. If F is a purely folded map with a transversal self-crossing of the fold, then $\dim(\Upsilon(F))$ equals the maximal multiplicity of the self-crossing.

Topology of the fibers. Let $F : X \to Y$ be a self-transversal folding of a closed n-manifold and denote by $\Sigma_k \subset Y$, $k = 0, \dots, \dim(Y)$, the set of points of multiplicity k, i.e. where $\Sigma(F)$ self-intersects with multiplicity k (agreeing that $\Sigma_0 = F(X) \setminus \Sigma$). Every $\Sigma_k \subset Y$ is a smooth locally closed $(m-k)$-dimensional submanifold for $m \dim(Y)$ and the fiber $F^{-1}(y)$ over each point $y \in Y_k$ is a smooth $(n-m)$-manifold away from k isolated singular points; $F^{-1}(y)$ looks exactly like a singular level of a Morse function on an m-manifold over a critical point (value) of multiplicity k.

The map F is a fibration over every Σ_k but the topology of a fiber changes as one goes from one Σ_k to another. In fact, if a smooth path P of points y in the closure of Σ_k transversally passes through Σ_{k+1}, the fiber over y undergoes a Morse surgery where the relevant Morse function is the tautological map from $F^{-1}(P)$ to P. Thus, it follows from Morse theory that the sum of the Betti number of every non-singular as well as every singular fiber is bounded by $2\,\mathrm{dep}(\Sigma(F))$.

Remark. The homology of singular fibers of more general smooth maps may significantly exceed that of the smooth fibers. However, for generic smooth maps F there is a universal bound on this excess: when a generic fiber degenerates to a singular one, the Betti numbers may jump up by at most const_n for $n \dim(X)$. In fact, each fiber of a generic map has at most m isolated singularities and by the Thom stability theorem, the topology of a smooth map at each point in X is the same as of its s-jet where s depends only on n. Such a jet is a certain polynomial map of degree s and thus all its homological invariants are bounded in terms of $s = s_n$. (Probably, const_n can be replaced by $\mathrm{const}_m \sim m$, something like $\max_{y\in\Sigma}\mu(\Sigma, y)$.)

Leray integration over Y. Consider all connected components of all Σ_k, call them S_j, denote by $\hat{X}_j$ their pullbacks and observe that the sum of the Betti numbers of X is bounded by the sum of these of all $\hat{X}_j$, that is $\mathrm{rank}(H^*(X)) \leq \sum_j \mathrm{rank}(H^*(\hat{X}_j))$ where the cohomology may be understood over an arbitrary coefficient field. On the other hand, if an S_j is simply connected, then, by Leray spectral sequence, $\mathrm{rank}(H^*(\hat{X}_j)) \leq (\mathrm{rank}(H^*(S_j)) \cdot (\mathrm{rank}(H^*(F^{-1}(x_j)))$, $y_j \in S_j$, where, in general, this product needs to be replaced by the rank of the cohomology of S_j in the local system of the cohomology of the fibers varying over S_j. (The latter can be bounded in terms of the cohomology of finite coverings of S_j). Thus

> *the sum N_B of the Betti numbers of X is bounded by the "integral" of those of the fibers over the (co)homology of S_j. In particular, N_B is bounded by the product of* $\mathrm{dep}(\Sigma)$ *by the sum of the Betti numbers of all* Σ_k (understood in the twisted sense if there are non-trivial local systems).

This Leray bound, however, applied to functions, is no match for the Morse inequalities: it gives us $\frac{1}{2}N_{\mathrm{crit}}^2 \geq \sum \mathrm{rank}\, H_i(X)$ instead of $N_{\mathrm{crit}} \geq \sum \mathrm{rank}\, H_i(X)$.

Also the monodromy of the action of the fundamental groups of the strata on the homologies of the fibers may significantly complicate the picture. For example, take the double X_0 of the complement of a knotted torus T in the 3-space naturally folded at T. The homology of X_0 is bounded by that of T via the Alexander duality, but some finite covering X of X_0, with mildly self-intersecting immersed torus as the fold, may become homologically large.

Question. Denote by $N_\sigma = N_\sigma(\Sigma)$ the minimal number of cells in the cell decomposition of Y (or of the image $F(X) \subset Y$) that are compatible with the $\Sigma(F)$-stratification. What is the best bound on $\mathrm{rank}\, H_*(X)$ in term of N_σ? How much do we gain (if at all) with the minimal number $N_\Delta \geq N_\sigma$ of simplices of the triangulations of Y compatible with Σ?

We shall see in 2.1 that $\mathrm{rank}(H_*(X)) \leq N_\sigma$ for generic maps $X \to \mathbb{R}^2$ but it remains unclear if, in general, $\mathrm{rank}\, H_*(X) \leq \mathrm{const}(n)N$ for maps to $\mathbb{R}^{m\geq 3}$. (Apparently, there is such a linear bound with the minimal number N_{shell} of simplices in the *shellable* triangulations of Y compatible with Σ, where, observe, N_{shell} admits a universal bound by some function $\Omega_m(N_\Delta)$; yet, Ω_m cannot be bounded by a recursive function for $m \geq 4$.)

1.3 Inequalities with characteristic classes. The above indicates that one cannot bound all of the topology of X in terms of Σ for $\dim(Y) = m > 2$ and our purpose is to identify particular invariants of X properly reflected in Σ. One knows in this respect that the singularity $\hat{\Sigma}(F) \subset X$, along with the differential of F on the tangent bundle of X restricted to $\hat{\Sigma}(F)$, carries an essential information on the characteristic classes of X. Here is the classical expression of the Euler class in terms of singularities.

The second differential Q of F at a folding point $x \in X$, is a quadratic form on the $(n-m)$-dimensional kernel of the first differential, for $\dim X = n \geq m = \dim(Y)$, where Q takes values, strictly speaking, not in $\mathbb{R}$ but in the coimage of the first differential, where this coimage is conveniently identified with (co)normal bundle of $\Sigma \subset Y$. Since the index $\mathrm{ind}(Q) = \mp 1$ of Q, defined by the parity of the number of positive squares in Q, satisfies $\mathrm{ind}(Q) = (-1)^{m-n}\,\mathrm{ind}(-Q)$, this index behaves as a locally constant function on the folding part of the singularity for odd $m-n$ and it gives a co-orientation to $\Sigma \subset Y$ (i.e. orients the normal bundle) if $m-n$ is even.

(A) *Hopf formula for* $\dim(Y)$ *odd.* Let X be a closed manifold of even dimension n and $F : X \to Y$ be a purely folded map. Denote by $\hat{\Sigma}^{+}(F) \subset X$ the union of those components of $\hat{\Sigma}(F)$ where $\mathrm{ind}(Q) = +1$ and let $\hat{\Sigma}^{-}(F)$ be made of the components where $\mathrm{ind}(Q) = -1$. Then the Euler characteristics of $\Sigma^{\pm}$ satisfy,

$$\chi(X) = \chi\big(\hat{\Sigma}^{+}(F)\big) - \chi\big(\hat{\Sigma}^{-}(F)\big)\,.$$

Proof. Take a generic non-vanishing covector field φ on Y (odd-dimensional manifolds carry non-vanishing fields); compare the sum of the indices of its zeros on Σ with those of the lift of φ to X and equate these sums to the Euler characteristic according to Poincaré formula.

(B) $\dim(Y)$ *is even.* The same argument applies if Y admits a non-vanishing field (more generally, if the Euler class of Y pulls back to zero in the cohomology of X) but the right-hand side of Hopf formula needs to be modified as the Euler characteristics of all components of the singularity vanish for $\dim(\hat{\Sigma}(F))$ odd. The sum of the indices of the zeros of φ on Σ defines some kind of a *tangential degree* $\delta(\Sigma)$ depending on the co-orientation of $\hat{\Sigma}$; if $Y = \mathbb{R}^m$ this is twice the degree of the tangential (Gauss) map of $\hat{\Sigma}$ to the sphere S^{m-1}.

Corollary. *The Euler characteristics of an even-dimensional manifold X folded in $Y = \mathbb{R}^m$ along Σ, is bounded, by the k-weighted sum of the Betti numbers b_k of the subsets (strata) Σ_k, $k = 1, 2, \dots,$ of k-multiple self-intersections of Σ,*

$$|\chi(X)| \leq \sum_k k b_k\,.$$

Proof. An immersed cooriented transversally self-intersecting hypersurface Σ in an m-manifold Y decomposes into "simple cycles" represented by *almost embedded* submanifolds in Y as follows. Take some k-multiple point $y \in \Sigma$ and order the k branches of Σ at y according to the order in which they meet a line L_y directed along the sum of the co-orienting vectors. Thus Σ locally decomposes into k parts and these local decompositions obviously fit together into certain cycles C_j. These C_j are by no means smooth; yet they are all manifolds immersed into $\mathbb{R}^m$ since everywhere each of them meets L at a single point. They are not necessarily embedded, but they cannot cross each other, i.e. one can make them embedded by an arbitrarily small perturbation. Furthermore these C_j they can be easily smoothed in $\mathbb{R}^m$, such that their tangential degrees add up to that of Σ. This yields the inequality since the tangential degree of an *embedded* Euclidean odd-dimensional hypersurface, equals one half of the sum of its Betti numbers by another formula of Hopf.

EXAMPLE. If $\dim(Y) = 2$, then the C_j's are (obviously) made of $N = 2N_2 + 2N_{\mathrm{cmpt}}$ embedded circles where N_2 and N_{cmpt} denote the numbers of the self-crossings and of connected components of Σ; thus, for $Y = \mathbb{R}^2$, the bound on χ improves to $|\chi(X)| \leq N$. Moreover, if χ is non-positive, then $-\chi(X) + 2 \leq N$ since one of the circles must have positive degree. In particular, if X is a connected orientable surface of genus g then $N_2 + N_{\mathrm{cmpt}} \geq g + 1$.

REMARKS. (a) According to 1(A) (also see 2.1), the full sum of the Betti numbers of an X folded in the plane is bounded by $N = 2N_2 + 2N_{\mathrm{cmpt}}$.

(b) This inequality for surface folds in the plane was pointed out to me by Yasha Eliashberg who later told me that Larry Guth had constructed folded surfaces in the plane with the minimal numbers of the fold double points for all genera g.

(c) There is a (well-known, I guess) generalization of (A) to simplicial maps between *Eulerian manifolds* (the Euler characteristics of the links are same as for manifolds), where the index comparison is replaced by Dehn–Sommerfeld counting argument.

(d) If one is allowed to use the geometry of Σ one is able, following Hopf, to express $\chi(X)$ as the integral of the Gauss curvature of the folded part of Σ that is valid for all (not only purely folded) generic maps F; there are similar formulas for the Pontryagin numbers as well (see [Gr15, 2.1.4]).

Formulas with "crossing" invariants" of singularities. Such invariants of maps as the number $N_m(F)$ of maximal self-crossings of the fold, unlike those coming from $\hat{\Sigma}$, are rather global in nature; yet, the stratified cobordism theory developed by A. Szücs [Sz] provides some relations between these and certain characteristic classes of X.

For example, according to *the Herbert formula* [H] the number N of (maximal) self-crossings of multiplicity $2k+1$ counted with appropriate $\pm$ signs of an *immersed* oriented $4k$-dimensional manifold in $\mathbb{R}^{4k+2}$, equals $(p_1^\perp)^k[X]$ for the first normal Pontryagin class $p_1^\perp$ of X.

This provides, for an arbitrary large N, a closed (simply connected, if you wish) manifolds $X = X(N)$ of a given dimension $n = 4k$ that admits an immersion to $\mathbb{R}^{n+2}$ and where

> *every generic immersion* $X \to \mathbb{R}^{n+2}$ *necessarily has at least* N *self-intersection points of (maximal possible) multiplicity* $2k+1$.

(See [Sz] and references therein for results on maps $X^n \to Y^m$ with $m > n$.)

A result nearest to our paper (where $m < n$) is the *signature formula* by Saeki–Yamamoto [SY] (based on a detailed analysis of modifications of singular fibers of maps under cobordisms) that implies that *the number N_3 of the 3-multiple fold self-intersection points of a smooth generic map of a closed orientable 4-manifold X into a 3-manifold is bounded by $|\sigma(X)|$.*

(I am indebted to András Szücs who explained this and the Herbert formula to me.)

We shall prove in 3.3 (compare with 1(F)) (generalizing the corresponding result by Costantino–Thurston for $n = 3$ [CosT]) the inequality

$$N_m(F) \geq \text{const}(n)\|X\|_\Delta$$

for the manifolds X of dimension $n \geq 3$, say with *hyperbolic* fundamental groups, and generic maps $F : X \to \mathbb{R}^m$, where $m = n - 1$, and where $\|X\|_\Delta$ denotes the *simplicial volume* (see section 3) of X (in place of a characteristic number).

This generalizes the corresponding result Costantino–Thurston for $n = 3$ [CosT]) and provides, for a given $n \geq 3$, closed (stably parallelizable, if you wish) n-manifolds $X^n(N)$, $N = 1, 2, 3, \ldots$, where

every generic maps $X^n(N) \to \mathbb{R}^m$ for $m = n - 1$ has the number N_m of m-multiple fold self-intersection points bounded from below by N.

Notice that the simplicial volume vanishes for simply connected manifolds and it remains unclear if there are odd-dimensional simply connected manifolds X where generic maps to $\mathbb{R}^m$, $m \dim(X) - 1$, necessarily have large N_m. (My bet is on "No" with the simply connected surgery, compare below.)

On the other hand, $\|X\|_\Delta$ has an advantage over the characteristic numbers of X: the lower bound on N_m passes from X to all X' that admit maps of non-zero degree to X, since $\|X'\|_\Delta \geq \|X\|_\Delta$ in this case (see 3.1).

Yet, it remains unclear if there are similar lower bounds on N_m for $m = n$ and/or for $m < n - 1$. For example, one has the following unresolved

Topological version of Bogomolov's question. Does there exist, for every closed oriented n-manifold X_0, a closed oriented n-manifold X that admits a map $X \to X_0$ of positive degree and, at the same time, can be smoothly fibered over some Y with $\dim(Y) = n - 2$? (Bogomolov's original question concerns parametrization of *complex algebraic* manifolds X_0 by algebraic manifolds fibred by surfaces.)

On maps $X^n \to \mathbb{R}^{n+1}$. If X_0 is an *arbitrary* oriented pseudomanifold, then, by Serre theorem, a multiple of the fundamental class of the suspension of X_0 is spherical. It follows, that

there exists a smooth closed hypersurface $X \subset \mathbb{R}^{n+1}$ (with no self-intersection at all) that admits a map of positive degree to X_0.

On "simply connected" surgery (compare [Gr15, p. 52])***.*** If X is a stably parallelism *simply connected* manifold of dimension $n \geq 5$ then it can be obtained by adding handles of dimension $k \leq n/2 + 1$ to some "special" manifolds X^n_j, $j = 1, 2, \ldots, s(n) = \text{order}(\pi_{n+N}(S^N)) < \infty$, $N > n + 1$, and, if $m > k$, one, probably, can (it is straightforward for $m = n+1$) perform the corresponding surgery of some purely folded maps $X^n_j \to \mathbb{R}^m$ that transforms X^n_j to X and does not increase N_m; thus, one would obtain an upper bound for N_m by a constant const(n) for the simply connected stably parallelism X and $m > n/2 + 1$, but the actual values of const(n) (e.g. when are they > 0?) are unclear.

Something like this may work for all simply connected manifolds of dimension≥ 5 and all m. On the other hand, one may expect non-trivial lower bounds for N_m by Donaldson's (and similar) invariants of (possibly simply connected) 4-manifolds.

1.4 Construction of maps of low topological complexity. According to the Eliashberg folding theorem [E] there is no significant constrains on the topology of the *singularity* $\hat{\Sigma}(F)$ up in M except for those coming from the characteristic classes. For example, if X is *stably parallelism* (e.g. realizable by a hypersurface in the Euclidean space or being an orientable 3-manifold), then (amazingly!) it admits a purely folded map to each $\mathbb{R}^m$, $2 \leq m \leq n \dim(X)$ with $\hat{\Sigma}(F) \subset M$ consisting of $\leq 2m$ small concentric $(m-1)$-spheres in X. However, and this is the point of the present paper,

> *there are non-trivial lower bounds on the topology of the* critical set $\Sigma(F)$ *downstairs in the manifold* Y, e.g. on the number of the self-crossings of folds of n-manifolds X in the plane.

Controlled genericity and upper bounds on the topological complexity of maps. If X comes with some local geometric structure, one considers (spaces of) functions distinguished by this structure, where the topological complexity of these may be further controlled with Yomdin's *quantitative transversality theory* (see [YoG], and also in [Gr15, pp. 123,124] for a brief introduction).

Examples (compare [Ki]). (a) Let X be a simplicial polyhedron and the distinguished functions are those linear (or, rather, affine) on the simplices. Then the obvious upper bounds on the topology of maps are provided by generic simplicial maps. For example, if X contains N_m simplices of dimension $m-1$, then a generic simplex-wise linear map $X \to \mathbb{R}^m$ has at most $(N_m)^m$ of the m-multiple intersection points of these simplices in $\mathbb{R}^m$, where these points play the role of the m-multiple fold self-intersection points in the smooth category.

Such upper bounds provide us with guidelines for possible lower bounds, where the latter depend on particular combinatorial classes of X and categories of maps (simplicial, piecewise smooth, continuous, etc.) as we shall see in Part 2 of this paper,

(b) If X carries a real algebraic structure then the distinguished functions are (generic) polynomials of given degree and these are sometimes optimal for geometric/topological lower bounds (compare 5.1, 5.2).

(c) Let X be a closed Riemannian n-manifold with 1-*bounded geometry*: the sectional curvatures and the injectivity radius are bounded by $|\mathrm{curv}(X)| \leq 1$ and $\mathrm{Inj\,Rad}(X) \geq 1$. Every such X admits a triangulation with at most $N \leq \mathrm{const}(n)\,\mathrm{vol}(X)$ simplices and also supports "many" "topologically simple" smooth functions.

Indeed, take a covering of X by balls B_i of radii $1/2$, such that the multiplicity of the covering is bounded by $\mathrm{const}(n)$ and let f_{ij} be some standard C^∞-functions supported in the concentric unit balls and equal local coordinates x_j on the $2/3$-balls. Then, for example, m generic linear combinations of f_{ij} make a map $F : X \to \mathbb{R}^m$ such that

the number N_m of the m-multiple self-crossing points of the fold of F is bounded by

$$N_m(F) \le \mathrm{const}'(n)\,\mathrm{vol}(X)^m,$$

while the Betti numbers of all fibers X_y of F and the depth of the critical set are bounded by

$$\mathrm{rank}\big(H_*(X_y)\big) \le 2\,\mathrm{dep}(\Sigma(F)) \le \mathrm{const}''(n)\,\mathrm{vol}(X)\,.$$

This shows that our inequalities 1(C) and 1(D) are qualitatively sharp but it remains unclear if there are similar results for manifolds X of dimensions $n > 3$.

On the other hand, we shall see in Part 2 of the paper that

there exist, for every $n \ge 6$, closed simply connected Riemannian n-manifolds X with 1-bounded geometries and arbitrarily large volumes, where the homologies of the fibers (levels) of all Morse functions $F : X \to \mathbb{R}$ satisfy

$$\max_{y\in\mathbb{R}} \mathrm{rank}\big(H_*(F^{-1}(y))\big) \ge \varepsilon(n)\,\mathrm{vol}(X)$$

for some positive constant $\varepsilon(n) > 0$.

Notice in this respect that every closed simply connected 5-manifold decomposes into the connected sums of manifolds X_i with a uniform bound on the homology of X_i by the *Smale–Barden theorem*. Thus,

every simply connected 5-manifold X admits a Morse function $F : X \to \mathbb{R}$, where $\mathrm{rank}(H_*(F^{-1}(y)) \le const < 100$ for all $y \in \mathbb{R}$.

(d) The collapsing theory for spaces with curvature bounds suggests that manifolds (and singular Alexandrov spaces), with, say non-negative, curvature also support large families of functions of low topological complexity and this may be true, up to a limited extent, for spaces with a lower bound on the Ricci curvature, where the volume of X should be replaced by the minimal number of unit balls needed to cover X.

For instance, "topologically simple" maps of hyperbolic 3-manifolds (of curvature -1 and, possibly, with small injectivity radii at some points) into the plane were constructed in [CosT] where the reader also finds references to the earlier results.

On the other hand, even (almost flat) infra-nil manifolds are not fully understood in this respect.

(e) Possibly, some non-Riemannian locally homogeneous spaces (e.g. affine flat manifolds X) are also amenable to a similar treatment.

2 Homological Lower Bounds on $\Sigma(F)$ and the Fibers of Maps F

2.1 Planar Morse inequality for cusps and fold crossings.

Let X be a closed manifold and $F : X \to \mathbb{R}^2$ be a smooth map, where the critical set $\Sigma = \Sigma(F) \subset \mathbb{R}^2$ has N_2 transversal self-crossings (double points), N_{cs} cusps and N_{cmpt} connected components. Then X admits a Morse function with $N = 2N_2 + N_{cs} + 2N_{\mathrm{cmpt}}$ critical points. Consequently, the sum of the Betti numbers of X is bounded by

$$\mathrm{rank}\big(H_*(X)\big) \le N = 2N_2 + N_{cs} + 2N_{\mathrm{cmpt}}\,.$$

Proof. Orient (the closed possibly disconnected curve) $\hat\Sigma \subset X$, let $\hat\Sigma_{cs} \subset \hat\Sigma$ be the (finite) set of cusps of F and let $\hat\tau : \hat\Sigma \to S^1 \subset \mathbb{R}^2$ be the tangential (Gauss) map (or, rather, the composition of F with the tangential map of $\Sigma \subset \mathbb{R}^2$). Notice that

- the map $\hat\tau$ sends each cusp $\sigma \in \hat\Sigma_{cs}$ to a pair of $\pm$-symmetric points in the circle S^1;

•• $\hat\tau(\hat\sigma) \ne \pm\tau(\hat\sigma')$ for every pair of double points $\hat\sigma, \hat\sigma \subset \hat\Sigma$ (i.e. for $F(\hat\sigma) = F(\hat\sigma'))$.

Denote by $\hat T = \hat T(\Sigma) = T(F(\hat\Sigma))$ the space of maps $\hat\Sigma \to S^1$ which satisfy • and ••, which are continuous away from $\hat\Sigma_{cs}$ and which lie in the connected components of $\hat\tau$ (i.e. homotopic to $\hat\tau$). Let $N_\circ = N_\circ(\Sigma) = N_\circ(F)$ be the minimal number such that some map $\hat\tau_\circ \in T$ has at most $N_\circ$ preimages of a point $s_0 \in S^1$.

The existence $\hat\tau_\circ$ with some $N_\circ$ is equivalent to the existence of a smooth generic *non-vanishing* 1-form $\varphi_\circ$ on the plane $\mathbb{R}^2$ with (y_1, y_2) coordinates, such that $\varphi_\circ$ equals dy_1 at infinity and such that its lift to $\hat\Sigma \subset X$ by the differential of F, has $N_\circ$ zeros.

Then, by the *Poincaré–Benedicson theorem*, there is a non-vanishing function ρ that equals one at infinity and such that $\rho\varphi$ is exact, say $\rho\varphi = d\psi$. The lift of ψ to X is a Morse function, say $f = \psi \circ F : X \to \mathbb{R}$, where the critical points of f coincide with the zeros of the lift of $\varphi_\circ$ to $\hat\Sigma$. Thus $\mathrm{rank}(H_*(X)) \le N_\circ$.

It remains to show that $N_\circ \le N = 2N_2 + N_{cs} + 2N_{\mathrm{cmpt}}$. To do this temporarily assume that the critical set $\Sigma \subset \mathbb{R}^2$ contains no cusps, i.e. $F : \hat\Sigma \to \mathbb{R}^2$ is an immersion. Decompose a slightly perturbed Σ into smooth cycles C_j that are tangent to each other at the double points of Σ and have *equal* orientations at the tangency points as in the proof of the corollary in 1.3. Denote the so modified Σ by $\Sigma' \subset \mathbb{R}^2$ (the subset $\Sigma' \subset \mathbb{R}^2$ is obtained from Σ by a homeomorphism of $\mathbb{R}^2$ which is a diffeomorphism away from the double points of Σ) and let $\tau' : \Sigma' \to S^1$ be the tangential map. All we need to show is that τ' is homotopic to map $\tau'_\circ$ which covers the circle with the average multiplicity strictly less than $2N_2 + 2N_{\mathrm{cmpt}} + 1$.

Observe that the general case trivially reduces to that where the graph Σ' (or, equivalently, Σ) is connected (i.e. $N_{\mathrm{cmpt}} = 0$) and contains no loops.

Denote by c the real 1-cocycle on Σ' induced from the fundamental cocycle on the circle. Clearly, the value of c on each simple 1-cycle in Σ' is at most one, and since there are no loops, the value of c on a cycle made of k edges is at most $k/2$.

It follows, by the Hahn–Banach theorem, that there is cocycle $c_\circ$ cohomologous to c, such that the value of $c_\circ$ at every edge in Σ' is $\le 1/2$; thus, the map $\tau'_\circ : \Sigma' \to S^1$ implementing $c_\circ$ has the required property.

This (together with Poincaré–Benedicson theorem) delivers the desired function ψ for maps F without cusps; then the (standard) cusps can be arranged at given points in $\hat{\Sigma}$, with every cusp contributing one critical point. QED

REMARKS. (a) I failed to find any reference to this result and I apologize to the author who was the first to observe it.

(b) Instead of the Poincaré–Benedicson theorem one may use *Steinitz' theorem* on realization of spherical triangulations by convex polytopes but this gives a non-sharp inequality.

2.2 A lower bound on the homologies of the fibers and on dep(Σ) by $\smile_m$-rank of $H^*(X)$.

Cup-rank of graded algebras. Let H^* be a graded algebra and let $\cap I_r \subset H^*$ denote the intersection of the two-sided graded ideals $I \subset H$ with $\mathrm{rank}(H^*/I) < r$ (that are the kernels of homomorphisms of H^* of ranks $< r$ to other graded algebras).

Define $\mathrm{rank}_k^{\smile}(H^*)$, as the maximal number r, such that the k-multiple product map $(H^*)^{\otimes k} \to H^*$ is not identically zero on $(\cap I_r)^{\otimes k} \subset (H^*)^{\otimes k}$.

Observe that $\smile_k$-rank is monotone decreasing in k and increasing under extension of algebras.

EXAMPLES. (a) Let $h_i \in H^*$, $i = 1, 2, \dots, k$ and let $L_i \subset H_*$ be linear subspaces of ranks r_i, such that, for every i, all non-zero $l \in L_i$ divide h_i. Then $h_i \in \cap I_r$. It follows that if the product of h_i does not vanish, then $\mathrm{rank}_k^{\smile}(H^*) \geq \min_i r_i$.

(b) If H^* is isomorphic to the cohomology algebra of the product X of k closed connected manifolds X_i (orientable, unless we work with $\mathbb{Z}_2$ coefficients) then, by the Poincaré duality, the fundamental classes h_i of X_i are divisible by all $0 \neq l \in L_i = H^*(X_i)$; therefore,

$$\mathrm{rank}_k^{\smile}\big(H^*(X)\big) \geq \min_i \mathrm{rank}\big(H^*(X_i)\big).$$

$\smile_{m+1}$-Inequality.

Let X be a compact topological space and $F : X \to Y$ be a continuous map. Then there exists a point $y \in Y$, such that the (Cech) cohomology (with arbitrary coefficients) restriction map to the y-fiber, denoted ρ_y^* : $H^*(X) \to H^*(F^{-1}(y))$, has

$$\mathrm{rank}(\rho_y^*) \geq \mathrm{rank}_{m+1}^{\smile}\big(H^*(X)\big) \qquad \smile_{m+1}$$

for $m \dim(Y)$.

Proof. If X is covered by k open subsets U_i, then the k-multiple cup-product map vanishes on the tensor product $\otimes_i I_i \subset (H^*(X))^{\otimes k}$ of the kernels $I_i = \ker \rho_i^*$ of the restriction homomorphisms $\rho_i^* : H^*(X) \to H^*(U_i)$ by the Lyusternik–Schnirelmann theorem.

Denote by U_{ij} the connected components of U_i and observe that the kernels $I_{ij} = \ker \rho_{ij}^* \subset H^*(X)$ of the restriction homomorphisms $\rho_{ij}^* : H^*(X) \to H^*(U_{ij})$ satisfy,

$$\cap_j I_{ij} = I_i$$

for each $i = 1, 2, \ldots, k$. Therefore, if $\mathrm{rank}(\rho^*_{ij}) < r$ for all $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots,$ then $\mathrm{rank}_{\smile k}(H^*(X)) < r$.

Since $F(X)$ is compact, we assume that Y is compact and consider the coverings $\{U_i(\varepsilon) = \cup_j U_{ij}(\varepsilon)\}$ of X, $i = 1, 2, \ldots, m + 1 \dim(Y) + 1$, which are the pullbacks of the ε-covers $\{\underline{U}_{ij}(\varepsilon)\}$ of Y with disjoint $\underline{U}_{ij}(\varepsilon)$ and $\underline{U}_{ij'}(\varepsilon)$ for all $i = 1, 2, \ldots, m+1$ and all pairs $j \neq j'$. The above shows that

$$\max_{i,j} \mathrm{rank}\big(\rho^*_{ij}(\varepsilon)\big) \geq \mathrm{rank}_{\smile m+1}\left(H^*(X)\right),$$

and, since the subsets $U_{ij}(\varepsilon) \subset X$ "shrink" to the fibers $F^{-1}(y) \subset X$, $y \in Y$, of F for $\varepsilon \to 0$, the inequality $\smile_{m+1}$ follows by the continuity of the Cech cohomology.

Torus Example. Let X be the n-torus. If $n \geq p(m+1)$, then every continuous map $X \to R^m$ has a fiber, say $X_y \subset X$, such that $\mathrm{rank}(\rho^*_y) \geq 2^p$.

On the other hand, if $n < \sum_{0 \leq i \leq m} p_i$, then every n-dimensional polyhedron can be covered by open subsets U_i where each U_i homotopy retracts to a $(p_i - 1)$-dimensional subpolyhedron. It follows that if $p > n/(m+1)$, then the n-torus admits a map to $\mathbb{R}^m$, such that the restriction homomorphisms $\rho^*_y : H^{\geq p}(X) \to H^{\geq p}(X_y)$ vanish for all $y \in \mathbb{R}^m$; thus $\mathrm{rank}(\rho^*_y) \leq n^{(p-1)}$.

It remains unclear what the *sharp* $\smile$-inequality is except for the case $m = 1$ (see section 4 in Part 2 for related results).

Related Questions. (a) When a graded linear space H^* is acted upon by an amenable group Γ exhausted by Følner sets Λ_i one denotes $\Lambda_i(A) =_{\mathrm{def}} \mathrm{span}_{\gamma \in \Lambda_i}\{\gamma(A)\}$, $A \subset H^*$ and defines, for all $\beta \geq 0$, *the β-rates of growth*

$$GR_\beta(H^k) = \sup_A \limsup_{i \to \infty} |\Lambda_i|^{-1} \big(\mathrm{rank}\, \Lambda_i(A)\big)^\beta$$

where $|\Lambda_i|$ denotes the cardinality of Λ_i and where "sup" is taken over all finite-dimensional linear subspaces $A \subset H^k$.

Consider a Γ-equivariant continuous map $F : X^\Gamma \to [0,1]^\Gamma$ for the natural action of Γ on the Cartesian Γ-powers. What is the maximum of the rates of growth of the cohomologies of the fibers of F? (This question makes sense for equivariant maps between arbitrary compact Γ-spaces, $F : \mathcal{X} \to \mathcal{Y}$, where the expected lower bound should depend on the upper bound, on the *mean dimension* of $\mathcal{Y}$, and the *log-multiplicative* growth rate of the cohomology of $\mathcal{X}$, where instead of the linear span of Λ_i-orbits of A one takes the logarithms of the ranks of the subalgebras generated by these orbits, compare [B], [Gr12].)

(b) Given a map $F : X \to Y$, denote by X^d/Y the *d-th Cartesian power of X over Y* that is mapped to Y by F^d/Y with the fibers $(F^d/Y)^{-1}(y) = (F^{-1}(y))^{\times d}$ and observe that the Betti numbers of X^d/Y, and/or the Poincaré polynomials $p_d(t) = \sum_i B_i t^i$ for $B_i = \mathrm{rank}(H^i(X^d/Y))$ carry most of the information contained in the cohomologies of the fibers of F.

If F is a Morse function then the topology of X^d/Y is transparent: k-handles that are the cones over the $(k-1)$-spheres become the cones over the d-th Cartesian powers of these spheres. Thus one sees, in particular, that the generating function $P(t, s) = \sum_d s^d p_d(t)$ is rational. (This may be true and known for all, say simplicial, maps F, but even the picture of the invariant subspaces of the d-th tensorial powers, of the two monodromy operators in the cohomologies of the fibers in fibrations over the figure ∞ is not clear to me for $d \to \infty$.)

On the other hand, the $\smile_{m+1}$-inequality implies that $\operatorname{rank}(H^*(X^d/Y)) \geq (\operatorname{rank}_{m+1}^{\smile}(H^*(X))^d$. (Probably, $\operatorname{rank}(H^*(X^d/Y)) \geq 2^d$ for all maps of closed n-dimensional manifolds X to $\mathbb{R}^m$ with $m \leq n \geq 1$. See 4.11 in Part 2 of this paper for a continuation of this discussion.)

Depth inequalities.

Let $F : X \to Y$ be a generic smooth map, where X is a closed n-dimensional manifold and Y is an open m-dimensional manifold Y (e.g. $Y = \mathbb{R}^m$). Then the depth of the critical set $\Sigma = \Sigma(F) \subset Y$ satisfies,

$$\operatorname{dep}(\Sigma) \geq \operatorname{const}(n) \operatorname{rank}_{m+1}^{\smile} \big(H^*(X)\big)$$

for the cohomology with arbitrary coefficients, where, moreover, $\operatorname{const}_n \geq 1/2$ *for purely folded maps F.*

Furthermore,

$$\operatorname{dep}(\Sigma) \geq \operatorname{const}(n)\mu(\Sigma)(m+1) \operatorname{rank}_{m}^{\smile} \big(H^*(X)\big) ,$$

where $\mu(\Sigma) = \max_{y\in\Sigma} \mu(\Sigma, y)$ for the local multiplicities $\mu(\Sigma, y)$ defined in 1.1 and where $\operatorname{const}(n) \geq 1/2$ *for purely folded (all generic?) maps F.*

Proof. Since the Betti numbers of all fibers of F (including the singular ones) are bounded by $2 \operatorname{const}_n \operatorname{dep}(\Sigma)$ (see 1.2) the proof of the first inequality follows from the above $\smile_{m+1}$-inequality.

Next we need the following

Retraction Lemma. *There exists a homotopy retraction $R : Y \to Y_\bullet \subset Y$, such that the image $Y_\bullet = R(Y) \subset Y$ (that is kept fixed by R) is a (piecewise) smooth subpolyhedron in Y of dimension $m-1$ and the pullback $Y_y = R^{-1}(y) \subset Y$, for every $y \in Y_\bullet$ consists of at most $m+1$ rays $R_+(y) \in Y$ in Y issuing from y and disjoint away from y, such that each ray (i.e. $[0, \infty)$ smoothly embedded to Y) crosses Σ at most $\mu(\Sigma) \operatorname{dep}(\Sigma)$ times.*

(I had a helpful conversation with Fedya Bogomolov at this point.)

Proof of the lemma. Take a sufficiently fine triangulation of Y in general position with respect to Σ and consider the dual cell subdivision of Y where the m-cells are denoted C_i. Observe, that no $m+2$ cells C_i come together at any point in Y.

Each C_i admits a radial projection to its boundary, from an interior point $y_i \in C_i \setminus \Sigma$, say $P_i : C_i \setminus \{y_i\} \to \partial C_i$ such that P_i is at most μ_i-to-1, on $\Sigma \cap C$ for $\mu_i = \max_{y \in C_i \cap \Sigma} \mu(\Sigma, y)$.

Thus we obtain a homotopy retraction P of $Y' = Y \setminus \cup_i \{y_i\}$ to the $(m-1)$-skeleton $Y'' \subset Y$ of our cell partition and we extend it to all of Y as follows. Take disjoint rays $[0, \infty)_i \subset Y$ starting from y_i and going to infinity, where every ray meets Σ only transversally at the pure folding locus and at most d times for $d = \operatorname{depth}(\Sigma)$. Let Q_i be the obvious homotopy retractions from small tubular neighborhoods U_i of these rays to the boundaries of U_i. The resulting map $Q : \cup_i U_i \to \partial(Y \setminus \cup_i U_i)$

is, clearly, at most d-to-1 and the composed map $R = P \circ Q$, that is a homotopy retraction on its image $Y_\bullet \subset Y'$, does the job.

The proof of the second depth inequality is concluded by applying the $\smile_m$-inequality to the map $R \circ F : X \to Y_\bullet$, where the cohomologies of the fibers of this map, that are the F-pullbacks $X_y \subset X$ of the R-pullbacks Y_y, are bounded by $\mu(\Sigma)(m+1)\,\mathrm{rank}_m^{\smile}(H^*(X))$ for generic points $y \in Y_\bullet$ by the Morse inequalities applied to the (at most) $(m+1)$ "branches" of the maps $X_y \to Y_y$ with the branches corresponding to the $\mathbb{R}_+(y)$-components constituting Y_y.

Cartesian Power Examples. Every closed surface X_0 of any genus g admits a Morse function where all levels (i.e. fibers) have their Betti numbers ≤ 4: circles, pairs of circles or figures ∞. But the $\smile_2$-inequality shows that every generic $F : X^{\times k} = X_0 \times X_0 \times \ldots \times X_0 \to \mathbb{R}^m$ necessarily has a fiber $X_y = F^{-1}(y)$, $y \in \mathbb{R}^m$, with

$$\mathrm{rank}\big(H_1(X_y)\big) \geq 2g \quad \text{if } k \geq m+1\,,$$

and every generic map $X^{\times m} \to \mathbb{R}^{m+1}$ has

$$\mathrm{dep}(\Sigma) \geq \mathrm{const}(m)g \quad \text{if } k \geq m\,.$$

This X is no match for the upper bound from 1.4 since any metric ϱ with 1-bounded geometry on this X has

$$\mathrm{vol}(X,\varrho) \geq \mathrm{const}(m)\,\mathrm{rank}\big(H_*(X)\big) = \mathrm{const}(m)(2g+2)^{m+1}.$$

On the other hand, these inequalities are qualitatively optimal for Cartesian powers.

Indeed, given a function F_0 on X_0 with N critical points, its m-th Cartesian power $F_0^{\times m} : X = X_0 \times X_0 \times \ldots \times X_0 \mathbb{R}^m$ has $\mathrm{dep}(\Sigma(F_0^{\times m}))$ about $\frac{1}{2}N$. This is close to our lower bound $\mathrm{dep}(\Sigma(F)) \geq \mathrm{const}(m,n_0)\,\mathrm{rank}(H^*(X_0))$, $n_0 \dim X_0$.

If the ranks of the homologies of the fibers of F_0 are bounded by N_0, then those of $F_0^{\times m}$ are bounded by N_0^m. For instance, if X_0 is a surface (of an arbitrarily large genus g) all fibers X_y of some $F = F_0^{\times m}$ have $\mathrm{rank}(H_*(X_y)) \leq 4^m$.

Power maps are non-generic. But the depth inequality remains valid for maps with "products of generic" singularities by an obvious generalization of our "generic" argument. Alternatively, such maps can be perturbed to generic ones, (sometimes to purely folded maps) with roughly the same depth of Σ and complexity of the fibers. Thus, our lower bounds on the fibers and critical sets are sharp for the Cartesian powers up to constants depending on dimension. (The asymptotic behavior of these constants for $\dim(X) \to \infty$ remains unclear.)

Question. Given numbers n, k and m, does every closed n-dimensional k-connected manifolds X admit a generic smooth map $X \to \mathbb{R}^m$ where every fiber has the sum of the Betti numbers bounded by a constant $\mathrm{const}(n,k,m)$ independent of X? (This may happen when n is not very large compared to k and/or m and, probably, can be proved in some cases by surgery in the category of k-connected manifolds.)

2.3 Slice inequalities. The depth of a Σ can seen by how Σ intersects with 1-dimensional subsets (the above rays) in Y, with the subsequent topological applications depending on the Morse inequalities.

One can proceed similarly by intersecting Σ with families of submanifolds of dimension > 1 in Y. Here is a particular such invariant adjusted to the planar Morse inequality in 2.1 .

Given three subsets $\Sigma, \Sigma_2, \Sigma_{11} \subset Y$ and a map $\alpha_0 : S_0 \to Y$ denote by N, N_2 and N_{11} the numbers of the connected components in the α_0-pullbacks of these sets and let $M(\alpha_0) = 2N(\alpha_0) + 2N_2(\alpha_0) + N_{11}(\alpha_0)$.

If $\Sigma = \Sigma(F)$ is a critical set of a generic map F, then we take the set of the double points of the fold for Σ_2 and the cuspidal locus for Σ_{11}. Observe that these have codimensions 2 in Y.

Given a family α of surfaces mapped to Y, say $\alpha_t : S_t \to Y$ and set $M(\alpha) = \sup_t M(\alpha_t)$.

Finally, we define $M(\Sigma(F)) = \inf_\alpha M(\alpha)$ where the infimum is taken over all *substantial* planar surface families α defined below.

Let P be a smooth manifold, T a polyhedron, and $f : P \to T$ a continuous map, where the fibers are planer surfaces $S_t = f^{-1}(t) \subset P$, $t \in T$, which smoothly foliate P, e.g. T is a smooth manifold with $\dim(T) \dim(P) - 2$ and f is a submersion.

A substantial family for an $F : X \to Y$ is a smooth generic equidimensional map $\alpha : P \to Y$ which sends P onto some open subset $U \subset Y$ containing the image of F (and, hence, Σ), such that the map $\alpha : P \to U$ is proper and has non-zero degree (over the coefficient field $\mathbb{F}$ for the cohomology in question).

EXAMPLE. Let $F : X = X^4 \to \mathbb{R}^3$ be a purely folded map with no triple points on the fold. The fold self-intersection Σ_2 is a smooth closed (for X is assumed a closed manifold) curve, i.e. a link in $\mathbb{R}^3$, and the contribution of Σ_2 to $M(\Sigma(F))$ with the maps $\alpha : P \to Y$ which are proper onto Y is something like the minimal number of braids needed to represent this link.

2D-slice inequality.

Let F be a smooth generic map of a closed n-dimensional manifold X to an m-dimensional manifold Y. Then

$$M(\Sigma(F)) \geq \mathrm{const}(n)\, \mathrm{rank}_{\smile (m-1)} \left(H^*(X)\right).$$

Proof. The fiber product $X \times_Y P$ of X and P over Y, mapped to Y by F and by α correspondingly, is a smooth manifold with $\mathrm{rank}_{\smile (m-1)}(H^*(X \times_Y P)) \geq \mathrm{rank}_{\smile (m-1)}(H^*(X))$ for substantial maps (families) α.

On the other hand, the ranks of the cohomologies of the generic fibers of the obvious map $X \times_Y P \to T$, say $Q_t \subset X \times_Y P$, are bounded by $M(\alpha)$ according to the planar Morse inequality from 2.1 applied to the tautological maps $Q_t \to S_t$.

Thus the $\smile_{m-1}$-inequality from 2.2 applied to the map $X \times_Y P \to T$ furnishes the proof.

2.4 A lower bound on dep(Σ) for hyperbolic manifolds. Let Γ be a countable group and $Z = K(\Gamma, 1)$ an aspherical (Eilenberg–MacLane) space with a base point $z_0 \in Z$, that is the classifying space for for Γ, i.e. $\pi_1(Z, z_0) = \Gamma$ and $\pi_i(Z, z_0) = 0$) for $i \geq 2$. Recall, that for every connected cellular space X with a base point $x_0 \in X$, the space $M_0(h)$ of *based* (i.e. $x_0 \mapsto z_0$) continuous maps $(X, x_0) \to (Z, z_0)$ with a given homomorphism $h : \pi_1(X, x_0) \to (Z, z_0)$ is contractible. Furthermore,

> *the space $M(h) \supset M_0(h)$ of the non-based maps $X \to Z$ that are (freely) homotopic to those in $M_0(h)$ is homotopy equivalent to the classifying space $K(cnt_h, 1)$ for the centralizer $cnt_h \subset \Gamma$ of the image of h.*

This implies, by a standard argument, that the universality (classifying property) of $K(\Gamma, 1)$ extends from individual spaces to families as follows.

> A. *Let $X \to Y$ be a fibration with connected fibers X_y, where Y and X_y are cellular spaces. Then there exists a (unique up to fiberwise homotopy equivalence) fibration $Z \to Y$ with aspherical fibers Z_y and a continuous (classifying) map $X \to Z$, where $X_y \to Z_y$ for all $y \in Y$ and where the induced homomorphisms $\pi_1(X_y) \to \pi_1(Z_y)$ are isomorphisms.*

More generally, let $F : X \to Y$ be "glued" of fibrations according to the following

Definition of stratified singular (quasi)fibrations over $Y = \cup_i Y^i$. Let $\{Y^i\}$ be a filtration of Y by closed cellular subsets $Y^0 \subset Y^1 \subset Y^2 \subset \cdots \subset Y$ (e.g. by the skeleta of some triangulation of Y) with $\cup_i Y^i = Y$, where F is a fibration with connected fibers over Y_0 and all open strata $Y^i \setminus Y^{i-1}$ (the fibers may be different over different connected components of $Y^i \setminus Y^{i-1}$) and where, moreover, there are homotopies of the identity maps, $h_i(t) : Y_i \to Y_i$, such that $h_i(1)$ send some neighborhoods of $Y_{i-1} \subset Y_i$ to Y_{i-1} and such that these homotopies lift to homotopies of the identity maps of $F^{-1}(Y_i)$.

> B. *Let $\alpha : X \to Z = K(\Gamma, 1)$ be a continuous map ant let Γ_y, $y \in Y$, denote the images (defined up to conjugation in Γ) of the fundamental groups of the fibers $F^{-1}(y) \subset X$ in $\Gamma = \pi_1(Z)$. Then there exist a stratified fibration $\Phi : Z_Y \to Y$ with aspherical fibers $\Phi^{-1}(y) = K(\Gamma_y, 1)$ and continuous maps $\alpha_Y : X \to Z_Y$ and $\beta : Z_Y \to Z$, such that the composed map $\beta \circ \alpha_Y$ is homotopic to α.*

Remark. If we are free to choose of the topology of Z_y, which are, a priori, defined only up to a (weak) homotopy equivalence, then Z_Y can be taken in agreement with the original stratification $(Y_i, h_i(t))$ of Y. But if the topologies of the fibers of Φ over the connected components of the open strata $Y^i \setminus Y_{i-1}$ are prescribed beforehand, one may need to refine the original stratification in order to have the fibrations trivial over the connected components of the open strata of such refinement.

Corollary. *If the cohomology of the subgroups $\Gamma_y \subset \Gamma$ with the coefficients in some Γ-module $\mathcal{F}$ (e.g. with constant coefficients in some $\mathbb{F}$) vanish for dimensions $> k$, then the induced homomorphisms $\alpha^* : H^j(Z, \mathcal{F}) \to H^j(X, \alpha^*\mathcal{F})$ vanish for $j > k + m$, $m = \dim(Y)$. Furthermore, if each $K(\Gamma_y, 1)$ is homotopy equivalent to a k-dimensional cellular space, then the map α is homotopic to a map of X to the $(k + m)$-skeleton of $Z = K(\Gamma, 1)$.*

In order to use this we need a bound on the cohomology dimensions of the images $\Gamma_y \subset \Gamma$ of the fundamental groups $\pi_1(X_y)$ in terms of these groups themselves, e.g. by the cardinalities of minimal generating subsets in $\pi_1(X_y)$.

For instance, let $N_{fr}(\Gamma)$ be the maximal number N, such that every subgroup with N generators in Γ is free.

Basic Example.

Let a group Γ *act on a geodesic* δ*-hyperbolic metric space* Z *and let* $Rad = \mathrm{Rad}(Z/\Gamma)$ *denote the minimum displacements by the non-trivial* $\gamma \in \Gamma$,

$$\mathrm{Rad} =_{\mathrm{def}} \inf_{id \neq \gamma \in \Gamma} \inf_{z \in Z} \mathrm{dist}\,\big(z, \gamma(z)\big)\,.$$

Then

$$N_{fr}(\Gamma) \geq N(\mathrm{Rad}\,/\delta)$$

for some universal function $N(r) \to \infty$ *for* $r \to \infty$.

This property of hyperbolic groups is stated in [Gr3] with $N(r) \geq (1+\varepsilon)^r$ but I realize now that the argument suggested in [Gr3] only gives the bound $N(r) \geq \varepsilon r/\log(r)$, with some $\varepsilon > 10^{-6}$ (see below). Two other proofs appear in [Ar], [KaW] (where the authors establish something stronger than the freedom of the subgroups with $N < N(r)$ generators). The bound $N(r) \geq (1+\varepsilon)^r$ remain conjectural even for the 3-dimensional hyperbolic manifolds of constant curvature and for the small cancellation group.

Proof of $N(r) \geq \varepsilon r/\log(r)$. Normalize to $\delta = 1$, assume $R > 10000$ and consider a connected graph G (1-dimensional cell complex) with at most N independent cycles mapped to Z/Γ, such that the image of the fundamental group of G generates Γ and then deform this map to the one with the minimal length of the image $G_{\min} \subset Z/\Gamma$, such that the image of the fundamental group of $G_{\min} \subset Z/\Gamma$ also generates Γ and $G_{\min}$ has no more than N cycles. (Strictly speaking, this $G_{\min}$ is immersed rather than imbedded to Z/Γ but we treat it as if it were embedded to save notation.)

Then, for $N \leq \varepsilon R/\log(R)$, $R = \mathrm{Rad}$, every simple infinite path P in the universal cover of $G_{\min}$ *quasi-isometrically embeds* into Z. Otherwise, P would contain a segment $S = [p_1, p_2] \subset P$ of length in the interval $[R/100, R/2]$ with the distance between its ends in Z, or equivalently, in Z/Γ, bounded by $300\log(R)$ (see 7.1.B and 7.2 in [Gr3]). Join these ends by a shortest geodesic segment L in Z/Γ and observe that the loop $\Lambda = L \cup S$ in Z/Γ represents id $\in \Gamma$, since length$(\Lambda) = 300\log(R) + R/2 < R$ for $R > 10000$.

On the other hand, the loop Λ, which consists of at most $3N+1$ geodesic segments, has length $\geq R/100$; therefore, for our small ε, it contains a segment L' of length $> 300\log(R)$. Then the graph $G' = (G_{\min} \setminus L') \cup L$ would be shorter than $G_{\min}$.

Thus we see that the (free!) fundamental group of $G_{\min}$ *injects* into Γ and is, moreover is quasi-convex. QED

Clarifying remarks suggested by a referee. One may assume that every vertex in $G_{\min}$ has degree ≥ 3; therefore the number E of edges is $\leq 3N-1$.

The shortest cycle in $G_{\min}$ has length$\geq R$; therefore, the loop Λ consists of at most $E+1 \leq 3N$ geodesic segments.

Combining this example with the above corollary, we conclude to the following

Lower bound on $\pi_1(X_y)$.

Let $F : X \to Y$ be a stratified map between connected cellular spaces, such that the fundamental group Γ_y of each connected component of every fiber $X_y = F^{-1}(y) \subset X$ can be generated by N elements and let $\alpha : X \to Z = K(\Gamma, 1)$ be a continuous map for a group Γ which admits an action on δ-hyperbolic space Z' with $\operatorname{Rad}/\delta \geq 10^6 N \log(\operatorname{Rad}/\delta)$ (where $Z' = Z$ is an essential example). Then the map α is $m+1$-contractible, i.e. homotopic to a map to the $m+1$-dimensional skeleton of Z for $m \dim(Y)$.

For example, if $X = Z$ is a closed n-dimensional manifold of sectional curvature ≤ -1, then the identity map $X \to Z$ is not contractible to the $n-1$-skeleton; therefore, if the radius of injectivity of X at all points bounded from below by R, then

every stratified (e.g. smooth generic or piecewise linear) map of X to an $(n-2)$-dimensional space Y necessarily has a fiber, where the fundamental group cannot be generated by less than $\varepsilon R/log(R)$ elements. If Y is an open manifold of dimension $m \leq n-1$, then every smooth generic map $F : X \to Y$ satisfies

$$\operatorname{dep}(\Sigma(F)) \geq \operatorname{const}(n) R/\log(R)\,.$$

Generalization. The above argument allows some γ with small displacements, provided the regions with small displacement are far apart. Namely,

Let $\Gamma_i \subset \Gamma$ be elementary subgroups and $Z_i \subset Z$ be Γ_i-invariants subsets that satisfy the following two conditions:

$$\operatorname{dist}\big(z, \gamma(z)\big) \geq R \quad \textit{for all } \gamma \neq \operatorname{id} \textit{ and all } z \in \bigcup_i \bigcup_{\gamma \in \Gamma} \gamma(Z_i)\,,$$

and Z_i/Γ_i inject into Z/Γ for all i and their images are pairwise R-far apart.

Then, if Γ is generated by $\leq \varepsilon R/\log(R)$ elements, then it is isomorphic to the free product of Γ_i and some cyclic groups.

It follows, that if the inequality $\operatorname{Rad}/\delta \geq 10^6 N \log(\operatorname{Rad}/\delta)$ is replaced by the above two conditions where, moreover, the groups Γ_i are virtually free (that is automatic if the action has no parabolic and no infinite isotropy subgroups) then

the induced homomorphism $\alpha^* : H^*(Z;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ vanishes above dimension $m+1$.

Convexity and virtual malnormality. Given a subgroup in a word hyperbolic group, say $\Gamma_0 \subset \Gamma$, denote by $\Gamma_0^+ \supset \Gamma_0$ the maximal subgroup in Γ with the same limit set as that of Γ_0. If Γ_0 is quasi-convex, then so is Γ_0^+ and Γ_0 has finite index in Γ_0^+. Furthermore, Γ_0^+ is *malnormal* in Γ: if two elements of Γ_0^+ are conjugate in Γ than they are conjugate in Γ_0^+ itself. Moreover, if Γ_0 is free and Γ is torsion free, then Γ_0^+ is free.

Also one knows that if Γ is torsion free word hyperbolic, then the centralizer of every non-Abelian subgroup Γ_0 is trivial and if Γ_0 is a non-trivial Abelian subgroup then its centralizer is free cyclic and equals Γ_0^+. (The centralizer of the trivial subgroup $\{id\} \subset \Gamma$ equals Γ.)

Then we easily see that

> *If Γ in the above lower bound on $\pi_1(X_y)$ is a torsionless word hyperbolic group and the subgroups $\Gamma_y \subset \Gamma$ are non-trivial free quasiconvex subgroups, then the classifying map $\alpha : X \to Z = K(\Gamma, 1)$ is $(d+1)$-contractible for $d \dim \Upsilon(F)$: it factors, up to homotopy, through a map to a stratified fibration over the adjacency complex $\Upsilon(F)$, where the fibers are the classifying spaces for the groups Γ_y^+.* (If we admit torsion in Γ, then the $d+1$-contractibility conclusion is relaxed to the vanishing of the induced homology homomorphism $\alpha_* : H_{>d+1}(X;\mathbb{Q}) \to H_{>d+1}(Z;\mathbb{Q})$.)

This becomes more useful if we allow some $\Gamma_y = \{id\}$:

if $\dim(\Upsilon(F)) < m \dim(Y)$, then the map α is m-contractible.

This can be applied, for instance, to smooth purely folded maps $X \to Y$ with $m \dim(Y) = \dim(X) - 1$, provided the fold has no self-crossings of multiplicity m. (We shall evaluate he contribution of the m-crossings to the topology of X in the next section.)

Remarks. (a) One can replace in some cases the above homotopy argument with Z_Y by a geometric consideration based on the fact that the intersection of every metric sphere S in a geodesic hyperbolic space with a Δ-neighborhood U of an orbit of a co-convex subgroup with the zero-dimensional limit set decomposes into the union of disjoint subset with diameters $\leq D$, where D depends on Δ but not on S.

In fact, let Γ be an isometry group acting on a complete contractible hyperbolic space $\tilde{X}$ and let $\hat{U}$ be an open invariant subset contained in the orbit of a bounded subset in $\tilde{X}$ by a subgroup of Γ. Observe that the limit set of $\hat{U}$ in the ideal boundary of $\tilde{X}$ equals (almost by definition) the limit set $L = L(\Gamma) \subset \partial_\infty(\tilde{X})$ of Γ. If $\tilde{X}$ is homeomorphic to $\mathbb{R}^n$ and Γ is co-compact, one needs at least n such U (associated to several subgroups where there may be several U's for the same subgroups) to cover sufficiently large S. Thus we arrive at the same lower bound on the fundamental groups of the fibers of maps of $X \to Y$ as earlier. However, it is unclear if this remains true in general for $X = \tilde{X}/\Gamma$, where Γ is convex co-compact with the topological dimension of the limit set equal $n-1$. Here are related

Questions. Let $\tilde{X}$ be a contractible δ-hyperbolic space with n-dimensional ideal boundary. Can one ever contract $\tilde{X}$ to a $(n-1)$-dimensional subset in X by a *bounded* homotopy? Can one contract the boundary S of bounded domain containing a ball of radius R to something $(n-2)$-dimensional in $\tilde{X}$ by a homotopy bounded by εR for large R and small $\varepsilon > 0$?

(b) One can sometimes reverse the arrow *Geometry* $\Rightarrow$ *Topology*, e.g. for the cyclic coverings X_k of an X associated to a non-contractible map $X \to S^1$, where all admit maps $X_k \to S^1$ with a uniform bound on the topology of the fibers; thus, every metric with curvature ≤ -1 on every X_k must have $\inf_x \mathrm{Inj\,.Rad}(X_k, x \in X_k) \leq \mathrm{const}(X)$. (In fact, $\sup_x \mathrm{Inj\,.Rad}(X_k, x \in X_k) \leq \mathrm{const}(X)$, see 5.9, 5.11.)

(c) The $\smile_{m+1}$-inequality can be expressed in terms similar to those in this section with the (classifying) map of X to the product of $K(\Pi, i)$ spaces that represents the cohomology of X. Thus, one, probably, can take into account the (Steenrod) cohomology operations and/or (by going up on the rational Postnikov tower) the multiplicative structure of the minimal model of X.

3 Lower Bounds on the Fold Maximal Self-Crossing Number N_m by the Simplicial Volume

Let $F : X \to Y$ be a stratified (or stratum-wise) fibration, e.g. a generic smooth maps $F : X \to Y$. We are concerned with lower bounds on the number N of the "most singular" isolated points of F in Y such as the number of the m-multiple self-intersection points of the fold of a purely folded map $X \to Y$, $\dim(Y) = m$.

Notice, that a generic smooth F may have "most singular" points that are quite different from the transversal m-crossing points of the fold. For example, some may come from the isolated "most singular" points of the singularity $\hat{\Sigma}(F) \subset X$, where the number of these is bounded from below by the characteristic numbers of X, and where such a bound is (essentially) sharp by Eliashberg's h-principle for maps with prescribed singularities.

In general, points $y \in \Sigma^0(F)$ of F are isolated transversal crossings in Y of some (canonical) strata of $\hat{\Sigma}(F) \subset X$ mapped by F to Y, for example, the intersections of the folding surfaces with the cuspidal curves of maps of manifolds of dimension≥ 3 to $\mathbb{R}^3$.

Yet, all these kinds of points are "less singular" than the transversal fold m-crossing points of the fold and they will not enter the Δ-inequality in 3.3.

3.1 Recollection on $\|X\|_\Delta$. Given an n-dimensional real homology class h in a topological space X, consider all singular cycles $c = \sum_i r_i\sigma_i$ representing this $h \in H_n(X;\mathbb{R})$, where $r_i \in \mathbb{R}$ and σ_i are singular simplices in X that a continuous maps of the standard n-simplex Δ to X. Let $\|c\|_{l_1} = \sum_i |r_i|$ and define the *simplicial (semi)norm* of h as the infimum

$$\|h\|_\Delta = \inf_c \|c\|_{l_1} .$$

This (semi)-norm on H_n is, obviously, *functorial*: it is monotone decreasing under continuous maps $f : X_1 \to X_2$.

Moreover, [Gr13],

if f induces an isomorphism on the fundamental groups, then the induced homomorphism $H_n(X_1;\mathbb{R}) \to H_n(X_2;\mathbb{R})$ is an isometry with respect to this simplicial (semi)norm. Thus,

$$\|h\|_\Delta = \|h_*\|_\Delta ,$$

where h_* denotes the image of h in $H_n(\pi_1(X)) =_{\mathrm{def}} H_n(K(\pi_1(X), 1);\mathbb{R})$ under the classifying map $X \to K(\pi_1(X), 1)$.

If X is closed oriented n-manifold (or pseudo-manifold), then the simplicial volume $\|X\|_\Delta$ is defined as the simplicial norm of the fundamental class of X and if X is non-orientable then $\|X\|_\Delta$ is defined as 1/2 of the simplicial norm of the oriented double cover of X. Thus, $\|X\|_\Delta$ is bounded by the minimal number N of the n-simplices of any triangulation of X. Furthermore, if X receives a map of degree d from a pseudo-manifold P triangulated into N simplices of dimension n, then, obviously, $\|X\|_\Delta \leq N/|d|$. In fact, one can *define* essentially equivalent $\|X\|_\Delta$ as the infimum of N/d over all such $P \to X$. (It is unclear by how much the norm would increase if one dropped "pseudo" in such definition).

One knows [Gr13] that the simplicial volume is additive under connected sums of n-dimensional (pseudo)manifolds for $n \geq 3$ and almost multiplicative under Cartesian products,

$$\|X_1 \# X_2\|_\Delta = \|X_1\|_\Delta + \|X_2\|_\Delta\,, \tag{$\#_\Delta$}$$

$$C_n^{-1}\|X_1\|_\Delta \cdot \|X_2\|_\Delta \leq \|X_1 \times X_2\|_\Delta \leq C_n\|X_1\|_\Delta \cdot \|X_2\|_\Delta\,, \tag{$\times_\Delta$}$$

for some constant $C_n \leq n^n$, $n = n_1 + n_2 \dim(X_1) + \dim(X_2)\dim(X)$.

Remarks and Questions. (a) It is unclear if $\|X_1 \times X_2\|_\Delta = C_{n_1 n_2}\|X_1\|_\Delta \cdot \|X_2\|_\Delta$.

(b) One can modify the definition of the norm by using cubical (rather than simplicial) singular chains; then, the resulting cubical norm, is obviously, equivalent to $\|X\|_\Delta$,

$$n^{-n} \leq \|X\|_\Box / \|X\|_\Delta \leq n^n,$$

and it satisfies

$$\|X_1 \times X_2\|_\Box \leq \|X_1\|_\Box \cdot \|X_2\|_\Box\,.$$

But it is unclear if it is multiplicative, i.e. if

$$\|X_1 \times X_2\|_\Box = \|X_1\|_\Box \cdot \|X_2\|_\Box\,.$$

In fact, one does not know if there is any other norm equivalent to $\|X\|_\Delta$ (or, at least, is not identically zero) that *is* multiplicative.

(c) One can define norm on homology of every topological spaces with a use of a functor from the homotopy category of connected spaces X to the category of metric spaces M and 1-Lipschitz maps, where one needs some geometrically defined norms on the homologies of metric spaces M.

For example, take the semi-simplicial complex $M = M_\Delta(X)$ constructed along with a homotopy equivalence $M \to X$ as follows (compare [Gr13]). Let Δ^∞ be the simplex with the integers $\mathbb{N}$ as the vertex set; the faces of Δ^∞ correspond to finite subsets of integers. Take $\mathbb{N}$ for the 0-skeleton (the vertex set) M^0 of M and let $\mathbb{N} \ni v \mapsto x = x(v) \in X$ be an arbitrary map.

Take representatives in all homotopy classes of paths in X between the points $x = x(v) \in X$ (recall, X is assumed connected) and attach edges to the pairs (v_1, v_2), $v_1 > v_2$, corresponding to the paths from $x(v_1)$ to $x(v_2)$; thus we get the 1-skeleton $M^1 \supset M^0$ mapped to X (and also mapped to the 1-skeleton of Δ^∞ injectively on every edge.) Next make the 2-skeleton $M^2 \supset M^1$ out of 2-simplices mapped to X and filling the triangles of edges in $M^1 \to X$ in all homotopy classes. (Such a set of fillings is empty for each non-contractible edge triangle; if a triangle is contractible, then the set of the homotopy classes of such fillings is a principal homogeneous space of the second homotopy group of X.) Then make M^3 out of the $3D$-filling of the boundaries of 3-simplices in M^2 mapped to X, etc. Thus one obtains a space homotopy equivalent to X that maps to Δ^∞ injectively on all faces.

Alternatively, one can start with the cube $\square^\infty$ and construct a (semi-cubical) space $M = M_\square(X)$ homotopy equivalent to X that maps to $\square^\infty$ injectively on the faces.

Granted such $M = \cup_k M^k$, take some length metric dist_Δ on the underlying Δ^∞ (or $\mathrm{dist}_\square$ on $\square^\infty$) (preferably, invariant under affine automorphisms of the simplex or the cube) and endow M with the length metric dist induced by the map $M \to \Delta^\infty$ from dist_Δ (or from $\mathrm{dist}_\square$).

Then every geometric invariant of (M, dist) serves as a homotopy invariant of X. For example, every choice of n-mass (e.g. n-dimensional Hausdorff measure) on the n-cycles in M leads to a ($\mathbb{R}$-mass) norm on $H_n(M;\mathbb{R})$ and, hence, on $H_n(X;\mathbb{R}) = H_n(M;\mathbb{R})$.

In fact, in order to have a norm on the n-dimensional homology, one does not need a length structure on M itself, but rather a volume (measure) structure on the n-dimensional chains in M, e.g. given by a norm on the n-th exterior power Λ^n of the tangent bundle of the space Δ^∞ (or $\square^\infty$) underlying the above $M = M_\Delta$ (or $M = M_\square$). In particular, one recaptures the simplicial norm on $H_n(X) = H_n(M_\Delta(M))$ with the l_1-norm on the space $\Lambda^n(T(\Delta^\infty))$, that is the maximal norm, such that its value on every n-frame of vectors (parallel to) edges of Δ^∞ at every vertex equals 1.

This applies, in particular to $K(\Gamma,1)$ spaces $Z = \tilde{Z}/\Gamma$, where Γ acts discretely and freely on the contractible universal cover $\tilde{Z}$. Here there are alternative constructions of metric Γ-spaces $\tilde{M}$, where $\tilde{M}/\Gamma$ may serve as metric models of Z, whenever $\tilde{M}$ is contractible and the action is free discrete as well as isometric.

For instance, the action of Γ on the space of measures on Γ or on $\Gamma \times B$ for the standard Borel space B (isomorphic to $[0,1]$), leads again to the simplicial norm.

Also one may use some (canonical) actions of Γ on infinite-dimensional symmetric spaces $\tilde{M}$, where the spaces of positive as well as of negative curvature lead to interesting possibilities (see [BeCG] for sharp inequalities associated to such norms) and where the case of a flat $\tilde{M}$ (locally Hilbertian) is pertinent for Haagarup (a-T-menable) groups.

Yet another possibility is the action of Γ on the (contractible!) unitary group $\tilde{M} = U(\infty)$ corresponding to some infinite-dimensional unitary representations of Γ, e.g. a regular representation for an infinite Γ.

Then the metric Γ-invariants of such a metric model serve as algebraic (topological) invariants of Γ.

Questions. When does the norm on homologies of spaces X associated to such a model M (e.g. $M = \tilde{M}?\Gamma$) *not* equal zero for at least one space X (or group Γ)? What are relations between these norms?

(d) There is another norm on $H_*(\Gamma;\mathbb{Q})$ (apparently of a different nature) defined via the *assembly homomorphism* from H_* to the (rational) Wall–Grothendieck surgery group $L_*(\Gamma)$, roughly, as follows (compare $8\frac{1}{2}$ in [Gr6]). Every $\lambda \in L_*$ is represented by a finite diagram δ of free $\mathbb{Q}(\Gamma)$-moduli, where we denote by rank(δ) the sum of the ranks of these moduli and by rank(λ) the minimum of rank(δ) for all δ representing λ. Then we define the (semi)norm $\|\lambda\|$ as $\lim_{d\to\infty} d^{-1}\,\mathrm{rank}(d\lambda)$ and pass this norm to H_* via the assembly homomorphism.

Geometrically speaking, one represents d-multiples of $h \in H_n(X)$ by maps of closed oriented manifolds, $\sigma : M \to X$ (where, in some version of the definition, one restricts oneself to stably parallelizable M, or, in the opposite direction, allows rational homology manifolds for M) and set

$$\|h\|_{\mathrm{Mor}} =_{\mathrm{def}} \inf_{M,\sigma,d} \tfrac{1}{d}|M|_{\mathrm{Mor}}\,,$$

where $|M|_{\mathrm{Mor}}$ denotes the minimal number of the critical points a Morse function may have on M.

Instead of $|M|_{\mathrm{Mor}}$, one can use the minimal number of cells in all cell decompositions of M, or of some space M' homotopy equivalent to M, or else the minimal number of simplices needed to triangulate M.

The latter norm (with triangulations) obviously majorizes the simplicial volume but $\|h\|_{\mathrm{Mor}}$ vanishes, for example, for the fundamental classes of hyperbolic 3-manifolds X where $\|X\|_\Delta \neq 0$. It is unclear if, in general, $\|h\|_{\mathrm{Mor}} \leq \mathrm{cost}_n\, \|h\|_\Delta$.

(e) There is yet another quantity, denoted $\mathrm{rank}_{l_2}(h)$, the von Neumann rank of the cup-pairing of the l_2-cohomology of Γ on $h \in H_*(\Gamma)$; this, conjecturally, is majorized, up to a constant, by the simplicial norm.

3.2 Averaging singular chains over amenable subgroups. Let $c = \sum_i r_i\sigma_i$ be a singular n-cycle in X, denote by $V_i \subset X$ the image of the vertices of the the standard simplex Δ mapped by σ_i to X and let $V = V(c) = \cup_i V_i \subset X$.

Let $\alpha : X \to Z = K(\Gamma,1)$ be a continuous map $\alpha : X \to Z = K(\Gamma,1)$ that sends all vertices $v \in V$ of c to a point $z_0 \in Z$

Let $P_l : [0,1] \to X$, $l = 1,2,\dots$, be sets of *edge paths* of σ_i between some of these vertices, where an edge path of a $\sigma : \Delta \to X$ is the restriction of σ to an oriented edge of Δ identified with the unit segment $[0,1]$. (Every σ defines $n(n+1)$ edge paths where some among them may be equal.)

Denote by $I \ni i$ the set of (the indices of) the singular simplices $\sigma = \sigma_i$ composing c and let $E \subset I$ be the set of (the indices of) the *essential* simplices $\sigma : \Delta \to X$ among σ_i, which means that none of the edge paths of this σ equals some $p \in P_l$ for any l.

Amenable Reduction Lemma. *If the groups Γ_l are amenable, then the simplicial norm of the α-image of the homology class $h = [c] \in H_n(X;\mathbb{R})$ represented by the cycle $c = \sum_{i\in I} r_i\sigma_i$ satisfies*

$$\big\|\alpha_*(h)\big\|_\Delta \leq \sum_{i\in E} |r_i| \qquad [ess]$$

for an arbitrary choice of the sets of edge paths P_l.

Proof. We average away all non-essential simplices as follows (compare [Gr13, 47-48]).

Assume without loss of generality that $z_{l_1} \neq z_{l_2}$ for $l_1 \neq l_2$, and realize every singular simplex in Z with the vertices in the set $\{z_j\} \subset \alpha(V) \subset Z$ by the homotopy classes of its edges.

Each group Γ_j acts on the resulting complex, say C_* complex in the obvious way: a loop at z_j is represented by a self-homotopy of Z moving z_j around this loop, where the time one map of this homotopy sends the complex C_* into itself.

Thus the Cartesian product Γ_J of Γ_j, $j \in J$, and the product $A_J \subset \Gamma_J$ of A_j act on C_*.

Since the groups A_j are amenable, each of them supports an ε-invariant probability measure, say $\mu_{j,\varepsilon}$ and we denote by μ_ε the product of these measures on A_J. Then, for every singular simplex $\alpha \circ \sigma_i$ in C_*, we consider the singular chains $\mu_\varepsilon * \sigma_i =_{\text{def}} \mu_\varepsilon * (\alpha \circ \sigma_i)$ in Z that are the μ_ε-weighted convex combinations of the singular simplices $\gamma * (\alpha \circ \sigma_i)$, for $\gamma \in A_J$, and set

$$\mu_\varepsilon * c = \sum_{i \in I} r_i \mu_\varepsilon * \sigma_i \, .$$

Clearly, the singular chain $\mu_\varepsilon * c$ is a cycle representing the same homology class in Z as $\alpha_*(c) = \sum_i r_i \alpha \circ \sigma_i$.

On the other hand,

$$\|\mu_\varepsilon * c\|_{l_1} \leq |\Sigma|_{ess} + |\Sigma|_\varepsilon \, ,$$

where

$$|\Sigma_{ess}| \underset{\text{def}}{=} \Big\| \sum_{i \in E} r_i \mu_\varepsilon * \sigma_i \Big\|_{l_1}$$

and

$$|\Sigma_\varepsilon| \underset{\text{def}}{=} \Big\| \sum_{i \in I \setminus E} r_i \mu_\varepsilon * \sigma_i \Big\|_{l_1} \, .$$

It is obvious, that $|\Sigma_{ess}| \leq \sum_{i \in E} |r_i|$, while

$$|\Sigma_\varepsilon| \leq \varepsilon \sum_{i \in I \setminus E} |r_i| \, ,$$

due to the ε-invariance of the measure μ_ε.

Then the lemma follows with $\varepsilon \to 0$.

Remark. One can slightly improve the $[ess]$-inequality by counting only those essential simplices, that contain no edge with both end vertices in V and such that the corresponding loop in Z is contractible: the simplices with the "α-contractible" edges average away if we use ε-invariant measures on A_j that are symmetric for $\gamma \leftrightarrow \gamma^{-1}$.

Stratification and pre-stratifications. Consider a partition $\mathcal{S}$ of a locally compact topological space X into locally closed subsets $S \subset X$, called *strata of* $\mathcal{S}$. *The boundary* ∂S of a stratum S is defined as closure$(S) \setminus S$.

A stratum $S_1 \subset X$ is called *adjacent* to another stratum $S_2 \subset X$ if S_1 has non-empty intersection with the boundary ∂S_0; this relation is denoted by $S_1 \prec S_2$, albeit it is *not* always a partial order relation.

A partition $\mathcal{S}$ is called a (topological) *stratification* if the boundary ∂S of each stratum S equals the union of the strata S' adjacent to S. In this case the adjacency *is* a strict partial order relation.

$\mathcal{S}$ is called a *pre-stratification* if the adjacency extends to a strict partial order relation. In other words, there is no chain $S_1 \prec S_2 \prec \cdots \prec S_n = S_1$.

Define the *corank* of a stratum S_0 as the maximal length k of the chains of strata $S_0 \prec S_1 \prec S_2 \prec \cdots \prec S_k$.

Define the covering number $N_{\mathrm{cov}}(S_0)$ as the minimal number l, such that there exit neighborhoods $U_i = U(S_i) \subset X$ of S_0 and all strata $S_i \succ S_0$, $i \neq 0$, such that no point in X is contained in more then $l+1$ subsets U_i.

Thin covering property. Let X be a compact pre-stratified space and S_i, $i = 0, 1, 2, \ldots$, a set of strata, where $S_0 \prec S_i$ for all $i = 1, 2, \ldots, m$. Then there exist open subsets $U_i \supset S_i$ in X, $i = 0, 1, 2, \ldots, m$, such that no point $x \in X$ is contained in more than $\mathrm{corank}(S_0) + 1$ of these U_i.

Indeed, if arbitrarily small neighborhoods of two non-intersecting locally closed subsets, say S and $S' \neq S$ in X intersect, then either $S \prec S'$ or $S' \prec S$. Therefore, if arbitrarily small neighborhoods of $l+1$ pairwise non-intersecting locally closed subsets in X have a common point, then there is an ordering of these subsets, say by $i = 0, 1, \ldots, l$, such that $S_i \prec S_j$ for all $0 \leq i < j \leq l$.

In what follows, we assume that X is a locally compact space and the (pre)stratifications are locally finite: every compact subset in X meets only finitely many strata.

EXAMPLES. (a) Start with some non-intersecting open subsets $S_{0i} \subset U$, called the *principal strata* and let $X^{-1} \subset X$ be the complement to their union. Then take non-intersecting subsets $S_{1i} \subset X^{-1}$ that are open in X^{-1}, let $X^{-2} = X^{-1} \setminus \cup_i S_{1i}$, etc. If this process (locally) terminates we obtain a (locally) finite pre-stratification of X, where $\mathrm{corank}(S_{ji}) \leq j$ for all i and j.

(b) *F-Pre-stratification of the Base.* This is canonically constructed with a continuous map, $F : X \to Y$, where $S_{0i} \subset Y$ are maximal connected open subsets such that F is a locally trivial fibration over S_{0i}, then $S_{1i} \subset Y^{-1}$ are similarly defined for $F : F^{-1}(Y^{-1}) \to Y^{-1}$, etc.

If F is a generic smooth map and X is a compact manifold, then $\mathrm{codim}(S_{ji}) \leq j$ and thus all strata have coranks $\leq \dim Y$.

(c) *$\hat{F}$-pre-stratification of the critical set.* Let $F : X \to Y$ be a generic smooth map, and let us take the above F-stratification of the critical set $\Sigma = \Sigma(F) \subset Y$, for the map $\hat{F}$ that equals F restricted to the singularity $\hat{\Sigma} = \hat{\Sigma}(F) \subset X$ of F, that is $\hat{F} : \hat{\Sigma} \to \Sigma$.

Since the map $\hat{F}$ is finite-to-one, "fibration" amounts to "finite covering map" and since $\hat{F}$ is one-to one over the simple pure folding points as well as over the pure cuspidal locus C (that has codimension one in Σ), the principal strata S_{0i} cover all these points. Observe that C has corank 2 for the F-stratification of $Y \supset \Sigma$, where $\mathrm{codim}_Y(\Sigma) = 1$ (we assume here as ever that $n \dim(X) \geq m+1$ for $m \dim(Y)$) but, yet, C is contained in the *principal* strata of $\hat{F}$ of corank 0 in Σ that is strictly less than $\mathrm{codim}_\Sigma(C) = 1$.

It follows that $\Sigma^{-1} \subset \Sigma$ equals the union of the set $\Sigma_\times \subset \Sigma^{-1}$ of the multiple points in the pure folding locus that has $\mathrm{codim}_\Sigma(\Sigma_\times) = 1$ unless it is empty, and of a subset $\Sigma_2 \subset \Sigma^{-1}$ that has $\mathrm{codim}_\Sigma(\Sigma_2) \geq 2$. Thus, $\Sigma^{-(m-1)} \cap \Sigma_2 = \varnothing$ for $m \dim(Y) \dim(\Sigma)+1$, while $\Sigma^{-(m-1)} \cap \Sigma_\times$ consists of the m-multiple self-intersections of the pure folding locus in Σ.

m-Crossing Corollary. *The only strata of corank $m-1$ in the $\hat{F}$-pre-stratification of $\Sigma(F)$ are the m-multiple self-crossing points of the pure folding locus.*

This simple property, along with the localization of the simplicial volume (see below), underlies the bound on the simplicial volume of X^{m+1} generically mapped to Y^m by the number of the m-multiple fold self-crossing points of the map. (See Δ-inequality in the next section.)

Call a stratum $S \subset X$ (or any subset in X for that matter) *α-amenable* for a given map $\alpha : X \to Z$ if the α-image of the fundamental group of each connected component of S in $\pi_1(Z)$ is amenable. (Here and below we deal with (sub)spaces that have no set theoretic pathologies, e.g. being locally contractible and thus having well defined fundamental groups.)

Define the *stratified simplicial norm* on the real singular homology classes $h \in H_n(X)$ of a stratified X by taking the infimum of the l_1-norms of the singular cycles $c = \sum_i r_i \sigma_i$ representing h, where all σ_i satisfy the following *ord*(er) and the *int*(ernality) conditions on the constituent singular simplices σ_i.

(*ord*) The image $\sigma_i(\Delta) \subset X$ of each σ_i is contained in an *interval of strata*, denoted $[S_1, S_2] \subset X$ and signifying the union of all strata S, such that $S_1 \preceq S \preceq S_2$, (where "$\preceq$ " signifies "$\prec$ or $=$").

(*int*). If a σ_i sends the boundary of some face Δ' of Δ to a stratum S then all of Δ' goes to S by this σ_i.

Clearly, the resulting norm, denoted $\|h\|^{\mathcal{S}}_\Delta$, is $\geq \|h\|_\Delta$.

Example. Let X be a triangulated n-dimensional (pseudo)manifold stratified into the open simplices of the triangulation. Then the stratified simplicial volume $\|[X]\|^{\mathcal{S}}_\Delta$ of X equals $n!N$, where N denotes the number of the n-simplices in the triangulation. On the other hand, the stratification with a single stratum $= X$, gives the stratified norm equal the ordinary one.

Define the stratified simplicial norm on the relative homology classes $h \in H_n(U, \partial U; \mathbb{R})$ for open subsets $U \subset X$ by taking the infimum of the l_1-norms of the relative cycles representing h and satisfying the (*ord*) and (*int*) conditions.

Localization Lemma. *Let X be a pre-stratified space and $\alpha : X \to Z = K(\Gamma, 1)$ a continuous map, such that the strata of corank $< n$ in X are α-amenable. Denote by $X_{-n} \subset X$ the union of the strata with corank $\geq n$, take a neighborhood $U \subset X$ of X_{-n} an assume that the restriction of the pre-stratification of X to U (obtained by intersecting the starts with U) is a stratification. Then the α-image of every homology class $h \in H_n(X; \mathbb{R})$ to U, satisfies*

$$\big\|\alpha_*(h)\big\|_\Delta \leq \|h_U\|^{\mathcal{S}}_\Delta ,$$

where $h_U \in H_n(U, \partial U; \mathbb{R})$ denotes the restriction of h to U (obtained from h by composing the homomorphism $H_n(X) \to H_n(X, X \setminus U)$ with the excision isomorphism $H_n(X, X \setminus U) \to H_n(U, \partial U)$).

Proof. It suffices to extend a given relative cocycle c_U representing h_0 to a c representing h with no new essential simplices with respect to some map $\alpha' : X \to Z$ homotopic to α.

Assume without loss of generality that X is a compact metric space and take a sequence of small fast decreasing positive numbers,

$$\varepsilon_{-n} = \varepsilon_{-n}(\mathcal{S}, c_U) > \varepsilon_{-n+1} = \varepsilon_{-n+1}(\varepsilon_{-n}) > \ldots > \varepsilon_{-1}(\varepsilon_{-2}) > \varepsilon_0 = 0\,.$$

Consider paths issuing from a point $x \in X$, $i = 0, 1, \ldots, n$, having diameters $\leq \varepsilon_{-i-1}$ and terminating in strata $S \subset X_{-i-1}$. Take such a path, say $p_x : [0,1] \to X$, with the maximal possible i and denote by S_x the strata, where this path terminates, i.e. $S_x \ni p_x(1)$. Choose a point $x_S \in S$ in each strata S and continue p_x by a path p'_x terminating in $x_S = x_{S_x}$.

Take an h-extension c of c_U, such that $c - c_U = c_1 + c_\delta$, where

- • c_1 equals a subdivided cylinder $\partial c_U \times [0,1]$ mapped to X via the projection $\partial c_U \times [0,1] \to \partial c_U = \partial c_U \times 0$;
- •• all singular simplices in c_δ have diameters (of their images in X) $\leq \delta < 2^{-n}\varepsilon_{-1}$.

Take the paths p'_v for the vertices $v \in X$ of the singular simplices in $c - c_U$ and homotope α in two steps as follows. First we take a homotopy of the identity map $X \to X$ by extending the homotopies $v \rightsquigarrow x_{S_v} \in S_v \subset X$, compose this with α and then continue with a homotopy of maps $X \to Z$ bringing all points x_{S_v} to the same $z_0 \in Z$. Then one sees (with the Lebesgue covering lemma as in the text books construction of the simplicial approximation of continuous maps by simplicial ones) that c has no new essential simplices with respect to the resulting map $\alpha' : X \to Z$, provided $\varepsilon{-}i$ are sufficiently small and fast decreasing; thus the reduction lemma applies and the proof follows.

3.3 Amenable stratifications of maps with drop one in dimension and $\boldsymbol{\Delta}$-inequality. Consider as a generic smooth map $F : X \to Y$ where $\dim(X) = n = m+1 = \dim(Y)+1$. If X is a manifold, assumed closed unless stated otherwise, then each non-empty connected component of the fiber $X_y = F^{-1}(y) \subset X$ is either a single point or a 1-dimensional space, a graph, obtained from a circle by gluing some pairs of points in it.

In particular, if F is purely folded with self-transversal fold, then all singular points of such graph have valency 4 and the number of these in X_y is at most k, where $k = 0, 1, \ldots, m$ is the multiplicity of the self-crossing of the fold at $y \in Y$. For example, the pullback of a generic $y \in Y$ (i.e. away from the fold $\Sigma(F) \subset Y$) consists of several copies of S^1, the fiber over a simple folding point may have the figure ∞, i.e. once pinched S^1, denoted G_1, or a single point component, the pullback of a double-crossing point y of Σ may have a twice pinched S^1 as a component, that is a graph G_2 with two 4-valent vertices, which either has four edges between the vertices or it is the necklace of three circles (two edges between the vertices and and a loop at each of the two) etc.

Consider, besides $F : X \to Y$, a continuous map $\alpha : X \to Z = K(\Gamma, 1)$ and, assuming X is oriented, let us estimate the simplicial norm of the α-image of the fundamental class $[X] \in H_n(X;\mathbb{R})$ in terms of a suitable (pre)stratifications of X and Y associated with F as follows.

Factor F via $\tilde{F} : X \to \tilde{Y} \to Y$, where the points $\tilde{y} \in \tilde{Y}$ represent the connected components of the fibers of F. Thus, all fibers $X_{\tilde{y}} = \tilde{F}^{-1}(\tilde{y}) \in X$ are identified with the connected components of the fibers over $y \in Y$, for y that lie under $\tilde{y}$.

Denote by $\tilde{Y}_{am} \subset \tilde{Y}$ the set of points where the fibers $X_{\tilde{y}}$ are *α-amenable* which means the α-images of the fundamental groups of these fibers in $\Gamma = \pi_1(Z)$ are amenable. Then let

$$X_{am} = \tilde{F}^{-1}(\tilde{Y}_{am})\,.$$

Observe that $\tilde{Y}_{am} \subset \tilde{Y}$ and $X_{am} \subset X$ are open subsets, where X_{am} contains the fibers of F that do not meet the singularity $\hat{\Sigma} = \hat{\Sigma}(F) \subset X$. The complements of these two subsets are denoted by $\tilde{Y}_{nam} \subset \tilde{Y}$ and $X_{nam} \subset X$.

Consider the intersection $\hat{\Sigma}_{nam}$ of the singularity $\hat{\Sigma}$ with X_{nam} and pre-stratify it into the maximal connected subsets S such that the restrictions of F to these sets, $F|S; S \to F(S)$, are covering maps.

Call the resulting (canonical) pre-stratification $\mathcal{S}(\hat{\Sigma}_{nam})$ and its strata $S = S(\hat{\Sigma})_{nam}$. Then pre-stratify X_{nam} by adding to these strata the maximal connected subsets $S \subset X_{nam} \setminus \hat{\Sigma}$, such that every fiber $\tilde{F}^{-1}(\tilde{y})$, $\tilde{y} \in \tilde{Y}_{nam}$, passing through $s \in S$ meets $\hat{\Sigma}$ over the same strata $S(\hat{\Sigma})_{nam}$. (Although the maps $\tilde{F}|S \to \tilde{F}(S)$ are not always fibrations for generic maps F, the images of the fundamental groups of their fibers in X behave as if they were fibrations.) The resulting (canonical) pre-stratification is called $\mathcal{S}(X_{nam})$.

Finally, extend $\mathcal{S}(X_{nam})$ to all of X by (this time non-canonically) stratifying X_{am} into strata S_{am}, such that they are either of the form $S_{am} = \tilde{F}^{-1}(\tilde{S})$ for contractible subsets $\tilde{S} \subset \tilde{Y}_{am}$ (amenability of $\pi_1(\tilde{S})$ will do), where the maps $F|S_{am} : S_{am} \to \tilde{S}$ are Serre fibrations, or such that the inclusion maps $S_{am} \to X$ are homotopic to maps sending S_{am} to some (possibly non-amenable) strata of the stratification $\mathcal{S}(X_{nam})$.

This is done with the (canonical) pre-stratification $\mathcal{S}(\tilde{\Sigma}_{nam})$ of $\tilde{\Sigma}_{nam} = Y_{nam} \cap \tilde{\Sigma}$ for $\tilde{\Sigma} = \tilde{F}(\hat{\Sigma}) \subset \tilde{Y}$ as follows.

Extend (non-canonically) the pre-stratification $\mathcal{S}(\tilde{\Sigma}_{nam})$ to a pre-stratification of $\tilde{Y}$ by stratifying $\tilde{Y}_{am}$, where each stratum $\tilde{S}_{am} \subset \tilde{Y}_{am}$ is either small and, thus, contractible in $\tilde{Y}$ ($\pi_1(\tilde{S})$ amenable suffices), or $\tilde{S}_{am}$ can be moved by a (small) homotopy into some stratum of $\mathcal{S}(\tilde{\Sigma}_{nam})$ (where the former implies the latter but it is convenient to distinguish the two cases). Then stratify X_{am} by the pullbacks $\tilde{F}^{-1}(\tilde{S}_{am})$.

If F is a generic map, than all strata of this (non-canonical) pre-stratification $\mathcal{S}$ of X have corank $\leq n$ and those of $corank = n$ are isolated points say $v \in X$, all contained in $\hat{\Sigma}_{nam} \subset X$ and having corank $n-1$ with respect to the (canonical) pre-stratification $\mathcal{S}(\hat{\Sigma}_{nam})$ of $\hat{\Sigma}_{nam}$.

Important Remark. The extension of the canonical pre-stratification $\mathcal{S}(X_{nam})$ to the above $\mathcal{S}$ can be (obviously) made canonical near $\hat{\Sigma}$, and, in particular, at the points of corank m. Thus the relative $\mathcal{S}$-stratified simplicial volumes of small neighborhoods of these points are bounded by a universal constant const(n) for all generic maps F. (In fact, const$(n) \leq n^{2n}$)

***The case* $\dim(X) = 3$.** The critical set $\Sigma \subset Y$ of the generic map $F : X^3 \to Y^2$ is a curve whose singular points (if any) are double points and cusps and the bound on $\|X\|_\Delta$, due to Costantino–Thurston in a sharper form [CosT], can be seen as follows.

The connected components of the pullbacks of the regular points, $y \in Y \setminus \Sigma$, as well as of the cusps on Σ are topological circles and have amenable ($= \mathbb{Z}$) fundamental groups. The non-circular components of the pullbacks of the non-singular points $y \in \Sigma$ are figures ∞, called G_1-fibers, and the pullback of a double points $Y \in \Sigma$ may contain one connected component that is a graph, call it *a G_2-fiber*, with two vertices of valency 4 on $F^{-1}(y)$. Such a graph is either made of four edges between the two vertices, or there are two edges between the vertices and a loop at each vertex.

The images of the fundamental groups of the G_1- and G_2-fibers in $\pi_1(X)$ may be non-amenable. (Those adjacent to the cusps are always amenable). Yet, all strata of the above pre-stratification $\mathcal{S}$ of X (obviously) generate amenable (even cyclic) subgroups in $\pi_1(X)$ and, by the localization lemma,

The simplicial volume of X is bounded by the number of the G_2-fibers and, thus, by the number of the double points of the fold,

$$\|X\|_\Delta \leq \mathrm{const} \cdot N_{G_2} \leq \mathrm{const} \cdot N_2$$

for some universal constant const $=$ const$(3) \leq 24$.

(In fact, const ≤ 10 according to [CosT].)

Example. Let X be a hyperbolic manifold that fibers over S^1. Then the N-sheeted cyclic coverings X_N of X, induced by the N-sheeted self-coverings of S^1, have $\|X_N\|_\Delta = N\|X\|_\Delta > N$; thus every generic map $X_N \to \mathbb{R}^2$ must have $> 00.1N$ fold double points. On the other hand, there obviously exist generic maps $F_N : X_N \to S^1 \times [0,1] \subset \mathbb{R}^2$, that are the "suspensions" of the fibrations $X_N \to S^1$ induced by the N-sheeted self-covering of S^1 with the pullbacks to X_N of some Morse function $X \to [0,1]$, which have $N_2(F_N) \leq \mathrm{const} \cdot N$ for some const $=$ const(X).

Δ-inequality for generic maps $F : X^n \to Y^m$ for $m = n-1 \geq 2$. The pullback $F^{-1}(y) \subset X$ of a k-multiple fold intersection point $y \in \Sigma \subset Y$ may have (at most) one connected component with k vertices of valencies four, that is called a *G_k-fiber of F*. Thus, there are finitely many G_m-fibers, each lying over a transversal m-multiple intersection point $y \in \Sigma \subset Y$ of the fold of F in Y.

Let X be a closed oriented n-manifold, $F : X \to Y$, $\dim(Y) = m = n-1$, a generic smooth map and $\alpha : X \to Z = K(\Gamma, 1)$ a continuous map. Then the simplicial norm of the α-image of the fundamental class of X is bounded by the number of the G_m-fibers and, thus, by the number of the m-multiple self-intersection points of the folding locus of F,

$$\big\|\alpha_*[X]\big\|_\Delta \leq \mathrm{const}(n) N_{G_m} \leq \mathrm{const}(n) N_m\,,$$

provided the fundamental group of Z (i.e. the group Γ) contains no pair of commuting non-amenable subgroups.

Proof. Observe that if a connected component G of a fiber of F has a non-amenable fundamental group, then G is a connected graph with at least one vertex of degree > 2. Since the group of homeomorphisms of such G is homotopy equivalent to a finite group, the monodromy group of such fiber over every subset $\underline{S} \subset Y$ where the map F is fibration is finite. Therefore, if the fundamental group of some of $S \subset X$ *injects* into that of $\underline{S}$ then this group *almost commutes* with the fundamental group of G, i.e. some subgroups of *finite indices* in the two groups commute.

This applies to the above "semi-canonical" pre-stratification $\mathcal{S}$ of $X = X_{am} \cup X_{nam}$: the image in $\pi_1(X)$ of the fundamental group of each stratum S of $\mathcal{S}(X_{nam})$ *almost commutes* with the images of the fundamental groups of the connected components of the fibers of the map F which meet S . Therefore, all strata S of $\mathcal{S}(X_{nam})$, and, hence, of $\mathcal{S}$ are α-amenable and the localization lemma applies.

Remark. The absence of commuting non-amenable subgroups in Γ is essential: every product manifold $X_1 \times X_2$, where X_2 is a surface, admits a purely folded map $X \to X_1 \times [0,1]$, which is the identity map on X_1 times a Morse function $X_2 \to [0,1]$, where the fold has no self-crossing at all. Furthermore, there exists an X that admits a map of positive degree to $X_1 \times X_2$, as well as a purely folded map F to $\mathbb{R}^m$, $m = n-1 = \dim(x) - 1$, where the fold has no self-crossing.

Examples. The word hyperbolic groups have no pairs of commuting non-amenable (and just, infinite) subgroups and the same is true for many other (non-cocompact) isometry groups of hyperbolic spaces.

Every virtually non-split cocompact lattice Γ in the product $L = \times_i L_i$ of semi-simple Lie groups L_i with $\operatorname{rank}_{\mathbb{R}} L_i = 1$ (where the corresponding symmetric spaces have strictly negative curvatures) also contains no pair of commuting non-amenable (just virtually unsolvable) subgroups and the Δ-inequality applies. (It is unclear what happens here to the Δ-inequality for $\operatorname{rank}_{\mathbb{R}} L_i \geq 2$.)

In particular, the Δ-inequality applies to compact Hilbert complex modular surfaces (where $L = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$) that have the same universal covers as the product of (real) surfaces, where the inequality does not apply.

Questions. (a) What happens if we only assume that the class $\alpha_*[X] \in H_n(Z;\mathbb{R})$ is *not* a linear combinations of homology classes $h = h_1 \bigotimes h_2 \in H_n(Z;\mathbb{R})$ coming from $H_n(\Gamma_1 \times \Gamma_2;\mathbb{R})$ for all pairs of infinite commuting subgroups Γ_1 and Γ_2 in $\Gamma = \pi_1(Z)$?

(b) Are there instances of a non-trivial lower bounds on N_m where the lower bound on the simplicial norm $\|\alpha_*[X]\|_\Delta$ in Γ is unavailable?

For example, if Γ is "anti-Abelian", i.e. the centralizers of all $\gamma \neq id$ are free cyclic, then every purely folded map $F : X \to Y$, $\dim(Y) = m = \dim(X) - 1$ necessarily has an m-multiple self-crossing point of the fold in Y. In fact, we shall see in 5.7 that some fiber of such a map has at least m singular points.

But it is unclear if there is a lower bound on the number of such points even if one assumes that F is a purely folded map, such that the singularity $\hat{\Sigma} \subset X$ is the union of $2n$ disjoint concentric spheres $S^{m-1} \subset \mathbb{R}^m \subset \mathbb{R}^n \subset X$. (Such maps are abundant for stably parallelizable manifolds by the Eliashberg h-principle.) Also it is unclear if "purely folded" can be replaced by "smooth generic".

(c) *On symplectization and stabilization.* Is there a symplectic version of the Δ-inequality?

For example, let $\Pi : X = X^n \to P = P^{n-2}$ be a smooth fibration with connected fibers, and consider maps $F = \Pi \times f : X \to P \times \mathbb{R}$ for a Morse function $f : X \to \mathbb{R}$. This defines a family of Lagrangian manifolds in the cotangent bundles of the (surface) fibers, $L_p \subset T^*(X_p = \Pi^{-1}(p))$, $p \in P$, and the appearance of a crossing point of the fold of F over a point $p \in P$ corresponds to the presence of some integer relation between the relative homology classes in $H^2(T^*(X_p), L_p \cup X_p)$, where $X_p \subset T^*(X_p)$ is embedded by the zero section.

Does the counterpart of the Δ inequality hold true for families of exact Lagrangian submanifolds $L_p \subset T^*(X_p)$?

It is unclear how to formulate a symplectic conjecture generalizing the Δ-inequality for general X and F but the above question can be generalized in another direction.

Let (T_p, ω_p, L_p) be a family of compact symplectic manifolds parametrized by a space $P \ni p$, where $L_p \subset X_p$ are Lagrangian submanifolds (possibly empty). When can one bound from below the number of points $p \in P$, such that the relative cohomology class $[\omega_p] \in H^2(X_p, L_p; \mathbb{R}) = H^2(X_p, L_p; \mathbb{Z}) \bigotimes \mathbb{R}$ is contained in a $\mathbb{Q}$-affine subspace (defined by linear equations with rational coefficients) of codimension $\geq m$? (In order to encompass the previous example one needs Lagrangian manifolds with self-intersections, and to go beyond fibrations one may try singular symplectic manifolds.)

This can be approached with the known techniques when the symplectic family (T_p, ω_p, L_p) is obtained by a deformation from another family, where linear $\mathbb{Z}$-relations are associated to degenerations (e.g. bubbling) of pseudo-holomorphic curves.

On the other hand, looking from the perspective of generating functions makes a symplectic version of the Δ-inequality rather improbable, since this inequality is unstable: The Cartesian product $X \times X_1$, where X is arbitrary and where X_1 can be realized by a co-oriented hypersurface in Y (e.g. X_1 is the sphere S^{m-1} and $Y = \mathbb{R}^m$) admits a purely folded map $X \times X_1 \to Y$, such that the fold has no self-crossing at all.

(d) *On universal singular fibrations.* One can regard smooth maps $F : X \to Y$ as fibrations over Y with variable fibers $X_y = F^{-1}(y)$, where, observe, every map $f : Y_1 \to Y$ induces such a fibration over Y_1, denoted $f^*(F) : f^*(X) \to Y_1$ with the fibers $X_{f(y_1)}$. If f is transversal (relative) to F, i.e. the map $F \times f : X \times Y_1 \to Y \times Y$ is transversal to the diagonal in $Y \times Y$, then $F^*(X)$ is a manifold of dimension $\dim(X) + \dim(Y_1) - \dim(Y)$ and the map $f^*(F)$ is smooth and "as generic" as F. In particular, if F is purely folded, then so is $f^*(F)$.

This suggests the existence of a universal such fibration with fibers in a given class $\mathcal{G}$ of (singular) spaces G. (Beware: singularities of generic smooth maps can make non-trivial C^∞-*moduli*, i.e. continuous families, albeit this does not happen for purely folded maps.)

Alternatively, one may start with a small category $\mathcal{G}$ of topological spaces G and certain admissible (continuous) maps between them. and construct the universal fibration with fibers G over the classifying space $Y_\mathcal{G}$ of the category $\mathcal{G}$ (compare 5.6).

For example, the space $Y_\mathcal{G}$ classifying purely folded maps $X^{m+k} \to Y^m$ is associated to the category of k-dimensional spaces with isolated quadratic (Morse) singularities, where the admissible maps correspond to the limit maps of one parameter families of fibers.

The classifying spaces $Y_\mathcal{G}$ have particular geometric models for $n-m = \dim(G) = 0, 1, 2$, and, possibly, for $\dim(G) = 3$, where one may(?) use the Hamilton–Perelman flow in the "space" of singular Riemannian 3-manifolds.

The Δ-inequality provides a necessary condition on X to appear as a global space of such a fibration, with $\mathcal{G}$ consisting of the graphs G with all connected components having at most $m-1$ vertices of valency ≥ 4. Can this be seen in the light of the classifying spaces $Y_\mathcal{G}$?

4 *p.l.*-Families of Point/Edge Singular Surfaces and Minimal Maps

We study here manifolds (and more general spaces) X represented as singular fibrations with 2-dimensional fibers $G_y \subset X$ parametrized by a space (e.g. a manifold) $Y \ni y$, and establish lower bounds on the topologies of Y_y in terms of some geometric invariants (called "waists" in [Gr5]) of metric spaces Z receiving X via continuous maps $\alpha : X \to Z$.

The main example is where X comes with a metric of negative curvature and α is the identity map $X \to Z = X$; this is used for lower bounds on the numbers of the crossing (double) points of the folding curves of 3-manifolds mapped to surfaces, such as

$$N_2 \geq \varepsilon(X)s^2$$

for generic smooth maps $F : X_s \to \mathbb{R}^2$ of certain infinite sequences of s-sheeted coverings X_s of hyperbolic 3-manifolds X.

4.1 Length and area minimization. Let Z be a *(geodesic) length space*, i.e. a metric space where every two points within distance d can be joint by a (minimal geodesic) path of length d. Consider a continuous map $\alpha_0 : G \to Z$ and the *induced length "metric"* ϱ_0 on G by first assigning to each curve in G the length of the corresponding curve in Z and then defining the distance between each pair of points in G as the infimum of the length of the path in G joining them.

This ϱ_0 may be degenerated in two ways: it may be infinite for some pairs of points, or it may be zero. The former does not happen if α_0 is Lipschitz with respect to some background length metric in G; the latter is unavoidable for maps that shrink rectifiable curves to points but we still treat ϱ_0 as a metric.

The n-dimensional Hausdorff measure of (G, ϱ_0) is called the *n-volume* of α_0. (If $\dim(Z) > \dim(G)$, then in most cases $\mathrm{vol}_n(\alpha_0)$ equals the n-volume of the image $\alpha_0(G) \subset Z$ and we use the two notations interchangeably. Also we often use "*length*" for vol_1 and "area" for vol_2.)

For example, if G and Z are Riemannian manifolds G and α is smooth (Lipschitz is enough), then this volume equals the integral of the Jacobian, i.e. the norm of the top exterior power of the differential, of α,

$$\mathrm{vol}_n(\alpha_0) = \int_G \left\|\Lambda^n D(\alpha_0)\right\| .$$

A map $\alpha_0 : G \to Z$ is called *length extremal* if it admits no *shortening* homotopy that, by definition, strictly decreases (i.e. such that $\varrho_t \lneqq \varrho_0$ for $t > 0$) the induced length structure. A map α_0 is called *length extremal* at a point $g \in G$ if there is a neighborhood $U \subset G$ of g, such that α_0 admits no shortening homotopy that is fixed outside U.

If G is a graph, then every locally extremal map at an interior point g of an edge is geodesic on this edge near g if, moreover, Z is a Riemannian manifold, then α_0 is extremal at a vertex g then the tangent vectors to the geodesic edges at the points $z = \alpha_0(g) \in Z$ are *not* contained in open half-spaces of the tangent spaces $T_z(Z)$ for all vertices $g \in G$.

These maps will be used later on with the following definition.

G^1-extremality. Let G^1 be a subgraph in a space G, e.g. the 1-skeleton of some triangulation of G. A map from G to Z is called *edge extremal (locally edge extremal)* on G^1 if its restriction to $G^1 \subset G$ is length extremal (locally length extremal at all points in G^1).

In what follows, plain "edge extremal" means "edge extremal on the 1-skeleton some (sometimes specified) triangulation of G".

Further examples of extremal maps. (a) Every *(piecewise) ruled* map is locally length extremal, where "ruled" signifies each point $g \in G$ is contained in a simple open ark, $A \subset G$, i.e. the open interval $(0, 1)$ topologically embedded to G, such that A is *isometrically* sent by α_0 to a minimal geodesic segment in Z. (Such an A is necessarily geodesic in G.)

(b) Every map between Riemannian manifolds locally (in the space of maps) minimizing some energy $\int_G \sigma(D(\alpha))$ for a continuous *positive* (i.e. $\sigma(D) > 0$ for $D \neq 0$) function of D, e.g. for the p-energy $\int \|D(\alpha)\|^p$ (that is a convex as well as positive functional for $p > 1$) is length extremal.

(c) A volume minimizing map is locally length extremal at the points $g \in G$ where $\mathrm{rank}\, D_g(\alpha_0) = \dim(G)$. (Notice that the volume of a map is a non-convex functional, but the rank constrain is due to its non-positivity rather than to non-convexity.)

It is obvious that every Lipschitz map α_0 between compact spaces, where the receiving space Z is locally contractible, can be *shortened* by some homotopy to a length extremal map $\alpha_{\min}$. This length shortening is most transparent for maps of graphs into CAT($\kappa \leq 0$) *spaces* Z that is used below for the *area shortening* of surfaces in such Z.

4.2 Recollection on CAT(0)-spaces. A length (e.g. Riemannian) space Z is called *Alexandrov of curvature* $\leq \kappa$ or *(local)* CAT(κ)-*space* for $\kappa \leq 0$, if the geodesic triangles in Z are "smaller" than those in the spaces of constant curvature κ in the following sense.

Let Δ^2_κ be a triangle in the (Euclidean or hyperbolic) plane of curvature κ and $\alpha_0 : \Delta^2_\kappa \to Z$ a continuous map that is locally isometric on each edge e_i of Δ^2_κ, $i = 1, 2, 3$ (i.e. every e_i goes to a geodesic between the images of the corresponding vertices of Δ^2_κ). Then there is a homotopy α_t, $t \in [0, 1]$, of $\alpha_0 = \alpha_{t=0}$ that is fixed on the boundary of Δ^2_κ (that is on the three edges) and such that $\alpha_1 = \alpha_{t=1}$ is 1-Lipschitz, i.e. distance (non-strictly) decreasing map. (If $\kappa > 0$ this is required only of the triangles Δ^2_κ in the hemisphere of curvature κ.)

REMARK. This definition is usually stated in terms of the universal covering $\tilde{Z}$ of Z, by saying that every *geodesic triangle in* $\tilde{Z}$, i.e. the union of three geodesic edges between three points in $\tilde{Z}$, denoted, $\partial = \cup_{i=1,2,3} e_i \subset \tilde{Z}$, can be filled-in by 1-*Lipschitz* map $\tilde{\alpha} : \Delta^2_\kappa \to \tilde{Z}$ isometric on the edges.

In fact, one can take this $\tilde{\alpha}$ in a (rather) canonical way. For example

the geodesic cone map over ∂ *from any given point* $z_0 \in \partial$ *is* 1-*Lipschitz.*

Moreover, one can use a filling by any ruled surface spanning ∂, since, according to Alexandrov,

the induced length metrics on ruled (and hence, length extremal) surfaces in CAT(κ) *spaces are* CAT(κ).

Geodesic interpolation. If $\tilde{Z}$ is a simply connected CAT($\kappa \leq 0$) space, then every two points $z_0, z_1 \in \tilde{Z}$ are joint by a *unique* minimizing geodesic segment of length $d = \mathrm{dist}(z_0, z_1)$, denoted $[z_0, z_1] \subset \tilde{Z}$, where $(1-t)z_0 + tz_1 \subset [z_0, z_1]$, $0 \leq t \leq 1$, denotes the point dividing this segment in the proportion $(1-t) : t$. Moreover,

$$\mathrm{dist}\big(((1-t)z_0 + tz_1), ((1-t)z_0' + tz_1')\big) \leq (1-t)\,\mathrm{dist}(z_0, z_0') + t\,\mathrm{dist}(z_1, z_1')$$

for all quadruples of points in $\tilde{Z}$. (This trivially follows from the above definition of CAT($\kappa \leq 0$).) Therefore,

the geodesic interpolation $((1-t)z_0(p) + tz_1(p))$ *between two* λ-*Lipschitz maps* $z_0(p), z_1(p)$ *of some metric space* P *to* $\tilde{Z}$ *is* λ-*Lipschitz.*

In particular, the space of the above 1-Lipschitz fillings of a geodesic triangle in $\tilde{Z}$ by Δ^2_κ is contractible.

It follows that if Z is not simply connected, then every path in a homotopy $z_t(p)$ between two maps $z_0(p), z_1(p)$ can be homotoped to a unique geodesic segment with the same end points and

the geodesic interpolation $((1-t)z_0(p)+tz_1(p))$ *is defined in the universal covering of every connected component of the space of maps of any* P *to* Z *and it preserves the class of "*λ*-Lipschitz" maps. In particular, the space of* 1-*Lipschitz* Δ^2_κ*-fillings of geodesic triangles in* Z *is either empty or contractible.*

Two CAT(κ) Examples. (a) Every complete Riemannian manifold with sectional curvatures $\leq \kappa$ is CAT(κ).

(b) Let G be a triangulated surface with a length metric ϱ, where every 2-simplex of the triangulation is isometric to a triangle in the plane of constant curvature κ. Then G is CAT(κ) if and only if the sum of the angles of the triangles at every vertex is $\geq 2\pi$.

Remark. These kind of examples, where the singular metrics can be approximated by regular ones, apparently, were the major motivation for A.D. Alexandrov. One knows now-a-days, since the work by Bruhat–Tits, that the class of singular CAT(κ)-spaces is much wider than that of non-singular (i.e. Riemannian) spaces even on the homotopy level.

Here are some relevant properties of CAT(κ) spaces going back, I guess, to Alexandrov:

1. *Gauss–Bonnet area inequality.* Every compact connected surface G with CAT($\kappa \leq 0$)-metric ϱ where the boundary $\partial(G)$ is geodesic (i.e. locally distance minimizing) has non-positive Euler characteristic and

$$\mathrm{area}(G) \leq 2\pi|\kappa|^{-1}\chi(G)\,;$$

moreover, there exists a metric ϱ_κ on S of constant curvature κ with geodesic boundary and a 1-Lipschitz homeomorphism $\varphi : (G, \varrho_\kappa) \to (S, \varrho)$. (This 1-Lipschitz map φ, for a suitable ϱ_κ, can be chosen conformal.)

Let $G = (G, \varrho)$ be a surface as in Example (b) triangulated into Δ^2_κ-triangles and let G_1 be the 1-skeleton of the triangulation. Let $\alpha_0 : G \to Z$ be a continuous locally edge extremal map that is locally isometric on each edge of G^1. Then

2. *The above sums of the angles are all $\geq 2\pi$ and the metric ϱ has curvature $\leq \kappa$.*

Consequently,

3. *There is a shortening homotopy α_t of α_0, that is fixed on the graph G^1 and such that the map $\alpha_{t=1} : G \to Z$ is 1-Lipschitz and, therefore, has* $\mathrm{area}(\alpha_{t=1}) \leq 2\pi\kappa^{-1}\chi(G)$ *if $\kappa \leq 0$.*

4. *Every map α_0 of a closed surface G into a compact* CAT($\kappa < 0$)*-space can be homotoped (by a family of distance decreasing maps α_t) to a map α_1 with* $\mathrm{area}(\alpha_1) \leq 2\pi\kappa^{-1}\chi(G)$ *for $\chi(G) < 0$ and to a map of zero area if $\chi(G) \geq 0$.*

The same conclusion (the existence of a homotopic map of small area) can also be achieved with a homotopy of α_0 to edge extremal α'_1 for the 1-skeleton G^1 of some triangulation of G with the consecutive filling-in all triangles by the Δ^2_κ-triangles with the edge lengths equal to the corresponding length of the edges in Z. We shall apply this below for shortening of *families* of such maps $\alpha_y : G_y \to Z$, where the topology of G_y may change at certain values of $y \in Y$. and we shall be using the following.

5. **Petrunin Theorem** [Pe]. *The metrics on surfaces G induced by locally length extremal maps into* CAT(κ)*-spaces are* CAT(κ).

This is reduced to the above by showing that such metrics ϱ can be approximated by those induced by edge extremal maps for subgraphs G^1 in G incorporating "sufficiently many" ϱ-minimal paths in G. (This approximation property is valid for all locally compact locally contractible spaces G.)

6. Reparametrization Corollary. *Let $\alpha : \Delta^2_\kappa \to Z$ be a Lipschitz map that is locally isometric on the three edges of Δ^2_κ and is locally length extremal at all points except the three vertices of Δ^2_κ. Then α can be reparametrized to a 1-Lipschitz map $\alpha_1 : \Delta^2_\kappa \to Z$, where "reparametrization" means that $\alpha_1 = \alpha \circ \beta$ for some map $\beta : \Delta^2 \to \Delta^2_\kappa$ that is the identity on the boundary of Δ^2_κ (where one uses β that is 1-Lipschitz for the length metric induced by α.)*

Remark. This is useful for controlling the areas of families of length extremal maps of surfaces by the (more manageable) Lipschitz constants of reparametrized maps. In fact, we shall work for most part with edge extremal maps for the 1-skeleta G^1 of triangulations of $G = \Delta^2_\kappa$, where the Petrunin theorem reduces to the above property 2.

4.3 Area shortening by harmonic flow. Given a continuous maps of a 2-dimensional space into a metric space, $\alpha : G \to Z$, let

$$\sup_y \mathrm{area}(\alpha) = \sup_{y\in Y} \; area(F|G_y)$$

and define the min-area of the homotopy class of α, denoted as the infimum of the $\sup_y$-areas of all maps $X \to Z$ homotopic to α

$$\text{min-area}[\alpha] = \inf_{\alpha'} \sup_y \mathrm{area}(\alpha')$$

over all maps $\alpha' : X \to Z$ in the homotopy class of α.

The simplest way to bound min-area for smooth spaces is to use the harmonic flow (see below) but we shall later adopt a combinatorial approach that avoid technicalities of harmonic maps between singular spaces.

min-area ***inequality for surface fibrations.*** Let $F : X \to Y$ be a locally trivial fibration where the fiber G is a smooth closed connected surface and and the structure group is $\mathrm{Diff}(G)$ and let Z be a compact manifold with sectional curvatures$\leq \kappa < 0$.

> *If G has non-negative Euler characteristics then every homotopy class of continuous maps $\alpha : X \to Z$ has* min-area$[\alpha] = 0$*; if $\chi(G) < 0$, then*
> $$\text{min-area}[\alpha] \leq 2\pi\kappa^{-1}\chi(G)\,.$$

Proof. Assign a smooth Riemannian metric ϱ_y to each fiber $G = G_y = F^{-1}(y)$ continuously depending on y and follow the (Eels–Samson) harmonic flows $\alpha_{y,t}$ starting from the restricted maps $\alpha_{y,0} = \alpha|G_y : (G_y, \varrho_y) \to Z$. (The harmonic flow on the space of maps $\alpha(G, \varrho) \to Z$ is the minus gradient flow for the energy $E_2(\alpha) = \int_G \|D(\alpha)\|^2$.) This converges to a continuous map $\alpha_\infty : X \to Z$ that is harmonic (i.e. E_2-minimizing) and, hence, length extremal, on every G_y. Since the induced length metric on the fibers are $\mathrm{CAT}(\kappa)$ by Petrunin's theorem (classically known in this case), the Gauss–Bonnet inequality applies.

REMARK. If the map α is a smooth *immersion* on each fiber G_y, then one can take the induced metrics in the fibers for ϱ_y. Then

> *the harmonic flow decreases the areas of the maps, since it decreases their energies.*

In general, the Riemannian "metric" ϱ induced by a smooth map $\alpha_y : G_y \to Z$ is singular, i.e. only *semi*definite, but it can be ε-*regularized* and, thus, made definite with an arbitrarily small perturbation $\varrho'_y = \varrho_y + \varepsilon$; then the harmonic flow applies to this perturbed metric and and almost (i.e. up to an arbitrarily small $\varepsilon > 0$) decreases the area of α_y.

COROLLARY. *Let M denote the space of continuous maps $\alpha : G \to Z$, where G is a closed connected surface and Z is a compact Riemannian* CAT(κ) *manifold and let $M_a \subset M$, $a \geq 0$, consist of the smooth maps with areas $\leq a$.*

> *If $\kappa < 0$ and $a \geq 2\pi\kappa^{-1}\chi(G)$, then the inclusion $M_a \subset M$ is a homotopy equivalence.*

It is easy to see that this remains true for complete (not necessarily compact) Z and, with a little work generalizes to singular (i.e. non-Riemannian) CAT($\kappa \leq 0$)-spaces Z. However, this is harder to implement for singular 2-polyhedra G and we shall be using below edge-extremal rather than harmonic maps (albeit we could use harmonic maps as well).

REMARK. If one successively applies the operation

$$\varepsilon_{i+1}\text{-regularization of } \alpha_i + \text{harmonic flow} \rightsquigarrow \alpha_{i+1}$$

to an initial smooth map $\alpha_0 : G \to Z$, one obtains in the limit a locally (in the space of maps) area minimizing map $\alpha = \lim_{i\to\infty} \alpha_i$.

If $\alpha_0 : G \to Z$ is *incompressible*, i.e. the α_0-image of no simple closed non-contractible curve in G is contractible in Z, then, at least for compact (complete with an extra stability assumptions on α_0) space Z, this minimal map is smooth; otherwise the image of α may decompose into several smooth components joined by some graphs of geodesics in Z.

QUESTIONS. (a) The above min-area inequality is rarely sharp. In fact,

$$\text{min-area}[\alpha] \leq 2\pi\kappa^{-1}\chi(G) - C\,,$$

where $C > 0$, unless α is homotopic to an immersion that is locally isometric (geodesic) with respect to some metric of constant curvature κ on G.

If, for example, Z is a compact manifold of constant curvature -1 then, by an easy limit argument, $C \geq 00.1 \min(1, (\operatorname{Inj}\operatorname{Rad}(Z)^2))$, unless α induces an isomorphism of the fundamental group of G onto a convex cocompact subgroup in $\Gamma = \pi_1(Z)$.

Can a presence of short closed geodesics $\gamma \in Z$ make C uncontrollably small? Are there minimal surfaces G in Z that are for the most part almost flat (geodesic) in Z and the lifts of which to the universal covering $\tilde{Z}$ are almost helical (i.e. invariant under one-parameter loxodromic subgroups of isometries) around lifts of some short geodesics γ to $\tilde{Z}$?

What condition can ensure min-area$[\alpha] \leq (1-C)2\pi\kappa^{-1}\chi(G)$?

(b) What happens for $k = \dim(G) \geq 3$? For example, let G be a compact connected k-dimensional manifold (the case of singular 2-polyhedra is also not fully trivial, compare 4.6, 4.7, 5.11) and consider the space M of smooth maps $\alpha : G \to Z$ in a fixed homotopy class of maps. What is a possible *Morse profile* of the function $\alpha \mapsto \mathrm{vol}_k(\alpha)$, $\alpha \in M$, for CAT(-1) spaces Z, i.e. (the behavior of) the relative homology $H_(M_b, M_a)$, $b \geq a$, of the family of subspaces $M_a \subset M$ of maps with k-volumes $\leq a$. For instance, when, for given numbers $0 \leq a \leq b \leq c < \infty$, every map $\alpha \in M_b$ can be homotoped within M_c to M_a? (If $k = 2$ then the relative homotopy groups $\pi_i(M_b, M_a)$, $i = 0, 1, 2, \dots$, vanish for all $a \leq b > 2\pi\kappa^{-1}\chi(G)$ by the harmonic flow argument.)

What is the asymptotic, for $i \to \infty$, behavior of the above α_i taken to be k-*harmonic* maps that minimize $\int_G \|D(\alpha)\|^k dg$, where the norm of the differential and the measure dg are taken with some ε_i-regularization of the metric induced by α_{i-1}?

Harmonic flow with singular fibers. If $F : X \to Y$ is a generic smooth maps $F : X \to Y$ with 2-dimensional fibers G_y, then the harmonic flow applies to the *non-singular fibers* G_y, but the resulting fiber-wise harmonic map α_∞ becomes discontinuous as one goes across the critical set $\Sigma(F) \subset Y$. However, one can recapture the continuity with an interpolation with harmonic maps with respect to intermediate induced metrics.

EXAMPLES. (a) Let some fiber G_1 be obtained by attaching a 1-handle to G_0 and let $\alpha_0 : G_0 \to Z$ be a length extremal (e.g. harmonic for some metric) map. Then this handle can be implemented by a narrow tube mapped to Z, so that the area of the resulting map $\alpha_{0,1} : G_2 \to Z$ does not exceed area$(\alpha_0) + \varepsilon$ for an arbitrarily small $\varepsilon > 0$ and the harmonic flow for the (regularized by an arbitrarily small perturbation) metric $\varrho_{0,1}$ induced by $\alpha_{0,1}$ provides the needed (1=parameter in this example) family of maps across the critical set.

(b) Let $\alpha_0 : G \times [0,1] \to Z$, $t \in [0,1]$, be a Heegard decomposition of a closed 3-manifold Z, i.e. the map α_0 is one-to-one on $G \times (0,1)$ and it sends $G \times 0$ and $G \times 1$ onto two disjoint subgraphs in Z. Then the harmonic flows for the metrics induced on $G \times t$ for $0 < t < 1$ deliver a family of harmonic maps interpolating between two maps that collapse G to graphs (different from the original ones and not even necessarily disjoint) in Z.

4.4 Area shortening with κ-maps. We work here with compact finite dimensional *p(iecewise) l(inear)* spaces G, where a *p.l.* structure is defined with a triangulation of G, and where two triangulation define the same *p.l.* structure if they admit a common subdivision.

A *p.l. family* G_y parametrized by a locally compact stratified space $Y \ni y$, i.e. represented by the fibers $G_y = F^{-1}(y)$ of a map $F : X = \cup_y G_y \to Y$, where X is called the total space of the family, is given by the following data:

- a triangulation of each G_y,
- •• *p.l.*-maps $P_{y_1,y_2} : G_{y_1} \to G_{y_2}$ for all pairs of points $y_1 \succeq y_2$, where this partial order signifies that the stratum containing y_2 lies in the boundary of the stratum of y_1.

These must satisfy the following conditions:

0. $P_{y,y} = id$ for all $y \in Y$.
1. $P_{y_1,y_3} = P_{y_1,y_2} \circ P_{y_2,y_3}$ whenever $y_1 \preceq y_2 \preceq y_3$.
2. If y_1 and y_2 belong to the same stratum $S \subset Y$, then the maps $P_{y_1,y_2} : G_{y_1} \to G_{y_2}$ are simplicial isomorphisms continuously depending on y_1, y_2. Thus, the spaces G_y make a flat fibration over each stratum $S \subset Y$.
3. If $y_1 \succ y_2$, then the map $P_{y_1,y_2} : G_{y_1} \to G_{y_2}$ is simplicial with respect to the triangulation in Y_{y_1} and some refinement of the triangulation of Y_{y_2}, where this refinement depends only on the $S_1 \ni y_1$ and $S_2 \ni y_2$ and where the map P_{y_1,y_2} is continuous in $(y_1, y_2) \in S_1 \times S_2$.
4. X is given a topology (often associated to a metric), that makes X locally compact and that is compatible with the maps P as follows.

The map $P_{y_1,y_2} : G_{y_1} \to X \supset G_{y_2}$ for y_1 belonging to a stratum $S \subset Y$ and y_2 to the closure $Cl(S) \subset Y$, continuously depends on (y_1, y_2) for the ordinary (compact open) topology in the space of maps. Furthermore, we require the map $F : X \to Y$ to be continuous.

***κ*-maps and their areas.** Let G be a triangulated 2-dimensional space, Z metric length space and $\alpha : G \to Z$ a continuous map that is geodesic (i.e. locally isometric) on every edge of the graph G and is length minimizing in its homotopy class (with given end points). Identify each 2-simplex in G with the Δ^2_κ-triangle (of constant curvature κ) the with edge lengths equal to the lengths of the corresponding geodesic segments in Z, denote by ϱ^α_κ the resulting length metric in G and call such α a *(triangulated) κ-map* tacitly assuming the presence of the underlying triangulation and of the metric ϱ^α_κ.

The κ-area of a κ-map is, by definition, the area of G with respect to the metric ϱ^α_κ.

If Z is CAT(κ) then every κ-map can be homotoped, keeping it fixed on the 1-skeleton of G, to a 1-Lipschitz map α. Such 1-Lipschitz maps, that are locally isometric on the edges of some triangulation of G are called *triangulated κ-short maps*. Clearly, every such map satisfies

$$\text{area}(\alpha) \leq \kappa\text{-}\,\text{area}(\alpha)\,.$$

Length shortening of κ-maps. Every continuous map α of a finite graph G^1 to Z can be first canonically deformed to a geodesic (i.e. locally isometric) map on the edges of G^1 without changing α at the vertices. If Z is CAT($\kappa \leq 0$), such geodesic map α_{geo} is unique on every connected component of G^1 containing at least one vertex but if G equals the circle S^1 with no (marked) vertex this α_{geo} is non-unique, but a canonical choice is (obviously) possible. Then, we deform α_{geo} in the class of geodesic maps but moving the vertices one by one and thus obtain

a length extremal map, that continuously depends on the initial α.

In fact, local non-extremality of a geodesic map at a vertex is an open condition in the space of geodesic maps that implies the continuity. Thus,

> *every continuous map $X \to Z$, where $X = \cup_{y \in Y} G_y$ is the total space of a p.l.-family G_y, is homotopic to a continuous map $\alpha_{\min}$ that is triangulated κ-minimal on every G_y with respect to some of the triangulations implied by the p.l.-structure on this G_y.*

Proof. Every G_y comes with a (coarsest) triangulation T_y and several subdivisions of it, say T'_y, T''_y, etc. We use maps $\alpha : X \to Z$ that are geodesic on the edges of T_y for all $y \in Y$ and such that every α is geodesic on the edges of some T'_y and also is G^1_y-extremal at the vertices of T'_y that lie in the interiors of the 2-simplices of T_y.

The short (i.e. 1-Lipschitz) Δ^2_κ- fillings of an edge triangle Δ in T_y depends on the subdivision and, due to this ambiguity, such an α may be discontinuous. But since the space of short Δ^2_κ-fillings of every geodesic triangle (with the edges coming from T^1_y) is contractible (by the geodesic interpolation, see 4.2), every such α can be deformed, to the required *continuous* $\alpha_{\min}$.

4.5 Evaluation of min-area$_\kappa$ via $|\chi|_{\rm hyp}$ of *p.l.* families. Define min-area$_\kappa$, for $\kappa \le 0$, of a *p.l.*-family $X = \cup_y G_y$ with 2-dimensional fibers G_y as the infimum of the numbers $a > 0$, such that every continuous map of X to a complete CAT(κ) space Z is homotopic to a continuous map α that is a triangulated κ-map on each G_y with respect to some subdivision of the original *p.l* structure on this G_y with κ-area$(\alpha|G_y) \le a$.

Define $|\chi|_{\rm hyp}(G)$ for (possibly non-compact and disconnected) surface G as the absolute value of sum of the Euler characteristics of all its hyperbolic connected components, i.e. those with negative Euler characteristic. (This equals one half of the simplicial volume for closed surfaces G.)

Example. If $X \to Y$ is a surface fibration, i.e. all fibers G_y are homeomorphic to a closed surface G and all maps $P_{y_1,y_2} : G_{y_1} \to G_{y_2}$ homeomorphisms then, the above shortening with the Gauss–Bonnet inequality imply that

$$\text{min-area}_\kappa(X) \le 2\pi|\kappa|^{-1}|\chi|_{\rm hyp}(G)\,.$$

(If Z is complete non-compact, then our shortening process may diverge, but it makes κ-area $\le 2\pi|\kappa|^{-1}|\chi|_{\rm hyp}(G) + \varepsilon$ for every $\varepsilon > 0$.)

For our applications, we need a similar inequality for *point-singular surface fibrations*, e.g. generic smooth maps $F : X \to Y$, defined as follows.

A locally compact polyhedral (possibly disconnected) space G is called a *point-singular surface* if it is locally homeomorphic to $\mathbb{R}^2$ away from a discrete subset $G^{\rm sing} \subset G$ (thus allowing point components in G). Every such G admits a (unique up to a reparametrization) *regularization* by a (non-singular!) surface $\hat{G}$, that is a locally homeomorphic map $H : \hat{G} \to G$ which is a homeomorphism over the non-singular locus $G \setminus G^{\rm sing}$.

A *p.l*-family of such G, say $X = \cup_y G_y$, is called (at most) *point-singular* (surface (quasi)fibration) if every map $P_{y_1,y_2} : G_{y_1} \to G_{y_2}$ is proper and one-to-one over G_{y_1} minus a discrete subset $S \subset G$. (If $X \to Y$ is a generic smooth map, then $S \subset G^{\rm sing}_{y_1}$.)

area$_\kappa$-*inequality with point singularities.*

Every point-singular surface fibration $X = \cup_y G_y$ with compact fibers G_y satisfies,

$$\text{min-area}_\kappa(X) \leq 2\pi|\kappa|^{-1} \max_y |\chi|_{\text{hyp}}(\hat{G}_y)$$

for the regularizing surfaces $\hat{G}_y$ of G_y.

Proof. Let us replace each G_y by a space $\hat{G}_y^+ \supset \hat{G}$ as follows. Consider the singular points $g \in G_y^{\text{sing}} \subset G_y$, that are *not* point components of G_y, i.e. those g that have ≥ 2 preimages $\hat{g}_i \in \hat{G}_y$. Attach the unit segment $[0,1]$ by the 0-end to each pre-image point $\hat{g}_i \in \hat{G}$, $i = 1, 2, \dots,\ j = j(g) \geq 2$, and identify the 1-ends of the segments whenever the corresponding points from $\hat{G}$ go to the same singular point $g \in G_y$.

There is a natural extension of H_y to a continuous map $H_y^+ : \hat{G}^+ \to G_y$, where the pullback of every singular g, denoted $J_g^+ \subset \tilde{G}^+$, is the union of $j(g)$-copies of the $[0,1]$-segment identified at the 1-ends. This map is a homotopy equivalence where the homotopically inverse map $H_y^- : G_y \to \hat{G}_y^+$ can be taken homeomorphic away from a union of small balls $B_g \subset G_y$ around the singular points $g \in G_y$ and such that each ball B_g is smashed in the obvious way onto J_g^+.

Given a continuous map $G_y \to Z$, we slightly deform it to another map that factors through H_y^- and thus we replace each G_y by $\hat{G}_y^+$ mapped to Z. We shorten this map on (the 1-skeleton of some triangulation of) the (non-singular!) surface $\hat{G}_y$, and, after this map has been made edge extremal, we shorten it (which is not truly necessary) on the remaining 1-dimensional (hence, not contributing to the area) J_+-parts. Then the Gauss–Bonnet inequality applies to the induced metrics on $\hat{G}^+$ that have the same areas as all of G_y. QED.

area$_\kappa$-inequality with point/edge singularities. Let the link $L(g)$ of every point g be the disjoint union of copies of the graphs L_k, where each L_k has two vertices with k edges between them.

Every such G admits a (unique up to a reparametrization) regularization by a surface $\hat{G}$ *with* boundary where the regularizing map $H : \hat{G} \to G$ sends the boundary of $\hat{G}$ onto the set $E(G) \subset G$ of the *edge points* $g \in G$, where the link has some L_k with $k \neq 2$ and where the map is one-to-one on each connected component of $\hat{G}$ away from the pullback of the points g of G where the link $L(g)$ is disconnected.

A *p.l.*-family $X = \cup_y G_y$ of the above surfaces is called (at most) *point/edge singular* if every map $P_{y_1,y_2} : G_{y_1} \to G_{y_2}$ is proper, it sends the edge points of G_{y_1} to the edge points of G_{y_2} and it is one-to-one over G_2 minus a discrete subset in G. (A relevant example is the map $R \circ F$ delivered by the Retraction Lemma in 2.2.)

Every point/edge singular family $X = \cup_y G_y$ with compact fibers G_y satisfies,

$$\text{min-area}_\kappa(X) \leq 2\pi|\kappa|^{-1} \max_y |\chi|_{\text{hyp}}(\hat{G}_y)\,,$$

where $|\chi|_{\text{hyp}}$ is defined as earlier by discounting the components of the regularizing surfaces $\hat{G}_y$ that have positive Euler characteristics.

Proof. We proceed as earlier by disengaging branches of each fiber G_y at the points g where the link $L(g)$ is disconnected and thus passing to ("normalization") $\overline{G}_y \to G$ where all links in $\overline{G}$ *are* connected. Then, as the only novelty, we insist on maps α that are locally isometric (geodesic) along the edge singularities of G_y and thus locally edge extremal at these points. Since this property of each G_{y_2} passes to the nearby fibers G_{y_1} via the maps P_{y_1,y_2}, an arbitrary continuous family deforms to an edge-extremal one; then the Gauss–Bonnet inequality applies.

4.6 Further bounds on the κ-area. The above cover most of our applications; to have a better perspective we present below several more general but less precise area inequalities.

Simplicial area and the inequality $\mathrm{area}_\kappa \le \pi|\kappa|^{-1}\,\mathrm{area}_\Delta$. Given a *p.l.* family $X = \cup_y G_y$, consider the triangulations $\mathcal{T} = \{T_y\}$ of the fibers G_y for which the maps $P_{y_1,y_2} : G_{y_1} \to G_{y_2}$ are simplicial, let $\mathrm{area}_\Delta(\mathcal{T})$ denote the supremum over y of the numbers of the 2-simplices in the triangulations T_y and set

$$\mathrm{area}_\Delta(X) = \inf_{\mathcal{T}} \mathrm{area}_\Delta(\mathcal{T}) .$$

Observe that if $\dim(G_y) = 2$ and $\kappa \le 0$, then

$$\mathrm{area}_\kappa(X) \le \pi|\kappa|^{-1}\,\mathrm{area}_\Delta(X) .$$

Proof. Geodesically straighten in Z the edges of triangulations T_y in G_y (mapped to Z) and recall that the triangles Δ^2_κ (in the hyperbolic plane of curvature κ) have areas $\le |\kappa|^{-1}\pi$.

The usefulness of this depends upon bounds on $\mathrm{area}_\Delta(X = \cup_y G_y)$ in terms of $\sup_y \mathrm{area}_\Delta(G_y)$.

Example. Let $F : X \to Y$ be a fibration where all fibers $G_y = F^{-1}(y)$ are homeomorphic to a connected surface G of genus ≥ 2. Then

> *there exists another family $F' : X' \to Y$, where all $G'_y = (F')^{-1}(y)$ are also homeomorphic to G and that is (not being a fibration) fiber-wise homotopy equivalent to $X \to Y$ and such that*
>
> $$\mathrm{area}_\Delta(X') \le C^{1+\dim(Y)}|\chi(G)| ,$$
>
> *for some constant $C \le 100$; furthermore, there always exits such X' (for every Y) with*
>
> $$\mathrm{area}_\Delta(X') \le C^{1+3|\chi(G)|}|\chi(G)| .$$

Proof. Let $F : X \to Y$ be a smooth fibration with closed k-dimensional manifold fibers G_y and let ρ_y be a continuous family of Riemannian metrics on the fibers with 1-bounded geometry, i.e. with $|\,\mathrm{curv}\,| \le 1$ and $\mathrm{Inj\,Rad} \ge 1$. Then, by the standard triangulation argument, there is a *p.l.* structure on the family $X = \cup_y\{G_y\}$ associated to some stratification (actually, a triangulation) of Y with all $P_{y_1,y_2} : G_{y_1} \to G_{y_2}$ being homeomorphisms, where the numbers N_y of the simplices in the implied triangulations of the fibers G_y are bounded by

$$N_y \le C_k^{1+\dim(Y)}\,\mathrm{vol}(G_y,\rho_y)$$

for some constant $C_k \approx k^k$. (Probably, there is a *p.l.* structure with $N_y \le C_k \dim(Y)\cdot \mathrm{vol}(G_y,\rho_y)$ and possibly with $C_k(\dim(Y) + \mathrm{vol}(G_y,\rho_y))$.)

If $k = 2$ and $\chi = \chi(G_y) < 0$, then, by the uniformization theorem, the fibers G_y of an arbitrary fibration carry metrics of constant curvature -1 and these can be "thickened" around short geodesics to a family of metrics ρ_y with bounded geometries and areas about $|\chi|$. Then we take the above *p.l.* structure and pass to X' by approximating the (homeomorphic) maps $P_{y_1,y_2} : G_{y_1} \to G_{y_2}$ by simplicial (not necessarily homeomorphic) maps $P'_{y_1,y_2} : G_{y_1} \to G_{y_2}$.

Thus we obtain the first bound on $area_\Delta$, while the second one follows by a reduction to the universal family G_u of surfaces over the Riemann modular space U that has dimension $\dim(U) = 3|\chi|$.

Remarks and Questions. (a) The above generalizes to the point and point/edge singular surface fibrations, where the bound on the numbers N_y of the 2-simplices depends, besides the Euler characteristics of the "normalizations" $\hat{G}_y$, also on the numbers of the singular points and of the edges in G_y.

(b) The area_Δ-approach extends from $\text{CAT}(\kappa < 0)$ to all aspherical finite dimensional simplicial polyhedra Z with word hyperbolic fundamental group Γ:

> *every continuous map $\alpha : X \to Z$ is homotopic to a continuous map α', such that* $\text{area}(\alpha') \leq \text{const}(Z)\,\text{area}_\Delta(X)$, *for the piecewise Euclidean metric associated to the triangulation of Z.*

In fact, there exists Γ-equivariant Lipschitz maps

$$G_k : (\tilde{Z})^{k+1} \times \Delta^k \to \tilde{Z}\,,$$

for all $k = 1, 2, \dots$, where $(\tilde{Z})^{k+1}$ is the $(k+1)$th Cartesian power of the universal covering $\tilde{Z}$ of Z and Δ^k is the k-simplex, that is invariant under the $(k+1)$-permutation group diagonally acting on $(\tilde{Z})^{k+1} \times \Delta^k$ and that has the following properties:

- • The k-volume of G_k restricted to every k-simplex $z_0 \times z_1 \times \dots \times z_k \times \Delta^k$ is bounded by a constant $\text{const} = \text{const}(Z, k)$ for all $k = 2, 3, \dots$.
- •• The restriction of G_k to the l-face of each simplex $z_0 \times z_1 \times \dots \times z_k \times \Delta^k$ is equal, up to a reparametrization, to G_l on this face, where "reparametrization" is a self-homeomorphism of this face (that is Δ^l) that preserves all subfaces (the faces of Δ^l).

Such G_k are constructed starting with G_1, where one uses parametrized geodesic lines in $\tilde{Z}$ and where • depends on the hyperbolicity of a suitable parametrization (see [Gr3][MMS]).

Combinatorial harmonic maps. The *combinatorial (quadratic) energy E* of a continuous and geodesic on the edges map of a graph G^1 to a length space Z is the sum of the squares of the lengths of the edges mapped to Z. If Z is (locally) $\text{CAT}(\kappa \leq 0)$, then every continuous map is homotopic to a unique harmonic map that minimizes the energy.

The area of every κ-triangle is bounded by $1/2\sqrt{3}$ times the sum of the squares of the three edges, and this bound is sharp for the regular flat triangles and it is almost sharp for the small almost regular triangles for all κ. Thus, if we start with a κ-map α of a two-polyhedron G into Z, where all triangles are small and where the ε regular ones cover all area of G up-to ε, then the corresponding *edge harmonic* κ-map α_{harm}, i.e. harmonic on the 1-skeleton $G^1 \subset G$, has

$$\mathrm{area}_\kappa(\alpha_{\mathrm{harm}}) \leq \mathrm{area}_\kappa(\alpha) + a(\varepsilon)\,,$$

where $a(\varepsilon) \to 0$ for $\varepsilon \to 0$.

Then we observe that every Δ^2_κ-triangulation of G can be subdivided into arbitrarily small Δ^2_κ-triangles, where almost all the area is covered by almost regular ones. Thus, as in the smooth case, we obtain, for every $a > 0$,

> *a deformation α_t of an arbitrary κ-map α_0 and every $a > 0$ to an extremal map α_1 (edge harmonic on some subdivision), where $area_\kappa(\alpha_t) \leq area_\kappa(\alpha_0) + a$. for all $0 \leq t \leq 0$, where one may use the geodesic interpolation $\alpha_t = (1-t)\alpha_0 + t\alpha_1$.* (The energy, unlike the area, is convex in t for the geodesic interpolation in CAT($\kappa \leq 0$)-spaces).

It follows, for example, that

> *if all maps $P_{y_1,y_2} : G_{y_1} \to G_{y_2}$ in some p.l. family $X = \cup_y G_y$ are one-to-one over the interiors of the 2-simplices in G_{y_2}, then every continuous map $\alpha_0 : X \to Z$ can be homotoped to α_1, such that the κ-area of α_1 on each G_y does not exceed, up to an arbitrarily small $a > 0$, the κ-area of an edge extremal map $G_{y'} \to Z$ for some $y' = y'(y) \in Y$.*

Bounds on the $\boldsymbol{\kappa}$-area in regular spaces $\boldsymbol{Z}$. Let G be 2-dimensional space which is, at each point $g \in G$, is topologically the cone from g over a finite graph, called the link L_g. We assume that all L_g are connected (otherwise, disengage them as earlier) and non-contractible (otherwise, G contracts to its proper subspace). A point $g \in G$ is called *regular* if L_g is the circle, it is *an edge point* if L_g consists of $k \geq 3$ edges between two points and g is called a *vertex* for the other topologies of the link L_g.

Let G be endowed with a length metric that has curvature $\leq \kappa$ at the regular points and where the edges are geodesics. The graphs L_g carry natural *angular metrics* (corresponding to the tangent cones metrics of G at g) where the length of an edge $l \subset L_g$ is denoted $\angle(l)$.

Then the Gauss–Bonnet inequality bounds the area of G in terms of these angles and the Euler characteristic of the set $\mathrm{reg}(G) \subset G$ of the regular points in G,

$$\kappa \cdot \mathrm{area}(S_i) + \sum_{g \in G^0} \sum_{l \in L^1_g} \big(\pi - \angle(l)\big) \geq 2\pi\chi(\mathrm{reg}(G))\,,$$

where G^0 denotes the set of vertices of G and L^1_g the set of edges in the link L_g.

For example, if $\kappa \leq 0$ and all connected components of $reg(G)$ are triangles, each bounded by three geodesic edges and where we allow vertices of such triangle merge to a single vertex point in G, then

$$\mathrm{area}(G) \leq |\kappa|^{-1} \sum_{g \in G^0} \sum_{l \in L^1_g} \big(\pi/3 - \angle(l)\big)\,.$$

In particular, these angles should be $\leq \pi/3$ on the average.

If the metric on G is induced by an extremal map α of G to a CAT(κ) space Z, which, moreover, minimizes the total length (i.e. l_1-energy) of the edges, then, depending on the combinatorics of the graphs L_g, $g \in G^0$, and the singularity types of Z at the points $\alpha(g) \in Z$, one can bound from below the sums $\sum_{l\in L^1_g} \angle(l)$ and, thus, bound area(α).

Since the extremal map that minimizes the total length of the edges may collapse some edges to points, it is convenient to make such a collapse beforehand and arrive at the case where G has a single vertex g. Such G gives the *triangular* presentation of the group $\Pi = \pi_1(G)$; it is obtained by attaching triangles to G^1 that is a joint of circles, where each edge of a triangle goes around a single loop in G^1 and where no loop in Z corresponding to an edge-loop in G is contractible in Z (otherwise, we just remove such loops from G).

Given a graph L and a metric space S with the distance denoted by $\angle$, (e.g. the base S_z of the tangent cone of Z at $z = \alpha(g)$), consider the *centered* maps $\tau : L^0 \to S$, such that the cone cone(τ) of this map, sending the unit cone cone(L) to the unit metric cone cone(S), minimizes the total length of cone$(L) \to$ cone(S) with the fixed ends in S.

For example, if S equals the sphere $S^{N-1} \subset \mathbb{R}^n$, then cone$(S^{N-1})$ equals the unit N-ball and the centered maps $\tau : L^0 \to S^{N-1} \subset \mathbb{R}^N$ are those having $\sum_{l\in L^0} \tau(l) = 0$.

Define
$$\angle_{\min}(L,S) = \frac{1}{\mathrm{card}(L^1)} \inf_\tau \sum_{l\in L^1} \angle\big(\tau(\partial_+ l), \tau(\partial_- l)\big)$$
where L^1 denotes the set of the edges l of the graph L, with the ends called $\partial_+ l$ and $\partial_- l$ and where the infimum is taken over all centered τ. (For example, if S equals the sphere $S^{N-1} \subset \mathbb{R}^N$, then cone$(S^{N-1})$ equals the unit N-ball and the centered maps $\tau : L^0 \to S^{N-1} \subset \mathbb{R}^N$ are those having $\sum_{l\in L^0} \tau(l) = 0$.)

In particular, if S equals the two point set $\{-1,+1\}$ regarded as the unit sphere in $\mathbb{R}^1 \supset [-1,+1] = \mathrm{cone}\{-1,+1\}$, then the extremal maps $\tau : L^0 \to \{-1,+1\}$ for $L = L_g$ are those where $\tau(v) \neq \tau(v')$ for the vertices v and v' in L_g representing the pairs of the ends of the edge-loops in G, and
$$\angle\big(\tau(\partial_+ l), \tau(\partial_+ l)\big) = \tfrac{1}{2}\pi \big|\tau(\partial_+ l) - \tau(\partial_- l)\big| \,.$$
Thus,
$$\angle_{\min}\big(L, \{-1,+1\}\big) = \pi \frac{1}{\mathrm{card}(L^1)} \inf_\tau \mathrm{card}(L^1_\tau)\,,$$
where L^1_τ is the set of the edges in L going from the subset $\tau^{-1}(-1) \subset L^0$ to $\tau^{-1}(-1) \subset L^0$.

Summing up, we conclude:

Let G be a triangular presentation of $\Pi = \pi_1(G)$, i.e. G has a single vertex $g \in G$ and all connected components of reg(G) are triangles, and let Z be a complete CAT$(\kappa \leq 0)$ space. Then

> *every continuous map $\alpha : G \to Z$, where the α-image of no edge-loop in G^1 is contractible in Z, is homotopic to a continuous map $\alpha_{\min} : G \to Z$, such that*
> $$\mathrm{area}(\alpha_{\min}) \leq 3\,\mathrm{card}(G^2)\Big(\pi/3 - \inf_{z\in Z} \angle_{\min}(L_g, S_z)\Big) + \varepsilon\,,$$
> *where* card(G^2) *denotes the number of triangles in G (that are connected components of* reg(G)*), where L_g is the link at the vertex of G and S_z*

denote the bases of the tangent cones of Z at the points $z \in Z$ and where ε is a given positive number. (If Z is compact one may take $\varepsilon = 0$.)

Remark. If S is the Hilbert sphere, $S = S^\infty \subset \mathbb{R}^\infty$, then $\angle(L, S)$ is of the same order of magnitude as the first eigenvalue of the combinatorial Laplacian on L and the above inequality is similar to the Garland–Borel criterion for the vanishing of the cohomology of groups Π with coefficients in Hilbert moduli. (Here one allows *not necessarily free* actions of Π on Hilbert spaces, and, in general, on simply connected $\mathrm{CAT}(\kappa \leq 0)$ spaces $\tilde{Z}$.)

Possibly, $\angle(L, S^\infty) \geq \pi/2 - \varepsilon > \pi/3$ for most (expander) graphs L (this would yield the Kazhdan T-property of the corresponding groups Π, for example); but even if $\angle(L_g, S) < \pi/3$ the above area inequality is useful in conjunction with a *lower bound* on the area of minimal 2-subvarieties in Z (see 6.3, 6.4).

4.7 Volume bounds for simplicial families. A *p.l. family* G_y is called *simplicial* if the adjacency maps $P_{y_1,y_2} : G_{y_1} \to G_{y_2}$ are simplicial with respect to the triangulations in G_y.

Example. Start with a simplicial map $F_0 : X_0 \to Y$ where, observe, the fibers $G_y = F^{-1}(y)$ are naturally partitioned into convex cells contained in the simplices of the triangulation of X_0. If one subdivides these cells into simplices and approximate the adjacency maps between the fibers by simplicial maps, one obtains a simplicial family, say $F : X \to Y$ that is fiberwise homotopy equivalent to F. In fact, there exists a *p.l.*-map $h : X_0 \to X$, such that $h \circ F = F_0$ where $h^{-1}(x)$ is a cell in X for all $x \in X$.

Observe that, unlike the *p.l.* case, one cannot in general achieve this with a homeomorphism h. For example, if all G are finite simplicial complexes and the adjacency maps $P_{y_1,y_2} : G_{y_1} \to G_{y_2}$ are simplicial and bijective, then F is a fibration with a finite structure group.

If Z is $\mathrm{CAT}(\kappa < 0)$ space, then the Thurston straightening argument (see 5.43 in [Gr9]) shows that

every continuous map of a simplicial family $\alpha_0 : X = \cup_y G_y \to Z$ is homotopic to an α, such that the k-volume of each $\alpha_y : G_y \to Z$ for $k \dim G_y$ is bounded by

$$\mathrm{vol}_k(G_y) \leq \delta_{\kappa,k} \, \mathrm{vol}_\Delta(G_y)\,,$$

where $\mathrm{vol}_\Delta(G_y)$ denotes the number of the k-simplices in G_y and δ_k is the volume of the ideal regular simplex of constant curvature -1.

Remarks and Questions. (a) The above generalizes to δ-hyperbolic spaces Z with the discussion in 4.6.

(b) It is unclear what happens to *p.l.*-families $X = \cup_y G_y$ for $k \geq 3$, where the corresponding geometric problem concerns smooth fibration $X \to Y$ where one looks for continuous families ρ of Riemannian metrics ρ_y (instead of triangulations) on the fibers G_y with some bounds on the geometries of (G_y, ρ_y). For example, one is concerned with $\inf_\rho \sup_y \mathrm{vol}(G_y)$, where inf may be taken over the following families of metrics (on the fibered spaces over Y that are fiber-wise diffeomorphic to $X \to Y$, or, alternatively, fiber-wise homotopy equivalent to $X \to Y$).

(1) The families ρ with fiber-wise bounded geometries,
(2) ρ with $|\operatorname{curv}(\rho_y)| \le 1$,
(3) ρ with $\operatorname{curv}(\rho_y) \ge -1$.

These and similar curvature/size bounds (see 5.10 for a fuller list) have been studied for individual manifolds G (see 5.41 in [Gr9] and $5\frac{5}{7}$ in [Gr6]); one wonders if something essentially new happens to the families.

One expects conclusive results for families of 3-manifold (with the 2-dimensional uniformization replaced by the Hamilton–Perelman flow) but there is little hope for something comparable in higher dimension (except, possibly, $k = 4$) due to the absence of canonical Riemannian metrics on smooth high-dimensional manifolds.

5 Morse Spectra in the Spaces of Cycles and the Lower Bounds on min-area

Given a topological space Z, denote by $ch_k = ch_k(Z, \mathbb{Z}_p)$ the set of the singular k-chains in Z with $(\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z})$-coefficients and let $cl_k = cl_k(Z, \mathbb{Z}_p) \subset ch_k(Z, \mathbb{Z}_2)$ be the set of the cycles.

A *path* between two k-cycles $c_0, c_1 \in cl_k$ is, by definition, a chain $\Delta^1_{01} \in ch_{k+1}$ such that $\partial(\Delta_{01}) = c_1 - c_0$. Given three k-cycles c_0, c_1, c_2, define the *triangle of paths* ∂^1_{012} by $\partial^1_{012} = c_{01} + c_{12} + c_{20}$ that is, clearly, a $(k+1)$-cycle. A *filling triangle* for this cycle is a $(k+2)$-chain $\Delta^2_{012} \in ch_{k+2}$ with $\partial(\Delta^2_{012}) = \partial^1_{012}$. Similarly, quadruples of filling triangles for a quadruple of cycles make $(k+2)$-cycles ∂^2_{0123} that are "potential boundaries" of *filling* 3*-simplices* (that may not exit in Z) $\Delta^3_{0123} \in CH_{k+3}$ that are defined by the condition $\partial(\Delta^3_{0123}) = \partial^3_{0123}$, etc.

Thus the set cl_k is endowed with a structure of a (semi)simplicial space where a potential boundary $\partial^i_{0123\dots}$ can actually be filled in if and only if the $(k+i)$-cycle $\partial^i_{0123\dots} \in cl_{k+i}$ is homologous to zero. Therefore, this space is (almost canonically) isomorphic to the (semi)simplicial representative of the product of the Eilenberg–MacLane spaces associated to the homology of the space Z. This implies

Dold–Thom–Almgren Theorem. *The space of cycles is homotopy equivalent to the product of the Eilenberg–MacLane spaces,*

$$cl_k(Z; \mathbb{Z}_p) \simeq \times_j K\big(H_j(Z; \mathbb{Z}_p), j-k\big) .$$

Therefore, the $\mathbb{Z}_p$-cohomology of $cl_k(Z; \mathbb{Z}_p)$ equals the tensor product of $N = \sum_{j \ge k} \operatorname{rank}(H_j(Z; \mathbb{Z}_p))$ standard (Cartan–Serre) moduli over the p-Steenrod algebra.

If Z is a connected (oriented for $p > 2$) manifold (or a $\mathbb{Z}_p$-pseudo-manifold), then $[Z]_{-k} \in H^{(N-k)}(cl_k(Z); \mathbb{Z}_p)$ denotes the fundamental class of $K(Z_p, N-k)$ corresponding to the fundamental class of Z.

Remarks. (a) The above definition and the Dold–Thom–Almgren theorem obviously extend to the spaces of relative cycles as well as to cycles with coefficients in arbitrary local systems over Z. We shall mostly deal with $\mathbb{Z}_2$-cycles.

(b) The functor $Z \rightsquigarrow cl_*(Z)$ is equivalent to the homology functor. In particular it commutes with suspensions and "twisted suspensions" that are Thom spaces of vector bundles over Z. If Z is an N-dimensional manifold, there is an intersection pairing $cl_k \times cl_l \to cl_{k+l-N}$. Also there is a pairing $cl_k(S) \times cl_l(Z^S) \to cl_{k+l}(Z)$, where S is an arbitrary space and S^Z denotes the space of continuous maps $S \to Z$. If Z and S are manifolds, there is the dual (Gysin intersection) pairing $cl_k(Z) \times cl_l(Z^S) \to cl_{M-N+k+l}(S)$ for $M \dim(S)$.

If $Z \to Z_0$ is a fibration, then, probably, the Serre filtration on the singular chains and/or the spectral sequence can be adequately represented by some spaces of "filtered cycles".

On the other hand, the topology and/or geometry of cycles in Z gives an additional structure to cl_k that cannot be accounted for by the homology of Z as we shall see below.

Smooth representation of $\mathbb{Z}_2$-cycles. A D-dimensional cycle C in $cl_k(Z;\mathbb{Z}_p)$ representing a D-dimensional class $h = [C] \in H_D((cl_k;\mathbb{Z}_p);\mathbb{Z}_p)$ can be thought of as a $(D+k)$-dimensional $\mathbb{Z}_p$-pseudomanifold mapped to Z, say $\alpha : X \to Z$, where X is simplicially mapped to ("fibered" over) a D-dimensional $\mathbb{Z}_p$-pseudomanifold, $F : X \to Y = Y^D$, where the fibers of F called the *fibers* $G_y \subset X$, when mapped by α to Z, make the Y*-family* of k-dimensional $\mathbb{Z}_p$-cycles that constitute C. The cycle C is homologous to zero in $cl_k(Z;\mathbb{Z}_p)$ if and only if the image of C under the map $\alpha \times F$ is homologous to zero in $Z \times Y$.

If $p = 2$, then, by Thom's theorem, the fundamental class of Y can be represented by a smooth manifold mapped to Y, say $\tilde{a} : \tilde{Y} \to Y$. Then we take the fibered product $X' = X \times_Y \tilde{Y}$, that is, for a generic $\tilde{a}$, a $\mathbb{Z}_2$-pseudomanifold and represent its fundamental class by a manifold, $\tilde{b}' : \tilde{X}' \to X'$. Thus, we could use *smooth manifolds* for X and Y to start with

> *every homology class $h \in H_D((cl_k;\mathbb{Z}_2);\mathbb{Z}_2)$ can be represented by a smooth $(D+k)$-dimensional manifold $\alpha : X \to Z$, where X is smoothly generically mapped to a D-dimensional manifold by $F : X \to Y$. In particular, every h can be represented by a Y-family of k-cycles $\alpha_y : G_y \to Z$, where each G_y is a smooth manifold apart form a finite subset in G_y.* (If $k = 2$, these G_y are just point-singular non-oriented surfaces.)

Fiber complexity and Σ-complexity of an $h \in H_*(cl_1(Z))$. The topology of possible *parameter spaces*, manifolds or pseudomanifolds Y representing h, is essentially independent of Z, since the homotopy type of the space $cl_k(Z;\mathbb{Z}_2)$ is the same for all Z with a given homology $H_*(Z;\mathbb{Z}_2)$. On the other hand, the minimal possible "topological complexity" of the critical set $\Sigma(F) \subset Y$, called Σ*-complexity* of h, and/or of the topology of the "maximally complicated" fiber G_y expressing the *fiber complexity*, provide additional homotopy invariants of Z that are not captured by $H_*(Z)$.

Algebraic representation of cycles. Recall that every m-dimensional smooth manifold X carries a *Nash algebraic structure* given by a smooth embedding $X \subset \mathbb{R}^M$, such that X is contained in the real locus of an m-dimensional algebraic variety $\mathbb{C}X \subset \mathbb{C}^M = \mathbb{C}\mathbb{R}^M \supset \mathbb{R}^M$ defined over $\mathbb{R}$.

Given Nash structures on the above manifolds X and Y, the map F can be approximated by a real algebraic map F_{alg}. Thus

every class $h \in H_D((cl_k; \mathbb{Z}_2); \mathbb{Z}_2)$ is represented by a D-dimensional algebraic family of k-dimensional algebraic varieties $G_y = F_{\mathrm{alg}}^{-1}(y) \subset X \subset \mathbb{R}^M$ mapped by α to Z.

Furthermore, if Z is a smooth manifold, we give it a Nash structure and take an algebraic map for α. Thus,

every homology class $h \in H_D(cl_k(Z; \mathbb{Z}_2); \mathbb{Z}_2)$ can be implemented by a D-dimensional real algebraic family of real algebraic subvarieties $\underline{G}_y = \alpha(G_y)$ of dimensions k in Z.

Questions. (a) What are the relations between the singularities of the map F and the action of the Steenrod algebra on the cohomologies of Z and $cl_k(Z)$? For example, which classes h admit a smooth representation with a *purely folded* map $F : X \to Y$? When does there exit a representation where F is a fibration with a finite structure group? (This must be known to homotopy theorists.)

(b) The minimal possible algebraic degree $d = d(h)$ of the above algebraic varieties $G_y = F_{\mathrm{alg}}^{-1} \subset Z \subset \mathbb{R}^{M(h)}$ in a family representing a given class $h \in H_*(cl_k(Z; \mathbb{Z}_2); \mathbb{Z}_2)$ is an amusing homotopy invariant of Z. Can one evaluate it in terms of traditional invariants of Z and h?

For example, the function $d(h)$ is bounded for Z equal the N-sphere S^N and $k = N - 1$. In fact, $d(h) = 2$ for all $0 \neq h \in H_D(cl_{N-1}(S^N; \mathbb{Z}_2); \mathbb{Z}_2) = \mathbb{Z}_2$, $D \geq N - 1$. But it seems that $d(h)$ is unbounded in many (most?) cases.

A similar problem concerns the minimal possible degree $\underline{d} = \underline{d}(h, Z)$ of $\underline{G}_y \subset Z$ for a given algebraic structure in Z, (e.g. for the standard N-sphere S^N for Z). Here one has an easy bound $\underline{d}(h) \geq \mathrm{const}(Z) D^{\frac{1}{k+1}}$ for all $0 \neq h \in H_D(cl_{N-1}(S^N; \mathbb{Z}_2); \mathbb{Z}_2) = \mathbb{Z}_2$, since the dimension of the space of k-dimensional real algebraic subvarieties of degree d in $\mathbb{R}P^N$ is bounded (roughly) by $N^2 d^{k+1}$ by an elementary argument. (Larry Guth conjectured that such an estimate remains valid with $\mathrm{vol}_k(\underline{G}_y)$ instead of $\deg(\underline{G}_y)$ and he proved this up to a subexponential error term, see 5.1 below.

(c) Can every $\mathbb{Z}$-cycle of 2-dimensional $\mathbb{Z}$-cycles be represented by a family G_y of oriented point-singular surfaces? (One cannot, in general, have the parameter space Y smooth, but one needs here a Y with orbifold-like singularities, induced from those of the compactified moduli spaces of Riemann surfaces.)

There is another view on the space of cycles, where Z is a smooth manifold (or a piecewise smooth space) and *geometric cycles* are defined as a piecewise-Lipschitz $\mathbb{Z}_p$-sub-pseudo-manifold in Z with an additional cycle structure. We shall deal with particular families of such $\mathbb{Z}_2$-cycles coming from the following

Geometric Examples. (a) Let Z be be a smooth closed connected N-dimensional manifold (or a piecewise smooth pseudo-manifold) generically mapped to the Euclidean space $\mathbb{R}^D$ and consider the intersections of Z with the affine subspaces of codimension $N-k$. This gives a family of k-cycles G_t in Z parametrized by the Thom space T_D^{N-k} of the canonical $(N-k)$-vector bundle over the Grassmannian Gr_D^{N-k}, i.e. a map $\tau : T_D^{N-k} \to cl_k(Z;\mathbb{Z}_2)$.

Almost all G_t are non-singular, i.e. they are smooth submanifolds in Z; every singular G_t is a pseudo-manifold that is smooth apart from finitely many singular points (where some of these points can be isolated points in G_t).

The map τ sends the the fundamental cohomology class

$$[Z]_{-k} \in H^{N-k}\big(K(\mathbb{Z}_2, N-k)\big) \subset H^*\big(cl_k(Z);\mathbb{Z}_2\big)$$

to the Thom class $[T] \in H^{N-k}(T_D^{N-k}; Z_2)$ and the $(D-N+k+1)$th $\smile$-power $[Z]_{-k}^{D-N+k+1}$ goes to the fundamental cohomology class

$$[T_D^{N-k}]^\bullet \in H^{(D-N+k+1)(N-k)}(T_D^{N-k}; Z_2)\,.$$

Thus,

$$\tau_*[T_D^{N-k}] \neq 0\,,$$

i.e. *the fundamental homology class of the Thom space,*

$$[T_D^{N-k}] \in H_{(D-N+k+1)(N-k)}(T_D^{N-k};\mathbb{Z}_2)\,,$$

goes to a non-trivial class in $H_{(D-N+k+1)(N-k)}(cl_k(Z))$.

The entire homomorphism

$$\tau_* : H_*(T_D^{N-k};\mathbb{Z}_2) \to H_*\big(K(\mathbb{Z}_2, N-k);\mathbb{Z}_2\big)$$

is far from being onto for $N-k \geq 2$, since $H_*(T_D^{N-k}) \simeq H_{*-N+k}(Gr_D^{N-k})$, where the cohomology algebra $\oplus_i H^i(Gr_D^{N-k};\mathbb{Z}_2)$, that is generated by $N-k$ (Stiefel–Whitney) classes, has polynomial growth, i.e. $\mathrm{rank}(H^i(Gr_D^{N-k};\mathbb{Z}_2)) \leq \mathrm{const}(N,k) i^{N-k}$, while the cohomology of $K(\mathbb{Z}_2, N-k)$ – a polynomial algebra in infinitely many (Cartan–Serre) generators for $N-k \geq 2$ – grows super-polynomially.

(b) Let $Z \subset \mathbb{R}^D$ be an irreducible (over $\mathbb{R}$) real algebraic subvariety of dimension N that is not contained in a hyperplane. Then almost all intersections G_t of Z with the affine $(D-N+k)$-planes are k-dimensional real algebraic subvarieties making a family of $\mathbb{Z}_2$-cycles in Z. There may exist some exceptional G_t with $\dim(G_t) > k$ but these are "homologically insignificant".

It follows that if we start with an algebraic (sub)variety $Z \subset \mathbb{R}^{2N+1}$ of degree D_0 and consider complete intersection subvarieties in Z of dimension k and degree $(D_0 d)^{N-k}$ that are viewed as intersections of Z with affine $(D-N+k)$-planes in the linear space of dimension $D(\approx N^d)$ of polynomial maps $Z \to \mathbb{R}^{N-k}$ of degree d, we obtain

a homology class h of dimension $\approx N^d$ in $H_*(K(\mathbb{Z}_2, N-k); Z_2) \subset H_*(cl_k;\mathbb{Z}_2)$ that is represented by a family $G_t \subset Z$, $t \in T_D^{N-k}$, of algebraic subvarieties in Z of degrees $(D_0 d)^{N-k}$, such that

$$[Z]_{-k}^d(\tau_*(h)) \neq 0. \qquad \text{“} \neq 0\text{”}$$

Examples, Remarks, Questions. The classes coming from the maps $K(\mathbb{Z}_2, 1)^{N-k} \to K(\mathbb{Z}_2, N-k)$ classifying $H^{N-k}(K(\mathbb{Z}_2,1)^{N-k};\mathbb{Z}_2)$ can be obtained by intersecting hypersurfaces in Z of degrees d_i, $i = 1, 2, \dots, N-k$. Their images in $H_*(K(\mathbb{Z}_2), N-k)$ are computed in [Gu] where the author also indicates the problem concerning such representation of the homology of integer cycles (i.e. for $\mathbb{Z}_{p=\infty} = \mathbb{Z}$) by intersections of complex and real algebraic subvarieties.

Are there some classes of cycles in Z representing all (much) of the homology of $cl_*(Z;\mathbb{Z}_p)$ with the efficiency comparable to that of the algebraic cycles?

5.1 Evaluation of Morse–Steenrod spectra of $cl_*(Z)$. Let Z be a metric space, e.g. a Riemannian manifold and let $Lcl_* = Lcl_*(Z;\mathbb{Z}_p)$ stand for the space of the *Lipschitz* cycles, where all singular simplices are represented by *Lipschitz* maps $\sigma : \Delta^k \to Z$. We assume that the inclusion $Lcl_* \subset cl_*$ is a homotopy equivalence and observe that this assumption is satisfied in many cases, e.g. for Riemannian manifolds Z. (A notable exception is presented by the Carnot–Carathéodory spaces, where one needs a finer control on the metric properties of the maps $\Delta \to Z$.)

From this point on, we do not distinguish between Lcl_* and cl_* assuming that the cycles are Lipschitz whenever this property is needed.

The *volume* $\mathrm{vol}(c)$ of a Lipschitz cycle (or chain) $c = \sum_j r_j\sigma_j$ is the sum $\sum_j |r_j| \mathrm{vol}_k(\sigma_j)$. We regard this volume as a (Morse) function $\mathrm{vol} = \mathrm{vol}_k : cl_k \to \mathbb{R}_+$ and define the corresponding homological *volume spectrum* as the function also called *the min-max (Morse-Steenrod) volume spectral function on homology*, $\mathrm{vol}_\circ(h)$ on $H_*(cl_k)$, that is the infimum of numbers $\lambda \geq 0$ such that h is representable by a Lipschitz family of Lipschitz cycles of volumes $\leq \lambda$.

In other words, the inequality $\mathrm{vol}_\circ(h) < v$ for $h \in H_i(cl_k)$ says that h lies in the image of the inclusion homology homomorphism for $\mathrm{vol}^{-1}[0,\lambda) \subset cl_k$; moreover, this h must be representable by a *Lipschitz* $(k+i)$-cycle in Z. (The latter condition is purely technical, it is not always needed and it is usually satisfied achieved in our examples.)

Then the *min-max volume spectrum on cohomology*, *vol*$^\circ(h')$, $h' \in H^*(cl_k)$ is the dual to $\mathrm{vol}_\circ$, that is $\mathrm{vol}^\circ(h') < \lambda$ if and only if there exists a Lipschitz family of k-cycles of volumes $< \lambda$ such that h' does not vanish on this family.

The $\mathrm{vol}_\circ$ and vol° spectra carry the same information and we use them interchangeably.

If $f : Z_1 \to Z_2$ is a λ-Lipschitz map, then, obviously, $\mathrm{vol}_\circ(f_*(h)) \leq \lambda^k \mathrm{vol}_\circ(h)$ for all $h \in H_k(cl_k(Z_1))$. Thus the asymptotic, up to a multiplicative constant, growth rate of the function $\mathrm{vol}_\circ(h)$ (for sequences $h_i \in H_*(cl_*)$ with $\deg(h_i) \to \infty$) is a homotopy invariant in the category of compact (piecewise) Riemannian manifolds Z, while the constant(s) is a metric invariant.

If Z is a connected pseudomanifold, then the asymptotic growth rate (up to a multiplicative constant depending on the metric) of $\mathrm{vol}_\circ$ on $H_*(\mathbb{Z}_p, N-k) \subset H_*(cl_k(Z))$ is independent of Z and is the same as for $Z = S^N$, since the fundamental cohomology class of Z is spherical. (This growth rate is evaluated in [Gu] for $p = 2$ in terms of the Cartan–Serre basis in $H^*(K(\mathbb{Z}_2, N-k);\mathbb{Z}_2)$, compare below.)

QUESTION. Let the homology homomorphism $f_* : H_i(Z_1,\mathbb{Z}_p) \to H_i(Z_2;\mathbb{Z}_p)$ be injective for some $i > k$. Is then $\mathrm{vol}_\circ(f_*(h)) \geq \mathrm{const}(Z_1, Z_2, f)\, \mathrm{vol}_\circ(h)$ for all $h \in H_*(K(H_i(Z_1;\mathbb{Z}_p),\mathbb{Z}_p);\mathbb{Z}_p) \subset H_*(cl_k;\mathbb{Z}_p)$? (Here and below all constants "const" are *strictly* positive.)

REMARKS. (a) Our "spectral terminology" follows the variational/homological definition of the spectra of selfadjoint operators (compare [Gr10]), where the corresponding Morse (energy) functions are defined on projective spaces P^∞ (of functions) and where the spectrum is determined by a sequence of numbers λ_i at which the ranks of the sublevel (linear) spaces (that are $\mathrm{vol}_\circ^{-1}[0,\lambda] \subset H_*(cl_k)$ in the present case) on the homology $H_*(P^\infty)$ *strictly* increase (jump up by 1 or more).

(b) The Steenrod algebra action on $H^*(cl_k)$, expressed, for example, via the Cartan–Serre basis in $H^*(cl_k)$, significantly enriches the linear algebraic (grading by dimension) structure on the cohomology and thus enlarges the information content of the spectrum as was demonstrated by the evaluation of the Morse–Steenrod spectra of the volume functions on the spaces of $\mathbb{Z}_2$-cycles by Larry Guth (see [Gu] and 5.2 below).

(c) We are mainly concerned in this paper with lower bounds on the area (that is vol_2) of families of 2-cycles and, eventually, of families of point/edge singular surfaces that are *not* cycles, see 5.11, but since there there is nothing special to dim = 2 at this stage, we tell the story for all k with the peculiarities of dim = 1 and dim = 2 indicated in 5.5–5.9.

5.2 Geometric lower bounds on the volume spectrum. Let Z be a closed N-dimensional Riemannian (pseudo)manifold, Z_i, $i = 1, 2, \dots, D$, open subsets and $f_i : Z_i \to S^N_+(v_i)$ be vol_k-*contracting* Lipschitz maps of *odd* degrees,

where $S^N_+(v_i) \subset S^N(v_i) \subset \mathbb{R}^{N+1}$ are half spheres in the spheres, with the radii normalized to have the k-equators of volumes v_i,

where the boundary of each Z_i is sent by f_i to the boundary of $S^N_+(v_i)$ plus, possibly, a subset of dimensions $\le k-1$, so that the degrees are defined,

and where "vol_k-contracting" means that $\mathrm{vol}_k(f(V)) \le \mathrm{vol}_k(V)$ for all k-dimensional submanifolds $V \subset Z$.

Let G_t, $t \in T$ be a family of Lipschitz $\mathbb{Z}_2$-cycles of dimension k, such that the cohomology class $[Z]^D_{-k} \in H^{D(N-k)}(K(\mathbb{Z}_2, N-k) \subset cl_k(Z;\mathbb{Z}_2))$ goes to a *non-vanishing* class in $H^{D(N-k)}(T;\mathbb{Z}_2)$. Then this family abides

Almgren's *D*-inequality.

There exists $t_0 \in T$ such that the intersections of G_{t_0} with Z_i satisfy

$$\mathrm{vol}_k(Z_i \cap G_{t_0}) \ge v_i\,.$$

Therefore, if Z_i are mutually disjoint, then

$$\mathrm{vol}^\circ[Z]^D_{-k} \ge \sum_i v_i\,.$$

See [Gr5], [Gu] for the proof and the related discussion.

$D^{\frac{N-k}{N}}$-COROLLARY.

$$\mathrm{vol}^\circ[Z]^D_{-k} \ge \mathrm{const}(Z) D^{\frac{N-k}{N}}$$

for all compact Riemannian manifolds Z, where, moreover, $\mathrm{const}(Z) \ge (\mathrm{const}(N,k) - D^{-1/N}\,\mathrm{const}(Z)) \cdot \mathrm{vol}_N(Z)^{k/N}$.

This lower bound is asymptotically sharp:

$$\mathrm{vol}^\circ[Z]^D_{-k} \le \mathrm{const}'(Z) D^{\frac{N-k}{N}}\,.$$

Indeed, algebraic k-cycles of degree d^k, for some real algebraic structure in Z, have their volumes bounded by const $\cdot d^k$ by Crofton's formula and some D'-dimensional families of the (complete intersection) cycles, where $D' = (N-k)D \approx d^N$, detect *non-zero* $\smile$-powers in $H^{D'}(cl_*(Z;\mathbb{Z}_2);\mathbb{Z}_2)$ according to "$\ne 0$" from the previous section (compare [Gr5], [Gu]).

$\smile$-Superadditivity. Let a connected manifold Z (possibly with boundary) be subdivided by a hypersurface into two connected subdomains, $Z = Z_1 \cup Z_2$, and let $h = h_1 \smile h_2$, for

$$h_1, h_2 \in H^*\big(K(\mathbb{Z}_p, N-k);\mathbb{Z}_p\big) \subset H^*\big(cl_k(Z;\mathbb{Z}_p);\mathbb{Z}_p\big)\,.$$

Denote by $h_{ii} \in H^*(cl_k(Z_i;\mathbb{Z}_p);\mathbb{Z}_p)$, $i = 1,2$, the corresponding "restriction" classes of h_i to Z_i and observe that the above (or rather just the Lyusternik–Schnirelmann theorem, compare [Gr5], [Gu]) implies that

$$\mathrm{vol}^\circ(h) \ge \mathrm{vol}^\circ(h_{11}) + \mathrm{vol}^\circ(h_{22})\,.$$

It follows that for every $h \in H^*(K(\mathbb{Z}_p, N-k);\mathbb{Z}_p) \subset H^*(cl_k(Z;\mathbb{Z}_p);\mathbb{Z}_p)$ the limit

$$\lim_{D\to\infty} D^{-\frac{N-k}{N}}\,\mathrm{vol}^\circ(h^D)$$

exists and can be written as $\Omega^k_N(h)\,\mathrm{vol}_N(Z)^{k/N}$.

Thus, for every prime p a pair $(N, k<N)$, one gets a universal (not depending on the geometry of Z) function on the Steenrod algebra,

$$\Omega^k_N : \mathbb{S}_p \to [0,\infty] \quad \text{by } s \mapsto \Omega^k_N\big(s([Z]_{-k})\big)$$

for the Steenrod action of $s \in \mathbb{S}_p$ on the fundamental class

$$[Z]_{-k} \in H^*\big(K(\mathbb{Z}_p, N-k);\mathbb{Z}_p\big) \subset H^*\big(cl_k(Z;\mathbb{Z}_p);\mathbb{Z}_p\big)\,.$$

In fact, $\Omega^k_N(s) < \infty$ for all s, provided $p \le \infty$. (If $p = \infty$, then $\mathbb{S}^n_\infty$ equals $H^*((\mathbb{Z}, n);\mathbb{Z})$ by definition).

This is shown (in a sharper form) in [Gu] for $p=2$ by a "bending" argument that, in fact, applies to all $p = 2,\dots,\infty$ and allows the following:

Localization of deformations. Let Z be subdivided into D cubes of edge size $\le \varepsilon(\approx D^{-1/N})$ and be given the piecewise linear metric corresponding to this subdivision. Then

> *every class $h' \in H_*(cl_k(Z))$ can be represented by a family of G_y, $y \in Y$ of k-cycles in Z, such that every G_y decomposes into the sum of chains, $G_y = G^\varepsilon_y + G'_y$, where the support of G'_y lies in the k-skeleton of Z while* $\mathrm{vol}_k(G_y) \le \mathrm{const}\cdot\varepsilon^k$, *where the constant depends on h (and, thus, on the topology of Z) but not on the geometry of Z (i.e. the combinatorics of the subdivision).*

Then, it follows by "cancellation" as in [Gu], that if a cohomology class h does not vanish on h' then h^D is detected by the D-th power family G_t, $t \in Y^D$, where all G_t that are like cycles are the sums $G_t = \sum_i G_{y_i}$ for $t = (y_1,\dots,y_i,\dots,y_D)$, and have volumes $\le D\cdot \mathrm{vol}_k(G^\varepsilon_y) + N^N pD\varepsilon^k$.

Questions. Can one compute the function $\Omega_N^k(s)$ explicitly? Is this function still finite for $p=\infty$? Are there further terms in the asymptotic expansion of $\mathrm{vol}^\circ(h^D)$, $D\to\infty$, expressible with the (covariant derivatives of the) curvature of Z?

If N and k are even, then the function Ω_N^k is finite on the fundamental class $[Z]_{-k}\in H^*(K(\mathbb{Z},N-k);\mathbb{Z})$ as is seen by looking at the complex algebraic subvarieties. (See [Gu], where it is also pointed out that there is no known non-trivial upper bound on the volumes $\mathrm{vol}^\circ[Z]_{-k}^D$ for odd N, starting from $N=3$ and $k=1$.)

Følner spectra. Let $\tilde Z$ be a Galois covering of compact Riemannian manifold with *amenable* Galois (deck transformation) group Γ, consider an exhaustion of $\tilde Z$ by open connected Følner sets $U_i\subset\tilde Z$ with volumes D_i, take a cohomology class $h\in H^*(K(\mathbb{Z}_p,N-k);\mathbb{Z}_p)\subset H^*(cl_k(U_i);\mathbb{Z}_p)$ and let

$$\tilde\Omega(h;\beta)=\tilde\Omega_N^k(h;\beta)=\lim_{i\to\infty}\tfrac{1}{D_i}\,\mathrm{vol}_k^\circ\left(h^{[\beta D_i]}\right),$$

where vol_k° is taken with the geometry of U_i and where $[\beta D_i]$ denotes the integer part of βD_i. (Plugging in β amounts to rescaling the metric in U_i by $U_i\mapsto\beta^{\frac{N-k}{N}}U_i$.)

The asymptotics of $\tilde\Omega$ for $\beta\to\infty$, does not depend on the geometry of $\tilde Z$,

$$\tilde\Omega_N^k(h;\beta)\sim\beta^{\frac{N-k}{N}}\Omega_N^k(h)\,,$$

and we get nothing new. But the asymptotics of $\tilde\Omega(h;\beta)$ for $\beta\to0$ reflects the behavior of $\tilde Z$ at infinity that would be interesting to evaluate for particular groups Γ.

Questions, Remarks, Examples. (a) If Γ is homeomorphic to the N-torus, then

$$\tilde\Omega_N^k(h;\beta)\sim\mathrm{const}(Z,h)\beta^{\frac{N-k}{N}}\quad\text{for }\beta\to0\,.$$

Are the flat Riemannian manifolds uniquely characterized by the equality $\mathrm{const}(Z)=\lim_{\beta\to0}\beta^{-\frac{N-k}{k}}\tilde\Omega_N^k(h)=\Omega_N^k(h)$?

(b) How is $\tilde\Omega(h;\beta)$ related to the mean dimension of minimal subvarieties in $\tilde Z$ defined in [Gr12]?

(c) The counterpart of $\tilde\Omega$ can be associated with every (linear or non-linear) geometric spectrum on Z (see [Gr10]) abiding by the $\smile$-superadditivity, e.g. for the Laplace operator spectrum, where one considers the projective spaces $\mathbb{R}P^\infty(U_i)$ of not identically zero function φ on U_i instead of $cl_k(U_i;\mathbb{Z}_p)$ and the normalized energy

$$E(\varphi)=\int_{U_i}\|d\varphi\|^2\Big/\int_{U_i}|\varphi|^2,$$

where the role of $\mathrm{vol}^\circ[U_i]_{-k}^m$ is played by the m-th non-zero eigenvalue λ_m of the Laplace operator on U_i with the Neumann boundary condition.

The behavior of $\tilde\Omega$ associated to this E is similar to that of $\tilde\Omega_N^{N-1}$: both invariants encode to the asymptotic isoperimetry of (Følner domains in) $\tilde Z$ and, possibly, they can be expressed in terms of the Følner function of the group Γ. In fact the proof of the inequality (λ_m) in 6.1 adapted to manifolds with boundary (that are U_i in the present case that may need a mild regularization, e.g. by taking their 1-neighborhoods) shows that

$$\lambda_m(U_i)\le\mathrm{const}(Z)\tfrac{1}{D_i}\min\left(D_i,\mathrm{vol}_{N-1}^\circ[U_i]_{-(N-1)}^m\right).$$

(d) The $\tilde{\Omega}$ for the Laplacian on differential forms on the universal covering $\tilde{Z}$ recaptures, for $\beta \to 0$, the von Neumann Betti numbers of Z as well as the Novikov–Shubin invariants.

(e) Let $\tilde{Z}$ be a simply connected nilpotent Lie group. Then the Pansu–Varopoulos isoperimetric inequality implies, for Følner domains $U_i \subset \tilde{Z}$ of volumes D_i, that

$$\mathrm{vol}^\circ_{N-1}\left([\tilde{U}_i]^{[\beta D_i]}_{-(N-1)}\right) \sim \mathrm{const}(\tilde{Z})\beta^{\frac{M-1}{M}} D_i \quad \text{for } D_i \to \infty \text{ and } \beta \to 0\,,$$

where $M \geq N$ denotes the Hausdorff dimension of the limit space $\varepsilon\tilde{Z}$ for $\varepsilon \to 0$.

(f) Let $\tilde{Z}$ be the $(2n+1)$-dimensional Heisenberg group. Then

$$\tilde{\Omega}^k_N(\beta) \approx \beta^{\frac{M-k}{M}}$$

for $M = 2N+2$, $k \leq n$ and $\beta \to 0$.

This follows from to the finiteness of the function vol°_k on bounded domains $U \subset \tilde{Z}$ where vol_k is taken with respect to the Carnot–Carathéodory metric on $\tilde{Z}$ (see [Gr8] and references therein)

(g) The definition of $\tilde{\Omega}$ extends to the *(sofic) LEF groups* of Vershik–Gordon also called *sofic groups*. Also it makes sense for all homogeneous Riemannian manifolds. Can one evaluate the asymptotics of $\tilde{\Omega}(\beta)$, $\beta \to 0$, for symmetric spaces of non-compact type, for instance?

Guth's ε-inequality. The function vol° was evaluated for $p = 2$ in [Gu] on the (Cartan–Serre) monomials in the iterated Steenrod squares of $[Z]_{-k}$, where it is shown, in particular, that

> *Every non-zero class $h \in H^d(cl_k(Z);\mathbb{Z}_2)$ for the unit Euclidean N-ball Z satisfies,*
>
> $$\mathrm{vol}^\circ(h) \geq \mathrm{const}(N,\varepsilon) d^{\frac{1}{k+1}-\varepsilon} \quad \textit{for all } \varepsilon > 0\,.$$
>
> *Moreover, there exists a sequence of classes $h_i \in H^{d_i}(cl_1(Z);\mathbb{Z}_2)$ with $d_i \to \infty$ (where h_i are certain composed Steenrod squares of $[Z]_{-k} \in H^{N-k}(cl_1(Z);\mathbb{Z}_2)$), such that*
>
> $$\mathrm{const}(N,\varepsilon) d_i^{\frac{1}{k+1}-\varepsilon} \leq \mathrm{vol}^\circ(h_i) \leq \mathrm{const}(N) d_i^{\frac{1}{k+1}}\,.$$

Remarks. (a) It is conjectured (in a sharper and more general form) in [Gu] that the ε-inequality holds true with $\varepsilon = 0$.

(b) The Guth ε-inequality for balls trivially implies such an inequality for the "spherical part" of the cohomology $H^*(cl_k(Z);\mathbb{Z}_2)$ of all manifolds Z, i.e. for the images of the cohomology $H^*(cl_k(S^n))$ induced by the maps $cl_k(Z) \to cl_k(S^n)$ that are associated to continuous maps of Z to the spheres S^n, $n = 1, 2, \dots$.

It seems that Guth's "moving balls" argument from [Gu] extends to non-spherical classes. We indicate below the proof of such generalization of the D-Corollary for the product classes $h \in H^*(cl_k(Z);\mathbb{Z}_2)$ without the "sphericity" condition,

$\smile$-Inequality.

> *Let $h \neq 0$ decompose into the cup-product of some classes $h = \smile_{i_j} h_{i_j}$ for*
>
> $$h_{j_i} \in H^{m_i > 0}(K(H_i(Z;\mathbb{Z}_2), i-k);\mathbb{Z}_2) \subset H^*(cl_k;\mathbb{Z}_2)\,,$$

where $i = k+1, k+2, \dots, N \dim Z$, and $j_i = 1, 2, \dots, d_i$ (i.e. d_i denote the numbers of the classes in the cup-product corresponding to $H_i(Z)$). Then

$$\mathrm{vol}^\circ(h) \geq \mathrm{const}^\circ(Z) \sum_i d_i^{\frac{i-k}{i}}.$$

The idea is to replace disjoint ε-balls in Z (this is *not* Guth's ε, just a small positive number) in the proof of the D-inequality by ε-neighborhoods of some cycles in Z with the following

Semi-local Lemma. Let Z be a Riemannian manifold, $P \subset Z$ be a compact piecewise smooth polyhedron and $U_\varepsilon \supset P$ be the ε-neighborhood of P in Z and $U'_\varepsilon \supset U_\varepsilon$ be a larger *regular* neighborhood, where "regular" signifies that P is a homotopy retract of $U'_\varepsilon \supset P$. Then

the subspace $cl_k(\lambda) \subset cl_k(Z;\mathbb{Z}_p)$ of relative k-cycles in U'_ε ("relative" = "relative to the boundary $\partial U'_\varepsilon$") of volumes $\leq \lambda$ is contractible in $cl_k(Z;\mathbb{Z}_p)$ for $\lambda \leq \mathrm{const}(Z,P)\varepsilon^k$.

This can be seen either with the Federer–Fleming filling (see [Gr2], [Gu]) or with Almgren's variational arguments, where the latter shows, at least for the smooth ("piecewise smooth" may be enough) P, that $\mathrm{const}(Z,P) \geq (v_k - \mathrm{const}(Z,P)\varepsilon^{1/N})$ for v_k denoting the volume of the unit ball in $\mathbb{R}^k$. (The partition argument from [Gr5] may also apply to smooth submanifolds $P \subset Z$).

Since every Z can be embedded as neighborhood (homotopy) retract to a Euclidean space and, thus, with the induced inclusion homology homomorphism from Z to some Euclidean neighborhood *injective*, one may assume that Z itself is a compact smooth domain in $\mathbb{R}^N$. Then one can move the skeleta $P^i \subset \mathbb{R}^N$ of a triangulation of Z (or, rather of a slightly larger domain) by parallel translations with the following

Obvious Lemma. *Let $P \subset \mathbb{R}^N$ be a compact m-dimensional simplicial (i.e. made of finitely many affine simplices in $\mathbb{R}^N$) polyhedron and let $L \subset \mathbb{R}^N$ be a linear subspace of dimension $N-m$ that is transversal to all m-faces of P. Then there exists a constant $C = C(P,L)$, such that if some points $x_j \in L$ are δ-separated in L, then the translates $U_\varepsilon + x_j \subset \mathbb{R}^N$ of the ε-neighborhood $U_\varepsilon \supset P$ with $\varepsilon \leq C^{-1}\delta$ cover no point in $\mathbb{R}^N$ with the multiplicity $\geq C$.*

Now the proof of the $\smile$-inequality follows as in [Gr5], [Gu] from the following fact:

Let $U_{j_i} \subset Z \subset \mathbb{R}^N$ be domains, where each U_{i_j} contains the $(N-i)$-skeleton of some triangulation of Z, and let G_t, $t \in T$, be a family of k-cycles, such that the $\smile$-product class $h = \smile_{j_i} h_{j_i}$ does not vanish on T. Then none of the intersections families $G_t \cap U_{j_i}$ is contractible in the space $cl_k(U_{j_i})$ of (relative) cycles in the domain U_{j_i}.

Questions. (a) Does there exist a direct proof (not using the Steenrod algebra as in [Gu] of the above stated (special case of) Guth ε-inequality? Do such inequalities hold true for $cl_k(Z;\mathbb{Z}_p)$ for $p \geq 3$?

(b) Can one evaluate $\mathrm{const}^\circ(Z)$ for $\sum_i d_i \to \infty$ in terms of some "i-dimensional (co)volumes" of Z (or, rather, on spaces of cocycles on Z in the spirit of the comass norms on $H^i(Z;\mathbb{R})$)?

(c) Let $P \subset Z$ be a piecewise smooth subpolyhedron, e.g. a smooth submanifold (algebraic-like P, more singular than "piecewise smooth", are also of interest) and consider the ε neighborhoods $U_\varepsilon \subset Z$ of P for. Does there exist an asymptotic limit of the $\mathrm{vol}_\circ$-spectra on the spaces of (absolute and relative) cycles in U_ε for $\varepsilon \to \infty$?

Similarly, let $Z \to Z_0$ be a Riemannian fibration (submersion) and let Z_ε be obtained by shrinking the fibers by ε. (If one renormalizes to $\varepsilon^{-1}Z_\varepsilon$, one speaks of the "adiabatic limit"). What is the asymptotics of the $\mathrm{vol}_\circ$-spectra on the spaces of cycles in Z_ε? Are the extremal families of cycles detecting a given cohomology class in $H^*(cl_*)$ asymptotic to families of geometrically filtered cycles with the filtration reflecting that in the Serre spectral sequence?

For example, do the cycles in the families coming from the homology of a fiber, call it $F \subset Z$, asymptotically localize at the fibers? Does the part of the $\mathrm{vol}_\circ$-spectrum in Z_ε coming from F converges to the (ε-scaled) $\mathrm{vol}_\circ$-spectrum of F, provided the homology homomorphism $H_*(F) \to Z$ is injective? (This is so for the "bottom of the spectrum" for the (round) sphere bundles $S^M \subset Z \to Z_0$, i.e. for the families representing the fundamental classes in the homology $H_{M-k}(K(\mathbb{Z}_p, M-k)) \to H_{M-k}(cl_k(Z_\varepsilon;\mathbb{Z}_p))$ as we mentioned earlier.)

(d) The $\mathrm{vol}_\circ$-spectrum does not, a priori, account for the full information on the homologies of the sublevels $\mathrm{vol}_k^{-1}[0,v] \subset cl_k$ of the function vol_k on cl_k but only the the images of this in the whole space cl_k. Is there something else there or, on the contrary, the function $\mathrm{vol}_k : cl_k(Z) \to \mathbb{R}_+$ is *quasi-perfect*? The latter means that there exists a constant $C = C(Z)$, such that the kernels $\ker(v_1, v_2) \subset H_*(\mathrm{vol}_k^{-1}[0,v_1])$ of the homology inclusion homomorphisms $H_*(\mathrm{vol}_k^{-1}[0,v_1]) \to H_*(\mathrm{vol}_k^{-1}[0,v_2])$, $v_1 \le v_2$, do not depend on v_2 for $v_2 \ge Cv_1$.

5.3 Lower bounds on vol° in CAT($\kappa \le 0$) manifolds. Let Z be a complete N-dimensional CAT(κ) space and $B = B_z(v,k,\kappa) \subset Z$ a simply connected ball centered at $z \in Z$ of radius $R = R(v,k,\kappa)$ such that the R-balls in the k-dimensional simply connected space of constant curvature κ have volumes $= v$. Assume that the base $S_z(Z)$ of tangent cone $T_z(Z)$ is vol_{k-1}-*greater* than the unit sphere S^{N-1} in the following sense:

> *there exists a* vol_{k-1}*-contracting map* $S_z(Z) \to S^{N-1}$ *such that the fundamental cohomology class of* S^{N-1} *goes to a non-zero class in* $H^{N-1}(S_z(Z);\mathbb{Z}_2)$.

Example. Smooth CAT-spaces Z, where $S_z(Z) = S^{N-1}$, trivially, have this property. Also, this is true for the cubical CAT(0)-spaces. Possibly (but not very likely), this is true for all $\mathbb{Z}_2$-pseudo-manifolds Z.

A simple argument (used for smooth spaces in [Gr5]) shows that such a large ball B admits a vol_k-contracting map f of B to the hemisphere $S^N_+(v)$ of non-vanishing $\mathbb{Z}_2$-degree (i.e. f^* does not vanish on the fundamental relative cohomology class of $S^N_+(v)$). Thus one obtains *the lower bound*

$$\mathrm{vol}^\circ[Z]^d_{-k} \geq \sum_i v_i$$

for an Z *containing disjoint simply connected balls* $B_{z_i}(v_i,k,\kappa)$, $i = 1,2,\dots,d$, *provided the tangent "spheres"* $S_{z_i}(Z)$ *at the points* $z_i \in Z$ *are* vol_{k-1}*-greater than* S^{N-1}.

REMARKS. (a) There is a better bound for the CAT(0) locally symmetric spaces Z of $\mathrm{rank}_{\mathbb{R}}(Z) < k$, where the simply connected R-balls $B \subset Z$ are as "vol_k-large" as the R-balls in $\mathrm{CAT}(\kappa < 0)$ manifolds for (an easily computable) $\kappa = \kappa(Z)$.

(b) Guth's bounds on the vol_k° spectrum for Euclidean balls imply similar bounds for our Z. In particular, if Z contains a ball $B_{z_0}(v_0,k,\kappa)$, then, for each $\varepsilon > 0$, every non-zero class h in $H^D(K(\mathbb{Z}_2, N-k);\mathbb{Z}_2) \subset H^D(cl_k(Z;\mathbb{Z}_2);\mathbb{Z}_2)$ satisfies

$$\mathrm{vol}_k^\circ(h) \geq \mathrm{const}(N,\varepsilon) D^{\frac{1}{k+1}-\varepsilon} v_0 \,.$$

5.4 Higher-order volumes. The geometry of families of real algebraic cycles (that serve as a major source of upper spectral bounds for our functions on the spaces of $\mathbb{Z}_2$-cycles) suggests the following finer invariants incorporating the smooth structure of Z.

Let a chain from $ch_k(Z;\mathbb{Z}_2)$ be represented by an almost everywhere smooth (e.g. algebraic or semialgebraic) subset $G \in Z$ and consider the set of all tangent k-planes to G at the regular points. This is a subset in the Grassmann manifold of the tangent k-planes in Z, called the *tangential lift of* G and denoted $G'_{\mathrm{reg}} \subset Z' = Gr_k(Z)$. Let $G' \subset Z'$ be the closure of the lift G'_{reg}. We call our chain C^1*-regular*, if G' is almost everywhere regular in Z' with the $\mathbb{Z}_2$-boundary $\partial(G') = (\partial(G))'$. Then we go to $G'' \subset Z''$, define C^2-regularity etc.

One similarly defines regular D-parametric families of k-chains for $D < N+k$ as regular $k+D$-chains in Z and asks, what is the homotopy type of the resulting spaces of C^r-regular cycles in $Z \times \mathbb{R}^M$ for $M \to \infty$? (Alternatively, one may use the flat topology in the space of the regular cycles.)

EXAMPLE. The fibers of a generic smooth map $Z = Z^N \to \mathbb{R}^D$, $D = N-k$, make such a regular family of k-cycles $G_t \in Z$, $t \in \mathbb{R}^D$. Observe that if Z is oriented, then this family makes a D-dimensional $\mathbb{Z}$-cycle of k-dimensional $\mathbb{Z}$-cycles, but after the "blow-up", the "differential" $G'_t \subset Z'$ is not a $\mathbb{Z}$-cycle of $\mathbb{Z}$-cycles anymore; yet, it is a $\mathbb{Z}_2$-cycle of $\mathbb{Z}_2$-cycles.

If Z comes with a smooth Riemannian metric μ_Z, then the fibration $Z' \to Z$ inherits the Levi-Civita connection from μ_Z, while the fiber, that is the Grassmann manifold Gr^k_N, $N \dim(Z)$, carries the standard metric denoted μ_{Gr}. One takes the convex combinations of this metric with the Levi-Civita lift μ'_Z of μ_Z to Z', that is $(1-\varrho_1)\mu'_Z + \varrho_1\mu_{Gr}$, and abbreviates $(Z', (1-\varrho_1)\mu'_Z + \varrho_1\mu_{Gr})$ to $Z'(\varrho_1)$. Then one defines $Z''(\varrho_1,\varrho_2),\dots$, and denotes by $\mathrm{vol}_k^{(r)}(G) = \mathrm{vol}_k^{(r)}(G;\varrho_1,\varrho_2,\dots,\varrho_r)$ the k-volume of $G^{(r)} = G''^{\cdots}$ in $Z^{(r)}(\varrho_1,\varrho_2,\dots,\varrho_r)$.

The most significant of these is $\mathrm{vol}_k^{(1)}(G,\varrho_1)$ that represents some combination of the ordinary k-volume with the integral curvature of G; it reduces to vol_k for $\varrho_1 \to 0$ and becomes the pure Gaussian curvature for $\varrho_1 \to 1$.

Observe that every "r-th order volume functions" satisfies $\mathrm{vol}_k^{(r)}(G) = \mathrm{vol}_k(G_{\mathrm{reg}}^{(r)})$; thus, it extends to all cycles with possibly infinite values at some G and the (joint) spectrum evaluation problem for $\mathrm{vol}_k^{(r)}$ makes sense in the space $cl_k(Z;\mathbb{Z}_2)$ of all (not only regular) cycles.

Remarks and Questions. (a) The Betti numbers of a smooth submanifold G in a compact Z are bounded by $\mathrm{const}(Z)\,\mathrm{vol}_k^{(1)}(G)$, as can be seen, for example, by embedding Z into some Euclidean space where the total curvature of G is bounded by $\mathrm{vol}_k^{(1)}(G)$, but it is unclear what else of the topology of G (besides the Pontryagin numbers) can be bounded by (some function of) $\mathrm{vol}_k^{(1)}(G)$.

(b) Can one approximate a submanifold $G \subset \mathbb{R}^N$ by a component of an algebraic submanifold of degree d, with d bounded in terns of $\mathrm{vol}_k^{(N)}(G)$?

5.5 Diagram spectra. Represent cycles by pseudomanifolds mapped to Z, say, $\alpha : G \to Z$, and let $\lambda(G,\alpha)$ be some measure of the topological complexity of the pseudomanifold G and of the map α, e.g. the sum of the Betti numbers of G or the rank of the homology homomorphism α_*.

If $p = 2$ and every $h \in H^d(cl_k(Z;\mathbb{Z}_2);\mathbb{Z}_2)$ is detectable by a continuous map $\alpha : X \to Z$, where X is an (irreducible component of a) $(d+k)$-dimensional algebraic variety $X \subset \mathbb{R}^M$, "fibered" by a regular map F over a d-dimensional variety Y, then an attractive measure of complexity of h is offered by the minimum of the algebraic degrees of such X, Y and F and/or the minimum of the degrees of the fibers of F (compare section 5).

An instance of a finer topological (rather than just homotopical) invariant is provided by the pattern of m-multiple points of G mapped by α to Z. (If $d \geq (m-1)N - mk$, then the cycles G in d-dimensional families, may have stable m-multiple points.)

Given such λ, regard it as a function on the space $cl_*(Z)$, define the associated $\lambda_\circ$ and λ° spectra on the homology and cohomology of the space $cl_*(Z)$ and try to bound these from below in terms of the topology of Z.

Example. The inequality $\smile_{m+1}$ from 2.2 provides a lower bound on the bottom of the spectrum of $\lambda_\circ = (\mathrm{rank}(\alpha_*))_\circ$ (with X in the place of Z) and the other inequalities in section 2 also have the spectral interpretation.

Let us look more closely at 1-cycles in Z with $\mathbb{Z}_2$-coefficients. Every such cycle is represented by a *diagram* that is a finite *even* graph G, i.e. with all vertices $g \in G$ of *even* degrees (valencies) and a continuous map $G \to Z$. Thus each cycle $\in cl_1(Z;\mathbb{Z}_2)$ is tagged with an (isomorphism class of a) graph G and we denote by $cl_G \subset cl_1$, the set of the cycles with the tag G.

The partition of $cl_1(Z)$ into the "strata" $cl_G(Z)$ (albeit far from being a stratification) is functorial for continuous maps $Z_1 \to Z_2$ and it is invariant, up to partitions preserving homotopies, under the homotopy equivalences $Z_1 \simeq Z_2$. Thus, the space $cl_1(Z;\mathbb{Z}_2) \simeq \times_j K(H_j(Z;\mathbb{Z}_2), j-k)$ acquires more structure depending on the homotopy type of Z, than provided by the plain homology $\oplus_j H_j(Z;\mathbb{Z}_2)$.

QUESTION. Given a class $\mathcal{G}_0$ of graphs G, when does a given homology class $h \in H_d(cl_1(Z;\mathbb{Z}_2);\mathbb{Z}_2)$ contain a cycle C represented by a d-parametric family of graphs G_c mapped to Z with all $G_c \in \mathcal{G}$?

Put it another way, let $\lambda(G)$ be some function measuring the "complexity" of G and set $\lambda(C) = \lambda(G(C))$ $C \in cl_1$. What can one say about the corresponding spectrum $\lambda_\circ$ on the homology of cl_1?

SPHERICAL EXAMPLE. Let $Z = S^N$ and observe that every $\smile$-power of the fundamental class $[Z]_{-1}$ can be represented by a cycle in the stratum cl_{S^1}, i.e. made of maps $S^1 \to Z = S^N$.

On the other hand, if $N \geq 3$, then the space $cl_1(Z;\mathbb{Z})$ has Betti numbers $B_D = \mathrm{rank}(H_D(cl_1;\mathbb{Z})) \sim D^{\log(D)}$, while every "stratum" $cl_G \subset cl_1$, that is roughly the Cartesian product of several copies of the loop space of S^N, has polynomial growth of the Betti numbers.

QUESTION. Denote by $r(D,l)$ the rank of the subgroup in $H_D(cl_1(Z);\mathbb{Z}_2)$ generated by the classes h that admit smooth representations $\alpha : X^{D+1} \to Z$, with smooth generic maps $F : X^{D+1} \to Y^D$, such that all fibers $G_y = F^{-1}(y)$ are graphs with $\lambda(G_y) \leq l$, where $\lambda(G)$ is one of the following (measures of topological complexity) of G

- $\lambda(G) = \mathrm{rank}(H_*(G))$;
- $\lambda(G) = \mathrm{rank}(H_1(G))$;
- $\lambda(G)$ equals the maximum of the ranks of H_1 of the connected components of G.

Does $r(D,l)$ for $Z = S^N$ grow at most polynomially in D for each l? What is the minimal $l = l(D)$ for which $r(D,l) = \mathrm{rank}(H_D(cl_1(Z);\mathbb{Z}_2))$?

Additional structure in $cl_1(Z)$ is associated with the fundamental group Γ of Z that is most informative for aspherical spaces Z. Namely, every map $\alpha : G \to Z$ define a subset $\mathrm{conj}(\alpha)$ in the set $\mathrm{conj}(\Gamma)$ of the conjugacy classes of subgroups of Γ that are the images of the fundamental groups of the connected components of G; thus every set $C \subset \mathrm{conj}(\Gamma)$ define a subset $cl_C \subset cl_1(Z)$ of the cycles representable by maps α with $\mathrm{conj}(\alpha) \subset C$.

For instance, if C equals (the class of) the identity element in Γ, then the corresponding set of cycles, i.e. the space of 1-cycles represented by contractible maps of graphs to Z is contractible.

If C consists of the conjugacy class of some infinite cyclic subgroup $A = A(\alpha) \subset \Gamma$ that equals its own centralizer, then cl_A is homotopy equivalent to $cl_1(S^1;\mathbb{Z}_2)$ and, thus, consists of the two contractible components corresponding to the two elements in $H_1(S^1;\mathbb{Z}_2)$.

Denote by $cl_{\mathcal{A}} \subset cl_1$ the space the cycles corresponding to C equal to the set of conjugacy classes of Abelian subgroups in Γ. If Γ is anti-Abelian, i.e. every $id \neq \gamma \in \Gamma$ has infinite cyclic centralizer, then the space $cl_{\mathcal{A}}$ is homotopy equivalent to a disjoint union of contractible spaces corresponding to the conjugacy classes of the maximal Abelian subgroups in Γ and a component corresponding to $id \in \Gamma$. (The space $cl_{\mathcal{A}}$ for more general Γ can probably be used as 6.6 in [Gr2] for lower bounds on the injectivity radii and/or volumes of Riemannian manifolds, in the spirit of the Kazhdan–Margulis theorem.)

5.6 Semi-stable spaces $\mathcal{G}_{sst}$ of 1-cycles G and $|\chi|_{01}$-spectra. The partition of cl_1 into subsets cl_G needs a significant refinement to make it a (pre)stratification. For example, if Y is a family of the real algebraic curves $G_y \subset \mathbb{R}P^3$, $y \in Y$, the adjacency relation between subsets $Y_G \subset Y$ made of the curves G_y homeomorphic G may have "cycles" and thus does not extend to a partial order relation, not even for algebraic families.

We describe below a "subspace" $\mathcal{G}_{sst}(Z)$ in $cl_1(Z;\mathbb{Z}_2)$ consisting of "semi-stable" cycles, where the partition into subsets cl_G *is* a stratification.

Start with the following standard construction. Given a small subcategory $\mathcal{C}$ of the category of topological spaces and continuous maps, every string S of maps $G_0 \to G_1 \to \ldots \to G_n$ defines a S-simplex, denoted Δ_S, that is defined by induction as follows:

> *the $(n+1)$-simplex $\Delta_{\mathcal{G}'}$ for $S' = (G_0 \to G_1 \to \ldots \to G_n \to G_{n+1})$ equals the cylinder of the map $\Delta_S \to G_{n+1}$ composed of the tautological map $\Delta_S \to G_n$ (for the cylinder at the previous step) and $G_n \to G_{n+1}$.*

For example if all G_i are one point spaces, this is the ordinary (ordered) n-simplex obtained by the iterated cone construction.

Given a topological space Z define $\mathcal{C}(Z)$ as the (semi)simplicial space where the simplices are continuous maps $\alpha : \Delta_{\mathcal{G}} \to Z$ for all strings $\mathcal{G}$ with the obvious face relations between the $\mathcal{G}$-simplices mapped to Z.

Examples. (a) If $\mathcal{C}$ consists of a single point with the identity map, then $\mathcal{C}(Z)$ is the standard semi-simplicial model of Z and, thus, is homotopy equivalent to Z. Similarly, if $\mathcal{C}$ consists of a single space G with the identity map, then $\mathcal{C}(Z)$ is homotopy equivalent to the space Z^G of continuous maps $G \to Z$.

(b) If $\mathcal{C}$ is the category of finite sets and all maps or the category of finite sets and injective maps, then $\mathcal{C}(Z)$ is contractible for all Z. In fact this is, obviously, true for every category $\mathcal{C}$ closed under "disjoint unions" of spaces and maps.

(c) If $\mathcal{C}$ consists of a single object G and a group Γ of homeomorphisms of G left translation of a discrete group Γ, then $\mathcal{C}(Z)$ is (canonically) homotopy equivalent to the *homotopy quotient* $Z^\Gamma//\Gamma$ for the natural action of Γ in the space of maps $\Gamma \to Z$. (The homotopy quotient of a Γ-space X is the total space of the X-fibration (i.e. with the fiber X) associated to the principal Γ-fibration over the classifying space of Γ.)

If Z and G have all components of positive dimension and Γ is compact, then the action of Γ on Z^G is free on an open subset $U \subset Z$ (of generic maps $G \to Z$) and the inclusion $U//\Gamma \subset Z^G//\Gamma$ is a homotopy equivalence. Thus Z^G is homotopy equivalent to U/Γ, since the obvious map $U//\Gamma \to U/\Gamma$ is a homotopy equivalence for free actions.

(d) Let the objects of $\mathcal{C}$ be finite k-dimensional simplicial polyhedra P, each endowed with a distinguished simplicial k-dimensional $\mathbb{Z}_p$-cycle, and the morphisms are simplicial maps sending distinguished cycles to distinguished cycles. Then the natural map $\mathcal{C}(Z) \to cl_k(Z;\mathbb{Z}_p)$ is a homotopy equivalence. (This is, probably, true for the smaller category, where all P are unions of finite sets and oriented $\mathbb{Z}_p$-pseudomanifolds carrying their fundamental cycles and the morphisms $P_1 \to P_2$ are one-to-one over all of P_2 minus finitely many points.)

(e) *$\mathcal{G}$-Representation of $cl_1(Z;\mathbb{Z}_2)$.* Let $\mathcal{G}$ be the category, where the objects are finite *even graphs (diagrams)*, i.e. each vertex has even degree (valency), and where morphisms are maps $G_1 \to G_2$ that are one-to-one and onto over G_2 minus a finite subset. In other words, G_2 is obtained from G_1 by shrinking some even subgraphs (e.g. pairs of points) to points and adding isolated points. (Instances of these are limit maps between fibers of a generic smooth map $F : X^{D+1} \to Y^D$.)

It is not hard to show that $\mathcal{G}(Z)$ is (canonically) homotopy equivalent to $cl_1(Z;\mathbb{Z}_2)$. (For instance, every smooth representation of a D-dimensional homology class of $cl_1(Z;\mathbb{Z}_2)$ by $\alpha : X^{D+1} \to Z$ with a generic $F : X^{D+1} \to Y^D$ gives us a D-cycle in $\mathcal{C}(Z)$; which shows, by Thom's theorem, that the tautological map $\mathcal{G}(Z) \to cl_1(Z;\mathbb{Z}_2)$ is surjective on homology.)

$\mathcal{G}_{st}$ and $\mathcal{G}_{sst}$-spaces. Let $\mathcal{G}_{st}$ be the (stable) subcategory of $\mathcal{G}$, where the objects are graphs with all vertices of degree 2 and 4 and where the morphisms $G_1 \to G_2$ are those maps $G_1 \to G_2$, where the pullbacks of non-isolated points are finite and the pullback of every isolated point is a circle, a point, or the empty set. (The limit maps between fibers of a purely folded smooth map $F : X^{D+1} \to Y^D$ are stable in this sense.)

There is a slightly bigger space with equally nice properties where "stability" is relaxed to "semistability" by admitting *all finite even* graphs for the objects of $\mathcal{G}_{sst}$ (i.e. with all vertices of even degrees) and keeping the same restriction on the morphisms.

Questions. Is the tautological embedding of $\mathcal{G}_{sst}(Z) \to \mathcal{G}(Z)$ a homotopy equivalence? In other words, does (every diagram of morphisms in) $\mathcal{G}$ admit a semistable refinement/resolution? (This is similar to the Deligne–Mumford compactification of the moduli spaces of curves, and is possibly known.) When does a smooth generic (say, algebraic) map $\alpha : X^{D+1} \to Y^D$ admit a resolution to an $\hat{\alpha} : \hat{X} \to \hat{Y}$ that is homotopic to a purely folded map $X \to Y$? Does every homology class $h \in H_D(\mathcal{G}_{sst}(Z);\mathbb{Z}_2)$ admit a smooth representation $\alpha : X^{D+1} \to Z$ with a purely folded map $F : X^{D+1} \to Y^D$?

$\mathcal{S}_{[G]}$-stratification. The isomorphisms classes of graphs are partially ordered by the morphisms in $\mathcal{G}_{sst}$, since the "absolute Euler characteristics" ("total curvature"),

$$|\chi|_{01}(G) = \sum_i \left|\chi(G_i)\right|,$$

where G_i are the connected components of G, is strictly monotone increasing under non-isomorphism morphisms $G_1 \to G_2$. (If all vertices in G have valency 4, then $|\chi|_{01}(G)$ equals the number of the vertices.)

It follows that the partition of the space $\mathcal{G}_{sst}(Z)$ into the subsets $\mathcal{S}_{[G]} \subset \mathcal{G}_{sst}(Z)$ corresponding to the isomorphism classes $[G]$ of (even) graphs G is a *pre-stratification*, where the adjacency relation $\mathcal{S}_{[G_2]} \prec \mathcal{S}_{[G_1]}$ (i.e. $\mathcal{S}_{[G_2]} \subset \partial\mathcal{S}_{[G_1]}$) signifies that that there exists a non-isomorphism morphism $G_1 \to G_2$.

In fact, the subsets $\mathcal{S}_{[G]}$ *stratify* $\mathcal{G}_{sst}(Z)$, where each stratum $\mathcal{S}_{[G]}$ is (homotopy equivalent to) the space of continuous maps of the graph G to Z homotopy divided (i.e. $//\Gamma_G$) by the automorphism group Γ_G of G. (This group is finite unless G contains an S^1 connected component.) The "normal space" $\mathcal{S}_G^\perp$ to the stratum $\mathcal{S}_G$ at a generic point $s \in \mathcal{S}_G$, where the action of Γ_G is free (we assume that $\dim(Z) > 0$), called a *G-handle*, equals the simplicial complex associated to the partially ordered set of the graphs $G' \preceq G$ and, thus, has $\dim(\mathcal{S}_G^\perp) = |\chi|_{01}(G)$.

For example, if G consists of a single point, then the G-handle equals the segment $[0.1]$ "stemming" from 0 serving as the *center of the handle*. If G is a figure ∞, then the handle is the tripod T, three $[0,1]$-segments stemming from the (common) central 0-point.

If all vertices in G are either isolated points p or vertices v of valency 4, then the handle is the Cartesian product of the copies of $[0,1]$ and T_3 that is $\times_p[0,1]_p \times_v T_v$.

Every monotone increasing (for the partial order "$\succeq$") function $\lambda(G) \in \mathbb{R}_+$ (expressing a measure of the combinatorial complexity of G), regarded as a function $\lambda : \mathcal{G}_{sst} \to \mathbb{R}_+$ that is equal $\lambda(G)$ on the stratum $\mathcal{S}_G$, defines the functions $\lambda_\circ$ and λ° on the homology and the cohomology of the space $\mathcal{G}_{sst}$. Thus, $\lambda^\circ(h)$, $h \in H^*(\mathcal{G}_{sst};\mathbb{Z}_2)$, equals, by definition, the minimal number $\mu \in [0,\infty)$ such that the restriction cohomology homomorphism from (the cohomology of) $\mathcal{G}_{sst}$ to (the cohomology of) the sublevel $\lambda^{-1}[0,\mu] \subset \mathcal{G}_{sst}$ does not vanish on h and $\lambda_\circ$ is the dual of λ°.

The basic examples are $\lambda(G) = |\chi|_{01}(G)$ and

$$|\chi|_1(G) = \sum_i \left|\chi(G_i)\right|,$$

where G_i are the connected components of G of dimension 1 (i.e. isolated points are discarded.)

Remarks. The subsets $\mathcal{S}_{[G]}$ are also defined in the space $\mathcal{G}(Z) \supset \mathcal{G}_{sst}(Z)$ but the partition of $\mathcal{G}(Z)$ into these subsets is not a pre-stratification (nor is the function $|\chi|_1$ semicontinuous on $\mathcal{G}(Z)$).

Apparently, every d-cycle in cl_1 can be implemented by diagrams (graphs) G with $|\chi_1|(G) \leq d$ (but it is unclear if such an implementation is possible in cl_1^{sst}).

5.7 Evaluation of the homotopy dimension of $\mathcal{G}_d(Z)$ for aspherical spaces Z. The *homotopy dimension* of a continuous map between topological spaces, say $f : S_0 \to S$, is the minimal dimension of a polyhedron P such that f is homotopic to a map that factors via $S_0 \to P \to S$.

The relative homotopy dimension of a subspace $S_0 \subset S$, denoted $\dim_{\mathrm{hom}}(S_0 \subset S)$, is the homotopy dimension of the inclusion map of S_0 to S. The (absolute) homotopy dimension of S_0 is defined as $\dim_{\mathrm{hom}}(S_0)\,\dim_{\mathrm{hom}}(S_0 \subset S_0)$.

Let Z be an aspherical space and denote by $\mathcal{G}_d = \mathcal{G}_d(Z) \subset \mathcal{G}_{sst}(Z)$ the union of the strata $\mathcal{S}_{[G]}$ with $|\chi|_1(G) \leq d$.

If the fundamental group Γ of Z is "anti-Abelian", i.e. the centralizer of every $\gamma \neq id$ is free Abelian, then

$$\dim_{\mathrm{hom}}\big(\mathcal{G}_d \subset \mathcal{G}_{sst}(Z)\big) \leq d\,. \qquad (\dim \leq |\chi|_1)$$

Consequently,

$$\dim_{\mathrm{hom}}\big(\mathcal{G}_d \subset cl_1(Z;\mathbb{Z}_2)\big) \leq d\,;$$

thus, if $\alpha : X^{d+1} \to Z$, $F : X^{d+1} \to Y$ is a smooth representation of a non-zero homology class in $H_d(cl_1;\mathbb{Z}_2)$ where F is a purely folded map, then F has a d-multiple fold self-intersection point $y \in Y$.

Proof. If Z is contractible, then $\mathcal{G}_{sst}(Z)$ is also contractible, since every diagram of morphisms in $\mathcal{G}_{sst}$ can be included into a larger diagram with a unique terminal object.

Next, the space of maps of an arbitrary graph G to the circle satisfies

$$\dim_{\mathrm{hom}}\big((S^1)^G \subset \mathcal{G}_{sst}(S^1)\big) = 1\,.$$

Indeed, it is suffices to show that the "circle of maps" $s \mapsto \alpha_s = s \cdot \alpha : G \to S^1$ for all $s \in S^1$ and the obvious action $s\cdot$ on maps, is contractible in $\mathcal{G}_{sst}(S^1)$ for every $\alpha : G \to S^1$.

If $G = S^1$ this follows from the extendability of every smooth fibration $S^1 \times S^1 \to S^1$ to a purely folded map to the disc, $S^1 \times S^1 \to D^2 \supset S^1$, which exists by Eliashberg's h-principle for folded maps.

Since every connected graph G with the vertices of even degrees admits a morphism $S^1 \to G$ (by Euler's 7 bridges in Konigsberg's theorem), the space of maps $Z^G \subset \mathcal{G}_{sst}(Z)$ can be homotoped to $Z^{S^1} \subset \mathcal{G}_{sst}(Z)$.

Thus we see that all connected components in the strata $\mathcal{S}_{[G]}$ are contractible in $\mathcal{G}_{sst}(Z)$, and attaching $\mathcal{S}_{[G]}$ to the union of the strata $\mathcal{S} \succ \mathcal{S}_{[G]}$ has the same homotopy effect in $\mathcal{G}_{sst}$ as attaching the corresponding G-handle.

Since the dimension of the handle equals $|\chi|_{01}(G)$ this implies the required inequality with $|\chi|_{01}(G)$ instead of $|\chi|_1(G)$. But since adding the products of $\times_p[0,1]_p$-handles centered at $(0,0,\dots,0)$ has no effect on the homotopy type, these can be discarded and $(\dim \leq |\chi|_1)$ follows.

About examples. There are many stably parallelizable manifolds $X = X^{d+1}$, where the fundamental group $\Gamma = \pi_1(X)$ is anti-Abelian and where the $\mathbb{Z}_2$-fundamental class does not vanish in $H_d(K(\Gamma, d+1);\mathbb{Z}_2)$. These X admit purely folded maps $F : X \to \mathbb{R}^d$ by the Eliashberg h-principle and the above inequality says that the folding locus must have a d-multiple point.

Some Γ in these examples are non-word hyperbolic (e.g. in amalgamated products $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$, where Γ_1 and Γ_2 are word hyperbolic and Λ is distorted in both Γ_i) and deliver groups Γ, where $(\dim \leq |\chi|_1)$ does not *obviously* follow from the Δ-inequality in 3.3 But even if Λ is quasiconvex (at least in one of the two Γ_i that is sufficient for the hyperbolicity of Γ, say, in the malnormal case) the group $H_{d+1}(\Gamma;\mathbb{R})$ may be zero while $H_{d+1}(\Gamma;\mathbb{Z}_2) \neq 0$.

For instance, if a $(d+1)$-dimensional manifold Z_0 of negative curvature has two connected totally geodesic boundary components ∂_1 and ∂_2, where ∂_1 is isometric to a p-sheeted cover of ∂_2 with $p > 1$ odd, then the space Z obtained from Z_0 by attaching ∂_1 to ∂_2 by the covering map is CAT$(\kappa < 0)$ with $H_{d+1}(Z; \mathbb{R}) = 0$ and $H_{d+1}(Z;\ Z_2) = \mathbb{Z}_2$.

(The non-zero class $h \in H_{d+1}(Z; \mathbb{Z}_2)$ can be realized only by a *non-orientable* manifold X and the Eliashberg theorem needs to be applied to maps to non-orientable Y. Also the vanishing of $H_{d+1}(Z; \mathbb{R})$ does not exclude an indirect applicability of the Δ-inequality since the $\mathbb{R}$-homology may become non-zero in some hyperbolic subquotient of Γ, but I guess, one can arrange examples where this does not happen.)

Most (all?) apparent examples of anti-Abelian groups have some "residual hyperbolicity" and one does not know, for instance, if there are finitely generated (finitely presented) non-cyclic amenable anti-Abelian groups Γ.

Remarks and Questions. (a) There is a version of the $(\dim \leq |\chi|_1)$-inequality for the groups Γ with a given bound on the homotopy dimensions of the centralizer of its subgroups. For example if $X = X^{d+1}$ is an aspherical manifold and all $id \neq \gamma \in \Gamma = \pi_1(X)$ have centralizers of the homotopy dimension of their classifying spaces $\leq \delta$, then every purely folded map $X \to Y^d$ has an m-multiple fold self-intersection point for $m \geq d/\delta$.

(b) What is the counterpart of the $(\dim \leq |\chi|_1)$-inequality for the space $\mathcal{G}(Z)$? Which of the above lower fold multiplicity bounds generalizes (modifies) to generic smooth maps $X^{d+1} \to Y^d$ that are not purely folded?

(c) If Z is a CAT$(\kappa \leq 0)$-space, then the Δ-inequality from 3.3 gives a lower bound on the "geometric intersection number" of the cycles representing the fundamental homology class $[Z]_{-1}$ in $cl(Z; \mathbb{Z}_2)$ with $\mathcal{G}_d$. Is there a similar bound for other homology classes?

For instance, can one bound from below the number of the affine planes A that intersect a generic $Z \to \mathbb{R}^{m+1}$ along the graphs G with $|\chi|_1(G) = m(l+1)$? Can one say something more about the combinatorics of these G besides the estimates on $|\chi|_1(G)$?

The $(\dim \leq |\chi|_1)$-inequality bounds not only the sum $|\chi|_1(G) = \sum_i |\chi(G_i)|$ over the 1-dimensional components G_i of G of the "maximally singular" 1-cycle $G = G_y$, but also $\max_i |\chi(G_i)|$. Are there lower bounds on this $\max_i |\chi(G_i)|$ for some classes in the (co)-homology of $cl_1(Z)$ apart from the fundamental class?

(d) Are there other (monotone) invariants besides $|\chi|_1$ that can be controlled by the topology of Z, e.g. those incorporating the genera of graphs G? Does planarity of G have any effect on the homotopy role of the strata $\mathcal{S}_{[G]}$?

(e) If Z is a manifold then the 1-cycles can be represented by graphs topologically (or piecewise smoothly) *embedded* into Z. The spaces $cl_1^{\mathrm{emb}}(Z)$ and $\mathcal{G}_{sst}^{\mathrm{emb}}$ of embedded graphs are homotopy equivalent to their "continuously mapped" counterparts but their partitions into strata $\mathcal{S}_{[G]}^{\mathrm{emb}}$, that take into account self-intersections in families of "moving subgraphs" and the pattern of transformation of subgraphs at the self-intersection points, are not homotopy invariants of Z any more, but rather topological invariants (and $p.l.$-invariants for piecewise smoothly embedded subgraphs).

For example, if Z is the sphere S^N, then every $\smile$-power of the fundamental class in $H^{N-1}(cl_1(S^N;\mathbb{Z}))$ is detectable by a family Y of circles mapped to $cl_1(S^N)$, where Y is the Cartesian product of d-copies of the sphere $S^{(N-1)}$. (This, contrary to the statement made in the original version of this paper, is unlikely to be true for $cl_1(S^N;\mathbb{Z}_2)$ as was pointed out to me by Larry Guth.)

But such circles, as they move in an N-dimensional manifold Z along a non-trivial high-dimensional ($\dim \gg N$) cycle (parametrized by) Y in the space $cl_1(Z)$, necessarily acquire many intersection points in Z. For example, if generic smooth maps, $F : X = X^{D+1} \to Y = Y^D$ for $D = d(N-1)$ and $\alpha : X \to Z$, represent a D-dimensional cycle, that is $[Y]$, in $cl_1(Z;\mathbb{Z})$, such that the d-th $\smile$-power of the fundamental cohomology class $[Z]_{-1} \in H^{N-1}((cl_1;\mathbb{Z}_1);\mathbb{Z}_2)$ does not vanish on $[Y]$, then some fiber $G_y = F^{-1}(y)$ must acquire a d-multiple self-intersection point in Z under the map α.

(f) The homological study of the "self-intersections stratifications" in the spaces of curves was initiated (unless I missed an earlier paper) by Alexander Vinogradov [V] who was motivated by knot theory. Recently, an astoundingly rich algebraic structure in these spaces was revealed in [CS] where one can find references to many other papers.

(g) How much (all?) of the topology of a space (a manifold) Z can be reconstructed from $\mathcal{G}^{\mathrm{emb}}_{sst}(Z)$ stratified into $\mathcal{S}^{\mathrm{emb}}_{[G]}$? For instance, can one express the Pontryagin classes of a manifold Z in terms of $(\mathcal{G}^{\mathrm{emb}}_{sst}(Z), \{\mathcal{S}^{\mathrm{emb}}_{[G]}\})$?

Geometric examples. Let H_y, $y \in Y$, be a smooth D-dimensional family of smooth submanifolds of codimension $(N-k)$ in a manifold M, e.g. of the $(N-k)$-codimensional affine subspaces in the Euclidean space $M = \mathbb{R}^P$, and let $f : Z \to M$ be a smooth generic map. Then the pullbacks $G_y = f^{-1}(H_y) \subset Z$ make a D-dimensional cycle in $cl_k(Z;\mathbb{Z}_2)$ parametrized by Y and representing the homology class denoted by $f^![Y] \in H_D((cl_k;\mathbb{Z}_2);\mathbb{Z}_2)$.

Problem 1. Given M and Y, find lower bounds, applicable to all smooth generic f, on the "topological complexity" of the "maximally complicated" fiber G_y in terms of $(M, Y, f^![Y])$.

Next, denote by X the set of the pairs (y, g) for all $y \in Y$ and $g \in f^{-1}(H_y) \in Z$, and observe that it is a smooth manifold for generic maps f that is tautologically mapped to Y, say $F = F_f : X \to Y$, with the fibers $G_y = F^{-1}(y) = f^{-1}(H_y)$.

Problem 2. Find lower bounds on the topological complexity of the critical set $\Sigma(F) \subset Y$.

Problem 3. Describe "convex" (Z, f), i.e. those with the minimal topological complexity in the sense of 1 and 2. In particular, let M be a compact homogeneous (e.g. Riemannian symmetric) space under an action of a Lie group L and $H_y \subset M$ make the L-orbit of some submanifold $H_0 \subset M$. What are submanifolds (and general closed/open subsets for this matter) $Z \subset M$, such that the topology of the intersection $H_y \cap Z$ is constant in y, i.e. the map $F : X \to Y$ is a locally trivial fibration?

For instance let M be the Grassmannian Gr_n^2 of 2-planes in $\mathbb{R}^n$ acted upon by the linear group $L = GL_n$ and $H_0 = Gr_{n-1}^2 \subset Gr_n^2$ be the Grassmannian of 2-planes in a hyperplane in $\mathbb{R}^n$. Suppose that the intersections of H_y with some $Z \subset Gr_n^2$ have the topology constant in $y \in Y$ for $Y = Gr_n^{n-1}$ being the space of the hyperplanes in $\mathbb{R}^n$. Can the restriction of the Euler class $e \in H^2(Gr_n^2)$ to Z be zero? Does it help to assume that not only the topology but also the GL_n-geometry of $Z \cap H_y$ is constant in y? (If n is even, then $Z_0 = \mathbb{C}P^{\frac{n}{2}-1}$, the space of $\mathbb{C}$-lines in $\mathbb{R}^n = \mathbb{C}^{\frac{n}{2}}$, is transversal to all H_y. Equisingular (i.e. preserving transversality, e.g small) deformations of Z_0 and unions of their GL_n-translates are the apparent examples of Z's with the topology of $Z \cap H_y$ constant in $y \in Y$.

Returning to Problem 1, observe that if $\dim(Y) = D$ is much greater than $N \dim(Z)$, then the major contribution to the complexity of the fibers G_y is local: if a submanifold H_y is tangent to, say embedded, $Z \subset M$ at some point $z_0 \in Z$ with high order, then the intersection $G_y = H_y \cap Z$ must be locally complicated at z_0.

For example, let Y be the family of affine hyperplanes $H \subset M = \mathbb{R}^P$, where P equals the dimension of the space of polynomials in N variables of degree $d+1$. Then among the fibers G_y passing through a (generic) point $z_0 \in Z \subset R^P$ we find levels of generic functions on Z that vanish with order d at z_0 and (some of) such levels are as complicated as real algebraic hypersurfaces of degree $d+1$ may be.

On the other hand, a generic $Z \subset \mathbb{R}^P$ always has a supporting hyperplane $H \subset \mathbb{R}^P$ that is tangent to Z at m-points for $m \geq \frac{P}{N+1}$, since the boundary of the convex hull of Z has dimension $P-1$ while what you can get with m points is at most $mN + m - 1$. It follows that the map $F = F_f : X \to Y$ necessarily has an m-multiple fold self-intersection point.

Can one ever do better than that? For instance, let Z be a closed N-dimensional Riemannian manifold of negative curvature generically mapped (e.g. embedded) to $\mathbb{R}^P$. Then, does some linear projection of Z to a linear subspace of dimension $N-1$ have an m-multiple fold self-intersection point for $m = N - 1 + D$, for D $=\dim(Gr_P^{N-1}) = (N-1)(P-N+1)$?

The natural candidates for "convex" $f : Z \to \mathbb{R}^P$ are Y_k-*equisingular* deformations of the *Veronese maps* $f_d : S^N \to \mathbb{R}^{P_d}$ (these are embeddings for odd d) where P_d is the dimension of the linear space of homogeneous polynomials of degree d on $\mathbb{R}^{N+1} \supset S^N$, where Y_k denotes the space of $(N-k)$-codimensional affine subspaces in $\mathbb{R}^{P_d}$ and where "equisingularity" means that the topology of the critical set $\Sigma(F) \subset Y_k$ for the above $F = F_f$ does not change under the deformation.

What is the closure of the space of "convex" Z in some weak (e.g. Hausdorff) topology? Is there some version of the Blaschke compactness theorem for (the space of) "convex" $Z \subset \mathbb{R}^P$?

Global invariants of families of* $\mathbf{1}$*-cycles. We were primarily concerned so far with the lower bounds on the *fiber* complexity of homology classes $h \in H_*(cl_1(Z))$

(more specifically, the complexity expressed by $\max_y |\chi|_{01}(G_y)$ for D-dimensional Y-families of 1-cycles, $X = \cup_{y \in Y} G_y \to Z$, that represent h). Some lower bounds on the Σ-complexity, i.e. on the topology of the critical sets $\Sigma(F) \subset Y$ for smooth representations of h with generic $F : X \to Y$, will be derived in section 6 from bounds on the fiber complexity of auxiliary $(D-1)$-dimensional families of 2-cycles, similar to what was done in 2.2 with the Retraction Lemma.

5.8 Length spectra in cl_1 and cl_1^{sst}. The Almgren D-inequality from 5.1 provides the lower bound on the maximal length (1-volume) of 1-cycles in the families detecting the $\smile$-powers of the fundamental class $h = [Z]_{-1} \in H^*(cl_1(Z;\mathbb{Z}_2);\mathbb{Z}_2)$ for Riemannian manifolds Z. In particular, if $\dim(Z) = 2$, this implies that

$$\liminf_{D\to\infty} \frac{1}{\sqrt{D}} \operatorname{length}^\circ(h^D) \geq \frac{\sqrt{2}}{\sqrt[4]{3}} (\operatorname{area}(Z))^{\frac{1}{2}},$$

as seen with the hexagonal tessellation of the plane.

On the other hand, by looking at moving edges of the square tessellation, one sees that

$$\limsup_{D\to\infty} \frac{1}{\sqrt{D}} \operatorname{length}^\circ(h^D) \leq 2(\operatorname{area}(Z))^{\frac{1}{2}}.$$

Besides the total length of graphs G mapped to Z, one can encode the lengths of individual edges of G by considering the subsets $\mathcal{S}^{\mathrm{Lip}}_{[G]}(Z) \subset cl_1(Z)$, where G denote here *metric* graphs, i.e. graphs with the length structures and $\mathcal{S}^{\mathrm{Lip}}_{[G]}(Z)$ denotes the space of 1-Lipschitz maps $G \to Z$.

If we restrict to $\mathcal{G}_{sst}(Z) \subset cl_1(Z;\mathbb{Z}_2)$, this amounts to the partial order on $\mathcal{G}_{sst}(Z)$ corresponding to the partial order $G_1 \succeq_{\mathrm{Lip}} G_2$ on metric graphs signifying the existence of a semi-stable morphism represented by a 1-*Lipschitz* map $G_1 \to G_2$. The partially ordered space $(\mathcal{G}_{sst}(Z), \succeq_{\mathrm{Lip}})$ makes a comprehensive metric invariant of Z: it fully recaptures (by an easy argument) the topology and the Riemannian metric of Z.

Every monotone increasing function $\lambda : \mathcal{G}^{\mathrm{Lip}}_{sst} \to \mathbb{R}_+$ defines the corresponding function, also denoted λ, on $\mathcal{G}_{sst}(Z)$ (compare 5.1) and then the associated spectral functions $\lambda_\circ$ and λ° on the homology and the cohomology of $\mathcal{G}_{sst}$ as well as of $cl_1(Z)$. In particular, for each $0 < p \leq 1$, one has the *p-energy of G*, the sum of the p-th powers of the lengths of the edges of G. (If $p > 1$ the p-energy is not monotone, albeit a convex function.)

QUESTION. Can one reconstruct Z by the spectral functions *p-energy*$^\circ$ (on the cohomology of $\mathcal{G}_{sst}(Z)$ and/or of $cl_1(Z)$) for all $0 < p \leq 1$?

A lower spectral bound for $\lambda(G) = |\chi|_1(G) + p$-energy$(G)$. There is semblance of Morse theory for this (discontinuous) function λ on the space $\mathcal{G}_{sst}(Z)$ for $1 < p < 1+\varepsilon$, for small (depending on the total length of G) positive ε, where the critical points are graphs G that are edge geodesically mapped to Z.

If G is an N-dimensional CAT($\kappa \leq 0$)-space, then the Morse index ind(G), i.e. the dimension of the "handle" that is the normal section of the G-stratum in $\mathcal{G}_{sst}(Z)$, implemented in the space of cycles by the down-stream gradient orbits, is bounded by $(N-1)|\chi|_1(G)$, as was mentioned earlier.

If $\kappa > 0$, then the conjugate points along the geodesic edges of G also contribute to the index, where this contribution is bounded by $N\kappa\pi^{-1}\,\mathrm{length}(G)$.

It follows that the function $\lambda_\circ$ on the homology of $\mathcal{G}_{sst}(Z)$ for $\lambda(G) = |\chi|_1(G) + N\kappa\pi^{-1}\,\mathrm{length}(G)$ satisfies

$$\lambda_\circ(h) \geq \deg(h)$$

for all $0 \neq h \in H_d(\mathcal{G}_{sst}(Z); \mathbb{Z}_2)$, $d = \deg(h)$.

Question. Let Z be a closed surface of constant curvature κ and $cl_1(Z)$ is implemented by embedded cycles (subgraphs) in Z. Every connected component of this cl_1 is homotopy equivalent to the real projective space P^∞, and one wants to understand the Morse structure of the length function on these P^∞. For example, how does this structure behave under deformations of the metric in Z for $\kappa \leq 0$? What are the critical graphs of minimal length l_d for each of index d, that detect $0 \neq h \in H_d(P^\infty)$? ("Detect" means that h is contained in the image of the homology inclusion homomorphism for $\mathrm{length}^{-1}[0, l_d] \subset P^\infty = cl_1(Z; \mathbb{Z}_2)$.)

Questions concerning higher-order length spectra. The appearance of large curvature in non-singular deformations of singular algebraic curves near the singularities indicates a close relation between $\mathrm{vol}_1^{(1)}$ and $\mathrm{vol}_1 + |\chi|_{01}$. Is there a general spectral inequality implementing such a relation?

Does the spectrum of $\mathrm{vol}_1^{(1)}$ (which is a geometric invariant) majorizes (in some way, at least asymptotically, for D-dimensional families of cycles for $D \to \infty$) the (purely topological) $|\chi|_{01}$-spectrum?

For example, let Z be a closed N-dimensional manifold of negative curvature and $Z \to \mathbb{R}^{N-1}$ be a generic smooth map. It is obvious that some nonsingular (contractible) fiber has total curvature $> 2\pi$ but it is unclear if there a lower bound on the supremum of the integral geodesic curvatures of the non-singular fibers G_t of such map by $\mathrm{const} \cdot N$ for a universal or, even depending on Z) constant > 0. And it seems equally plausible that the opposite is true: given an $\varepsilon > 0$, every $\mathbb{Z}_2$-homology class in $\mathcal{G}_{sst}(Z)$ can be represented by a cycle Y of graphs $G_y \subset Z$, where the total curvature of the edges of G_y is $\leq \varepsilon$ for all $y \in Y$.

5.9 $|\chi|_{\mathrm{hyp}}$-spectra in spaces of 2-cycles. Consider the category $\mathcal{G}^2$, where the objects are, possibly disconnected, point-singular surfaces G (which are allowed to have isolated points among their connected components) and the morphisms are *p.l.*-maps $G_1 \to G_2$ that are one-to-one over G_2 minus finitely many points $g \in G_2$ (i.e. the pullback of almost every $g \in G_2$ is a one point set) and where the pullback of each exceptional point is a (possibly empty) union of a point-singular surface and an even graph that meet (at most) across a finite set.

Maps $Y \to \mathcal{G}^2(Z)$ can be regarded as *p.l.*-families $X = \cup_y G_y$, $y \subset Y$, of point-singular surfaces. In particular a smooth representation $[Y]$ of a D-cycle in the space $cl_2(Z; \mathbb{Z}_2)$ by $\alpha : X^{D+2} \to Y^D$ with $F : X \to Y$ a *generic* smooth map defines a map $Y \to \mathcal{G}^2(Z)$.

It does not seem hard to show that the natural map $\mathcal{G}^2(Z) \to cl_2(Z; \mathbb{Z}_2)$ is a homotopy equivalence. In any case, the induce homology homomorphism

$$H_*\big(\mathcal{G}^2(Z); \mathbb{Z}_2\big) \to H_*\big(cl_2(Z; \mathbb{Z}_2); \mathbb{Z}_2\big)$$

is surjective by the Thom theorem, and this is all we shall be using below.

The space $\mathcal{G}^2(Z)$ is naturally partitioned into subsets $\mathcal{S}_{[G]}$ according to topological types $[G]$ of surfaces G and this partition carries non-trivial (i.e. not reducible to $H_*(Z)$) information on the homotopy type of Z similar to the diagram partition of the space of 1-cycles. This partition, however is *not* a pre-stratification due to the possible appearance and disappearance of "spherical bubbles" in families of G_y, where a *bubble*, by definition, is topological 2-sphere that is not a connected component of G (and, thus contains a singular point of G.

The information encoded by the topology of a singular surface G is determined by the bipartite graph H where the vertices are the singular points $g \in G$ and the connected non-singular surfaces $S \subset G$ and where the edges correspond to the inclusions $g \in S$. Besides, each S carries a weight, the Euler characteristic $\chi(S)$, and the graph H with the function χ on the set of the S-vertices fully determines the topology of G.

There are (at least) two ways of turning $\mathcal{S}_{[G]}$ into a (pre)-stratification.

Firstly, as in the case of 1-cycles, one passes to the subspace $\mathcal{G}^2_{sst}(Z) \subset \mathcal{G}^2(Z)$, (that mimics smooth representations with purely folded maps $F : X \to Y$) which is associated to the smaller category $\mathcal{G}^2_{sst}$ with the same objects as $\mathcal{G}^2$ and where the morphisms are the *p.l.*-maps $G_1 \to G_2$ that are 1-dimensional over non-isolated exceptional points $g \in G_2$ (these are either circles or pairs of points for purely folded maps) and are either single points or 2-spheres over the isolated points $g \in G_2$. The partition of $\mathcal{G}^2_{sst}(Z)$ into $\mathcal{S}_{[G]}$ is a pre-stratification, since the bubbles carrying singular points cannot disappear.

Another possibility is to remove the bubbles from the point-singular surfaces G and divide the resulting bubble free surfaces into classes according to their topological types, denoted $[G]_\ominus$. Then the partition of $\mathcal{G}^2(Z)$ into the subsets $\mathcal{S}_{[G]_\ominus}$ is a pre-stratification of $\mathcal{G}^2(Z)$.

Then, given a positive function λ on the set of the classes $[G]$, which is monotone increasing for the order $[G_1] \succeq [G_2]$ signifying the existence of a morphism $G_1 \to G_2$ in $\mathcal{G}^2_{sst}$, one raises the question of evaluation of the corresponding spectra on the homology and the cohomology of the space $\mathcal{G}^2_{sst}(Z)$ defined as in 5.1. Furthermore, given such λ on the set of the $[G]_\ominus$-classes, one asks this question for both spaces, $\mathcal{G}^2(Z)$ and $\mathcal{G}^2_{sst}(Z)$.

Here is our basic example.

$|\chi|_{\mathrm{hyp}}$-spectrum on the space of 2-Cycles in Z. Let $\lambda[G]_\ominus = |\chi|_{\mathrm{hyp}}(G)$. Then we know that if Z comes along with a complete CAT$(\kappa \leq -1)$-metric, then the corresponding λ°-spectrum on $H^*(\mathcal{G}^2(Z);\mathbb{Z}_2)$ (where the restriction homomorphism from the $\mathbb{Z}_2$-cohomology of $cl_2(Z;\mathbb{Z}_2) \supset \mathcal{G}^2(Z)$ to that of $\mathcal{G}^2(Z)$ is injective by Thom's theorem) is bounded by below by area$_\kappa$ and, hence, by the vol$_2$-spectrum:

$$|\chi|^\circ_{\mathrm{hyp}} \geq \tfrac{1}{2\pi}\,\mathrm{vol}^\circ_2\ .$$

Important Remark. According to 4.4 families of surfaces G in Z can be deformed to other families of surfaces, say $G' \subset Z$, with negative ambient curvatures and thus having their intrinsic curvatures ≤ -1. If Z is only CAT$(\kappa \leq 0)$ but yet contains an open subset $B \subset Z$ where $\kappa \leq -1$, then these G' satisfy

$$|\chi|^{\circ}_{\mathrm{hyp}}(G) = |\chi|^{\circ}_{\mathrm{hyp}}(G') \geq \tfrac{1}{2\pi}\,\mathrm{vol}_2(G' \cap B)\,,$$

by the Gauss–Bonnet theorem.

It follows that the $|\chi|^{\circ}_{\mathrm{hyp}}$ of the $\smile$ powers of the fundamental class in the space of 2-cycles are bounded in terms of vol°_2 in Z_0,

$$|\chi|^{\circ}_{\mathrm{hyp}}[Z]^d_{-2} \geq \tfrac{1}{2\pi}\,\mathrm{vol}^{\circ}_2[Z_0]^d_{-2}\,.$$

This, combined with 5.2, 5.3, yields, for example, the following lower bound on $|\chi|^{\circ}_{\mathrm{hyp}}[Z]^d_{-2}$.

Let Z be a complete Riemannian $\mathrm{CAT}(\kappa \leq 0)$ *manifold . Then the d-th $\smile$-power $h = [Z]^d_{-2}$ of the fundamental class $[Z]_{-2} \in H^{N-2}(K(\mathbb{Z}_2, N-2);\mathbb{Z}_2) \subset H^{N-2}(cl_2(Z;\mathbb{Z}_2);\mathbb{Z}_2)$, $N\dim(Z)$, satisfies,*

$$|\chi|^{\circ}_{\mathrm{hyp}}(h) \geq \sum_i v_i\,,$$

provided Z contains disjoint simply connected balls $B_{z_i}(v_i)$, $i = 1, 2, \dots, d$, or radii $r(v_i)$, such that the curvature of the metric on these balls is ≤ -1 and where $r(v)$ denotes the radius of the hyperbolic disc of constant curvature -1 with area v. In particular,

$$|\chi|^{\circ}_{\mathrm{hyp}}(h) \geq \mathrm{const}(Z) d^{\frac{N-2}{N}}\,,$$

where $\mathrm{const}(Z) >)$ *for all connected* $\mathrm{CAT}(0)$*-manifolds where the sectional curvatures are strictly negative at some point.*

Furthermore,

every non-zero class h in $H^D(K(\mathbb{Z}_2, N-2);\mathbb{Z}_2) \subset H^D(cl_2(Z;\mathbb{Z}_2);\mathbb{Z}_2)$ satisfies

$$|\chi|^{\circ}_{\mathrm{hyp}}(h) \geq \mathrm{const}(N,\varepsilon) D^{\frac{1}{3}-\varepsilon} v_0\,,$$

provided Z contains a simply connected ball of radius $r(v_0)$ on which $\kappa \leq -1$.

This applies, in particular, to smooth Y-parametric representation $X = \cup_y G_y \to Z$, $y \in Y$, of homology classes in $cl_2(Z;\mathbb{Z}_2)$ and yields the bounds

$$\max_y |\chi|_{\mathrm{hyp}}(G_y) \geq \sum_i v_i \quad \text{and} \quad \max_y |\chi|_{\mathrm{hyp}}(G_y) \geq \mathrm{const}(N,\varepsilon) D^{\frac{1}{3}-\varepsilon} v_0\,,$$

whenever $h([Y]) \neq 0$.

Here is another

Important Example. Let Z_i be complete N-dimensional manifolds of constant curvature -1 and finite volume. Chop away the cusps, let $Z'_i \subset Z_i$ denote the remaining compact submanifolds with Riemannian flat (e.g. toric) boundaries and let Z be obtained by gluing Z_i by linear isomorphisms between (some of) these tori. Then

$$|\chi|^{\circ}_{\mathrm{hyp}}[Z]^d_{-2} \geq \tfrac{1}{2\pi} \max_i \mathrm{vol}^{\circ}_2[Z_i]^d_{-2}\,. \qquad (\Sigma_i Z'_i)$$

Indeed, Z carries a family ρ_t of $\mathrm{CAT}(0)$-metrics which converge for $t \to \infty$ to the disjoint union of Z_i.

Remarks and Questions. (a) The upper bound on $\max_y |\chi|_{\mathrm{hyp}}(G_y)$, say, for $h = [Z]^d_{-2}$, that is delivered by algebraic families G_y, comes with a greater power of d. For example, if $N = 4$, the complex algebraic curves of degree $\approx d^{1/2}$ provide the bound

$$|\chi|^{\circ}_{\mathrm{hyp}}[Z]^d_{-2} \leq \mathrm{const}(Z)d\,.$$

The gap between this and our lower bound, $|\chi|^{\circ}_{\mathrm{hyp}}[Z]^d_{-2} \geq \mathrm{const}(Z)d^{1/2}$ is due to the fact that the inequalities in 4.5 do not take into account the relative curvature (second fundamental form) of surfaces G in Z.

Can one improve the lower bounds on $|\chi|_{\mathrm{hyp}}(G)$ with a use of the tangential lift of G from Z to the Grassmann bundle $Z' = Gr_2(Z)$? (The area of the lifted G in Z' encodes the curvature of G in Z as well as the area, see 5.4.)

(b) The geometric assumptions of our inequalities have purely topological consequences. For example, if Z is a compact locally symmetric manifold of negative (e.g. constant) curvature then it admits a sequence of finite s_i-sheeted coverings, $s_i \to \infty$, such that the injectivity radius of Z_i is about $\log^{\varepsilon}(s_i)$. It follows that these Z_i contains simply connected $r(v_i)$-balls with $v_i \sim s_i^{\varepsilon}$ for some $\varepsilon = \varepsilon(N) > 0$.

Topological Corollary: Maps $Z^N \to \mathbb{R}^{N-2}$. *There exist, for each $N \geq 3$, a positive $\varepsilon = \varepsilon(N) > 0$, and a closed N-dimensional manifold Z that admits a sequence of finite s_i-sheeted coverings Z_i, $s_i \to \infty$, such that every generic smooth map $F : Z_i \to \mathbb{R}^{N-2}$ has a smooth fiber $G = G_y = F^{-1}(y), y \in \mathbb{R}^{N-2}$, with*

$$|\chi|_{\mathrm{hyp}}(G) \geq \mathrm{const}(N)s_i^{\varepsilon}\,.$$

(The hyperbolic characteristic $|\chi|_{\mathrm{hyp}}(G)$ of a closed surface G is the sum of the absolute values of the Euler characteristics of the hyperbolic connected components of G.)

We shall see in 6.3 that the maximal $\varepsilon(3) = 1$, i.e. if $N \dim(Z) = 3$, then there exits a sequence of coverings, where the above inequality is satisfied with $\varepsilon = 1$ (see 6.3). The maximal $\varepsilon(N)$ are unknown for $N \geq 4$, where the possibility of $\varepsilon(N) = 1$ for all $N \geq 3$ is not excluded. (It is obvious that $\varepsilon(N) \leq 1$ for all N.)

(c) Let Z be a closed aspherical manifold with the fundamental group Γ. How much of Γ can be seen in the partitions of the spaces $cl_2(Z)$, $\mathcal{G}^2(Z)$ and $\mathcal{G}^2_{sst}(Z)$ into the strata $\mathcal{S}_{[G]}$? For example, can one reconstruct Γ from the $|\chi|^{\circ}_{\mathrm{hyp}}$-spectrum for a hyperbolic group Γ?

(d) The above question also makes sense for $cl_1(Z)$ and the corresponding $\mathcal{G}$-spaces. Is there a good way to integrate the information coming from the spaces of 1- and 2-cycles?

For example, consider simultaneous representations of homologies in l different spaces of cycles (of equal or of different dimensions), with a single $\alpha : X \to Z$, where X is sliced into fibers G (pullbacks of points) by several maps, $F_k : X \to Y_k$, $k = 1, 2, \dots, l$, with the condition (connecting them) that the F_{k_1}-fibers are contained in F_{k_2}-fibers for $k_1 < k_2$. However, the geometric/topological information encoded by such joint families seems to bring little new compared to these representations looked at separately.

(e) Besides the total area, and the areas of the (regular) components $G_1, G_2 \dots$ (of the regularization $\hat{G}_y$ of G_y, compare section 4), a surface G in a Riemannian manifolds Z is characterized by the conformal structures of the induced metrics in $G_1, G_2, \dots$

To take these into account, consider a family $\mathcal{X} = \cap_y G_y \to \mathcal{Y}$ of point-singular surfaces with conformal structures, and look at families X induced by continuous maps $\beta : Y \to \mathcal{Y}$. Then every Y-family of surfaces in Z, $\alpha = \{\alpha_y\} : X = \cup_y G_y \to Z$ a map $\mathcal{E} : Y \to \mathcal{Y} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \dots$, for $y \mapsto (\beta(y), E_1(\alpha_y), E_2(\alpha_y), \dots)$, where $E_1(\alpha_y), E_2(\alpha_y), \dots$ denote the (conformally invariant Dirichlet quadratic) energies of α_y restricted to the regular components $G_{y1}, G_{y2} \dots$ of G_y for their conformal structures (and where we ignore a minor ambiguity due to the possible presence of isomorphic/indistinguishable regular components of G_y).

These maps $\mathcal{E}$ can be used to define geometric invariants of Z similarly to the space of metric graphs in Z (see 5.5–5.8), where, observe, every graph can be seen as a limit of point-singular surfaces and so the conformal surface invariants contain the full metric information about Z. Here is a more illuminating

Heegard Decomposition Example. Let Z be a 3-manifold and $\alpha : X = G \times [0.1] \to Z$ a Heegard decomposition with a surface G of the Euler characteristic c. Our $\mathcal{E}$ defines defines a path $[0,1] \to \mathcal{Y}(c) \times \mathbb{R}_+$ where $\mathcal{Y}(c)$ is a compactified moduli space of G's. The set $\mathcal{H}$ of the paths of *all* such decompositions with a given c is an instance of the metric invariant of Z defined via conformal surface families in Z.

Similarly, given a homology class $h \in cl_2(Z)$, we consider the set $\mathcal{H}$ of Y's along with the maps $Y \to \mathcal{Y} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \dots$ that appear as the maps $\mathcal{E}$ associated to smooth representations of h, where $\mathcal{Y}$ is the "universal moduli space" of point-singular surfaces with conformal structures.

Problem. Organize suitable "reductions" of these $\mathcal{H}$'s into a category of "nicely structured" objects.

(f) If we want to capture the differential topology (rather than the homotopy type) of Z, we need to consider families of point-singular surfaces that are *smoothly embedded* (rather than just mapped) to Z (as in the case of graphs, see 5.7). The most attractive case is that of 4 manifolds Z, where the (properly structured) space of such surfaces, probably, carries the the information similar to (greater than?) that contained in the Donaldson and/or Seiberg–Witten invariants.

5.10 Simplicial spectra. Every homology class in $H_k(cl_k(Z;\mathbb{Z}_p))$ can be represented by a simplicial family (see 4.7) $F : X \to Y$, with $\alpha : X \to Z$, where the fibers $G_y = F^{-1}(y)$ are k-dimensional $\mathbb{Z}_p$-pseudomanifolds. Let $\mathrm{vol}_\Delta(\alpha) = \max_y \mathrm{vol}_\Delta(G_y)$, where $\mathrm{vol}_\Delta(G_y)$ is the number of the k-simplices in G_y and let $\mathrm{vol}^\circ_\Delta(h)$, $h \in H^k((cl_k(Z;\mathbb{Z}_p));\mathbb{Z}_p)$ be the minimal number v such that h does not vanish on some homology class represented by a simplicial family α with $\mathrm{vol}_\Delta(\alpha) \le v$. Observe that the function $h \mapsto \mathrm{vol}_\Delta(h)$ is a homotopy invariant of Z

if Z is $\mathrm{CAT}(\kappa < 0)$ *then the k-volume of every cohomology class of $cl_k(Z)$ is bounded by*

$$\mathrm{vol}^\circ(h) \le \delta_{\kappa,k} \, \mathrm{vol}^\circ_\Delta(h) \,. \qquad (\mathrm{vol}_\Delta)$$

This follows by Thurston's straightening argument (see 4.7) and a similar inequality holds for all aspherical hyperbolic polyhedra.

Remarks and Questions. (a) This inequality generalizes to those CAT($\kappa = 0$)-spaces where there is a "sufficient pool" of k-simplices of bounded volume. These include Z that are locally isometric to the Riemannian product of CAT($\kappa < 0$) spaces Z_i of dimensions N_i, where

$$(\mathrm{vol}_\Delta) \textit{ holds as it stands for all } k > 1 + \textstyle\sum_i N_i - \min_i N_i.$$

(Probably, the sufficient condition, say for compact Z, is *nonexistence* of k-dimensional cylindrical geodesic subspaces, $Z_0 \times \mathbb{R}$, in the universal covering of Z, compare [LS], [Bu].)

For example, (vol_Δ) applies to families of 5-cycles in the products of two 3-dimensional manifolds of negative curvature but not to families G_y of 3-cycles in product of surfaces, where, however, a non-trivial lower bound on $\max_y \mathrm{vol}_\Delta(G_y)$ does exist by the $\smile$-inequality in 2.2. The true shape and the range of the applicability of (vol_Δ) remain unclear.

(b) The inequality (vol_Δ) when combined with Almgren and Guth inequalities (see 5.1) delivers a lower bound on the $\mathrm{vol}^\circ_\Delta$-spectrum of CAT($\kappa < 0$)-spaces and the above CAT(0)-spaces, and, as in the case with simplicial volume, provides, for example, obstructions for the existence of maps $Z_1 \to Z_2$ of non-zero degree. But the lower bound obtained via vol_k, unlike the corresponding bound for the simplicial volume, seems far from being sharp, not even asymptotically for $\deg(h) \to \infty$. For example, such lower bound on the $\smile$-powers of the fundamental class of an N-dimensional manifold Z of negative curvature,

$$\mathrm{vol}^\circ_\Delta[Z]^d_{-k} \geq \mathrm{const}(Z) d^{\frac{N-k}{N}}$$

does not match the obvious upper bound $\mathrm{vol}^\circ_\Delta[Z]^d_{-k} \leq \mathrm{const}'(Z) d$.

It remains unclear (even for $k = 2$ as we mentioned earlier) what the true asymptotics of $\mathrm{vol}^\circ_\Delta$-spectrum is in this case as well as for most other classes of (say aspherical) spaces Z.

(c) Is there an upper bound on $\mathrm{vol}_\Delta(h)$ by higher-order volumes, e.g. by $\mathrm{vol}^{(1)}_k(h)$? (It seems clear that $\mathrm{vol}^\circ_\Delta \leq \mathrm{const}(Z)(\mathrm{vol}^{(1)}_2)^\circ$ for $k = 2$, where $\mathrm{vol}_\Delta(G) \approx \mathrm{rank}(H_*(G))$ for surfaces G.)

(d) The number of simplices does not strike one as the right measure of topological complexity of G_y in the present context. Some algebraic degree of a family (e.g. where F maps simplices to simplices by algebraic maps of degrees $\leq d$) looks a better invariant but it is unclear how to bound it from below in specific examples. Another possibility is to use the k-volume in some canonical geometric model of Z as it goes with the simplicial volume (see 3.1).

Alternatively, one may use *Riemannian representation* $F : X \to Y$, $\alpha : X \to Z$, where X is endowed with a Riemannian metric with some bound on the local geometry, such that (compare 4.7, 5 $\frac{5}{7}$ in [Gr6] and 5.41 in [Gr9])

(A) The Ricci curvature of X is bounded from below by -1.

(B) The sectional curvatures are bounded from below, by -1.

(C) The sectional curvatures are bounded from above and below, $-1 \leq K(X) \leq -1$.
(C+) The above bound plus $\operatorname{Inj Rad} \geq 1$.

Next, one introduces some measures of the sizes of the fibers $G_y = F^{-1}(y)$ in (X, ρ), e.g.
$V_1(G_y) = \operatorname{Diam}(G_y)$;
$V_2(G_y) = \operatorname{vol}_k(G_y) \leq v$;
$V_3(G_y) = \operatorname{vol}_k^{(1)}(G_y) \leq v$;
$V_4(G_y)$, that is the N-Volume of the ε-neighborhoods of G_y in X, say for $\varepsilon = 1$.
Then one uses Riemannian representations with some of the above four conditions (A)–(C+), and take the minimal value of V_i needed to represent a homology class $h \in H_*(cl_k(Z))$ (probably, only few of the 16 possibilities would lead to something interesting).

What are the spectra of the resulting functions V_i° on $H^*(cl_k(Z))$?

Example. If X satisfies (C+) than, the family $F : X \to Y$ can be modified to a simplicial family with $\operatorname{vol}_\Delta \leq \operatorname{const}_N \min(V_2, V_4)$ and the bound on $\operatorname{vol}_\Delta^\circ$-spectrum yields similar bounds on the V_2° and V_4° spectra.

5.11 Homotopy min max-volumes. Families of *cycles* G_y are not sufficient for our topological application, where we need to deal, for example, with families of point/edge singular surfaces and where, we need a lower bound on the volumes $G_y \to Z$ under the homotopy rather than homology constrains on the family. Here is the general

Problem. Let Z be a metric space, $\mathcal{G}$ denote the space of compact k-dimensional subsets $G \subset Z$ and $\mathcal{Y}$ be a class of spaces Y along with maps $Y \to \mathcal{G}$ with the notation $y \mapsto G_y$. Evaluate

$$\inf_{Y \in \mathcal{Y}} \sup_{y \in Y} \operatorname{vol}_k(G_y)$$

in terms of the homotopy theoretic properties of the class $\mathcal{Y}$ (e.g. the homotopy dimension of some associated map,)

Another way to put it, let $F : X \to Y$ be a continuous map with compact k-dimensional fibers $G_y = F^{-1}(y)$, and $\alpha_0 : X \to Z$ a continuous map, such that the Hausdorff (alternatively Minkowski) measures of the α_0-images of all these fibers satisfy

$$\operatorname{vol}_k\big(\alpha_0(G_y)\big) < v$$

for some $v > 0$. Find a homotopy β_t of $\beta_0 = \alpha_0 \times F : X \to Z \times Y$ to a "topologically minimal" map $\beta_1 : X \to Z$ and and evaluate the maximal v for which such minimization is possible (where this evaluation may depend on a particular notion of "size" of the "minimal map" we look for).

If we are concerned with the variational problem we may restrict ourselves to homotopies β_t, that, when projected to $\alpha_t : X \to Z$, do not much enlarge the volumes of $G_y \to Z$, e.g. keeps $\operatorname{vol}_k(\alpha_t(G_y)) = 0$ for the fibers with $\operatorname{vol}_k(\alpha_0(G_y)) = 0$. Also we may allow such "minimization" to take place within a larger space $Y_+ \supset Y$ that is, in some sense, homotopically close to Y (such as an Eilenberg–MacLane space $\supset Y$ in the homological setting of the previous section).

The "topological size" we want to minimize is, typically, either the *dimension of a map* that is the dimension of its image or such dimension of some associated map(s). (In the homological setting this is the maximal dimension where a related (co)-homology homomorphism is non-zero.)

If $n\dim(X) \leq N\dim(Z)$ then the question make sense for α_0 itself, while looking at homotopies of β_0 is essential for $n > N$. Observe, that the projection of β_t to Z makes a homotopy α_t of α_0 with the same fibers $G_y \subset X$ for all t. Alternatively, one can change the fibers along with the homotopy β_t by defining $G_{y,t} \subset X$ being the pullbacks of $y \in Y$ under the map $P_Y \cdot \beta_t$ for the coordinate projection $P_Y : Z \times Y \to Y$. Then the minimax vol_k-problem for families of $G_y \to Z$ modifies to evaluating $\min_{\beta_1} \max_y \mathrm{vol}_k(P_Z \circ \beta_1(G_{y,1}))$.

If $n \gg N$ it is worthwhile considering (homotopies of) higher (absolute and, especially, relative) Cartesian products of maps α_0 and F. For example, let $F^d_{/Y} : X^d_{/Y} \to Y$ denote the d-th Cartesian power of X over Y (with the fibers being the Cartesian powers $(G_y)^{\times d}$ of G_y) and α_0^d be the obvious map $X^d_{/Y} \to Z^d$. Then the $\smile$-Inequality in 2.2 suggests the following:

PROBLEM. Evaluate the maximal v, such that, for every α_0 with $\max_y \mathrm{vol}_k(\alpha_0(G_y)) < v$, the map

$$\alpha_0^d \times F : X^d_{/Y} \to Z^d \times Y$$

is homotopic to a map of dimension $< \dim(X^d_{/Y}) = m + dk$ for $m\dim(Y)$ and $k\dim(G_y)$.

The simplest minimization results are as follows.

EQUIDIMENSIONAL EXAMPLE. Let Z be a connected manifold of dimension k and $G_{y,0} \subset Z$ be a compact family of proper (i.e. $\neq Z$) compact subsets that is (semi-)continuous in y in the Hausdorff topology. Then there is a homotopy $G_{y,t}$ of this family to $G_{y,1} \subset Z$, where $\dim(G_{y,t}) < k$ for all y.

1-DIMENSIONAL EXAMPLE. Let $F : X \to Y$ be a *p.l.*-family with $k\dim(G_y) = 1$ and $\alpha_0 : X \to Z$ be a Lipschitz map such that $\mathrm{length}(\alpha_0(G_y)) < v$, $G_y = F^{-1}(y)$, for all $y \in Y$. If the *convexity radius* of Z is everywhere $\geq v$ (e.g. Z is the round sphere of radius $\geq 2v/\pi$), then α_0 is homotopic to a map that is 0-dimensional on each fiber G_y. (This conclusion holds for all $k \geq 1$ whenever $\mathrm{diam}(\alpha_0(G_y)) \leq \mathrm{Conv\,Rad}(Z)$, while for $k = 1$ the condition $\mathrm{length}(\alpha_0(G_y)) \leq 2\,\mathrm{Conv\,Rad}$ and, probably, $\mathrm{length}(\alpha_0(G_y)) \leq 4\,\mathrm{Conv\,Rad}$ suffices.)

However, it is unclear if every family can be homotoped to α_1, where each image $\alpha_1(G_y)$ is a graph with at most d edges for d being a (reasonable) function on $v = \max_y \mathrm{length}(\alpha_0(G_y))$ (where the first choice for d, motivated by Guth's inequality stated in 5.1, would be $d = \mathrm{const}(Z)v^2$ or, at least, $d = \mathrm{const}(Z,\delta)v^{2+\delta}$.)

QUESTIONS. (a) Does there exist a positive $v = v(Z) > 0$, such that the bound $\max_y \mathrm{vol}_k(\alpha_0(G_y)) < v = v(Z)$ (possibly under some extra conditions, e.g. continuity of $G_y \to Z$ in the *flat topology*, that is satisfied with Lipschitz α_0 if F is *p.l.*-map, a piecewise real analytic or *generic* C^∞-map) implies that α_0 is homotopic to α_1, with $\dim(\alpha_1(G_y)) < k\dim(G_y)$? (One may insist on $\dim(\alpha_1(G_y)) < l$ for some $l < k$

if X is a Carnot–Carathéodory space, for example, where all smooth l-dimensional submanifolds $G \subset X$ have $\mathrm{vol}_k(G) > 0$ and where one may use the argument from [You].

An answer to a version of this question is available for integral (and/or $\mathbb{Z}_p$) currents $G_y \subset Z$ that are continuous in y with respect to the flat topology by a result of Almgren (see [Gu] and references therein), where Almgren's inequality for v is sharp for Euclidean spheres S^N: Almgren's v equals (to nobody's surprise) the volume of S^k.

(b) Does there exist a (reasonable) $d = d(Z, v)$ for all $v > 0$, such that every α_0 satisfying $\max_y \mathrm{vol}_k(\alpha_0(G_y)) \leq v$ is homotopic to α_1, such that the image $\alpha_1(G_y) \subset Z$ of every G_y admits a triangulation, "continuously depending" on y with at most d simplices of dimension k? (The "continuity" means that the triangulations can be organized into a simplicial family over Y.)

Here, similarly to (a), some information is available in the homological setting by the results of Guth cited in 5.1.

Codimension-1 Example. Let Z be a closed connected N-dimensional Riemannian manifold for $N = k + 1$ and let v be the supremum of the numbers such that every domain $U \subset Z$ with $\mathrm{vol}_N(U) \leq \mathrm{vol}(Z)/2$ and connected boundary satisfying $\mathrm{vol}_k(\partial U) < v$ can be homotoped within itself to the $N-2$-skeleton of some (and, hence, any) triangulation. This v is, clearly, > 0 and it can be pretty well evaluated with the isoperimetric profile of Z (see section 6). Probably, every α_0 with $\mathrm{vol}_k(\alpha_0(G_y)) < v$, can be homotoped with Almgren's technique to a map rendering (the images of) all G_y-fibers $(k-1)$-dimensional.

Despite the failure of flat continuity, there is a version of Almgren's theorem for arbitrary continuous maps F from X to $\mathbb{Z}_2$-homology manifolds Y, where one can bound from below another (a priori larger) waist defined with the Minkowski (rather than Hausdorff measure) volume vol_k, where such bound is sharp for S^n (see [Gr5]).

On the other hand, if the fibers G_y are not k-cycles in any sense, e.g. they are fibers of a map of S^n to a general $(n-k)$-polyhedron, one does not know if some G_y has $\mathrm{vol}_k(G_y) \geq \mathrm{vol}_k(S^k)$, even under the strongest regularity assumption on the map, that is to be *p.l.* in this case. Yet, the following *Federer–Fleming "pushing" construction* provides "simplifying" homotopies in some cases.

Let Z be either a compact Riemannian N-manifold without boundary or with a locally convex boundary, such that every geodesic ray eventually reaches the boundary. Take two points $z, z' \in Z$ that are joined by a *unique* minimizing geodesic segment $[z, z'] \subset Z$ and

if Z has no boundary, extend this segment to the *maximal* minimizing segment $[z, z''] \supset [z, z']$;

if Z has a boundary extend it till the point $z'' \in \partial Z$.

Thus we obtain a map $R_z : Z \setminus \{z\} \to Z$ for $z' \mapsto z''$ that sends $Z \setminus \{z\}$ either to the cut locus of Z with respect to z or to the boundary ∂Z.

Given a map $\alpha_0 : G \to Z$ let $R_z \bullet \alpha_0 : G \times [0,1] \to Z$ be the obvious geodesic homotopy between α_0 and $\alpha_1 = R_z \circ \alpha_0$.

Since the integral $\int |z|^k dz$ converges in $\mathbb{R}^N$ at 0 for $k < N$ one has, following Federer–Fleming, the following

Minimal volume inequality.

The volumes of $R_z \bullet \alpha_0$ and α_1 satisfy

$$\mathrm{vol}_N(Z)^{-1}\int_Z \mathrm{vol}_{k+1}\big(R_z \bullet \alpha(G\times[0,1])\big)dz \le \mathrm{const}_\bullet(Z)\,\mathrm{vol}_k\big(\alpha_0(G)\big)\,,$$

and, in the case with locally convex boundary,

$$\mathrm{vol}_N(Z)^{-1}\int_Z \mathrm{vol}_k\big(\alpha_1(G)\big)dz \le \mathrm{const}'(Z)\,\mathrm{vol}_k\big(\alpha_0(G)\big)\,.$$

Consequently, there is, in both cases, a point z, for which

$$\mathrm{vol}_{k+1}\big(R_z \bullet \alpha(G\times[0,1])\big) \le \mathrm{const}_\bullet(Z)\,\mathrm{vol}_k\big(\alpha_0(G)\big)\,.$$

Let G be embedded to Z (with $\alpha_0 = id$ and, thus, dropped from the notation), and apply $R_z\bullet$ to a k-plane $\tau \in Gr^k(Z)$ tangent to $G \subset Z$ at a point $g \in G$ and regarded as an infinitesimal element of G. Define

$$|R_z|(\tau) =_{\mathrm{def}} \mathrm{vol}_{k+1}\big(R_z \bullet (\tau\times[0,1])\big)\big/\mathrm{vol}_k(\tau)$$

and

$$|R_\bullet|(\tau) = \mathrm{vol}_N(Z)^{-1}\int_Z |R_z|(\tau)\,.$$

Then

$$\mathrm{vol}_N(Z)^{-1}\int_Z \mathrm{vol}_k\big(R_z \bullet (G\times[0,1])\big)dz = \int_G |R_\bullet|\big(\tau_g(G)\big)dg$$

for all rectifiable k-dimensional $G \subset Z$.

Equality for $Z = S^N$ and inequality for S^N_+. If Z equals the sphere $S^N(R) \subset \mathbb{R}^{N+1}$ of radius R, then the function $|R\bullet|(\tau)$ is constant on $Gr^k(S^N)$, since it is invariant under the isometry group of S^N that is transitive on the Grassmannian $Gr^k(S^N)$. It follows that

$$\mathrm{vol}_N(Z)^{-1}\int_Z \mathrm{vol}_{k+1}\big(R_z \bullet \alpha_0(G\times[0,1])\big)dz = \sigma R\cdot \mathrm{vol}_k\big(\alpha_0(G)\big)\,,$$

where, by looking on great k-spheres in Z for G, one sees that

$$\sigma = \sigma(k) = \tfrac{1}{2}\,\mathrm{vol}_{k+1}\big(S^{k+1}(1)\big)\big/\mathrm{vol}_k\big(S^k(1)\big)\,.$$

(Similar *equalities* hold for all *two-point homogeneous* spaces Z and $k = 1, N-1$. Also there are such equalities for complex analytic and for Lagrangian submanifolds G in the complex projective spaces.)

If τ belongs to a hemisphere $S^N_+ \subset S^N$ (i.e. $\tau \in Gr^N(S^N_+) \subset Gr^N(S^N)$), then

$$\int_{S^N_+} |R_z|(\tau)dz \ge \int_{S^N_-} |R_z|(\tau)dz$$

for $S^N_- = S^N \setminus S^N_+$, as is seen with the reflection of the sphere S^N in the equator ∂S^N_+. It follows that the averaged volume of the $R_\bullet$-cylinder pushing $\alpha_0(G)$ to the boundary ∂S^N_+ satisfies

$$\mathrm{vol}_N\big(S^N_+(R)\big)^{-1}\int_{S^N_+(R)} \mathrm{vol}_{k+1}\big(R_z \bullet \alpha_0(G\times[0,1])\big)dz \le \sigma(k)R\cdot \mathrm{vol}_k\big(\alpha_0(G)\big)$$

for the above $\sigma(k)$. Consequently

$$\inf_{z\in S^N_+(R)} \mathrm{vol}_{k+1}\big(R_z \bullet \alpha_0(G\times[0,1])\big) \le \sigma(k)R\cdot \mathrm{vol}_k\big(\alpha_0(G)\big)$$

for all Lipschitz maps α_0 of piecewise smooth k-dimensional spaces G to Z.

Homotopy Corollary. *Let Z be as in the above case with locally convex boundary, $F : X \to Y$ be a p.l. map between polyhedra with k-dimensional fibers $G_y = F^{-1}(y)$, where $n \dim(X) = k+m \leq N \dim(Z)$, $m \dim(Y)$, and let $\alpha_0 : X \to Z$ be a Lipschitz map. Then, for every $\varepsilon > 0$, there exists a (roughly, ε-fine) subdivision of Y and a Lipschitz homotopy of α_0 restricted to the F-pullback X_ε^{n-1} of the $(m-1)$-skeleton Y_ε^{m-1} of this subdivision of Y, say*

$$\alpha : X_\varepsilon^{n-1} \times [0,1] \to Z\,,$$

such that

1. *α is trivial (i.e. constant in t) over the boundary ∂Z,*
$$\alpha_t \big| (X_\varepsilon^{n-1} \cap \alpha_0^{-1}(\partial Z)) \big| = \alpha_0 \big| (X_\varepsilon^{n} \cap \alpha_0^{-1}(\partial Z)) \big|\,,$$
2. *α_1 sends X_ε^{n-1} into the boundary ∂Z.*
3. *The maps α_1 and α satisfy the following inequalities on the F-pullback of every (ε-small) l- simplex of the subdivided Y,*
$$\mathrm{vol}_{l+k}\left(\alpha_1(F^{-1}(\Delta_\varepsilon^l))\right) \leq \mathrm{const}_1(Z) \max_{y \in Y} \mathrm{vol}_k\left(\alpha_0(G_y)\right) + \varepsilon\,,$$
$$\mathrm{vol}_{l+k+1}\left(\alpha(F^{-1}(\Delta_\varepsilon^l) \times [0,1])\right) \leq \mathrm{const}_2(Z)\, \mathrm{vol}_{l+k}\left(\alpha_1(F^{-1}(\Delta_\varepsilon^l))\right);$$
hence,
$$\mathrm{vol}_{l+k+1}\left(\alpha(F^{-1}(\Delta_\varepsilon^l) \times [0,1])\right) \leq \mathrm{const}_3(Z) \max_{y \in Y} \mathrm{vol}_k\left(\alpha_0(G_y)\right) + \varepsilon\,.$$
Consequently, if
$$\max_{y \in Y} \mathrm{vol}_k\, \alpha_0(G_y) \leq \delta R^k$$
for a small positive $\delta = \delta(Z) \ll \mathrm{const}_3^{-1}$, then the homotopy α_t extends to a homotopy on all of X that is trivial over $\partial\Delta$ and with $\alpha_1(X) \subset \partial\Delta$. In particular, if $X = Z$ and α_0 is the identity map, then
$$\max_{y \in Y} \mathrm{vol}_k\, F^{-1}(y) \geq \delta(Z)\,.$$

Proof. Proceed by inductions on the l-skeleta of finely the subdivided Y, by first moving the F-fibers G_{y_i} over the vertices y_i to to ∂Z by geodesic radial projections with homotopies, say $\alpha_t^{y_i}$ of small vol_{k+1} that are provided by the minimal volume inequality. Then extend this to the F-pullback over the 1-skeleton of the subdivided Y with the inequality applied to the "unions" of pairs maps $\alpha_t^{y_i}$ for adjacent vertices with α_0 over the edges between these vertices, etc.

Remarks. (a) The induction by skeletons applies to a class of (non-simplicial) *k-coregular maps $F : X \to Y$* where $\mathrm{vol}_{i+k}(F^{-1}(\Delta_\varepsilon^i)) \to 0$ for i-dimensional simplices in Y of diameters $\varepsilon \to 0$.

(b) Let Z be a complete Riemannian manifold with 1-bounded geometry: the sectional curvatures of Z are pinched between -1 and 1 and the injectivity radius of Z is everywhere ≥ 1. Then the original Federer–Fleming argument (projecting a $(N-n-1)$-skeleton to the dual n-skeleton with the following contraction to the $(n-1)$-skeleton) applies to maps $\alpha_0 : X \to Y$ for all $n \dim(X) \leq N \dim(Z)$ that have $\max_y(\alpha_0(G_y)) \leq \mathrm{const}(N)$ for some universal $\mathrm{const}(N) > 0$, and provides a homotopy of α_0 where α_1 sends X to a $(n-1)$-dimensional skeleton of some triangulation of Z. (Probably, one can replace $\mathrm{const}(N)$ by $\mathrm{const}(n)$ with a suitable filling technique.)

The above (hemi)spherical example allows the following more precise version of the above lemma.

Contraction and retraction inequalities for round spheres and hemispheres.

If $Z = S^N(R)$ and

$$\max_{y\in Y} \mathrm{vol}_k\, \alpha_0(G_y) < \delta \cdot \mathrm{vol}_k \big(S^k(R)\big)$$

for some

$$\delta = \delta(m)\dim(Y) \geq 2^m/m!\,,$$

then the map α_0 is contractible. If $Z = S^N_+(R)$ and

$$\max_{y\in Y} \mathrm{vol}_k\, \alpha_0(G_y) < \delta\, \mathrm{vol}_k \big(S^k_+(R)\big)$$

with $\delta \geq 2^m/m!$, then there is a homotopy retraction (i.e. a homotopy that is trivial over the boundary of the hemisphere) of α_0 to (a map with the image in) the boundary of the hemisphere.

Remarks. (a) This inequality is sharp for $m\dim(Y) = 1$ but the sharp inequalities on δ for spheres and hemispheres remain problematic for $m \geq 2$. (Sharp inequalities with $\delta = 1$ are known in the homological setting as was mentioned earlier.)

(b) Similar inequalities hold for *all* manifold Z with a low bound on the *Uryson width* (see B_2' on p. 139 in [Gr2], where we use different terminology) but the above provides a better bound on δ for (hemi)spheres.

Retraction to the boundary of hyperbolic balls.

Let X, Y, F be as above, Z be a Riemannian R-ball with a metric of curvature $\leq \kappa \leq 0$ and with convex boundary ∂B and let a Lipschitz map $\alpha_0 : X \to Z$ satisfy

$$\max_{y\in Y} \mathrm{vol}_k \big(\alpha_0(G_y)\big) \leq \delta \cdot v_k(R,\kappa)$$

for some

$$\delta = \delta\big(m\dim(Y)\big) \geq 2^m/m!$$

and $v_k(R,\kappa)$ denoting the k-volume of the R-ball of constant curvature κ. Then there exists a homotopy α_t of α_0, constant over ∂B, such that α_1 maps X into ∂B.

Proof. Use a vol_k contracting diffeomorphism $B \to S^N_+(R_\kappa)$, where $\mathrm{vol}_k(S^N_+(R_\kappa)) = \mathrm{vol}_k(\kappa)$ and apply the above.

Questions. (a) One (naively?) expects that the inequality $\max_{y\in Y} \mathrm{vol}_k(\alpha_0(G_y)) < v_k(R,\kappa)$, i.e. $\delta < 1$ is sufficient for the existence of the retraction and, possibly, even a smaller δ will do.

This would follow if one could deform α_0 by vol_k-decreasing homotopy to an extremal map α, that had a differential $D_g(\alpha_0)$ with $\mathrm{rank}(D_g) = k$ at some point $g \in G$, then the volume of such α within every R-ball $B = B^N(z,R)$ at $z = \alpha(g)$ would be $\geq v_k(R)$, roughly, as large by the standard comparison argument of α with the conical maps over $\alpha^{-1}(\partial B)$. Possibly, a suitably modified Almgren's theory of varifolds and/or some multi-parametric version of the filling argument from [Gr2] can help in some cases.

(b) Let Z be a complete manifold of non-positive curvature with the injectivity radius of Z being *everywhere* $\geq R$ and let $v_k(R, \tilde{Z})$ be the infimum of the *filling k-volumes* of $(k-1)$-cycles $C \subset \tilde{Z}$ (i.e. the infimum of the k-volumes of the k-chains with the boundaries C) that are *not* homologous to zero in their R-neighborhoods in $\tilde{Z}$. Let $\alpha_0 : X \to Z$ be as above, but with no restriction on $m \dim(Y)$, such that

$$\max_y \mathrm{vol}_k \left(\alpha_0(G_y)\right) < \delta \cdot v_k(R, \tilde{Z}) .$$

Is there a bound $\delta \leq \delta(k, m)$ for $k \dim(G_y$ and $m \dim(Y)$ that guarantees that the map $\beta_0 = \alpha_0 \times F$ is homotopic to an $(n-1)$-dimensional map? Is it true with $\delta = 1$ (this is not apparent even in the homology setting) or, at least with $\delta = \delta(k)$? (A stronger but a less realistic request would be for a homotopy of α_0 to a map that is $(k-1)$-dimensional on each G_y.)

Hyperbolic $(N-1)$-contraction of surfaces.

Let $X = \cup_y G_y$ be a point/edge singular family with compact fibers G_y, Z be a closed connected manifold of negative curvature $\leq \kappa < 0$, and $\alpha_0 : X \to Z$ be a continuous map. Let Z contain a simply connected ball $B = B(R)$ of radius R, such that

$$v_2(R, \kappa) \geq 2\delta(m)\pi|\kappa|^{-1} \max_y |\chi|_{\mathrm{hyp}}(\hat{G}_y)$$

for $m = \dim(X) - 2\dim(Y)$ the above $\delta(m)$ and $v_2(R, \kappa)$ and $|\chi|_{\mathrm{hyp}}$ being the sum of the absolute values of the Euler characteristics of the hyperbolic (i.e. with $\chi < 0$) components of the regularized surface $\hat{G}_y$ (see section 4). Then, if $n \dim(X) = N \dim(Z)$, the map α_0 is $N-1$-contractible i.e. homotopic to a map into the $(N-1)$-skeleton of some (and, hence, any) triangulation of Z. In particular, if $X = Z$ and α_0 is the identity map then some fiber G_y has

$$|\chi|_{\mathrm{hyp}}(\hat{G}_y) \geq \frac{2^{m-1}|\kappa|}{m!\pi} v_2(R, \kappa)$$

(where, observe, $v_2(R, \kappa) \sim \exp R|\kappa|^{1/2}$).

Question. Let the injectivity radius of Z be everywhere $\geq R$ and $\max_y |\chi|_{\mathrm{hyp}}(\hat{G}_y) < \mathrm{const}_m\, v_2(R, \kappa)$ but allow $n \neq N$. What degree of "homotopical degeneracy" continuous maps $\alpha_0 : X \to Z$ must have under these assumptions? May Almgren's varifolds be of some help here?

Lower bound on depth(Σ) for maps of codimension -1 of CAT(κ)-manifolds.

Let X be a closed n-dimensional manifold of curvature $\leq \kappa < 0$ and $F : X \to Y$ be a smooth generic map where Y is an open manifold of dimension $m = n-1$. If X contains a simply connected ball of radius R, then the depth of the critical set of F satisfies

$$\mathrm{depth}(\Sigma(F)) \geq \mathrm{const}_m\, v_2(R, \kappa)$$

for some universal constant $\mathrm{const}_m > 0$*, where*

$$\mathrm{const}_m \geq \frac{2^{m-2}|\kappa|}{(m+1)!\pi}$$

for all purely folded (all generic?) maps F*.*

Proof. Use the Retraction Lemma from 2.2 and thus construct a map F' of X to an $(n-2)$-dimensional polyhedron Y' with point/edge singular surface fibers G_y. Then apply the above to this F' in place of F in the above hyperbolic $(N-1)$-contraction proposition.

By applying this to congruence covering of locally symmetric spaces (compare 6.3) we conclude that

there exists a closed manifold X_0 *(of constant negative curvature) of a given dimension* $n \geq 3$ *that admits a sequence of* s_i*-sheeted coverings* X_i *for* $s_i \to \infty$*, such that every generic smooth map* $F : X_i \to \mathbb{R}^{n-1}$ *has*

$$\mathrm{depth}(\Sigma(F)) \geq \mathrm{const}_n\, s_i^\gamma \qquad s^\gamma$$

for $\gamma \geq 1/n^2$*.*

Remarks. (a) We shall prove this in 6.3 with $\gamma = 1$ for $n = 3$, but the optimal $\gamma = \gamma(n)$ is unknown for $n \geq 4$. Also it is unclear what happens to maps into R^m with $m < n-1$.

(b) There is no true generalization of the above properties of families of (point/edge singular) surfaces, where $k = 2$, to the case $k > 2$; yet the definition of the homological simplicial spectra from 5.10 can be rendered homotopical in the same way as that for vol_k, where the corresponding inequality holds in $\mathrm{CAT}(\kappa < 0)$ spaces.

6 Isoperimetry and the Spectrum

The results of the previous section can be improved for cycles G of codimension 1 in connected manifolds Z, where there is an additional information on the spectrum of the Laplace operator on Z, and thus on the isoperimetric profile of Z.

6.1 Inverse Maz'ya–Cheeger inequality for families of hypersurfaces. Let Z be a closed N-dimensional Riemannian manifold with the Ricci curvature bounded from below by $\mathrm{Ricci}(Z) \geq -(N-1)$, e.g. sectional curvature $\kappa \geq -1$, and let $Z_+ \subset Z$ a domain with boundary $G = \partial Z_+$, where $v_+ =_{\mathrm{def}} \mathrm{vol}_N(Z_+) \leq \frac{1}{2}\mathrm{vol}_N(Z)$. Then, as is well known,

the first non-zero eigenvalue λ_1 *of the Laplace operator on* Z *satisfies*

$$\lambda_1^{-1} \geq \frac{v_+}{C \cdot \mathrm{vol}_{N-1}(G)} - 1$$

for some constant $C = C(N)$*.*

In fact, take a domain Z_+ with volume v_+ that minimizes $\mathrm{vol}_{N-1}(\partial Z_+)$, assume without loss of generality that the (constant) mean curvature vector on $G = \partial Z_+$ points outward. Let

$$\varphi(z) = \min\big(2, \mathrm{dist}(z, \partial(Z_+))\big) \quad \text{for } z \in Z_+$$

and

$$\varphi(z) = 0 \quad \text{for } z \text{ outside } Z_+ .$$

The volume of the (internal) 2-neighborhood of the boundary G of Z_+ in Z_+ is bounded by $C \cdot \mathrm{vol}_{N-1}(G)$ for $C = C(N)$ equal the volume of the cylinder made of the geodesics of length 2 normal to a domain G_1 with $\mathrm{vol}_{N-1}(G_1) = 1$ in a hyperplane in the hyperbolic N-space, since the mean curvature of G in Z_+ is negative (the possible singularities of G do not matter, compare App. C in [Gr9]); thus,

$$\int \big\| \mathrm{grad}(\varphi(z)) \big\|^2 dz \le C \cdot \mathrm{vol}_{N-1}(G)$$

and

$$\lambda_1^{-1} \ge \inf_{a \in \mathbb{R}} \int \big|\varphi(z) - a\big|^2 dz \Big/ \int \big\| \mathrm{grad}(\varphi(z)) \big\|^2 dz \ge \frac{v_+}{C \cdot \mathrm{vol}_{N-1}(G)} - 1 .$$

Remark. The first draft of the paper contained a more general inequality between (all of) the codimension-one volume spectrum (not only for the first eigenvalue) for codimension-one $\mathbb{Z}_2$-cycles and the Laplace spectrum; however, a referee pointed out that the proof was (to put it mildly) incorrect. It is already unclear what the relation(s) is between the second eigenvalues of these spectra.

Let us combine the above with the following.

Closel–Selberg Theorem [Cl]. *Let Z be a compact locally symmetric space and let the fundamental group Γ of Z be represented in the group $GL_M(\wp)$ of matrices of some (finite) order M with the entries in the ring $\mathcal{R}$ of S-integers in an algebraic number field, for S being a finite set of prime ideals (finite places). Let $\Gamma_\wp$ be the intersection of $\Gamma \subset G_S$ with the kernels of the natural homomorphism $GL_M(\wp) \to GL_M(\mathcal{R}/\wp)$ for the prime ideals $\wp$ away from S.*

Then the first eigenvalues of the Laplace operators on the corresponding congruence coverings $Z_\wp$ of Z satisfy

$$\lambda_1(Z_\wp) \ge \mathrm{const} = \mathrm{const}(Z) > 0 .$$

Main Corollary. *Let Z be a compact locally symmetric space that locally splits into irreducible symmetric spaces of dimensions ≥ 3. Then there exists an infinite sequence of s_i-sheeted coverings Z_i of Z, $s_i \to \infty$, such that*

$$\mathrm{vol}^\circ_\Delta [Z_i]_{-(N-1)} \ge \varepsilon(Z) s_i$$

and, if $N = 3$, then

$$|\chi|^\circ_{\mathrm{hyp}} [Z_i]_{-2} \ge \varepsilon(Z) s_i .$$

Since the simplicial and the $|\chi|_{\mathrm{hyp}}$ spectra are monotone decreasing under the maps of odd degree between manifolds, the above inequalities provide obstructions for existence of such maps. Here is the simplest example of a sequence of manifolds Z'_d, $d = 1, 2, \dots$, with an upper bound on these spectra which admit no maps of odd degree into the above Z_i with large i.

> *The cyclic d-sheeted coverings Z'_d of any given Z' with the fundamental groups $\pi_1(Z'_d) = h^{-1}(d\mathbb{Z}) \subset \pi_1(Z')$ for a homomorphism $\pi_1(Z') \to \mathbb{Z} \supset d\mathbb{Z}$ all have* $\mathrm{vol}^\circ_\Delta [Z'_d] \le \mathrm{const}(Z')$.

Remarks. (a) The vol°_{N-1}-spectrum of Z with Ricci $\geq -(N-1)$ is related to the *covering profile* $\mathrm{cov}_\rho(Z)$ that is the minimal number $m = m(\rho)$ of ρ-balls needed to cover Z,

$$\mathrm{vol}^\circ[Z]^m_{-(N-1)} \geq \mathrm{const}(N)\exp(-N\rho)\rho^{-1}\,\mathrm{vol}_N(Z)$$

(see App. C in [Gr9], where this is used for the lower bound on λ_m).

(b) The covering profile is especially informative for manifolds (and Alexandrov spaces) with non-negative sectional curvature. For example, if Z equals the rectangular solid, $Z = [0, l_1] \times [0, l_2] \times \dots . \times [0, l_N]$, then the edge lengths l_i and the ratios l_i/l_j can be reconstructed with a fair precision from $\mathrm{cov}_\rho(Z)$. In general, $\mathrm{cov}_\rho(Z)$ reflects the extend of collapse of Z and carries essentially the same information as the Uryson's widths of Z [Per]. Furthermore,

if the sectional curvature of Z is bounded from below, say by $\mathrm{curv}(Z) \geq -1$, *then*

$$\mathrm{vol}^\circ_k[Z]^m_{-k} \geq \mathrm{const}(N)\exp(-N\rho)\rho^{-(N-k)}\,\mathrm{vol}_N(Z)$$

for m equal the minimal number of ρ-balls needed to cover Z.

This is shown along the lines of the proof of the corresponding inequality with Ricci $\geq -(N-1)$ in App. C of [Gr9] by first estimating $\mathrm{vol}^\circ_k[B]_{-(N-k)}$ for balls $B \subset Z$ (or, rather, for pairs $B(\rho) \subset B(2\rho)$) by means of the Almgren–Morse theory and the (Buyalo–Hentze–Karcher comparison) bound on the volumes of ρ-neighborhoods of minimal k-dimensional subvarieties in Z, and then using the $\smile$-subadditivity (see 5.1). (This $\mathrm{const}(N)$, as well as the one above, is put for safety, probably, it is significantly greater than 1.)

(c) The first version of this paper contained the "proof" of the following:

The simplicial spectrum of $(N-1)$-dimensional $\mathbb{Z}_2$-cycles in an N-dimensional Riemannian manifold Z that is a local (e.g. global) Riemannian product of manifolds of dimension ≥ 3 with negative sectional curvatures pinched in the interval $[\kappa, \kappa^{-1}]$ with $\kappa < 0$, satisfies

$$\mathrm{vol}^\circ_\Delta[Z]^m_{-(N-1)} \geq \varepsilon(N,\kappa)\,\mathrm{vol}_N(Z)\lambda_m^{1/2}(Z) \qquad (\mathrm{vol}^\circ_\Delta)$$

(*where* $\varepsilon(N,\kappa) \approx \exp -N\kappa$) *and, if* $N = 3$,

$$|\chi|^\circ_{\mathrm{hyp}}[Z]^m_{-2} \geq \varepsilon(\kappa)\,\mathrm{vol}_N(Z)\lambda_m^{1/2}(Z)\,. \qquad (|\chi|^\circ_{\mathrm{hyp}})$$

A referee pointed out, however, that there was no proof, except for the above cases $m = 1$ and $m \gg \mathrm{cov}_1(Z)$. These inequalities remain conjectural. (In order to generalize the argument used for $m = 1$ one needs, at the very least, a developed Morse–Franks–Witten theory for the volume function. on the space of cycles.)

Further Questions. What is the rough asymptotics of $\mathrm{vol}^\circ_\Delta[Z]^m_{-(N-1)}$ and $|\chi|^\circ_{\mathrm{hyp}}[Z]^m_{-2}$ for $m \to \infty$? The upper bounds coming from obvious construction does not seem to match the above lower bounds.

Is there a refinement of Guth's (volume spectrum) inequality (see 5.1) taking into account the curvature of the underlying manifold?

Are there versions of Guth's inequality for $\mathrm{vol}^\circ_\Delta$ and $|\chi|^\circ_{\mathrm{hyp}}$ for the above Z?

What are relations between *the Lp, q spectra* (i.e. the spectra of $f \mapsto \|\mathrm{grad}(f)\|_{L_p}/\|f\|_{L_q}$ on the projective space of functions $f : Z \to \mathbb{R}$, see [Gr10]) for manifolds Z with $\mathrm{Ricci}(Z) \geq -(N-1)$? (A suitable inequality between $L1,1$- and $L2.2$- spectra would prove (c).)

6.2 Mappings to trees. Let Y be a finite tree and μ a probability measure on Y. It is obvious, that

if all leaves (ends) $y \in Y$ have $\mu(y) \leq \frac{1}{2}\mu(Y)$ (e.g. μ has no atoms $> \frac{1}{2}\mu(Y)$), then the exists a point $y_{\mathrm{mean}} \in Y$ such that all connected components of the complements $Y \setminus y_{\mathrm{mean}}$ have measures $\leq \frac{1}{2}\mu(Y)$.

Let Z be a connected Riemannian manifold (possibly with a boundary) of finite volume, where all domains $Z_+ \subset Z$ with $\mathrm{vol}_N(Z_+) \leq \frac{1}{2}\mathrm{vol}_N(Z)$ satisfy the following *isoperimetric inequality*:

$$\mathrm{vol}_N(Z_+) \leq f\big(\mathrm{vol}_{N-1}(\partial Z_+)\big)$$

for a monotone increasing *sublinear* function $f(v)$, i.e. $f(v_1 + v_+2) \leq f(v_1) + f(v_2)$. Then

an arbitrary continuous map $F : Z \to Y$ has a fiber $G_y = F^{-1}(y) \subset Z$, such that

$$f(\mathrm{vol}_{N-1}(G_y)) \geq \tfrac{1}{2}\mathrm{vol}_N(Z)\,.$$

Indeed, take G_y for $y = y_{\mathrm{mean}}$ with respect to the normalized pushforward measure $\mu = F_*(\nu_Z)$ of the Riemannian measure ν_Z on Z.

More generally, let X be a connected N-dimensional manifold, $F : X \to Y$ a continuous map and $\alpha : X \to Z$ a continuous proper (boundary $\to$ boundary, infinity $\to$ infinity) map of non-zero degree. (If Z is non-orientable, the degree is understood mod 2). Then

there exists an F-fiber $G_y = F^{-1}(y) \subset X$ such that all connected components of the complement $Z \setminus \alpha(G_y)$ have volumes $\leq \frac{1}{2}\mathrm{vol}_N(Z)$; hence,

$$f\big(\mathrm{vol}_{N-1}(\alpha(G_y))\big) \geq \tfrac{1}{2}\mathrm{vol}_N(Z)\,.$$

Proof. Given $y \in Y$, let $B \subset Y$ be a branch of Y, i.e. a connected component $B = B_i(y)$ of the complement of some point $y \in Y$, denote by $d(B,z)$ for $z \in Z \setminus \alpha(G_y)$, the local topological degree of $\alpha|F^{-1}(B)$ over z, and observe that $d(B,z)$ is constant on the connected component $C \subset Z$ of z in $Z \setminus \alpha(G_y)$ and the function $d(B,z)$ is additive on the branches at $y \in Y$, with $\sum_i d(B_i(y),z) = \deg(\alpha) \neq 0$ for every $y \in Y$ and $z \in Z \setminus \alpha(G_y)$ and d is locally constant in $B = B(y)$ as y moves along an edge of the graph in-so-far as d is defined, i.e. $z \notin \alpha(F^{-1}(y))$. Furthermore, if $y' \in B = B_{i_0}(y)$ approaches y, then the branch $B' = B'(y') \supset B$ satisfies

$$\lim_{y' \to y} d\big(B'(y',z)\big) = \sum_{i \neq i_0} d\big(B(y,z)\big)\,.$$

Thus, if $d(B'(y',z)) \neq 0$ and y is not a leaf (end point) of the graph, then there exists a branch $B_{i_1}(y) \neq B_{i_0}(y)$, such that $d(B_{i_1}(y,z)) \neq 0$. Consequently,

there is no "continuous" assignment $y \mapsto C(y) \subset Z$ of a connected component of $Z \setminus \alpha(G_y)$, for all $y \in Y$,

where "continuous" means that every $y \in Y$ has a neighborhood $U(y) \in Y$, such that the intersection $C(y') \cap C(y)$ is non-empty for $y' \in U(y)$.

Indeed, given $y \mapsto C(y)$, consider the "z-essential" branches $B = B(y)$ issuing from the points $y \in Y$, where " z-essential" means $d(B_{i_1}(y,z)) \neq 0$ for $z \in C(y)$. Move in Y, starting from a leaf $y_0 \in Y$ in Y, following the essential branches. The above properties of d show that thus we eventually arrive at an empty branch at another leaf $y_1 \in Y$ but no empty branch is "essential".

It follows, there exists a y, where there is no component $C(y)$ of $Z \setminus \alpha(G_y)$ with $\nu_Z(C(y)) > \nu_Z(Z)/2$. QED

Families of hypersurfaces parametrized by graphs.

Let X be an N-dimensional pseudomanifold, possibly with a boundary, let $F : X \to Y$ be a simplicial map, where Y is a finite graph and let $\alpha : X \to Z$ be a proper continuous map of non-zero degree, where Z is an N-dimensional Riemannian manifold with $\mathrm{Ricci}(Z) \geq -(N-1)$. *Then some fiber $G_y = F_{-1}(y) \subset X$ satisfies*

$$\mathrm{vol}_{N-1}\left(\alpha(G_y)\right) \geq \mathrm{const}_N \left(1 + b_1(Y)\right)^{-1} \lambda_1(Z)\,\mathrm{vol}_N(Z)\,,$$

where b_1 denotes the number of independent cycles in Y.
Consequently, if X is a 3-manifold and the fibers G_y make (at most) point/edge singular surface family, then some G_y has

$$|\chi|_{\mathrm{hyp}}(G_y) \geq \mathrm{const} \cdot \big(1 + b_1(Y)\big)^{-1} \lambda_1(Z)\,\mathrm{vol}_3(Z)\,. \qquad (\chi)$$

Proof. If Y is a tree, i.e. $b_1(Y) = 0$, the proof directly follows from above, and in general, we compose F with an obvious (b_1+1)-to-1 map of Y to a tree.

6.3 Maps of 3-manifolds into surfaces. If Z is a complete hyperbolic 3-manifold with a metric of curvature -1 and finite volume, let

$$\mu_1(Z) = \lambda_1(Z)(\mathrm{vol}(Z))^2,$$

where λ_1 is the first non-zero eigenvalue of Z.

Then define $\mu_1(Z)$ for the 3-manifolds Z that admit Thurston's decomposition as the maximum of μ_1 of the hyperbolic components (of finite volume) of Z.

(3→2)-Mapping inequalities.

Let X be a closed connected 3-manifold,m $\alpha : X \to Z$ a continuous map of non-zero degree, where Z is a closed connected 3-manifold, where all component Z_i of Thurston's decomposition are hyperbolic and let $F : X \to Y$ be a smooth generic map, where Y is a connected surface of genus b. Then the number N_2 of the double (self-crossing) points of the fold of the critical set $\Sigma = \Sigma(F) \subset Y$ satisfies,

$$N_2 \geq \varepsilon \max_i \mu_1(Z_i)/(b+1) \qquad (N_2)$$

for some universal positive $\varepsilon > 0.001$, where $\mu_1(Z_i)$ are taken for the complete hyperbolic metrics of finite volumes in Z_i.
Furthermore, if Y is open, then the depth of the critical set of F satisfy,

$$\mathrm{dep}(\Sigma) \geq \varepsilon \mu_1^{1/2}(Z)/b\,. \qquad (\mathrm{dep})$$

Consequently, there exits an infinite sequence of finite s_i-sheeted coverings of a closed (hyperbolic) 3-manifold, $X_i \to X$, such that every generic *smooth map* $X_i \to Y$, *has*

$$N_2 \geq \varepsilon(X) s_i^2/(b+1) \quad \text{and} \quad \text{dep}(\Sigma) \geq \varepsilon(X) s_i/b\,.$$

To prove these, we need the following (well-known)

GRAPHS OVER TREES LEMMA. *Every graph of valency (degree)* $\leq \delta$ *with* N *vertices, that is embedded into a surface of genus* b, *admits a simplicial map* f *into a binary tree* T *such that the cardinality of each* f*-fiber satisfies*

$$c = \text{card}\left(f^{-1}(t)\right) \leq \text{const}\cdot\delta^2 b^{1/2}\sqrt{N} \quad \textit{for all } t \in T\,.$$

We shall present the standard construction of f (that sends the vertices of our graph to the leaves of the tree and where the depth of the tree is of order $\log(N)$) in 6.6.

Proof of (N_2). Apply the lemma to the graph $\Sigma = \Sigma(F) \subset Y$, and slightly modify f in order to make it *generic*, where the genericity of maps of smooth manifolds to trees T (and similarly to simplicial polyhedra of all dimensions) is defined as follows. Imbed T into $\mathbb{R}^2$ linearly on the edges and let $R : U = U(T) \to T$ by the standard piecewise linear retraction of some neighborhood $U \subset \mathbb{R}^2$ of T. Then *generic maps to* T are, by definition, are the composition of generic C^∞-maps to U with R.

Every continuous map $Y \to T$ can be approximated by a generic one and, in the present situation, we can (obviously) find such an approximation that only slightly enlarges c, say by $c \mapsto c' \leq 4c$. We ignore this and assume f_1 is generic to start with.

The fibers of a generic f_1 are graphs (curves) $L_t \subset Y$, such that smooth curves, for all but finitely many $t \in T$, including the singular points of T, i.e. the vertices of valency 3.

If t is a singular point of T, then L_t may contain vertices of valency 3 and no other singularities. The map f_1 near L_t is locally topologically equivalent to R.

The map f_1 over each non-singular point (regarded as a real function) is Morse and thus L_t may have only finitely many Morse singularities.

The curves L_t meet Σ transversally for almost all t including all singular t, where, clearly, $\text{card}(L_t \cap \Sigma) = \text{card}\, F^{-1}(t))$ (where L_t may have connected circular components that do not meet Σ).

It follows that the fibers $G_t = F_1^{-1}(t)$, that are equal $F^{-1}(L_t)$, make a point/edge singular surface family with

$$|\chi|_{\text{hyp}}(G_t) \leq c$$

(compare the application of around the Retraction Lemma in 2.2).

Now the proof of (N_2) follows from 5.3. 5.9, 5.11 and 6.1 (where the F-pullbacks of the circular components of L_t are tori that do not contribute to $|\chi|_{\text{hyp}}$) and the proof of (dep) for hyperbolic Z is similar with the above lemma replaced by the Retraction Lemma.

REMARK. One may allow some non-hyperbolic Z_i in the Thurston decomposition, e.g. graph manifolds with $\text{CAT}(\kappa \leq 0)$ metrics and totally geodesic toric boundaries, and, probably, this is true in general.

6.4 Surface families in non-compact manifolds and the crossing numbers of links. Consider a smooth link $L \subset \mathbb{R}^3 \subset S^3$ and project it to S^2 via the Hopf map $S^3 \to S^2$. Denote by $N_{cr}(L)$ its *crossing number* i.e. the minimal number of crossing such map $L' \to S^2$ may have for all links $L' \subset R^3$ diffeotopic to L.

If all components Z_i of the Thurston decomposition of $Z = S^3 \setminus L$ are hyperbolic then

$$N_{cr} \geq \varepsilon \max_i \mu_1(Z_i)\,. \qquad (N_{cr})$$

Proof. Let $\underline{L} \subset S^2$ be the Hopf projection of L and consider a map of $\underline{L}$ to a binary tree T with small fibers provided by the above lemma. Extend this map to S^2 and then compose it with the Hopf map on Z. Thus we arrive at the earlier situation except that Z is now an open manifold with the standard cuspidal/toral geometry at infinity. We conclude the proof by observing that the above inequality (N_2) applied to the double $2Z$ of Z (obtained by gluing two copies of Z across the boundary) yields (N_{cr}).

Remark. This picture with links is similar to that with “slices” in 2.3 we shall revisit both of them in a more general context in Part 2 of the paper.

6.5 Problems with higher codimensional filling inequalities. Given a k-dimensional $\mathbb{Z}_p$-cycle G in a Riemannian manifold Z, let $f\,\mathrm{vol}_{k+1}(G)$ denote the infimum of the $(k+1)$-volumes of the $(k+1)$-chains with boundary G and define the *filling (or isoperimetric)* profile

$$\mathrm{fil}_{k+1}(v) = \mathrm{fil}_{k+1}(v, Z; \mathbb{Z}_p)\,, \quad v \in \mathbb{R}_+\,,\ p = 2, 3, 5, \dots, \infty\,,$$

as the supremum of $\mathrm{fil}\,\mathrm{vol}_{k+1}(G)$ over all cycles with $\mathrm{vol}_k(G) \leq v$.

Observe that $\mathrm{fil}(v) = \infty$ for $v \geq \mathrm{sys}_k(Z)$, where $\mathrm{sys}_k(Z; \mathbb{Z}_p)$, the *$k$-systole of Z with $\mathbb{Z}_p$ coefficients*, is the infimum of the volumes of the k-cycles G in Z that are *not* homologous to zero. In particular $\mathrm{fil}_N(v) = 0$ for $v < \mathrm{vol}_N(Z)$ and $\mathrm{fil}_N(v) = \infty$ for $v \geq \mathrm{vol}_N(Z)$ for closed connected manifolds Z (assumed orientable if $p \neq 2$).

Standard Examples. (a) If $Z = \mathbb{R}^N$ then $\mathrm{fil}_k(v)$ equals the volume of the Euclidean $(k+1)$-ball B with $\mathrm{vol}_k(\partial B) = v$ by a theorem of Almgren [A] and Almgren's proof yields a similar result for round spheres S^N as was pointed out to me by Bruce Kleiner.

(b) If Z is a simply connected CAT(0)-space one expects that $\mathrm{fil}_{k+1}(v, Z) \leq \mathrm{fil}_{k+1}(v, \mathbb{R}^{k+1})$. Apart from the simple case of $k = 1$ and and the classical one of $N = k+1$ and Z of constant curvature, such *sharp* inequality for variable curvature is known only for $k+1 = N = 3, 4$ (see [Kl], [Cr]).

On the other hand, the rough inequality,

$$\mathrm{fil}_{k+1}(v, Z) \leq \mathrm{const}(k) v^{\frac{k+1}{k}}\,,$$

follows from the filling inequality from [Gr2]. Furthermore, one knows (see section 6 in [Gr14]) that

the filling profile is (sub)linear for $k \geq \mathrm{rank}(Z)$,

$$\mathrm{fil}_{k+1}(v, Z) \leq \mathrm{const}(Z) \cdot v\,,$$

where the inequality $k \geq \mathrm{rank}(Z)$ *means that neither* Z *nor any Hausdorff (ultra)limit of pointed spaces* $(Z, z \in Z)$ *for* $z \to \infty$ *contain* $(k+1)$-*flats, i.e. isometric copies of* $\mathbb{R}^{k+1}$ *(e.g.* Z *itself contains no such flat and admits a cocompact isometry group).*

(c) The filling profile of the $(2N+1)$-dimensional Heisenberg (Lie) group $H^{2N=1}$ (with a left-invariant Riemannian metric) satisfies $\mathrm{fil}_k(v) \sim v^{\frac{k+1}{k}}$ for $k < N$ and $v \gg 1$, while $\mathrm{fil}_{k+1}(v) \sim v^{\frac{k+2}{k+1}}$ for $k > N$ and $\mathrm{fil}_{N+1}(v) \sim v^{\frac{N+2}{N}}$. (See [You], where this is proved, beside H^{2N+1}, for a rather large class of nilpotent Lie groups.)

Systoles and fillings profiles of compact $\mathbf{CAT}(\boldsymbol{\kappa} \leq \mathbf{0})$***-spaces.*** Let $\tilde{Z} \to Z$ be a covering map which is injective on the R-balls in $\tilde{Z}$ for some $R \geq 1$.

If $\mathrm{fil}_k(v, \tilde{Z}) \leq \tilde{c} \min\left(v, v^{\frac{k}{k-1}}\right)$, *then*

$$\mathrm{sys}_k(Z) \geq \mathrm{const}(k) \exp(\tilde{c}R)\,, \tag{sys_k}$$

and if $\mathrm{fil}_{k+1}(v, \tilde{Z})$ *is majorized by a sub-linear* $(f(v_1+v_2) \leq f(v_1)+f(v_2))$ *monotone increasing function* $f(v)$ *then*

$$\mathrm{fil}_{k+1}(v, Z) \leq \mathrm{const}(\tilde{c}, k) f(3v) \quad \text{for } v \leq \mathrm{const}''(k) \exp \tfrac{1}{2}\tilde{c}R\,. \tag{fil_{k+1}}$$

The proof is standard. Take a vol_k-minimizing k-cycle $G \subset Z$ non-homologous to zero (i.e. $\mathrm{vol}_k(G) = \mathrm{sys}_k(Z)$), consider the r-balls balls $B(r) = B(g, r) \subset Z$, $g \in G$, and let $v_k(r) = \mathrm{vol}_k(G \cap B(r))$ and $v_{k-1}(r) = \mathrm{vol}_k(G \cap \partial B(r))$. Then

$$\mathrm{fil}_k\left(v_{k-1}(r), \tilde{Z}\right) \geq \mathrm{fil}_k\left(v_{k-1}(r), Z\right) \geq \min\left(v_k(r), \mathrm{vol}_k(G) - v_k(r)\right)$$

for $r \leq R$, while the r-derivative of $v_k(r)$ satisfies

$$v_k'(r) \geq v_{k-1}(r)\,.$$

This implies the required lower bound on $\mathrm{sys}_k(Z)$, provided the minimal G exists.

In fact, "minimal" can be replaced by " q- quasi-minimal":

$$\mathrm{fil}_k\left(v_{k-1}(r), Z\right) \geq \tfrac{1}{q} v_k(r)$$

for *all* balls $B(g, r)$; this, say with $q = 2$, implies the same lower bound on $\mathrm{sys}_k(Z)$ but with a slightly smaller (yet strictly positive!) $\mathrm{const}(k)$. Therefore, the condition $v = \mathrm{vol}_k(G) \leq \mathrm{const}''(k) \exp \frac{1}{2}\tilde{c}R$ implies that G is *not* 2-quasi-minimal.

Now, to prove (fil_{k+1}), take a (minimal) filling $G_{\min}(r) \subset Z$ of the intersections $G \cap \partial B(g, r)$ for $g \in G$ and $r \leq R$ maximizing $v_k(r)$ and such that $\mathrm{vol}_k(G_{\min}(r)) \leq \frac{1}{2} v_k(r) = \frac{1}{2}\mathrm{vol}_k(G \cap B(r))$. Let $G'(r) = G - (G \cap B(r)) + G_{\min}(r)$ and observe that

$$\mathrm{vol}_k(G'(r)) \leq \mathrm{vol}_k(G) - \tfrac{1}{2} v_k(r)$$

and

$$\mathrm{fil\,vol}_{k+1}(G) \leq \mathrm{fil\,vol}_{k+1}(G'(r)) + \mathrm{fil}_{k+1}\left(\tfrac{3}{2} v_k(r), \tilde{Z}\right).$$

Thus, the required bound on $\mathrm{fil\,vol}_{k+1}$ for G is reduced to that for G' of smaller volume and the proof trivially follows by iterating: $G \rightsquigarrow G' \rightsquigarrow G'' \rightsquigarrow \ldots \to 0$.

Lower bound on $\mathbf{vol}^\circ[\boldsymbol{Z}]_{-\boldsymbol{k}}$ ***by the filling profiles.*** Define the functions $f_i(v)$ by induction on i,

$$f_1(v) = \mathrm{fil}_{k+1}(v) \quad \text{and} \quad f_i(v) = \mathrm{fil}_{k+i}\left(i f_{i-1}(v)\right) \quad \text{for } i = 2, 3, \ldots, m = N - k\,.$$

Then the Federer–Fleming induction by skeleton argument in 5.11 shows that

$$f_m\left(\mathrm{vol}^\circ[Z]_{-k}\right) = \infty\,.$$

This provides, for example, a lower bound on $\mathrm{vol}^\circ[Z]_{-k}$ for (possibly singular) CAT(κ) spaces and for δ-hyperbolic spaces that is equivalent to such bound proved in [Gr5] for *Riemannian* CAT(κ)-manifolds.

Simplicial filling. If Z is a simplicial complex, then one defines the combinatorial $\Delta\,\mathrm{fil}\,\mathrm{vol}$ by considering simplicial $\mathbb{Z}_p$-cycles $G = \sum_I A_i\Delta_i^k$ and chains $C = \sum_j b_j\Delta_j^{k+1}$ with $\partial C = G$, with the volume(s) substituted by the "l_1-norms" of these

$$\Delta\,\mathrm{fil}_{k+1}(v) = \sup_G \inf_C \sum |b_i|\,,$$

where the *sup* is taken over all G with

$$\mathrm{vol}_k(G) =_{\mathrm{def}} \sum_i |a_i| = v$$

where $|a|$, $a \in \mathbb{Z}/p\mathbb{Z}$, is the minimum of the absolute values of the integers representing a.

If Z is a manifold with 1-bounded geometry ($|\,\mathrm{curv}\,|(Z) \le 1$, $\mathrm{Inj\,Rad}(Z) \ge 1$) then it admits a triangulation into roughly unit simplices, i.e. l-bi-Lipschitz equivalent to the unit Euclidean simplices with roughness (i.e. the bi-Lipschitz constant l) depending on $n\dim(Z)$. Then the Federer–Fleming "pushing to the skeleta" argument implies that the two filling volumes are *equivalent*:

$$c_1(v)\Delta\,\mathrm{fil}\big(c_2(v)v\big) = \mathrm{fil}(v)$$

for two functions satisfying $0 < C_1^{-1}(N) \le c_1, c_2, \le C(N) < \infty$.

Filling in 0-homologous cycles. Extend $\mathrm{fil}_{k+1}(v)$ beyond $v = \mathrm{sys}_k$ by limiting the definition of the cycles G that are *required* to be homologous to zero. Since the resulting filling function, call it $\mathrm{fil}^0_{k+1}(v)$, is bounded for $k = N-1$ in terms of the first eigenvalue λ_1 of the Laplace operator on Z, one has

$$\mathrm{fil}^0_N(v, Z_\wp) \ge \mathrm{const}(Z)v$$

for the congruence coverings $Z_\wp$ of a compact locally symmetric Z by the Closel–Selberg theorem.

Questions. What is the asymptotic behavior of $\mathrm{vol}^\circ_k[Z_\wp]_{-k}$, $\mathrm{fil}_{k+1}(v, Z_\wp)$, $\mathrm{fil}^0_{k+1}(v, Z_\wp)$, and $\mathrm{sys}_k(Z_\wp)$, $|\wp| \to \infty$, of the congruence coverings $Z_\wp$ of compact locally symmetric (e.g. arithmetic) spaces Z for $2 \le k \le N-2$? Are there instances of the Selberg type inequalities $\mathrm{fil}^0_{k+1}(v, Z_\wp) \ge \mathrm{const}(Z)v$ for $k \le \dim(Z)-2$ or, pointing to the opposite direction, of *isosystolic inequalities* $\mathrm{sys}_k(Z_\wp) \le \mathrm{const}(Z)\,\mathrm{vol}_N(Z_\wp)^{k/N}$?

What are the asymptotics of the volumes and/or of the injectivity radii of the "Jacobian tori" $H^k(Z_\wp;\mathbb{R})/H^k(Z_\wp;\mathbb{Z})$, for the L_2-metrics (norms) on the vector spaces $H^k(Z_\wp;\mathbb{R})$ realized by harmonic forms on $Z_\wp$?

(The *stable isosystolic inequalities* see [Kat] appeal to the L_∞-metric on closed forms but the L_2-metric may display some asymptotic regularity of the "Jacobian volume",

$$v_\wp = \mathrm{vol}_{r_\wp}\big(H^k(Z_\wp;\mathbb{R})/H^k(Z_\wp;\mathbb{Z})\big)$$

for $r_\wp = \mathrm{rank}(H^k(Z_\wp))$. This is suggested by the behavior the Laplace–Hodge ζ-function associated with the *analytic torsion*; also one is tempted to take some

Euler product over $\wp$, where the most promising case is that of an arithmetic Z with non-zero Euler characteristic and $k\dim(Z)/2$, where $\mathrm{rank}(H^k(Z_\wp)) \sim \mathrm{vol}_N(Z_\wp)$.)

6.6 Appendix: Separation and maps of graphs to trees. The main ingredient of the proof of the Graphs over Trees Lemma from 6.2 is the following version of a more general result from [GHT] (also, see [AlST] for a far reaching combinatorial generalization). Our notation below somewhat deviates from the rest of the paper.

Rough Separator Theorem. *Every graph $G = (V, E)$ of valency $\leq d$ and genus $\leq g$ contains a subset $W \subset V$ of cardinality $r \in [r_0, 10dr_0]$, for any given positive integer r_0, such that* $\mathrm{card}(\partial^1(W)) \leq \mathrm{const}\cdot dg^{1/2}\sqrt{r}$, *where* "const" *is a universal positive constant.*

Proof by the "length-area" argument of Loewner–Hersch. Remove loops in G and observe this does not changes isoperimetry and disregard multiple edges in G (this will be justified in the course of the proof below). We also can assume that G is embedded into a closed connected orientable surface S of genus $\leq g$, such that each connected component of the complement $S \setminus G$ is a topological 2-cell. There is an obvious subdivision of each of these cells into triangles without introducing new vertices and at most tripling the valency of the vertices. Thus we arrive at the situation, where G equals the 1-skeleton of a triangulation of a surface S of genus $\leq g$ and we endow S with the piecewise Euclidean metric ρ, where each 2-simplex is isometric to the regular plane triangle of unit area.

Take a (ramified) conformal map φ of degree $h \leq 2g+1$ of S to the unit sphere S^2 that is guaranteed by the Riemann theorem. Denote by μ the φ-pushforward of the (area) measure on S corresponding to ρ and let $B_0 \subset S^2$ be the disk (ball) of minimal radius with $\mu(B_0) = r_0$. Observe that the concentric disc B_1 of radius $R_1 = 1.1R_0$ can be covered by 4 disks of radius R_0 and thus $\mu(B_1) \leq 4r_0$. Also observe that the annulus $A = (B_1 \setminus B_0) \subset S^2$ is conformally equivalent to the cylinder $S^1 \times [0, \delta]$, where S^1 is the circle of unit length and where $\delta \geq 0.05$. Identify A with this cylinder and denote by $|D\varphi|$ the conformal factor of φ, i.e. the norm of the differential of φ with respect to the metrics ρ on $\hat{A}$ and the cylindrical metric on A.

Consider the circles $C_t = S^1 \times t \subset A = S^1 \times [0, \delta]$ for $t \in [0, \delta]$ and denote by $\hat{C}_t \subset S$ their φ-pullbacks; use the parameters $\hat{s}$ in these lifted circles coming via φ^{-1} from the s-parameter in the circles $C_t = S^1$ and let $\hat{t}$ corresponds to $t \subset [0, \delta]$.

Observe that

$$\int_{\hat{C}_t} |D\varphi|^{-1} d\hat{s} = \mathrm{length}_\rho(\hat{C}_t) \quad \text{for all } t \subset [0, \delta]$$

and

$$\int_{\hat{A}} |D\varphi|^{-2} d\hat{s}d\hat{t} = \mathrm{area}_\rho(\hat{A}) = \mu(A) \leq 3r_0\,.$$

Then, by Schwartz inequality,

$$\int_{[0,\delta]} \mathrm{length}_\rho(\hat{C}_t)d\hat{t} = \int_{\hat{A}} |D\varphi|^{-1} d\hat{s}d\hat{t} \leq \sqrt{\delta h}\sqrt{\int_{\hat{A}} |D\varphi|^{-2} d\hat{s}d\hat{t}} \leq \sqrt{3r_0\delta h}\,.$$

Thus we established the following version of the local-to-global hyperbolicity criterion.

Non-hyperbolicity Cut Theorem (compare [P]). *Let S be an orientable surface of genus $\leq g$ (possibly non-compact) with a piecewise smooth Riemannian metric of finite area. Then there exists a smooth co-oriented curve $\hat{C} \subset S$ (among $\hat{C}_t$), closed in S as a subset, consisting of at most $h = 2g+1$ components of total length $\leq 6\delta^{-1/2}h^{1/2}\sqrt{r_0}$ that cuts S into two (possibly disconnected) parts, one of which has area between r_0 and $4r_0$, where r_0 is an arbitrarily chosen positive number and where $\delta \geq 0.05$.*

The existence of such cuts for (S, ρ) implies, by the Federer–Fleming "pushing argument" (see 5.11), the existence of the required combinatorial cuts and the proof of the separator theorem is concluded.

REMARKS. Hersch [H] used the Riemann mapping theorem in his 1970 proof of the following (sharp!) upper bound on the first eigenvalue of spheres:

> *among all orientable surfaces of genus zero with given (finite) area the round sphere has the largest first eigenvalue of the Laplacian.*

This was followed by similar bounds on the spectrum of other Riemann surfaces [YY], [Ko], [Gr1], but the sharp inequalities remain unknown.

QUESTION. Is there a link between the conformal geometry (used above and in [YY], [Gr1]) and the theory of graph minors exploited in [AlST]? (Notice that the arguments in [Ko] and [P] run along combinatorial rather than conformal lines. Probably, the filling/variational technique from [Gr2] and [NR], as well as those from [CoM], can be also applied here.)

The proof of the Graphs over Trees Lemma now follows from the

Isoperimetric mapping to trees criterion.

> *Let the vertex set V' of every subgraph X' of X can be partitioned into two subsets V_1' and V_2' of cardinalities N_1' and $N_2' = N' - N_1'$ for $N' =$ $\mathrm{card}(V')$, such that $N'^1 \geq N_2' \geq C \cdot N'$ for some constant C and such that the number of edges between V'^1 and V'^2 is bounded by $J(N')$ for some real function J vanishing for $N' < 1$. Then there exists a map f of X onto subtree Y in a binary tree Y_d of depth $d \leq C \cdot \log_2(N)$, such that the vertices of X go to the leaves of Y and the f-pullbacks of all points $y \subset Y$ have cardinalities $\leq C \cdot kN \cdot \sum_{i=1,2,\dots} J\big(1 - \frac{1}{C}\big)^i$, where N denotes the number of vertices in X.*

Proof. Divide the vertex set V of X into V_1 and V_2, then divide V_1 and V_2 and keep dividing until you arrive at one point sets. The resulting family of nested vertex sets naturally define Y as well as the required f.

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