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LOG CANONICAL THRESHOLDS OF DEL PEZZO SURFACES

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Dedicated to Yuri Manin on his seventieth birthday

Abstract. We study global log canonical thresholds of del Pezzo surfaces.

All varieties are assumed to be defined over $\mathbb{C}$.

1 Introduction.

The multiplicity of a nonzero polynomial $\phi \in \mathbb{C}[z_1, \cdots, z_n]$ at the origin $O \in \mathbb{C}^n$ is the nonnegative integer m such that $\phi \in \mathfrak{m}^m \setminus \mathfrak{m}^{m+1}$, where $\mathfrak{m}$ is the maximal ideal of polynomials vanishing at the point O in $\mathbb{C}[z_1, \cdots, z_n]$. It can be defined by derivatives, because the equality

$$m = \min \left\{ m \in \mathbb{N} \cup \{0\} \;\Big|\; \frac{\partial^m \phi(z_1, \ldots, z_n)}{\partial^{m_1} z_1 \partial^{m_2} z_2 \cdots \partial^{m_n} z_n}(O) \neq 0 \right\}.$$

holds. We have a similar invariant that is defined by integrations. This invariant is given by

$$c_0(\phi) = \sup \left\{ c \in \mathbb{Q} \;\Big|\; \text{the function } \tfrac{1}{|\phi|^c} \text{ is locally } L^2 \text{ near the point } O \in \mathbb{C}^n \right\},$$

and $c_0(\phi)$ is called the log canonical threshold of ϕ at the point O. The number $c_0(\phi)$ appears in many places. (The number $c_0(\phi)$ is also called the complex singularity exponent of ϕ (see [K]).) For instance, it is known that $c_0(\phi)$ is the same as the absolute value of the largest root of the Bernstein–Sato polynomial of ϕ (see [K]).

Even though the log canonical threshold was known implicitly, it was formally introduced in the paper [S] as follows. Let X be a variety with

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log terminal singularities, let $Z \subseteq X$ be a closed subvariety, and let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then the number

$$\mathrm{lct}_Z(X, D) = \sup\left\{\lambda \in \mathbb{Q} \mid \text{the log pair } (X, \lambda D) \text{ is log canonical along } Z\right\}$$

is said to be the log canonical threshold of D along Z. The number $\mathrm{lct}_Z(X, D)$ is known to be positive and rational. Moreover, if $X = \mathbb{C}^n$ and $D = (\phi = 0)$, then the equality

$$\mathrm{lct}_O(X, D) = c_0(\phi)$$

holds (see [K]). For the case $Z = X$ we use the notation $\mathrm{lct}(X, D)$ instead of $\mathrm{lct}_X(X, D)$. Then

$$\begin{aligned}\mathrm{lct}(X, D) &= \inf\left\{\, \mathrm{lct}_P(X, D) \mid P \in X\right\}\\ &= \sup\left\{\lambda \in \mathbb{Q} \mid \text{the log pair } (X, \lambda D) \text{ is log canonical}\right\}.\end{aligned}$$

Even though several methods have been invented in order to compute log canonical thresholds, it is not easy to compute them in general. However, the log canonical thresholds play a significant role in the study of birational geometry showing many interesting properties (see [K], [P]).

Thus far the log canonical threshold has a local character. In this paper we wish to develop its global analogue for Fano varieties. We shall see it is useful to consider the smallest of the log canonical thresholds of effective $\mathbb{Q}$-divisors numerically equivalent to an anticanonical divisor.

Let X be a Fano variety with log terminal singularities, and G be a finite subgroup in $\mathrm{Aut}(X)$.

DEFINITION 1.1. We define the global G-invariant log canonical threshold of X by the number

$$\mathrm{lct}(X, G) = \inf\Big\{\mathrm{lct}(X, D) \;\Big|\; \text{the effective } \mathbb{Q}\text{-divisor } D \text{ is } G\text{-invariant and } D \equiv -K_X\Big\}.$$

We put $\mathrm{lct}(X) = \mathrm{lct}(X, G)$ if the group G is trivial. Note that it follows from Definition 1.1 that

$$\mathrm{lct}(X, G) = \sup\left\{\lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l}\text{the log pair } (X, \lambda D) \text{ has log canonical singularities}\\ \text{for every } G\text{-invariant effective } \mathbb{Q}\text{-divisor } D \equiv -K_X\end{array}\right\} \geqslant 0.$$

EXAMPLE 1.2. It follows from Proposition 16.9 in [K et al.] that $\mathrm{lct}(\mathbb{P}(1, 1, n)) = 1/(2 + n)$ for $n \in \mathbb{N}$.

For a given Fano variety, it is usually very hard to compute its global log canonical threshold explicitly (see [C2]). For instance, the papers [H1] and

[H2] show that the global log canonical threshold of a rational homogeneous space of Picard rank 1 and Fano index r is $1/r$.

EXAMPLE 1.3. Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree $n \geq 3$. Then

$$\mathrm{lct}\big(X\big) \geqslant 1 - 1/n$$

due to [C1]. It is clear that the inequality $\mathrm{lct}(X) = 1 - 1/n$ holds if the hypersurface X contains a cone of dimension $n-2$. But the paper [Pu] shows that $\mathrm{lct}(X) = 1$ if X is general and $n \geqslant 6$.

Global log canonical thresholds of Fano varieties play an important role in geometry. (It follows from [CS, Append. A] that global log canonical thresholds of Fano varieties are algebraic counterparts of α-invariants introduced in [T1].)

EXAMPLE 1.4. Let X be a general well-formed quasismooth hypersurface in $\mathbb{P}(1, a_1, \ldots, a_4)$ of degree $\sum_{i=1}^{4} a_i$ with terminal singularities such that $-K_X^3 \leqslant 1$. Then $\mathrm{lct}(X) = 1$ by [C2], which implies that

$$\mathrm{Bir}(\underbrace{X \times \cdots \times X}_{m \text{ times}}) = \left\langle \prod_{i=1}^{m} \mathrm{Bir}(X), \mathrm{Aut}(\underbrace{X \times \cdots \times X}_{m \text{ times}}) \right\rangle,$$

the variety $X \times \cdots \times X$ is not rational and not birational to a conic bundle (see [C2]).

One of the most interesting applications of global log canonical thresholds of Fano varieties is the following result proved in [DK] (see also [N], [T1] and [CS]).

Theorem 1.5. *Let X be a Fano variety with quotient singularities, and let G be a finite subgroup on $\mathrm{Aut}(X)$ such that the inequality*

$$\mathrm{lct}(X, G) > \frac{\dim(X)}{\dim(X) + 1}$$

holds. Then X has a G-invariant orbifold Kähler–Einstein metric.

The following conjecture is inspired by [T3, Question 1].

CONJECTURE 1.6. *For a given Fano variety X with log terminal singularities and finite subgroup $G \subset \mathrm{Aut}(X)$, the equality*

$$\mathrm{lct}(X, G) = \mathrm{lct}(X, D)$$

holds for some G-invariant effective $\mathbb{Q}$-divisor D on the variety X such that $D \equiv -K_X$.

The main purpose of this paper is to prove the following result.

Theorem 1.7. *Let X be a smooth del Pezzo surface. Then*

$$\operatorname{lct}\big(X\big)=\begin{cases}1 \text{ when } K_X^2=1 \text{ and } |-K_X| \text{ has no cuspidal curves,}\\ 5/6 \text{ when } K_X^2=1 \text{ and } |-K_X| \text{ has a cuspidal curve,}\\ 5/6 \text{ when } K_X^2=2 \text{ and } |-K_X| \text{ has no tacnodal curves,}\\ 3/4 \text{ when } K_X^2=2 \text{ and } |-K_X| \text{ has a tacnodal curve,}\\ 3/4 \text{ when } X \text{ is a cubic surface in } \mathbb{P}^3 \text{ without Eckardt points,}\\ 2/3 \text{ when } K_X^2=4 \text{ or } X \text{ is a cubic surface in } \mathbb{P}^3\\ \qquad\qquad\qquad\qquad\qquad\qquad \text{with an Eckardt point,}\\ 1/2 \text{ when } X\cong\mathbb{P}^1\times\mathbb{P}^1 \text{ or } K_X^2\in\{5,6\},\\ 1/3 \text{ in the remaining cases.}\end{cases}$$

Taking the paper [P] and Theorem 1.7 into consideration, we see that the assertion of Conjecture 1.6 holds for smooth del Pezzo surfaces with trivial group action. Also, in this paper, we prove the following result.

Theorem 1.8. *Let X be a del Pezzo surface with ordinary double points such that $K_X^2=1$. Then*

$$\operatorname{lct}\big(X\big)=\begin{cases}1 \text{ when } |-K_X| \text{ does not have cuspidal curves,}\\ 3/4 \text{ when } |-K_X| \text{ has a cuspidal curve } C \text{ such that}\\ \qquad\qquad\qquad\qquad\qquad\qquad \operatorname{Sing}(C)\subseteq\operatorname{Sing}(X),\\ 5/6 \text{ in the remaining cases.}\end{cases}$$

We see that Theorems 1.5 and 1.8 imply the existence of an orbifold Kähler–Einstein metric on every del Pezzo surface of degree 1 that has at most ordinary double points. (The problem of the existence of a Kähler–Einstein metric on smooth del Pezzo surfaces is solved in [T2].)

Further we will study global G-invariant log canonical thresholds of some smooth del Pezzo surfaces admitting an action of a finite group G. Let us consider two examples.

Example 1.9. The simple group $\mathrm{PGL}(2,\mathrm{F}_7)$ is a group of automorphisms of the quartic

$$x^3y+y^3z+z^3x=0\subset\mathbb{P}^2\cong\operatorname{Proj}\big(\mathbb{C}[x,y,z]\big)\,,$$

which induces $\mathrm{PGL}(2,\mathrm{F}_7)\subset\mathrm{Aut}(\mathbb{P}^2)$. Then $\operatorname{lct}(\mathbb{P}^2,\mathrm{PGL}(2,\mathrm{F}_7))=4/3$ by Lemma 5.1.

Example 1.10. Let X be a del Pezzo surface with ordinary double points that is given by

$$\sum_{i=0}^{4}x_i^2=\sum_{i=0}^{4}\lambda_ix_i^2=0\subseteq\mathbb{P}^4\cong\operatorname{Proj}\big(\mathbb{C}[x_0,\dots,x_4]\big)\,,$$

where $\lambda_1,\dots,\lambda_4\in\mathbb{C}$. Then $\operatorname{lct}(X,\mathbb{Z}_2^4)=1$ by Lemma 5.1.

There is a crucial difference between the two and higher-dimensional cases: in the latter case, we usually assume that G is trivial. For surfaces, it is not so, and applications are more special.

Example 1.11. Let X be a smooth cubic surface in $\mathbb{P}^3$ that is given by the equation

$$x^2y + xz^2 + zt^2 + tx^2 = 0 \subset \mathbb{P}^3 \cong \mathrm{Proj}\big(\mathbb{C}[x, y, z, t]\big)\,,$$

and let X' be a smooth del Pezzo surface such that $K^2_{X'} = 5$. Then $\mathrm{Aut}(X) \cong \mathrm{Aut}(X') \cong \mathrm{S}_5$ (see [DoI]). It follows from Lemma 5.1 and Example 5.5 that $\mathrm{lct}(X, \mathrm{S}_5) = \mathrm{lct}(X', \mathrm{S}_5) = 2$. There is a classical embedding $\mathrm{A}_5 \subset \mathrm{Aut}(\mathbb{P}^1)$ such that the induced embeddings $\mathrm{Aut}(\mathbb{P}^1 \times X) \supset \mathrm{A}_5 \times \mathrm{S}_5 \subset \mathrm{Aut}(\mathbb{P}^1 \times X')$ induce the embeddings

$$\mathrm{A}_5 \times \mathrm{S}_5 \cong \Omega \subset \mathrm{Bir}(\mathbb{P}^3) \supset \Gamma \cong \mathrm{A}_5 \times \mathrm{S}_5\,,$$

respectively. Then Ω and Γ are not conjugated in $\mathrm{Bir}(\mathbb{P}^3)$ by Lemma 6.2 and Theorem 6.4.

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2 Basic Tools

Let S be a surface with canonical singularities, and D be an effective $\mathbb{Q}$-divisor on it.

Remark 2.1. Let B be an effective $\mathbb{Q}$-divisor on S such that (S, B) is log canonical. Then

$$\left(S, \frac{1}{1-\alpha}(D - \alpha B)\right)$$

is not log canonical if (S, D) is not log canonical, where $\alpha \in \mathbb{Q}$ such that $0 \leqslant \alpha < 1$.

Let $\mathrm{LCS}(S, D) \subsetneq S$ be a subset such that $P \in \mathrm{LCS}(S, D)$ if and only if (S, D) is not log terminal at the point P. The set $\mathrm{LCS}(S, D)$ is called the locus of log canonical singularities.

Lemma 2.2. *Suppose that $-(K_S + D)$ is ample. Then the set* $\mathrm{LCS}(S, D)$ *is connected.*

Proof. See Theorem 17.4 in [K et al.]. □

Let P be a smooth point of the surface S. Suppose that (S, D) is not log canonical at P.

REMARK 2.3. The inequality $\mathrm{mult}_P(D) > 1$ holds (see [K]).

Let C be an irreducible curve on the surface S. Put $D = mC + \Omega$, where m is a non-negative rational number, and Ω is an effective $\mathbb{Q}$-divisor such that $C \not\subseteq \mathrm{Supp}(\Omega)$.

REMARK 2.4. Suppose that $C \subseteq \mathrm{LCS}(S, D)$. Then $m \geqslant 1$ (see [K]).

Suppose that the inequality $m \leqslant 1$ holds and $P \in C$.

LEMMA 2.5. *Suppose that C is smooth at P. Then $C \cdot \Omega > 1$.*

Proof. See Theorem 17.6 in [K et al.]. □

Let $\pi\colon \bar{S} \to S$ be a birational morphism, and $\bar{D}$ is a proper transform of D via π. Then

$$K_{\bar{S}} + \bar{D} + \sum_{i=1}^{r} a_i E_i \equiv \pi^*(K_S + D)\,,$$

where E_i is a π-exceptional curve, and a_i is a rational number.

REMARK 2.6. The log pair (S, D) is log canonical if and only if $(\bar{S}, \bar{D} + \sum_{i=1}^{r} a_i E_i)$ is log canonical.

Suppose that π is a blow up of the point P. Then $r = 1$ and $\pi(E_1) = P$. The log pair

$$\big(\bar{S}, \bar{D} + (\mathrm{mult}_P(D) - 1)E_1\big)$$

is not log canonical at some point $\bar{P} \in E_1$ by Remark 2.6. But $a_1 = \mathrm{mult}_P(D) - 1 > 0$.

COROLLARY 2.7. *The inequality* $\mathrm{mult}_{\bar{P}}(\bar{D}) + \mathrm{mult}_P(D) > 2$ *holds.*

Most of the described results are valid in much more general settings (see [K et al.] and [K]).

3 Smooth surfaces.

In this section we prove Theorem 1.7. Let X be a smooth del Pezzo surface. Putting

$$\omega = \begin{cases} 1/3 \text{ when } X \cong \mathbb{F}_1 \text{ or } K_X^2 \in \{7,9\}, \\ 1/2 \text{ when } X \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_X^2 \in \{5,6\}, \\ 2/3 \text{ when } K_X^2 = 4 \text{ or } X \text{ is a cubic surface in } \mathbb{P}^3 \text{ with an Eckardt point}, \\ 3/4 \text{ when } X \text{ is a cubic surface in } \mathbb{P}^3 \text{ without Eckardt points}, \\ 3/4 \text{ when } K_X^2 = 2 \text{ and } |-K_X| \text{ has a tacnodal curve}, \\ 5/6 \text{ when } K_X^2 = 2 \text{ and } |-K_X| \text{ has no tacnodal curves}, \\ 5/6 \text{ when } K_X^2 = 1 \text{ and } |-K_X| \text{ has a cuspidal curve}, \\ 1 \text{ when } K_X^2 = 1 \text{ and } |-K_X| \text{ has no cuspidal curves}, \end{cases}$$

we see that we must show that $\mathrm{lct}(X) = \omega$ to prove Theorem 1.7. But $\mathrm{lct}(X) \leqslant \omega$ by [P].

Suppose that the inequality $\mathrm{lct}(X) < \omega$ holds. To prove Theorem 1.7, we must show that this assumption leads to a contradiction. There is an effective $\mathbb{Q}$-divisor D on the surface X such that the equivalence $D \equiv -K_X$ holds, and $(X, \omega D)$ is not log canonical at some point $P \in X$.

LEMMA 3.1. *The inequality $K_X^2 \neq 1$ holds.*

Proof. Suppose that $K_X^2 = 1$. Take $C \in |-K_X|$ such that $P \in C$. Then C is an irreducible curve, and $(X, \omega C)$ is log canonical. We may assume that $C \not\subseteq \mathrm{Supp}(D)$ by Remark 2.1. Then

$$1 = C \cdot D \geqslant \mathrm{mult}_P(D) > 1/\omega \geqslant 1,$$

which is a contradiction. The obtained contradiction completes the proof. □

LEMMA 3.2. *The inequality $K_X^2 \leqslant 7$ holds.*

Proof. The equalities $\mathrm{lct}(\mathbb{P}^2) = 1/3$ and $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^1) = 1/2$ follow from Remarks 2.1 and 2.3, which implies that we may assume that $X = \mathbb{F}_1$ to complete the proof. Then $\omega = 1/3$.

Let L and C be irreducible curves on X such that $L^2 = 0$ and $C^2 = -1$. Then

$$-K_X \equiv 2C + 3L\,,$$

and the singularities of the log pair $(X, \omega(2C + 3L))$ are log canonical.

It follows from Remark 2.3 that $L \subseteq \mathrm{Supp}(D)$, because $L \cdot D = 2$. Therefore, we may assume that $C \not\subseteq \mathrm{Supp}(D)$ by Remark 2.1. Let Z be a general curve in $|C + L|$ such that $P \in Z$. Then

$$3 = Z \cdot D \geqslant \mathrm{mult}_P(D) > 1\big/\omega = 3\,,$$

which is a contradiction. The contradiction obtained completes the proof. □

LEMMA 3.3. *The inequality $K_X^2 \leqslant 4$ holds.*

Proof. Suppose that $K_X^2 \geqslant 5$. Then there is a birational morphism $\pi\colon X \to S$ such that

- The morphism π is an isomorphism in a neighborhood of P;
- Either $S \cong \mathbb{F}_1$ or $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $S \cong \mathbb{P}^2$,

and we may assume that $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ whenever $K_X^2 \leqslant 6$. Then the log pair $(S, \omega\pi(D))$ is not log canonical at $\pi(P)$. But $\pi(D) \equiv -K_S$, which is impossible by Lemma 3.2. $\square$

LEMMA 3.4. *The inequality $K_X^2 \neq 4$ holds.*

Proof. Suppose that $K_X^2 = 4$. Then X is an intersection of two quadrics in $\mathbb{P}^4$, and

$$D = \sum_{i=1}^{r} a_i C_i \equiv -K_X\,,$$

where C_i is an irreducible curve on the surface X, and $0 \leqslant a_i \in \mathbb{Q}$.

The equality $\omega = 2/3$ holds. Suppose that $a_k > 1/\omega = 3/2$. Then

$$4 = -K_X \cdot D = \sum_{i=1}^{r} a_i \deg(C_i) \geqslant a_k \deg(C_k) > \frac{3\deg(C_k)}{2}\,,$$

which implies that $\deg(C_k) \leqslant 2$. Let Z be an irreducible curve on X such that $C_k + Z$ is cut out by a general hyperplane section of $X \subset \mathbb{P}^4$ that passes through C_k. Then

$$3 \geqslant 4 - \deg(C_k) = Z \cdot D = \sum_{i=1}^{r} a_i (Z \cdot C_i) \geqslant a_k (Z \cdot C_k) = 2a_k > 3\,,$$

which is a contradiction. Therefore, we see that $\omega a_i \leqslant 1$ for every $i = 1, \ldots, r$.

There is $\lambda \in \mathbb{Q}$ such that $0 < \lambda < \omega = 2/3$ and $(X, \lambda D)$ is not log canonical at P. Then

$$\mathrm{LCS}(X, \lambda D) = \{P\}$$

by Lemma 2.2. But there is a birational morphism $\pi\colon X \to \mathbb{P}^2$ such that π is an isomorphism in a neighborhood of the point P. Then $\pi(D) \equiv -\lambda K_{\mathbb{P}^2}$. Let L be a general line on $\mathbb{P}^2$. Then

$$\pi(P) \cup L \subseteq \mathrm{LCS}\big(\mathbb{P}^2, \pi(D) + L\big)\,,$$

which is impossible by Lemma 2.2. The obtained contradiction completes the proof. $\square$

Let $\pi\colon U \to X$ be a blow up of the point P, and E be the exceptional curve of π. Then
$$\bar{D} \equiv \pi^*(-K_X) - \mathrm{mult}_P(D)E\,,$$
where $\bar{D}$ is the proper transform of D on the surface U. It follows from Remark 2.6 that
$$\big(U, \omega\bar{D} + \omega(\mathrm{mult}_P(D) - 1)E\big)$$
is not log canonical at some point $Q \in E$. Then $\mathrm{mult}_Q(\bar{D}) + \mathrm{mult}_P(D) > 2/\omega$ by Corollary 2.7.

Lemma 3.5. *The inequality $K_X^2 \neq 2$ holds.*

Proof. Suppose that $K_X^2 = 2$. There is a double cover $\psi\colon X \to \mathbb{P}^2$ such that ψ is branched over a smooth quartic curve $C \subset \mathbb{P}^2$. Then either $\psi(P) \in C$ or $\psi(P) \notin C$.

Suppose that $\psi(P) \in C$. There is a curve $L \in |-K_X|$ that is singular at P, and we may assume that at least one irreducible component of the curve L is not contained in the support of the divisor D by Remark 2.1, because $(X, \omega L)$ is log canonical (see [P]). Then
$$2 = L \cdot D \geqslant \mathrm{mult}_P(D)\,\mathrm{mult}_P(L) \geqslant 2/\omega > 2$$
in the case when L is irreducible. So, we must have $L = L_1 + L_2$, where L_1 and L_2 are irreducible smooth curves such that $L_1 \cdot L_2 = 2$ and $L_1^2 = L_2^2 = -1$. Without loss of generality, we may assume that $L \not\subset \mathrm{Supp}(D)$. Put $D = mL_1 + \Omega$, where $0 \leqslant m \in \mathbb{Q}$, and Ω is an effective $\mathbb{Q}$-divisor such that $L_1 \not\subseteq \mathrm{Supp}(\Omega)$. Then
$$m + 1 < 2m + \Omega \cdot L_2 = D \cdot L_2 = 1\,,$$
which is a contradiction. Therefore, we see that $\psi(P) \notin C$.

In particular, the log pair $(X, \omega D)$ is log canonical outside of finitely many points.

There is a unique curve $Z \in |-K_X|$ such that $P \in Z$ and $Q \in \bar{Z}$, where $\bar{Z}$ is the proper transform of the curve Z on the surface U. Then Z consists of at most two components.

Suppose that Z is irreducible. We may assume $Z \not\subseteq \mathrm{Supp}(D)$. Hence, we have
$$2 - \mathrm{mult}_P(D) = \bar{Z} \cdot \bar{D} \geqslant \mathrm{mult}_Q(\bar{D}) > 2/\omega - \mathrm{mult}_P(D)$$
which is a contradiction. So, we must have $Z = Z_1 + Z_2$, where Z_1 and Z_1 are irreducible smooth curves such that $Z_1 \cdot Z_2 = 2$ and $Z_1^2 = Z_2^2 = -1$. We may assume that $P \in Z_1$ and $P \notin Z_2$.

It is easy to see that the log pair $(X, \omega Z_1 + \omega Z_2)$ is log canonical. Thus, we may assume that either $Z_1 \not\subseteq \mathrm{Supp}(D)$ or $Z_2 \not\subseteq \mathrm{Supp}(D)$ by Remark 2.1. But

$$1 = Z_1 \cdot D \geqslant \mathrm{mult}_P(D) \geqslant 1/\omega > 1\,,$$

which implies that $Z_2 \not\subseteq \mathrm{Supp}(D)$. Then $Z_1 \subseteq \mathrm{Supp}(D)$. Put $D = \bar{m}Z_1 + \Upsilon$, where $0 < \bar{m} \in \mathbb{Q}$, and Υ is an effective $\mathbb{Q}$-divisor on the surface X such that $Z_1 \not\subseteq \mathrm{Supp}(\Upsilon)$. Then

$$2\bar{m} \leqslant 2\bar{m} + \Upsilon \cdot Z_2 = D \cdot Z_2 = 1\,,$$

which gives $\bar{m} \leqslant 1/2$. But $Q \in \bar{Z}_1$, where $\bar{Z}$ it the proper transform of Z_1 on the surface U. Then

$$2 - \mathrm{mult}_P(D) \geqslant 1 - \mathrm{mult}_P(D) + 2\bar{m} = \bar{Z}_1 \cdot \bar{\Upsilon} > 2/\omega - \mathrm{mult}_P(D) > 2 - \mathrm{mult}_P(D)$$

by Lemma 2.5. The obtained contradiction completes the proof. □

It follows from Lemmas 3.2, 3.3, 3.4, 3.1, 3.5 that X is a smooth cubic surface in $\mathbb{P}^3$.

Lemma 3.6. *The cubic surface X does not have Eckardt points.*

Proof. There is a birational morphism $\pi\colon X \to S$ such that

- The morphism π is an isomorphism in a neighborhood of the point P;
- The surface S is a smooth del Pezzo surface and $K_S^2 = 4$.

Suppose that X has an Eckardt point. (A point of a cubic surface is an Eckardt point if the cubic contains 3 lines passing through this point.) Then $\pi(D) \equiv -K_S$ and $(S, \omega\pi(D))$ is not log canonical at the point $\pi(P)$, which is impossible by Lemma 3.4. □

Therefore, we see that $\omega = 3/4$ and $\mathrm{mult}_P(D) > 4/3$ by Remark 2.3.

Lemma 3.7. *The log pair $(X, \omega D)$ is log canonical on $X \setminus P$.*

Proof. Arguing as in the proof of Lemma 3.4, we see that the locus $\mathrm{LCS}(X, \omega D)$ contains finitely many points. Then the log pair $(X, \omega D)$ is even log terminal on $X \setminus P$ by Lemma 2.2. □

Let T be the unique hyperplane section of X that is singular at P. We may assume that the support of the divisor D does not contain at least one irreducible component of the curve T, because $(S, \omega T)$ is log canonical (see [P]). The following cases are possible:

- The curve T is irreducible and U is a del Pezzo surface;
- The curve T is a union of a line and an irreducible conic intersecting at P;

- The curve T consists of 3 lines such that one of them does not pass through P;

where T is reduced and $-K_U$ is nef and big. We exclude these cases one by one.

LEMMA 3.8. *The curve T is reducible.*

Proof. Suppose that T is irreducible. There is a double cover $\psi\colon U \to \mathbb{P}^2$ branched over a quartic curve. Let $\tau \in \mathrm{Aut}(U)$ be an involution induced by ψ. (The involution τ induces an involution in $\mathrm{Bir}(X)$ that is called the Geiser involution.) It follows from [M] that $\tau(\bar{T}) = E$ and

$$\tau^*\big(\pi^*(-K_X)\big) \equiv \pi^*(-2K_X) - 3E\,.$$

Let $\bar{T}$ be the proper transform of T on the surface U. Suppose that $Q \in \bar{T}$. Then

$$\begin{aligned}3 - 2\,\mathrm{mult}_P(D) = \bar{T}\cdot\bar{D} &\geqslant \mathrm{mult}_Q(\bar{T})\,\mathrm{mult}_Q(\bar{D})\\ &> \mathrm{mult}_Q(\bar{T})\big(8/3 - \mathrm{mult}_P(D)\big) \geqslant 8/3 - \mathrm{mult}_P(D)\,,\end{aligned}$$

which implies that $\mathrm{mult}_P(D) \leqslant 1/3$. But $\mathrm{mult}_P(D) > 4/3$. Thus, we see that $Q \notin \bar{T}$.

Put $\breve{Q} = \pi\circ\tau(Q)$. Let H be the hyperplane section of X that is singular at $\breve{Q}$. Then $T \neq H$, because $P \neq \breve{Q}$ and T is smooth outside of the point P. Then $P \notin H$, because otherwise

$$3 = H\cdot T \geqslant \mathrm{mult}_P(H)\,\mathrm{mult}_P(T) + \mathrm{mult}_{\breve{Q}}(H)\,\mathrm{mult}_{\breve{Q}}(T) \geqslant 4\,.$$

Let $\bar{H}$ be the proper transform of H on the surface U. Put $\bar{R} = \tau(\bar{H})$ and $R = \pi(\bar{R})$. Then

$$\bar{R} \equiv \pi^*(-2K_X) - 3E\,,$$

and the curve $\bar{R}$ must be singular at the point Q.

Suppose that R is irreducible. The singularities of the log pair $\big(X, \frac{3}{8}R\big)$ are log canonical, which implies that we may assume that $R \not\subseteq \mathrm{Supp}(D)$ by Remark 2.1. Then

$$6 - 3\,\mathrm{mult}_P(D) = \bar{R}\cdot\bar{D} \geqslant \mathrm{mult}_Q(\bar{R})\,\mathrm{mult}_Q(\bar{D}) > 2\big(8/3 - \mathrm{mult}_P(D)\big)\,,$$

which implies that $\mathrm{mult}_P(D) < 2/3$. But $\mathrm{mult}_P(D) > 4/3$. The curve R must be reducible

The curves R and H are reducible. So, there is a line $L \subset X$ such that $P \notin L \ni \breve{Q}$.

Let $\bar{L}$ be the proper transform of L on the surface U. Put $\bar{Z} = \tau(\bar{L})$. Then $\bar{L}\cdot E = 0$ and

$$\bar{L}\cdot\bar{T} = \bar{L}\cdot\pi^*(-K_X) = 1\,,$$

which implies that $\bar{Z} \cdot E = 1$ and $\bar{Z} \cdot \pi^*(-K_X) = 2$. We have $Q \in \bar{Z}$. Then

$$2 - \mathrm{mult}_P(D) = \bar{Z} \cdot \bar{D} \geqslant \mathrm{mult}_Q(\bar{D}) > 8/3 - \mathrm{mult}_P(D) > 2 - \mathrm{mult}_P(D)$$

in the case when $\bar{Z} \not\subseteq \mathrm{Supp}(\bar{D})$. Hence, we see that $\bar{Z} \subseteq \mathrm{Supp}(\bar{D})$.

Put $Z = \pi(\bar{Z})$. Then Z is a conic and $P \in Z$. Let F be a line on X such that $F + Z$ is cut out by a hyperplane passing through Z. Then $P \notin F$, because $T \neq F + Z$.

Put $D = \epsilon Z + \Upsilon$, where ϵ is a positive rational number, and Υ is an effective $\mathbb{Q}$-divisor whose support does not contain the conic Z. We may assume that $F \not\subseteq \mathrm{Supp}(\Upsilon)$ by Remark 2.1. Then

$$1 = F \cdot D = 2\epsilon + F \cdot \Upsilon \geqslant 2\epsilon \,,$$

which implies that $\epsilon \leqslant 1/2$. Let $\bar{\Upsilon}$ be the proper transform of Υ on the surface U. Then

$$2 - \mathrm{mult}_P(D) + \epsilon = \bar{Z} \cdot \bar{\Upsilon} > 8/3 - \mathrm{mult}_P(D)$$

by Lemma 2.5, which implies that $\epsilon > 2/3$. But $\epsilon \leqslant 1/2$. □

Therefore, there is a line $L_1 \subset X$ such that $P \in L_1$.

Lemma 3.9. *There is a line $L_2 \subset X$ such that $L_1 \neq L_2$ and $P \in L_2$.*

Proof. Suppose that there is no line $L_2 \subset X$ such that $L_1 \neq L_2$ and $P \in L_2$. Then $T = L_1 + C$, where C is an irreducible conic that passes through the point P.

Let $\bar{L}_1$ and $\bar{C}$ be the proper transforms of L_1 and C on the surface U, respectively. Then

$$\bar{L}_1^2 = -2 \,, \quad -K_U \cdot \bar{L}_1 = 0 \,, \quad \bar{C}^2 = -1 \,, \quad -K_U \cdot \bar{C} = 1 \,,$$

but the divisor $-K_U$ is nef and big. There is a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\zeta"] \arrow[d, "\pi"'] & W \arrow[d, "\psi"] \\ X \arrow[r, dashed, "\rho"'] & \mathbb{P}^2 \end{tikzcd}$$

where ζ is the contraction of the curve $\bar{L}_1$ to an ordinary double point, ψ is a double cover branched over a quartic curve, and ρ is the projection from the point P.

Let τ be the biregular involution of U induced by ψ. Then $\tau(E) = \bar{C}$ and

$$\tau^*(\bar{L}_1) \equiv \bar{L}_1 \,, \quad \tau^*(E) \equiv \bar{C} \,, \quad \tau^*\big(\pi^*(-K_X)\big) \equiv \pi^*(-2K_X) - 3E - \bar{L}_1 \,.$$

Note that we assumed earlier that the support of the divisor D does not contain at least one irreducible component of the curve T. Then either $L_1 \not\subseteq \mathrm{Supp}(D)$ or $C \not\subseteq \mathrm{Supp}(D)$. But

$$\bar{L}_1 \cdot \bar{D} = 1 - \mathrm{mult}_P(D) < 0\,,$$

which implies that $C \not\subseteq \mathrm{Supp}(D) \supseteq L_1$. Put $D = mL_1 + \Omega$, where m is a positive rational number, and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the line L_1. Then

$$m\bar{L}_1 + \bar{\Omega} \equiv \pi^*(-K_X) - \big(m + \mathrm{mult}_P(\Omega)\big)E \equiv \pi^*(-K_X) - \mathrm{mult}_P(D)E\,,$$

where $\bar{\Omega}$ is the proper transform of Ω on the surface U. We have

$$0 \leqslant \bar{C} \cdot \bar{\Omega} = 2 - \mathrm{mult}_P(\Omega) + 2m < 2/3 - m\,,$$

which implies that $m < 2/3$. Then $\mathrm{mult}_P(D) = \mathrm{mult}_P(\Omega) + m$, which implies that

$$\mathrm{mult}_Q(\bar{\Omega}) > 8/3 - \mathrm{mult}_P(\Omega) - m\big(1 + \mathrm{mult}_Q(\bar{L}_1)\big)\,. \tag{3.10}$$

Suppose that $Q \in \bar{L}_1$. Then it follows from Lemma 2.5 that

$$1 - \mathrm{mult}_P(\Omega) + m = \bar{L}_1 \cdot \bar{\Omega} > 8/3 - \mathrm{mult}_P(\Omega) - m\,,$$

which implies that $m > 5/6$. But $m < 2/3$. Hence, we see that $Q \notin \bar{L}_1$.

Suppose that $Q \in \bar{C}$. Then it follows from the inequality 3.10 that

$$2 - \mathrm{mult}_P(\Omega) - 2m = \bar{C} \cdot \bar{\Omega} > 8/3 - \mathrm{mult}_P(\Omega) - m\,,$$

which implies that $m < 0$. Hence, we see that $Q \notin \bar{C}$.

We have $\tau(E) = \bar{C}$. Let H be the hyperplane section of the cubic surface X that is singular at the point $\pi \circ \tau(Q) \in C$. Then $P \notin H$, because C is smooth.

Let $\bar{H}$ be the proper transform of H on the surface U. Put $\bar{R} = \tau(\bar{H})$ and $R = \pi(\bar{R})$. Then

$$\bar{R} \equiv \pi^*(-2K_X) - 3E - \bar{L}_1\,,$$

and the curve $\bar{R}$ is singular at the point Q by construction.

Suppose that R is irreducible. Then $R + L_1 \equiv -2K_X$, but $\big(X, \frac{3}{8}(R + L_1)\big)$ is log canonical, which implies that we may assume that $R \not\subseteq \mathrm{Supp}(D)$ by Remark 2.1. The inequality 3.10 gives

$$5 - 2\big(m + \mathrm{mult}_P(\Omega)\big) - m = \bar{R} \cdot \bar{\Omega} \geqslant 2\,\mathrm{mult}_Q(\bar{\Omega}) > 2\big(8/3 - m - \mathrm{mult}_P(\Omega)\big)\,,$$

which implies that $m < 0$. Hence, there is a line $L \subset X$ such that $P \notin L$ and $\pi \circ \tau(Q) \in L$.

Let $\bar{L}$ be the proper transform of the line L on the surface U. Then

$$\bar{L} \cdot \bar{C} = \bar{L} \cdot \pi^*(-K_X) = 1 \quad \text{and} \quad \bar{L} \cdot E = \bar{L} \cdot \bar{L}_1 = 0\,,$$

but τ preserves the intersection form. Put $\bar{Z} = \tau(\bar{L})$. Then $\bar{Z} \cdot E = 1$, $\bar{Z} \cdot \bar{L}_1 = 0$, $\bar{Z} \cdot \pi^*(-K_X) = 2$.

Suppose that the support of $\bar{\Omega}$ does not contain $\bar{Z}$. Then the inequality (3.10) implies that

$$2 - m - \mathrm{mult}_P(\Omega) = \bar{Z} \cdot \bar{\Omega} > 8/3 - m - \mathrm{mult}_P(\Omega)\,,$$

which is impossible. Thus, the support of $\bar{\Omega}$ must contain the curve $\bar{Z}$.

Put $Z = \pi(\bar{Z})$. Then Z is a conic that passes through the point P. The line L is the line on X such that the curve $L+Z$ is cut out by a hyperplane passing through Z. We have $P \not\in F$. Put

$$D = \epsilon Z + mL_1 + \Upsilon\,,$$

where ϵ is a positive rational number, and Υ is an effective $\mathbb{Q}$-divisor on the surface X such that the support of the divisor Υ does not contain the curves Z and L_1.

We may assume that $L \not\subseteq \mathrm{Supp}(\Upsilon)$, because $(X, \omega(L+Z))$ is log canonical. Then

$$1 = L \cdot D = 2\epsilon + mL \cdot L_1 + L \cdot \Upsilon = 2\epsilon + L \cdot \Upsilon \geqslant 2\epsilon\,,$$

which implies that $\epsilon \leqslant 1/2$. But $\bar{Z} \cap \bar{L}_1 = \varnothing$. Then it follows from Lemma 2.5 that

$$2 - \mathrm{mult}_P(D) + \epsilon = \bar{Z} \cdot \bar{\Upsilon} > 8/3 - \mathrm{mult}_P(D)\,,$$

where $\bar{\Upsilon}$ is a proper transform of Υ on the surface U. We deduce that $\epsilon > 2/3$. But $\epsilon \leqslant 1/2$. □

We have $T = L_1 + L_2 + L_3$, where L_3 is a line such that $P \not\in L_3$. Then

$$\bar{L}_1^2 = \bar{L}_2^2 = -2\,, \quad E \cdot \bar{L}_1 = E \cdot \bar{L}_2 = -K_U \cdot \bar{L}_3 = 1\,,$$
$$-K_U \cdot \bar{L}_1 = -K_U \cdot \bar{L}_2 = E \cdot \bar{L}_3 = 0\,, \quad \bar{L}_3^2 = -1\,,$$

where $\bar{L}_i$ is the proper transform of L_i on the surface U. There is a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\zeta"] \arrow[d, "\pi"'] & W \arrow[d, "\psi"] \\ X \arrow[r, dashed, "\rho"'] & \mathbb{P}^2, \end{tikzcd}$$

where ζ is the contraction of the curves $\bar{L}_1$ and $\bar{L}_2$ to ordinary double points, ψ is a double cover branched over a quartic curve, and ρ is the projection from the point P.

Let τ be the biregular involution of the surface U induced by ψ. Then

$$\tau^*\big(\pi^*(-K_X)\big) \equiv \pi^*(-2K_X) - 3E - \bar{L}_1 - \bar{L}_2\,,$$

and $\tau(\bar{L}_1) = \bar{L}_1$, $\tau(\bar{L}_2) = \bar{L}_2$, $\tau(\bar{L}_3) = E$. Recall that $\mathrm{mult}_P(D) > 4/3$ by Remark 2.3.

We assume that $T \not\subseteq \mathrm{Supp}(D)$. Then $\mathrm{Supp}(D)$ does not contain one of L_1, L_2, L_3. But

$$\bar{L}_1 \cdot \bar{D} = \bar{L}_2 \cdot \bar{D} = 1 - \mathrm{mult}_P(D) < 0\,,$$

which implies that $L_2 \subseteq \mathrm{Supp}(D) \supseteq L_2$ and $L_3 \not\subseteq \mathrm{Supp}(D)$. Put

$$D = m_1 L_1 + m_2 L_2 + \Omega\,,$$

where $0 < m_i \in \mathbb{Q}$, and Ω is an effective $\mathbb{Q}$-divisor such that $L_2 \not\subseteq \mathrm{Supp}(\Omega) \not\supseteq L_2$.

The inequality $m_1 + m_2 \leqslant 1$ holds, because $1 - m_1 - m_2 = L_3 \cdot \Omega \geqslant 0$.

Let $\bar{\Omega}$ be the proper transform of Ω on the surface U. Then

$$m_1 \bar{L}_1 + m_2 \bar{L}_2 + \bar{\Omega} \equiv \pi^*(-K_X) - \big(m_1 + m_2 + \mathrm{mult}_P(\Omega)\big)E\,,$$

where $m_1 + m_2 + \mathrm{mult}_P(\Omega) = \mathrm{mult}_P(D)$. The latter equality implies that

$$\mathrm{mult}_Q(\bar{\Omega}) > 8/3 - \mathrm{mult}_P(\Omega) - m_1\big(1 + \mathrm{mult}_Q(\bar{L}_1)\big) \\ - m_1\big(1 + \mathrm{mult}_Q(\bar{L}_2)\big)\,. \quad (3.11)$$

LEMMA 3.12. *The curves $\bar{L}_1$ and $\bar{L}_2$ do not contain the point Q.*

Proof. Suppose that $Q \in \bar{L}_1 \cup \bar{L}_2$. Without loss of generality we may assume that $Q \in \bar{L}_1$. Then

$$1 - \mathrm{mult}_P(\Omega) - m_2 + m_1 = \bar{L}_1 \cdot \bar{\Omega} > 8/3 - \mathrm{mult}_P(\Omega) - m_1 - m_2$$

by Lemma 2.5. We have $m_1 > 5/6$. Then

$$1 - m_1 + m_2 = \Omega \cdot L_2 > 4/3 - m_1 - m_2\,,$$

which implies the inequality $m_2 > 1/6$. The latter contradicts the inequality $m_1 + m_2 \leqslant 1$. □

Therefore, the point $\pi \circ \tau(Q)$ is contained in the line L_3, but $\pi \circ \tau(Q) \notin L_1 \cup L_2$.

LEMMA 3.13. *The line L_3 is the only line on X that passes through the point $\pi \circ \tau(Q)$.*

Proof. Suppose that there is a line $L \subset X$ such that $L \neq L_3$ and $\pi \circ \tau(Q) \in L$. Then

$$\bar{L} \cdot \bar{L}_1 = \bar{L} \cdot \bar{L}_2 = \bar{L} \cdot E = 0\,, \quad \bar{L} \cdot \pi^*(-K_X) = \bar{L} \cdot \bar{L}_3 = 1\,,$$

where $\bar{L}$ is the proper transform of the line L on the surface U.

The involution τ preserves the intersection form. Put $\bar{Z} = \tau(\bar{L})$ and $Z = \pi(\bar{Z})$. Then

$$\bar{Z} \cdot E = 1\,, \quad \bar{Z} \cdot \bar{L}_3 = 0\,, \quad \bar{Z} \cdot \pi^*(-K_X) = 2\,,$$

which implies that the curve $\pi(\bar{Z})$ is a conic passing through the point P.

The support of the divisor Ω contains the conic Z, because otherwise

$$2 - m_1 - m_2 - \mathrm{mult}_P(\Omega) = \bar{Z} \cdot \bar{\Omega} > 8/3 - m_1 - m_2 - \mathrm{mult}_P(\Omega)\,,$$

which is impossible. Put $D = \epsilon Z + m_1 L_1 + m_2 L_2 + \Upsilon$, where ϵ is a positive rational number, and Υ is an effective $\mathbb{Q}$-divisor on X whose support does not contain Z, L_1, L_2.

The line L is the line on the surface X such that the curve $L + Z$ is cut out by a hyperplane that passes through the conic Z. We may assume that the support of Υ does not contain the line L by Remark 2.1, because the log pair $(X, \omega(L + Z))$ is log canonical. Then

$$1 = L \cdot D = 2\epsilon + m_1 L \cdot L_1 + m_2 L \cdot L_2 + L \cdot \Upsilon = 2\epsilon + L \cdot \Upsilon \geqslant 2\epsilon\,,$$

which implies that $\epsilon \leqslant 1/2$. But $Q \notin \bar{L}_1$ and $Q \notin \bar{L}_2$ by Lemma 3.12. Thus, the log pair

$$\big(U, \epsilon\bar{Z} + \omega\bar{\Upsilon} + (\omega\,\mathrm{mult}_P(D) - 1)E\big)$$

is not log canonical at the point Q, where $\bar{\Upsilon}$ is a proper transform of Υ on the surface U. Then

$$\begin{aligned} 2 - \mathrm{mult}_P(D) + \epsilon &= 2 - \mathrm{mult}_P(D) + \epsilon - m_1\bar{L}_1 \cdot \bar{Z} - m_2\bar{L}_2 \cdot \bar{Z} \\ &= \bar{Z} \cdot \bar{\Upsilon} > 8/3 - \mathrm{mult}_P(D) \end{aligned}$$

by Lemma 2.5, which implies that $\epsilon > 2/3$. But $\epsilon \leqslant 1/2$. □

Let $C \subset X$ be a conic such that $C + L_3$ is cut out by the hyperplane tangent to X at $\pi \circ \tau(Q)$, and let $\bar{C}$ be the proper transform of C on the surface U. Put $\bar{Z} = \tau(\bar{C})$ and $Z = \pi(\bar{Z})$. Then

$$\bar{Z} \equiv \pi^*(-2K_X) - 4E - \bar{L}_1 - \bar{L}_2\,,$$

and Z is singular at P. We have $\bar{Z} \cdot E = 2$ and $\bar{Z} \cdot \bar{L}_1 = \bar{Z} \cdot \bar{L}_2 = 0$, because $C \cap L_1 = C \cap L_2 = \varnothing$.

Lemma 3.14. *The support of the divisor D contains Z.*

Proof. Suppose that $Z \not\subseteq \mathrm{Supp}(D)$. Then it follows from Corollary 2.7 that

$$4 - 2\,\mathrm{mult}_P(D) = \bar{Z} \cdot \bar{D} \geqslant \mathrm{mult}_Q(\bar{D}) > 8/3 - \mathrm{mult}_P(D)\,,$$

which implies that $\mathrm{mult}_P(D) < 4/3$. But $\mathrm{mult}_P(D) > 4/3$. □

Put $D = \epsilon Z + m_1 L_1 + m_2 L_2 + \Upsilon$, where $0 < \epsilon \in \mathbb{Q}$, and Υ is an effective $\mathbb{Q}$-divisor whose support does not contain the curves Z, L_1, L_2. Then $L_1 + L_2 + Z \equiv -2K_X$ and

$$D \cdot L_1 = m_2 - m_1 + 2\epsilon + L_1 \cdot \Upsilon = D \cdot L_2 = m_1 - m_2 + 2\epsilon + L_2 \cdot \Upsilon = 1\,,$$

which implies that $\epsilon \leqslant 1/2$. Let $\bar{\Upsilon}$ be a proper transform of Υ on the surface U. Then

$$4 - 2\,\mathrm{mult}_P(D) = \bar{Z} \cdot \bar{\Upsilon} > 8/3 - \mathrm{mult}_P(D)$$

by Lemma 2.5, which implies that $\mathrm{mult}_P(D) < 4/3$. But $\mathrm{mult}_P(D) > 4/3$.

The contradiction obtained completes the proof Theorem 1.7.

4 Singular Surfaces

Let X be a del Pezzo surface with Du Val singularities such that $K_X^2 = 1$, and singularities of the surface X consist of finitely many points of type $\mathbb{A}_1$ or $\mathbb{A}_2$. Put

$$\omega = \begin{cases} 1 \text{ when } |-K_X| \text{ does not have cuspidal curves,} \\ 2/3 \text{ when } |-K_X| \text{ has a cuspidal curve } C \text{ such that} \\ \qquad\qquad\qquad\qquad \mathrm{Sing}(C) \text{ is a point of type } \mathbb{A}_2, \\ 5/6 \text{ when } |-K_X| \text{ has cuspidal curves, but their cusps} \\ \qquad\qquad\qquad\qquad \text{are not contained in } \mathrm{Sing}(S), \\ 3/4 \text{ in the remaining cases.} \end{cases}$$

LEMMA 4.1. *The equality* $\mathrm{lct}(X) = \omega$ *holds.*

Proof. Taking into a consideration curves in $|-K_X|$, we see that $\mathrm{lct}(X) \leqslant \omega$. Thus, to conclude the proof, we may assume that $\mathrm{lct}(X) < \omega$. Then there is an effective $\mathbb{Q}$-divisor D on the surface X such that $D \equiv -K_X$, but $(X, \lambda D)$ is not log terminal and for some $\omega > \lambda \in \mathbb{Q}$.

Suppose that $\mathrm{LCS}(X, \lambda D)$ is not zero-dimensional. There is an irreducible curve C such that

$$D = mC + \Omega$$

where $1 < 1/\lambda \leqslant m \in \mathbb{Q}$, and Ω is an effective $\mathbb{Q}$-divisor such that $C \not\subseteq \mathrm{Supp}(\Omega)$. Then

$$1 = H \cdot D = mH \cdot C + H \cdot \Omega > m > 1\,,$$

where H is a general curve in the pencil $|-K_X|$. Thus, the locus $\mathrm{LCS}(X, \lambda D)$ is zero-dimensional.

It follows from Lemma 2.2 that the locus $\mathrm{LCS}(X, \lambda D)$ consists of a single point $P \in X$.

Let Z be the curve in $|-K_X|$ such that $P \in Z$. Arguing as in the proof of Lemma 3.1, we see that we may assume that $P \in \mathrm{Sing}(X)$.

We may assume that $Z \not\subseteq \mathrm{Supp}(D)$, because $(X, \omega Z)$ is log canonical, and Z is irreducible.

Suppose that P is a point of type $\mathbb{A}_1$. Let $\pi\colon U \to X$ be a blow up of the point P. Then

$$\begin{cases} \bar{D} \equiv \pi^*(-K_X) - aE\,, \\ \bar{Z} \equiv \pi^*(-K_X) - E\,, \end{cases}$$

where $\bar{D}$ and $\bar{Z}$ are proper transforms of D and Z on the surface U, respectively, E is the exceptional curve of π, and a is a positive rational number. Then $a \leqslant 1/2$, because $1 - 2a = \bar{Z} \cdot \bar{D} \geqslant 0$.

The log pair $(U, \lambda\bar{D} + \lambda aE)$ is not log terminal at some point $Q \in E$ by Remark 2.6. Then

$$1 \geqslant 2a = E \cdot \bar{D} > 1/\lambda > 1$$

by Lemma 2.5, which is a contradiction. Thus, the point P is a singular point of type $\mathbb{A}_2$.

There is a birational morphism $\zeta\colon W \to X$ such that ζ contracts two irreducible smooth rational curves E_1 and E_2 to the point P, the morphism ζ induces an isomorphism

$$W \setminus (E_1 \cup E_2) \cong X \setminus P\,,$$

and W is smooth along E_1 and E_2. Then $E_1^2 = E_2^2 = -2$ and $E_1 \cdot E_2 = 1$. But

$$\begin{cases} \grave{D} \equiv \zeta^*(-K_X) - a_1E_1 - a_2E_2\,, \\ \grave{Z} \equiv \zeta^*(-K_X) - E_1 - E_2E\,, \end{cases}$$

where $\grave{D}$ and $\grave{Z}$ are proper transforms of D and Z on the surface W, respectively, and $0 \leqslant a_i \in \mathbb{Q}$.

The inequalities $\grave{Z} \cdot \grave{D} \geqslant 0$, $E_1 \cdot \grave{D} \geqslant 0$, $E_1 \cdot \grave{D} \geqslant 0$ imply that

$$a_1 + a_2 \leqslant 1\,, \quad 2a_1 \geqslant a_2\,, \quad 2a_2 \geqslant a_1\,,$$

respectively. Thus, we see that $a_1 \leqslant 2/3$ and $a_2 \leqslant 2/3$. But the equivalence

$$K_W + \lambda\grave{D} + \lambda a_1E_1 + \lambda a_2E_2 \equiv \zeta^*(K_X + \lambda D)$$

implies the existence of a point $O \in E_1 \cup E_2$ such that $(W, \lambda\grave{D} + \lambda a_1E_1 + \lambda a_2E_2)$ is not log terminal at the point O (see Remark 2.6). Without loss of generality, we may assume that $O \in E_1$.

Suppose that $O \notin E_2$. Then $(W, \lambda\grave{D} + E_1)$ is not log terminal at Q. We have

$$2a_1 - a_2 = E_1 \cdot \grave{D} > 1/\lambda > 1\,,$$

by Lemma 2.5, which implies that $a_1 > 2/3$, because $2a_2 \geqslant a_1$. But $a_1 \leqslant 2/3$.

Thus, we see that $O = E_1 \cap E_2$. Then

$$\begin{cases} 2a_1 - a_2 = E_1 \cdot \grave{D} \geqslant 1/\lambda - a_2 > 1 - a_2\,, \\ 2a_2 - a_1 = E_1 \cdot \grave{D} \geqslant 1/\lambda - a_1 > 1 - a_1\,, \end{cases}$$

by Lemma 2.5, which implies that $a_1 > 1/2$ and $a_2 > 1/2$. But $a_1 + a_2 \leqslant 1$. □

The assertion of Theorem 1.8 follows from Lemma 4.1.

5 Invariant Thresholds

Let X is a smooth del Pezzo surface, let H be a Cartier divisor on X, let G be a finite subgroup in $\mathrm{Aut}(X)$ such that the G-invariant subgroup of the group $\mathrm{Pic}(X)$ is $\mathbb{Z}H$, and

- let r be the biggest natural number such that $-K_X \sim rH$,
- let k be the smallest natural number such that $k = |\Sigma|$, where $\Sigma \subset X$ is a G-orbit,
- let m be the smallest natural number such that there is a G-invariant divisor in $|mH|$.

It follows from Definition 1.1 that $\mathrm{lct}(X, G) \leqslant m/r$.

LEMMA 5.1. *Suppose that* $h^0(X, \mathcal{O}_X((m-r)H)) < k$. *Then* $\mathrm{lct}(X, G) = m/r$.

Proof. We suppose that $\mathrm{lct}(X, G) < m/r$. Then there is an effective G-invariant $\mathbb{Q}$-divisor D on the surface X such that $\mathrm{LCS}(X, \lambda D) \neq \varnothing$ and $D \equiv -K_X$, where $0 < \lambda \in \mathbb{Q}$ such that $\lambda < m/r$.

It follows from the Nadel vanishing theorem (see [L, Th. 9.4.8]) that the sequence

$$H^0\big(X, \mathcal{O}_X((m-r)H)\big) \longrightarrow H^0\big(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_X((m-r)H)\big) \longrightarrow 0 \qquad (5.2)$$

is exact, where $\mathcal{J}(\lambda D)$ is the multiplier ideal sheaf of λD, and $\mathcal{L}$ is the corresponding subscheme.

Suppose that $\mathcal{L}$ is zero-dimensional. Then the exact sequence (5.2) implies that

$$k > h^0\big(X, \mathcal{O}_X((m-r)H)\big) \geqslant h^0\big(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_X((m-r)H)\big) = h^0(\mathcal{O}_{\mathcal{L}}) \geqslant |\,\mathrm{Supp}(\mathcal{L})| \geqslant k\,,$$

because the subscheme $\mathcal{L}$ is G-invariant. Hence, the subscheme $\mathcal{L}$ is not zero-dimensional.

Thus, there is a G-invariant reduced curve C on the surface X such that

$$\lambda D = \mu C + \Omega\,,$$

where $\mu \geqslant 1$, and Ω is an effective one-cycle on the surface X, whose support does not contain any component of the curve C. Then $C \sim lH$ for some natural number l. We have $l \geqslant m$. But

$$m > \lambda r \geqslant \mu l \geqslant l \geqslant m\,,$$

because the G-invariant subgroup of the group $\mathrm{Pic}(X)$ is generated by the divisor H. □

Let us show how to apply Lemma 5.1.

EXAMPLE 5.3. Suppose that $K_X^2 = 5$ and $k \neq 1$. Then X has 6 curves $E_1, \ldots, E_6$ such that

$$\sum_{i=1}^{6} E_i \sim -K_X$$

and $E_i^2 = -1$. The divisor $\sum_{i=1}^{6} E_i$ is G-invariant. Then $\mathrm{lct}(X, G) = 1$ by Lemma 5.1.

EXAMPLE 5.4. Suppose that $X = \mathbb{P}^2$ and $G = \mathrm{A}_5$ such that the subgroup G leaves invariant a smooth conic on $\mathbb{P}^2$. Then $\mathrm{lct}(X, G) = 2/3$ by Lemma 5.1, because $r = 3$, $k = 6$, $m = 2$.

EXAMPLE 5.5. Suppose that $K_X^2 = 6$ and $G = \mathrm{Aut}(X) \cong \mathrm{S}_5$ (see [RS]). Then $r = 1$ and $k > 6$, because the stabilizer of every point induces a faithful two-dimensional linear representation in its tangent space. Then $\mathrm{lct}(X, G) = 2$ by Lemma 5.1, because $m = 2$ (see [RS]).

Even if $h^0(X, \mathcal{O}_X((m-r)H)) \geqslant k$, we still may be able to show that $\mathrm{lct}(X, G) = m/r$.

LEMMA 5.6. *Suppose that X be the cubic surface in $\mathbb{P}^3$ that is given by the equation*

$$x^3 + y^3 + z^3 + t^3 = 0 \subset \mathbb{P}^3 \cong \mathrm{Proj}\big(\mathbb{C}[x, y, z, t]\big)\,,$$

and $G = \mathrm{Aut}(X)$. Then $\mathrm{lct}(X, G) = 4$.

Proof. We have $r = 1$ and $G \cong \mathbb{Z}_3^3 \rtimes \mathrm{S}_4$ (see [DoI]). Then it is easy to check that $m = 4$ and $k = 18$, which implies that we are unable to apply Lemma 5.1 to deduce the equality $\mathrm{lct}(X, G) = 4$.

Suppose that $\mathrm{lct}(X, G) < 4$. Then there is an effective G-invariant $\mathbb{Q}$-divisor D on the cubic surface X such that $\mathrm{LCS}(X, \lambda D) \neq \varnothing$ and $D \equiv -K_X$, where $0 < \lambda \in \mathbb{Q}$ such that $\lambda < 4$.

Arguing as in the proof of Lemma 5.1, we see that the locus $\mathrm{LCS}(X, \lambda D)$ consists of 18 points, because every G-orbit containing at most 20 points must consist of 18 points. Then

$$\mathrm{LCS}(X, \lambda D) = \{O_1, \ldots, O_{18}\}\,,$$

where $O_1, \ldots, O_{18}$ are all Eckardt points of the surface X (see [DoI]).

Let R be a curve on the surface X that is cut out by $xyzt = 0$. Then R is G-invariant, and the log pair (X, R) is log canonical. We may assume that $R \not\subseteq \mathrm{Supp}(D)$ by Remark 2.1. Then

$$12 = R \cdot D \geqslant \sum_{i=1}^{18} \mathrm{mult}_{O_i}(R)\,\mathrm{mult}_{O_i}(D) = \sum_{i=1}^{18} 2\,\mathrm{mult}_{O_i}(D) \geqslant 36\,\mathrm{mult}_{O_i}(D)\,,$$

which implies that $\mathrm{mult}_{O_i}(D) \leqslant 1/3$.

Let $\pi\colon U \to X$ be a blow up of the points $O_1, \dots, O_{18}$. Then

$$K_U + 4\bar{D} + \sum_{i=1}^{18} \big(4\,\mathrm{mult}_{O_i}(D) - 1\big)E_i \equiv \pi^*(K_X + 4D)\,,$$

where E_i is the π-exceptional curve such that $\pi(E_i) = O_i$, and $\bar{D}$ is the proper transform of D on the surface U. Then there is $Q_i \in E_i$ such that $\mathrm{mult}_{Q_i}(\bar{D}) > 1/2 - \mathrm{mult}_{O_i}(D)$ for $i = 1, \dots, 18$.

Let Σ be the G-orbit of the point Q_i. Then $\Sigma \cap E_i \neq Q_i$, because the representation induced by the action of the stabilizer of O_i on its tangent space is irreducible. We have

$$\mathrm{mult}_{O_i}(D) = E_i \cdot \bar{D} > |\Sigma \cap E_i|\big(1/2 - \mathrm{mult}_{O_i}(D)\big)\,,$$

which implies that $|\Sigma \cap E_i| = 1$, because $\mathrm{mult}_{O_i}(D) \leqslant 1/3$. $\square$

Lemma 5.7. *Suppose that $K_X^2 = 5$ and $G = \mathrm{A}_5$. Then* $\mathrm{lct}(X, G) = 2$.

Proof. The surface X is embedded in $\mathbb{P}^5$ by the linear system $|-K_X|$, and X contains 10 lines, which we denote as $L_1, \dots, L_{10}$. Then $r = 1$ and $\mathrm{Aut}(X) \cong \mathrm{S}_5$ (see [RS]).

The divisor $\sum_{i=1}^{10} L_i \sim -2K_X$ is S_5-invariant, which implies that $\mathrm{lct}(X, G) \leqslant 2$.

The surface X can be obtained as a blow up $\pi\colon X \to \mathbb{P}^2$ of the four points

$$P_1 = (1 : -1 : -1)\,, \quad P_2 = (-1 : 1 : -1)\,,$$
$$P_3 = (-1 : -1 : 1)\,, \quad P_4 = (1 : 1 : 1)\,,$$

of the plane $\mathbb{P}^2$. Let W be the curve in $\mathbb{P}^2$ that is given by the equation

$$x^6 + y^6 + z^6 + (x^2 + y^2 + z^2)(x^4 + y^4 + z^4) = 12x^2y^2z^2 \subset \mathbb{P}^2 \cong \mathrm{Proj}\big(\mathbb{C}[x, y, z]\big)\,,$$

and Z be its proper transform on X. Then Z is S_5-invariant (see [IK]) and $Z \sim -2K_X$.

The curves Z and $\sum_{i=1}^{10} L_i$ are the only S_5-invariant curves in $|-2K_X|$.

Let $\mathcal{P}$ be the pencil generated by Z and $\sum_{i=1}^{10} L_i$. It follows from [E] that $\mathcal{P}$ is A_5-invariant, and there are exactly 5 singular curves in $\mathcal{P}$, which can be described in the following way:

- the curve $\sum_{i=1}^{10} L_i$;
- two irreducible rational curves R_1 and R_2 that have 6 nodes;
- two fibers F_1 and F_2 each consisting of 5 smooth rational curves.

We have $m = 2$ and $k = 6$ by [RS]. The smallest G-orbit are $\mathrm{Sing}(R_1)$ and $\mathrm{Sing}(R_2)$ (see [IK]).

Suppose that $\mathrm{lct}(X,G) < 2$. Then there is an effective G-invariant $\mathbb{Q}$-divisor D on the quintic surface X such that $\mathrm{LCS}(X,\lambda D) \neq \varnothing$ and $D \equiv -K_X$, where $0 < \lambda \in \mathbb{Q}$ such that $\lambda < 2$.

We may assume that the support of D does not contain R_1 and R_2 due to Remark 2.1, because both log pairs (X,R_1) and (X,R_2) are log canonical. Now arguing as in the proof of Lemma 5.1, we see that either $\mathrm{LCS}(X,\lambda D) = \mathrm{Sing}(R_1)$ or $\mathrm{LCS}(X,\lambda D) = \mathrm{Sing}(R_2)$.

Without loss of generality we may assume that the locus $\mathrm{LCS}(X,\lambda D)$ consists of the singular points of the curve R_1. Denote them as $O_1,\ldots,O_6$. Then $\mathrm{mult}_{O_i}(D) \leqslant 5/6$, because

$$10 = R_1 \cdot D \geqslant \sum_{i=1}^{6} \mathrm{mult}_{O_i}(D)\,\mathrm{mult}_{O_i}(R_1) \geqslant 12\,\mathrm{mult}_{O_i}(D)\,.$$

Let $\pi\colon U \to X$ be a blow up of the points $O_1,\ldots,O_6$. Then

$$K_U + 2\bar{D} + \sum_{i=1}^{6}\big(2\,\mathrm{mult}_{O_i}(D) - 1\big)E_i \equiv \pi^*(K_X + 2D)\,,$$

where E_i is the π-exceptional curve such that $\pi(E_i) = O_i$, and $\bar{D}$ is the proper transform of D on the surface U. Then $\mathrm{mult}_{Q_i}(\bar{D}) > 1 - \mathrm{mult}_{O_i}(D)$ for some point $Q_i \in E_i$, where $i = 1,\ldots,6$.

Let Σ be the G-orbit of the point Q_i. Then $|\Sigma \cap E_i| \geqslant 2$, because the stabilizer of O_i acts faithfully on its tangent space. We have $|\Sigma \cap E_i| = 2$, because $\mathrm{mult}_{O_i}(D) \leqslant 5/6$ and

$$\mathrm{mult}_{O_i}(D) = E_i \cdot \bar{D} > |\Sigma \cap E_i|\big(1 - \mathrm{mult}_{O_i}(D)\big)\,.$$

Let $\bar{R}_1$ be the proper transform of the curve R_1 on the surface U. Then

$$\Sigma = \bar{R}_1 \bigcap (E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_5)\,,$$

because the orbit of length 2 of the action on E_i of the stabilizer of O_i is unique. We have

$$12\big(1 - \mathrm{mult}_{O_i}(D)\big) = 10 - 2\sum_{i=1}^{6}\mathrm{mult}_{O_i}(D) = \bar{R}_1 \cdot \bar{D} \geqslant 2\bigg(\sum_{i=1}^{6}\mathrm{mult}_{Q_i}(\bar{D})\bigg)$$
$$> 12\big(1 - \mathrm{mult}_{O_i}(D)\big)\,,$$

which is a contradiction. □

Lemma 5.8. *Suppose that $K_X^2 = 5$ and $G = \mathbb{Z}_5$. Then $\mathrm{lct}(X,G) = 4/5$ holds.*

Proof. It is well known that the group G fixes exactly two points of the surfaces X (see [RS]), which we denote as O_1 and O_2. There are five conics $Z_1,\ldots,Z_5\subset X$ that passes through O_1, and the divisor $\sum_{i=1}^{5}Z_i\sim -2K_X$ is G-invariant, which implies that $\mathrm{lct}(X,G)\leqslant 4/5$.

Suppose that $\mathrm{lct}(X,G)<4/5$. Then there is an effective G-invariant $\mathbb{Q}$-divisor D on the quintic surface X such that $\mathrm{LCS}(X,\lambda D)\neq\varnothing$ and $D\equiv -K_X$, where $0<\lambda\in\mathbb{Q}$ such that $\lambda<4/5$.

The proof of Lemma 5.1 implies that $\mathrm{LCS}(X,\lambda D)=\{O_1\}$ or $\mathrm{LCS}(X,\lambda D)=\{O_1\}$.

Without loss of generality, we may assume that $\mathrm{LCS}(X,\lambda D)=\{O_1\}$, and we may assume that the support of the divisor D does not contain the conics $Z_1,\ldots,Z_5$ by Remark 2.1. Then

$$2=Z_1\cdot D\geqslant \mathrm{mult}_{O_1}(D)\,.$$

Let $\pi\colon U\to X$ be a blow up of the point O_1, and E be the π-exceptional curve. Then

$$\mathrm{mult}_Q(\bar{D})\geqslant 2/\lambda-\mathrm{mult}_{O_1}(D)>5/2-\mathrm{mult}_{O_1}(D)$$

for some point $Q\in E$ by Corollary 2.7, where $\bar{D}$ is the proper transform of D on the surface U.

The point Q must be G-invariant, because otherwise

$$\mathrm{mult}_{O_1}(D)=E\cdot\bar{D}>5\big(5/2-\mathrm{mult}_{O_1}(D)\big)\,,$$

which is impossible, because $\mathrm{mult}_{O_1}(D)\leqslant 2$.

Let $\bar{Z}_i$ be the proper transform of the conic Z_i on the surface U. Then $Q\not\in\cup_{i=1}^{5}\bar{Z}_i$, and there is a birational morphism $\phi\colon U\to\mathbb{P}^2$ that contracts the curves $\bar{Z}_1,\ldots,\bar{Z}_5$.

The curve $\phi(E)$ is a conic that contains $\phi(\bar{Z}_1),\ldots,\phi(\bar{Z}_5)$. Let T_i be the proper transform on the surface U of the line in $\mathbb{P}^2$ that passes through the points $\phi(Q)$ and $\phi(\bar{Z}_i)$. The log pair

$$\left(X,\frac{\lambda}{3}\sum_{i=1}^{5}\pi(T_i)\right)$$

has log terminal singularities, and $\sum_{i=1}^{5}\pi(T_i)\equiv 3D$. Thus, we may assume that the support of the divisor $\bar{D}$ does not contain any of the curves $T_1,\cdots,T_5$ due to Remark 2.1. Then

$$3-\mathrm{mult}_{O_1}(D)\geqslant T_i\cdot\bar{D}\geqslant\mathrm{mult}_Q(\bar{D})\,,$$

which implies that $\mathrm{mult}_{O_1}(D)+\mathrm{mult}_Q(\bar{D})\leqslant 3$.

Let $\xi\colon V\to U$ be a blow up of the point Q, and F be the ξ-exceptional divisor. Then

$$K_W + \lambda \grave{D} + \big(\lambda\,\mathrm{mult}_{O_1}(D) - 1\big)\grave{E} + \big(\lambda\,\mathrm{mult}_{O_1}(D) + \lambda\,\mathrm{mult}_Q(\bar{D}) - 2\big)F \equiv (\pi\circ\xi)^*(K_X + \lambda D)\,,$$

where $\grave{D}$ and $\grave{E}$ are proper transforms of D and E on the surface V, respectively. The log pair

$$\big(W, \lambda \grave{D} + (\lambda\,\mathrm{mult}_{O_1}(D) - 1)\grave{E} + (\lambda\,\mathrm{mult}_{O_1}(D) + \lambda\,\mathrm{mult}_Q(\bar{D}) - 2)F\big)$$

is not log terminal at some point $P \in F$ by Remark 2.6, because $\mathrm{mult}_{O_1}(D) \leqslant 2$.

Suppose that $P \in \grave{E}$. Let $\grave{T}$ be the proper transform on V of the line on $\mathbb{P}^2$ that is tangent to the conic $\phi(E)$ at the point $\phi(Q)$. Then $P \in \grave{T}$, which implies that

$$5 - 2\,\mathrm{mult}_{O_1}(D) - \mathrm{mult}_Q(\bar{D}) = \grave{T}\cdot\grave{D} \geqslant \mathrm{mult}_P(\grave{D}) > 5 - 2\,\mathrm{mult}_{O_1}(D) - \mathrm{mult}_Q(\bar{D})\,,$$

because we may assume that $\grave{T} \not\subseteq \mathrm{Supp}(\grave{D})$ by Remark 2.1. Hence, we have $P \notin \grave{E}$.

The log pair $(W, \lambda\grave{D} + (\lambda\,\mathrm{mult}_{O_1}(D) + \lambda\,\mathrm{mult}_Q(\bar{D}) - 2)F)$ is not log terminal at P. But

$$\lambda\grave{D} + \big(\lambda\,\mathrm{mult}_{O_1}(D) + \lambda\,\mathrm{mult}_Q(\bar{D}) - 2\big)F$$

is an effective divisor, because $\mathrm{mult}_Q(\bar{D}) \geqslant 2/\lambda - \mathrm{mult}_{O_1}(D)$. Then

$$\mathrm{mult}_P(\grave{D}) \geqslant 3/\lambda - \mathrm{mult}_{O_1}(D) - \mathrm{mult}_Q(\bar{D}) > 15/4 - \mathrm{mult}_{O_1}(D) - \mathrm{mult}_Q(\bar{D})\,.$$

Let $\grave{T}_i$ be the proper transform of T_i on the surface V. Suppose that $P \in \grave{T}_k$. Then

$$3 - \mathrm{mult}_{O_1}(D) - \mathrm{mult}_Q(\bar{D}) = \grave{T}_k\cdot\grave{D} > 15/4 - \mathrm{mult}_{O_1}(D) - \mathrm{mult}_Q(\bar{D})\,,$$

which is a contradiction. Thus, we see that $P \notin \cup_{i=1}^5 \grave{T}_i$.

Let M be an irreducible curve on V such that $P \in M$, the curve $\phi\circ\xi(M)$ is a line that passes through the point $\phi(Q)$. Then $\pi\circ\xi(M)$ has an ordinary double point at O_1, and $\pi\circ\xi(M) \equiv -K_X$, because $P \notin \cup_{i=1}^5 \grave{T}_i$. We may assume that $M \not\subseteq \mathrm{Supp}(\grave{D})$ by Remark 2.1. Then

$$5 - 2\,\mathrm{mult}_{O_1}(D) - \mathrm{mult}_Q(\bar{D}) = M\cdot\grave{D} > 15/4 - \mathrm{mult}_{O_1}(D) - \mathrm{mult}_Q(\bar{D})\,,$$

which implies that $\mathrm{mult}_{O_1}(D) \leqslant 5/4$. But $\mathrm{mult}_{O_1}(D) > 5/4$. □

We did not prove that groups in Example 5.5 and Lemmata 5.6, 5.7 and 5.8 act on X in such a way that the G-invariant subgroup in $\mathrm{Pic}(X)$ is $\mathbb{Z}$. But the latter is well known (see [DoI]).

6 Direct Products

Let X be an arbitrary smooth Fano variety, and let G be a finite subgroup in $\mathrm{Aut}(X)$ such that the G-invariant subgroup of the group $\mathrm{Pic}(X)$ is $\mathbb{Z}$.

DEFINITION 6.1. The variety X is said to be G-birationally superrigid if for every G-invariant linear system $\mathcal{M}$ on the variety X that does not have any fixed components, the singularities of the log pair $(X, \lambda\mathcal{M})$ are canonical, where $\lambda \in \mathbb{Q}$ such that $\lambda > 0$ and $K_X + \lambda\mathcal{M} \equiv 0$.

The following result is well known (see [M], [DoI]).

LEMMA 6.2. *Suppose that X is a smooth del Pezzo surface such that*

$$|\Sigma| \geqslant K_X^2$$

for any G-orbit $\Sigma \subset X$. Then X is G-birationally superrigid.

Proof. Suppose that the surface X is not G-birationally superrigid. Then there is a G-invariant linear system $\mathcal{M}$ on the surface X such that $\mathcal{M}$ does not have fixed curves, but $(X, \lambda\mathcal{M})$ is not canonical at some point $O \in X$, where $\lambda \in \mathbb{Q}$ such that $\lambda > 0$ and $K_X + \lambda\mathcal{M} \equiv 0$.

Let Σ be the G-orbit of the point O. Then $\mathrm{mult}_P(\mathcal{M}) > 1/\lambda$ for every point $P \in \Sigma$. Then

$$K_X^2/\lambda^2 = M_1 \cdot M_2 \geqslant \sum_{P\in\Sigma} \mathrm{mult}_P^2(\mathcal{M}) > |\Sigma|/\lambda^2 \geqslant K_X^2/\lambda^2,$$

where M_1 and M_2 are sufficiently general curves in $\mathcal{M}$. □

EXAMPLE 6.3. Let X be a smooth del Pezzo surface such that $K_X^2 = 5$. Then $\mathrm{Aut}(X) \cong \mathrm{S}_5$, and the proof of Lemma 5.7 implies that the surface X is A_5-birationally superrigid by Lemma 6.2.

Let X_i be a smooth G_i-birationally superrigid Fano variety, where G_i is a an arbitrary finite subgroup of $\mathrm{Aut}(X_i)$ such that the G_i-invariant subgroup of $\mathrm{Pic}(X_i)$ is $\mathbb{Z}$, and $i = 1, \ldots, r$.

Theorem 6.4. *Suppose that $\mathrm{lct}(X_i, G_i) \geqslant 1$ for every $i = 1, \ldots, r$. Then*

- *there is no $G_1 \times \cdots \times G_r$-equivariant birational map $\rho\colon X_1 \times \cdots \times X_r \dashrightarrow \mathbb{P}^n$;*
- *every $G_1 \times \cdots \times G_r$-equivariant birational automorphism of $X_1 \times \cdots \times X_r$ is biregular;*
- *for any $G_1 \times \cdots \times G_r$-equivariant dominant map $\rho\colon X_1 \times \cdots \times X_r \dashrightarrow Y$, whose general fiber is rationally connected, there a commutative*

diagram

$$\begin{tikzcd} X_1 \times \cdots \times X_r \arrow[d, "\pi"'] \arrow[drr, dashed, "\rho"] & & \\ X_{i_1} \times \cdots \times X_{i_k} \arrow[rr, dashed, "\xi"'] & & Y \end{tikzcd}$$

where ξ is a birational map, π is a natural projection, and $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$.

Proof. The required assertion follows from the proof of Theorem 1 in [Pu]. $\square$

EXAMPLE 6.5. The simple group A_6 is a group of automorphisms of the sextic

$$10x^3y^3 + 9zx^5 + 9zy^5 + 27z^6 = 45x^2y^2z^2 + 135xyz^4 \subset \mathbb{P}^2 \cong \mathrm{Proj}\,\big(\mathbb{C}[x, y, z]\big)$$

and there is an embedding $\mathrm{A}_6 \subset \mathrm{Aut}(\mathbb{P}^2)$ such that $\mathrm{lct}(\mathbb{P}^2, \mathrm{A}_6) = 2$ by Lemma 5.1 (see [Cr]), and $\mathrm{A}_6 \times \mathrm{A}_6$ acts naturally on $\mathbb{P}^2 \times \mathbb{P}^2$. There is an induced embedding $\mathrm{A}_6 \times \mathrm{A}_6 \cong \Omega \subset \mathrm{Bir}(\mathbb{P}^4)$ such that Ω is not conjugated to a subgroup of $\mathrm{Aut}(\mathbb{P}^4)$ by Lemma 6.2 and Theorem 6.4.

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