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SUPERRIGIDITY, GENERALIZED HARMONIC MAPS AND UNIFORMLY CONVEX SPACES

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Abstract. We prove several superrigidity results for isometric actions on Busemann non-positively curved uniformly convex metric spaces. In particular we generalize some recent theorems of N. Monod on uniform and certain non-uniform irreducible lattices in products of locally compact groups, and we give a proof of an unpublished result on commensurability superrigidity due to G.A. Margulis. The proofs rely on certain notions of harmonic maps and the study of their existence, uniqueness, and continuity.

Ever since the first superrigidity theorem for linear representations of irreducible lattices in higher rank semisimple Lie groups was proved by Margulis in the early 1970s, see [M3] or [M2], many extensions and generalizations were established by various authors, see for example the exposition and bibliography of [J] as well as [P]. A superrigidity statement can be read as follows:
Let

- G be a topological group,
- Γ a subgroup of G,
- H another topological group, and
- $f : \Gamma \to H$ a homomorphism.

Then, under some certain conditions on G, Γ, H and f, the homomorphism f extends uniquely to a continuous homomorphism $F : G \to H$. In case $H = \mathrm{Isom}(X)$ is the group of isometries of some metric space X, the

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conditions on H and f can be formulated in terms of X and the action of Γ on X.

In the original superrigidity theorem [M1] it was assumed that G is a semisimple Lie group of real rank at least two and $\Gamma \leq G$ is an irreducible lattice. (Superrigidity theorems were proved later also for lattices in the rank one Lie groups $\mathrm{SP}(n,1), F_4^{-20}$ see [C] and [GS]. It seems however that the same phenomenon holds in these cases for different reasons.) It is not clear how to define a rank for a general topological group. One natural extension, although not a generalization, of the notion of higher rank is the assumption that G is a non-trivial product. Margulis [M1] also proved a superrigidity theorem for commensurability subgroups in semisimple Lie groups. The target in these superrigidity theorems was the group of isometries of a Riemannian symmetric space of non-compact type or an affine building.

It was later realized in an unpublished manuscript of Margulis [M4] which was circulated in the 1990s (cf. [J]), that superrigidity for commensurability subgroups extends to a very general setting: a general locally compact, compactly generated G and a target group being the isometry group of a complete Busemann non-positively curved uniformly convex metric space.

In this paper we establish quite general superrigidity theorems for actions of irreducible lattices in products of locally compact groups on Busemann non-positively curved uniformly convex metric spaces. We also include a proof of the unpublished result for commensurability subgroups mentioned above, since the methods are similar.

Our method relies on certain notions of generalized harmonic maps. The main part is the proof of their existence (Theorem 3.2, Proposition 8.2) which is of independent interest and may have other applications. Once the existence is established the superrigidity results follow from the nice properties of these maps.

Our results for lattices in products generalize recent theorems of N. Monod [Mo] for actions on CAT(0) spaces. Loosely speaking, the argument of Monod [Mo] is divided into three steps: 1. Inducing the lattice action to an action of the ambient group on the space of square integrable equivariant maps. 2. Proving a splitting theorem for actions of product groups on CAT(0) spaces (generalizing earlier results for Riemannian manifolds and general proper CAT(0) spaces, cf. [BrH, p. 239]). 3. Using the splitting of the induced space to obtain an invariant subset of the original

space on which the lattice action extends. The proof we give here is in a sense more direct and therefore applies in a more general setup where the splitting result does not hold. The spaces considered in the current paper, namely Busemann non-positively curved uniformly convex metric spaces, generalize CAT(0) spaces in a similar manner as (uniformly convex) Banach spaces generalize Hilbert spaces and (uniformly convex NPC) Finsler manifolds generalize NPC Riemannian manifolds. In particular, CAT(0) spaces are BNPC and UC, but these conditions are much weaker than CAT(0), for instance Hilbert spaces, while being the least convex among CAT(0) spaces, are the most convex among UC Banach spaces (we refer the reader to [H], [N], [U] for examples). There are several technical difficulties that could be avoided by requiring stronger assumptions on the spaces considered (for example, Proposition 2.13 with $p = 2$ is obvious for CAT(0) spaces). In particular, when assuming the CAT(0) condition, our proof is significantly simplified. Note that we also obtain some new results for actions on CAT(0) spaces (cf. Theorem 1.4).

Sections 1 through 7 deal with lattices in products, and section 8 deals with commensurability subgroups. The argument in section 8 is slightly simpler and more detailed than the original proof given in [M4], however we assume here a slightly stronger convexity assumption on the space X.

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1 Assumptions and Conclusion

We shall first discuss some properties of the groups, the spaces and the actions under consideration, and then state our main results.

By saying that a lattice Γ in a locally compact group with a specified decomposition $G = G_1 \times \dots G_n$ is *irreducible*, we mean that $\Gamma\big(\prod_{j\neq i} G_j\big)$ is dense in G for each $1 \leq i \leq n$. (Note that the projection of Γ to a sub-product $\prod_{j\in J} G_j$ with $|J| > 1$ is not necessarily dense.)

A complete geodesic metric space X is said to be *Busemann non-positively curved* (BNPC), if the distance between any two constant speed geodesics is a convex function. In particular, X is a uniquely geodesic space, i.e. any two points $x, y \in X$ are joined by a unique arc whose length is $d(x, y)$. We shall denote the midpoint of x and y by $\frac{x+y}{2}$.

A uniquely geodesic metric space X is said to be *strictly convex* if $d\big(x, \frac{y_1+y_2}{2}\big) < \max\{d(x,y_1), d(x,y_2)\}$, $\forall x, y_1, y_2 \in X$ with $y_1 \neq y_2$. We shall say that X is *weakly uniformly convex* (WUC) if additionally for any $x \in X$, the modulus of convexity function

$$\delta_x(\epsilon, r) := \inf\left\{ r - d\left(x, \frac{y_1+y_2}{2}\right) : y_i \in X\,,\ d(x,y_i) \leq r\,,\ d(y_1,y_2) \geq \epsilon r \right\}$$

is positive for any $\epsilon, r > 0$. We shall say that X is *uniformly convex* (UC) if

$$\forall \epsilon > 0\,,\ \exists \delta(\epsilon) > 0 \quad \text{such that} \quad \forall r > 0\,,\ x \in X\,,\ \delta_x(\epsilon, r) \geq \delta(\epsilon) \cdot r\,.$$

We do not know an example of a BNPC WUC space which is not UC, and it is conceivable that BNPC and WUC imply UC. We will allow ourselves to assume whenever we find it convenient that the metric space under consideration is UC, although some of our results can be proved under weaker convexity assumptions. For instance Theorem 8.1 was proved in [M4] under the assumption that X is BNPC and WUC. All the main results stated below remain true if one assumes only that X is uniformly convex with respect to some point $x_0 \in X$, i.e. that $\inf_{r>0} \frac{1}{r}\delta_{x_0}(\epsilon, r) > 0$, $\forall \epsilon > 0$ (and the proofs require only minor changes).

The *projection* of a point $x \in X$ to a closed convex set $C \subset X$ is the point $p \in C$ closest to x. It is a standrd fact that projections exists uniquely in spaces which are weakly uniformly convex and complete.

Suppose that Γ acts on X by isometries. The action is called *C-minimal* if there is no non-empty closed convex proper Γ-invariant subset of X.

For any subset $\Sigma \subset \Gamma$ we associate the displacement function,

$$d_\Sigma(x) := \max_{\sigma \in \Sigma} d(\sigma \cdot x, x)\,.$$

We shall say that d_Σ *goes to infinity* and write $d_\Sigma \to \infty$ if $\lim_{x\to\infty} d_\Sigma(x) = \infty$, where $x \to \infty$ means that x eventually gets out of any ball.

Following [Mo], we shall say that an action $\Gamma \circlearrowleft X$ is *reduced* if there is no unbounded closed convex proper subset $Y \subset X$ which is of finite Hausdorff distance from $\gamma \cdot Y$ for any $\gamma \in \Gamma$.

Recall that a *Clifford isometry* T of a metric space X is a surjective isometry $T : X \to X$ for which $d(x, T(x))$ is constant on X. A necessary assumption in some of the superrigidity theorems below is that X has no non-trivial Clifford isometries. However, for typical spaces, this assumption follows from reduceness of $\mathrm{Isom}(X)$ (cf. Corollary 2.6).

The assumption that the action is reduced is very strong, and by requiring it we can prove a quite general statement.

Theorem 1.1. *Let $G = G_1 \times \ldots \times G_n$, $n > 1$ be a locally compact compactly generated topological group, and Γ an irreducible uniform lattice in G. Let X be a complete Busemann non-positively curved uniformly convex metric space without non-trivial Clifford isometries. Assume that Γ acts on X by isometries and that this action is reduced and has no global fixed point. Then the action of Γ extends uniquely to a continuous G-action, and this G-action factors through one of the G_i's.*

REMARK 1.2. (i) In section 7 we shall give a generalization of Theorem 1.1 for non-uniform lattices which are p-integrable and weakly cocompact.

(ii) The theorem also remains true when G is not assumed to be compactly generated but only a σ-compact group if we add the assumption that $d_\Sigma \to \infty$ for some finite subset $\Sigma \subset \Gamma$. (One should only choose the function h in the definition of the energy E below more carefully so that the energy of some Γ-equivariant map (in $L_2(\Gamma\backslash G, X)$) would be finite.) Note however that when G is not compactly generated, Γ is not finitely generated. Hence this additional assumption is perhaps not very natural.

Even when an action is reduced, it is not clear how to verify this. Moreover, in many cases, the action can be extended under weaker assumptions. By requiring stronger assumptions on the space X we can drop this assumption. The following two theorems imply (and generalize) Margulis' original theorem in the case where G is not simple. For an explanation why Theorem 1.3 implies Margulis' theorem, we refer to [Mo] where the same result is proved under the assumption that X is CAT(0). Note that Remark 1.2 applies also for Theorems 1.3 and 1.4.

Theorem 1.3. *Let G, Γ be as in Theorem 1.1, let X be a complete BNPC uniformly convex space, and assume further that X is proper (i.e. the closed balls in X are compact). Assume that Γ acts on X without a global fixed point. Then there is a non-empty closed invariant subset $\mathcal{L}$ of the visual boundary ∂X (cf. [Pa] for definition) on which the Γ action extends to a continuous G action which factors through some G_i.*

We will say that the space X is *geodesically complete* (resp. *uniquely geodesically complete*) if any geodesic segment in X is contained in a (unique) two-sided infinite geodesic.

Theorem 1.4. *Let G and Γ be as above, and let Σ be a finite generating set of Γ. Let X be a complete uniquely geodesically complete CAT(0) space without Euclidean factors with the additional assumption that if two geodesic segments are parallel then the corresponding geodesics are parallel.*

Suppose that Γ acts on X by isometries with $d_\Sigma \to \infty$. Then there is a Γ-invariant geodesically complete closed subset $Y \subset X$, such that the Γ-action on Y extends to a continuous G-action. Moreover, Y can be decomposed to an invariant direct product $Y = Y_1 \times \ldots \times Y_n$ such that each G_i acts trivially on each Y_j with $j \neq i$.

REMARK 1.5. (1) Recall that a CAT(0) space X admits a non-trivial Clifford isometry iff it has an Euclidean factor, i.e. can be decomposed as a direct product $X' \times \mathbb{R}$ (cf. [BrH, p. 235]).

(2) If X is CAT(0) and the distance function between any two geodesics is analytic (except at intersection points) then all assumptions on X (in 1.4) are satisfied.

(3) Trees and buildings are usually not uniquely geodesically complete.

2 Some Remarks about the Assumptions

2.1 Parallel segments. We shall say that two segments $[a, b], [x, y] \subset X$ are parallel, and write $[a, b] \parallel [x, y]$, if

$$d(a, x) = d(b, y) = d\left(\frac{a+b}{2}, \frac{x+y}{2}\right).$$

Recall the following fact:

LEMMA 2.1 (Busemann [Bu, Th. 3.14]). *Whenever X is BNPC*

$$[a, b] \parallel [x, y] \Leftrightarrow [a, x] \parallel [b, y].$$

2.2 Intersection property for convex sets. A real valued function on X is called *convex* if it is convex in the usual sense when restricted to any geodesic segment, i.e. if its sub-level sets are convex.

LEMMA 2.2. *Let X be a weakly uniformly convex complete metric space. Then any collection of closed bounded convex sets has the finite intersection property. If f is a convex function on X which satisfies $f(x) \to \infty$ when $x \to \infty$, then f attains a minimum in X.*

Proof. To prove the first claim let C_α be a descending net of closed convex sets which are contained in some ball. Fix $x_0 \in X$ and let x_α be the projection of x_0 to C_α. Then $d(x_0, x_\alpha)$ is a non-decreasing bounded net of non-negative numbers and hence has a limit. If this limit is 0 then x_0 belongs to the intersection $\cap C_\alpha$, and if it is positive one shows that x_α must be a Cauchy net as follows: if $\beta > \alpha$ then x_β belongs to C_α and if α is "large" then $d(x_0, x_\beta)$ is "almost" the same as $d(x_0, x_\alpha)$ and hence $d(x_\beta, x_\alpha)$ must

be "small" for otherwise $\frac{x_\alpha+x_\beta}{2}$ would be closer to x_0 than x_α, by positivity of the modulus of convexity function. Clearly $\lim x_\alpha \in \cap C_\alpha$, and hence $\cap C_\alpha \neq \emptyset$.

The second claim follows from the first by taking the convex sets to be non-empty sub-level sets of f. □

2.3 Linear growth of convex functions. The following lemma will be used in the proof of the existence of a harmonic map.

LEMMA 2.3. *Let X be a geodesic metric space and let $f : X \to \mathbb{R}$ be a convex function. Assume that $f(x) \to \infty$ when $x \to \infty$. Then there exists $b > 0$ such that $f(x) \geq b \cdot d(x, x_0) - \frac{1}{b}$, $\forall x \in X$. Moreover if X is weakly uniformly convex and $f(x) > 0$, $\forall x \in X$, then there exist $b > 0$ such that $f(x) > b \cdot d(x_0, x)$, $\forall x \in X$.*

Proof. The second statement follows from the first using Lemma 2.2. Now, assuming the contrary, there must be a sequence x_n in X such that

- $f(x_n) \leq \frac{1}{n} d(x_n, x_0)$, and
- $d(x_n, x_0) \geq n^2$.

Take y_n to be the point of distance n from x_0 on the geodesic segment $[x_0, x_n]$. Then $y_n \to \infty$ but

$$f(y_n) \leq \frac{d(y_n, x_n) f(x_0) + d(y_n, x_0) f(x_n)}{d(x_0, x_n)} \leq f(x_0) + \frac{\frac{d(y_n, x_0)}{n} d(x_0, x_n)}{d(x_0, x_n)} = f(x_0) + 1 .$$

This however contradicts the assumption that $f(x) \to \infty$ when $x \to \infty$. □

2.4 Spaces with Clifford isometries. For spaces with Clifford isometries, the analog of Theorem 1.1 is not true as shown by the simple example from [Mo] where G is the discrete group $(\mathbb{Z}/(2) \ltimes \mathbb{Z}) \times (\mathbb{Z}/(2) \ltimes \mathbb{Z})$ and Γ the index 2 irreducible subgroup $\mathbb{Z}/(2) \ltimes (\mathbb{Z} \oplus \mathbb{Z})$, and Γ acts on $\mathbb{R}$, each $\mathbb{Z}$ by translation and the order 2 element by reflection through 0; this action does not extend to G.

The typical example of a space with many Clifford isometries is a Banach space. A weaker superrigidity result for isometric and, more generally, for uniformly bounded affine actions of irreducible lattices on uniformly convex Banach spaces was proved in [BGFM] (cf. Theorem D there).

On the other hand, spaces with non-trivial Clifford isometries admit a canonical non-trivial invariant metric foliation and one can prove superrigidity theorems for the induced action on the space of leaves. Moreover, an action on such a space cannot be reduced, unless it is a Banach space.

PROPOSITION 2.4. *Let X be a complete BNPC metric space, and assume that the set $\mathrm{CL}(X)$ of all Clifford isometries of X form a subgroup of $\mathrm{Isom}(X)$. Then $\mathrm{CL}(X)$ is normal and abelian. The orbits of $\mathrm{CL}(X)$ are all isometric to some fixed uniformly convex real Banach space. The quotient space $X/\mathrm{CL}(X)$ is BNPC. The induced action of $\mathrm{Isom}(X)$ on $X/\mathrm{CL}(X)$ is by isometries. Furthermore, if X is uniformly convex then so is $X/\mathrm{CL}(X)$.*

Proof. It follows from Lemma 2.1 that $S^{-1}T^{-1}ST(x) = x$, $\forall x \in X$, and for arbitrary $S, T \in \mathrm{CL}(X)$, hence $\mathrm{CL}(X)$ is abelian. Clearly $\mathrm{CL}(X) \triangleleft \mathrm{Isom}(X)$. To give the structure of a Banach space to $\mathrm{CL}(X) \cdot x$ note that it follows from our assumptions and Lemma 2.1 that $T^{1/2}(x) := \frac{x+T(x)}{2}$ is also a Clifford isometry and hence we can define a multiplication by a dyadic number, and by continuity we can define a multiplication by any real number. It is also easy to verify that $X/\mathrm{CL}(X)$ is BNPC, and also uniformly convex if X is. Any isometry T induces a 1-Lipschitz map on $X/\mathrm{CL}(X)$, which forces it, as T^{-1} is also a 1-Lipschitz, to be an isometry. □

REMARK 2.5. The assumption that $\mathrm{CL}(X)$ is a group holds in many cases, e.g. it holds for CAT(0) spaces (cf. [BrH, p. 235]), as well as spaces for which parallelity of segments is a transitive relation, i.e. $([a_1, b_1] \parallel [a_2, b_2]) \wedge ([a_2, b_2] \parallel [a_3, b_3]) \Rightarrow [a_1, b_1] \parallel [a_3, b_3]$.

COROLLARY 2.6. *Let X be a BNPC complete metric space which is not isometric to a Banach space, and assume that $\mathrm{CL}(X)$ is a group. Suppose that $\mathrm{CL}(X)$ is non-trivial. Then the action of $\mathrm{Isom}(X)$ on X is not reduced.*

Proof. Choose $y \in X$ and let $Y = \mathrm{CL}(X) \cdot y$. Then Y is a closed convex unbounded proper subset which is equidistant to its image under any isometry. □

Still, for spaces with Clifford isometries one can prove the following:

Theorem 2.7. *Let Γ and G be as in Theorem 1.1, and let X be a complete BNPC uniformly convex metric space such that $\mathrm{CL}(X)$ is a group. Assume that Γ acts on X with $d_\Sigma \to \infty$ where $\Sigma \subset \Gamma$ is a finite generating set, and that the induced action of Γ on $X/\mathrm{CL}(X)$ is reduced. Then the Γ action on $X/\mathrm{CL}(X)$ extends uniquely to a continuous G action which factors through some G_i.*

We shall not elaborate on the proof of Theorem 2.7, which requires only a small modification in step (2) of the proof of Theorem 1.1.

2.5 Reduced actions, displacement functions and C-minimality. Suppose that a group Γ is generated by a finite set Σ and acts by isometries on a space X.

LEMMA 2.8. *Assume that X is a BNPC space which is not isomorphic to a Banach space and that* $\mathrm{CL}(X)$ *is trivial. If the action is reduced then the displacement function* $d_\Sigma \to \infty$.

Proof. Since $\mathrm{CL}(X)$ is trivial, if $\gamma \in \Sigma$ acts non-trivially on X, $d_\gamma(x) = d(x, \gamma\cdot x)$ is non-constant. This implies that the function d_Σ is non-constant. To see this one can argue by contradiction as follows: Suppose $d_\Sigma(x) \equiv m$, let $x \in X$ be a point where $J(x) = |\{\gamma \in \Sigma : d_\gamma(x) = m\}|$ is minimal, let $\gamma \in \Sigma$ be such that $d_\gamma(x) = m$ and let $y \in X$ be another point where $d_\gamma(y) < m$. Then $J\big(\frac{x+y}{2}\big) \leq J(x) - 1$ in contrary to the minimality of $J(x)$.

Now, since X is BNPC the function d_Σ is convex, being the maximum of the convex functions $\{d_\gamma : \gamma \in \Sigma\}$. Since the action is reduced, each proper sub-level set $\{x \in X : d_\Sigma(x) \leq a\}$, $a > 0$, is bounded, being convex and of bounded Hausdorff distance from its translation by any $\gamma \in \Gamma = \langle\Sigma\rangle$. On the other hand, one can show that a bounded non-constant convex function on an unbounded geodesic space must have an unbounded proper sub-level set. We may of course assume that X is unbounded for otherwise there is nothing to prove. It follows that d_Σ is unbounded with bounded sub-level sets, i.e. that $d_\Sigma \to \infty$. □

REMARK 2.9. In view of Corollary 2.6, the assumption that $\mathrm{CL}(X)$ is trivial in Lemma 2.8 can be replaced by the assumption that it is a group.

LEMMA 2.10. *Assume that X is complete, BNPC and uniformly convex. If $d_\Sigma \to \infty$ then there exists a minimal closed convex invariant subset.*

Proof. We may assume that there is no global fixed point. Let C_α be a descending chain of closed convex invariant sets. We claim that the intersection $\cap C_\alpha$ is non-empty. By Lemma 2.2 it is enough to show that they all intersect some given ball. Fix $x_0 \in X$ and let x_α be the projection of x_0 to C_α. Let b be the constant from Lemma 2.3 applied to the convex function d_Σ and let $a = d_\Sigma(x_0)$. Since X is uniformly convex there is $R > 0$ such that $\delta_{x_0}(b, r) > a$ for all $r \geq R$. We claim that $d(x_0, x_\alpha) \leq R$. Suppose in contrary that $d(x_0, x_\alpha) > R$, and let $\gamma \in \Sigma$ be such that $d(\gamma \cdot x_\alpha, x_\alpha) \geq b \cdot d(x_0, x_\alpha)$. Since $d(x_0, \gamma\cdot x_\alpha) \leq a + d(x_0, x_\alpha)$ this implies that $d\big(x_0, \frac{x_0+\gamma\cdot x_\alpha}{2}\big) < d(x_0, x_\alpha)$, contrary to the definition of x_α. Thus, we can apply Zorn lemma and conclude that there is a minimal invariant closed convex set. □

From the existence and uniqueness of circumcenters in weakly uniformly convex, complete spaces, it follows that a bounded minimal invariant convex set must be a point. Indeed, the projection of the circumcenter of a closed

bounded invariant convex set to the set is a fixed point. (Note that if X is not CAT(0), the circumcenter of a closed bounded convex set may lie outside the set.)

Therefore,

COROLLARY 2.11. *Assume that X is complete, BNPC and uniformly convex. If the action of Γ on X is reduced, and has no global fixed points, then it is also C-minimal.*

By the exact same argument as in the proof of Lemma 2.10 one can show

LEMMA 2.12. *Assume that X is a complete, uniquely geodesically complete, BNPC and uniformly convex. If $d_\Sigma \to \infty$ then there exists a minimal closed geodesically complete invariant subset.*

2.6 Spaces of p-integrable maps. Let (Ω, μ) be a probability space and let $1 < p < \infty$. We denote by $L_p(\Omega, X)$ the space of measurable maps $\varphi : \Omega \to X$ which satisfy $\int_\Omega d(\varphi(w), x_0)^p < \infty$ for $x_0 \in X$ with the distance

$$\rho(\varphi, \psi) = \left(\int_\Omega d\big(\varphi(w), \psi(w)\big)^p \right)^{1/p}.$$

Obviously, the finiteness of this integral is independent of the choice of x_0.

We shall make use of the following proposition whose technical proof might be skipped at first reading.

PROPOSITION 2.13. *Suppose that X is complete BNPC and uniformly convex. Then, for any $1 < p < \infty$, the space $L_p(\Omega, X)$ is also complete BNPC and uniformly convex.*

Proof. The completeness of $L_p(\Omega, X)$ follows from that of X by a straightforward argument. Similarly the BNPC property follows from that of X since f is the midpoint of $g, h \in L_p(\Omega, X)$ iff $f(w)$ is the midpoint of $g(w)$ and $h(w)$ in X for almost every $w \in \Omega$.

Let us show that $L_p(\Omega, X)$ is uniformly convex. Let $\delta(\epsilon) > 0$ be the associated constant in a linear lower bound for the modulus of convexity $\delta_x(\epsilon, r)$. We may assume that $\delta(\epsilon/4)$ is sufficiently small compared with ϵ to satisfy inequality (3), as well as the last calculation in the proof. Let β_p be the modulus of convexity function of the Banach space $L_p(0, 1)$. Set

$$\tau(\epsilon) = \beta_p\big(\delta^4(\epsilon/4)\big).$$

We claim that if $\delta(\epsilon/4)$ is chosen small enough then, for any $\psi, \varphi_1, \varphi_2 \in L_p(\Omega, X)$ which satisfy $\rho(\varphi_i, \psi) \le r$, $i = 1, 2$ and $\rho(\varphi_1, \varphi_2) \ge \epsilon r$, we have

$\rho\big(\frac{\varphi_1+\varphi_2}{2},\psi\big) \le r(1-\tau(\epsilon))$. To see this, note first that if

$$\left(\int_\Omega \big|d(\varphi_1(w),\psi(w)) - d(\varphi_2(w),\psi(w))\big|^p\right)^{1/p} \ge \delta^4(\epsilon/4)r\,, \tag{1}$$

then the this claim follows from the uniform convexity of $L_p(0,1)$, since by BNPC,

$$d\left(\frac{\varphi_1(w)+\varphi_2(w)}{2},\psi(w)\right) \le \frac{d(\varphi_1(w),\psi(w)) + d(\varphi_2(w),\psi(w))}{2}\,, \quad \forall w\in\Omega\,.$$

We shall therefore assume below that inequality (1) does not hold. Set

$$\Omega' := \big\{w\in\Omega : |d(\varphi_1(w),\psi(w)) - d(\varphi_2(w),\psi(w))| \ge \delta^2(\epsilon/4)d(\varphi_1(w),\psi(w))\big\}\,,$$

then the negation of inequality (1) implies

$$\left(\int_{\Omega'} d\big(\varphi_1(w),\psi(w)\big)^p\right)^{1/p} < \delta^2(\epsilon/4)r\,, \tag{2}$$

and hence, using the negation of (1) again,

$$\left(\int_{\Omega'} d\big(\varphi_2(w),\psi(w)\big)^p\right)^{1/p} < \left(\delta^2\left(\tfrac{\epsilon}{4}\right) + \delta^4\left(\tfrac{\epsilon}{4}\right)\right) r\,.$$

Thus

$$\int_{\Omega\setminus\Omega'} d\big(\varphi_1(w),\varphi_2(w)\big)^p \ge \left(\epsilon^p - \left(2\delta^2\left(\tfrac{\epsilon}{4}\right) + \delta^4\left(\tfrac{\epsilon}{4}\right)\right)^p\right) r^p \ge \left(\tfrac{\epsilon}{2}r\right)^p\,. \tag{3}$$

Now let

$$\Omega'' = \big\{w\in\Omega\setminus\Omega' : d\big(\varphi_1(w),\varphi_2(w)\big) \ge \tfrac{\epsilon}{4}d\big(\varphi_1(w),\psi(w)\big)\big\}\,.$$

Then from inequality (3) it follows that

$$\left(\int_{\Omega''} d(\varphi_1(w),\psi(w))^p\right)^{1/p} \ge \frac{\epsilon}{4}r\,. \tag{4}$$

Thus, by uniform convexity of X

$$\left(\int_{\Omega''} d\left(\frac{\varphi_1+\varphi_2}{2},\psi\right)^p\right)^{1/p} \le \left(\int_{\Omega''} d(\varphi_1,\psi)^p\right)^{1/p} - \left(\frac{\epsilon}{4}\delta\left(\frac{\epsilon}{4}\right) - \delta^4\left(\frac{\epsilon}{4}\right)\right) r\,.$$

Therefore, assuming $\delta(\epsilon/4)$ is small enough, we have

$$\begin{aligned}
&\int_\Omega \left(d(\varphi_1,\psi)^p - d\left(\frac{\varphi_1+\varphi_2}{2},\psi\right)^p\right)\\
&\quad= \left(\int_{\Omega''} + \int_{\Omega\setminus\Omega''}\right)\left(d(\varphi_1,\psi)^p - |, d\left(\frac{\varphi_1+\varphi_2}{2},\psi\right)^p\right)\\
&\quad\ge \int_{\Omega''}\left(d(\varphi_1,\psi)^p - d\left(\frac{\varphi_1+\varphi_2}{2},\psi\right)^p\right) - \int_{\Omega\setminus\Omega''}\big(d(\varphi_2,\psi)^p - d(\varphi_1,\psi)^p\big)\vee 0
\end{aligned}$$

$$\geq \left(1-\left[1-\tfrac{\epsilon}{4}\delta\left(\tfrac{\epsilon}{4}\right)+\delta^4\left(\tfrac{\epsilon}{4}\right)\right]^p\right)\left(\tfrac{\epsilon}{4}r\right)^p - p\delta^4\left(\tfrac{\epsilon}{4}\right)r^p \geq \delta^2\left(\tfrac{\epsilon}{4}\right)r^p.$$

It follows that $\left(\int_\Omega d\left(\frac{\varphi_1+\varphi_2}{2},\psi\right)^p\right)^{1/p} \leq \left(1-\delta^2\left(\frac{\epsilon}{4}\right)\right)^{1/p} r$. Finally, if $\delta\left(\frac{\epsilon}{4}\right)$ is sufficiently small then $\left(1-\delta^2\left(\frac{\epsilon}{4}\right)\right)^{1/p} \leq 1-\delta^4\left(\frac{\epsilon}{4}\right)$, and since $\beta_p\left(\delta^4\left(\frac{\epsilon}{4}\right)\right) \leq \delta^4\left(\frac{\epsilon}{4}\right)$ this completes the proof. □

Arguing similarly, and avoiding the part of Ω where both φ_i are close to ψ one can show:

PROPOSITION 2.14. *If X is uniformly convex for large distances, i.e. for some (and hence any) $x \in X$, $\lim_{r\to\infty}\frac{1}{r}\delta(\epsilon,r) > 0$ for any $\epsilon > 0$, then so is $L_p(\Omega, X)$.*

Since we will not make use of this variant of Proposition 2.13, we shall not elaborate on its proofs.

REMARK 2.15. It is straightforward that if X is CAT(0) then so is $L_2(\Omega, X)$. Thus for CAT(0) spaces one can avoid the technical Proposition 2.13.

3 Definition and Existence of Generalized Harmonic Maps

Let $G = G_1 \times G_2$ be a compactly generated locally compact group and $\Gamma \leq G$ a uniform lattice. Let Ω be a measurable relatively compact right fundamental domain for Γ, i.e. $G = \bigsqcup_{\gamma\in\Gamma}\gamma\cdot\Omega$. Assume that X is a complete BNPC uniformly convex metric space, and that Γ acts by isometries on X with $d_\Sigma \to \infty$, where Σ is a finite generating set for Γ containing the identity. (Note that in this section, we neither require that the action $\Gamma \circlearrowleft X$ is reduced, nor that $\mathrm{CL}(X)$ is trivial, nor that $\Gamma \leq G$ is irreducible.)

A function $\varphi : G \to X$ is said to be Γ-*equivariant* if $\varphi(\gamma g) = \gamma\cdot\varphi(g)$ for any $\gamma \in \Gamma, g \in G$. Since such a function is determined by its restriction to Ω we will abuse notation and make no distinction between Γ-equivariant functions and their restriction to Ω. In particular, by $L_2(\Omega, X)$ we mean the space of Γ-equivariant measurable maps $\Omega \to X$ whose restriction to Ω is square integrable (see 2.6).

Fix a compact generating set K of G_1 and define $h : G_1 \to \mathbb{R}$ by $h(g_1) := e^{-|g_1|_K^2+1}$ where $|\ |_K$ is the word norm associated to K, i.e.

$$|g_1|_K = \min\{k : K^k \ni g_1\}, \quad g_1 \in G_1.$$

For convenience we will assume that K contains the compact set $U_1 \subset G_1$ defined in the proof of Proposition 3.3 below.

We define the (leafwise G_1-) energy of a Γ-equivariant function $\varphi : G \to X$ to be

$$E(\varphi)=\int_{\Omega\times G_1} h(g_1)d\big(\varphi(\omega),\varphi(\omega g_1)\big)^2=\int_{(\Gamma\backslash G)\times G_1} h(g_1)d\big(\varphi(g),\varphi(gg_1)\big)^2.$$

Note that the energy E is convex and G_2-invariant from the right, i.e. $E(\varphi)=E(\varphi(\cdot g_2))$ for any $g_2\in G_2$. It is also easy to check that if $\varphi\in L_2(\Omega,X)$ then $E(\varphi)<\infty$ and that E is continuous on $L_2(\Omega,X)$. Let $M=\inf\{E(\varphi):\varphi\in L_2(\Omega,X)\}$.

DEFINITION 3.1. $\varphi\in L_2(\Omega,X)$ is called harmonic if $E(\varphi)=M$.

Theorem 3.2. *There exists a harmonic map.*

If Γ has a global fixed point y_0 then the constant map $\varphi(g)\equiv y_0$ is Γ-equivariant with energy 0, and hence harmonic. For the rest of this section, we will assume that Γ has no global fixed point in X.

Let us fix a point $x_0\in X$ and denote by x_0 also the element in $L_2(\Omega,X)$ which sends Ω to x_0. For $\varphi\in L_2(\Omega,X)$ let

$$\|\varphi\|:=\rho(\varphi,x_0)=\left[\int_\Omega d\big(\varphi(\omega),x_0\big)^2\right]^{1/2}.$$

For each n, let $\varphi_n\in L_2(\Omega,X)$ be a map satisfying:

- $E(\varphi_n)\le M+\frac{1}{n}$, and
- $\|\varphi_n\|\le\inf\{\|\varphi\|:\varphi\in L_2(\Omega,X),\ E(\varphi)\le M+\frac{1}{n}\}+\frac{1}{n}$.

PROPOSITION 3.3. *The maps φ_n are uniformly bounded, i.e.* $\sup\|\varphi_n\|<\infty$.

Proof. Let

$$\mathcal{F}_n=\inf\left\{\|\varphi\|:\varphi\in L_2(\Omega,X)\,,\ E(\varphi)\le M+\tfrac{1}{n}\right\}.$$

Note that $\mathcal{F}_n\le\|\varphi_n\|\le\mathcal{F}_n+\frac{1}{n}$.

Let $\tilde\Omega=\Sigma\cdot\Omega$, $U=\Omega^{-1}\tilde\Omega$, and $U_i=\pi_i(U)$, $i=1,2$. For $u\in U$ let $u_i=\pi_i(u)\in U_i$, where $\pi_i:G\to G_i$ is the canonical projection. We may normalize the Haar measure μ_G of G so that $\mu_G(\Omega)=1$.

Let b be the constant from Lemma 2.3 applied to the function d_Σ, then

$$\begin{aligned}\frac{(b\|\varphi_n\|)^2}{4}\le\frac{1}{4}\int_\Omega d_\Sigma\big(\varphi_n(\omega)\big)^2&\le\inf_{y\in X}\int_\Omega\max_{\gamma\in\Sigma}d\big(y,\gamma\cdot\varphi_n(\omega)\big)^2\\&\le\inf_{y\in X}\int_\Omega\sum_{\gamma\in\Sigma}d\big(y,\gamma\cdot\varphi n(\omega)\big)^2,\end{aligned}$$

where the second $\le$ follows from the triangle inequality since Σ contains 1. By the definition of $\tilde\Omega$, the last term is equal to

$$\inf_{y\in X}\int_{\tilde\Omega}d\big(y,\varphi_n(\tilde\omega)\big)^2\le\int_{\Omega\times\tilde\Omega}d\big(\varphi_n(\omega),\varphi_n(\tilde\omega)\big)^2\le\int_{\Omega\times U}d\big(\varphi_n(\omega),\varphi_n(\omega u)\big)^2;$$

here the first $\leq$ holds since $\mu_G(\Omega) = 1$, and the second since $\tilde{\Omega} \subset wU$, $\forall w \in \Omega$. By the triangle inequality, the last term is bounded by

$$\int_{\Omega\times U} \big[d(\varphi_n(\omega), \varphi_n(\omega u_2)) + d(\varphi_n(\omega u_2), \varphi_n(\omega u_2 u_1))\big]^2$$
$$\leq 2\int_{\Omega\times U} d\big(\varphi_n(\omega), \varphi_n(\omega u_2)\big)^2 + 2\int_{\Omega\times U_2\times U_1} d\big(\varphi_n(\omega u_2), \varphi_n(\omega u_2 u_1)\big)^2.$$

Now the second summand is bounded above by $2\mu_2(U_2)E(\varphi_n) \leq 2\mu_2(U_2)(M+1)$, since $h|_{U_1} \geq 1$. Hence, if we assume that $\frac{b^2}{4}\|\varphi_n\|^2 \geq 4\mu_2(U_2)(M+1)$ then we have

$$\int_{\Omega\times U} d\big(\varphi_n(\omega), \varphi_n(\omega u_2)\big)^2 \geq \frac{(b\|\varphi_n\|)^2}{16}$$

and therefore, for some $u_2 \in U_2$

$$\rho\big(\varphi_n, \varphi_n(\cdot u_2)\big)^2 = \int_\Omega d\big(\varphi_n(\omega), \varphi_n(\omega u_2)\big)^2 \geq \frac{1}{\mu(U)}\frac{b^2}{16}\|\varphi_n\|^2.$$

On the other hand the relatively compact set ΩU_2 is contained in $\tilde{\Sigma}\cdot\Omega$ for some finite set $\tilde{\Sigma} \subset \Gamma$, and we have $\|\varphi_n(\cdot u_2)\| \leq \|\varphi_n\| + d_{\tilde{\Sigma}}(x_0)$, and if we assume further that $\|\varphi_n\| \geq d_{\tilde{\Sigma}}(x_0)$ and take $\epsilon_0 := b/\big(8\sqrt{\mu(U)}\big)$ then we also have $\rho(\varphi_n, \varphi_n(\cdot u_2)) \geq \epsilon_0(\|\varphi_n\| + d_{\tilde{\Sigma}}(x_0))$.

Let $\varphi_n' = \frac{\varphi_n + \varphi_n(\cdot u_2)}{2}$ then

$$E(\varphi_n') \leq E(\varphi_n) \leq M + \tfrac{1}{n}\,,$$

since E is convex and G_2-invariant from the right. Besides, if we let $\tau(\epsilon) > 0$ be a linear lower bound for the modulus of convexity function of $L_2(\Omega, X)$, as in the definition of uniform convexity (see Proposition 2.13), then

$$\|\varphi_n'\| \leq \big(\|\varphi_n\| + d_{\tilde{\Sigma}}(x_0)\big) - \tau(\epsilon_0)\big(\|\varphi_n\| + d_{\tilde{\Sigma}}(x_0)\big)$$
$$\leq \big(\mathcal{F}_n + 1 + d_{\tilde{\Sigma}}(x_0)\big) - \tau(\epsilon_0)\big(\|\varphi_n\| + d_{\tilde{\Sigma}}(x_0)\big)\,,$$

and since $\mathcal{F}_n \leq \|\varphi_n'\|$, this implies that $\tau(\epsilon_0)(\|\varphi_n\| + d_{\tilde{\Sigma}}(x_0)) \leq d_{\tilde{\Sigma}}(x_0) + 1$. Thus $\|\varphi_n\|$ is bounded independently of n. □

We are now able to finish the proof of Theorem 3.2. Note that we may assume that the positive function $\tau(\epsilon)$ is monotonic (non-decreasing) in ϵ. Let $\mathcal{F} = \lim \mathcal{F}_n$ and let $m > n$, then

$$\mathcal{F}_n \leq \left\|\frac{\varphi_n + \varphi_m}{2}\right\| \leq (\mathcal{F} + 1/n)\left(1 - \tau\left(\frac{\rho(\varphi_n, \varphi_m)}{\mathcal{F} + 1/n}\right)\right),$$

therefore $\tau\big(\frac{\rho(\varphi_n,\varphi_m)}{\mathcal{F}+1/n}\big) \to 0$, which force $\rho(\varphi_n, \varphi_m) \to 0$ when $n, m \to \infty$. Thus φ_n is a Cauchy sequence. Since $L_2(\Omega, X)$ is complete we can take $\varphi := \lim \varphi_n$ and by continuity of E, φ is a harmonic map. □

4 The Proof of Theorem 1.1

4.1 The case $n = 2$. Let G, Γ, X be as in Theorem 1.1, and suppose that we are given a reduced Γ-action on X without global fixed points. We will first deal with the case $n = 2$, i.e. $G = G_1 \times G_2$.

Observe the following facts about our harmonic maps:

- If φ is a harmonic map then so is $\varphi(\cdot g_2)$ for any $g_2 \in G_2$, because the energy E is G_2-invariant from the right.
- If φ and ψ are harmonic maps then so is $\frac{\varphi+\psi}{2}$ (by convexity of E).
- If φ and ψ are harmonic maps then $[\varphi(g), \varphi(gg_1)] \parallel [\psi(g), \psi(gg_1)]$, $\forall g \in G$, $g_1 \in G_1$ (by BNPC of X).

Let us now fix a harmonic map φ. From the first and the third facts we see that

$$\big[\varphi(g), \varphi(gg_1)\big] \parallel \big[\varphi(gg_2), \varphi(gg_1g_2)\big] ,$$

and by Lemma 2.1

$$\big[\varphi(g), \varphi(gg_2)\big] \parallel \big[\varphi(gg_1), \varphi(gg_1g_2)\big] ,$$

for almost all $g \in G$, $g_1 \in G_1$, $g_2 \in G_2$.

LEMMA 4.1. *For any fixed $g_1 \in G_1$, the function $g \mapsto d(\varphi(g), \varphi(gg_1))$ is essentially constant. Similarly, for any fixed $g_2 \in G_2$ the function $g \mapsto d(\varphi(g), \varphi(gg_2))$ is essentially constant.*

Lemma 4.1 is similar to a statement implanted in the proof of the main theorem in [Mo].

Proof. The first (resp. second) function is measurable, Γ-invariant from the left and G_2 (resp. G_1) invariant from the right. Since Γ is irreducible, G_2 (resp. G_1) acts ergodically on $\Gamma\backslash G$ from the right. The result follows. □

COROLLARY 4.2. *The harmonic map φ is essentially continuous.*

Proof. It follows from Lemma 4.1 that

$$d\big(\varphi(g), \varphi(gg_1)\big) = \rho\big(\varphi, \varphi(\cdot g_1)\big) \text{ and } d\big(\varphi(g), \varphi(gg_2)\big) = \rho\big(\varphi, \varphi(\cdot g_2)\big) ,$$

for almost any $g \in G, g_i \in G_i$. Since the (right) action of G on $L_2(\Gamma\backslash G, X)$ is continuous (this is well known when X is replaced by $\mathbb{R}$, and is easily seen to be true for any metric space X), φ is essentially uniformly (on G) continuous from the right along G_1 and G_2, and hence along G. It follows that φ is essentially continuous. □

By changing φ on a set of measure 0, we can assume that it is actually continuous. All harmonic maps considered below will be assumed to be continuous rather than essentially continuous.

For the proof of Theorem 1.1 we will distinguish between two cases:

(1) $\varphi(G_2)$ is bounded.
(2) $\varphi(G_2)$ is unbounded.

In case (1) we will show that there is a G_2-invariant harmonic map φ_0 (perhaps different from φ). In case (2), we will show that φ is G_1-invariant. We will then conclude that in case (i) (where $i = 1$ or 2) the Γ action extends to a continuous G-action which factors through G_i.

Before we start let us note that in a uniquely geodesic metric space X', the closed convex hull $\overline{\mathrm{conv}(Y)}$ of a subset $Y \subset X'$, which by definition is the minimal closed convex subset of X' containing Y (which is also the intersection of all such sets) can be constructed recursively as follows: Define $Y_0 = Y$, and $Y_n = \cup\left\{\frac{x+y}{2} : x, y \in Y_{n-1}\right\}$. Then $Y_n \supset Y_{n-1}$ because we can take $x = y$, and $\overline{\mathrm{conv}(Y)} = \overline{\cup_{n=0}^{\infty} Y_n}$.

(1) If $\varphi(G_2)$ is bounded, then, as follows from Lemma 4.1, the set

$$G_2 \cdot \varphi = \left\{\varphi(\cdot g_2) : g_2 \in G_2\right\}$$

is bounded in $L_2(\Omega, X)$. By the constructive description of $\overline{\mathrm{conv}(G_2 \cdot \varphi)}$ we see that it consists of harmonic maps. Since $\overline{\mathrm{conv}(G_2 \cdot \varphi)}$ is bounded and convex and since $L_2(\Omega, X)$ is uniformly convex, there is a unique relative circumcenter, i.e. a unique point $\varphi_0 \in \overline{\mathrm{conv}(G_2 \cdot \varphi)}$ which minimizes

$$\sup\left\{\rho(\varphi_0, \varphi') : \varphi' \in \overline{\mathrm{conv}\ (G_2 \cdot \varphi)}\right\}.$$

(The existence and uniqueness of a relative circumcenter of a closed bounded convex subset of a complete WUC space follow from the existence and uniqueness of the usual circumcenter by ignoring the complement of the set.)

Since $\overline{\mathrm{conv}(G_2 \cdot \varphi)}$ is G_2-invariant, the function φ_0 is G_2-invariant.

(2) Suppose that $\varphi(G_2)$ is unbounded, and let $Y = \overline{\mathrm{conv}(\varphi(G_2))}$. Then the Hausdorff distance $\mathrm{Hd}(\gamma \cdot Y, Y)$ is finite for any $\gamma \in \Gamma$. Indeed $\gamma \cdot \varphi(g_2) = \varphi(\gamma_2 g_2 \gamma_1)$ where $\gamma = (\gamma_1, \gamma_2)$. Hence $d(\gamma \cdot \varphi(g_2), \varphi(\gamma_2 g_2))$ is a constant depending on γ_1 as follows from Lemma 4.1. It follows, since the action $\Gamma \circlearrowleft X$ is assumed to be reduced, that $Y = X$.

By the constructive description of the closed convex hull we see that $X = Y = \overline{\cup_{n=0}^{\infty} Y_n}$ where each point of $\cup_{n=0}^{\infty} Y_n$ is of the form $\psi(1)$ for some harmonic function ψ in the convex hull of $G_2 \cdot \varphi$. Let $g_1 \in G_1$. Since the segments $[\psi(1), \psi(g_1)]$, $\psi \in \mathrm{conv}(G_2 \cdot \varphi)$ are all parallel to each other, the map $\psi(1) \mapsto \psi(g_1)$ extends to a Clifford isometry on $\overline{\{\psi(1) : \psi \in \mathrm{conv}(G_2 \cdot \varphi)\}}$, namely on X. Since X has no non-trivial Clifford isometries, we get that $\varphi(g_2) = \varphi(g_1 g_2)$ for any $g_2 \in G_2$. Since g_1 is arbitrary, we get that φ is G_1-invariant.

We showed that there is a harmonic map φ_0 which is either G_1 or G_2 invariant. Suppose it is G_2-invariant. Then, since φ_0 is Γ-equivariant and continuous, the orbit map $\gamma \mapsto \gamma \cdot x$ is continuous for any $x \in \varphi_0(G)$, where the topology on Γ is the (not necessarily Hausdorff) one induced from G_1 (equivalently, we can consider Γ as a dense subgroup of G_1). It follows that the set

$$\{x \in X : \text{the obit map } \ \gamma \mapsto \gamma \cdot x \ \text{ is continuous with respect to the } \ G_1\text{-topology}\}$$

is nonempty. Since that set is also closed convex and Γ-invariant, it follows from C-minimality that it is the whole space X. Thus the orbit map is continuous for any point $x \in X$, when Γ is considered with the topology induced from G_1, so we can define the action of G on X by

$$g \cdot x := \lim_{\pi_1(\gamma) \to \pi_1(g)} \gamma \cdot x \,.$$

4.2 The case $n > 2$. Let now $n \geq 2$ be general, $G = \prod_{i=1}^n G_i$, and define $n-1$ energies $E_1, E_{1,2}, \ldots, E_{1,\ldots,n-1}$ as follow:

$$E_{1,\ldots,k}(\varphi) := \int_{(\Gamma\backslash G)\times G_1 \times \ldots \times G_k} h_1(g_1) \ldots h_k(g_k) d\big(\varphi(g), \varphi(g g_1 \ldots g_k)\big)^2,$$

where $h_i(g_i) = e^{-|g_i|^2}$, $|\ |$ being the norm with respect to the word metric on G_i associated with some compact generating set.

The set H_1 of E_1-harmonic maps is closed convex, and by Theorem 3.2 non-empty. Furthermore, there exist some $\varphi \in H_1$ which minimize $E_{1,2}$. To see this, argue as in the proof of Theorem 3.2, taking the φ_n to be E_1-harmonic and letting $\prod_{i>2} G_i$ play the rule of G_2 in 3.2. Call such a function $E_{1,2}$-harmonic. More generally, for all $k < n$ call a Γ-equivariant map $E_{1,\ldots,k}$-harmonic if it minimizes $E_{1,\ldots,k}$ among the $E_{1,\ldots,k-1}$ harmonic maps. By repeating the argument above finitely many times, one proves the existence of an $E_{1,\ldots,n-1}$-harmonic map φ. Since $\varphi(G)$ is Γ-invariant it must be unbounded. Let $1 \leq k \leq n$ be the smallest integer such that $\varphi(G_{k+1} \times \ldots \times G_n)$ is bounded. Then $\varphi(G_k)$ is unbounded, and by replacing φ by the relative circumcenter of

$$\overline{\operatorname{conv}\big\{\varphi(\cdot g) : g \in G_{k+1} \times \ldots \times G_n\big\}}$$

one gets an $E_{1,\ldots,k'}$-harmonic map φ_0 which is G_i invariant for all $i > k$, where $k' = \min\{k, n-1\}$. As in case (2) of the proof of 3.2 one deduces, since φ_0 is also $E_{1,\ldots,k-1}$-harmonic (in case $k > 1$), that φ is also G_i-invariant for all $i < k$. Therefore one can use φ_0 to extend the Γ-action to a continuous G action which factors through G_k. □

5 The Proof of Theorem 1.3

For simplicity assume again that $n = 2$. If $d_\Sigma \not\to \infty$ where Σ is a finite generating set, then Γ has a fixed point in ∂X. Assume therefore that $d_\Sigma \to \infty$. By Lemma 2.10 there is an unbounded closed convex Γ-invariant subset on which the action is C-minimal. Replacing X with such a subset we may assume that the action on X is C-minimal. Moreover, since $d_\Sigma \to \infty$ we have a harmonic map φ, and we can argue as in subsection 4.1. In case (1), when $\varphi(G_2)$ is bounded, we obtain that the Γ-action extends to a continuous G-action on X which factors through G_1. In case (2), when $\varphi(G_2)$ is unbounded, then for any $g_1 \in G_1$ the map $\varphi(g_2) \mapsto \varphi(g_1 g_2)$ extends to a parallel translation from $\overline{\mathrm{conv}(\varphi(G_2))}$ onto $\overline{\mathrm{conv}(\varphi(g_1 G_2))}$. Hence, in that case Γ preserves $\partial\overline{\mathrm{conv}(\varphi(G_2))}$ and the action on it factors through G_2. Finally, since φ is continuous and X is proper, the Γ-action on $\partial\overline{\mathrm{conv}(\varphi(G_2))}$ is continuous with respect to the G_2-topology, and extends to a continuous G_2-action.

When the number of factors n is greater than two, one argues as in subsection 4.2. □

6 The Proof of Theorem 1.4

By Lemma 2.12, up to replacing X with a closed non-empty geodesically complete subset, we may assume that X is a minimal complete uniquely geodesically complete for the Γ-action.

For a subset $A \subset X$ of cardinality ≥ 2 let $\mathrm{span}(A)$ denote the minimal set containing A with the geodesic extension property. One can construct $\mathrm{span}(A)$ recursively by defining $A_0 = A$ and $A_{m+1} = \cup\{\overline{xy} : x, y \in A_m\}$, where $\overline{xy}$ is the geodesic containing x, y, and taking the union $\mathrm{span}(A) = \cup A_m$.

For the sake of simplicity let us assume again that $n = 2$, i.e. $G = G_1 \times G_2$. One can extend the argument below to any $n \geq 2$ using the energies $E_{1,\dots,k}$, $k = 1, \dots n-1$ as in subsection 4.2.

Let $\varphi : G \to X$ be a harmonic map. If $\varphi(G_i)$ is a single point, for $i = 1$ or 2, then φ is G_i-invariant and the action extends to a G-action which factors through G_{3-i}. Thus we may assume that $|\varphi(G_i)| \geq 2$, for $i = 1, 2$.

Let $H \leq L_2(\Omega, X)$ be the subset of harmonic maps.

LEMMA 6.1. *$L_2(\Omega, X)$ is complete* CAT(0) *and uniquely geodesically complete, and so is its subset H.*

Proof. The first statement is straightforward. The second one follows, using Lemma 2.1, from the fact that a Γ-equivariant map $\psi : G \to X$ is harmonic if and only if for almost all $g \in G$, $g_1 \in G_1$ the segment $[\psi(g), \psi(gg_1)]$ is parallel to $[\varphi(g), \varphi(gg_1)]$. □

Let

$$X_1 = \overline{\operatorname{span}\big(\varphi(G_1)\big)} \quad \text{and} \quad X_2 = \big\{\psi(1) : \psi \in H\big\}.$$

By definition X_1 is closed and geodesically complete, and it follows from Lemma 6.1 that also X_2 is. We will show that $X = X_1 \times X_2$. For this we need

LEMMA 6.2. *Let φ_1, φ_2 be two harmonic maps, then*

$$\operatorname{conv}\big(\overline{\operatorname{span}(\varphi_1(G_1))} \cup \overline{\operatorname{span}(\varphi_2(G_1))}\big) \cong \overline{\operatorname{span}\big(\varphi(G_1)\big)} \times \big[0, d(\operatorname{span}(\varphi_1(G_1)), \operatorname{span}(\varphi_2(G_1)))\big].$$

Proof of Lemma 6.2. Since X is CAT(0), parallelity is a transitive relation on the set of two sided infinite geodesics in X (cf. [BrH, p. 183]), and we can use this property to extend the map $\varphi_1(g_1) \mapsto \varphi_2(g_1)$ to a parallel isometry $T : \overline{\operatorname{span}(\varphi_1(G_1))} \to \overline{\operatorname{span}(\varphi_2(G_1))}$, parallel means that

$$\big[x, T(x)\big] \parallel \big[y, T(y)\big], \quad \forall x, y \in \overline{\operatorname{span}\big(\varphi_1(G_1)\big)}.$$

The lemma would follow from the flat strip theorem (see [BrH, p. 183]; the flat strip theorem is stated for parallel geodesics, but the proof extends straightforwardly to parallel geodesically complete sets) once we show that $d(x, \overline{\operatorname{span}(\varphi_2(G_1))})$ is constant over $\overline{\operatorname{span}(\varphi_1(G_1))}$. Suppose in contradiction that there are $x, y \in \overline{\operatorname{span}(\varphi_1(G_1))}$ with

$$d\big(x, \overline{\operatorname{span}(\varphi_2(G_1))}\big) < d\big(y, \overline{\operatorname{span}(\varphi_2(G_1))}\big).$$

Let $c(t)$ be the geodesic with $c(0) = x, c(d(x, y)) = y$. Since the function

$$t \mapsto d\big(c(t), \overline{\operatorname{span}(\varphi_2(G_1))}\big)$$

is convex on $\mathbb{R}$, $d(c(t), \overline{\operatorname{span}(\varphi_2(G_1))}) \to \infty$ when $t \to +\infty$, contradicting the fact that $d(c(t), T(c(t)))$ is constant. □

Let

$$Y = \overline{\operatorname{conv}\Big\{\bigcup \psi(G) : \psi \in H\Big\}} = \overline{\operatorname{conv}\Big\{\bigcup \psi(G_1) : \psi \in H\Big\}}.$$

Using again the fact that parallelity is a transitive relation and the flat strip theorem, one sees that every point in Y is contained in an isometric parallel copy of X_1. It follows that X_1 is a direct factor of Y. (In case X_1 is a single geodesic, this is the content of [BrH, II.2.14] which again extends straightforwardly to the more general case where X_1 is geodesically

complete). Furthermore, since Y is Γ-invariant it follows that $X = \overline{\mathrm{span}(Y)}$, which allows one to show that every point in X is contained in an isometric parallel copy of X_1. It follows that X_1 is a direct factor of X.

Next we claim that, for any $\varphi_1, \varphi_2 \in H$, the projection

$$p : \overline{\mathrm{span}\left(\varphi_1(G_1)\right)} \to \overline{\mathrm{span}\left(\varphi_2(G_1)\right)}$$

extending the parallel translation $\varphi_1(g_1) \mapsto \varphi_2(g_1)$ is an orthogonal projection. Indeed, if that was not the case then the orthogonal projection $\pi : \overline{\mathrm{span}(\varphi_2(G_1))} \to \overline{\mathrm{span}(\varphi_1(G_1))}$ composed with p would be a non-trivial Clifford isometry on $\overline{\mathrm{span}(\varphi_1(G_1))}$. To see this note that for any $x_1, x_2 \in \overline{\mathrm{span}(\varphi_1(G_1))}$

$$[x_1, x_2] \parallel \big[p(x_1), p(x_2)\big] \parallel \big[\pi(p(x_1)), \pi(p(x_2))\big]$$

which, by transitivity (since X is uniquely geodesically complete, parallelity is a transitive relation on segments as well) and Lemma 2.1 implies that $[x_1, \pi(p(x_1))] \parallel [x_2, \pi(p(x_2))]$. This however would imply that $\overline{\mathrm{span}(\varphi_1(G_1))}$ has a euclidian factor. Now since $\overline{\mathrm{span}(\varphi_1(G_1))}$ is isometric to X_1 and since X_1 is a direct factor of X, this contradicts our assumptions on X.

It follows in particular that the set $Z = \overline{\big\{ \bigcup \mathrm{span}(\psi(G_1)) : \psi \in H \big\}}$ decomposes as a direct product

$$Z = \mathrm{span}\left(\varphi(G_1)\right) \times \big\{\psi(1) : \psi \in H\big\} = X_1 \times X_2\,,$$

and hence it is geodesically complete. Since Z is also Γ-invariant, we derive from minimality that $X = Z$ and hence $X = X_1 \times X_2$.

Finally we claim that G_i acts by isometries on X_i, for $i = 1, 2$. First note that, if $\psi \in H$ and $g_2 \in G_2$, then also $\psi(\cdot g_2) \in H$, and since $X_2 = \{\psi(1) : \psi \in H\}$ we can define an action of G_2 on X_2 by

$$g_2 \cdot \psi(1) = \psi(g_2)\,.$$

We need to show that this action is well defined and is by isometries. These two claims follow from the fact that Γ is irreducible in G. Indeed, any $g_2 \in G_2$ can be approximated by γg_1 where $\gamma \in \Gamma$, $g_1 \in G_1$, and we know that for any two harmonic maps φ_1, φ_2 the segments $[\varphi_1(1), \varphi_1(g_1)]$ and $[\varphi_2(1), \varphi_2(g_1)]$ are parallel. Since γ is an isometry and φ_i are Γ-equivariant, we derive that

$$d\big(\varphi_1(\gamma g_1), \varphi_2(\gamma g_2)\big) = d\big(\varphi_1(g_1), \varphi_2(g_1)\big) = d\big(\varphi_1(1), \varphi_2(1)\big)\,,$$

and by passing to limit, $d(\varphi_1(g_2), \varphi_2(g_2)) = d(\varphi_1(1), \varphi_2(1))$.

Similarity one shows that G_1 acts on X_1 by isometries (with $g_1 \cdot \varphi(g_1') = \varphi(g_1 g_1')$) by approximating g_1 with $\gamma g_2 \in \Gamma G_2$.

This produces an action of G on $X = X_1 \times X_2$ which extends the Γ-action and is clearly continuous. □

7 Non-Uniform Weakly Cocompact p-Integrable Lattices

Let G be a locally compact group, Γ a lattice in G, and Ω a right fundamental domain. We define a map $\chi : G \to \Gamma$ by the rule $g \in \chi(g)\Omega$. Suppose that Γ is generated by a finite set Σ and let $|\ | : \Gamma \to \mathbb{N}$ be the word norm associated to Σ. Let $p > 1$. In analogy to [S], we will say that Γ is p-integrable if for any element $g \in G$ the function $\omega \mapsto |\chi(\omega g)|$ belongs to $L_p(\Omega)$. This assumption ensures that whenever Γ acts by isometries on a metric space X, the space $L_p(\Omega, X)$ is invariant under G, i.e. if $\varphi : G \to X$ is Γ-equivariant and $\varphi|_\Omega \in L_p(\Omega, X)$ then also $\varphi(\cdot g) \in L_p(\Omega, X)$ for any $g \in G$. Note that the property of being p-integrable is independent of the generating set Σ, however, it does depend on the choice of Ω. When this condition is satisfied we shall say that Ω is *p–admissible*.

Let $L^0_p(\Gamma\backslash G)$ denote the codimension-one subspace of $L_p(\Gamma\backslash G)$ of function with 0 mean. Following [M2, III.1.8], we will say that Γ is weakly cocompact if the right regular representation of G on $L^0_p(\Gamma\backslash G)$ does not almost have invariant vectors. This is equivalent to each of the following:

(1) If $f_n \in L_p(\Gamma\backslash G)$ are normalized asymptotically invariant positive functions then (f_n) converges to a constant function.
(2) If $f_n \in L_p(\Gamma\backslash G)$ are normalized positive functions such that for any compact $K \subset G$, $\int_K \|f_n - f_n(\cdot k)\|_p \to 0$ then (f_n) converges to a constant function.

Moreover, using the Mazur map $M_{p,q} : L_p(\Gamma\backslash G) \ni f \mapsto |f|^{p/q}\operatorname{sign}(f) \in L_q(\Gamma\backslash G)$ which intertwines the G actions and is uniformly continuous (cf. [BeL, Th. 9.1]) one can show that this property is independent of $1 < p < \infty$.

The following extends the superrigidity results from the previous sections to non-uniform lattices which are p-integrable and weakly cocompact. This is analogous to [Mo, Th. 7] which gives a similar statement for actions of weakly cocompact 2-integrable non-uniform lattices on CAT(0) spaces.

Theorem 7.1. *Theorems 1.1, 1.3, 1.4 and 2.7 remain true for finitely generated non-uniform lattices Γ provided they are weakly cocompact and p-integrable for some $1 < p < \infty$.*

For the sake of simplicity, let us assume that $G = G_1 \times G_2$ is a product of two factors. Fix $1 < p < \infty$ such that Γ is p-integrable with respect to Ω, and normalize the Haar measure so that $\mu(\Omega) = 1$. Then G acts measurably and hence continuously from the right on $L_p(\Omega, X)$. It also follows from p-integrability that one can choose a measurable function $h : G_1 \to \mathbb{R}_{>0}$ such that the energy $E(\varphi) := \int_{\Omega \times G_1} h(g_1) d(\varphi(g), \varphi(gg_1))^p$ is finite for every

$\varphi \in L_p(\Omega, X)$. Since G acts continuously on $L_p(\Omega, X)$, we can take h with $\inf_{k\in K_1} h(k) > 0$ for every compact $K_1 \subset G_1$.

The only place in the previous sections that compactness of Ω was used is the proof of Proposition 3.3. Hence, we need only justify why the function φ_n, chosen as in section 3, are uniformly bounded when Ω is not relatively compact but Γ is p-integrable and weakly cocompact. Let ρ denote the distance on $L_p(\Omega, X)$, φ_{x_0} the Γ-equivariant function sending Ω to $x_0 \in X$, and for $\varphi \in L_p(\Omega, X)$ set $\|\varphi\| = \rho(\varphi, \varphi_{x_0})$.

LEMMA 7.2. *There exists a compact subset $K \subset G$ and a positive constant $\beta > 0$ such that any $\varphi \in L_p(\Omega, X)$ satisfies $\int_K \rho(\varphi, k \cdot \varphi) > \beta\|\varphi\| - \frac{1}{\beta}$.*

Proof. Assuming the contrary, as G is σ-compact, one can find a sequence $\psi_n \in L_p(\Omega, X)$ with $\|\psi_n\| \to \infty$ and $1/\|\psi_n\| \int_{K'} \rho(\psi_n, k \cdot \psi_n) \to 0$ for every compact K'. Let $f_n'(g) = d(\psi_n(g), \varphi_{x_0}(g))$ and $f_n = f_n'/\|f_n'\|_p$. It is straightforward to verify that these (f_n) satisfy the condition (2) above and hence (f_n) converges to a constant function. Let $K_0 \subset \Omega$ be a compact subset with positive measure. Then if n is sufficiently large, for every k in some subset $K_n \subset K_0$ of at least half the measure of K_0, $d(x_0, \psi_n(k)) > \frac{1}{2}\|\psi_n\|$. Taking $K = K_0^{-1}\Sigma \cdot K_0$ and using Lemma 2.3 one gets a constant $\beta > 0$ for which $\int_K \rho(\psi_n, k\cdot\psi_n) > \beta\|\psi_n\|$ for all sufficiently large n, contrary to the assumptions on ψ_n. □

Now consider the functions φ_n defined as in section 3. Replacing K by a larger compact set, we may assume it is of the form $K = K_1 \times K_2$ with $K_i \subset G_i$. As in section 3 the energy of φ_n bounds its variation along the G_1 factor, so we conclude that for some positive constant β'

$$\int_{K_2} \rho(\varphi_n, k_2 \cdot \varphi_n) > \beta'\|\varphi_n\| - \tfrac{1}{\beta'}\,,$$

and hence for some $k_2 \in K_2$ and another positive constant ϵ' we have $\rho(\varphi_n, \varphi_n(\cdot k_2)) > \epsilon'\|\varphi_n\| - \frac{1}{\epsilon'}$. Finally since k_2 belongs to the compact set K_2 the norm of $\varphi_n(\cdot k_2)$ is bounded by the norm of $\varphi_n +$ some constant. Define $\varphi_n' = \frac{\varphi_n + \varphi_n(\cdot k_2)}{2}$ then $E(\varphi_n') \le E(\varphi_n)$ since E is G_2-invariant and convex, and $\|\varphi_n\| - \|\varphi_n'\| \ge \delta'\|\varphi_n\| - \frac{1}{\delta'}$ for some positive constant δ', by uniform convexity of $L_p(\Omega, X)$. This, together with the second property of φ_n implies that its norm must be bounded independently of n.

REMARK 7.3. (i) It is possible to show that if G is a semisimple real Lie group without compact factors not locally isomorphic to $SL(2, \mathbb{R})$, then any lattice in G is p-integrable for some $p > 1$. Similarly, all irreducible lattices

in higher-rank groups over local fields are 2-integrable (cf. [S]). Rémy [R] showed that all Kac–Moody lattices are 2-integrable.

(ii) Every lattice in a semisimple Lie group over a local field is weakly cocompact [M2, III.1.12]. Clearly, if G has Kazhdan property (T) then any lattice in G is weakly cocompact.

8 Superrigidity for Commensurability Subgroups

Let G be a locally compact, compactly generated group and Γ a cocompact lattice in G. Let Λ be a subgroup of

$$\mathrm{Comm}_G(\Gamma) := \{g \in G : g\Gamma g^{-1} \text{ and } \Gamma \text{ are commensurable}\},$$

containing Γ which is dense in G. Let X be a complete uniformly convex BNPC metric space. Assume that Λ acts by isometries on X such that any subgroup $\Gamma_0 \leq \Delta$ commensurable to Γ satisfies $d_{\Sigma_0} \to \infty$ where Σ_0 is a finite generating set of Γ_0, and has no parallel orbits, i.e. for any two distinct points $x, y \in X$, there is $\gamma \in \Gamma_0$ (equivalently, there is $\gamma \in \Sigma_0$) for which $[\gamma \cdot x, \gamma \cdot y] \not\parallel [x, y]$. (Note that if X is uniquely geodesically complete then the assumption that there are no parallel orbits follows from the assumption that $d_{\Sigma_0} \to \infty$.) Assume moreover that the action is C-minimal. The following superrigidity theorem was proved in [M4] (under the weaker assumption that X is WUC rather than UC).

Theorem 8.1. *Under the above assumptions, the Λ-action extends uniquely to a continuous isometric G-action.*

For a subgroup $\Gamma_0 \leq \Delta$ commensurable to Γ we define a Γ_0-*harmonic map* to be a Γ_0-equivariant map in $L_2(\Omega_0, X)$ which minimizes

$$\begin{aligned} I_{\Gamma_0}(\varphi) &= \frac{1}{\mu(\Omega_0)} \int_{\Omega_0 \times G} h(\omega^{-1}g) d\big(\varphi(\omega), \varphi(g)\big)^2 \\ &= \frac{1}{\mu(\Omega_0)} \int_{\Gamma_0 \backslash (G \times G)} h(g_1^{-1}g_2) d\big(\varphi(g_1), \varphi(g_2)\big)^2, \end{aligned}$$

where Ω_0 is a measurable relatively compact fundamental domain for Γ_0 in G and h is a function on G similar to the ones in the previous sections. Then $I_{\Gamma_0}(\varphi)$ is finite for any $\varphi \in L_2(\Omega_0, X)$. Denote the infimum value of this functional by M_{Γ_0} and call φ Γ_0-harmonic if $I_{\Gamma_0}(\varphi) = M_{\Gamma_0}$.

Proposition 8.2. *There exists a Γ_0-harmonic map.*

This is actually a special case of Theorem 3.2, where G can be considered as a product $G \times 1$. However, since Proposition 8.2 is much simpler than the general case of Theorem 3.2 we shall give an alternative simpler proof.

Proof. It is straightforward to check that

$$I_{\Gamma_0}(\varphi) = \frac{1}{\mu(\Omega_0)} \int_{\Omega_0 \times \Omega_0} \sum_{\gamma \in \Gamma_0} h(\omega_1^{-1}\gamma\omega_2) d\big(\varphi(\omega_1), \gamma\varphi(\omega_2)\big)^2.$$

Let Σ be finite generating set for Γ_0. By Lemma 2.3 $d_\Sigma(x) \to \infty$ in at least a linear rate. Since $h \geq \beta > 0$ on $\Omega_0(\Sigma \cup \{1\})\Omega_0$, we have

$$I_{\Gamma_0}(\varphi) \geq \frac{\beta}{\mu(\Omega_0)} \int_{\Omega_0} \left(\int_{\Omega_0} \sum_{\gamma \in \Sigma \cup \{Id\}} d\big(\varphi(\omega_1), \gamma\varphi(\omega_2)\big)^2 d\mu(\omega_2) \right) d\mu(\omega_1)\,.$$

We divide the integration in ω_2 into two parts according to whether $d(\varphi(\omega_2), x_0) \geq d(\varphi(\omega_1), x_0)/2$ or not. Taking in account only the $\gamma \in \Sigma \cup \{1\}$ which gives the largest contribution, in view of Lemma 2.3, we get for some constant $c > 0$

$$I_{\Gamma_0}(\varphi) \geq c \int_\Omega d\big(\varphi(\omega_1), x_0\big)^2 = c\|\varphi\|_2^2\,.$$

This means that for any minimizing sequence φ_n the L_2-norm is uniformly bounded. As in the proof of Theorem 3.2, we can find such a sequence which is cauchy and hence converges to a harmonic map. □

PROPOSITION 8.3. *The Γ_0-harmonic map is unique.*

Proof. Take two Γ_0-harmonic maps φ and ψ. BNPC implies that

$$d\left(\frac{\varphi(g_1) + \psi(g_1)}{2}, \frac{\varphi(g_2) + \psi(g_2)}{2}\right)^2 < \frac{d(\varphi(g_1), \varphi(g_2))^2 + d(\psi(g_1), \psi(g_2))^2}{2}\,,$$

unless $[\varphi(g_1), \psi(g_1)] \parallel [\varphi(g_2), \psi(g_2)]$. Since both φ and ψ minimize I_{Γ_0} and are Γ_0-equivariant, and since there are no parallel orbits for the Γ_0-action, it follows that $\varphi = \psi$. □

LEMMA 8.4. *Let φ be the (unique) Γ_0-harmonic map and $\lambda \in \Lambda$. Then $\varphi_\lambda(g) := \lambda^{-1}\varphi(\lambda g)$ is the $\lambda^{-1}\Gamma_0\lambda$-harmonic map.*

Proof. Indeed

$$\varphi_\lambda(\lambda^{-1}\gamma\lambda g) = \lambda^{-1}\varphi(\gamma\lambda g) = \lambda^{-1}\gamma\varphi(\lambda g) = \lambda^{-1}\gamma\lambda\varphi_\lambda(g)\,,$$

and hence φ_λ is $\lambda^{-1}\Gamma_0\lambda$-equivariant. It is also straightforward to verify that $I_{\Gamma_0}(\varphi) = I_{\lambda^{-1}\Gamma_0\lambda}(\varphi_\lambda)$. Note that $\lambda^{-1}\Omega_0$ is a fundamental domain for $\lambda^{-1}\Gamma_0\lambda$. □

LEMMA 8.5. *Let Γ_1 be a finite index normal subgroup of Γ_0 and let φ be the Γ_1-harmonic map. Then φ is also Γ_0-harmonic.*

Proof. By normality and Lemma 8.4 we get $\varphi_\gamma(g) = \varphi(g)$ for any $\gamma \in \Gamma_0$, which proves the Γ_0-equivariance. Additionally, choosing the fundamental domain for Γ_1 to be the union of finitely many translations of Ω_0 one can easily verify that $I_{\Gamma_1}(\varphi') = I_{\Gamma_1}(\varphi')$ for any Γ_0-equivariant map. □

LEMMA 8.6. *Let $\Gamma_i \leq \Delta$, $i = 1, 2$, be two subgroups commensurable to Γ with associated harmonic maps φ_i. Then $\varphi_1 = \varphi_2$.*

Proof. Take $\Gamma_4 \leq \Gamma_3$ of finite index in $\Gamma_1 \cap \Gamma_2$ such that Γ_3 is normal in Γ_1 and Γ_4 is normal in Γ_2. Lemma 8.5 implies that $\varphi_1 = \varphi_3 = \varphi_4 = \varphi_2$, where φ_i is the Γ_i-harmonic map, $1 \leq i \leq 4$. □

We conclude

PROPOSITION 8.7. *The Γ harmonic map φ is Λ-equivariant.*

Proof. Let $\lambda \in \Lambda$. By Lemma 8.6 φ is also $\lambda^{-1}\Gamma\lambda$-harmonic, and hence by Lemma 8.4 $\varphi = \varphi_\lambda := \lambda^{-1}\varphi(\lambda\cdot)$. Since $\lambda \in \Delta$ is arbitrary, φ is Δ-equivariant. □

Since Λ is dense in G, and hence acts ergodically on G and since φ is measurable and Λ-equivariant, we conclude

COROLLARY 8.8. *The harmonic map φ is essentially continuous.*

Proof. Indeed, for each $g' \in G$ the function $g \mapsto d(\varphi(g), \varphi(gg'))$ is measurable and Δ-invariant, hence constant. The result follows as the action by right translations of G on $L_2(\Omega, X)$ is continuous. □

Changing φ on a set of measure 0, we can assume that it is actually continuous. We derive from continuity and Λ-equivariance that the set of points $x \in X$ for which the orbit map $\lambda \mapsto \lambda \cdot x$ from Λ to X is continuous, is non-empty, indeed it contains $\varphi(G)$. Since this set is also convex and closed, it follows from C-minimality that the orbit map is continuous for every $x \in X$, and hence that the isometric action extends continuously to G.

REMARK 8.9. (i) As in section 7, also Theorem 8.1 can be generalized to the setting of finitely generated $(1 < p < \infty)$-integrable non-uniform weakly cocompact lattices. The only part that needs new justification is Proposition 8.2. However, since Proposition 8.2 is a special case of Theorem 3.2 this generalization can be derived from the discussion in the previous section. Moreover, the assumption that Γ is weakly cocompact is actually not required here. One can see this by arguing as in the proof given in section 3, for the special case that $G_2 = 1$. In order to keep this section as simple as possible we chose to state and give a complete proof of Theorem 8.1 under

the assumption that Γ is uniform and only remark on how this can be generalized to p-integrable lattices using arguments that appeared in earlier sections.

(ii) In [Mo] it is not assumed that there are no parallel orbits, but the conclusion is much weaker: the Γ-action extends but not necessarily the Λ-action as in Theorem 8.1. The following example shows that this assumption is required here: Let G be the full isometry group of $\mathbb{R}$ including the reflection f in 0. Let $\Gamma = \langle f \rangle \ltimes \mathbb{Z}$ and $\Lambda = \langle f \rangle \ltimes \mathbb{Z}[\sqrt{2}]$, and let Λ act on $\mathbb{R}$ where f acts by reflection, $\mathbb{Z}$ by positive and $\mathbb{Z} \cdot \sqrt{2}$ by negative translations. In this example also the displacement (of the index 2 subgroup $\mathbb{Z}$ of Γ) does not go to infinity (see the note in parentheses on page 1546), however, one can still prove that harmonic maps do exist, the lack of uniqueness is what prevents us from extending the Λ action to G.

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