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A DUALITY THEOREM FOR RIEMANNIAN FOLIATIONS IN NONNEGATIVE SECTIONAL CURVATURE

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Using a new type of Jacobi field estimate we will prove a duality theorem for singular Riemannian foliations in complete manifolds of nonnegative sectional curvature. Recall that a transnormal system $\mathcal{F}$ is a subdivision of M into C^∞ immersed connected complete submanifolds without boundary, called leaves, such that geodesics emanating perpendicularly to one leaf stay perpendicular to the leaves. If M is complete the leaf $\mathcal{L}(p)$ of each point $p \in M$ is intrinsically complete as well. A transnormal system $\mathcal{F}$ is called a singular Riemannian foliation if there are vector fields X_i $(i \in I)$ in M such that $T_p\mathcal{L}(p) = \mathrm{span}_{\mathbb{R}}\{X_{i|p} \mid i \in I\}$ for all $p \in M$, see [M]. Examples of singular Riemannian foliations are the fiber decomposition of a Riemannian submersion or the orbit decomposition of an isometric group action.

A piecewise smooth curve c is called horizontal with respect to a transnormal system $\mathcal{F}$, if $\dot{c}(t)$ is in the normal bundle $\nu_{c(t)}(\mathcal{L}(c(t)))$ of the leaf $\mathcal{L}(c(t))$. One can define a dual foliation $\mathcal{F}^\#$ by defining

$$\mathcal{L}^\#(p) := \big\{q \in M \mid \text{there is a piecewise smooth horizontal curve from } p \text{ to } q\big\}$$

as dual leaf of a point $p \in M$. We will see that $\mathcal{L}^\#(p)$ is a smooth immersed submanifold of M, see section 2. The double dual is not always equal to the original foliation. But the triple dual foliation is usually isomorphic to the dual foliation. In general one cannot expect that the dual foliation has too many reasonable properties. We will see that this is different in nonnegative curvature. The main results can be interpreted as rigidity versions of the following:

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Theorem 1. *Suppose that M is a complete positively curved manifold with a singular Riemannian foliation $\mathcal{F}$. Then the dual foliation has only one leaf.*

In other words one can connect two arbitrary points in M by a horizontal curve. It should be noted that the theorem is global in nature. If one considers a cohomogeneity one action on a sphere then of course the horizontal distribution is one dimensional in the generic part and hence integrable. However, a horizontal curve can run into singular orbits and then with different directions out of singular orbits. This way one can reach more than just a one-dimensional subset and in fact every point on the sphere.

Theorem 1 suggests to introduce a new length metric on M by defining the distance of two points as the infimum over the length of all horizontal curves connecting these two points. The previous example shows that one cannot expect that the two metrics induce the same topology, but it would be interesting to know whether one can say more about the latter metric, other than that M stays connected.

We prove Theorem 1 together with the following rigidity result in section 3.

Theorem 2. *Suppose that M is a complete nonnegatively curved manifold with a singular Riemannian foliation $\mathcal{F}$. Suppose the leaves of the dual foliation are complete. Then $\mathcal{F}^{\#}$ is a singular Riemannian foliation as well.*

In many cases it is actually possible to remove the assumption on the completeness of the dual leaves.

Theorem 3. *Suppose that M is a complete nonnegatively curved manifold with a singular Riemannian foliation $\mathcal{F}$. Then the dual foliation has intrinsically complete leaves if in addition one of the following holds:*

(a) *$\mathcal{F}$ is given by the orbit decomposition of an isometric group action.*
(b) *$\mathcal{F}$ is a non-singular foliation and M is compact.*
(c) *$\mathcal{F}$ is given by the fibers of the Sharafutdinov retraction.*

We recall that an open nonnegatively curved manifold M is by the soul theorem of Cheeger and Gromoll [ChG] diffeomorphic to the normal bundle of a compact totally geodesic submanifold Σ, the soul of M. Sharafutdinov showed that there is a distance nonincreasing retraction $P\colon M \to \Sigma$. By Perelman's [P] solution of the soul conjecture, P is a Riemannian submersion of class C^1. Guijarro [Gu] improved the regularity to C^2. Before we prove Theorem 3 in section 5, we will use Theorem 2 to establish the following regularity result in section 4.

Corollary 4. *Let (M,g) be an open nonnegatively curved manifold, Σ a soul of M. Then the Sharafutdinov retraction $P\colon M \to \Sigma$ is of class C^∞.*

Cao and Shaw [CS] proposed an independent proof of Corollary 4. They showed that the fibers of the Sharafutdinov retraction admit locally a one-dimensional foliation by geodesics. However, the tangent fields of these geodesics are obtained from the convex exhaustion in the soul construction. This vector field is continuous and in general not differentiable. Thus the map G constructed in the proof of Proposition 6 of [CS] is also just a C^1 parametrization and not of class C^∞ as claimed. Subsequently the author heard that Cao and Shaw think they can fix the problem, by somehow reducing it to showing that the vector field is "horizontally of class C^∞". However, this announcement contains no definition or argument and thus the author is not able to comment on its validity.

Recall that a map between metric spaces $\sigma\colon X \to Y$ is a submetry if $\sigma(B_r(p)) = B_r(\sigma(p))$ for any metric ball $B_r(p)$ in X. If both X and Y are Riemannian manifolds then σ is a Riemannian submersion of class $C^{1,1}$ by a result of Berestovskii and Guijarro [BG]. If X is a Riemannian manifold and Y is arbitrary, then the fibers of σ give rise to a generalized singular Riemannian foliation. In general the fibers can have boundary and might be only of class $C^{1,1}$.

Corollary 5. *Let (M,g) be an open nonnegatively curved manifold, and let Σ be a soul of M. Then there is a noncompact Alexandrov space A and a submetry*

$$\sigma\colon M \to \Sigma \times A$$

where $\Sigma \times A$ is endowed with the product metric. Moreover the fibers of σ are compact smooth submanifolds without boundary.

In particular, any non-contractible open manifold of nonnegative sectional curvature has a nontrivial product as a metric quotient, for the proof see section 7.

Corollary 6. *Let (M,g) be an open nonnegatively curved manifold, Σ a soul of M, and $P\colon M \to \Sigma$ the Sharafutdinov retraction. Suppose $x \in T_pM$ is horizontal with respect to P, and suppose that $v \in T_pM$ is a vertical vector perpendicular to the holonomy orbit. Then v and x span a totally geodesic flat.*

We should mention that the family of totally geodesic flats in Corollary 6 is at least as big as the family obtained from Perelman's proof of the soul conjecture. Equality occurs precisely if the normal holonomy group of Σ

acts transitively on the normal sphere. From a metric point of view this is a somewhat special case which is better understood than the general situation. For example by Walschap [W] the normal exponential map of the soul is a diffeomorphism and the cone at infinity is a ray. In general the diffeomorphism between $\nu(\Sigma)$ and M is not given by the exponential map. However using our results we can show that the diffeomorphism can be chosen such that it respects the structure of M as a doubly foliated space, see section 8.

Corollary 7. *There is a diffeomorphism $f\colon \nu(\Sigma) \to M$ satisfying*

(a) *$P \circ f = \pi$, where $\pi\colon \nu(\Sigma) \to \Sigma$ denotes the natural projection.*
(b) *f_* maps the horizontal geodesics in $\nu(\Sigma)$ onto horizontal geodesics in M, where $\nu(\Sigma)$ is endowed with the natural connection metric.*

It was shown in [Wi] that if a group G acts isometrically on a positively curved manifold with a nontrivial principal isotropy group, then the orbit space M/G has boundary. In nonnegative sectional curvature we have the following rigidity result (section 9).

Corollary 8. *Let (M,g) be a nonnegatively curved complete manifold, and suppose a Lie group G acts isometrically and effectively on (M,g) with principal isotropy group $\mathsf{H} \neq 1$. If the orbit space M/G has no boundary, then there is a closed subgroup K with $\mathsf{H} \lhd \mathsf{K}$, an invariant metric on G/K, and a G equivariant Riemannian submersion $\sigma\colon M \to \mathsf{G}/\mathsf{K}$ with totally geodesic fibers.*

The main new tool used to prove the above results is a simple and general observation which may very well be useful in different context as well. It allows us to give what we call transversal Jacobi field estimates. Let $c\colon I \to (M,g)$ be a geodesic in a Riemannian manifold (M,g), and let Λ be an $(n-1)$-dimensional family of normal Jacobi fields for which the corresponding Riccati operator is self adjoint. Recall that the Riccati operator $L(t)$ is the endomorphism of $(\dot c(t))^\perp$ defined by $L(t)J(t) = J'(t)$ for $J \in \Lambda$. Suppose we have a vector subspace $\Upsilon \subset \Lambda$. Put

$$T^v_{c(t)}M := \big\{J(t) \;\big|\; J \in \Upsilon\big\} \oplus \big\{J'(t) \mid J \in \Upsilon\,,\ J(t) = 0\big\}\,.$$

Observe that the second summand vanishes for almost every t and that $T^v_{c(t)}M$ depends smoothly on t. We let $T^\perp_{c(t)}M$ denote the orthogonal complement of $T^v_{c(t)}M$, and for $v \in T_{c(t)}M$ we define $v^\perp$ as the orthogonal projection of v to $T^\perp_{c(t)}M$. If L is non-singular at t we put

$$A_t\colon T^v_{c(t)}M \to T^\perp_{c(t)}M\,, \quad J(t) \mapsto J'(t)^\perp \quad \text{for } J \in \Upsilon\,.$$

It is easy to see that A can be extended continuously on I. For a vector field $X(t) \in T^{\perp}_{c(t)}M$ we define $\nabla^{\perp}X/\partial t = (X'(t))^{\perp}$. The following observation which is proved in section 1 is key.

Theorem 9. *Let $J \in \Lambda - \Upsilon$ and put $Y(t) := J^{\perp}(t)$. Then Y satisfies the following Jacobi equation*

$$\frac{(\nabla^{\perp})^2}{\partial t^2}Y(t) + \big(R(Y(t),\dot c(t))\dot c(t)\big)^{\perp} + 3A_tA_t^*Y(t) = 0\,.$$

One should consider $(R(\cdot,\dot c(t))\dot c(t))^{\perp} + 3A_tA_t^*$ as the modified curvature operator. The crucial point in the equation is that the additional O'Neill type term $3A_tA_t^*$ is positive semidefinite. We will denote the family of all vector fields Y obtained from Λ and Υ by Λ/Υ.

Corollary 10. *Consider an $(n-1)$-dimensional family Λ of normal Jacobi fields with a self adjoint Riccati operator along a geodesic $c\colon \mathbb{R} \to M$ in a nonnegatively curved manifold. Then*

$$\Lambda = \mathrm{span}_{\mathbb{R}}\big\{J \in \Lambda \mid J(t) = 0 \textit{ for some } t\big\} \oplus \big\{J \in \Lambda \mid J \textit{ is parallel}\big\}\,.$$

It should be understood that this does not follow from the usual Rauch or Riccati comparison for the family Λ since this fails after the first conjugate point. Instead one considers

$$\Upsilon := \big\{J \in \Lambda \mid J(t) = 0 \text{ for some } t\big\}\,.$$

Then for any $J \in \Lambda - \Upsilon$ and any $t \in \mathbb{R}$ the vector $J(t)$ is transversal to

$$T^{v}_{c(t)}M := \big\{J(t) \mid J \in \Upsilon\big\} \oplus \big\{J'(t) \mid J \in \Upsilon\,,\ J(t) = 0\big\}\,.$$

By Theorem 9 the family Λ/Υ satisfies again a Jacobi equation with nonnegative curvature operator, and as explained for this family the selfdual Riccati operator is non-singular everywhere. By the usual Riccati comparison (see for example [EH]) the Riccati operator of the family Λ/Υ vanishes and Λ/Υ consists of parallel Jacobi fields. Clearly Corollary 10 follows. We conclude the introduction with a few open problems.

Problem (Berestovskii and Guijarro). Let $\sigma\colon M \to B$ a submetry between complete nonnegatively curved manifolds. Is σ of class C^{∞}?

If one assumes in addition that M is compact and that σ is of class C^{∞} on some open subset $U \subset M$, then it is conceivable that one can modify the proof of Corollary 4 to show that σ is smooth. Similarly Theorem 3 might be viewed as support for the following

Conjecture. *Suppose $\mathcal{F}$ is a singular Riemannian foliation of a nonnegatively curved complete manifold M. Then the dual foliation has complete leaves.*

In singular spaces one can define metric foliations as subdivisions into connected subsets which are given locally by the fibers of a submetry. One can still define horizontal curves in this setting and it is natural to ask

PROBLEM. Suppose X is an Alexandrov space of nonnegative curvature and suppose that $\mathcal{F}$ is a metric foliation. Is the dual foliation also a metric foliation?

The idea for this paper came when the author thought about a problem posed by V. Kapovitch proposing that collapse of manifolds with lower curvature bound should in a suitable sense occur along the fibers of a submetry. If M is an open manifold of nonnegative sectional curvature, then the cone at infinity $C(M)$ of M is isometric to the cone at infinity of A where A is the Alexandrov space from Corollary 5. By combining with Perelman's stability theorem, if $\dim(C(M)) = \dim(A)$, then the collapse of M to $C(M)$ indeed occurs along the fibers the submetry $\mathrm{pr}_2 \circ \sigma \colon M \to A$ from Corollary 5.

REMARK 11. Corollary 10 also gives obstructions for invariant positively curved metrics on cohomogeneity one manifolds. In fact, if $c(t)$ is a normal geodesic in a positively curved cohomogeneity one G–manifold, then the Killing fields of the action give rise to an $(n-1)$–dimensional family of Jacobi fields along c with a self adjoint Riccati operator. Applying Corollary 10 gives that the Lie algebras of the isotropy groups along c generate the Lie algebra of G as a vectorspace. For more details we refer the reader to [GWZ].

I would like to thank one of the referees for useful comments.

1 The Transversal Jacobi Field Equation

In this section we prove Theorem 9. It suffices to prove the equality for a generic t_0, i.e. we may assume that the Riccati operator is non-singular at t_0 or equivalently $(\dot{c}(t_0))^{\perp} = \{J(t_0) \mid J \in \Lambda\}$. Since we can add a Jacobi field of Υ to J without changing Y, we can without loss of generality assume that $J(t_0) \in T^{\perp}_{c(t_0)}M$. Let $X_1(t), \dots, X_d(t) \in T^{\perp}_{c(t)}M$ be orthonormal vector fields with $\frac{\nabla^{\perp}}{\partial t} X_i = 0$. We may assume $J(t_0) = X_1(t_0)$. We claim

$$\big(J'(t_0)\big)^v = A^* J(t_0) \quad \text{and} \quad X_i'(t) = -A^* X_i(t)\,. \tag{1}$$

To prove these equations let $V(t)$ denote a Jacobi field in Υ. Then

$$\big\langle J'(t_0), V(t_0)\big\rangle = \big\langle J(t_0), V'(t_0)\big\rangle = \big\langle J(t), AV(t)\big\rangle\,,$$

where we used that the Riccati operator of the family Λ is self adjoint. The second equation of (1) follows from

$$\begin{aligned}0 = \tfrac{d}{dt}\big\langle X_i(t), V(t)\big\rangle &= \big\langle X_i'(t), V(t)\big\rangle + \big\langle X_i(t), V'(t)\big\rangle \\ &= \big\langle X_i'(t), V(t)\big\rangle + \big\langle X_i(t), AV(t)\big\rangle .\end{aligned}$$

Thus we can finish the proof of Theorem 9 as follows:

$$\begin{aligned}&\left\langle \frac{(\nabla^\perp)^2}{\partial t^2} J^\perp, X_k(t_0)\right\rangle + \big\langle R(J(t_0), \dot c(t_0))\dot c(t_0), X_k(t_0)\big\rangle \\ &\quad = \tfrac{d^2}{dt^2}{}_{t=t_0}\langle J, X_k\rangle - \big\langle J''(t_0), X_k(t_0)\big\rangle \\ &\quad = 2\big\langle J'(t_0), X_k'(t_0)\big\rangle + \big\langle J(t_0), X_k''(t_0)\big\rangle \\ &\quad = 2\big\langle J'(t_0), X_k'(t_0)\big\rangle + \big\langle X_1(t_0), X_k''(t_0)\big\rangle \\ &\quad = 2\big\langle J'(t_0), X_k'(t_0)\big\rangle - \langle X_1'(t_0), X_k'(t_0)\big\rangle + \tfrac{d}{dt}{}_{|t=t_0}\big\langle X_1(t), X_k'(t)\big\rangle \\ &\quad = -3\big\langle AA^* J(t_0), X_k(t_0)\big\rangle ,\end{aligned}$$

where we used $\langle X_1(t), X_k'(t)\rangle \equiv 0$ and equation (1) for the last equality. Clearly the theorem follows.

2 Some General Remarks on Dual Foliations

Proposition 2.1. *Let (M, g) be a Riemannian manifold with a transnormal system $\mathcal{F}$, and let $\mathcal{F}^\#$ denote the dual foliation. There is a family of C^∞ vector fields $(X_i)_{i\in I}$ with compact supports such that any dual leaf $\mathcal{L}^\#$ is a C^∞ immersed submanifold with $T_p\mathcal{L}^\# = \mathrm{span}_{\mathbb{R}}\{X_{i|p} \mid i \in I\}$.*

Remark 12. (a) Even if the ambient manifold and the leaves of $\mathcal{F}$ are complete it is in general not clear that the dual leaves are complete. The dual foliation could for example have open leaves.

(b) The proof below also shows that one can connect two points of one dual leaf by a piecewise horizontal geodesic.

Proof of Proposition 2.1. Let $\mathcal{L}$ be a leaf of $\mathcal{F}$. By assumption, geodesics emanating perpendicularly to $\mathcal{L}$ stay horizontal. We consider all C^∞ vector fields X in M which can be obtained as follows: there is a relatively compact open subset K of the normal bundle of $\mathcal{L}$ such that $\exp_K$ is a diffeomorphism onto its image, the image contains the support of X and $\exp_*^{-1}(X)$ is a vector field tangential to the fiber direction, that is $\exp_*^{-1}(X)$ is in the kernel of π_* where $\pi\colon \nu(\mathcal{L}) \to \mathcal{L}$ denotes the natural projection. Since the set $\exp(\nu_p(\mathcal{L}))$ is contained in a dual leaf for each $p \in \mathcal{L}$, it follows that the flow lines of such a vector field stay in a dual leaf. In fact two points on a flow line can be connected by a piecewise horizontal geodesic.

We let C denote the collection of all these vector fields where $\mathcal{L}$ runs as well. Let D denote the diffeomorphism group generated by the flows of all vector fields in C. Finally put

$$C_2 := \{\phi_* X_{\circ\phi^{-1}} \mid X \in C\,,\ \phi \in \mathsf{D}\}\,.$$

Since $\phi \in \mathsf{D}$ maps the integral curves of $X \in C$ to the integral curves of $\phi_* X_{\circ\phi^{-1}}$, the group D is also the group generated by the flows of the vector fields in C_2. By construction the singular distribution spanned by C_2 has constant dimension along orbits of D. Using the description of the Lie bracket as a Lie derivative, we see that Lie brackets of vector fields in C_2 are tangential to the distribution. As in the proof of Frobenius' theorem we see that the orbits of D are smooth submanifolds whose tangent space at each point is spanned by vector fields in C_2. Since these tangent spaces contain all vectors which are horizontal with respect to $\mathcal{F}$, it is clear that the orbits of D coincide with the dual leaves. □

3 The Dual Foliation in Nonnegative Curvature

In this section we prove Theorem 1 and Theorem 2 simultaneously. Let $\mathcal{F}$ be a singular Riemannian foliation of a complete nonnegatively curved manifold. Suppose the dual foliation has more than one leaf. Then there is a dual leaf $\mathcal{L}^\#$ which is not open. By Remark 12 any two points in $\mathcal{L}^\#$ can be connected by a piecewise horizontal geodesic.

We first plan to show that for any $\mathcal{F}$-horizontal geodesic $c\colon \mathbb{R} \to \mathcal{L}^\#$ the normal space of $\mathcal{L}^\#$ along c is spanned by parallel Jacobi fields.

We consider the family Λ of normal Jacobi fields along c that correspond to variations of c by geodesics emanating perpendicularly to $\mathcal{L}(c(0))$ at time 0. Clearly, the Riccati operator corresponding to Λ is self adjoint.

Consider a Jacobi field $J \in \Lambda$ with $J(t_0) = 0$ for some t_0. We want to prove $J'(t_0) \in \nu(\mathcal{L}(c(t_0)))$. By assumption J is the variational vector field a variation $c_s(t)$ of c by horizontal geodesics. Let Y_i $(i \in I)$ be a family of vector fields satisfying $\mathrm{span}_{\mathbb{R}}\{Y_{i|p} \mid i \in I\} = T_p\mathcal{L}(p)$ for all p. Since $\dot c_s(t_0)$ is perpendicular to $Y_{i|c_s(t_0)}$ and $\frac{d}{ds}_{|s=0} c_s(t_0) = J(t_0) = 0$, we get

$$0 = \tfrac{\partial}{\partial s}_{|s=0}\langle Y_{i|c_s(t_0)}, \dot c_s(t_0)\rangle = \langle Y_{i|c(t_0)}, J'(t_0)\rangle$$

for all i. Thus $J'(t_0) \in \nu(\mathcal{L}(c(t_0)))$. This shows that J can be written also as a variation of c by horizontal geodesics with a fixed value $c(t_0)$ at time t_0.

Therefore $J(t)$ is tangential to the dual leaf $\mathcal{L}^\#$ for all t. Hence the vector fields in

$$\Upsilon := \mathrm{span}_{\mathbb{R}} \left\{ J \in \Lambda \mid J(t) = 0 \text{ for some } t \right\}$$

are everywhere tangential to the dual leaf $\mathcal{L}^\#$.

We deduce from Corollary 10 that Λ contains a nontrivial subfamily of parallel Jacobi fields and this completes the proof of Theorem 1.

We proceed with the proof of Theorem 2. Each Jacobi field $J \in \Lambda$ with $J(0) \in T\mathcal{L}^\#$ is everywhere tangential to $\mathcal{L}^\#$. Therefore, the subspace $V \subset \Lambda$ of normal Jacobi fields which are everywhere tangential to $\mathcal{L}^\#$ has the maximal possible dimension $\dim(\mathcal{L}^\#) - 1$. Notice that the decomposition of Corollary 10 necessarily defines two pointwise orthogonal families of Jacobi fields. Because of $\Upsilon \subset V$ we see that the normal bundle of $\mathcal{L}^\#$ along c is spanned by parallel Jacobi fields. The rest of the proof is divided into three steps.

Step 1. Let $\mathcal{L}_0^\#$ be a dual leaf of maximal dimension. Then $\mathcal{F}^\#$ induces a (non-singular) Riemannian foliation in the r-tube $B_r(\mathcal{L}_0^\#)$ around $\mathcal{L}_0^\#$ for a suitable small $r > 0$.

By Proposition 2.1 there is an open neighborhood U of $\mathcal{L}_0^\#$ such that $\mathcal{F}^\#$ induces an actual foliation on U. We may assume that U decomposes into dual leaves. In fact otherwise we can replace U by $\bigcup_{\phi \in \mathsf{D}} \phi(U)$, where D denotes the group of diffeomorphisms defined in the proof of Proposition 2.1. Let $\mathcal{L}^\# \subset U$ be a dual leaf, and let $N \subset \nu(\mathcal{L}^\#)$ be an induced leaf of the normal bundle of $\mathcal{L}^\#$. The natural projection $N \to \mathcal{L}^\#$ is a covering and along $\mathcal{F}$-horizontal geodesics in $\mathcal{L}^\#$, the submanifold N develops by parallel Jacobi fields. Since any two points in a dual leaf can be connected by a piecewise horizontal geodesic, it follows that N consists of vectors of the same length. Clearly this shows that $\mathcal{F}^\#$ is a Riemannian foliation in U.

Choose a point $p \in \mathcal{L}_0^\#$ and a number $r > 0$ such that $B_r(p) \subset U$. Since the Riemannian foliation $\mathcal{F}_{|U}$ decomposes into intrinsically complete dual leaves, it follows that $B_r(\mathcal{L}_0^\#) \subset U$. This completes the proof of Step 1.

Step 2. Let $\mathcal{L}_0^\#$ be a dual leaf of maximal dimension, and let $N \subset \nu(\mathcal{L}_0^\#)$ be an induced leaf of the normal bundle of $\mathcal{L}_0^\#$. There is a unique maximal $s_0 \in (0, \infty]$ such that $\exp(sN)$ is a dual leaf of maximal dimension for all $0 < s < s_0$. If $s_0 < \infty$, then $\exp(s_0 N)$ is a dual leaf whose dimension is not maximal. Furthermore, the map $N \to \exp(s_0 N)$, $v \mapsto \exp(s_0 v)$ is a submersion.

By Step 1 $\exp(sN)$ is a dual leaf of maximal dimension for small $s > 0$. Suppose $s_0 < \infty$. The inclusion $\mathcal{L}^\#(\exp(s_0x)) \subset \exp(s_0N)$ follows from $\mathcal{L}^\#(\exp(sx)) \subset \exp(sN)$ for $s < s_0$ and Proposition 2.1.

We next want to prove that $\exp(s_0N)$ is contained in a dual leaf. Here the definition of $\mathcal{F}^\#$ enters the proof once more. Fix $x \in N$ and let $y \in N$ be any other point. Choose a piecewise horizontal geodesic $\tilde{c}$ from the foot-point of x to the foot-point of y such that x and y are parallel along $\tilde{c}$. Let $X(t)$ be the parallel vector field along $\tilde{c}$ with $X(0) = x$. By the previous considerations $c_s = \exp(sX(t))$ is a variation of curves that maps to the trivial variation on a local quotient, $s < s_0$. Since c_0 projects to a locally minimizing curve in a local quotient we deduce from the Rauch II comparison theorem and the equality discussion that $c_s = \exp(sX(t))$ is a piecewise horizontal geodesic as well, $s < s_0$. By continuity the same holds for $s = s_0$ and $\exp(s_0y)$ is contained in the same dual leaf as $\exp(s_0x)$. In other words, $\exp(s_0N) \subset \mathcal{L}^\#(\exp(s_0x))$. Thus $\exp(s_0N) = \mathcal{L}^\#(\exp(s_0x))$. Since s_0 was chosen maximal it is clear that $\mathcal{L}^\#(\exp(s_0x))$ cannot have maximal dimension. It remains to check that the map $\psi\colon N \to \exp(s_0N)$, $v \mapsto \exp(s_0v)$ is a submersion. Put $\mathcal{L}_1^\# := \exp(s_0N)$. We define a map $\varphi\colon N \to \nu(\mathcal{L}_1^\#)$ by assigning to $x \in N$ the normal vector $y = -\frac{d}{ds}_{|s=s_0}\exp(sx)$. Clearly ι is an injective immersion. If we let $\pi\colon \nu(\mathcal{L}_1^\#) \to \mathcal{L}_1^\#$ denote the natural projection, then $\psi = \pi \circ \varphi$. Thus it suffices to prove that $\pi_{|\varphi(N)}$ is a submersion.

Consider vector fields $(X_i)_{i\in I}$ as in Proposition 2.1, and let D denote the diffeomorphism group generated by the flows of these vector fields. In particular the orbits of D are dual leaves. If we identify $\nu_p(\mathcal{L}_1^\#)$ with $T_pM/T_p\mathcal{L}_1^\#$, we get a natural action of D on the normal bundle $\nu(\mathcal{L}_1^\#)$. It is clear that $\varphi(N)$ is invariant under this action. Since $\pi_{|\varphi(N)}$ is equivariant with respect to the D-action it follows that it is a submersion.

Step 3. $\mathcal{F}^\#$ is a singular Riemannian foliation.

Consider again a dual leaf $\mathcal{L}_0^\#$ of maximal dimension. Notice that the closure F of the immersed submanifold $\mathcal{L}_0^\#$ in M is contained in the tubular neighborhood $B_r(\mathcal{L}_0^\#)$ from Step 1. In particular we deduce that F decomposes into dual leaves of maximal dimension.

We claim that the set of points in M for which the dual leaves have maximal dimension is open and dense. In fact for $q \in M$ choose a minimal geodesic $c\colon [0,1] \to M$ from F to q. We have seen that $\mathcal{L}^\#(c(0))$ has maximal dimension and clearly $\dot{c}(0) \in \nu(\mathcal{L}^\#(c(0)))$. If we let N denote the leaf of $\dot{c}(0)$ in $\nu(\mathcal{L}^{\#}(c(0)))$, then $\exp(sN)$ is a leaf of maximal dimension for $s \in [0,1)$. In fact for each $s \in [0,1)$ the map $N \to \exp(sN)$, $x \mapsto \exp(sv)$ is injective because otherwise c would not be a minimal geodesic from $\mathcal{L}^{\#}(c(0)) \subset F$ to q.

We are now ready to verify that $\mathcal{F}^{\#}$ is a singular Riemannian foliation. Let $q_0 \in M$. It suffices to show that each geodesic emanating perpendicularly to $\mathcal{L}^{\#}(q_0)$ at q_0 stays for a short time perpendicular to the dual leaves. We let L_r denote the component of $\mathcal{L}^{\#}(q_0) \cap B_{5r}(p)$ with $q_0 \in L_r$ for small r. Since $\mathcal{L}(q_0)$ is an immersed submanifold it is clear that L_r is Lipschitz continuous in $r \in (0, r_0]$ with respect to the Hausdorff distance between subsets of M. We also may assume that the normal exponential map of L_r has an injectivity radius $> 3r$.

Clearly we can establish our claim by verifying the following statement for some $r > 0$: for any dual leaf $\mathcal{L}^{\#}$ the distance function $d(\cdot, L_r)$ is locally constant on $B_r(q_0) \cap \mathcal{L}^{\#}$. As above, we can find a leaf $N \subset \nu(\mathcal{L}^{\#}_h)$ in the normal bundle of a dual leaf of maximal dimension such that $\exp(sN)$ is a dual leaf of maximal dimension for all $s \in [0,1)$ and $\mathcal{L}^{\#}(q_0) = \exp(N)$.

We choose an element $u \in N$ with $\exp(u) = q_0$ and let for $\delta \ll r$, $L_r(\delta)$ denote the connected component of $\mathcal{L}^{\#}(\exp(1-\delta)u) \cap B_{5r}(q_0)$. We have seen above that the map $N \mapsto \mathcal{L}(\exp(1-\delta)u)$, $x \mapsto \exp((1-\delta)x)$ is injective and thus there is a local submersion $L_r(\delta) \to L_{r+\delta}$ that maps $\exp((1-\delta)x)$ to $\exp(x)$. In summary, we can say that the Hausdorff distance between L_r and $L_r(\delta)$ is proportional to δ.

Therefore it suffices to check that the following holds. Let L_1 denote a component of $\mathcal{L}^{\#}_1 \cap B_{5r}(q_0)$ that intersects $B_r(q_0)$ where $\mathcal{L}^{\#}_1$ is a dual leaf of maximal dimension. Then for any other dual leaf $\mathcal{L}^{\#}_2$ the function $q \mapsto d(L_1, q)$ is locally constant on $\mathcal{L}^{\#}_2 \cap B_r(q_0)$.

Fix a point in $q' \in \mathcal{L}_2 \cap B_r(q_0)$. We plan to show that $\mathcal{L}_2 \cap B_r(q_0) \to \mathbb{R}$, $q \mapsto d(q, L_1)$ attains a local maximum at q'. Since q' was arbitrary, this will imply that the function is locally constant. Choose a vector $v \in TM$ of minimal length with a foot point in the closure of L_1 and with $\exp(v) = q'$. The foot point p_v of v is clearly contained in $B_{3r}(q_0)$ and the dual leaf $\mathcal{L}^{\#}(p_v)$ has maximal dimension as well. Furthermore an intrinsic open neighborhood L' of $p(v)$ in $\mathcal{L}^{\#}(p_v)$ is contained in the closure of L_1 in M. In particular $d(q, L_1) \le d(q, L')$ for all $q \in M$. Therefore it suffices to prove that $L_2 \cap B_r(q_0) \to \mathbb{R}$, $q \mapsto d(q, L')$ attains a local maximum at q'. Let N be the induced leaf in the normal bundle of $\mathcal{L}^{\#}(p_v)$ with $v \in N$, and let N' be the connected component of N intersected with the normal bundle of L'

with $v \in N'$. For all $s < 1$ the set $\exp(sN')$ is not contained in a singular dual leaf because otherwise the geodesic $\exp(\tau v)$ would not be a minimal connection from L' to q'. By our previous considerations it follows that for all $s \leq 1$ the set $\exp(sN')$ is an open subset of a dual leaf. In particular, $L_2 \cap B_r(q_0) \to \mathbb{R}$, $q \mapsto d(q, L')$ attains a local maximum at q'.

4 Smoothness of the Sharafutdinov Retraction

The aim of this section is to prove Corollary 4.

We consider the Sharafutdinov retraction $P\colon M \to \Sigma$. By Perelman P is a Riemannian submersion of class $C^{1,1}$. Moreover, $P \circ \exp\colon \nu(\Sigma) \to \Sigma$ equals the natural projection π from the normal bundle $\nu(\Sigma)$ to the soul Σ. We let $\mathcal{F}$ denote the fiber decomposition given by P and $\mathcal{F}^{\#}$ the dual foliation. There is a distance tube $B_r(\Sigma)$ of radius r around Σ on which P is of class C^∞. Also any horizontal curve in M has constant distance to the soul. Thus there is a natural subdivision of $B_r(\Sigma)$ into dual leaves. These submanifolds are of class C^∞ and for suitable small r they are also intrinsically complete since they are via the exponential map diffeomorphic to the corresponding dual leaves in $\nu(\Sigma)$.

Theorem 4.1. *Consider the dual foliation $\mathcal{F}^{\#}$ of an open nonnegatively curved manifold M. Suppose $\mathcal{L}^{\#}$ is a dual leaf of class C^∞, and assume that $P_{|\mathcal{L}^{\#}}$ is smooth as well.*

(a) *For each $v \in \nu(\mathcal{L}^{\#})$, the curve $P(\exp(tv))$ is constant in t.*

(b) *Let $c(t) \in \mathcal{L}^{\#}$ be a piecewise geodesic which is horizontal with respect to P, and let $X(t)$ be a parallel vector field along c with $X(0) \in \nu(\mathcal{L}^{\#})$. Then $\exp(X(t))$ is a piecewise horizontal geodesic with respect to P as well.*

(c) *Let $F_1 = P^{-1}(p_0)$, $F_2 = P^{-1}(q_0)$ be fibers of the Sharafutdinov retraction. Consider a broken geodesic in Σ from p_0 to q_0 and $l\colon F_1 \to F_2$, $p \mapsto c_p(1)$, where c_p denotes the unique horizontal lift of c with $c_p(0) = p$. For $p \in F_1 \cap \mathcal{L}^{\#}$ and $q = c_p(1)$ the diagram*

$$\begin{array}{ccc} \nu_p(\mathcal{L}^{\#}) & \xrightarrow{Par_{c_p}} & \nu_q(\mathcal{L}^{\#}) \\ \exp \downarrow & & \downarrow \exp \\ F_1 & \xrightarrow{l} & F_2 \end{array}$$

commutes, where Par_{c_p} denotes the parallel transport along c_p.

SUBLEMMA 4.2. *Let $c(t)$ be a horizontal geodesic in M. Then there is an $(n-1)$-dimensional family Λ of normal Jacobi fields along c with a self-adjoint Riccati operator such that each Jacobi field in Λ is the variational vector field of a variation of horizontal geodesics.*

If P is of class C^∞ in a neighborhood of $c(0)$, then the sublemma is a general statement on Riemannian submersions. Since any horizontal geodesic is a limit of such geodesics, the result follows.

Proof of Theorem 4.1. Let $c(t)$ be a horizontal geodesic in $\mathcal{L}^\#$. Choose a family of Jacobi fields Λ as in the sublemma. As in the proof of Theorem 2 one can show for each $J \in \Lambda$ that if $J(t) \in T\mathcal{L}^\#$ for some t, then $J(t) \in T\mathcal{L}^\#$ for all t. As before we deduce that the normal bundle of $\mathcal{L}^\#$ along c is spanned by parallel Jacobi fields contained in Λ.

Therefore each normal vector v of $\mathcal{L}^\#$ has the property that parallel transport along any broken P-horizontal geodesic maps v to a vector which is perpendicular to the dual leaf and hence vertical with respect to P. Using Theorem 3.1 in [Gu] part (a) and (b) follow. Part (c) is just a simple consequence of (b). □

The proof of Theorem 3.1 in [Gu] is a generalization of Perelman's proof of the soul conjecture. A similar generalization will be given in section 5.

LEMMA 4.3. *The dual leaves are immersed submanifolds of class C^∞, and the restriction of P to each dual leaf $\mathcal{L}_1^\#$ is of class C^∞.*

Proof. As before we choose $r > 0$ such that P is of class C^∞ in $B_r(\Sigma)$. Let $\mathcal{L}^\#$ a generic dual leaf in $B_r(\Sigma)$. In other words the intersection of $\mathcal{L}^\#$ with a fiber of P corresponds to a principal orbit of the action of the normal holonomy group on the fiber. Then $\mathcal{L}^\#$ is of class C^∞. Furthermore the trivialization of the normal bundle $\nu(\mathcal{L}^\#)$ of $\mathcal{L}^\#$ which is given by Bott parallel vector fields is of class C^∞ as well. We recall that Bott parallel vector field in $\nu(\mathcal{L}^\#)$ is locally given as the horizontal lift of a fixed vector in a local quotient $U/\mathcal{F}^\#$ space of the dual foliation. Consider a Bott parallel vector field X in the normal bundle of $\mathcal{L}^\#$. Then X is parallel along any horizontal geodesic in $\mathcal{L}^\#$ and by Theorem 4.1 the image of the map

$$h\colon \mathcal{L}^\# \to M,\ p \mapsto \exp(X(p))$$

is a dual leaf $\mathcal{L}_1^\#$. Of course it is also clear that all dual leaves arise in this way. Moreover the map is of class C^∞. In order to show that $\mathcal{L}_1^\#$ is of class C^∞ it suffices to show the map h has constant rank.

The differential of h at p gives rise to a family of Jacobi fields along the geodesic $s \to \exp(sX(p))$. By Theorem 4.1 this family is the sum of a subfamily of parallel Jacobi fields which are horizontal with respect to P and a vertical family. Thus the kernel of h_{*p} is vertical with respect to P. Let q be another point in $\mathcal{L}^{\#}$, c_p be a horizontal broken geodesic from p to q, and put $c = P \circ c_p$. Finally we define

$$l\colon P^{-1}(P(p)) \to P^{-1}(P(q))$$

as in the Theorem 4.1 (c). Since P is of class $C^{1,1}$ the map l is locally bilipschitz. Furthermore P and l are of class C^∞ in a neighborhood of p. By Theorem 4.1 the diagram

$$\begin{array}{ccc} \mathcal{L}^{\#} \cap P^{-1}(P(p)) & \xrightarrow{\ l\ } & \mathcal{L}^{\#} \cap P^{-1}(P(q)) \\ h \downarrow & & \downarrow h \\ F_1 & \xrightarrow{\ l\ } & F_2 \end{array}$$

commutes. Thus the kernel of h_{*q} is given by the image of the kernel of h_{*p} under l_{*p}. In particular, h is a map of constant rank. Thus $\mathcal{L}_1^{\#}$ is of class C^∞. In order to show that $P_{|\mathcal{L}_1^{\#}}$ is of class C^∞, we observe that $P \circ h = P_{|\mathcal{L}^{\#}}$ by Theorem 4.1. Since $\mathcal{L}^{\#}$ is of class C^∞ and $h\colon \mathcal{L}^{\#} \to \mathcal{L}_1^{\#}$ is a smooth submersion, it follows that $P_{|\mathcal{L}_1^{\#}}$ is of class C^∞ as well. □

Proof of Corollary 4. Let $p \in M$. By Lemma 4.3 $\mathcal{L}^{\#}(p)$ is of class C^∞ and $P_{|\mathcal{L}^{\#}(p)}$ is of class C^∞ as well. Because of Theorem 4.1 $P \circ \exp_{\nu(\mathcal{L}^{\#}(p))} = P \circ \pi$, where $\pi\colon \nu(\mathcal{L}^{\#}(p)) \to \mathcal{L}^{\#}(p)$ is the natural projection. Since $P \circ \pi$ is of class C^∞ and $\exp_{\nu(\mathcal{L}^{\#}(p))}$ is a local diffeomorphism in a neighborhood of 0_p, it follows that P is of class C^∞ in a neighborhood of p. □

5 Completeness of Dual Leaves.

This section is devoted to the proof of Theorem 3. We first consider the case (a). This case is in fact rather obvious. Since the group acts transitively on the space of dual leaves, all dual leaves have the same dimension and, by Proposition 2.1, the dual foliation is an actual non-singular foliation.

(b) Let $\mathcal{F}$ be a Riemannian foliation by k-dimensional leaves of a nonnegatively curved n-dimensional compact manifold M. We choose a finite foliated atlas consisting of maps $x_i\colon U_i \to \mathbb{D}^k \times \mathbb{D}^{n-k}$ where $\mathbb{D}^k, \mathbb{D}^{n-k}$ are unit discs in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ respectively.

Notice that for each i the disc $\mathbb{D}^{n-k}$ carries a natural quotient metric g_i. We let $\sigma_i\colon U_i \to (\mathbb{D}^{n-k}, g_i)$ denote the Riemannian submersion.

Choose $\varepsilon > 0$ such that the injectivity radius of the normal exponential map of each of these k-dimensional discs is larger than 2ε and for each point $p \in M$ there is an i with $\bar{B}_{2\varepsilon}(p) \subset U_i$. We let $H \subset TM$ denote the set of all unit vectors which are perpendicular to the dual leaves. By Proposition 2.1 it is clear that H is compact.

We claim that for $v \in H$ the geodesic $\exp(sv)$ ($s \in [0, \varepsilon]$) stays in one leaf of $\mathcal{F}$.

As in [Gu] we modify Perelman's proof of the soul conjecture to establish our claim. We define a displacement function as follows. For $v \in H$ consider the foot point p of v and choose an i with $B_{2\varepsilon}(p) \subset U_i$. Put

$$\operatorname{dis}(s, v) := d\big(\sigma_i(p), \sigma_i(\exp(sv))\big)\,,$$

where $\sigma_i \colon U_i \to (\mathbb{D}^{n-k}, g_i)$ is the Riemannian submersion. It is an important and elementary fact that $\operatorname{dis}(s, v)$ is independent of the choice of i. We consider

$$f(s) := \max\big\{ \operatorname{dis}(s, v) \mid v \in H \big\}\,.$$

Clearly it suffices to prove that the function $f_{[0,\varepsilon]}$ is monotonously decreasing. Suppose $f(t) > 0$ for some $t \in [0, \varepsilon]$. Choose $v \in H$ with $f(t) = \operatorname{dis}(t, v)$ and i with $\bar{B}_{2\varepsilon}(p) \subset U_i$, where p is the foot point of v.

$$f(t) = d\big(\sigma_i(\exp(tv), \sigma_i(p))\big)\,.$$

Let $c \colon [0, 1] \to (\mathbb{D}^{n-k}, g_i)$ be the unique minimal geodesic from $\sigma_i(p)$ to $\sigma_i(\exp(tv))$, and let $c^h(s)$ be the unique horizontal lift of c starting at p. By construction there is a $\delta > 0$ such that the extended geodesics $c^h_{[-\delta,1]}$ and $c_{[-\delta,1]}$ are minimal. Furthermore by choosing δ sufficiently small we may assume that $B_{2\varepsilon+\delta f(t)}(p) \subset U_i$. Extend v to a parallel vector field X along c^h. From the proof of Theorem 2 we know that X stays perpendicular to the dual leaf.

By applying the Rauch comparison we see that the curve $\exp(tX(s))$ ($s \in [-\delta, 0]$) is not longer than the curve $c_{|[-\delta,0]}$. Thus

$$d\big(\sigma_i(\exp(tX(-\delta))), c(1)\big) \le d\big(\exp(tX(-\delta)), \exp(tX(0))\big) \le d\big(c(0), c(-\delta)\big)\,.$$

Therefore

$$\begin{aligned} d\big(\sigma_i(\exp(tX(-\delta))), c(-\delta)\big) &\ge d\big(c(1), c(-\delta)\big) - d\big(c(0), c(-\delta)\big) \\ &= d\big(c(1), c(0)\big)\,. \end{aligned}$$

Using $c(-\delta) = \sigma_i(c^h(-\delta))$ our choice of v implies that equality must hold. By the equality discussion in Rauch II the strip $\exp(\tau X(s))$, $s \in [-\delta, 0]$, $\tau \in [0, t]$ is flat. Thus

$$d\big(\sigma_i(\exp((t-h)X(-\delta))), c(1)\big)^2 \le d\big(\exp((t-h)X(-\delta)), \exp(tX(0))\big)^2$$

$$\leq d\big(c(0), c(-\delta)\big)^2 + h^2$$

and

$$\begin{aligned} f(t-h) &\geq d\big(\sigma_i(\exp((t-h)X(-\delta))), c(-\delta)\big) \\ &\geq d\big(c(1), c(-\delta)\big) - d\big(c(-\delta), c(0)\big) - \frac{h^2}{2d(c(0), c(-\delta))} \\ &= f(t) - \frac{h^2}{2d(c(0), c(-\delta))} \,. \end{aligned}$$

Therefore

$$\lim_{h \uparrow 0} \frac{f(t) - f(t-h)}{h} \leq 0 \,.$$

Consequently $f_{[0,\varepsilon]}$ is monotonously decreasing and thereby constant.

Using the equality discussion in Rauch II we see that for a piecewise $\mathcal{F}$-horizontal geodesic c in a dual leaf $\mathcal{L}^{\#}$ and a parallel unit vector field X along c which is normal to $\mathcal{L}^{\#}$ the curves $t \mapsto \exp(sX(t))$ are piecewise $\mathcal{F}$-horizontal geodesics as well ($s \in [0, \varepsilon]$).

Suppose there is a dual leaf $\mathcal{L}^{\#}$ which is not complete. We may assume that $\mathcal{L}^{\#}$ has minimal dimension among all non-complete leaves. Since the intrinsic boundary of $\mathcal{L}^{\#}$ in M is a union of dual leaves, we can find a dual leaf $\mathcal{L}_1^{\#}$ in the closure of $\mathcal{L}^{\#}$ whose dimension with $\dim(\mathcal{L}_1^{\#}) < \dim(\mathcal{L}^{\#})$. From the previous claim it is clear that for any $\varepsilon' \leq \varepsilon$ the ε' neighborhood around $\mathcal{L}_1^{\#}$ is the union of dual leaves. By construction $\mathcal{L}_1^{\#}$ is in the closure of $\mathcal{L}^{\#}$. Since $\mathcal{L}^{\#}$ and $\mathcal{L}_1^{\#}$ have different dimensions, we can employ Proposition 2.1 to see that $\mathcal{L}^{\#}$ and $\mathcal{L}_1^{\#}$ have positive Hausdorff distance in M. Combining the last three statements gives a contradiction.

For the proof of (c) notice that we can apply Theorem 4.1 to see that the ε-neighborhood of a complete dual leaf decomposes into dual leaves for all small $\varepsilon > 0$. As in the previous paragraph this gives the completeness of all dual leaves.

6 Totally Geodesic Flats in Foliated Manifolds with Nonnegative Sectional Curvature

In this section we prove Corollary 6. In fact it clearly follows from the following more general result.

PROPOSITION 6.1. *Let $\mathcal{F}$ be a singular Riemannian foliation of a nonnegatively curved manifold M and suppose the dual foliation $\mathcal{F}^{\#}$ has* *complete leaves. Let $x \in T_pM$ be a vector that is horizontal with respect to $\mathcal{F}$ and $v \in T_pM$ a vector that is horizontal with respect to $\mathcal{F}^\#$. Then x and v span a totally geodesic flat.*

Proof. Let $c(t) = \exp(tx)$ and let $V(t)$ be parallel along c with $V(0) = v$. We have seen that $V(t)$ stays perpendicular to the dual leaf $\mathcal{L}^\#$. Furthermore for each t the curve $s \mapsto \exp(sV(t))$ is a horizontal geodesic with respect to $\mathcal{F}^\#$ and hence it is vertical with respect to $\mathcal{F}$. For each t one can find an $\varepsilon > 0$ such that $c_{|[t,t+\varepsilon]}$ is a local minimal connection between $\mathcal{L}(c(t))$ and $\mathcal{L}(c(t+\varepsilon))$. By Rauch II the parallel curves $t \mapsto \exp(sV(t))$ are not longer. Since these curves connect the same leaves equality must hold in Rauch's comparison theorem and thus c and V generate a totally geodesic flat. □

7 Non-Contractible, Nonnegatively Curved Open Manifolds have Nontrivial Products as Metric Quotients

In this section we prove Corollary 5.

PROPOSITION 7.1. *Let $\mathcal{F}$ be a singular Riemannian foliation of a nonnegatively curved manifold M and suppose the dual foliation $\mathcal{F}^\#$ has complete leaves. We define a singular foliation $\mathcal{F} \cap \mathcal{F}^\#$ by the property that the leaf of a point p is given by the p-component of $\mathcal{L}(p) \cap \mathcal{L}^\#(p)$. Then $\mathcal{F} \cap \mathcal{F}^\#$ is a transnormal system.*

Proof. Notice that $\mathcal{L}(p)$ and $\mathcal{L}^\#(p)$ intersect transversely. So $\mathcal{F} \cap \mathcal{F}^\#$ is indeed a subdivision into intrinsically complete immersed submanifolds. Let $u \in T_pM$ be perpendicular to $\mathcal{L}(p) \cap \mathcal{L}^\#(p)$. Then $u = x + v$ with $x \in \nu_p(\mathcal{L}(p))$ and $v \in \nu_p(\mathcal{L}^\#(p))$. By Proposition 6.1 x and v span a totally geodesic flat. Moreover it is clear form the proof of Proposition 6.1 that at each point the flat is spanned by one $\mathcal{F}$-horizontal and one $\mathcal{F}^\#$-horizontal vector. Therefore all tangent vectors of the flat are $\mathcal{F} \cap \mathcal{F}^\#$-horizontal and hence the same holds for the curve $\exp(tu)$. □

Proof of Corollary 5. We let $\mathcal{F}$ denote the foliation induced by the Sharafutdinov retraction and $\mathcal{F}^\#$ its dual. We define the leaves of $\overline{\mathcal{F}}^\#$ as the closures of leaves of $\mathcal{F}^\#$. Clearly the leaves of $\overline{\mathcal{F}}^\#$ are the fibers of a globally defined proper submetry $\sigma_2 \colon M \to A$, where A is a noncompact Alexandrov space.

Furthermore there is a distance tube $B_r(\Sigma)$ around the soul such that the leaves are via the exponential map isomorphic to the corresponding closures of dual leaves in $\nu(\Sigma)$. In particular the leaves of $\overline{\mathcal{F}}^\#$ in $B_r(\Sigma)$ are of class C^∞. Analogously to the proof of Lemma 4.3 one can now show that all leaves in $\overline{\mathcal{F}}^{\#}$ are of class C^∞.

Thus $\overline{\mathcal{F}}^{\#}$ is a transnormal system. As in the proof of Proposition 7.1 one can show that $\overline{\mathcal{F}}^{\#}$ intersected with the fibers of P gives a transnormal system as well. Hence the map $\sigma := (P, \sigma_2) \colon M \to \Sigma \times A$ is a submetry. Clearly the fibers of σ are compact and smooth. □

Notice that the fibers of σ are given by the closures of orbits of the normal holonomy group of the soul acting on the fibers of P.

8 The Horizontal Distribution of an Open Nonnegatively Curved Manifold is Linear

In this section we prove Corollary 7. We start with an observation that is somewhat related to the construction of the Sharafutdinov retraction.

Lemma 8.1. *Let M be an open nonnegatively curved manifold and let $\mathcal{F}^{\#}$ denote the dual foliation of the Sharafutdinov retraction $P \colon M \to \Sigma$.*

(a) *The convex exhaustion obtained from the soul construction is invariant under the dual foliation.*

(b) *For each dual leaf $\mathcal{L}^{\#} \neq \Sigma$, there is a sequence of dual leaves $\mathcal{L}^{\#}_n$ converging to $\mathcal{L}^{\#}$ with $\dim(\mathcal{L}^{\#}_n) = \dim(\mathcal{L}^{\#})$, $\dim(\bar{\mathcal{L}}^{\#}_n) = \dim(\bar{\mathcal{L}}^{\#})$ and $d(\Sigma, \mathcal{L}^{\#}_n) < d(\Sigma, \mathcal{L}^{\#})$.*

Proof. (a) We start by considering the Busemann function of a point $p_0 \in M$, $b(x) = \lim_{r\to\infty} d(\partial B_r(p_0), x) - r$.

Let $c \colon \mathbb{R} \to M$ be a horizontal geodesic. Then c is contained in a relatively compact set and thus $b \circ c$ is bounded. On the other hand $b \circ c$ is concave and hence $b \circ c$ is constant. This simple observation shows that the levels of b decompose into dual leaves.

Let C denote the maximal level of the Busemann function, and let ∂C denote the intrinsic boundary of C. Let Σ denote the soul of C. If ∂C is empty then $C = \Sigma$ and we are done.

We have just seen that C decomposes into dual leaves. Notice that the dual leaf Σ has constant distance to ∂C. Put

$$G := \left\{ p \in C - \partial C \mid \mathcal{L}^{\#}(p) \text{ has constant distance to } \partial C \right\}.$$

We have just seen that G is not empty and clearly G is closed in $C - \partial C$. We claim that G is open in C as well. Let $\mathcal{L}^{\#}$ denote a dual leaf in G and let r denote the distance to ∂C. Then the set $B_r(\mathcal{L}^{\#}) \cap C$ decomposes into dual leaves. Let $c(t)$ be a horizontal geodesic in $B_r(\mathcal{L}^{\#}) \cap C$. As before it is clear that $t \mapsto d(\partial C, c(t))$ is both bounded and concave and thereby constant. Thus all dual leaves in $B_r(\mathcal{L}^\#) \cap C$ have constant distance to ∂C and this in turn shows that G is open in C.

We have proved that the level sets of $d(\partial C, \cdot)$ decompose into dual leaves as well. A simple induction argument shows that the whole convex exhaustion is invariant under the dual foliation.

(b) For each point $p \in M \setminus \Sigma$ there is a unique convex set C in the convex exhaustion such that p is contained in the intrinsic boundary ∂C of C. By (a) $\mathcal{L}^\#(p) \subset \partial C$. We let T_pC denote the tangent cone of C, and let c be a horizontal geodesic in $\mathcal{L}^\#(p)$. Since C decomposes into dual leaves we can employ Theorem 4.1 to see that $T_{c(t)}C \cap \nu_{c(t)}(\mathcal{L}^\#(p))$ is parallel along c. For each $q \in \mathcal{L}^\#(p)$ we define X_q as the unique unit in the tangent cone T_qC with maximal distance to the boundary of T_qC. Clearly X is normal to the dual leaf and parallel along any horizontal geodesic in $\mathcal{L}^\#(p)$. This proves that for each $s > 0$, the image of $\mathcal{L}^\#(p) \to M, q \mapsto \exp(sX_{|q})$ is a dual leaf as well. Clearly its distance to the soul is smaller than the distance of $\mathcal{L}^\#(p)$. Moreover its dimension constant in s for small s. □

We recall that the normal holonomy group of the soul does not need to be compact even in the simply connected case.

Proposition 8.2. *Let M be an open manifold of nonnegative sectional curvature, Σ a soul of M and $p \in \Sigma$. Consider a fiber $F := P^{-1}(p) = \exp(\nu_p(\Sigma))$ of the Sharafutdinov retraction. The normal holonomy group H of Σ acts on F by diffeomorphisms and the image of the induced homomorphism $\mathsf{H} \to \mathrm{Diff}(F)$ has a compact Lie group as its closure.*

Proof. Let F_r denote the ball of radius r in F around p. Notice that the action of H leaves F_r invariant. For small r it is clear that the homomorphism $\mathsf{H} \to \mathrm{Diff}(F_r)$ has a relatively compact image, since the action of H is via the exponential map isomorphic to a linear orthogonal action.

Choose $r \in (0, \infty]$ maximal such that the image of the above homomorphism is relatively compact. Suppose, on the contrary, that $r < \infty$.

Even though the action of H is not isometric, we can use Theorem 4.1 to see that $\|L_{h*}v\| = \|v\|$ and $h \exp(v) = \exp(L_{h*}v)$ for any vector $v \in \nu_q(Hq)$ in the normal bundle of an orbit, where $L_h(q) := hq$. Consider an orbit $\bar{\mathsf{H}} \star q$ of the closure $\bar{\mathsf{H}} \subset \mathrm{Diff}(F_r)$. Let ε denote the focal radius of the normal exponential map of $\bar{\mathsf{H}} \star q$. The above discussion shows that the H action in the tubular neighborhood $B_\varepsilon(\bar{\mathsf{H}} \star q) = B_\varepsilon(\mathsf{H} \star q)$ extends naturally to an action of $\bar{\mathsf{H}}$. Thus it suffices to show that the union of all tubes $B_\varepsilon(\mathsf{H} \star q)$ covers the closure of F_r.

By Lemma 8.1 each closure of an H orbit in the boundary of F_r can be approximated by a sequence of closures of H orbits in F_r which have the same dimension as the given one. Since the closure of these orbits are the smooth fibers of a submetry on F, it follows that the focal radii of these submanifolds stay bounded below and thus each orbit in the boundary of F_r is contained in some $B_\varepsilon(\mathsf{H} \star q)$ with $q \in F_r$. □

Proof of Corollary 7. Consider a fiber $F = P^{-1}(p)$ of the Sharafutdinov retraction. Recall that the distance function of p has no critical points in F. Thus we can find a gradient-like unit vector field X in $F \setminus p$.

By Proposition 8.2 the closure $\bar{\mathsf{H}}$ acts on F. For any vector $v \in \nu_q(\bar{\mathsf{H}} \star q)$ and any $h \in \bar{\mathsf{H}}$ we have $\|L_{h*}v\| = \|v\|$ and $h \exp(v) = \exp(L_{h*}v)$, see Theorem 4.1. Using that the orbits of $\bar{\mathsf{H}}$ induce a singular Riemannian foliation we see furthermore that all minimal geodesics from q to $p \in \Sigma$ are perpendicular to $\bar{\mathsf{H}} \star q$.

It is now easy to see that for any $h \in \bar{\mathsf{H}}$ the vector field $\tilde{X}_{|q} := L_{h*}X_{|h^{-1}q}$ is a again a gradient like vector field. A simple averaging argument shows that we can find a gradient like vector field Y of bounded length that commutes with the action of $\bar{\mathsf{H}}$. We can also assume Y coincides in a small pointed neighborhood $B_\delta(p) \setminus p$ of p with the actual gradient of the distance function.

Since Y commutes with the action of the holonomy group, there is a unique way to extend Y to a vertical gradient like vector field Z on M, by pushing Y with diffeomorphism as in Theorem 4.1 to different fibers. Notice Z is given by Jacobi fields along horizontal geodesics and the flow of Z maps horizontal geodesics to horizontal geodesics.

We now consider the diffeomorphism $f : \nu(\Sigma) \to M$ given as follows: for $rv \in \nu(\Sigma)$ with $\|v\| = 1$ and $r \geq 0$ consider the integral curve γ of Z with $\gamma(0) = \exp(\delta v)$ and put $f(rv) = \gamma(r - \delta)$. Notice that $f(rv) = \exp(rv)$ for $r \leq \delta$. Since the vector field Z is vertical, f satisfies (a). Since the flow of Z maps horizontal geodesics to horizontal geodesics, f maps parallel vector fields along geodesics in Σ onto horizontal geodesics in M, as claimed in (b). □

Remark 13. (a) Of course Corollary 7 implies that the Alexandrov space A from Corollary 5 is bilipschitz equivalent to $\nu_p(\Sigma)/\bar{\mathsf{H}}$, where $\bar{\mathsf{H}}$ denotes the closure of the normal holonomy group of the soul, there is of course no global bilipschitz constant though.

(b) Because of Lemma 8.1 the convex exhaustion obtained from the soul construction in M is just the inverse image of the convex exhaustion of the soul construction in A under the submetry $\sigma_2 : M \to A$.

9 Rigidity of Non-Primitive Actions in Nonnegative Sectional Curvature

This section is devoted to the proof of Corollary 8.

PROPOSITION 9.1. *Suppose a Lie group* G *acts isometrically on a nonnegatively curved manifold* M. *Let* $\mathcal{F}$ *denote the singular Riemannian foliation induced by the orbit decomposition of* G. *Suppose the dual foliation* $\mathcal{F}^{\#}$ *has a leaf which is not dense. Then there is a closed subgroup* $\mathsf{K} \subsetneq \mathsf{G}$, *an invariant metric* G/K *and a* G*-equivariant Riemannian submersion* $\sigma\colon M \to \mathsf{G}/\mathsf{K}$. *Furthermore the fibers of* σ *are closures of leaves of* $\mathcal{F}^{\#}$.

Proof. Let $\bar{\mathcal{F}}^{\#}$ denote the foliation whose leaves are given by the closures of leaves in $\mathcal{F}^{\#}$. Clearly the group G acts transitively on the space of dual leaves and hence also on the space of leaves of $\bar{\mathcal{F}}^{\#}$. Since $\mathcal{F}^{\#}$ is a singular Riemannian submersion by Theorem 2 and Theorem 3 the leaves of $\bar{\mathcal{F}}^{\#}$ are the fibers of a submetry $\sigma\colon M \to X$. The action of G on M induces a transitive isometric action of G on X and hence X is a homogeneous space G/K.

Consider the fiber $F := \sigma^{-1}(\mathsf{K})$. Then $F = \exp(\nu(\mathsf{K}p) \cap \nu(\mathsf{G} \star p))$ for each $p \in F$ and hence F is smooth. Thus σ is a Riemannian submersion. □

Proof of Corollary 8. Consider a fixed-point component N of the principal isotropy group H which intersects a principal orbit. Using that M/G has no boundary we deduce from Proposition 11.3 in [Wi] that any isotropy group of a point $p \in N$ is contained in the normalizer $N(\mathsf{H})$ of H. In particular it follows that for all $p \in N$ the normal space $\nu_p(\mathsf{G} \star p)$ of the orbit $\mathsf{G} \star p$ is contained in T_pN. Therefore for each $p \in N$ the dual leaf $\mathcal{L}^{\#}(p)$ of p is contained in N. By Proposition 9.1 there is a Riemannian submersion $\sigma\colon M \to \mathsf{G}/\mathsf{K}$. Furthermore one fiber F of σ is contained in N. This in turn shows $\mathsf{K} \subset N(\mathsf{H})$.

We next plan to show that each fiber $F := \sigma^{-1}(x)$ is totally geodesic. For that we consider a dense dual leaf $\mathcal{L}^{\#} \subset F$. Let M' denote the union of all principal orbits in M, and let $p \in M' \cap \mathcal{L}^{\#}$. Using that M/G has no boundary it is not hard to check

$$M' \cap \mathcal{L}^{\#} := \{q \in M \mid \text{there is piecewise horizontal curve in } M' \text{ from } p \text{ to } q\}.$$

Consider a Killing field X which is perpendicular to $\mathcal{L}^{\#}$ at p. We claim that X is perpendicular to $\mathcal{L}^{\#}$ for all $q \in \mathcal{L}^{\#}$. To prove this we may assume that $q \in \mathcal{L}^{\#} \cap M'$. Since there is a piecewise horizontal geodesic in M' from p to q it suffices to prove the following. If $c(t)$ ($t \in [0,1]$) is a horizontal geodesic in $M' \cap \mathcal{L}^{\#}$ and $X_{|c(0)} \in \nu(\mathcal{L}^{\#})$, then $X_{|c(1)} \in \nu(\mathcal{L}^{\#})$. But this is clear since by the proof of Theorem 2 X is a parallel Jacobi field along c which is perpendicular to $\mathcal{L}^{\#}$.

This shows that each Bott parallel vector field X along $\mathcal{L}^{\#}$ is the restriction of a Killing field. By restricting attention to those Bott parallel vector fields along $\mathcal{L}^{\#}$ which are perpendicular to the closure F of $\mathcal{L}^{\#}$, we see that the Bott parallel vector fields along F with respect to σ are given by Killing fields. Of course this implies that the holonomy maps of the submersion $\sigma\colon M \to \mathsf{G}/\mathsf{K}$ are isometries and hence the fibers of σ are totally geodesic. □

Remark 14. Notice that the fibers of the Riemannian submersion σ of Proposition 9.1 are pairwise isometric. It would be interesting to know whether the fibers have nonnegative curvature.

10 A Slice Theorem for Dual Foliations

In nonnegative curvature dual foliations have an additional remarkable property.

Theorem 10.1. *Let $\mathcal{F}$ be a singular Riemannian foliation of a nonnegatively curved manifold M. Suppose the dual foliation has closed leaves. Then there there is subgroup $\mathsf{D} \subset \mathrm{Diff}(M)$ such that the leaves of $\mathcal{F}^{\#}$ are orbits of the D–action and for each dual leaf $\mathcal{L}^{\#}$ the D action on a suitable tubular neighborhood $B_r(\mathcal{L}^{\#})$ is orbit equivalent to the natural action of D on $\nu(\mathcal{L}^{\#})$.*

We recall that the action of D on $\nu(\mathcal{L}^{\#})$ is induced by the identification $\nu_q(\mathcal{L}^{\#}) = T_qM/T_q\mathcal{L}^{\#}$ for all $q \in \mathcal{L}^{\#}$. In particular it follows that each tangent cone of the orbit space M/D is isometric to $\mathbb{R}^d/\mathsf{H}$, where H is a suitable subgroup of $\mathrm{O}(d)$. In other words, the singularities of M/D look like singularities on an orbit space of an isometric group action.

Proof. Define D as in the proof of Proposition 2.1. Consider a unit normal vector $v \in \nu_p(\mathcal{L}^{\#})$, a piecewise horizontal geodesic in $\mathcal{L}^{\#}$ starting at p, and the parallel vector field X along c with $X(0) = v$. By Proposition 6.1 and Proposition 7.1 the curve $t \mapsto \exp(s(X(t))$ is a piecewise horizontal geodesic as well for each $s \in \mathbb{R}$.

Let N denote the subset of $\nu(N)$ consisting of all unit vectors which are parallel to v along some piecewise horizontal geodesic. It follows that for all s the set $\exp(sN)$ is a dual leaf. Hence it suffices to prove that N is an orbit of D with respect to the induced action of D on $\nu(\mathcal{L}^{\#})$.

Consider a horizontal geodesic c in $\mathcal{L}^{\#}$ and a parallel vector field H normal to $\mathcal{L}^{\#}$. We extend $\dot{c}(0)$ to a vector field X along $\mathcal{L} := \mathcal{L}(c(0))$ in neighborhood of $c(0)$ by using radially normal parallel translation. We now choose a vector field Y in the normal bundle $\nu(\mathcal{L})$ with compact support such that Y contained in the kernel of π_* and $Y_{sX_p} = \frac{d}{dt}_{t=s} tX$ in a neighborhood of $(p, s) = (c(0), 0)$. We may assume that there is an open set $U \subset \nu(\mathcal{L})$ containing the support of Y such that $\exp_U$ is a diffeomorphism onto its image.

Let Z be the vector field in M which is exp-related to Y. By construction the flow of Z is contained in D. The induced action of the flow of Z in $\nu(\mathcal{L}^{\#})$ has $H_{[-\varepsilon,\varepsilon]}$ as an integral curve for a suitable small $\varepsilon > 0$. In summary it follows that there is an $\varepsilon > 0$ such that $H([-\varepsilon, \varepsilon])$ is contained in a D-orbit in $\nu(\mathcal{L}^{\#})$. A simple compactness argument shows that N is a D-orbit in $\nu(\mathcal{L}^{\#})$. □

Final Remarks

REMARKS. (a) One can show that for a transnormal system there are Lipschitz continuous vector fields $\{X_i \mid i \in I\}$ such that $T_p\mathcal{L}(p) = \mathrm{span}_{\mathbb{R}}\{X_{i|p} \mid i \in I\}$. Using this it is clear that Theorems 1 and 2 remain valid if one just assumes that $\mathcal{F}$ is a transnormal system.

(b) If $\mathcal{F}$ is a transnormal system such that all leaves have the same dimension, then $\mathcal{F}$ is a (non-singular) Riemannian foliation.

(c) To the best of the authors knowledge it is not known whether there is any transnormal system which is not a singular Riemannian foliation. There are claims in the literature that examples exist, but these claims would also imply that part (b) of this remark is false.

The proofs of these remarks are elementary but not trivial. Maybe the details will be carried out somewhere else.

References

[BG] V.N. Berestovskii, L. Guijarro, A metric characterization of Riemannian submersions, Ann. Global Anal. Geom. 18 (2000), 577–588.

[CS] J. Cao, M. Shaw, The smoothness of Riemannian submersions with nonnegative sectional curvature, Communication in Contemporary Math. 7 (2005), 1–8.

[ChG] J. Cheeger, D. Gromoll, On the structure of complete manifolds of nonnegative sectional curvature, Ann. of Math. 96 (1972), 413–443.

[EH] J. Eschenburg, E. Heintze, Comparison theory for Riccati equations, Manuscripta Math. 68:2 (1990), 209–214.

[GWZ] K. Grove, B. Wilking, W. Ziller, Positively curved cohomogeneity one manifolds and 3 Sasakian geometry, JDG, to appear.

[Gu] L. Guijarro, On the metric structure of open manifolds with nonnegative curvature, Pacific J. Math. 196 (2000), 429–444.

[M] P. Molino, Riemannian Foliations, Birkhäuser Boston, Inc., Boston, MA, 1988.

[P] G. Perelman, Proof of the soul conjecture of Cheeger and Gromoll, J. Differential Geom. 40 (1994), 209–212.

[W] G. Walschap, Nonnegatively curved manifolds with souls of codimension 2, J. Differential Geom. 27:3 (1988), 525–537.

[Wi] B. Wilking, Positively curved manifolds with symmetry, Ann. of Math. 163:2 (2006), 607–668.

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