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TRANSPORTATION TO RANDOM ZEROES BY THE GRADIENT FLOW

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Abstract. We consider the zeroes of the random Gaussian entire function

$$f(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}$$

($\xi_0, \xi_1, \ldots$ are Gaussian i.i.d. complex random variables) and show that their basins under the gradient flow of the random potential $U(z) = \log|f(z)| - \frac{1}{2}|z|^2$ partition the complex plane into domains of equal area.

We find three characteristic exponents 1, 8/5, and 4 of this random partition: the probability that the diameter of a particular basin is greater than R is exponentially small in R; the probability that a given point z lies at a distance larger than R from the zero, it is attracted to decays as $e^{-R^{8/5}}$; and the probability that, after throwing away 1% of the area of the basin, its diameter is still larger than R decays as e^{-R^4}.

We also introduce a combinatorial procedure that modifies a small portion of each basin in such a way that the probability that the diameter of a particular modified basin is greater than R decays as $e^{-cR^4(\log R)^{-3/2}}$.

1 Introduction and Main Results

Let $\mathcal{Z}$ be a random point process in $\mathbb{R}^d$ with the distribution invariant with respect to the isometries of $\mathbb{R}^d$. Suppose that $\mathcal{Z}$ has intensity 1; that is, the mean number of points of $\mathcal{Z}$ per unit volume equals 1. The *transportation* (a.k.a. “matching”, “allocation”, “marriage”, etc.) of the Lebesgue measure m_d to $\mathcal{Z}$ is a (random) measurable map $T : \mathbb{R}^d \to \mathcal{Z}$

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that pushes forward the Lebesgue measure m_d to the counting measure $n_{\mathcal{Z}} = \sum_{a\in\mathcal{Z}} \delta_a$ of the set $\mathcal{Z}$ (δ_a is the unit mass at a). In other words, the whole space $\mathbb{R}^d$ is split into disjoint random sets $B(a)$ of the Lebesgue measure 1 indexed by $a \in \mathcal{Z}$. Because of the invariance of the process $\mathcal{Z}$, it is natural to assume that the transportation T has an invariant distribution; i.e. that the distribution of the vector $T(x) - x$ does not depend on x. The better T is localized, the more uniformly the process $\mathcal{Z}$ is spread over $\mathbb{R}^d$. Thus it is interesting to know the optimal rate of decay of the probability tails $\mathbb{P}\{|T(x) - x| > R\}$ as $R \to \infty$. A constructive counterpart is to find an *explicit* and well-localized way to transport the Lebesgue measure m_d to the point process $\mathcal{Z}$.

The transportation of the Lebesgue measure m_d to the Poisson process in $\mathbb{R}^d$ was recently developed by Hoffman, Holroyd and Peres [HoHP1,2] (a finite volume version was studied earlier by Ajtai, Komlós and Tusnády [AjKT], Leighton and Shor [LeS], and Talagrand [T]). In this paper, we consider the random zero point set $\mathcal{Z}_f = f^{-1}(0)$ of a Gaussian entire function f in $\mathbb{C}$ and study the transportation of the two-dimensional Lebesgue measure m_2 to $\mathcal{Z}_f$.

Let

$$f(z) = \sum_{k\geqslant 0} \xi_k \frac{z^k}{\sqrt{k!}}$$

where ξ_k are independent standard complex Gaussian random variables (i.e. the density of ξ_k on the complex plane $\mathbb{C}$ is $\frac{1}{\pi}e^{-|z|^2}$). We shall call such a random function a Gaussian entire function (GEF).

The (random) zero set $\mathcal{Z}_f$ of this function is known as *flat chaotic analytic zero points* [H1,2], [L]. It is distinguished by the invariance of its distribution with respect to the isometries of $\mathbb{C}$; i.e. rotations and translations, see [ST, Part I] for details and references. Note that the intensity of the zero process $\mathcal{Z}_f$ equals $1/\pi$. In [ST, Part II], the question about the existence of a well-localized transportation of the area measure to the zero set of the GEF in $\mathbb{C}$ was studied. Using the Hall matching lemma and some potential theory, the authors of [ST] proved the existence of a transportation with sub-Gaussian decay of the tail probability. Unfortunately, the proof one obtains in this way is a pure existence proof giving no idea of what the transportation in question looks like.

The aim of this paper is to carry out another approach that was suggested but not followed in [ST, Part II], namely, the transportation by the gradient flow of a random potential. The main advantage of this approach is that it provides a quite natural and explicit construction for the desired transportation.

Let $U(z) = \log|f(z)| - \frac{1}{2}|z|^2$ be the random potential corresponding to the GEF f. The distribution of U is also invariant with respect to the isometries of the complex plane, see [ST, Part I] or section 2.2 below. We shall call any integral curve of the differential equation

$$\frac{dZ}{dt} = -\nabla U(Z)\,,$$

a *gradient curve* of the random potential U.

We orient the gradient curves in the direction of decay of U (this is the reason for our choice of the minus sign in the differential equation above). If $z \notin \mathcal{Z}_f$, and $\nabla U(z) \neq 0$, by Γ_z we denote the (unique) gradient curve that passes through the point z.

Definition 1.1 (The basin). *Let a be a zero of the GEF f. The basin of a is the set*

$$B(a) = \left\{z \in \mathbb{C} : \nabla U(z) \neq 0\,,\ \textit{and}\ \Gamma_z\ \textit{terminates at}\ a\right\}.$$

The picture below may help the reader to visualize this definition. It shows the random zeroes and the trajectories of various points under the gradient flow.

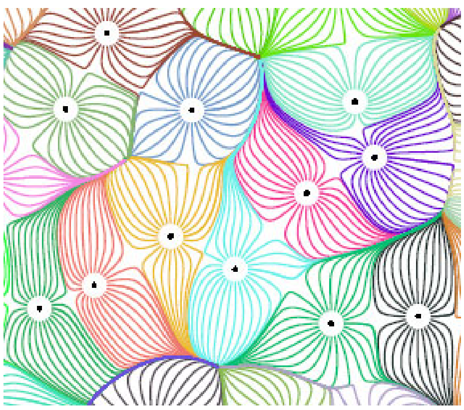

Figure 1: The basins $B(a)$ and trajectories of the gradient field

Clearly, each basin $B(a)$ is a connected open set, and $B(a') \cap B(a'') = \varnothing$ if a' and a'' are two different zeroes of f. If the basin $B(a)$ is bounded and the boundary of $B(a)$ is nice, then $\partial U/\partial n = 0$ on $\partial B(a)$ and therefore, applying the Green formula and observing that the distributional Laplacian of U equals $\Delta U = 2\pi \sum_{a\in\mathcal{Z}_f} \delta_a - 2m_2$, one gets

$$1 - \frac{m_2 B(a)}{\pi} = \frac{1}{2\pi} \iint_{B(a)} \Delta U(z)\, dm_2(z) = \frac{1}{2\pi} \int_{\partial B(a)} \frac{\partial U}{\partial n}(z)\, |dz| = 0\,;$$

i.e. $m_2 B(a) = \pi$.

Now we are ready to formulate our main results:

Theorem 1.2 (Partition). *Almost surely, each basin is bounded by finitely many smooth gradient curves (and, thereby, has area π), and*

$$\mathbb{C} = \bigcup_{a\in\mathcal{Z}_f} B(a)$$

up to a set of measure 0 *(more precisely, up to countably many smooth boundary curves).*

Consider the random set

$$S = \bigcup_{a\in\mathcal{Z}_f} \partial B(a)\,;$$

that is, the union of all " singular" gradient curves; i.e. the curves that do not terminate at $\mathcal{Z}_f$. Due to the translation invariance of the random potential U, the probability $\mathbb{P}\{z_0 \in S\}$ does not depend on the choice of the point $z_0 \in \mathbb{C}$, hence vanishes:

$$\mathbb{P}\{0 \in S\} = \tfrac{1}{\pi} \iint_{\mathbb{D}} \mathbb{P}\{z \in S\} dm_2(z) = \int_{\Omega} m_2(S \cap \mathbb{D})\, d\mathbb{P} = 0$$

(here Ω is the probability space and $\mathbb{D}$ is the unit disk). Thus, almost surely, any given point $z \in \mathbb{C}$ belongs to some basin.

By B_z we denote the basin that contains the point z. By $\operatorname{diam}(A)$ we denote the diameter of a set $A \subset \mathbb{C}$. We denote by C and c absolute (numerical) constants that may change from one line to another.

Theorem 1.3 (Diameter of the basin). *For any point $z \in \mathbb{C}$ and any $R \geqslant 1$,*

$$ce^{-CR(\log R)^{3/2}} \leqslant \mathbb{P}\big\{\operatorname{diam}(B_z) > R\big\} \leqslant Ce^{-cR(\log R)^{3/2}}\,.$$

The proof of Theorem 1.3 relies on the following auxiliary theorem. Let $Q(w,s)$ be the square centered at w with side length $2s$ and let $\partial Q(w,s)$ be its boundary.

Theorem 1.4 (Long gradient curve). *Let $R \geqslant 1$. The probability of the event that there exists a gradient curve joining $\partial Q(0,R)$ with $\partial Q(0,2R)$ does not exceed $Ce^{-cR(\log R)^{3/2}}$.*

The proof of this theorem is, unfortunately, quite involved. For a weaker upper bound $Ce^{-cR\sqrt{\log R}}$ that has a simpler proof, see the first version of this work posted in the `arxiv` [NSV]. The approach in [NSV] may be more suitable for extensions to point processes of a different nature: recently, using a similar approach, Chatterjee, Peled, Peres, and Romik found counterparts of Theorems 1.3 and 1.4 for the Poisson process in $\mathbb{R}^d$ with $d \geqslant 3$ [CPPR]. It might be helpful for the reader to look at [NSV] prior to reading the proof of the long gradient-curve theorem given here.

Let a_z be the random zero whose basin contains a given point $z \in \mathbb{C}$. In other words, the gradient curve Γ_z terminates at a_z. It appears that the probability $\mathbb{P}\{|z - a_z| > R\}$ is much smaller than the probability $\mathbb{P}\{\operatorname{diam}(B_z) > R\}$:

Theorem 1.5 (Distance to the sink)**.** *For any point $z \in \mathbb{C}$ and any $R \geqslant 1$,*

$$ce^{-CR^{8/5}} \leqslant \mathbb{P}\big\{|z - a_z| > R\big\} \leqslant Ce^{-cR^{8/5}}.$$

This is related to long, thin *tentacles* seen on the picture around some basins. They increase the typical diameter of basins though the probability that a given point z lies in such a tentacle is very small.

Let $D(w, r)$ be the disk of radius r centered at w.

Theorem 1.6 (Diameter of the core)**.** *For any $z \in \mathbb{C}$, any $\varepsilon > 0$, and any $R \geqslant 1$,*

$$c(\varepsilon)e^{-C(\varepsilon)R^4} \leqslant \mathbb{P}\big\{m_2(B_z \setminus D(a_z, R)) \geqslant \varepsilon\big\} \leqslant C(\varepsilon)e^{-c(\varepsilon)R^4}.$$

Here $c(\varepsilon)$, $C(\varepsilon)$ are positive constants that depend only on ε.

There exists a combinatorial procedure that allows one to cut the tentacles off and to get an almost optimal estimate for the diameters of the modified basins.

Theorem 1.7 (Modified basins)**.** *Given $\varepsilon > 0$, there exist open pairwise disjoint sets $B'(a)$ with the following properties:*

(i) $m_2 B'(a) = \pi$;

(ii) $\mathbb{C} = \bigcup_{a \in \mathcal{Z}_f} B'(a)$ *(up to a set of measure* 0*);*

(iii) $m_2\left(B(a) \bigcap B'(a)\right) \geqslant \pi - \varepsilon$;

(iv) *For any $z \in \mathbb{C}$, and any $R \geqslant 2$,*

$$\mathbb{P}\{\operatorname{diam}(B')_z > R\} \leqslant C(\varepsilon)e^{-cR^4(\log R)^{-3/2}}.$$

Here $(B')_z$ is the modified basin that contains the point z.

The estimate in item (iv) is not as good as the tail estimate $e^{-cR^4(\log R)^{-1}}$ that can be obtained by modification of the proof in [ST, Part II], but it comes fairly close.

Now, a few words about the tools we use in the proofs. First of all, it is the "almost independence" of the localizations of a GEF to distant disks (Theorem 3.2), which may be useful in other problems as well. In the proof of the long gradient curve theorem, we use lower bounds for the determinants of large covariance matrices of some Gaussian complex random variables. These bounds are proved in section 5. The proofs of the distance to the sink theorem 1.5, the diameter of the core theorem 1.6, and the modified basins theorem 1.7 are based on a version of the length and area principle (Proposition 8.2).

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2 Preliminaries

2.1 Basic facts about complex Gaussian random variables. We fix some probability space $(\Omega, \mathbb{P})$ and some (very big) family $\{\Xi_j\}_{j\in J}$ of independent standard complex Gaussian random variables on that probability space (i.e. the density of Ξ_j on the complex plane $\mathbb{C}$ is $\frac{1}{\pi}e^{-|z|^2}$). Every complex Gaussian random variable in this paper will be just a (possibly infinite) linear combination of Ξ_j with square summable coefficients. Such a complex Gaussian random variable ξ is standard if $\mathbb{E}|\xi|^2 = 1$.

A useful remark is that if η_k are standard complex Gaussian random variables and $a_k \in \mathbb{C}$ satisfy $\sum_k |a_k| < +\infty$, then $\sum_k a_k\eta_k$ can be represented as $a\eta$ where $0 \leqslant a \leqslant \sum_k |a_k|$ and η is some standard Gaussian random variable.

We shall start with simple probabilistic estimates.

LEMMA 2.1. *Let η_k be standard complex Gaussian random variables (not necessarily independent). Let $a_k > 0$, $a = \sum_k a_k$. Then, for every $t > 0$,*

$$\mathbb{P}\Big\{ \sum_k a_k |\eta_k| > t \Big\} \leqslant 2e^{-\frac{1}{2}a^{-2}t^2}.$$

Proof. Without loss of generality, $a = 1$. We have

$$\begin{aligned} \mathbb{P}\Big\{ \sum_k a_k|\eta_k| > t \Big\} &\leqslant e^{-\frac{1}{2}t^2} \mathbb{E} \exp \Big\{ \tfrac{1}{2} \Big(\sum_k a_k |\eta_k| \Big)^2 \Big\} \\ &\leqslant e^{-\frac{1}{2}t^2} \sum_k a_k \mathbb{E} \exp \big\{ \tfrac{1}{2} |\eta_k|^2 \big\} = e^{-\frac{1}{2}t^2} \mathbb{E} \exp \big\{ \tfrac{1}{2} |\eta|^2 \big\} \end{aligned}$$

where η is a standard complex Gaussian random variable. But

$$\mathbb{E} \exp \big\{ \tfrac{1}{2}|\eta|^2 \big\} = \tfrac{1}{\pi} \iint_{\mathbb{C}} e^{-\frac{1}{2}|z|^2} dm_2(z) = 2 \int_0^{+\infty} r e^{-\frac{1}{2}r^2} dr = 2\,. \qquad \square$$

LEMMA 2.2. *Let $\{\xi_i\}_{1 \leqslant i \leqslant n}$ be complex Gaussian random variables, and let $\Gamma = (\gamma_{ij})$ be their covariance matrix; i.e. $\gamma_{ij} = \mathbb{E}\xi_i \bar{\xi}_j$. Suppose $\det \Gamma \geqslant 1$. Then*

$$\mathbb{P}\big\{ |\xi_i| \leqslant \varepsilon\,,\ 1 \leqslant i \leqslant n \big\} \leqslant \varepsilon^{2n}.$$

Proof. The joint density function of the variables ξ_i is

$$\frac{1}{\pi^n \det \Gamma} e^{-\langle \Gamma^{-1}\xi, \xi \rangle} \leqslant \pi^{-n}.$$

Thus

$$\mathbb{P}\big\{ |\xi_i| \leqslant \varepsilon\,,\ 1 \leqslant i \leqslant n \big\} \leqslant \tfrac{1}{\pi^n} \int \cdots \int_{|\xi_1| < \varepsilon,\, \dots,\, |\xi_n| < \varepsilon} dm_2(\xi_1) \dots dm_2(\xi_n) = \varepsilon^{2n}. \quad \square$$

Now we want to elaborate on the well-known fact that a family $\{\xi_i\}_{i \in I}$ of complex Gaussian random variables is independent if and only if the covariances $\mathbb{E}\xi_i\bar{\xi}_j$ vanish for $i \neq j$.

LEMMA 2.3. *Let ξ_k be standard complex Gaussian random variables whose covariances $\gamma_{ij} = \mathbb{E}\xi_i\bar{\xi}_j$ satisfy*

$$\sum_{j: j \neq i} |\gamma_{ij}| \leqslant \sigma \leqslant \tfrac{1}{3} \quad \textit{for all } i\,.$$

Then $\xi_k = \zeta_k + b_k \eta_k$ where ζ_k are independent standard complex Gaussian random variables, η_k are standard complex Gaussian random variables, and $b_k \in [0, \sigma)$.

Proof. Note that $\|M\| = \sup_i \sum_j |m_{ij}|$ defines a norm on matrices $M = (m_{ij})$ (more precisely, it is the norm of M as an operator in ℓ^∞). Now let $\Gamma = (\gamma_{ij})$ be the covariance matrix of the family ξ_k. Note that $\Gamma = I - \Delta$ where I is the identity matrix and $\|\Delta\| \leqslant \sigma$. Then $\Gamma^{-1/2} = I - \Delta'$ with

$\|\Delta'\| \leqslant \sigma$. Indeed, using the Taylor series $(1-z)^{-1/2} = 1 + \frac{1}{2}z + \sum_{\ell\geqslant 2}\alpha_\ell z^\ell$ and observing that $|\alpha_\ell| < 1$ for all $\ell \geqslant 2$, we get

$$\|\Delta'\| = \Big\|\frac{1}{2}\Delta + \sum_{\ell\geqslant 2}\alpha_\ell \Delta^\ell\Big\| \leqslant \frac{\sigma}{2} + \sum_{\ell\geqslant 2}\sigma^\ell = \frac{\sigma}{2} + \frac{\sigma^2}{1-\sigma} \leqslant \sigma\,.$$

It remains to put $\zeta_k = \sum_j (\Gamma^{-1/2})_{kj}\xi_j$, $b_k\eta_k = \sum_j \Delta'_{kj}\xi_j$. □

2.2 Operators T_w and shift invariance. The main thing we need from f and U is their shift invariance. It is literally true that U is shift invariant (as a random process) but U is a little bit less convenient than f to work with because, firstly, it is not a Gaussian process and, secondly, it has singularities. The random function f itself is not shift invariant, but there is a simple transformation that makes a shift of f a GEF again.

For a function $f : \mathbb{C} \to \mathbb{C}$ and a complex number $w \in \mathbb{C}$, define

$$T_w f(z) = f(w+z)e^{-z\overline{w}}e^{-\frac{1}{2}|w|^2}.$$

LEMMA 2.4. *Let $f : \mathbb{C} \to \mathbb{C}$ be an arbitrary function and let $w \in \mathbb{C}$. Let $U(z) = \log|f(z)| - \frac{1}{2}|z|^2$ and let $U_w(z) = \log|T_w f(z)| - \frac{1}{2}|z|^2$. Then $U(w+z) = U_w(z)$.*

Proof.

$$\begin{aligned} U_w(z) = \log\big|T_w f(z)\big| - \tfrac{1}{2}|z|^2 &= \log\big|f(w+z)\big| - \operatorname{Re} z\overline{w} - \tfrac{1}{2}|w|^2 - \tfrac{1}{2}|z|^2 \\ &= \log\big|f(w+z)\big| - \tfrac{1}{2}|w+z|^2 = U(w+z)\,. \end{aligned}$$ □

LEMMA 2.5. *For any $w', w'' \in \mathbb{C}$, we have*

$$T_{w'+w''}f = e^{i\operatorname{Im} w'\overline{w''}}T_{w'}T_{w''}f\,.$$

Proof.

$$\begin{aligned} (T_{w'}T_{w''}f)(z) &= (T_{w''}f)(w'+z)e^{-z\overline{w'}}e^{-\frac{1}{2}|w'|^2} \\ &= f(w''+w'+z)e^{-(z+w')\overline{w''}}e^{-\frac{1}{2}|w''|^2}e^{-z\overline{w'}}e^{-\frac{1}{2}|w'|^2} \\ &= f(w'+w''+z)e^{-i\operatorname{Im} w'\overline{w''}}e^{-z\overline{(w'+w'')}}e^{-\frac{1}{2}|w'+w''|^2} \\ &= e^{-i\operatorname{Im} w'\overline{w''}}(T_{w'+w''}f)(z)\,. \end{aligned}$$ □

LEMMA 2.6. *Let f be a GEF Then $T_w f$ is also a GEF.*

Proof. It suffices to check that the covariances of these two complex Gaussian processes are the same. Recalling that, for a GEF f, we have $\mathbb{E}f(z')\overline{f(z'')} = e^{z'\overline{z''}}$, we get

$$\begin{aligned} \mathbb{E}(T_w f)(z')\overline{(T_w f)(z'')} &= e^{-z'\overline{w}-\overline{z''}w}e^{-|w|^2}\mathbb{E}f(w+z')\overline{f(w+z'')} \\ &= e^{-z'\overline{w}-\overline{z''}w}e^{-|w|^2}e^{(w+z')\overline{(w+z'')}} = e^{z'\overline{z''}} = \mathbb{E}f(z')\overline{f(z'')}\,. \end{aligned}$$ □

Note that one can give another proof of this lemma using the fact that the functions $z^n/\sqrt{n!}$ form an orthonormal basis in the Fock–Bargmann space $\mathcal{H}$ (that is, in the closure of the analytic polynomials in the weighted space $L^2_{\mathbb{C}}\big(\frac{1}{\pi}e^{-|z|^2}dm_2\big)$) and that T_w is a unitary operator on $\mathcal{H}$.

Lemma 2.6 together with Lemma 2.4 immediately imply that the random potential U corresponding to a GEF f is shift invariant (as a random process).

3 Almost Independence

Now let $T_wf(z) = \sum_{k\geqslant 0}\frac{\xi_k(w)}{\sqrt{k!}}z^k$ be the (random) Taylor series of T_wf at 0. Lemma 2.6 implies that, for a fixed $w\in\mathbb{C}$, $\xi_k(w)$ are independent standard Gaussian random variables, but, of course, the covariances between $\xi_k(w')$ and $\xi_j(w'')$ may be nontrivial for $w'\neq w''$.

Lemma 3.1.

$$\big|\mathbb{E}\xi_j(w')\overline{\xi_k(w'')}\big| \leqslant 5^{\frac{1}{2}(j+k)}e^{-\frac{1}{4}|w'-w''|^2}.$$

Proof. Let $w = w'' - w'$. Since, according to Lemma 2.5, $T_{w''}f = e^{i\,\mathrm{Im}\,w\overline{w'}}T_wT_{w'}f$, the random variable $\xi_k(w'')$ equals $e^{i\,\mathrm{Im}\,w\overline{w'}}\sqrt{k!}$ times the k-th Taylor coefficient of the function $\sum_\ell \xi_\ell(w')\frac{(w+z)^\ell}{\sqrt{\ell!}}e^{-z\overline{w}}e^{-\frac{1}{2}|w|^2}$. Hence the absolute value of the covariance in question is just $\sqrt{k!}$ times the absolute value of the k-th Taylor coefficient of the function $\frac{1}{\sqrt{j!}}(w+z)^je^{-z\overline{w}}e^{-\frac{1}{2}|w|^2}$. According to the Cauchy inequality, this coefficient does not exceed

$$\frac{1}{\sqrt{j!}}\rho^{-k}\max_{|z|=\rho}\big|(w+z)^je^{-z\overline{w}}e^{-\frac{1}{2}|w|^2}\big| \leqslant \frac{1}{\sqrt{j!}}\rho^{-k}\big(|w|+\rho\big)^je^{\rho|w|}e^{-\frac{1}{2}|w|^2}$$

for any $\rho>0$. Choosing $\rho=\frac{1}{4}|w|$, we get the estimate

$$\big|\mathbb{E}\xi_j(w')\overline{\xi_k(w'')}\big| \leqslant \frac{\sqrt{k!}}{\sqrt{j!}}4^k\left(\tfrac{5}{4}\right)^j|w|^{j-k}e^{-\frac{1}{4}|w|^2}.$$

Exchanging the roles of w' and w'', we get the symmetric inequality

$$\big|\mathbb{E}\xi_j(w')\overline{\xi_k(w'')}\big| \leqslant \frac{\sqrt{j!}}{\sqrt{k!}}4^j\left(\tfrac{5}{4}\right)^k|w|^{k-j}e^{-\frac{1}{4}|w|^2}.$$

Taking the geometric mean of these two estimates, we get the statement of the lemma. □

Our next aim is to show that the GEF $T_{w_j}f$ can be simultaneously approximated by independent GEF in the disk $|z|\leqslant r$ if all distances between the points w_j are much greater than r. More precisely, the following statement holds.

Theorem 3.2 (Almost independence). *For every $N>0$, there exists $A=A(N)>0$ such that, for all $r>1$, and for all families of points $w_j\in\mathbb{C}$*

satisfying $|w_i - w_j| \geqslant Ar$, $i \neq j$, *we can write*

$$T_{w_j} f = f_j + h_j \,,$$

where f_j *are independent GEF and* h_j *are random analytic functions satisfying*

$$\mathbb{P}\Big\{ \max_{|z|\leqslant r} |h_j(z)| > e^{-Nr^2} \Big\} \leqslant 2\exp\left\{ -\tfrac{1}{2}\exp\{Nr^2\} \right\} \,.$$

Proof. Fix two constants $A \gg B \gg 1$ to be chosen later. Consider the standard complex Gaussian random variables $\xi_k(w_j)$ with $k \leqslant B^2r^2$. We want to apply Lemma 2.3. To this end, we need to estimate the sum of covariances $\sum_{(k,j):(k,j)\neq(\ell,i)} |\mathbb{E}\xi_k(w_j)\overline{\xi_\ell(w_i)}|$. Recall that $\mathbb{E}\xi_k(w_j)\overline{\xi_\ell(w_i)} = 0$ if $i = j$. For $j \neq i$, we can use Lemma 3.1, which yields

$$\sum_{(k,j):j\neq i} \left|\mathbb{E}\xi_k(w_j)\overline{\xi_\ell(w_i)}\right| \leqslant (B^2r^2+1)5^{B^2r^2} \sum_{j:j\neq i} e^{-\frac{1}{4}|w_j-w_i|^2}.$$

It remains to estimate $\sum_{j:j\neq i} e^{-\frac{1}{4}|w_j-w_i|^2}$. Let μ be the counting measure of the set $\{w_j\}_{j\neq i}$. We have

$$\mu(D(w_i,s)) \leqslant \begin{cases} 0\,, & \text{if } s < Ar\,, \\ 9A^{-2}r^{-2}s^2, & \text{if } s \geqslant Ar\,. \end{cases}$$

(The second estimate follows from the observation that the disks $D\big(w_j, \frac{1}{2}Ar\big)$ are pairwise disjoint and contained in the disk $D\big(w_i, s+\frac{1}{2}Ar\big) \subset D\big(w_i, \frac{3}{2}s\big)$ if $|w_j - w_i| \leqslant s$ and $s \geqslant Ar$). Now, write

$$\begin{aligned} \sum_{j:j\neq i} e^{-\frac{1}{4}|w_j-w_i|^2} &= \iint_{\mathbb{C}} e^{-\frac{1}{4}|z-w_i|^2} d\mu(z) = \int_0^\infty \frac{s}{2} e^{-\frac{1}{4}s^2} \mu\big(D(w_i,s)\big) ds \\ &\leqslant 9\int_{Ar}^\infty A^{-2}r^{-2}\frac{s^3}{2} e^{-\frac{1}{4}s^2} ds = 9(1+4A^{-2}r^{-2})e^{-\frac{1}{4}A^2r^2} \\ &\leqslant 10e^{-\frac{1}{4}A^2r^2}, \end{aligned}$$

provided that $A \gg 1$ and $r > 1$. Using this estimate, we finally get

$$\sum_{(k,j):j\neq i} \left|\mathbb{E}\xi_k(w_j)\overline{\xi_\ell(w_i)}\right| \leqslant 10(B^2r^2+1)5^{B^2r^2} e^{-\frac{1}{4}A^2r^2} \leqslant e^{-\frac{1}{5}A^2r^2},$$

provided that $A \gg B \gg 1$ and $r > 1$. Applying Lemma 2.3, we conclude that $\xi_k(w_j) = \zeta_k(w_j) + b_{kj}\eta_k(w_j)$ where $\zeta_k(w_j)$ are independent standard Gaussian random variables, $\eta_k(w_j)$ are standard Gaussian random variables, and $0 \leqslant b_{kj} \leqslant e^{-\frac{1}{5}A^2r^2}$.

For $k > B^2r^2$, let $\zeta_k(w_j)$ be independent standard complex Gaussian random variables that are also independent with $\zeta_\ell(w_i)$ for all $\ell \leqslant B^2r^2$ and for all i. Put

$$f_j(z) = \sum_{k\geqslant 0} \zeta_k(w_j)\frac{z^k}{\sqrt{k!}}$$

and

$$h_j(z) = T_{w_j} f(z) - f_j(z)$$
$$= -\sum_{k>B^2r^2} \zeta_k(w_j)\frac{z^k}{\sqrt{k!}} + \sum_{k>B^2r^2} \xi_k(w_j)\frac{z^k}{\sqrt{k!}} + \sum_{k\leqslant B^2r^2} b_{kj}\eta_k(w_j)\frac{z^k}{\sqrt{k!}}\,.$$

The GEF f_j are, clearly, independent and all we need to do now is to show that h_j are small in the disk $|z| \leqslant r$. We shall use Lemma 2.1. It reduces our task to that of estimating the sum

$$2\sum_{k>B^2r^2} \frac{r^k}{\sqrt{k!}} + \sum_{k\leqslant B^2r^2} b_{kj}\frac{r^k}{\sqrt{k!}} \leqslant 2\sum_{k>B^2r^2} \frac{r^k}{\sqrt{k!}} + e^{-\frac{1}{5}A^2r^2} \sum_{k\leqslant B^2r^2} \frac{r^k}{\sqrt{k!}}\,.$$

Note that in the series $\sum_{k>B^2r^2} r^k/\sqrt{k!}$ the ratio of each term to the previous one equals $r/\sqrt{k+1} \leqslant r/Br = 1/B < 1/2$ if $B > 2$. Hence the sum does not exceed twice the first term of the series, which is

$$\frac{1}{\sqrt{k_0!}} r^{k_0} < \left(\frac{\sqrt{e}r}{\sqrt{k_0}}\right)^{k_0} \leqslant \left(\frac{\sqrt{e}r}{Br}\right)^{k_0} \leqslant \left(\frac{\sqrt{e}}{B}\right)^{B^2r^2} \leqslant e^{-B^2r^2}\,,$$

provided that $B > e\sqrt{e}$ (here k_0 is the smallest integer bigger than B^2r^2). On the other hand, the Cauchy–Schwarz inequality yields

$$\sum_{k\leqslant B^2r^2} \frac{r^k}{\sqrt{k!}} \leqslant \sqrt{B^2r^2+1}\sqrt{\sum_{k\geqslant 0} \frac{r^{2k}}{k!}} = \sqrt{B^2r^2+1}\, e^{\frac{1}{2}r^2}.$$

Thus, the sum we need to estimate does not exceed

$$4e^{-B^2r^2} + e^{-\frac{1}{5}A^2r^2}\sqrt{B^2r^2+1}\, e^{\frac{1}{2}r^2} \leqslant e^{-\frac{3}{2}Nr^2},$$

provided that $A \gg B \gg \sqrt{N}$.

It remains to apply Lemma 2.1 with $a \leqslant e^{-\frac{3}{2}Nr^2}$, $t = e^{-Nr^2}$. □

4 Size of the Potential U

First, we estimate the probability that the maximum of the random potential U over the disk of radius ρ is large positive.

LEMMA 4.1. *For $\rho \geqslant 1$ and $M > 0$,*

$$\mathbb{P}\Big\{\max_{|z|\leqslant\rho} U(z) > M\Big\} \leqslant C\rho^2 e^{-ce^{2M}}.$$

Proof. Since U is a stationary process, and since the disk $\{|z| \leqslant \rho\}$ can be covered by $C\rho^2$ copies of the unit disk, it suffices to show that

$$\mathbb{P}\Big\{\max_{|z|\leqslant 1} U(z) > M\Big\} \leqslant Ce^{-ce^{2M}}.$$

But this probability does not exceed $\mathbb{P}\{\max_{|z|\leqslant 1}|f(z)| > e^M\}$, which, in its turn, does not exceed $\mathbb{P}\big\{\sum_k \frac{1}{\sqrt{k!}}|\xi_k| > e^M\big\}$. Estimating the latter probability by Lemma 2.1, we get the desired result. □

LEMMA 4.2. *Suppose that* $\rho \geqslant 1$. *Then*

$$\mathbb{P}\Big\{ \max_{|z|\leqslant\rho} |f(z)| \leqslant e^{-3\rho^2} \Big\} < e^{-8\rho^4}.$$

Proof. Assume that $\max_{|z|\leqslant\rho} |f(z)| \leqslant e^{-3\rho^2}$. Then by Cauchy's inequalities for the Taylor coefficients of analytic functions, we have

$$|\xi_n| \leqslant \tfrac{\sqrt{n!}}{\rho^n} \max_{|z|\leqslant\rho} |f(z)| \leqslant \tfrac{n^{n/2}}{\rho^n} e^{-3\rho^2}\,, \quad n = 0, 1, 2, \dots .$$

The probabilities of these independent events do not exceed $(n\rho^{-2})^n e^{-6\rho^2}$. Thus

$$\mathbb{P}\Big\{ \max_{|z|\leqslant\rho} |f(z)| \leqslant e^{-3\rho^2} \Big\} \leqslant \prod_{0\leqslant n\leqslant 2\rho^2} \big[(n\rho^{-2})^n e^{-6\rho^2}\big] \leqslant \big(2^{2\rho^2} e^{-6\rho^2}\big)^{2\rho^2} < e^{-8\rho^4}.$$

□

Theorem 4.3. *Given* $\beta > 0$, *suppose that* ρ *is sufficiently large, and that* $\log^2 \rho \leqslant M \leqslant \rho^2$. *Then the probability of the event*

$$\Big\{ \textit{there exists a curve}\ \gamma \subset \rho^4\mathbb{D}\ \textit{with}\ \operatorname{diam}(\gamma) \geqslant \beta\rho\ \textit{such that}\ \max_\gamma U < -M \Big\}$$

does not exceed $e^{-c\rho M^{3/2}}$ *with the constant* c *depending on* β.

Recall that by $\mathbb{D}$ we denote the unit disk in the complex plane centered at the origin, $t\mathbb{D}$ is the disk of radius t concentric with $\mathbb{D}$.

Proof. We fix a sufficiently small constant $a < \min(1/2, \beta/4)$ and cover the disk $\rho^4\mathbb{D}$ by the disks $D_j = D\big(w_j, a\sqrt{M}\big)$, $j \in \mathcal{J}$, with bounded multiplicity of covering. Clearly, $\#\mathcal{J} \leqslant CM^{-1}\rho^8$.

Suppose that there exists a curve $\gamma \subset \rho^4\mathbb{D}$ with diameter at least $\beta\rho$ and such that $\max_\gamma U < -M$. Note that if γ enters the disk D_j, then it must exit the disk $2D_j = D\big(w_j, 2a\sqrt{M}\big)$; otherwise, $4a\sqrt{M}$ (the diameter of $2D_j$) is larger than $\beta\rho$ (the diameter of the curve γ), which is impossible due to our choice of a.

Let A be the constant corresponding to the value $N = a^{-2}$ in the almost independence theorem 3.2. Having the curve γ and the constants a and A, we choose a sub-collection of well-separated disks D_j, $j \in \mathcal{J}^*$, with the following properties:

- $|w_i - w_j| \geqslant 2Aa\sqrt{M}$ for $j \neq i$;
- the curve γ enters each of the disks D_j;
- $\#\mathcal{J}^* = \Big\lceil \frac{\beta\rho}{2Aa\sqrt{M}+2a\sqrt{M}} \Big\rceil = \Big\lceil \frac{\beta\rho}{2(A+1)a\sqrt{M}} \Big\rceil$.

By $\lceil x \rceil$ we denote the least integer $n \geqslant x$.

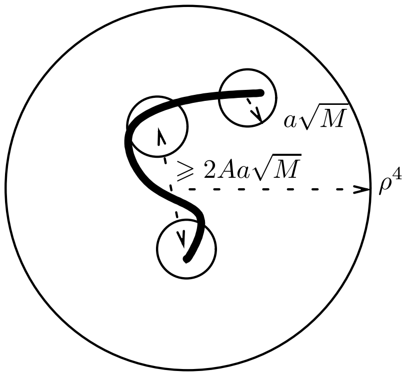

Figure 2: The curve γ and the disks $D_j = D(w_j, a\sqrt{M})$

Applying Theorem 3.2 with $r = 2a\sqrt{M}$, we get $T_{w_j} f = f_j + h_j$, $j \in \mathcal{J}^*$, where f_j are independent GEF and

$$\mathbb{P}\Big\{ \max_{2a\sqrt{M}\,\mathbb{D}} |h_j| > e^{-4M} \Big\} \leqslant 2 \exp\left[-\tfrac{1}{2} \exp(4M) \right] .$$

If $\max_{j\in\mathcal{J}^*} \max_{2a\sqrt{M}\mathbb{D}} |h_j| \leqslant e^{-4M}$, then, for $z \in (\gamma - w_j) \cap 2a\sqrt{M}\mathbb{D}$, and for big enough M,

$$\big|f_j(z)\big| \leqslant e^{-M} e^{\frac{1}{2}|z|^2} + e^{-4M} \leqslant e^{-M+2a^2M} + e^{-4M} < e^{-\frac{1}{2}M} .$$

Now, we introduce the independent events $(\star_j)$. We say that the event $(\star_j)$ occurs if there exists a curve γ_j that connects the circumferences $\big\{|z| = a\sqrt{M}\big\}$ and $\big\{|z| = 2a\sqrt{M}\big\}$ such that $|f_j(z)| < e^{-\frac{1}{2}M}$ everywhere on γ_j.

Claim 4.4. *If the constant a is small enough, then $\mathbb{P}\{(\star_j)\} \leqslant e^{-cM^2}$.*

Proof of Claim 4.4. Consider the function $\log|f_j|$ subharmonic in the disk $2a\sqrt{M}\mathbb{D}$. By Lemma 4.1, throwing away an event of probability less than

$$Ca^2Me^{-ce^{4a^2M}} < e^{-cM^2},$$

we have

$$\max_{z\in 2a\sqrt{M}\,\mathbb{D}} \left[\log|f_j(z)| - \tfrac{1}{2}|z|^2 \right] \leqslant 2a^2M$$

and hence

$$\max_{2a\sqrt{M}\,\mathbb{D}} \log|f_j| \leqslant 4a^2M, .$$

The curve γ_j connects the circumferences $\big\{|z| = a\sqrt{M}\big\}$ and $\{|z| = 2a\sqrt{M}\}$. Hence its harmonic measure with respect to $\big(2a\sqrt{M}\mathbb{D}\big)\setminus\gamma_j$ is bounded from below by a positive numerical constant c_0 uniformly in the disk $a\sqrt{M}\mathbb{D}$ (this well-known fact follows, for instance, from [A, Th. 3-6]). Thus

$$\max_{a\sqrt{M}\mathbb{D}} \log|f_j| \leqslant 4a^2M - \tfrac{c_0}{2}M < -\tfrac{c_0}{4}M\,,$$

if the constant a was chosen so small that $a^2 < \frac{1}{16}c_0$. Then $\frac{1}{4}c_0M > 3\big(a\sqrt{M}\big)^2$ and we can apply Lemma 4.2 to the function f_j in the disk $a\sqrt{M}\mathbb{D}$. The lemma yields that the probability that $(\star_j)$ happens does not exceed e^{-cM^2}. $\square$

We conclude that the existence of a curve γ satisfying the assumptions of the theorem implies existence of a subset $\mathcal{J}^* \subset \mathcal{J}$ with $\#\mathcal{J}^* = \left\lceil \frac{\beta\rho}{2(A+1)a\sqrt{M}} \right\rceil$, such that at least one of the following happens:

(i) For $j \in \mathcal{J}^*$, the *independent* events $(\star_j)$ occur with $\gamma_j = (\gamma - w_j) \cap 2a\sqrt{M}\mathbb{D}$;

(ii) $\max_{j\in\mathcal{J}^*} \max_{2a\sqrt{M}\,\mathbb{D}} |h_j| > e^{-4M}$.

In the case (i), the probability is bounded by $\binom{\#\mathcal{J}}{\#\mathcal{J}^*} \cdot \big[e^{-cM^2}\big]^{\#\mathcal{J}^*}$. Since $\binom{n}{k} \leqslant n^k$, $\#J \leqslant C\rho^8$, and $\#J^* \leqslant C\rho$, the first factor is bounded by $e^{C\rho\log\rho}$. The second factor does not exceed $e^{-cM^{3/2}\rho}$. Since $\log\rho \leqslant M^{1/2}$, the whole product is bounded by $e^{-cM^{3/2}\rho}$. In the case (ii), the probability does not exceed

$$\#\mathcal{J}^* \cdot \binom{\#\mathcal{J}}{\#\mathcal{J}^*} \cdot 2e^{-ce^{cM}} < e^{C\rho\log\rho - ce^{cM}} < e^{-ce^{cM}}\,,$$

which is much less than $e^{-c\rho M^{3/2}}$. This completes the proof. $\square$

5 Determinants of Covariance Matrices

In this section, we estimate from below the determinant of the covariance matrix of the complex Gaussian random variables $\{f'(z_i) - \bar{z}_i f(z_i)\}_{1\leqslant i\leqslant n}$. This estimate will be used in the next section when we apply Lemma 2.2 to the proof of the long gradient curve theorem 1.4.

To warm up, first, we estimate the determinant of the covariance matrix of random variables $\{f(z_i)\}_{1\leqslant i\leqslant n}$, which has a simpler structure.

LEMMA 5.1 (The 1st determinant estimate). *Let $\{z_i\}_{1\leqslant i\leqslant n} \subset \mathbb{C}$ be a well-separated sequence; i.e. for some $\lambda > 0$,*

$$|z_i - z_j| \geqslant \lambda|i-j|\,, \quad 1 \leqslant i\,,\ j \leqslant n\,,$$

and let $\Gamma = (\gamma_{ij})$ where $\gamma_{ij} = \mathbb{E} f(z_i)\overline{f(z_j)} = e^{z_i\bar{z}_j}$, $1 \leqslant i,j \leqslant n$. Then

$$\det\Gamma \geqslant \big(c\lambda\sqrt{n}\big)^{n(n-1)}\,.$$

Proof. Without loss of generality, we suppose that $n \geqslant 2$ (if $n = 1$, the statement is obvious). Since

$$e^{z_i \bar{z}_j} = \sum_{k=0}^{\infty} \frac{z_i^k}{\sqrt{k!}} \cdot \frac{\bar{z}_j^k}{\sqrt{k!}},$$

we have $\Gamma = AA^*$ with the matrix

$$A = \begin{pmatrix} 1 & \frac{z_1}{\sqrt{1!}} & \frac{z_1^2}{\sqrt{2!}} & \dots & \frac{z_1^k}{\sqrt{k!}} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{z_n}{\sqrt{1!}} & \frac{z_n^2}{\sqrt{2!}} & \dots & \frac{z_n^k}{\sqrt{k!}} & \dots \end{pmatrix}.$$

Hence, by the Cauchy–Binet formula,

$$\det \Gamma = \sum_t \left| m_t(A) \right|^2,$$

where the sum is taken over all principal minors $m_t(A)$ of the matrix A. We use only one principal minor

$$m_0(A) = \begin{vmatrix} 1 & \frac{z_1}{\sqrt{1!}} & \dots & \frac{z_1^{n-1}}{\sqrt{(n-1)!}} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{z_n}{\sqrt{1!}} & \dots & \frac{z_n^{n-1}}{\sqrt{(n-1)!}} \end{vmatrix} = \frac{1}{\sqrt{1!2!\dots(n-1)!}} \prod_{i<j} (z_j - z_i).$$

Since the points $z_1, \dots, z_n$ are well separated, we get

$$\left| m_0(A) \right| \geqslant \lambda^{n(n-1)/2} \sqrt{1!2!\dots(n-1)!}.$$

Since $k! \geqslant k^k e^{-k}$, $k \geqslant 1$, we have

$$\begin{aligned} 1!2!\dots(n-1)! &\geqslant \exp\left(\sum_{k=1}^{n-1} (k \log k - k) \right) \\ &\geqslant \exp\left(\int_0^n x \log x \, dx - n \log n - \frac{n(n-1)}{2} \right) \\ &\geqslant \exp\left(\tfrac{1}{2} n^2 \log n - n^2 - n \log n \right), \end{aligned}$$

and

$$\left| m_0(A) \right|^2 \geqslant \left(\lambda \sqrt{n} \right)^{n(n-1)} \cdot e^{-n^2 - n \log n} \geqslant \left(c \lambda \sqrt{n} \right)^{n(n-1)},$$

completing the proof of the lemma. □

In the second estimate, we fix the parameters $n \in \mathbb{N}$ and $r = B\sqrt{n}$ where $B \gg 1$.

LEMMA 5.2 (The 2nd determinant estimate). *Let $\{z_i\}_{1 \leqslant i \leqslant n} \subset \mathbb{C}$ be a collection of points such that $|z_i - z_j| \geqslant \frac{r}{n}|i - j|$, and let $\min_i |z_i| \geqslant r$. Let Γ be the covariance matrix of the complex Gaussian random variables $\xi_i = f'(z_i) - \bar{z}_i f(z_i)$, $1 \leqslant i \leqslant n$. If B is sufficiently big, then* $\det \Gamma \geqslant 1$.

The idea of the proof of this lemma is similar to that of Lemma 5.1, though the proof is more involved due to a more complicated structure of the covariance matrix.

Proof. First, we compute the values $\gamma_{ij} = \mathbb{E}\xi_i\bar{\xi}_j$:

CLAIM 5.3. $$\gamma_{ij} = \big(1 - |z_i - z_j|^2\big)\, e^{z_i \bar{z}_j}.$$

Proof of Claim 5.3.

$$\begin{aligned}\mathbb{E}\big(f'(z) - \bar{z}f(z)\big)\big(\overline{f'(w)} - w\,\overline{f(w)}\big) &= (\partial_z - \bar{z})\,(\partial_{\overline{w}} - w)\,\mathbb{E}f(z)\overline{f(w)} \\ = (\partial_z - \bar{z})\,(\partial_{\bar{w}} - w)\,e^{z\overline{w}} &= (1 + \overline{w}z - \bar{z}z - w\overline{w} + w\bar{z})\,e^{z\overline{w}} \\ &= \big(1 - |z-w|^2\big)e^{z\overline{w}}\,. \qquad \square\end{aligned}$$

Now, we suppose that $n \geqslant 2$ (if $n = 1$, the statement is obvious) and factor the matrix Γ. We have

$$\begin{aligned}\gamma_{ij} &= (\partial_{z_i} - \bar{z}_i)\,\big(\partial_{\bar{z}_j} - z_j\big)\left(\sum_{k=0}^{\infty}\frac{z_i^k}{\sqrt{k!}}\cdot\frac{\bar{z}_j^k}{\sqrt{k!}}\right) \\ &= \sum_{k=0}^{\infty}\frac{(k - |z_i|^2)z_i^{k-1}}{\sqrt{k!}}\cdot\frac{(k - |z_j|^2)\bar{z}_j^{k-1}}{\sqrt{k!}}\,.\end{aligned}$$

Put

$$A = \begin{pmatrix} -\bar{z}_1 & \frac{1-|z_1|^2}{\sqrt{1!}} & \frac{(2-|z_1|^2)z_1}{\sqrt{2!}} & \dots & \frac{(k-|z_1|^2)z_1^{k-1}}{\sqrt{k!}} & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots \\ -\bar{z}_n & \frac{1-|z_n|^2}{\sqrt{1!}} & \frac{(2-|z_n|^2)z_n}{\sqrt{2!}} & \dots & \frac{(k-|z_n|^2)z_n^{k-1}}{\sqrt{k!}} & \dots \end{pmatrix}.$$

Then $\Gamma = AA^*$, and by the Cauchy–Binet formula,

$$\det\Gamma \geqslant \sum_{t=1}^{n+1}\big|M_t(A)\big|^2$$

where the sum is taken over $n+1$ principal minors $M_t(A)$ of the matrix A:

$$M_t = \det\left(\frac{(k + t - |z_i|^2)z_i^{k+t-1}}{\sqrt{(k+t)!}}\right)_{1\leqslant i,k\leqslant n}, \quad t = 1, 2, \dots, n+1\,.$$

To estimate the sum of the squares of these determinants, we introduce the determinants of simpler structure:

$$\mu_t = \det\left(\frac{(k + t - |z_i|^2)z_i^{k-1}}{\sqrt{k!}}\right)_{1\leqslant i,k\leqslant n}, \quad t = 1, 2, \dots, n+1\,.$$

CLAIM 5.4. *For* $1 \leqslant t \leqslant n+1$, $|M_t| \geqslant |\mu_t|$.

Proof of Claim 5.4. This follows by a straightforward estimate of the ratio

$$\left|\frac{M_t}{\mu_t}\right| = \big(|z_1|\cdot|z_2|\cdot\ldots\cdot|z_n|\big)^t \cdot \sqrt{\frac{1!}{(t+1)!}\cdot\frac{2!}{(t+2)!}\cdot\ldots\cdot\frac{n!}{(t+n)!}}\,.$$

Since $|z_i| \geqslant r$, we have $(|z_1|\cdot|z_2|\cdot\ldots\cdot|z_n|)^t \geqslant r^{tn}$. For each integer t between 1 and $n+1$,

$$\sqrt{\frac{1!}{(t+1)!}\cdot\frac{2!}{(t+2)!}\cdot\ldots\cdot\frac{n!}{(t+n)!}} \geqslant \left(\frac{n!}{(t+n)!}\right)^{n/2}$$
$$= \left(\frac{1}{(n+1)\ldots(n+t)}\right)^{n/2} \geqslant \left(\frac{1}{n+t}\right)^{nt/2} \geqslant \left(\frac{1}{3n}\right)^{nt/2}.$$

Thus

$$\left|\tfrac{M_t}{\mu_t}\right| \geqslant \left(\tfrac{r^2}{3n}\right)^{nt/2} = \left(\tfrac{1}{3}B^2\right)^{nt/2} \geqslant 1\,,$$

provided that $B \geqslant \sqrt{3}$. This proves the claim. □

Now, we complete the proof of Lemma 5.2. Observe that μ_t is a polynomial of degree n in t. We use a version of the pigeonhole principle:

Claim 5.5. *Let $P(t)$ be a polynomial of degree n with the leading coefficient a. Then* $\max\{|P(t)| : t \in \{1,2,\ldots,n+1\}\} \geqslant |a|2^{-n}$.

Proof of Claim 5.5. We have

$$\max_{t\in\{1,2,\ldots,n+1\}} |P(t)| = |a| \cdot \max_{t\in\{1,2,\ldots,n+1\}} |t-\tau_1|\ldots|t-\tau_n|\,,$$

where $\tau_1,\ldots,\tau_n$ are the zeroes of $P(t)$. We have $n+1$ disjoint $\frac{1}{2}$-neighbourhoods of the points $1,2,\ldots,n+1$ in $\mathbb{C}$. At least one of them is free of the zeroes of $P(t)$. Hence at the center of this neighbourhood, the absolute value of P cannot be smaller than $|a|2^{-n}$; whence the claim. □

We apply this claim to the polynomial μ_t. Its leading coefficient equals

$$a_n = \lim_{t\to\infty} t^{-n}\mu_t = \det\left(\frac{z_i^{k-1}}{\sqrt{k!}}\right)_{1\leqslant i,k\leqslant n}.$$

We've already estimated this determinant in the proof of the model lemma 5.1. We get $|a_n| \geqslant \big(c\lambda\sqrt{n}\big)^{n(n-1)/2}$ with $\lambda = r/n$ and $n = (r/B)^2$. Then $|a_n| \geqslant (cB)^{n(n-1)/2}$, and

$$\max_{t\in\{1,2,\ldots,n+1\}} |\mu_t| \geqslant |a_n|2^{-n} \geqslant \left(\tfrac{1}{2}cB\right)^{n(n-1)/2},$$

whence

$$\det\Gamma \geqslant \sum_{t=1}^{n+1}|M_t|^2 \geqslant \sum_{t=1}^{n+1}|\mu_t|^2 \geqslant \left(\tfrac{1}{2}cB\right)^{n(n-1)} \geqslant 1\,,$$

provided that the constant B is chosen sufficiently large. This completes the proof of Lemma 5.2. □

6 The Long Gradient Curve Theorem

Till the end of the proof, we fix $\delta = 1/20$. Everywhere below we shall assume that $R \gg 1$. In the proof, we work with three scales: starting with the macroscopic R-scale, we move to the intermediate $\sqrt{\log R}$-scale, and then to the microscopic $1/R$-scale.

6.1 Bad squares. Suppose that there exists a gradient curve Γ connecting $\partial Q(0,R)$ and $\partial Q(0,2R)$. Due to Lemma 4.1, we can assume that $U \leqslant R^\delta$ everywhere on $Q(0,2R)$, hence on Γ: the probability of the opposite event does not exceed $CR^2 e^{-ce^{2R^\delta}}$. Suppose that there is a point on Γ where $U = -R^\delta$. Since Γ is a gradient curve, if such a point exists, then it is unique. This point splits Γ into two parts: Γ_1 where $U < -R^\delta$, and Γ_2 where $U \geqslant -R^\delta$. If $U \neq -R^\delta$ on Γ, then one of these parts is empty. One of the curves Γ_1, Γ_2 must connect either $\partial Q(0,R)$ with $\partial Q\big(0,\sqrt{2}R\big)$, or $\partial Q\big(0,\sqrt{2}R\big)$ with $\partial Q(0,2R)$. If this is the curve Γ_1, then its diameter is larger than cR. By Theorem 4.3, the probability of this event does not exceed $e^{-cR^{1+\frac{3}{2}\delta}}$, and we are done.

Thus the proof boils down to the case when the gradient curve Γ connects $\partial Q(0,R)$ with $\partial Q(0,\sqrt{2}R)$ and $-R^\delta \leqslant U \leqslant R^\delta$ everywhere on Γ. In this case,

$$\int_\Gamma \big|\nabla U(z)\big||dz| \leqslant 2R^\delta\,; \tag{6.1}$$

that is, the gradient ∇U is small in the mean on Γ. We will not use that Γ is a gradient curve anymore; from now on, it is an arbitrary curve connecting $\partial Q(0,R)$ with $\partial Q\big(0,\sqrt{2}R\big)$ such that (6.1) happens.

We take $r = \frac{1}{4}\sqrt{\delta \log R}$ and fix the standard partition of the complex plane $\mathbb{C}$ into squares $Q(w_j, r)$ with side length $2r$. Let $\mathcal{J}$ be the set of indices j for which the square $Q(w_j, 2r)$ is entirely contained in the "square annulus" $Q\big(0,\sqrt{2}R\big) \setminus Q(0,R)$. Note that $\#\mathcal{J} \leqslant (R/r)^2$.

Definition 6.2 (Bad squares). *Let $j \in \mathcal{J}$. We shall call the standard square $Q(w_j, r)$* bad *if there exists a curve γ_j joining $\partial Q(w_j, r)$ with $\partial Q(w_j, 2r)$ such that*

$$\int_{\gamma_j} \big|\nabla U(z)\big||dz| < rR^{2\delta-1}. \tag{6.3}$$

We shall call the square $Q(w_j, r)$ good *if it is not bad.*

By $\mathcal{F} \subset \mathcal{J}$ we denote the family of all indices j such that the square $Q(w_j, r)$ intersects the curve Γ.

LEMMA 6.4. *At most $8R^{1-\delta}/r$ of the squares $\{Q(w_j, r)\}_{j\in\mathcal{F}}$ are good.*

Proof. Let N denote the number of good squares $Q(w_j, r)$. By γ_j we denote a connected part of $\Gamma \cap (Q(w_j, 2r) \setminus Q(w_j, r))$ that joins $\partial Q(w_j, r)$ with $\partial Q(w_j, 2r)$. Since almost every point of the curve Γ belongs to at most 4 squares $Q(w_j, 2r)$, we can write

$$NrR^{2\delta-1} \leqslant \sum_{j:Q(w_j,r) \text{ is good}} \int_{\gamma_j} \big|\nabla U(z)\big||dz| \leqslant 4\int_\Gamma \big|\nabla U(z)\big||dz| \leqslant 8R^\delta\,,$$

whence the estimate. □

The immediate consequence of Lemma 6.4 is that the existence of a curve Γ connecting $\partial Q(0, R)$ with $\partial Q\big(0, \sqrt{2}R\big)$ such that (6.1) happens implies the existence of a family $\mathcal{F}$ of squares $Q(w_j, r)$ of cardinality $L \geqslant cR/r$ and a subfamily $\mathcal{F}' \subset \mathcal{F}$ of bad squares of cardinality at least $L - \frac{8R^{1-\delta}}{r} \geqslant \frac{1}{2}L$ in that family.

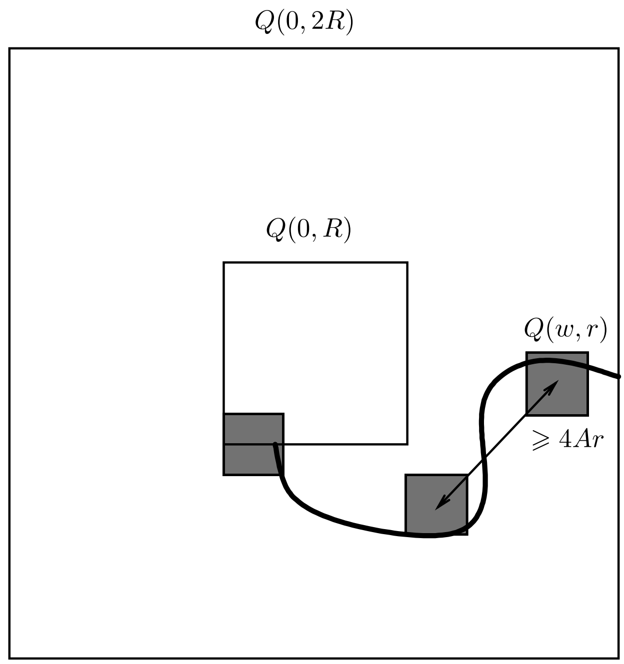

Figure 3: The gradient curve Γ generates separated bad squares

Let A be the constant corresponding to $N = 4\delta^{-1}$ in the almost independence theorem 3.2. Let $\mathcal{J}' \subset \mathcal{J}$ satisfy $|w_i - w_j| \geqslant 4Ar$ for $i, j \in \mathcal{J}'$, $i \neq j$. According to Theorem 3.2, applied to $4r$ instead of r, we can represent $T_{w_j} f$ as $f_j + h_j$ where f_j are independent GEF and all the functions h_j are small in the disk $|z| \leqslant 4r$. We set

$$\Omega_* = \Big\{ \max_{j\in\mathcal{J}'} \max_{|z|\leqslant 4r} \big|h_j(z)\big| > R^{-4} \Big\}\,.$$

Then for any $j \in \mathcal{J}'$, $\mathbb{P}\{\max_{|z|\leqslant 4r} |h_j(z)| > R^{-4}\} \leqslant 2e^{-\frac{1}{2}R^4}$ (recall that $e^{-16Nr^2} = R^{-4}$ for our choice of r and N). Therefore, $\mathbb{P}\{\Omega_*\} \leqslant 2R^2 e^{-\frac{1}{2}R^4} < e^{-cR^4}$.

The next proposition is the central part in the proof of the long gradient curve theorem.

PROPOSITION 6.5. *There exist events Ω_j with $\mathbb{P}\{\Omega_j\} \leqslant e^{-cr^4}$ depending only on f_j (and, thereby, independent), and such that, for any $j \in \mathcal{J}'$,*

$$\big\{Q(w_j, r) \textit{ is bad}\big\} \subset \Omega_j \cup \Omega_*\,.$$

Now, using this proposition, we complete the proof of the long gradient curve theorem. We choose a family $\mathcal{F}'' \subset \mathcal{F}'$ of $cA^{-2}r^{-1}R$ *$4Ar$-separated squares* (that is, all the distances between the centers of these squares are not less than $4Ar$), and discard the rest of $\mathcal{F}'$. From Proposition 6.5 we see that the probability that a *given* subfamily $\mathcal{F}'' \subset \mathcal{J}$ of $cA^{-2}r^{-1}R$ squares is bad does not exceed

$$\big(e^{-cr^4}\big)^{cA^{-2}r^{-1}R} + e^{-cR^4} = e^{-cR(\log R)^{3/2}} + e^{-cR^4} \leqslant 2e^{-cR(\log R)^{3/2}}\,,$$

provided that $R \gg 1$. At last, we have at most

$$(\#\mathcal{J})^{\#\mathcal{F}''} \leqslant (CR^2)^{CR} \leqslant e^{CR\log R}$$

ways to choose $\mathcal{F}''$ in $\mathcal{J}$. This does not harm the previous upper bound. Hence the long gradient curve theorem is proved (modulo the proposition). $\square$

6.2 Proof of the proposition. Assume that the event Ω_* does not occur. Then $T_{w_j} f = f_j + h_j$ where $\max_{4r\mathbb{D}} |h_j| \leqslant R^{-4}$. We fix $j \in \mathcal{J}'$ and aim at building an event Ω_j depending only on f_j of probability $\mathbb{P}\{\Omega_j\} \leqslant e^{-cr^4}$ and such that, if the square $Q(w_j, r)$ is bad and h_j is small as above, then Ω_j must occur. To simplify the notation, we set $w = w_j$.

Fix the partition of the complex plane $\mathbb{C}$ into standard squares with side length $2/R$.

Definition 6.6 (Black squares). *We shall call a standard square* $Q(\zeta, 1/R) \subset Q(w, 2r)$ black *if* $\inf_{Q(\zeta,1/R)} |\nabla U| \leqslant R^{3\delta-1}$. *Otherwise, the square* $Q(\zeta, 1/R)$ *is called* white.

First, we check that if the square $Q(\zeta, 1/R)$ is black (i.e. the gradient ∇U is small somewhere in this square), and the functions f_j and h_j are not too large, then the function $f_j'(z) - \bar{z} f_j(z)$ must be small at the center $\zeta - w$ of the shifted square.

Lemma 6.7. *Suppose that*

(i) *The square* $Q(\zeta, 1/R)$ *is black;*
(ii) $\max_{4r\mathbb{D}} |h_j| \leqslant R^{-4}$;
(iii) $\max_{4r\mathbb{D}} |f_j| \leqslant R^{\delta}$.

Then

$$\left| f_j'(\zeta - w) - (\overline{\zeta - w}) f_j(\zeta - w) \right| < R^{6\delta-1} . \tag{6.8}$$

Proof. We have $U(w+z) = \log |T_w f(z)| - \frac{1}{2}|z|^2$, whence $|\nabla U(w+z)| = \left| \frac{(T_w f)'(z)}{(T_w f)(z)} - \bar{z} \right|$. Thereby,

$$\left| f_j'(z) - \bar{z} f_j(z) \right| \leqslant \left| h_j'(z) - \bar{z} h_j(z) \right| + \left| \nabla U(w+z) \right| \left(|f_j(z)| + |h_j(z)| \right) . \tag{6.9}$$

Since the square $Q(\zeta, 1/R)$ is black, there exists a point z such that $w + z \in Q(\zeta, 1/R)$ and $|\nabla U(w+z)| \leqslant R^{3\delta-1}$. The other terms on the RHS of (6.9) are readily estimated using assumptions (ii) and (iii) and Cauchy's inequality for the derivative of an analytic function. We get

$$\left| f_j'(z) - \bar{z} f_j(z) \right| \leqslant 2rR^{-4} + R^{3\delta-1}(R^{\delta} + R^{-4}) < R^{5\delta-1} .$$

It remains to replace z by $\zeta - w$ on the LHS.

By Cauchy's inequalities,

$$\max_{Q(0,2r)} |f_j'| \leqslant r^{-1} R^{\delta} , \quad \text{and} \quad \max_{Q(0,2r)} |f_j''| \leqslant 2r^{-2} R^{\delta} .$$

Hence the operator norm of the differential of $f_j'(z) - \bar{z} f_j(z)$ does not exceed

$$2r^{-2} r R^{\delta} + 2rr^{-1} R^{\delta} + 2R^{\delta} < 5R^{\delta}$$

everywhere in $Q(0, 2r)$. Since $|z - (\zeta - w)| \leqslant \sqrt{2}/R$ and $R \gg 1$, we are done. □

Assume that the square $Q(w, r)$ is bad; i.e. there exists a curve γ joining $\partial Q(w, r)$ with $\partial Q(w, 2r)$ such that

$$\int_{\gamma} \left| \nabla U(z) \right| |dz| < rR^{2\delta-1} .$$

We fix an integer $n = (r/B)^2$ with $B \gg 1$. For any $t \in [0, 1]$, we put

$$r_i(t) = r + \frac{(i-1) + t}{n} r , \quad 1 \leqslant i \leqslant n .$$

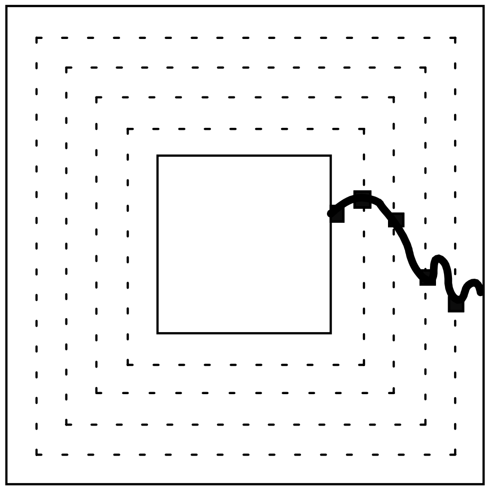

Figure 4: The curve γ and a sequence of black squares it generates

For each t, the squares $\partial Q(w, r_i(t))$ form a "chain of n fences", and the curve γ crosses this chain at least n times. It may happen that, for some value t, the gradient ∇U is not small at most of the crossing points, or even at all of them. However, as we shall see, for a large subset of $t \in [0,1]$, the gradient ∇U is sufficiently small at n crossing points $\gamma \cap \partial Q(w, r_i(t))$, $1 \leqslant i \leqslant n$, to guarantee that the corresponding $1/R$-squares containing these points are black. For each $t \in [0,1]$, we denote by $B(t) \subset \{1, 2, \dots, n\}$ the subset of those i's such that at least one point from the set $\gamma \cap \partial Q(w, r_i(t))$ is covered by a black square. By m_1 we denote the one-dimensional Lebesgue measure.

LEMMA 6.10. *Suppose that the square $Q(w, r)$ is bad. Then*

$$m_1\big\{t \in [0,1] : \#B(t) = n\big\} \geqslant \tfrac{1}{2}\,. \tag{6.11}$$

Proof: . Let L be the measure of the set of $\rho \in [r, 2r]$ such that the intersection $\gamma \cap \partial Q(w, \rho)$ is contained in white squares. Then

$$\int_\gamma \big|\nabla U(z)\big||dz| \geqslant R^{3\delta - 1} L\,.$$

Since the square $Q(w, r)$ is bad, the LHS does not exceed $rR^{2\delta-1}$, and we see that $L \leqslant rR^{-\delta}$. On the other hand,

$$L = \tfrac{r}{n}\int_0^1 \big(n - \#B(t)\big)dt \geqslant \tfrac{r}{n} \cdot m_1\big\{t \in [0,1] : \#B(t) \leqslant n-1\big\}\,,$$

whence $m_1\{t \in [0,1] : \#B(t) \leqslant n-1\} \leqslant n/R^\delta \leqslant r^2/B^2R^\delta \ll 1$ if $R \gg 1$. Hence the lemma. □

For each $t \in [0,1]$, consider the collection $\mathfrak{Z}(t)$ of "configurations": $\mathfrak{z} = \{z_1, \dots, z_n\}$ of n points such that each point z_i is a center of a standard square $Q(z_i, 1/R)$ from our partition that has a non-void intersection with $\partial Q(w, r_i(t))$. Let us introduce the events

$$\Upsilon_j(\mathfrak{z}) = \Big\{ \max_{1\leqslant i\leqslant n} \big| f_j'(z_i - w) - (\overline{z_i - w}) f_j(z_i - w)\big| \leqslant R^{6\delta - 1}\Big\}$$

$$\text{and } \Omega_j(t) = \bigcup_{\mathfrak{z}\in\mathfrak{Z}(t)} \Upsilon_j(\mathfrak{z}) \,,$$

and estimate their probabilities. Our estimate is based on the lower bound for the determinant of the covariance matrix of complex Gaussian random variables $\{f_j'(z_i) - \bar{z}_i f_j(z_i)\}$ given in Lemma 5.2.

Lemma 6.12. *Given* $t \in [0,1]$, $\mathbb{P}\{\Omega_j(t)\} \leqslant e^{-cr^4}$.

Proof. First, we estimate the probability of the event $\Upsilon_j(\mathfrak{z})$.

Claim 6.13. *For any configuration* $\mathfrak{z} \in \mathfrak{Z}(t)$, $\mathbb{P}\{\Upsilon_j(\mathfrak{z})\} \leqslant R^{-2(1-6\delta)n}$.

Proof. This is a straightforward combination of Lemmas 2.2 and 5.2. □

Next, we estimate the cardinality of the collection $\mathfrak{Z}(t)$ (recall that $t \in [0,1]$ is fixed).

Claim 6.14. $\#\mathfrak{Z}(t) \leqslant C^n(Rr)^n$.

Proof. For each i, there are at most CRr standard $1/R$-squares that intersect $\partial Q(w, r_i(t))$. Therefore, there are at most CRr choices for the centers z_i of these squares and the number of the corresponding configurations $\mathfrak{z}$ cannot exceed $(CrR)^n$. Hence the claim. □

Using Claims 6.14 and 6.13, we get

$$\mathbb{P}\big\{\Omega_j(t)\big\} \leqslant C^n(Rr)^n \cdot R^{-2(1-6\delta)n} \leqslant C^n R^{-(1-13\delta)n}\,,$$

provided that R is sufficiently big. Recalling that $13\delta = 13/20 < 1$, $n = \frac{r^2}{B^2}$, and $r^2 = \frac{1}{16}\delta \log R$, we see that $\mathbb{P}\{\Omega_j(t)\} \leqslant e^{-cr^4}$. This proves Lemma 6.12. □

Define the events

$$\Omega_j' = \big\{\omega \in \Omega : m_1\{t \in [0,1] : \omega \in \Omega_j(t)\} \geqslant \tfrac{1}{2}\big\}\,,$$

and

$$\Omega_j = \Omega_j' \cup \Big\{ \max_{4r\mathbb{D}} |f_j| \geqslant R^\delta \Big\}\,.$$

Note that the event Ω_j depends only on f_j. By Lemmas 6.7 and 6.10,

$$\big\{Q(w,r) \text{ is bad}\big\} \subset \Omega_j \cup \Big\{\max_{4r\mathbb{D}} |h_j| \geqslant R^{-4}\Big\}.$$

Lemma 6.15. *The probability of the event* $\{\max_{4r\mathbb{D}} |f_j| \geqslant R^\delta\}$ *does not exceed* e^{-cr^4}.

Proof. If $|f_j| \geqslant R^\delta$ somewhere in the disk $4r\mathbb{D}$, then (at the same point) the corresponding potential U is not less than $\delta \log R - \frac{1}{2}(4r)^2 = \frac{1}{2}\delta \log R$ (due to the choice of r). By Lemma 4.1, the probability of this event does not exceed $Cr^2 e^{-cR^\delta} \ll e^{-cr^4}$ if $R \gg 1$. □

Hence to complete the proof of Proposition 6.5, we need to estimate the probability of the event Ω_j'.

Lemma 6.16. $\mathbb{P}\{\Omega_j'\} \leqslant e^{-cr^4}$.

Proof: . Define the random set $A = \{t \in [0,1] : \Omega_j(t) \text{ occurs}\}$, and let $X = m_1(A)$. Then, by Chebyshev's inequality,

$$\mathbb{P}(\Omega_j') = \mathbb{P}(X \geqslant 1/2) \leqslant 2\mathbb{E}(X) = 2\int_0^1 \mathbb{P}\big(\Omega_j(t)\big)dt \leqslant 2 \max_{t\in[0,1]} \mathbb{P}\big(\Omega_j(t)\big).$$

By Lemma 6.12, the maximum on the right-hand side does not exceed e^{-cr^4}. □

This completes the proof of Proposition 6.5.

7 Proof of the Partition Theorem

Set $\Delta(z) = U_{xx}(z)U_{yy}(z) - U_{xy}^2(z)$.

Lemma 7.1. *For* $z \in \mathbb{C} \setminus \mathcal{Z}_f$, $\Delta(z) = 1 - |(f'/f)'(z)|$.

Proof. The proof is a straightforward computation. Since $\partial_x = \partial_z + \partial_{\bar z}$, $\partial_y = i(\partial_z - \partial_{\bar z})$, we have

$$\partial_{xx} = \partial_{zz} + 2\partial_{z\bar z} + \partial_{\bar z\bar z}\,, \quad \partial_{yy} = -\partial_{zz} + 2\partial_{z\bar z} - \partial_{\bar z\bar z}\,, \quad \partial_{xy} = i(\partial_{zz} - \partial_{\bar z\bar z})\,.$$

Whence

$$\begin{aligned}\Delta(z) &= (U_{zz} + 2U_{z\bar z} + U_{\bar z\bar z})(-U_{zz} + 2U_{z\bar z} - U_{\bar z\bar z}) + (U_{zz} - U_{\bar z\bar z})^2 \\ &= 4(U_{z\bar z}^2 - U_{zz}\cdot U_{\bar z\bar z})\,.\end{aligned}$$

Taking into account that $U_{zz} = \frac{1}{2}\left(\frac{f'}{f}\right)'$, $U_{\bar z\bar z} = \frac{1}{2}\overline{\left(\frac{f'}{f}\right)'}$, and $U_{z\bar z} = -\frac{1}{2}$, we get the result. □

Denote by $\operatorname{Crit} U$ the set $\{z \in \mathbb{C} : \nabla U(z) = 0\}$ of critical points of the potential U.

LEMMA 7.2. *Almost surely, the following hold:*

(i) *Each critical point of U is non-degenerate; i.e.* $\Delta(w) \neq 0$ *for* $w \in \mathrm{Crit}(U)$*;*

(ii) *The critical set* $\mathrm{Crit}\, U$ *has no finite accumulation points.*

Proof. Note that the probability that $w = 0$ is a critical point of U is 0. At the critical points $w \neq 0$ of U, we have $f(w) = \overline{w}^{-1} f'(w)$. Hence

$$\Delta(w) = 1 - \left| \overline{w} \tfrac{f''}{f'}(w) - \overline{w}^2 \right|^2, \quad w \in \mathrm{Crit}\, U .$$

Now, let us set $f(z) = \xi_0 + \xi_1 z + h(z)$, where $h(z)$ is a random entire function determined by $\xi_2, \xi_3, \ldots$. Then, on $\mathrm{Crit}\, U$, the determinant $\Delta(w)$ coincides with

$$\Delta_1(w) := 1 - \left| \overline{w} \frac{h''(w)}{h'(w) + \xi_1} - \overline{w}^2 \right|^2 .$$

Observe that, for each w with $|w| \neq 1$ and each $\xi_2, \xi_3, \ldots$, the set of ξ_1 where the last expression is 0 has zero measure. Thus, using the Fubini theorem, we conclude that for almost all $\xi_1, \xi_2, \ldots$, the set $\{w \in \mathbb{C} : \Delta_1(w) = 0\}$ has zero measure. Now let $g(z) = \xi_1 z + h(z)$. If g is fixed (i.e. $\xi_1, \xi_2, \ldots$, are fixed) and $w \neq 0$ is a critical point of U, then ξ_0 is determined by equation

$$\xi_0 = \frac{g'(w)}{\overline{w}} - g(w) .$$

The right-hand side defines a real-analytic mapping of the punctured plane $\mathbb{C} \setminus \{0\}$ and, therefore, it maps sets of zero area in the w-plane to sets of zero area in the ξ_0-plane. Hence, for almost every choice of the independent coefficients $\xi_1, \xi_2, \ldots$, the set of ξ_0 for which there exist degenerate critical points of U has measure zero. Using Fubini's theorem once more, we get the conclusion of statement (i) of the lemma.

Statement (ii) follows from (i). The planar map given by ∇U is real-analytic outside the set where U equals $-\infty$. Note that, unless f identically equals 0 (which is an event of zero probability), the gradient $\nabla U(z) = \overline{\left(\frac{f'}{f}(z)\right)} - z$ tends to ∞ at every zero of f and, therefore, no point of $U^{-1}(-\infty) = \mathcal{Z}_f$ can be an accumulation point of $\mathrm{Crit}\, U$. Thus, if the set $\mathrm{Crit}\, U$ has a finite accumulation point, then this point itself belongs to $\mathrm{Crit}\, U$ and, by the inverse function theorem, the map given by ∇U is degenerate at this point. □

It is worth mentioning that there is another way to prove statement (ii) of Lemma 7.2 elaborating on the fact that, if g is an analytic function and the solutions of the equation $g(z) = \bar{z}$ have a finite accumulation point, then g must be a Möbius transformation.

LEMMA 7.3. *Almost surely, the following hold:*

(i) *Each oriented curve* Γ *has a starting point* $s(\Gamma) \in \mathrm{Crit}(U)$ *and a terminating point* $t(\Gamma) \in U^{-1}\{-\infty\} \cup \mathrm{Crit}(U)$*;*

(ii) *At any limiting point, the oriented gradient curve* Γ *is tangent to a straight line passing through that point.*

Proof. We refer the reader to [Hu, Ch. 4] for the facts from the standard ODE theory we use.

(i) It follows from the long gradient curve theorem that, almost surely, gradient curves cannot escape to or come from infinity. Now it remains to observe that the limiting set $\mathcal{L}$ of any gradient curve Γ is contained in the set of singular points of the gradient flow; that is, in the set $\mathrm{Crit}\, U \cup U^{-1}\{-\infty\}$. Hence, by Lemma 7.2, $\mathcal{L}$ consists of isolated points.

(ii) The critical points of U are either local maxima or saddle points. By Lemma 7.2, almost surely all of them are non-degenerate. The rest follows from the standard ODE theory: the behaviour of the integral curves in a neighbourhood of these points is the same as the behaviour of the integral curves for the linear ODE obtained by discarding the non-linear terms in the Taylor expansion of ∇U. □

LEMMA 7.4. *Each gradient curve is real analytic everywhere except at the limiting points.*

Proof. ∇U is real analytic everywhere except on the set where $U = -\infty$. Hence, by the Cauchy existence theorem, the gradient curves are real analytic at all points where $\nabla U \neq 0$. □

Now we are ready to prove the partition theorem 1.2. By the long gradient curve theorem 1.4, almost surely all the basins are bounded. We call a gradient curve Γ *singular* if $t(\Gamma) \in \mathrm{Crit}\, U$. Note that, almost surely every point that is not in one of the basins must lie on a singular curve. Moreover, with probability 1, for every compact K on the complex plane, there exists another compact $\widetilde{K}$ such that all gradient curves intersecting K are contained in $\widetilde{K}$. (Otherwise, there exists an $N \in \mathbb{N}$ such that, for any integer $M > N$, there is a gradient curve connecting $\partial Q(0,N)$ and $\partial Q(0,M)$. The probability of this event is 0.) Also, a gradient curve cannot terminate at a local maximum of U and each saddle point of U serves as a terminating point for 2 singular curves. This allows us to conclude that, almost surely, we may have only finitely many singular curves intersecting any compact subset of $\mathbb{C}$. In particular, almost surely each basin $B(a)$ is bounded by finitely many singular curves and their limiting points, which is enough to justify the area computation in the introduction. □

8 The Upper Bound in Theorem 1.5

First, we prove a useful "length and area estimate" of deterministic nature valid for Liouville vector fields, that is, the fields with constant divergence. Then we derive the upper bound for the probability that a given point z is far from its sink a_z.

8.1 The length and area estimate. Consider the disk $D=\{|z-a|<\varepsilon\}$. Since $\nabla U(z) = \frac{z-a}{|z-a|^2} + O(1)$, as $z \to a$, we can fix a sufficiently small $\varepsilon > 0$ such that each gradient curve hits the boundary circumference $\{|z-a| = \varepsilon\}$ only once. This gives us a one-to-one correspondence between the points of the circumference $\{a + \varepsilon e^{i\theta}\}$ and the gradient curves in $B(a)$; i.e. the gradient curves are parameterized by the *angular coordinate* θ.

By $D(t)$ we denote the pre-image of D under the gradient flow of ∇U for time t; i.e. if $dZ/dt = -\nabla U(Z(t))$, then $D(t) = \{z = Z(0) : Z(t) \in D\}$. By $A(t)$ we denote the area of $B(a) \setminus D(t)$. Since $\operatorname{div}(\nabla U) = -2$ on $B(a) \setminus \{a\}$, the evolution of the area is very simple: $dA/dt = -2A$. This is Liouville's theorem (which follows from the divergence theorem), see, for instance, [Ar, §16].

We will need an "infinitesimal version" of this equation. The boundary $\partial B(a)$ contains finitely many saddle points of U. By $\alpha_1 < \cdots < \alpha_s < \alpha_{s+1} = \alpha_1 + 2\pi$ we denote the angular coordinates of the gradient curves that connect the saddle points on $\partial B(a)$ with the sink a. Take any θ different from $\alpha_1, \dots, \alpha_s$, say $\alpha_l < \theta < \alpha_{l+1}$, and choose θ_1 and θ_2 such that $\alpha_l < \theta_1 < \theta < \theta_2 < \alpha_{l+1}$. The gradient curves $\Gamma(\theta_1)$, $\Gamma(\theta_2)$ must terminate at the same local maximum. They bound a "diangle" $Y(\theta_1, \theta_2)$ with the vertices at a and at a local maximum. Consider the "triangle" $T(t; \theta_1, \theta_2) = Y(\theta_1, \theta_2) \setminus D(t)$ and its area $A(t; \theta_1, \theta_2) = m_2 T(t; \theta_1, \theta_2)$.

By Green's theorem,

$$-2A(t;\theta_1,\theta_2) = \iint_{T(t;\theta_1,\theta_2)} \Delta U \, dm_2 = \int_{\partial T(t;\theta_1,\theta_2)} \langle \nabla U, n \rangle \, |dz| \qquad (8.1)$$

where n is the unit normal directed outward the triangle. The boundary $\partial T(t; \theta_1, \theta_2)$ consists of parts of the gradient curves $\Gamma(\theta_1)$ and $\Gamma(\theta_2)$, where $\partial U/\partial n = 0$, and of the part $I = I(t; \theta_1, \theta_2)$ of the curve $\partial D(t)$. If $\{(x(t,\theta), y(t,\theta)) : \theta_1 \leqslant \theta \leqslant \theta_2\}$ is the equation of the arc I, then at the point

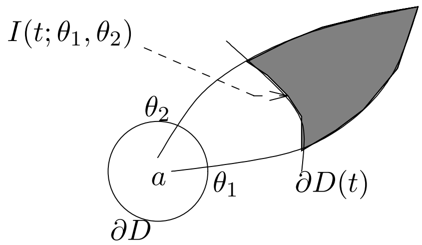

Figure 5: "Triangle" $T(t;\theta_1,\theta_2)$

θ the unit normal n is given by $-(y_\theta,-x_\theta)/\sqrt{x_\theta^2+y_\theta^2}$. Hence $A(t;\theta_1,\theta_2)=\frac{1}{2}\int_{\theta_1}^{\theta_2}(U_x y_\theta - U_y x_\theta)d\theta$, and we conclude that the area $A(t;\theta_1,\theta_2)$ has a smooth angular density $S(t,\theta)=\frac{1}{2}(U_x y_\theta - U_y x_\theta)$. By Liouville's theorem, $\partial A(t;\theta_1,\theta_2)/\partial t = -2A(t;\theta_1,\theta_2)$. Therefore, the density $S(t,\theta)$ satisfies the same differential equation $\partial S(t,\theta)/\partial t = -2S(t,\theta)$.

Now, we re-parameterize the gradient curve $\Gamma(\theta)$ by its length l starting at the sink a. We treat the restrictions of the density S and of the gradient ∇U to $\Gamma(\theta)$ as functions of the length l; i.e. $S(l)=S(t(l),\theta)$, and similarly for ∇U. Note that $dl/dt = |\nabla U|$. We arrive at the ordinary differential equation for the density S:

$$\tfrac{\partial S}{\partial l}\cdot|\nabla U| = -2S\,.$$

Solving this equation, we get

$$S(l)=S(l_0)\exp\left(-2\int_{l_0}^{l}\frac{dl}{|\nabla U|}\right).$$

Denote by z_l the point on the gradient curve $\Gamma(\theta)$ that cuts an arc of length l from that curve. By the Cauchy–Schwartz inequality,

$$\int_{l_0}^{l}\frac{dl}{|\nabla U|}\geqslant (l-l_0)^2\left(\int_{l_0}^{l}|\nabla U|dl\right)^{-1}=\frac{(l-l_0)^2}{U(z_l)-U(z_{l_0})}\,.$$

We arrive at the crucial

PROPOSITION 8.2. *In the same notation as above,*

$$S(l)\leqslant S(l_0)\exp\left(-\frac{2(l-l_0)^2}{U(z_l)-U(z_{l_0})}\right)\,.$$

8.2 Distance to the sink (the upper bound). Fix $\delta\in(0,2]$. We define the "tentacles" $T_R(a)$ of the basin $B(a)$ as follows. Given θ, we move along the gradient curve $\Gamma(\theta)$ in the direction of growth of the potential U, starting at the sink a, till we hit the point where $U = -R^\delta$. After that, we keep on moving along $\Gamma(\theta)$ the distance R (measured along $\Gamma(\theta)$), and then stop. The rest of the curve is called the θ-tentacle. The tentacles $T_R(a)$ are the union of all θ-tentacles. Of course, it may happen that the tentacles $T_R(a)$ are empty.

Now, we are ready to estimate the probability that $|z - a_z|$ is large. By translation invariance, this probability does not depend on the choice of z, so we choose $z = 0$. Suppose that $|a_0| > 2R$. We know that at least one of the following happens:

(i) Either the distance from 0 to the curve $\Gamma_0 \cap \{U < -R^\delta\}$ measured along Γ_0 is less than R;

(ii) Or $0 \in T_R(a_0)$.

(Recall that Γ_0 is the gradient curve that passes through the origin.)

In the first case, the curve $\gamma = \Gamma_0 \cap \{U < -R^\delta\}$ connects the circumferences $\{|z| = R\}$ and $\{|z| = 2R\}$. By Theorem 4.3, the probability of this event does not exceed $Ce^{-cR^{1+\frac{3}{2}\delta}}$.

Now, we estimate the probability of the event (ii). By translation invariance,

$$\begin{aligned}\pi\mathbb{P}\big\{0 \in T_R(a_0)\big\} &= \iint_{\mathbb{D}} \mathbb{P}\big\{w \in T_R(a_w)\big\} dm_2(w) \\ &= \int_\Omega m_2\big\{w \in \mathbb{D} : w \in T_R(a_w)\big\} d\mathbb{P}\,. \end{aligned} \tag{8.3}$$

Thus, we need to estimate the area of the random set $\{w \in \mathbb{D} : w \in T_R(a_w)\}$; that is, the area of the union of all possible tentacles within $\mathbb{D}$.

We throw away three exceptional events. Let Ω_1 be the event that there exists a gradient curve connecting the circumferences $\{|z| = 1\}$ and $\{|z| = R^2\}$. By the long gradient curve theorem, $\mathbb{P}\{\Omega_1\} \leqslant e^{-cR^2}$. If Ω_1 does not occur, then $|a| < R^2$, for any basin $B(a)$ that intersects the unit disk. Let Ω_2 be the event that there exists a gradient curve connecting the circumferences $\{|z| = R^2\}$ and $\{|z| = 2R^2\}$. Again, $\mathbb{P}\{\Omega_2\} \leqslant e^{-cR^2}$. If Ω_1 and Ω_2 do not occur, then any basin B that intersects the unit disk $\mathbb{D}$ is contained in the disk $2R^2\mathbb{D}$. Recalling that each basin has area π and comparing the areas, we see that the number of such basins does not exceed $4R^4$. At last, we exclude the event $\Omega_3 = \{\max_{2R^2\mathbb{D}} U > R^\delta\}$. By Lemma 4.1, $\mathbb{P}\{\Omega_3\} < CR^4 e^{-ce^{R^\delta}} < e^{-cR^4}$ if R is big enough.

Now, after throwing away these three events, we can estimate the area of the random set $\{w \in \mathbb{D} : w \in T_R(a_w)\}$. First, we bound the area of one tentacle $T_R(a)$. Since $U \leqslant R^\delta$ everywhere in $B(a)$, for each θ-tentacle, we can apply the length and area estimate from Proposition 8.2 with $l-l_0 \geqslant R$ and $U(z_l) - U(z_{l_0}) \leqslant 2R^\delta$. Integrating over θ, we get

$$m_2 T_R(a) \leqslant m_2 B(a) e^{-R^{2-\delta}} = \pi e^{-R^{2-\delta}} .$$

The number of tentacles coming from different basins and hitting the unit disk $\mathbb{D}$ does not exceed $4R^4$. We conclude that if the events Ω_i, $1 \leqslant i \leqslant 3$, do not occur, then $m_2\{w \in \mathbb{D} : w \in T_R(a_w)\} \leqslant 4\pi R^4 e^{-R^{2-\delta}}$. In view of (8.3), we see that the probability of the event $\{0 \in T_R(a_0)\}$ is bounded by $e^{-cR^{2-\delta}}$ if $R \gg 1$.

Thus,

$$\begin{aligned}\mathbb{P}\big\{|a_0| > 2R\big\} &< \mathbb{P}\big\{ \operatorname{diam}(\Gamma_0 \cap \{U < -R^\delta\}) > R\big\} + \mathbb{P}\big\{0 \in T_R(a_0)\big\} \\ &< Ce^{-cR^{1+\frac{3}{2}\delta}} + Ce^{-cR^{2-\delta}} .\end{aligned}$$

Choosing $\delta = 2/5$, we complete the proof. □

9 The Lower Bounds in Theorems 1.3 and 1.5

The proofs of the lower bounds for the diameter of the basin and the distance to the sink are based on the same idea. The function $z^n/\sqrt{n!}$ has a singular line $\{|z| = \sqrt{n}\}$ where the gradient of its potential vanishes. Then after any analytic perturbations small in the annulus $\{|z - \sqrt{n}| \leqslant 2\}$, this annulus still contains plenty of long gradient curves and of points that are far from their sinks.

9.1 Diameter of the basin (the lower bound). We choose a big $R \gg 1$ such that $n = R^2$ is an integer and consider the function $F(z) = \frac{z^n}{\sqrt{n!}}\big(1 + \frac{z}{10R}\big)$ in the domain

$$\mathcal{D} = \big\{z \in \mathbb{C} : R-1 < |z| < R+1\,,\ \big|\arg z - \tfrac{\pi}{2}\big| < \tfrac{1}{10}\big\} .$$

Note that, for the corresponding potential U, we have

$$\nabla U(z) = \overline{\frac{F'}{F}(z)} - z = \frac{n}{\bar z} - z + \frac{1}{10R}\,\frac{1}{1 + (\bar z/10R)} .$$

Since the vector $\frac{n}{\bar z} - z$ is purely radial and the sine of the angle between the vectors $1 + \frac{z}{10R}$ and z is at least $\frac{1}{2}$ for $z \in \mathcal{D}$, we see that the angular component of $-\nabla U$ is oriented counter-clockwise and its size is at least $1/30R$ in $\mathcal{D}$. Also, the gradient field $-\nabla U$ is directed outside the domain $\mathcal{D}$ on the boundary arcs $\big\{|z| = R \pm 1,\ |\arg z - \frac{\pi}{2}| < \frac{1}{10}\big\}$, and the radial component of $-\nabla U$ is at least 1 on these arcs.

Thus, there is a gradient curve that starts at the right boundary interval $\{\arg z - \frac{\pi}{2} = -\frac{1}{10},\ R-1 < |z| < R+1\}$, and hits the point iR. Thereby, its diameter must be at least $R/20$.

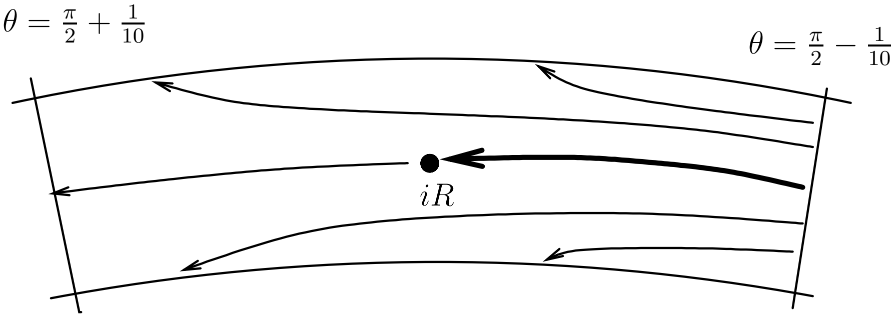

Figure 6: The field $-\nabla U$ in $\mathcal{D}$

This conclusion will be preserved if, instead of the function F, we consider its analytic perturbation $F+H$ with H satisfying $|H/F| \leqslant R^{-2}$ in the annulus $\{R-2 \leqslant |z| \leqslant R+2\}$. Indeed, the absolute value of the perturbation of ∇U the function H creates in $\mathcal{D}$ is only $\left|\frac{(F+H)'}{F+H} - \frac{F'}{F}\right| = \left|\frac{(H/F)'}{1+(H/F)}\right| \leqslant \frac{R^{-2}}{1-R^{-2}} < \frac{1}{60R}$ for $R \gg 1$, which is too small to change anything in the above picture.

Now it remains to estimate from below the probability of the event that a GEF f is such a perturbation of F.

Lemma 9.1. *If $R \gg 1$, then* $\mathbb{P}\{\max_{R-2\leqslant|z|\leqslant R+2}\left|\frac{f}{F}(z) - 1\right| \leqslant \frac{1}{R^2}\} \geqslant e^{-CR(\log R)^{3/2}}$.

Proof. We write $f(z) = F(z) + H(z)$ where

$$H(z) = \sum_{k:k\neq n,n+1} \xi_k \frac{z^k}{\sqrt{k!}} + (\xi_n - 1)\frac{z^n}{\sqrt{n!}} + \left(\xi_{n+1} - \frac{\sqrt{n+1}}{10R}\right)\frac{z^{n+1}}{\sqrt{(n+1)!}}\,.$$

Since $|F(z)| \geqslant \frac{1}{2}\frac{|z|^n}{\sqrt{n!}}$ in the annulus $R-2 \leqslant |z| \leqslant R+2$, it is enough to estimate from below the probability of the event that

$$\max_{R-2\leqslant|z|\leqslant R+2}\left[\sum_{k:k\neq n,n+1} |\xi_k|\frac{\sqrt{n!}}{\sqrt{k!}}|z|^{k-n} + |\xi_n - 1| + \left|\xi_{n+1} - \frac{\sqrt{n+1}}{10R}\right|\frac{|z|}{\sqrt{n+1}}\right] < R^{-3},$$

say. Now, let us handle ξ_n and ξ_{n+1} first. We just demand that the corresponding terms be both less than R^{-4}. It is not hard to see that the probability of this event is about R^{-16}. We may neglect it since the factor

R^{-16} does not affect the lower bound $e^{-CR(\log R)^{3/2}}$ we are trying to get. The remaining sum can be estimated as

$$\sum_{k:k<n} |\xi_k| \frac{\sqrt{n!}}{\sqrt{k!}} (R-2)^{k-n} + \sum_{k:k>n+1} |\xi_k| \frac{\sqrt{n!}}{\sqrt{k!}} (R+2)^{k-n} .$$

We shall show how to estimate from below the probability that the second sum is less than R^{-4}. The estimate for the first sum is very similar and we omit it (note that the corresponding events depend on different ξ_k and, therefore, are independent, so the probability that both sums are small is just the product of the probabilities that each of them is small). Let $k = n + m$, $m = 2, 3, \dots$. We choose some big constant $A \gg 1$ and split the sum into two: $S_1 = \sum_{2 \leqslant m \leqslant AR\sqrt{\log R}}$ and $S_2 = \sum_{m > AR\sqrt{\log R}}$. We shall show that the probability that $|S_2| < R^{-5}$ is very close to 1 and the probability that $|S_1| < R^{-5}\sqrt{\log R}$ is at least $e^{-CR(\log R)^{3/2}}$.

To estimate S_2, we would like to use Lemma 2.1. To this end, we need to estimate the sum

$$\sum_{m > AR\sqrt{\log R}} \frac{(R+2)^m}{\sqrt{(n+1)(n+2)\dots(n+m)}} .$$

Note that, starting with $m = n$, the terms in this sum decay like a geometric progression, more precisely, the ratio of each term to the previous one is $\frac{R+2}{\sqrt{n+m+1}} < \frac{R+2}{\sqrt{2n}} = \frac{R+2}{\sqrt{2}\,R} < \frac{3}{4}$ if R is large enough. Thus, it is enough to estimate the sum over m such that $AR\sqrt{\log R} < m \leqslant n$. Now, for $k = 1, 2, \dots, n$, we have $n + k \geqslant ne^{k/2n}$. Thus, the m-th term of our sum does not exceed

$$\left(1 + \frac{2}{R}\right)^m \prod_{k=1}^m e^{-\frac{k}{4n}} \leqslant e^{\frac{2m}{R}} e^{-\frac{m^2}{8n}} = e^{\frac{2m}{R}} e^{-\frac{m^2}{8R^2}} \leqslant Ce^{-\frac{m^2}{16R^2}} , \quad 1 \leqslant m \leqslant n , \tag{9.2}$$

and the whole sum does not exceed

$$\begin{aligned} C \sum_{m > AR\sqrt{\log R}} e^{-\frac{m^2}{16R^2}} &\leqslant C \int_{AR\sqrt{\log R}-1}^{\infty} e^{-\frac{t^2}{16R^2}}\, dt \\ &\leqslant CR \int_{\frac{1}{2}A\sqrt{\log R}}^{\infty} e^{-\frac{t^2}{16}}\, dt \leqslant CRe^{-\frac{A^2}{64}\log R} = CR^{1-\frac{A^2}{64}} < R^{-6} \end{aligned}$$

if R is large enough. Thus, according to Lemma 2.1, the probability that $|S_2| < R^{-5}$ is very close to 1 and, at least, greater than $1/2$.

As to S_1, we just demand that each term in S_1 be less than R^{-6} (then $|S_1| < AR^{-5}\sqrt{\log R}$). Since the coefficients $\frac{(R+2)^m}{\sqrt{(n+1)(n+2)\dots(n+m)}} \overset{(9.2)}{\leqslant} C$, it is enough to demand that $|\xi_k| < C^{-1}R^{-6}$ for $n+2 \leqslant k \leqslant n+AR\sqrt{\log R}$. But the probability of this event is at least $(cR^{-12})^{AR\sqrt{\log R}} \geqslant e^{-CR(\log R)^{3/2}}$. This proves the lemma. □

Thus, with probability $e^{-CR(\log R)^{3/2}}$, the point $z = iR$ belongs to a basin of diameter greater than $R/10$. It remains to note that, due to shift invariance of U, the same is true for any other point z on the complex plane. This proves the lower bound in the diameter of the basin theorem. □

9.2 Distance to the sink (the lower bound). We choose a big $R \gg 1$ such that $n = R^2$ is an integer. This time we start with the function $F(z) = \frac{z^n}{\sqrt{n!}} e^{zR^{\delta-1}-R^\delta}$ with $0 < \delta < 1$ (later, we'll choose $\delta = \frac{2}{5}$). The gradient of the corresponding potential U equals

$$\nabla U(z) = \overline{\frac{F'}{F}(z)} - z = \frac{n}{\bar{z}} - z + R^{\delta-1} = \frac{R^2 - r^2}{r}\, e^{i\theta} + R^{\delta-1}\,, \quad z = re^{i\theta}\,.$$

Let $\mathcal{A} = \{R-1 < |z| < R+1\}$, and let $\mathcal{D} = \left\{z \in \mathcal{A}, |\arg z - \frac{\pi}{2}| < \frac{1}{10}\right\}$ be the same sector as above. Note the following properties of the gradient field:

(i) On the boundary circumferences $|z| = R \pm 1$, the radial component of the field $-\nabla U$ is directed outward from $\mathcal{A}$, and its size is not less than 1; inside $\mathcal{A}$, the size of the radial component does not exceed 3;

(ii) The field $-\nabla U$ has the horizontal drift $R^{\delta-1}$ oriented to the left; in particular, inside the sector $\mathcal{D}$, the angular component of the field $-\nabla U$ is oriented counter-clockwise and its size is within the range $\left[\frac{1}{2}R^{\delta-1}, 2R^{\delta-1}\right]$.

By $\mathcal{G}$ we denote the set of points that hit the segment

$$J = \left[(R-1)e^{i\left(\frac{\pi}{2}+\frac{1}{10}\right)}, (R+1)e^{i\left(\frac{\pi}{2}+\frac{1}{10}\right)}\right]$$

when moving along their trajectories. Because of the "left-oriented horizontal drift" of the field $-\nabla U$, the points z with $\pi \geqslant |\arg z| > \frac{\pi}{2} + \frac{1}{10}$ cannot appear within $\mathcal{G}$. (In fact, it is easy to see that $\mathcal{G} \subset \left\{z : 0 < \arg z < \frac{\pi}{2} + \frac{1}{10}\right\}$ but we will not need this). By $\mathcal{G}(\theta)$ we denote the subset of $\mathcal{G}$ that is located clockwise with respect to the segment

$$J(\theta) = \left[(R-1)e^{i\left(\frac{\pi}{2}+\frac{1}{10}-\theta\right)}, (R+1)e^{i\left(\frac{\pi}{2}+\frac{1}{10}-\theta\right)}\right].$$

Note that $\mathcal{G}(\theta_2) \subset \mathcal{G}(\theta_1)$ for $\theta_2 > \theta_1$. By $A(\theta)$ we denote the area of $\mathcal{G}(\theta)$. We denote by $h(\theta)$ the length of the intersection of $\mathcal{G}$ with the segment $J(\theta)$; i.e. the length of the "left boundary wall" of the domain $\mathcal{G}(\theta)$.

Lemma 9.3. *If $R \gg 1$, then $h(\theta) \geqslant e^{-CR^{2-\delta}}$ for $0 \leqslant \theta \leqslant 1/5$, and*

$$m_2\left(\mathcal{G} \cap \left\{\frac{\pi}{2} - \frac{1}{10} < \arg z < \frac{\pi}{2}\right\}\right) \geqslant e^{-CR^{2-\delta}}\,.$$

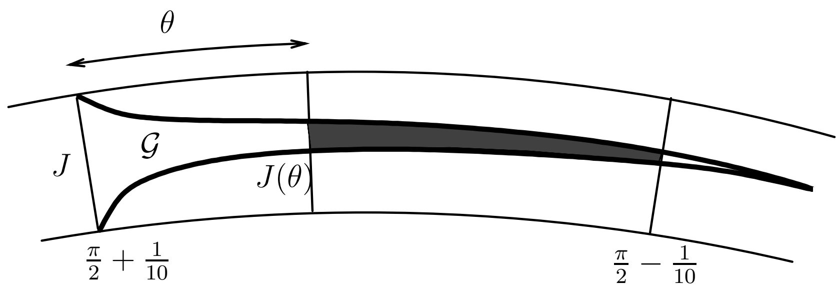

Figure 7: The sets $\mathcal{G}$ and $\mathcal{G}(\theta)$

Proof. Note that the second estimate follows from the first one by integration over θ. We have

$$A(\theta) = -\frac{1}{2}\iint_{\mathcal{G}(\theta)} \Delta U \, dm_2 = -\frac{1}{2}\int_{\partial\mathcal{G}(\theta)} \frac{\partial U}{\partial n}\,|dz| = -\frac{1}{2}\int_{\mathcal{G}\cap J(\theta)} \frac{\partial U}{\partial n}\,|dz|$$

(since the rest of the boundary of $\mathcal{G}(\theta)$ consists of gradient curves). In view of (ii),

$$\tfrac{1}{2}R^{\delta-1}h(\theta) \leqslant -\int_{\mathcal{G}\cap J(\theta)} \tfrac{\partial U}{\partial n}\,|dz| \leqslant 2R^{\delta-1}h(\theta)\,,$$

whence

$$\tfrac{1}{4}R^{\delta-1}h(\theta) \leqslant A(\theta) \leqslant R^{\delta-1}h(\theta)\,. \tag{9.4}$$

We notice that $|A'(\theta)| \leqslant (R+1)h(\theta) < 2Rh(\theta)$. Combining this with the lower bound in (9.4), we get the differential inequality $A'(\theta) \geqslant -8R^{2-\delta}A(\theta)$, whence $A(\theta) \geqslant A(0)e^{-8R^{2-\delta}\theta}$.

To estimate $A(0)$ from below, recall that it equals $1/2$ the flow of the field $-\nabla U$ through the interval J. Since the length of J is 2, $A(0)$ cannot be less than the minimum of the angular component of $-\nabla U$; i.e. $A(0) \geqslant \frac{1}{2}R^{\delta-1}$. Thus, $A(\theta) \geqslant \frac{1}{2}R^{\delta-1}e^{-8\theta R^{2-\delta}}$.

Now, using the upper bound in (9.4), we get

$$h(\theta) \geqslant R^{1-\delta}A(\theta) \geqslant e^{-8\theta R^{2-\delta}} > e^{-2R^{2-\delta}}\,, \quad \text{for } 0 \leqslant \theta \leqslant \tfrac{1}{5}\,.$$

Hence the lemma. □

We can replace the function F by its analytic perturbation $F+H$ with H satisfying $|H/F| \leqslant R^{-2}$ in the annulus $R-2 \leqslant |z| \leqslant R+2$. After this perturbation, the gradient field still satisfies the conditions (i) and (ii), and the previous lemma applies to the new gradient flow. The next lemma gives a lower bound for the probability of the event that a GEF f is such a perturbation.

LEMMA 9.5. *If* $0 < \delta < 1$ *and* $R \gg 1$, *then* $\mathbb{P}\bigl\{\max_{R-2\leqslant|z|\leqslant R+2} \bigl|\frac{f}{F}(z)-1\bigr| \leqslant \frac{1}{R^2}\bigr\} \geqslant e^{-CR^{1+\frac{3}{2}\delta}}$.

Proof. The proof we give is very similar to that of Lemma 9.1. Actually, we estimate from below the probability of the smaller event that $|f-F| \leqslant e^{-R^\delta}|F|$ everywhere in the annulus $R-2 \leqslant |z| \leqslant R+2$. Note that, in this annulus, $|F(z)| \geqslant \frac{1}{2}e^{-2R^\delta}\frac{|z|^n}{\sqrt{n!}}$.

First, we replace the exponent $e^{zR^{\delta-1}}$ by its Taylor polynomial of degree $M=[20R^\delta]$ in the disk $|z| \leqslant 2R$. It is easy to check that for $m \geqslant M$ and $|z| \leqslant 2R$, the m-th term in the Taylor expansion of the function $e^{zR^{\delta-1}}$ is bigger than twice the $m+1$-st term. Hence the absolute value of the tail that starts with the $M+1$-st term does not exceed the absolute value of the M-th term. In particular, the relative error we've made discarding the tail is at most

$$e^{2R^\delta}\frac{(2R^\delta)^M}{M!} \leqslant e^{2R^\delta}\left(\frac{2eR^\delta}{M}\right)^M < e^{-10R^\delta}\,.$$

Hence, for $|z| \leqslant 2R$,

$$\bigl|F(z)-P_M(z)\bigr| < e^{-10R^\delta}|F(z)|\,,$$

where

$$P_M(z) = \frac{z^n}{\sqrt{n!}}e^{-R^\delta}\sum_{m=0}^{M}\frac{R^{\delta m}}{m!}\left(\frac{z}{R}\right)^m = \sum_{m=0}^{M} a_m\frac{z^{n+m}}{\sqrt{(n+m)!}}$$

is the Taylor polynomial of F. Note that

$$a_m = \left(R^{-m}\sqrt{\frac{(n+m)!}{n!}}\right)\cdot\left(e^{-R^\delta}\frac{R^{\delta m}}{m!}\right)\,.$$

The second factor on the RHS is less than 1. If $R \gg 1$, then the first factor does not exceed 2:

$$\sqrt{\frac{(n+1)(n+2)\ldots(n+m)}{n^m}} < \left(1+\frac{M}{R^2}\right)^{M/2} < e^{\frac{1}{2}(M/R)^2} < 2\,.$$

Thus, $0 < a_m < 2$.

Note that

$$\left|\frac{f(z)}{F(z)}-1\right| \leqslant \left|\frac{f(z)-P_M(z)}{F(z)}\right| + \left|\frac{F(z)-P_M(z)}{F(z)}\right|\,,$$

and that we've already estimated the second term on the right-hand side. We write

$$\left|\frac{f(z)-P_M(z)}{F(z)}\right| \leqslant 2e^{2R^\delta} \max_{R-2\leqslant|z|\leqslant R+2} \Big[\sum_{n\leqslant k\leqslant n+M} |\xi_k - a_{k-n}| \sqrt{\tfrac{n!}{k!}} |z|^{k-n} + \sum_{k\neq n,n+1,\dots,n+M} |\xi_k| \sqrt{\tfrac{n!}{k!}} |z|^{k-n} \Big]$$

and show that with probability at least $e^{-CR^{1+\frac{3}{2}\delta}}$ the maximum of the brackets on the right-hand side does not exceed Ce^{-4R^δ}.

We start with the first sum and demand that

$$|\xi_{n+m} - a_m| < e^{-40R^\delta}, \quad m = 0, 1, \dots, M\,.$$

The probability of this event is not less than $\big(ce^{-80R^\delta}\big)^{M+1} > e^{-CR^{2\delta}} > e^{-CR^{1+\frac{3}{2}\delta}}$. For $|z| \leqslant R+2$, $R \gg 1$, and $k \geqslant n = R^2$, we have $\frac{|z|^{k+1}}{\sqrt{(k+1)!}} \leqslant 2\frac{|z|^k}{\sqrt{k!}}$. Hence $\sqrt{\frac{n!}{k!}}|z|^{k-n} \leqslant 2^{k-n}$, and the sum we are estimating does not exceed $2^{M+1}e^{-40R^\delta} < e^{-20R^\delta}$.

The second sum in the brackets does not exceed

$$\sum_{0\leqslant k<n} |\xi_k| \sqrt{\tfrac{n!}{k!}} (R-2)^{k-n} + \sum_{k>n+M} |\xi_k| \sqrt{\tfrac{n!}{k!}} (R+2)^{k-n}\,. \tag{9.6}$$

We estimate from below the probability that the first sum in (9.6) is less than $3e^{-4R^\delta}$. The estimate for the second sum is in the same spirit (cf. proof of Lemma 9.1) and we omit it. We choose a large constant $A \gg 1$ and split the first sum in (9.6) into two: $S_1 = \sum_{0\leqslant k<n-A\sqrt{Mn}}$ and $S_2 = \sum_{n-A\sqrt{Mn}\leqslant k<n}$.

As in the proof of Lemma 9.1, we apply Lemma 2.1 to estimate the sum S_1. For this, we need to estimate the sum

$$\sum_{0\leqslant k<n-A\sqrt{Mn}} \sqrt{n(n-1)\dots(n-(n-k-1))}(R-2)^{k-n} = \sum_{A\sqrt{Mn}<m\leqslant n} \sqrt{n(n-1)\dots(n-(m-1))}(R-2)^{-m}\,.$$

The m-th term of the sum on the right-hand side equals

$$\sqrt{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{m-1}{n}\right)}\left(1-\frac{2}{R}\right)^{-m}.$$

Using inequalities $1-\xi \leqslant e^{-\xi}$, $0 \leqslant \xi \leqslant 1$, and $(1-\xi)^{-1} \leqslant e^{2\xi}$, $0 \leqslant \xi \leqslant 1/2$, we bound the last expression by

$$e^{\frac{4m}{R}} \prod_{j=0}^{m-1} e^{-\frac{1}{2}\frac{j}{n}} = e^{\frac{4m}{R}-\frac{(m-1)m}{4n}} \leqslant Ce^{-\frac{m^2}{8R^2}}, \quad 1 \leqslant m \leqslant n\,.$$

Then the sum we are estimating does not exceed

$$C\int_{cAR^{1+\frac{\delta}{2}}}^{\infty} e^{-\frac{t^2}{8R^2}}\,dt \leqslant CR\int_{cAR^{\delta/2}}^{\infty} e^{-t^2}dt \leqslant CRe^{-cA^2R^{\delta}} \leqslant e^{-10R^{\delta}}\,,$$

if A is big enough.

Then, according to Lemma 2.1, the probability that

$$\sum_{0\leqslant k<n-A\sqrt{Mn}} |\xi_k|\sqrt{\tfrac{n!}{k!}}(R-2)^{k-n} > e^{-4R^{\delta}}$$

has a double exponential decay. We conclude modestly that $S_1 \leqslant e^{-4R^{\delta}}$ with probability at least 1/2.

Now, we look at the sum S_2. In this case, we demand that

$$|\xi_k| < e^{-10R^{\delta}}\,, \quad n - A\sqrt{Mn} \leqslant k < n\,.$$

The probability of this event is not less than

$$\left(\tfrac{1}{2}e^{-10R^{\delta}}\right)^{2(A\sqrt{Mn}+1)} \geqslant e^{-CR^{\delta}\cdot R^{1+\frac{\delta}{2}}} = e^{-CR^{1+\frac{3}{2}\delta}}\,.$$

Then the sum S_2 does not exceed

$$\begin{aligned} e^{-10R^{\delta}} &\sum_{n-A\sqrt{Mn}\leqslant k<n} \sqrt{\tfrac{n!}{k!}}(R-2)^{k-n} \\ &= e^{-10R^{\delta}} \sum_{1\leqslant m\leqslant A\sqrt{Mn}} \sqrt{n(n-1)\dots(n-(m-1))}(R-2)^{-m}\,. \end{aligned}$$

We know from the discussion above that each term of the latter sum is bounded by a constant. Hence

$$S_2 \leqslant CA\sqrt{Mn}e^{-10R^{\delta}} \leqslant e^{-9R^{\delta}}\,,$$

if R is big enough. This completes the estimate of expression (9.6) and proves the lemma. □

Now, let us fix the variables ξ_k such that the function $f(z)$ is a small perturbation of $F(z)$. For this function f, we consider the corresponding "tail" $\mathcal{G}$. If z belongs to the set $\mathcal{G}\cap\{z : \frac{\pi}{2}-\frac{1}{10} < \arg z < \frac{\pi}{2}\}$ (the area of this set was estimated in Lemma 9.3), then the trajectory Γ_z must traverse the whole set $\mathcal{G}\cap\{z : \frac{\pi}{2} < \arg z < \frac{\pi}{2}+\frac{1}{10}\}$ before it hits the radial interval J. Hence we expect that for such z's the distance from z to its sink a_z is comparable with R. We use this idea to prove the following lemma.

LEMMA 9.7. *Suppose $R \gg 1$. With probability at least $e^{-CR^{1+\frac{3}{2}\delta}}$,*

$$m_2\left\{z\in D(iR,R) : |z-a_z| \geqslant \tfrac{R}{100}\right\} \geqslant e^{-CR^{2-\delta}}\,.$$

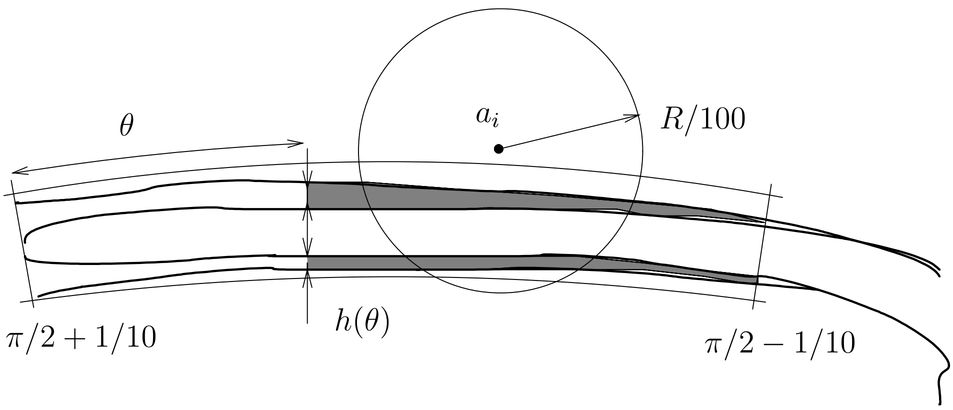

Figure 8: The *tail* $\mathcal{G}_i$. The grey area equals $A_i(\theta)$

Proof. After the trajectories from the tail $\mathcal{G}$ leave the sector $\mathcal{D}$, they are attracted by some of the zeroes of the function f. Let $a_1, \dots, a_N$ be the zeroes of f that lie in the disk $2R\mathbb{D}$ and attract these trajectories, and let $\mathcal{G}_i$ be the corresponding tails. We discard the event $N \geqslant 100R^2$ since, by Theorem 2 in [ST, Part III], its probability is bounded by e^{-CR^4} which is much less than $e^{-CR^{1+\frac{3}{2}\delta}}$. Hence we assume that $N \leqslant 100R^2$.

Let $A_i(\theta)$ be the area of the tail $\mathcal{G}_i \cap \bigl\{\frac{\pi}{2} - \frac{1}{10} < \arg z < \frac{\pi}{2} + \frac{1}{10} - \theta\bigr\}$, and let $h_i(\theta)$ be the length of the radial section of $\mathcal{G}_i$ by the ray $\bigl\{\arg z = \frac{\pi}{2} + \frac{1}{10} - \theta\bigr\}$; let $A_0(\theta)$, $h_0(\theta)$ be the similar quantities that correspond to the trajectories attracted by zeroes of f lying outside the disk $2R\mathbb{D}$. By Lemma 9.3,

$$\sum_{i=0}^{N} A_i\left(\tfrac{1}{10}\right) = m_2\left(\mathcal{G} \cap \left\{z : \tfrac{\pi}{2} - \tfrac{1}{10} < \arg z < \tfrac{\pi}{2}\right\}\right) \geqslant e^{-CR^{2-\delta}} ,$$

thereby, $A_i(1/10) \geqslant e^{-CR^{2-\delta}}$ for some i.

If $i = 0$, we are done: the points from the domain corresponding to A_0 are far from their sinks. If $i \neq 0$, then, as in the proof of Lemma 9.3, $h_i(\theta) \geqslant R^{1-\delta} A_i(\theta) \geqslant e^{-CR^{2-\delta}}$ for $0 \leqslant \theta \leqslant 1/10$. Hence, after deleting the disk $D(a_i, R/100)$, we still have a set of points within $\mathcal{D} \subset D(iR, R)$ of area at least $e^{-CR^{2-\delta}}$ that are attracted to a_i. This proves the lemma. □

Now, we apply the same "averaging trick" that we've already used in the proof of the upper bound for the distance to the sink. Consider the (random) set $\mathcal{C} = \{z : |z - a_z| \geqslant R/100\}$ and the event $\Omega^* = \{m_2(\mathcal{C} \cap D(z, R)) \geqslant e^{-CR^{2-\delta}}\}$. The probability of this event was estimated in the previous lemma (for convenience, we took there $z = iR$ but, due to the translation invariance, the probability of Ω^* does not depend on the choice of z).

We aim at estimating from below the probability $\mathbb{P}\{z \in \mathcal{C}\}$. We have

$$\begin{aligned}\pi R^2 \mathbb{P}\{z \in \mathcal{C}\} &= \iint_{R\mathbb{D}} \mathbb{P}\{z + w \in \mathcal{C}\} dm_2(w) \\ &= \int_\Omega m_2\big(\mathcal{C} \cap D(z,R)\big) d\mathbb{P} \geqslant \int_{\Omega^*} m_2\big(\mathcal{C} \cap D(z,R)\big) d\mathbb{P} \\ &= \mathbb{P}\{\Omega^*\} e^{-CR^{2-\delta}} \geqslant e^{-CR^{1+\frac{3}{2}\delta} - CR^{2-\delta}}.\end{aligned}$$

It remains to put $\delta = 2/5$ to balance the exponents. We are done. □

10 Diameter of the Core

Given $z \in \mathbb{C}$, we show that the probability of the event $\{m_2(B_z \setminus D(a_z, R)) > \varepsilon\}$ behaves as e^{-cR^4} when R is sufficiently large.

10.1 The upper bound. Given $z \in \mathbb{C}$, we show that the probability of the event $\{m_2(B_z \setminus D(a_z, R)) > \varepsilon\}$ cannot be bigger than e^{-cR^4} when $R \gg 1$.

We take a small positive η depending on ε only and assume that $U \leqslant \eta R^2$ everywhere in the basin B_z. It is not difficult to see that the probability of the opposite event does not exceed $Ce^{-c_\eta R^4}$. Indeed, the event $\big\{\max_{B_z} U > \eta R^2\big\}$ is contained in the union of the events $\{\mathrm{diam}(B_z) > R^4\}$ and $\big\{\max_{D(z,R^4)} U > \eta R^2\big\}$. By the long gradient curve theorem, the probability of the first event does not exceed Ce^{-cR^4}. By Lemma 4.1, the probability of the second event does not exceed $CR^8 e^{-ce^{2\eta R^2}}$.

Similarly, we also assume that $U \geqslant -\eta R^2$ everywhere in $B_z \setminus D(a_z, R/2)$. The opposite event is contained in the union of the events $\{\mathrm{diam}(B_z) > R^4\}$ and

$$\Big\{\text{there exists a curve } \gamma \subset R^4\mathbb{D} \text{ with } \mathrm{diam}(\gamma) \geqslant \tfrac{1}{2}R \text{ such that } \max_\gamma U < -\eta R^2\Big\},$$

and by Theorem 4.3, the probability of the second event is bounded by $e^{-c_\eta R^4}$.

Thus, discarding events of probability less than $e^{-c_\eta R^4}$, we may assume that $\max_{B_z \setminus D(a_z, R)} |U| \leqslant \eta R^2$. Then, by our length and area estimate

(Proposition 8.2),

$$m_2\big(B_z \setminus D(a_z,R)\big) \leqslant \pi \exp\left(-\frac{(R/2)^2}{\eta R^2}\right) = \pi \exp\left(-\frac{1}{4\eta}\right) < \varepsilon$$

if η is sufficiently small. This proves the upper bound. □

10.2 The lower bound. We fix a positive $\kappa < \pi$ and consider the random set $\mathcal{C} = \{z : m_2(B_z \setminus D(a_z,R)) \geqslant \kappa\}$. We need to estimate from below the probability $\mathbb{P}\{z \in \mathcal{C}\}$, which does not depend on the choice of z. We apply the averaging again, but this time we average over the disk of radius R^5. We get

$$\pi R^{10}\mathbb{P}\{0 \in \mathcal{C}\} = \iint_{R^5\mathbb{D}} \mathbb{P}\{w \in \mathcal{C}\}\, dm_2(w) = \int_\Omega m_2(\mathcal{C} \cap R^5\mathbb{D})\, d\mathbb{P}\,.$$

Introduce the event Ω^* that the following two conditions hold:

(i) $\#(\mathcal{Z}_f \cap R\mathbb{D}) \geqslant \frac{4\pi}{\pi-\kappa}\, R^2$;

(ii) there is no gradient curve connecting the circumferences $\{|z| = R\}$ and $\{|z| = R^5\}$.

The probability of the first event is not less than e^{-CR^4}. This estimate can be derived using the same techniques as in [ST, Part III] and in [K], though it was not explicitly proved in these papers. To get this estimate, denote by m the least integer that is not less than $\frac{4\pi}{\pi-\kappa}\, R^2$, and estimate from below the probability that

$$\left|\xi_m \frac{z^m}{\sqrt{m!}}\right| > \left|f(z) - \xi_m \frac{z^m}{\sqrt{m!}}\right|$$

everywhere on the circumference $\{|z| = R^2\}$. We skip the estimate since it repeats the one used in the proof of Theorem 3 in [K].

Next, by the long gradient curve theorem, the probability that the second event does not hold is less than e^{-cR^5}. Hence $\mathbb{P}\{\Omega^*\} \geqslant e^{-CR^4}$.

Now, assuming that Ω^* happens, we can easily give a lower bound for the area of the set $\mathcal{C} \cap R^5\mathbb{D}$. Actually, we need to find only *one basin* $B(a)$ with $|a| \leqslant R$ and $m_2\big(B(a) \setminus 2R\mathbb{D}\big) \geqslant \kappa$. Then, by assumption (ii), this basin lies within the disk $R^5\mathbb{D}$. Thereby, $m_2\big(\mathcal{C} \cap R^5\mathbb{D}\big) \geqslant \pi$, and we are done:

$$\mathbb{P}\{0 \in \mathcal{C}\} \geqslant R^{-10}\mathbb{P}\{\Omega^*\} \geqslant ce^{-CR^4}\,.$$

To find a basin $B(a)$ with $|a| \leqslant R$ and $m_2(B(a) \setminus 2R\mathbb{D}) \geqslant \kappa$, we do a simple counting. Consider the basins $B(a)$ with $|a| \leqslant R$ but $m_2(B(a) \cap 2R\mathbb{D}) > \pi - \kappa$. Let N be the number of such basins. Comparing the areas, we get

$$4\pi R^2 = m_2(2R\mathbb{D}) \geqslant \sum_a m_2\big(B(a) \cap 2R\mathbb{D}\big) > (\pi - \kappa)N\,;$$

that is, $N < 4\pi(\pi-\kappa)^{-1}R^2$. Hence, by assumption (i), there is at least one basin $B(a)$, with $|a| \leqslant R$ and $m_2(B(a) \setminus 2R\mathbb{D}) \geqslant \kappa$. This finishes the proof. □

11 Modified Basins

In this section, we prove the remaining Theorem 1.7. First, we describe a deterministic algorithm that "improves" partitions of the plane into domains of equal areas by cutting off the tentacles of the basins and reallocating them closer to the sinks. Then we'll prove the probabilistic estimates for the sizes of the modified basins of our random partition.

11.1 Cutting off the tentacles. Suppose we are given a partition of the plane $\mathbb{C} = \cup_i E_i$ into bounded open domains of equal area, say π, with marked points c_i, the *centers* of E_i. Let

$$R_i = \inf\left\{R : E_i \subset D(c_i, R)\right\}.$$

Clearly, $R_i \geqslant 1$.

Given $\varepsilon \in (0,1)$, we choose the least r_i satisfying the condition

$$m_2\big(E_i \setminus D(c_i, r_i)\big) \leqslant \tfrac{1}{AR_i^3}$$

with $A = 10^4\varepsilon^{-1}$ and define the *kernel* $K_i = E_i \cap D(c_i, r_i)$ and the *tentacle* $T_i = E_i \setminus D(c_i, r_i)$ of the domain E_i. Note that $m_2 T_i < 10^{-4}\varepsilon$. It is worth mentioning that this definition of the tentacle differs from the one we used in section 8.2. Later on, the factor R_i^{-3} will help us to avoid large tangles of different tentacles.

Proposition 11.1. *Given $\varepsilon > 0$, there exist open pairwise disjoint sets E_i' with the following properties:*

(i) $m_2 E_i' = \pi$;
(ii) $\mathbb{C} = \bigcup_i E_i'$ *(up to a set of measure* 0*);*
(iii) $m_2(E_i \cap E_i') \geqslant \pi - \varepsilon$;
(iv) $E_i' \subset D\big(c_i, r_i + \sqrt{5}\big)$.

This proposition is useful when some of the domains E_i have long tentacles; that is, $r_i \ll R_i$. The sets E_i may be assumed only measurable. Then the resulting sets E_i' will be measurable too.

Proof of Proposition 11.1. Split the plane $\mathbb{C}$ into standard unit squares. Suppose that Q is one of them. First, we check that the union of the tentacles T_i can cover only a small portion of the square Q:

LEMMA 11.2.
$$m_2\big(Q\cap(\cup_i T_i)\big)\leqslant \tfrac{1}{10}\varepsilon\,.$$

Proof. If the domain E_i with $R_i\leqslant R$ intersects the square Q, then E_i is contained in the square with side length $4R+1$ homothetic to Q. Hence, comparing the areas, we note that
$$N_Q(R)\stackrel{\rm def}{=}\#\{i: E_i\cap Q\neq\varnothing\,,\ R_i\leqslant R\}\leqslant \tfrac{1}{\pi}(4R+1)^2.$$
Thus
$$\begin{aligned}\sum_{i:E_i\cap Q\neq\varnothing} m_2T_i &= \tfrac{1}{A}\sum_{i:E_i\cap Q\neq\varnothing}\tfrac{1}{R_i^3}\\ &=\frac{3}{A}\int_1^\infty\frac{N_Q(R)}{R^4}\,dR\leqslant\frac{3}{\pi A}\int_1^\infty\frac{(4R+1)^2}{R^4}\,dR\leqslant\frac{1}{10}\varepsilon\,.\quad\square\end{aligned}$$

Now, let $\widehat{E}_i$ be a minimal square that is a union of several standard unit squares and that contains the set E_i.

LEMMA 11.3.
$$\sum_{i:Q\subset\widehat{E}_i} m_2T_i\leqslant\tfrac{1}{10}\varepsilon\,.$$

Proof. Proof Comparing the areas, we see that
$$\#\{i:Q\subset\widehat{E}_i, R_i\leqslant R\}\leqslant\tfrac{1}{\pi}(4R+3)^2\,.$$
The rest is the same as in the previous lemma. □

Let $\widehat{Q}_i$ be the square that contains the center c_i of E_i (if c_i lies on the grid, it does not matter which one of several squares containing c_i to choose). For each pair (i,Q) with $Q\subset\widehat{E}_i\setminus\widehat{Q}_i$, we choose a *store* $S_i(Q)\subset Q\cap(\cup_jK_j)$ according to the following rules:

(a) $m_2S_i(Q)=m_2T_i$;
(b) For different i's, the stores $S_i(Q)$ are mutually disjoint;
(c) For each pair (i,Q), the area of the store $S_i(Q)$ is distributed between the kernels $K_j\cap Q$ proportionally to their areas; i.e.
$$m_2\big(S_i(Q)\cap K_j\big)=m_2\big(S_i(Q)\big)\,\frac{m_2(Q\cap K_j)}{\sum_l m_2(Q\cap K_l)}\,.$$

By Lemma 11.3, the total area within Q that we need to allocate to all the stores does not exceed $\varepsilon/10$, while by Lemma 11.2, the area of $Q\cap(\cup_jK_j)$ is not less than $1-\frac{\varepsilon}{10}$. Hence, we can meet the requirements (a) and (b). The requirement (c) does not impose any additional restriction.

Now we describe the cut-off algorithm. It consists of countably many parallel processes, which are independent of each other. During the i-th process, for each square $Q \subset \widehat{E_i}$, the piece of the tentacle $T_i \cap Q$ is re-allocated to some centers c_l such that $K_l \cap Q \neq \varnothing$. At the same time, some subsets of $K_l \cap S_i(Q)$ are re-allocated to some centers c_m whose kernels K_m intersect one of the squares neighbouring Q.

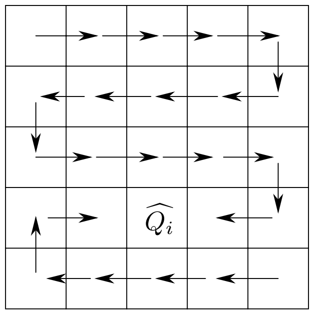

Figure 9: The square $\hat{E}_i$ and two sequences of unit squares

We split the unit squares from $\widehat{E}_i \setminus \widehat{Q}_i$ into two disjoint sequences $\{Q_1, Q_2, \dots, Q_{m_1}\}$ and $\{Q_{m_1+1}, Q_{m_1+2}, \dots, Q_{m_2}\}$ such that in each sequence any two consecutive squares Q_l and Q_{l+1} have a common boundary side, and the last squares Q_{m_1}, Q_{m_2} of each sequence have a common boundary side with the square $\widehat{Q}_i$ (see Figure 9).

Let us call $T_i \setminus \widehat{Q}_i$ the *grey area*. First, for each j, $1 \leqslant j \leqslant m_1$, we swap the set $T_i \cap Q_j$ with a part of the store $S_i(Q_j)$. More precisely, we

(i) Choose parts of the stores $G_{i,j} \subset S_i(Q_j)$ with $m_2 G_{i,j} = m_2(T_i \cap Q_j)$;
(ii) Decompose the tentacle $T_i \cap Q_j$ into disjoint union of subsets $T_{i,l,j}$, $l \neq i$, with $m_2 T_{i,l,j} = m_2(G_{i,j} \cap K_l)$;
(iii) For $1 \leqslant j \leqslant m_1$, re-allocate the grey area from $T_i \cap Q_j$ to $G_{i,j}$;
(iv) For each $l \neq i$, remove the set $K_l \cap \bigcup_{1 \leqslant j \leqslant m_1} G_{i,j}$ from E_l, and re-allocate the set $\bigcup_{1 \leqslant j \leqslant m_1} T_{i,l,j}$ of equal measure to E_l.

Now, the grey area occupies some parts of the stores $S_i(Q_j)$.

At the next step, starting with the square Q_1, square after square, we move the grey area from $S_i(Q_j)$ to $S_i(Q_{j+1})$, until the whole grey area appears in the last store $S_i(Q_{m_1})$ of the sequence of squares we are traversing. After that, we allocate the grey area to the center c_i. More formally:

(i) For $2 \leqslant j \leqslant m_1$, we choose subsets $G'_{i,j} \subset S_i(Q_j) \setminus G_{i,j}$ such that $m_2 G'_{i,j} = \sum_{k=1}^{j-1} m_2 G_{i,k}$, and set $G''_{i,j} = G'_{i,j} \cup G_{i,j}$, $G''_{i,1} = G_{i,1}$;
(ii) For $1 \leqslant j \leqslant m_1 - 1$, we decompose the sets $G''_{i,j}$ into disjoint union of subsets $G_{i,l,j}$ with $m_2 G_{i,l,j} = m_2(G'_{i,j+1} \cap K_l)$;

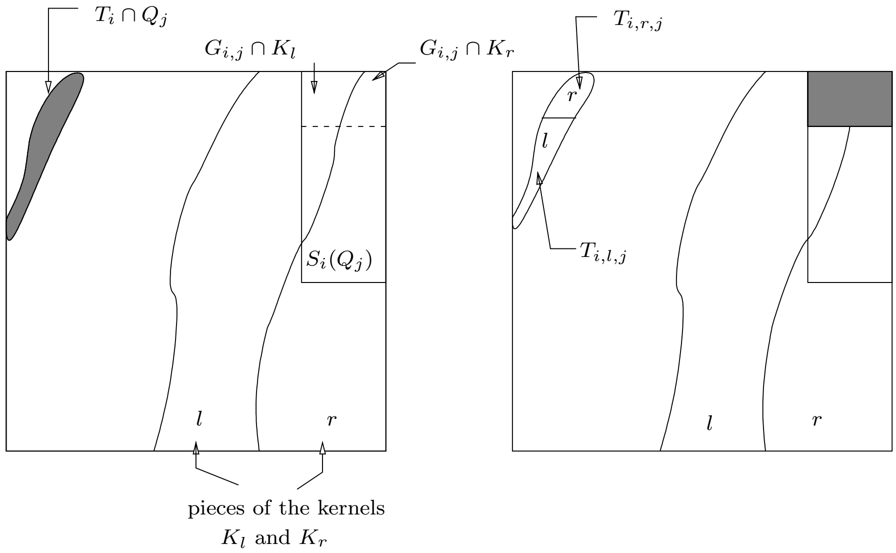

Figure 10: Putting the grey area $T_i \cap Q_j$ into the store $S_i(Q_j)$

(iii) Within each E_l, $l \neq i$, we replace the set $K_l \cap \bigcup_{1\leqslant j\leqslant m_1-1} G'_{i,j+1}$ by the set $\bigcup_{1\leqslant j\leqslant m_1-1} G_{i,l,j}$ of equal measure.

(iv) In the end, the tentacles $T_i \cap Q_j$, $1 \leqslant j \leqslant m_1$, are cut off from E_i, and the set G''_{i,m_1} with $m_2 G''_{i,m_1} = \sum_{1\leqslant j\leqslant m_1} m_2(T_i \cap Q_j)$ is added to E_i.

Then we apply the same process to the second sequence of squares $\{Q_{m_1+1}, \dots, Q_{m_2}\}$.

Note that all points re-allocated during the i-th process will appear either in T_i or in one of the stores S_i. Hence, due to the choice of the stores, these points are not displaced during the other steps. We see that for different i's the processes are independent of each other.

We conclude that the new sets E'_i are located in the $\sqrt{5}$-neighbourhoods of the kernels K_i and have the same area as E_i. ($\sqrt{5}$ is the length of the diagonal of the rectangle comprised of two adjacent standard squares.) By construction,

$$E_i \setminus E'_i \subset T_i \cup \bigcup_{(j,Q)} \big(S_j(Q) \cap K_i\big)\,.$$

Due to the choice of the stores and Lemmas 11.2 and 11.3,

$$\sum_{(j,Q)} m_2(S_j(Q) \cap K_i) \leqslant 2 \sum_{Q} \Big[\sum_{j:Q\subset \widehat{E}_j} m_2(T_j) \Big] m_2(Q \cap K_i) \leqslant \frac{\pi\varepsilon}{5}\,.$$

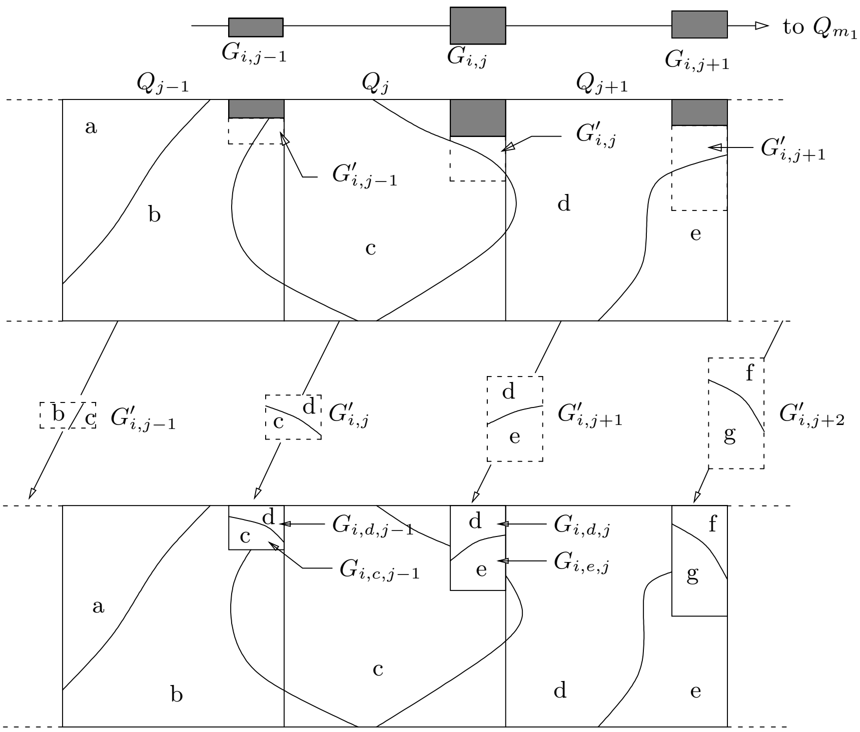

Figure 11: Moving the grey area from the stores $S_i(Q_j)$ to the store $S_i(Q_{m_1})$

Recall that $m_2 T_i \leqslant 10^{-4}\varepsilon$. Hence, for each i, $m_2(E_i \setminus E_i') < \varepsilon$. This proves Proposition 11.1. □

11.2 Probabilistic estimate. We fix $\varepsilon > 0$ and apply the cut-off algorithm to the basins $B(a)$. The sink a is the *center* of $B(a), R(a)$ is the least number R such that $B(a) \subset D(a, R)$. As above, we set $A = 10^4\varepsilon^{-1}$, and

$$r(a) = \min\left\{ r : m_2\big(B(a) \setminus D(a, r)\big) \leqslant \frac{1}{AR^3(a)} \right\}.$$

Proposition 11.1 gives us the modified basins $B'(a) \subset D\big(a, r(a) + \sqrt{5}\big)$ satisfying conditions (i)–(iii) of Theorem 1.7. Let $B'(\alpha)$ be the modified basin with center at α that contains the origin. Since $\operatorname{diam} B'(\alpha) \leqslant 2\big(r(\alpha) + \sqrt{5}\big)$, the proof of condition (iv) in Theorem 1.7 boils down to the estimate

$$\mathbb{P}\big\{r(\alpha) > R\big\} \leqslant e^{-cR^4/(\log R)^{3/2}} \tag{11.4}$$

for $R \gg A$.

CLAIM 11.5. $\mathbb{P}\big\{|\alpha| \geqslant \frac{1}{2}R^4\big\} \leqslant e^{-cR^4}$.

Proof. Assume that $|\alpha| \geqslant \frac{1}{2}R^4$. Since the origin lies at the distance at most $\sqrt{5}$ from the basin $B(\alpha)$, we know that there is a gradient curve that connects the circumferences $\big\{|z| = \sqrt{5}\big\}$ and $\big\{|z| = \frac{1}{2}R^4\big\}$. This gradient curve connects the boundaries of the squares $Q\big(0, \frac{1}{4\sqrt{2}}R^4\big)$ and $Q\big(0, \frac{1}{2\sqrt{2}}R^4\big)$. By the long gradient curve theorem, the probability of this event is less than e^{-cR^4}. □

Now, we prove (11.4). First, we suppose that $R(\alpha) > R^4$. In view of the claim, we also assume that $|\alpha| < \frac{1}{2}R^4$. We cover the disk $D\big(0, \frac{1}{2}R^4\big)$ by a bounded number of standard squares $Q\big(w, \frac{1}{2\sqrt{2}}R^4\big)$, and consider the square that contains the point α. We know that there is a gradient curve of diameter $R(\alpha)$ that terminates at the sink α. This gradient curve must connect $\partial Q\big(w, \frac{1}{4\sqrt{2}}R^4\big)$ with $\partial Q\big(w, \frac{1}{2\sqrt{2}}R^4\big)$. By the long gradient curve theorem, the probability of this event is less than e^{-cR^4}. Hence, $\mathbb{P}\{R(\alpha) > R^4\} < e^{-cR^4}$.

Now, we suppose that $R^4 \geqslant R(\alpha)$. Set $M = \frac{1}{52}\frac{R^2}{\log R}$. By Lemma 4.1, throwing away an event of probability much less than e^{-cR^4}, we may assume that $U \leqslant M$ everywhere in $D(0, 3R^4)$, in particular, everywhere in $D(\alpha, R(\alpha))$. By Claim 11.5, we may assume that $|\alpha| < \frac{1}{2}R^4$. Hence, if $\min_{B(\alpha)\setminus D(\alpha,\frac{1}{2}R)} U < -M$, then the disk $R^4\mathbb{D}$ contains a curve of diameter at least $\frac{1}{2}R$ where $U < -M$. By Theorem 4.3, the probability of this event does not exceed $e^{-cRM^{3/2}}$. Thus, discarding the event of probability at most $e^{-cR^4/(\log R)^{3/2}}$, we may assume that $|U| \leqslant M$ in $B(\alpha) \setminus D\big(\alpha, \frac{1}{2}R\big)$. Then by the length and area estimate (Proposition 8.2), the area of the set $B(\alpha) \setminus D(\alpha, R)$ cannot exceed

$$\pi e^{-2\frac{(R/2)^2}{2M}} = \pi R^{-13} < \frac{1}{AR^{12}} \leqslant \frac{1}{AR^3(\alpha)}\,,$$

provided that $R \geqslant \pi A$. Hence, after the events described above have been thrown away, we get $r(\alpha) \leqslant R$. Therefore, the probability of the event $\{r(\alpha) > R\}$ does not exceed the sum of probabilities of the events thrown away, and we are done. □

12 Discussion and Questions

12.1 Optimal transportation to the zero set of GEF.

QUESTION 12.1. Does there exists a transportation T of the Lebesgue measure $\frac{1}{\pi}m_2$ to the random zero set $\mathcal{Z}_f$ such that the tails $\sup_{z\in\mathbb{C}} \mathbb{P}\{|T(z) - z| > R\}$ decay as e^{-cR^4} as $R \to \infty$?

Recall that the estimate $e^{-cR^4(\log R)^{-1}}$ can be achieved by modification of the proof in [ST, Part II]. Note that, in view of the lower bound for the *hole probability* $\mathbb{P}\{\mathcal{Z}_f \cap R\mathbb{D} = \varnothing\} \geqslant ce^{-CR^4}$ proved in [ST, Part III], one cannot get a better estimate than ce^{-CR^4}.

12.2 Length of the gradient curve and the travel time. Given z, consider the gradient curve Γ_z that passes through the point z. Let ℓ_z be the length of the part of the curve Γ_z that starts at z and terminates at a_z.

QUESTION 12.2. Find the order of decay of the tails $\mathbb{P}\{\ell_z > R\}$ as $R \to \infty$.

An interesting characteristic of the "random landscape" of the potential U is the time τ_z needed for the point z to roll down to the sink a_z along the gradient curve Γ_z. By analogy with some models from astrophysics, Michael Douglas asked us about the *order of decay of the tails* $\mathbb{P}\{\tau_z > t\}$ *as* $t \to \infty$. Since $\operatorname{div}(\nabla U) = -2$ everywhere on $\mathbb{C} \setminus \mathcal{Z}_f$, one can show using Liouville's theorem that this probability *equals* e^{-2t} (cf. section 8). The length l measured along the gradient curve and the travel time τ are connected by relation $dl/d\tau = |\nabla U|$. Since we know the distribution of the gradient field (recall that $\nabla U = \overline{\frac{f'(z)}{f(z)}} - z$), it looks tempting to use this information to simplify the proofs of our main results and to achieve a better understanding of the properties of the random partition.

12.3 Statistics of the basins. There are several interesting questions related to the statistics of our random partition of the plane. We say that two basins are neighbours if they have a common gradient curve on the boundary. By N_z we denote the number of basins B neighbouring the basin B_z. Clearly, N_z equals the number of saddle points of the potential U connected with the sink a_z by gradient curves. Heuristically, since almost surely each saddle point is connected with two sinks,

$$\mathbb{E}N_z = 2\,\frac{\text{mean number of saddle points per unit area}}{\text{mean number of zeroes per unit area}}\,.$$

Douglas, Shiffman and Zelditch proved in [DSZ] that the mean number of saddle points of U per unit area is $4/3\pi$. (They proved this for another closely related "elliptic model" of Gaussian polynomials. It seems that their proof also works for GEF) Hence the question:

QUESTION 12.3. Prove that $\mathbb{E}N_z = 8/3$.

We are also interested in the behaviour of the tails of the random variable N_z:

QUESTION 12.4. Find the order of decay of $\mathbb{P}\{N_z > N\}$ as $N \to \infty$.

Another characteristic of the random partition is the number M of basins that meet at the same local maximum. Taking into account the result from [DSZ], we expect that its average equals 8. It is also interesting to look at the decay of the tails of M. Probably, some lower bound can be extracted from the analysis of perturbations of the polynomial $z^n - 1$ similar to the one we did in section 9.

12.4 The skeleton topology. By the *skeleton* of the gradient flow we mean the connected planar graph with vertices at local maxima of U and edges corresponding to the boundary curves of the basins. The graph may have multiple edges and loops. Our question is

QUESTION 12.5. Are there any non-trivial topological restrictions on finite parts of the skeleton that hold almost surely?

There is an interesting finite counterpart of this question. Choose N independent points $a_1, \dots, a_N$ uniformly distributed on the Riemann sphere $\widehat{\mathbb{C}}$ and consider the gradient flow of the random spherical potential

$$V(z) = \sum_i \log |z - a_i| - \tfrac{N}{2} \log \left(1 + |z|^2\right).$$

QUESTION 12.6. Describe all possible skeletons of the gradient flow on $\widehat{\mathbb{C}}$ of the potential V that are realized with positive probability.

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