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A LAURENT EXPANSION FOR REGULARIZED INTEGRALS OF HOLOMORPHIC SYMBOLS

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Abstract. For a holomorphic family of classical pseudodifferential operators on a closed manifold we give exact formulae for all coefficients in the Laurent expansion of its Kontsevich–Vishik canonical trace. This generalizes to all higher-order terms a known result identifying the residue trace with a pole of the canonical trace.

Introduction

Let M be a compact boundaryless Riemannian manifold of dimension n and E a smooth vector bundle based on M. For a classical pseudodifferential operator (ψdo) A of non-integer order acting on smooth sections of E one can define, following Kontsevich and Vishik [KV] and Lesch [L], the canonical trace of A

$$\mathrm{TR}(A) := \int_M dx\ \mathrm{TR}_x(A)\,, \quad \mathrm{TR}_x(A) := -\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}_x\big(\sigma_A(x,\xi)\big)\,đ\xi\,,$$

in terms of a local classical symbol σ_A and a finite-part integral $-\!\!\!\!\!\!\int_{T_x^*M}$ over the cotangent space T_x^*M at $x \in M$. Here, $đ\xi = (2\pi)^{-n}\,d\xi$ with $d\xi$ Lebesgue measure on $T_x^*M \cong \mathbb{R}^n$, while tr_x denotes the fibrewise trace. Since the work of Seeley [Se1] and later of Guillemin [Gu], Wodzicki [W] and then Kontsevich and Vishik [KV], it has been known that given a holomorphic family $z \mapsto A(z)$ of classical ψdos parametrized by a domain $W \subset \mathbb{C}$, with holomorphic order $\alpha : W \to \mathbb{C}$ such that α' does not vanish on

$$P := \alpha^{-1}\big(\mathbb{Z} \cap [-n, +\infty[\,\big)\,,$$

then the map $z \mapsto \mathrm{TR}(A(z))$ is a meromorphic function with no more than simple poles located in P. The complex residue at $z_0 \in P$ is given by a

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local expression [W], [Gu], [KV]

$$\mathrm{Res}_{z=z_0}\,\mathrm{TR}(A(z)) = -\frac{1}{\alpha'(z_0)}\,\mathrm{res}\big(A(z_0)\big)\,, \tag{0.1}$$

where for a classical pseudodifferential operator B with symbol σ_B

$$\mathrm{res}(B) := \int_M dx\,\mathrm{res}_x(B)\,, \quad \mathrm{res}_x(B) := \int_{S_x^*M} \mathrm{tr}_x\big((\sigma_B)_{-n}(x,\xi)\big) đ_S\xi$$

is the residue trace of B. Here, $đ_S\xi = (2\pi)^{-n} d_S(\xi)$ with $d_S(\xi)$ the sphere measure on $S_x^*M = \{|\xi| = 1 \mid \xi \in T_x^*M\}$, while the subscript refers to the positively homogeneous component of the symbol of order $-n$.

In this paper, extending the identification (0.1), we provide a complete solution to the problem of giving exact formulae for all coefficients in the Laurent expansion of $\mathrm{TR}(A(z))$ around each pole in terms of locally-defined canonical trace and residue trace densities.

For a meromorphic function G, define its finite-part $\mathrm{fp}_{z=z_0}G(z)$ at z_0 to be the constant term in the Laurent expansion of $G(z)$ around z_0. Let $A^{(r)}(z) = \partial_z^r A(z)$ be the derivative ψdo with symbol $\sigma_{A^{(r)}(z)} := \partial_z^r \sigma_{A(z)}$.

Theorem. *Let $z \mapsto A(z)$ be a holomorphic family of classical ψdos of order $\alpha(z) = qz + b$. If $z_0 \in P$ and $q \neq 0$, then $\mathrm{TR}(A(z))$ has Laurent expansion for z near z_0*

$$\mathrm{TR}(A(z)) = -\frac{\mathrm{res}(A(z_0))}{q}\,\frac{1}{(z-z_0)} + \sum_{k=0}^{\infty} \mathrm{fp}_{z=z_0}\mathrm{TR}\big(A^{(k)}(z)\big)\frac{(z-z_0)^k}{k!}\,. \tag{0.2}$$

Furthermore,

$$\left(\mathrm{TR}_x\big(A^{(k)}(z_0)\big) - \frac{1}{q(k+1)}\mathrm{res}_{x,0}\big(A^{(k+1)}(z_0)\big)\right) dx \tag{0.3}$$

defines a global density on M and

$$\mathrm{fp}_{z=z_0}\mathrm{TR}\big(A^{(k)}(z)\big) = \int_M dx \left(\mathrm{TR}_x\big(A^{(k)}(z_0)\big) - \frac{1}{q(k+1)}\mathrm{res}_{x,0}\big(A^{(k+1)}(z_0)\big)\right). \tag{0.4}$$

At a point $z_0 \notin P$ the function $\mathrm{TR}(A(z))$ is holomorphic near z_0 and the Laurent expansion (0.2) reduces to the Taylor series

$$\mathrm{TR}(A(z)) = \mathrm{TR}(A(z_0)) + \sum_{k=1}^{\infty} \mathrm{TR}\big(A^{(k)}(z_0)\big)\frac{(z-z_0)^k}{k!}\,.$$

It is to be emphasized here that $A^{(r)}(z)$ cannot be a classical ψdo for $r > 0$, but in local coordinates is represented for $|\xi| > 0$ by a log-polyhomogeneous symbol of the form

$$\sigma_{A^{(r)}(z)}(x,\xi) \sim \sum_{j\geq 0}\sum_{l=0}^{r} \sigma\big(A^{(r)}(z)\big)_{\alpha(z)-j,l}(x,\xi)\log^l|\xi|$$

with $\sigma(A^{(r)})_{\alpha(z)-j,l}(x,\xi)$ positively homogeneous in ξ of degree $\alpha(z)-j$. It follows that individually the terms in (0.3),

$$\mathrm{TR}_x\big(A^{(k)}(z_0)\big)\,dx := {-\!\!\!\!\!\!\int}_{T_x^*M} \mathrm{tr}_x\big(\sigma_{A^{(k)}(z_0)}(x,\xi)\big)\,đ\xi\,dx\,, \tag{0.5}$$

and

$$\mathrm{res}_{x,0}\big(A^{(k+1)}(z_0)\big)\,dx := \int_{S_x^*M} \mathrm{tr}_x\big(\sigma(A^{(k+1)}(z_0))_{-n,0}(x,\xi)\big)\,đ_S\xi\,dx\,, \tag{0.6}$$

do not in general determine globally-defined densities on the manifold M when $r>0$, rather it is then only the sum of terms (0.3) which integrates to a global invariant of M. (In particular, it is important to distinguish (0.6) from the higher residue trace density of [L], see Remark 1.5 here.) When $\alpha(z)=qz+b$ is *not* integer valued it is known that $\mathrm{TR}_x(A^{(k)}(z))dx$ does then define a global density on M; in this case, $\mathrm{res}_{x,0}(A^{(k+1)}(z))$ is identically zero and (0.4) reduces to the canonical trace $\mathrm{TR}(A^{(k)}(z)) = \int_M dx\,\mathrm{TR}_x(A^{(k)}(z))$ on non-integer order ψdos with log-polyhomogeneous symbol [L].

These results hold more generally when $\alpha(z)$ is an arbitrary holomorphic function with $\alpha'(z_0)\neq 0$ at $z_0\in P$. Then the local residue term in (0.4) is replaced by the local residue of an explicitly computable polynomial in the symbols of the operators $A^{(k+1)}(z_0),\dots,A(z_0)$. A general formula is given in Theorem 1.20, here we state the formula just for the constant term in the Laurent expansion of $\mathrm{TR}(A(z))$: one has

$$\mathrm{fp}_{z=z_0}\mathrm{TR}(A(z)) = \int_M dx\left(\mathrm{TR}_x\big(A(z_0)\big) - \frac{1}{\alpha'(z_0)}\mathrm{res}_{x,0}\big(A'(z_0)\big)\right) + \frac{\alpha''(z_0)}{2\alpha'(z_0)^2}\,\mathrm{res}\big(A(z_0)\big)\,. \tag{0.7}$$

Thus compared to (0.4), the constant term (0.7) in the expansion acquires an additional residue trace term. Moreover, the identification implies that

$$\left(\mathrm{TR}_x\big(A(z_0)\big) - \frac{1}{\alpha'(z_0)}\,\mathrm{res}_{x,0}\big(A'(z_0)\big)\right)dx \tag{0.8}$$

defines a global density on M independently of the order $\alpha(z_0)$ of $A(z_0)$ (for $z_0\notin P$ the residue term vanishes and (0.8) reduces to the usual canonical trace density). Though this follows from general properties of holomorphic families of canonical traces, we additionally give an elementary direct proof in Appendix A.

Applied to ψdo *zeta-functions* this yields formulae for a number of widely studied spectral geometric invariants. For Q an elliptic classical pseudodifferential operator of order $q > 0$ and with spectral cut θ, its complex powers Q_θ^{-z} are well defined [Se1], and to a classical pseudodifferential operator A of order $\alpha \in \mathbb{R}$ one can associate the holomorphic family $A(z) = AQ_\theta^{-z}$ with order function $\alpha(z) = \alpha - qz$. The generalized zeta-function

$$z \mapsto \zeta_\theta(A, Q, z) := \mathrm{TR}(AQ_\theta^{-z})$$

is meromorphic on $\mathbb{C}$ with at most simple poles in $P := \{\frac{\alpha-j}{q} | j \in [-n, \infty) \cap \mathbb{Z}\}$. It has been shown by Grubb and Seeley [GruS], [Gru1] that $\Gamma(s)\zeta_\theta(A, Q, s)$ has pole structure

$$\Gamma(s)\zeta_\theta(A, Q, s) \sim \sum_{j \geq -n} \frac{c_j}{s + \frac{j-\alpha}{q}} - \frac{\mathrm{Tr}\,(A\Pi_Q)}{s} + \sum_{l \geq 0} \left(\frac{c_l^{'}}{(s+l)^2} + \frac{c_l^{''}}{(s+l)} \right), \quad (0.9)$$

where the coefficients c_j and $c_l^{'}$ are locally determined, by finitely many homogeneous components of the local symbol, while the $c_l^{''}$ are globally determined. In particular, whenever

$$\frac{j-\alpha}{q} := l \in [0, \infty) \cap \mathbb{Z}$$

it is shown that the sum of terms

$$c_l^{''} + c_{\alpha+lq} \quad (0.10)$$

is defined invariantly on the manifold M, while individually the coefficients $c_l^{''}$ and $c_{\alpha+lq}$ (which contain contributions from the terms (0.5) and (0.6) respectively) depend on the symbol structure in each local trivialization. Here, in Theorem 2.2, we compute the Laurent expansion around each of the poles of the meromorphically continued Schwartz kernel

$$K_{AQ_\theta^{-z}}(x,x)|^{\mathrm{mer}} := \int\!\!\!\!\!\!- _{T_x^* M} \sigma_{AQ_\theta^{-z}}(x,\xi) đ\xi$$

giving the following exact formula for (0.10). One has

$$c_l^{''} + c_{\alpha+lq} = \frac{(-1)^l}{l!} \int_M dx \left(\mathrm{TR}_x(AQ^l) - \frac{1}{q} \mathrm{res}_{x,0}(AQ^l \log_\theta Q) \right). \quad (0.11)$$

The remaining coefficients in (0.9) occur as residue traces of the form (0.1). By a well-known equivalence, see for example [GruS], [Gru1], when Q is a Laplace-type operator these formulae acquire a geometric character as coefficients in the asymptotic heat trace expansion

$$\mathrm{Tr}\,(Ae^{-tQ}) \sim \sum_{j \geq -n} c_j t^{\frac{j-\alpha}{q}} + \sum_{l \geq 0} (-c_l^{'} \log t + c_l^{''}) t^l$$

as $t \to 0+$.

From (0.1) ([W], [Gu], [KV]) $\zeta_\theta(A,Q,z)$ has a simple pole at $z=0$ with residue $-\frac{1}{q}\,\mathrm{res}(A)$, which vanishes if $\alpha\notin\mathbb{Z}$. The coefficients of the full Laurent expansion of $\zeta_\theta(A,Q,s)$ around $z=0$ are given by the following formulae (Theorem 2.5).

Theorem. *For $k\in\mathbb{N}$, let $\zeta_\theta^{(k)}(A,Q,0)$ denote the coefficient of $z^k/k!$ in the Laurent expansion of $\zeta_\theta(A,Q,z)$ around $z=0$. Then*

$$\zeta_\theta^{(k)}(A,Q,0)=(-1)^k\int_M dx\left(\mathrm{TR}_x(A\log_\theta^k Q)-\frac{1}{q(k+1)}\,\mathrm{res}_{x,0}(A\log_\theta^{k+1} Q)\right)\\ +(-1)^{k+1}\mathrm{tr}(A\log_\theta^k Q\Pi_Q)\,, \quad (0.12)$$

where Π_Q is a smoothing operator projector onto the generalized kernel of Q. Specifically, for a classical ψdo A of arbitrary order

$$\left(\mathrm{TR}_x(A)-\tfrac{1}{q}\,\mathrm{res}_{x,0}(A\log_\theta Q)\right)dx \quad (0.13)$$

is a globally-defined density on M and, setting $\zeta_\theta(A,Q,0):=\zeta_\theta^{(0)}(A,Q,0)$, the constant term in the expansion around $z=0$ is

$$\zeta_\theta(A,Q,0)=\int_M dx\left(\mathrm{TR}_x(A)-\tfrac{1}{q}\,\mathrm{res}_{x,0}(A\log_\theta Q)\right)-\mathrm{tr}(A\Pi_Q)\,. \quad (0.14)$$

When A is a differential operator $\zeta_\theta(A,Q,0)=\lim_{z\to 0}\zeta_\theta(A,Q,z)$ and equation (0.14) becomes

$$\zeta_\theta(A,Q,0)=-\tfrac{1}{q}\,\mathrm{res}(A\log_\theta Q)-\mathrm{tr}(A\Pi_Q)\,. \quad (0.15)$$

When Q is a differential operator and m a non-negative integer, setting $\zeta_\theta(Q,-m):=\mathrm{fp}_{z=-m}\zeta_\theta(I,Q,z)$, one has

$$\zeta_\theta(Q,-m)=-\tfrac{1}{q}\,\mathrm{res}(Q^m\log_\theta Q)-\mathrm{tr}(Q^m\Pi_Q)\,. \quad (0.16)$$

If A is a ψdo of non-integer order $\alpha\notin\mathbb{Z}$ then $0\notin P$ and from [L] the canonical trace of $A\log_\theta^k Q$ is defined. Then (0.12) reduces to

$$\zeta_\theta^{(k)}(A,Q,0)=(-1)^k\mathrm{TR}(A\log_\theta^k Q)-(-1)^k\mathrm{tr}(A\log_\theta^k Q\Pi_Q) \quad (0.17)$$

and, in particular, in this case

$$\zeta_\theta(A,Q,0)=\mathrm{TR}(A)-\mathrm{tr}(A\Pi_Q)\,. \quad (0.18)$$

Notice that in (0.15) the term $\mathrm{res}(A\log_\theta Q)=\zeta_\theta(A,Q,0)+\mathrm{tr}(A\Pi_Q)$ is locally determined, meaning that it depends on only finitely many of the homogeneous terms in the local symbols of A and Q ([GruS, Th. 2.7], see also [S, Prop. 1.5]). In the case $A=I$ the identity (0.15) was shown for pseudodifferential Q in [S] and [Gru2], and in the particular case where Q is an invertible positive differential operator (0.16) can be inferred from [Lo]. The identity (0.18) is known from [Gru1, Rem. 1.6]. A resolvent proof of (0.14) has been given recently in [Gru3].

If, on the other hand, one considers, for example, $A(z) = AQ_\theta^{-\frac{z}{1+\mu z}}$, then the corresponding 'zeta function' $\mathrm{TR}\big(AQ_\theta^{-\frac{z}{1+\mu z}}\big)$ has simple and real poles in $\mathbb{C}\backslash\{-1/\mu\}$ and by (0.7) the constant term at $z=0$ has, compared to (0.14), an extra term

$$\mathrm{fp}_{z=0}\mathrm{TR}\big(AQ_\theta^{-\frac{z}{1+\mu z}}\big) = \int_M dx\left(\mathrm{TR}_x(A) - \frac{1}{q}\,\mathrm{res}_{x,0}(A\log_\theta Q)\right) - \mathrm{tr}(A\,\Pi_Q) + \frac{\mu}{q}\,\mathrm{res}(A)\,.$$

The appearance here of $\mu/q\,\mathrm{res}(A)$ corresponds to additional terms that occur as a result of a rescaling of the cut-off parameter when expectation values are computed from Feynman diagrams using a momentum cut-off procedure, see [Gr]. See also Remark 1.23.

One view point to adopt on (0.7) is that it provides a defect formula for regularized traces and indeed most well-known trace defect formulas [MN], [O1], [CDMP], [Gru2] are an easy consequence of it. On the other hand, new more precise formulae also follow. In particular, though TR is not in general defined on the bracket $[A,B]$ when the bracket is of integer order, we find (Theorem 2.18) that in this case the following exact global formula holds:

$$\int_M dx\left(\mathrm{TR}_x([A,B]) - \frac{1}{q}\,\mathrm{res}_{x,0}\big([A,B\log_\theta Q]\big)\right) = 0\,. \tag{0.19}$$

This holds independently of the choice of Q, when $[A,B]$ is not of integer order (0.19) reduces to the usual trace property of the canonical trace $\mathrm{TR}([A,B])=0$, see section 2.

Looking at the next term up in the Laurent expansion of $\mathrm{TR}(Q^{-z})$ at zero, equation (0.12) provides an explicit formula for the ζ-determinant

$$\det{}_{\zeta,\theta} Q = \exp\big(-\zeta'_\theta(Q,0)\big)\,,$$

where $\zeta'_\theta(Q,0) = \partial_z\zeta_\theta(Q,z))_{|z=0}$, of an invertible elliptic classical pseudodifferential operator Q of positive order q and with spectral cut θ. The zeta determinant is a complicated non-local invariant which has been studied in diverse mathematical contexts. From (0.12) one finds (Theorem 2.11):

Theorem.

$$\log\det{}_{\zeta,\theta}(Q) = \int_M dx\left(\mathrm{TR}_x(\log_\theta Q) - \frac{1}{2q}\,\mathrm{res}_{x,0}(\log^2_\theta Q)\right). \tag{0.20}$$

A slightly modified formula holds for non-invertible Q. Notice, here, that TR of $\log_\theta Q$ does not generally exist; if $\mathrm{res}_{x,0}(\log^2_\theta Q)=0$ pointwise it is defined, and then $\log\det_{\zeta,\theta}(Q) = \mathrm{TR}(\log_\theta Q)$, which holds, for example, for odd-class operators of even order, such as differential operators of even order on odd-dimensional manifolds [KV], [O2]. Equation (0.20) leads to explicit formulae for the multiplicative anomaly.

1 Finite-Part Integrals (and Canonical Traces) of Holomorphic Families of Classical Symbols (and Pseudodifferential Operators)

1.1 Classical and log-polyhomogeneous symbols. We briefly recall some notions concerning symbols and pseudodifferential operators and fix the corresponding notation. Classical references for the polyhomogeneous symbol calculus are, e.g., [G], [GruS], [Hö], [Se2], [Sh], and for the extension to log-polyhomogeneous symbols [L]. E denotes a smooth hermitian vector bundle based on some closed Riemannian manifold M. The space $C^\infty(M,E)$ of smooth sections of E is endowed with the inner product $\langle\psi,\phi\rangle := \int_M d\mu(x)\langle\psi(x),\phi(x)\rangle_x$ induced by the hermitian structure $\langle\cdot\,,\cdot\rangle_x$ on the fibre over $x\in M$ and the Riemannian measure μ on M. $H^s(M,E)$ denotes the H^s-Sobolev closure of the space $C^\infty(M,E)$.

Given an open subset U of $\mathbb{R}^n$ and an auxiliary (finite-dimensional) normed vector space V, the set of symbols $\mathrm{S}^r(U,V)$ on U of order $r\in\mathbb{R}$ consists of those functions $\sigma(x,\xi)$ in $C^\infty(T^*U,\mathrm{End}(V))$ such that $\partial_x^\mu\partial_\xi^\nu\sigma(x,\xi)$ is $O((1+|\xi|)^{r-|\nu|})$ for all multi-indices μ,ν, uniformly in ξ, and, on compact subsets of U, uniformly in x. We set $\mathrm{S}(U,V) := \bigcup_{r\in\mathbb{R}}\mathrm{S}^r(U,V)$ and $\mathrm{S}^{-\infty}(U,V) := \bigcap_{r\in\mathbb{R}}\mathrm{S}^r(U,V)$. A *classical* (1-step polyhomogeneous) symbol of order $\alpha\in\mathbb{C}$ means a function $\sigma(x,\xi)$ in $C^\infty(T^*U,\mathrm{End}(V))$ such that, for each $N\in\mathbf{N}$ and each integer $0\le j\le N$ there exists $\sigma_{\alpha-j}\in C^\infty(T^*U,\mathrm{End}(V))$ which is homogeneous in ξ of degree $\alpha-j$ for $|\xi|\ge 1$, so $\sigma_{\alpha-j}(x,t\xi)=t^{\alpha-j}\sigma_{\alpha-j}(x,\xi)$ for $t\ge1$, $|\xi|\ge1$, and a symbol $\sigma_{(N)}\in\mathrm{S}^{\mathrm{Re}(\alpha)-N-1}(U,V)$ such that

$$\sigma(x,\xi)=\sum_{j=0}^N\sigma_{\alpha-j}(x,\xi)+\sigma_{(N)}(x,\xi)\quad\forall(x,\xi)\in T^*U\,. \tag{1.1}$$

We then write $\sigma(x,\xi)\sim\sum_{j=0}^\infty\sigma_{\alpha-j}(x,\xi)$. Let $\mathrm{CS}(U,V)$ denote the class of classical symbols on U with values in V, and let $\mathrm{CS}^\alpha(U,V)$ denote the subset of classical symbols of order α. When $V=\mathbb{C}$, we write $\mathrm{S}^r(U)$, $\mathrm{CS}^\alpha(U)$, and so forth; for brevity we may omit the V in the statement of some results. A ψdo which, for a given atlas on M, has a classical symbol in the local coordinates defined by each chart is called *classical*, this is independent of the choice of atlas. Let $\mathrm{Cl}(M,E)$ denote the algebra of classical ψdos acting on $C^\infty(M,E)$ and let $\mathrm{Ell}(M,E)$ be the subalgebra of elliptic operators. For any $\alpha\in\mathbb{C}$, let $\mathrm{Cl}^\alpha(M,E)$, resp. $\mathrm{Ell}^\alpha(M,E)$, denote the subset of operators in $\mathrm{Cl}(M,E)$, resp. $\mathrm{Ell}(M,E)$, of order α. With $\mathbb{R}_+=(0,\infty)$, set $\mathrm{Ell}_{\mathrm{ord}>0}(M,E):=\bigcup_{r\in\mathbb{R}_+}\mathrm{Ell}^r(M,E)$.

To deal with derivatives of complex powers of classical ψdos one considers the larger class of ψdos with log-polyhomogeneous symbols. Given an open subset $U\subset M$, a non-negative integer k and a complex number α, a symbol σ lies in $\mathrm{CS}^{\alpha,k}(U,V)$ and is said to have order α and log degree k if

$$\sigma(x,\xi)=\sum_{j=0}^{N}\sigma_{\alpha-j}(x,\xi)+\sigma_{(N)}(x,\xi)\quad\forall(x,\xi)\in T^*U\,,\tag{1.2}$$

where $\sigma_{(N)}\in \mathrm{S}^{\mathrm{Re}(\alpha)-N-1+\epsilon}(U,V)$ for any $\epsilon>0$, and

$$\sigma_{\alpha-j}(\xi)=\sum_{l=0}^{k}\sigma_{\alpha-j,l}(x,\xi)\log^l[\xi]\quad\forall\xi\in T_x^*U\,,$$

with $\sigma_{\alpha-j,l}$ homogeneous in ξ of degree $\alpha-j$ for $|\xi|\geq 1$, and (in the notation of [Gru1]) $[\xi]$ a strictly positive C^∞ function in ξ with $[\xi]=|\xi|$ for $|\xi|\geq 1$. As before, in this case we write

$$\sigma(x,\xi)\sim\sum_{j=0}^{\infty}\sigma_{\alpha-j}(x,\xi)=\sum_{j=0}^{\infty}\sum_{l=0}^{k}\sigma_{\alpha-j,l}(x,\xi)\log^l[\xi]\,.\tag{1.3}$$

Then $\mathrm{CS}^{*,*}(U,V):=\bigcup_{k=0}^\infty\mathrm{CS}^{*,k}(U,V)$, where $\mathrm{CS}^{*,k}(U,V)=\bigcup_{\alpha\in\mathbb{C}}\mathrm{CS}^{\alpha,k}(U,V)$, defines the class filtered by k of log-polyhomogeneous symbols on U. In particular, $\mathrm{CS}(U,V)$ coincides with $\mathrm{CS}^{*,0}(U,V)$.

Given a non-negative integer k, let $\mathrm{Cl}^{\alpha,k}(M,E)$ denote the space of pseudodifferential operators on $C^\infty(M,E)$ which in any local trivialization $E_{|U}\cong U\times V$ have symbol in $\mathrm{CS}^{\alpha,k}(U,V)$. Set $\mathrm{Cl}^{*,k}(M,E):=\bigcup_{\alpha\in\mathbb{C}}\mathrm{Cl}^{\alpha,k}(M,E)$.

The following subclasses of symbols and ψdos will be of importance in what follows.

Definition 1.1. *A log-polyhomogeneous symbol* (1.3) *with integer order* $\alpha\in\mathbb{Z}$ *is said to be* even-even (*or, more fully, to have* even-even alternating parity) *if, for each* $j\geq 0$,

$$\sigma_{\alpha-j,l}(x,-\xi)=(-1)^{\alpha-j}\sigma_{\alpha-j,l}(x,\xi)\quad\text{for }|\xi|\geq 1\,,\tag{1.4}$$

and the same holds for all derivatives in x *and* ξ. *It is said to be* even-odd (*or, more fully, to have* even-odd alternating parity) *if, for each* $j\geq 0$,

$$\sigma_{\alpha-j,l}(x,-\xi)=(-1)^{\alpha-j-1}\sigma_{\alpha-j,l}(x,\xi)\quad\text{for }|\xi|\geq 1\,,\tag{1.5}$$

*and the same holds for all derivatives in x and ξ. A ψ*do $A \in \mathrm{Cl}^{\alpha,k}(M,E)$ *will be said to be* even-even (*resp.* even-odd) *if in each local trivialization any local symbol $\sigma_A(x,\xi) \in \mathrm{CS}^{\alpha,k}(U,V)$ representing A (modulo smoothing operators) has even-even (resp. even-odd) parity.*

Thus, an even-even symbol with even-integer degree is even in ξ, while an even-odd symbol with even-integer degree is odd in ξ; a similar statement holds if the symbol has odd-integer degree.

REMARK 1.2. The terminology in Definition (1.1) follows [Gru4]. Kontsevich–Vishik [KV] studied even-even classical ψdos on odd-dimensional manifolds, calling them *odd-class* operators. Odd-class operators (or symbols) form an algebra and include differential operators and their parametrices. The class of operators with even-odd parity symbols on even-dimensional manifolds, which includes the modulus operator $|A| = (A^2)^{1/2}$ for A a first-order elliptic self-adjoint differential operator, was introduced and studied by Grubb [Gru1]; this class admits similar properties with respect to traces on ψdos as the odd-class operators, though they do not form an algebra. In [O2] Okikiolu uses the terminology 'regular parity' and 'singular parity' for (1.4) and (1.5).

1.2 Finite part integrals of symbols and the canonical trace. In order to make sense of $\int_{T_x^*M} \sigma(x,\xi) đ\xi$ when $\sigma \in \mathrm{CS}^{\alpha,*}(U,V)$ is a log-polyhomogeneous symbol (the integral diverges a priori if $\mathrm{Re}(\alpha) \geq -n$) on an open subset $U \subset \mathbb{R}^n$, one can extract a *finite part* when $R \to \infty$ from the integral $\int_{B_x^*(0,R)} \sigma(x,\xi) đ\xi$ where $B_x^*(0,R) \subset T_x^*U$ denotes the ball centered at 0 with radius R for a given point $x \in U$.

First, though, we introduce the local residue density on log-polyhomogeneous symbols, which acts as an obstruction to the finite-part integral of a classical symbol defining a global density on M and measures the anomalous contribution to the Laurent coefficients at the poles of the finite-part integral when evaluated on holomorphic families of symbols.

DEFINITION 1.3. *Given an open subset $U \subset \mathbb{R}^n$, the local Guillemin–Wodzicki residue is defined for $\sigma \in \mathrm{CS}^{\alpha}(U,V)$ by*

$$\mathrm{res}_x(\sigma) = \int_{S_x^*U} \mathrm{tr}_x\big(\sigma_{-n}(x,\xi)\big) đ_S\xi\,,$$

and extends to a map $\mathrm{res}_{x,0} : \mathrm{CS}^{\alpha,k}(U,V) \to \mathbb{C}$ by the same formula

$$\mathrm{res}_{x,0}(\sigma) = \int_{S_x^*U} \mathrm{tr}_x\big(\sigma_{-n}(x,\xi)\big) đ_S\xi$$

$$= \sum_{l=0}^{k} \int_{S_x^*U} \operatorname{tr}_x\big(\sigma_{-n,l}(x,\xi)\big) \log^l |\xi| đ_S\xi$$
$$= \int_{S_x^*U} \operatorname{tr}_x\big(\sigma_{-n,0}(x,\xi)\big) đ_S\xi \,.$$

When $k > 0$ the extra subscript is included in the notation $\operatorname{res}_{x,0}(\sigma)$ as a reminder that it is the residue of the log degree zero component of the symbol that is being computed. The distinction is made because when $k > 0$ the local densities $\operatorname{res}_{x,0}(\sigma)dx$ do not in general define a global density on M, due to cascading derivatives of powers of logs when changing local coordinates. When $k = 0$, Guillemin [Gu] and Wodzicki [W] showed the following properties.

PROPOSITION 1.4. *Let $A \in \mathrm{Cl}^\alpha(M,E)$ be a classical ψdo represented in a local coordinate chart U by $\sigma \in \mathrm{CS}^\alpha(U,V)$. Then $\operatorname{res}_x(\sigma)dx$ determines a global density on M, that is, an element of $C^\infty(M,|\Omega|)$, which defines the projectively unique trace on $\mathrm{Cl}^{*,0}(M,E)$.*

Proofs may be found in *loc. cit.*, and in section 2 here. The first property means that $\operatorname{res}_x(\sigma)dx$ can be integrated over M. The resulting number,

$$\operatorname{res}(A) := \int_M \operatorname{res}_x(\sigma)dx = \int_M dx \int_{S_x^*M} \operatorname{tr}_x\big(\sigma_{-n}(x,\xi)\big) đ_S\xi \,, \tag{1.6}$$

is known as the residue trace of A. The terminology refers to the trace property in Proposition 1.4 that if the manifold M is connected and has dimension larger than 1, then up to a scalar multiple (1.6) defines on $\mathrm{Cl}^*(M,E)$ the unique linear functional vanishing on commutators

$$\operatorname{res}([A,B]) = 0 \,, \quad A, B \in \mathrm{Cl}^*(M,E) \,.$$

Notice, from its definition, that the residue trace also vanishes on operators of order $< -n$ and on non-integer order operators.

REMARK 1.5. The residue trace was extended by Lesch [L] to $A \in \mathrm{Cl}^{\alpha,k}(M,E)$ with $k > 0$ by defining $\operatorname{res}_k(A) := (k+1)! \int_M dx \int_{S_x^*M} \operatorname{tr}_x(\sigma_{-n,k}(x,\xi))đ_S\xi$. For an operator with log-polyhomogeneous symbol of log degree $k > 0$ the form $\sigma_{-n,k}(x,\xi)dx$ defines a global density on M, a property which is not generally true for the lower log degree densities $\sigma_{-n,0}(x,\xi)dx, \dots,$ $\sigma_{-n,k-1}(x,\xi)dx$ which depend on the symbol structure in each local coordinate chart. We emphasize that the higher residue is not being used in the Laurent expansions we compute here, rather the relevant object is the locally defined form $\sigma_{-n,0}(x,\xi)dx$ which for suitable A defines one component of a specific local density which does determine an element of $C^\infty(M, \operatorname{End}(E) \otimes |\Omega|)$.

It was, on the other hand, observed by Kontsevich and Vishik [KV] that the usual L^2-trace on ψdos of real order $< -n$ extends to a functional on the space $\mathrm{Cl}^{\mathbb{C}\setminus\mathbb{Z}}(M,E)$ of ψdos of non-integer order and vanishes on commutators of non-integer order. Lesch [L] subsequently showed that the resulting canonical trace can be further extended to

$$\mathrm{Cl}^{\mathbb{C}\setminus\mathbb{Z},*}(M,E) := \bigcup_{\alpha\in\mathbb{C}\setminus\mathbb{Z}} \mathrm{Cl}^{\alpha,*}(M,E)$$

in the following way.

Lemma 1.6. *Let U be an open subset of $\mathbb{R}^n$ and let $\sigma \in \mathrm{CS}^{\alpha,k}(U,V)$ be a log-polyhomogeneous symbol of order α and log-degree k. Then for any $x \in U$ the integral $\int_{B_x^*(0,R)} \sigma(x,\xi)đ\xi$ has an asymptotic expansion as $R \to \infty$,*

$$\int_{B_x^*(0,R)} \sigma(x,\xi)đ\xi \sim_{R\to\infty} C_x(\sigma) + \sum_{j=0,\alpha-j+n\neq 0}^{\infty} \sum_{l=0}^{k} P_l(\sigma_{\alpha-j,l})(\log R)R^{\alpha-j+n} + \sum_{l=0}^{k} \frac{1}{l+1}\int_{S_x^*U} \sigma_{-n,l}(x,\xi)đ_S\xi \log^{l+1} R\,, \quad (1.7)$$

where $P_l(\sigma_{\alpha-j,l})(X)$ is a polynomial of degree l with coefficients depending on $\sigma_{\alpha-j,l}$. Here $B_x^*(0,R)$ stands for the ball of radius R in the cotangent space T_x^*M and S_x^*U the unit sphere in the cotangent space T_x^*U.

Discarding the divergences, we can therefore extract a finite part from the asymptotic expansion of $\int_{B(0,R)} \sigma(x,\xi)đ\xi$:

Definition 1.7. *The finite-part integral of $\sigma \in \mathrm{CS}^{\alpha,k}(U,V)$ is defined to be the constant term in the asymptotic expansion* (1.7)

$$-\!\!\!\!\!\!\int_{T_x^*U} \sigma(x,\xi)đ\xi := \mathrm{LIM}_{R\to\infty} \int_{B_x^*(0,R)} \sigma(x,\xi)đ\xi := C_x(\sigma). \quad (1.8)$$

(This concept is closely related to *partie finie* of Hadamard [H], hence the terminology used here. However in the physics literature this is also known as "cut-off regularization".)

The proof of the following formula [Gru1], [P] and of Lemma 1.6 is included in Appendix B.

Lemma 1.8. *For* $\sigma \in \mathrm{CS}^{\alpha,k}(U,V)$

$$\begin{aligned}-\!\!\!\!\!\!\int_{T_x^*U} \sigma(x,\xi)đ\xi &= \sum_{j=0}^{N} \int_{B_x^*(0,1)} \sigma_{\alpha-j}(x,\xi)đ\xi + \int_{T_x^*U} \sigma_{(N)}(x,\xi)đ\xi \\ &\quad + \sum_{j=0,\alpha-j+n\neq 0}^{N} \sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha-j+n)^{l+1}} \int_{S_x^*U} \sigma_{\alpha-j,l}(x,\xi)đ_S\xi\,. \qquad (1.9)\end{aligned}$$

It is independent of $N > \mathrm{Re}(\alpha)+n-1$.

The residue terms on the right side of (1.9) measure anomalous behaviour in the finite-part integral. Specifically, (1.9) implies that, for a rescaling $R \to \mu R$,

$$\begin{aligned}\mathrm{LIM}_{R\to\infty} \int_{B_x^*(0,\mu R)} \sigma(x,\xi)đ\xi &= \mathrm{LIM}_{R\to\infty} \int_{B_x^*(0,R)} \sigma(x,\xi)đ\xi \\ &\quad + \sum_{l=0}^{k} \frac{\log^{l+1}\mu}{l+1} \int_{S_x^*U} \sigma_{\alpha-j,l}(x,\xi)đ_S\xi \qquad (1.10)\end{aligned}$$

(cf. Appendix B) and hence that the finite-part integral is independent of a rescaling if $\int_{S_x^*U} \sigma_{-n,l}(x,\xi)đ_S\xi$ vanishes for each integer $0 \le l \le k$. More generally, just as ordinary integrals obey the transformation rule

$$|\det C| \cdot \int_{\mathbb{R}^n} f(C\xi)đ\xi = \int_{\mathbb{R}^n} f(\xi)đ\xi\,,$$

one hopes for a similar transformation rule for the regularized integral $-\!\!\!\!\!\int_{\mathbb{R}^n}\sigma(\xi)đ\xi$ when σ is a log-polyhomogeneous symbol in order to obtain a globally-defined density on M. That, however, is generally not the case in the presence of a residue, as the following proposition shows.

Proposition 1.9 [L]. *The finite-part integral of* $\sigma \in \mathrm{CS}^{*,k}(U)$ *is generally not invariant under a transformation* $C \in Gl_n(T_x^*U)$. *One has,*

$$\begin{aligned}|\det C| \cdot -\!\!\!\!\!\!\int_{T_x^*U} \sigma(x,C\xi)đ\xi &= -\!\!\!\!\!\!\int_{T_x^*U} \sigma(x,\xi)đ\xi \\ &\quad + \sum_{l=0}^{k} \frac{(-1)^{l+1}}{l+1} \int_{S_x^*U} \sigma_{-n,l}(x,\xi)\log^{l+1}|C^{-1}\xi|đ\xi\,. \qquad (1.11)\end{aligned}$$

Proof. We refer the reader to the proof of Proposition 5.2 in [L]. □

As a consequence, whenever $\int_{S_x^*U} \sigma_{-n,l}(x,\xi)\log^{l+1}|C^{-1}\xi|đ\xi$ vanishes for each integer $0 \le l \le k$ and $x \in U$, one then recovers the usual transformation property

$$|\det C| \cdot -\!\!\!\!\!\!\int_{T_x^*U} \sigma(x,C\xi)đ\xi = -\!\!\!\!\!\!\int_{T_x^*U} \sigma(x,\xi)đ\xi\,. \qquad (1.12)$$

With respect to a trivialization $E_{|U} \cong U \times V$, a localization of $A \in \mathrm{Cl}^{\alpha,k}(M,E)$ in $\mathrm{Cl}^{\alpha,k}(U,V)$ can be written

$$Af(x) = \int_{\mathbb{R}^n} \int_U e^{i(x-y).\xi} \mathsf{a}(x,y,\xi) f(y) dy \, đ\xi$$

with amplitude $\mathsf{a} \in \mathrm{CS}^\alpha(U \times U, V)$. Then with

$$\sigma_A(x,\xi) := \mathsf{a}(x,x,\xi) \in \mathrm{CS}^\alpha(U,V)$$

we define

$$\mathrm{TR}_x(A)dx := ⨍_{T_x^*M} \mathrm{tr}_x\big(\sigma_A(x,\xi)\big) đ\xi \, dx\,.$$

If (1.12) holds for $\sigma = \sigma_A$ in each localization it follows that $\mathrm{TR}_x(A)dx$ is independent of the choice of local coordinates. This is known in the following cases.

Proposition 1.10. *Let $A \in \mathrm{Cl}^{\alpha,k}(M,E)$. In each of the following cases $\mathrm{TR}_x(A)dx$ defines an element of $C^\infty(M,|\Omega|)$, that is, a global density on M:*

1. *$\alpha \notin [-n,\infty) \cap \mathbb{Z}$;*
2. *A (of integer order) is even-even and M is odd-dimensional;*
3. *A (of integer order) is even-odd and M is even-dimensional.*

Cases (1) and (2) were shown in [KV], where the canonical trace was first introduced, in terms of homogeneous distributions. Case (1) was reformulated in [L] in terms of finite-part integrals and extended to log-polyhomogeneous symbols $k \geq 0$. Case (3) was introduced in [Gru1] where it was shown that (2) and (3) may be included in the finite-part integral formulation. We refer there for details. Notice though that it is easily seen that the integrals $\int_{S_x^*U} \sigma_{-n,l}(x,\xi) \log^{l+1}|C^{-1}\xi| đ\xi$ vanish in each case; for (1), there is no homogeneous component of the symbol of degree $-n$ and so the integrals vanish trivially, while setting $g(\xi) := \sigma_{-n,l}(x,\xi)\log^{l+1}|C^{-1}\xi|$, for cases (2) and (3) one has $g(-\xi) = -g(\xi)$ and so the vanishing is immediate by symmetry.

Definition 1.11. *For a ψdo $A \in \mathrm{Cl}^{\alpha,*}(M,E)$ satisfying one of the criteria (1),(2),(3) in Proposition 1.10, the canonical trace is defined by*

$$\mathrm{TR}(A) := \int_M dx \, \mathrm{TR}_x(A)\,.$$

The case of ψdos of non-integer order is all that is needed for the general formulae we prove here, cases (2) and (3) of Proposition 1.10 will be relevant only for applications and refinements. Case (2) in particular includes differential operators on odd-dimensional manifolds, though this holds by default in so far as TR_x vanishes on differential operators in any dimension (noted also in [Gru1]):

Proposition 1.12. *Let $A \in \mathrm{Cl}(M,E)$ be a differential operator with local symbol σ_A, then for any $x \in M$*

$$\mathrm{TR}_x(A) := -\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}_x\big(\sigma_A(x,\xi)\big) đ\xi = 0\,.$$

Proof. Since A is a differential operator, $\sigma_A(x,\xi) = \sum_{|k|=0}^{\mathrm{ord}A} \sigma_k(x,\xi)$ with $k = (k_1, \dots, k_n)$ a multi-index with $k_i \in \mathbb{N}$ and $\sigma_k(x,\xi) = a_k(x)\xi^k$ positively homogeneous (with the previous notation we have $\sigma_{(N)} = 0$ provided $N \geq \mathrm{ord}A$). Its finite-part integral on the cotangent space at $x \in M$ therefore reads

$$\begin{aligned} -\!\!\!\!\!\!\int_{T_x^*M} \sigma_A(x,\xi) đ\xi &= \sum_{|k|=0}^{\mathrm{ord}A} a_k(x) \mathrm{LIM}_{R\to\infty} \int_{B_x^*(0,R)} \xi^k đ\xi \\ &= \sum_{|k|=0}^{\mathrm{ord}A} a_k(x) \mathrm{LIM}_{R\to\infty} \bigg(\int_0^R r^{|k|+n-1} dr \bigg) \int_{S_x^*M} \xi^k đ\xi \end{aligned}$$

which vanishes since $\mathrm{LIM}_{R\to\infty} \frac{R^{|k|+n}}{|k|+n} = 0$. □

On commutators the canonical trace has the following more substantial vanishing properties [KV], [MN], [L], [Gru1], providing some justification for its name.

Proposition 1.13. *Let $A \in \mathrm{Cl}^{a,k}(M,E)$, $B \in \mathrm{Cl}^{b,l}(M,E)$. In each of the cases,*

1. $\alpha + \beta \notin [-n, \infty) \cap \mathbb{Z}$,
2. *A and B are both even-even or are both even-odd and M is odd-dimensional,*
3. *A is even-even, B is even-odd, and M is even-dimensional,*

the canonical trace is then defined on the commutator $[A,B]$ and is equal to zero,

$$\mathrm{TR}([A,B]) = 0\,.$$

The canonical trace extends the usual operator trace defined on the subalgebra $\mathrm{Cl}^{\mathrm{ord}<-n}(M,E)$ of ψdos of real order $\mathrm{Re}(\alpha) < -n$, in so far as for ψdos with (real) order less than $-n$, finite-part integrals coincide with ordinary integrals. More precisely, if $K_A(x,y)$ denotes the Schwartz kernel of $A \in \mathrm{Cl}^{\mathrm{ord}<-n}(M,E)$ in a given localization, then $\sigma_A(x,\xi)$ is integrable in ξ and $K_A(x,x)dx = \big(\int_{T_x^*M} \sigma_A(x,\xi)đ\xi\big)dx$ determines a global density on M, and one has

$$\mathrm{tr}(A) = \int_M dx\, \mathrm{tr}_x\big(K_A(x,x)\big) = \int_M dx -\!\!\!\!\!\!\int_{T_x^*M} \mathrm{tr}_x\big(\sigma_A(x,\xi)\big) đ\xi = \mathrm{TR}(A)\,.$$

1.3 Holomorphic families of symbols. We consider next families of symbols depending holomorphically on a complex parameter z. The definition is somewhat more delicate than that used in [KV] (or [L]) since growth conditions must be imposed on each z-derivative of the symbol. This is in order to maintain control of the full Laurent expansion.

First, the meaning here of holomorphic dependence on a parameter is as follows. Let $W \subset \mathbb{C}$ be a complex domain, let Y be an open subset of $\mathbb{R}^m$, and let V be a vector space. A function $p(z,\eta) \in C^\infty(W \times Y, \mathrm{End}(V))$ is holomorphic at $z_0 \in W$ if, for fixed η with

$$p^{(k)}(z_0,\eta) = \partial_z^k\big(p(z,\eta)\big)\big|_{z=z_0},$$

there is a Taylor expansion in a neighbourhood N_{z_0} of z_0,

$$p(z,\eta) = \sum_{k=0}^{\infty} p^{(k)}(z_0,\eta)\,\frac{(z-z_0)^k}{k!}, \tag{1.13}$$

which is convergent, uniformly on compact subsets of N_{z_0}, with respect to the (metrizable) topology on $C^\infty(W \times Y, \mathrm{End}(V))$ associated with the family of semi-norms,

$$\|q\|_{m,K_1,K_2} = \sup_{\substack{(z,\eta)\in K_1\times K_2\\ r+|\mu|\le m}} \left|\partial_z^r \partial_\eta^\mu q(z,\eta)\right|, \tag{1.14}$$

defined for $m \in \mathbf{N}$ and compact subsets $K_1 \subset W$, $K_2 \subset \mathbb{R}^m$.

Definition 1.14. *Let m be a non-negative integer, let U be an open subset of $\mathbb{R}^n$, and let W be a domain in $\mathbb{C}$. A holomorphic family of log-polyhomogeneous symbols parametrized by W of order $\alpha \in C^\infty(W,\mathbb{C})$ and of log-degree m means a function*

$$\sigma(z)(x,\xi) := \sigma(z,x,\xi) \in C^\infty(W \times U \times \mathbb{R}^n, \mathrm{End}V)$$

for which

(1) *$\sigma(z)(x,\xi)$ is holomorphic at $z \in W$ as an element of $C^\infty(W\times U\times\mathbb{R}^n, \mathrm{End}V)$ and*

$$\sigma(z)(x,\xi) \sim \sum_{j\ge 0} \sigma(z)_{\alpha(z)-j}(x,\xi) \in \mathrm{CS}^{\alpha(z),m}(U,V), \tag{1.15}$$

where the function $\alpha : W \to \mathbb{C}$ is holomorphic;

(2) *for any integer $N \ge 1$, the remainder*

$$\sigma_{(N)}(z)(x,\xi) := \sigma(z)(x,\xi) - \sum_{j=0}^{N-1} \sigma_{\alpha(z)-j}(z)(x,\xi)$$

is holomorphic in $z \in W$ as an element of $C^\infty(W \times U \times \mathbb{R}^n, \mathrm{End}V)$ with k^{th} z-derivative,

$$\sigma^{(k)}_{(N)}(z)(x,\xi) := \partial_z^k\big(\sigma_{(N)}(z)(x,\xi)\big) \in \mathrm{S}^{\alpha(z)-N+\epsilon}(U,V), \tag{1.16}$$

for any $\epsilon > 0$.

A family $z \mapsto A(z)$ of log-classical ψdos on $C^\infty(M,E)$ parametrized by a domain $W \subset \mathbb{C}$ is holomorphic if in each local trivialisation of E one has

$$A(z) = \mathrm{Op}(\sigma_{A(z)}) + R(z)$$

with $\sigma_{A(z)}$ a holomorphic family of log-polyhomogeneous symbols and $R(z)$ a smoothing operator with Schwartz kernel $R(z,x,y) \in C^\infty(W \times X \times X, \mathrm{End}(V))$ holomorphic in z.

There are, of course, other ways to express these conditions; for example, in terms of the truncated kernel $K^{(N)}(z)(x,y) := \int_{T_x^*U} e^{i\xi\cdot(x-y)}\sigma_{(N)}(z)(x,\xi)đ\xi$ with large N, and its derivatives $\partial_z^k K^{(N)}(z)(x,y)$, used in the case $k=0$ in [KV] to compute the pole of $\mathrm{Tr}\,(A(z))$ at $z_0 \in P$. When dealing with the full Laurent expansion the essential requirement is that a balance be preserved between the Taylor expansion in z, in terms of the growth rates of the z-derivatives of the symbol, and the asymptotic symbol expansion in ξ.

Proposition 1.15. *If $\sigma(z)(x,\xi) \in \mathrm{CS}^{\alpha(z),m}(U,V)$ is a holomorphic family of log-classical symbols, then so is each derivative*

$$\sigma^{(k)}(z)(x,\xi) := \partial_z^k\big(\sigma(z)(x,\xi)\big) \in \mathrm{CS}^{\alpha(z),m+k}(U,V)\,. \tag{1.17}$$

Precisely, $\sigma^{(k)}(z)(x,\xi)$ has an asymptotic expansion

$$\sigma^{(k)}(z)(x,\xi) \sim \sum_{j\geq 0} \sigma^{(k)}(z)_{\alpha(z)-j}(x,\xi)\,, \tag{1.18}$$

where as elements of $\bigcup_{l=0}^{m+k} \mathrm{CS}^{\alpha(z)-j,l}(U,V)$

$$\sigma^{(k)}(z)_{\alpha(z)-j}(x,\xi) = \partial_z^k\big(\sigma(z)_{\alpha(z)-j}(x,\xi)\big)\,. \tag{1.19}$$

That is,

$$\big(\partial_z^k\sigma(z)\big)_{\alpha(z)-j}(x,\xi) = \partial_z^k\big(\sigma(z)_{\alpha(z)-j}(x,\xi)\big)\,. \tag{1.20}$$

Proof. We have to show that

$$\partial_z^k\big(\sigma(z)(x,\xi)\big) \sim \sum_{j\geq 0} \partial_z^k\left(\sigma(z)_{\alpha(z)-j}(x,\xi)\right)\,, \tag{1.21}$$

where the summands are log-polyhomogeneous of the asserted order. First, the estimate

$$\partial_z^k\big(\sigma(z)(x,\xi)\big) - \sum_{j=0}^{N-1} \partial_z^k\left(\sigma(z)_{\alpha(z)-j}(x,\xi)\right) \in \mathrm{S}^{\alpha(z)-N+\epsilon}(U,V)$$

for any $\epsilon > 0$, needed for (1.21) to hold is equation (1.16) of the definition. It remains to examine the form of the summands in $\sum_{j=0}^{N-1}\partial_z^k(\sigma(z)_{\alpha(z)-j}(x,\xi))$. Taking differences of remainders $\sigma_{(N)}(z)(x,\xi)$ implies that each term $\sigma(z)_{\alpha(z)-j}(x,\xi)$ is holomorphic. In order to compute $\partial_z(\sigma(z)_{\alpha(z)-j}(x,\xi))$

one must compute the derivative of each of its homogeneous components; for $|\xi| \geq 1$ and any $l \in \{0, \cdots, m\}$

$$\begin{aligned}\partial_z\left(\sigma_{\alpha(z)-j,l}(z)(x,\xi)\right) &= \partial_z\left(|\xi|^{\alpha(z)-j}\sigma_{\alpha(z)-j,l}(z)\left(x, \tfrac{\xi}{|\xi|}\right)\right) \\ &= \left(\alpha'(z)|\xi|^{\alpha(z)-j}\sigma_{\alpha(z)-j,l}(z)\left(x, \tfrac{\xi}{|\xi|}\right)\right)\log|\xi| \\ &\quad + |\xi|^{\alpha(z)-j}\partial_z\left(\sigma_{\alpha(z)-j,l}(z)\left(x, \tfrac{\xi}{|\xi|}\right)\right).\end{aligned}$$

Since $\sigma_{\alpha(z)-j,l}(z)(x,\xi|\xi|^{-1})$ is a symbol of constant order zero, so is its z-derivative. Hence,

$$\partial_z\left(\sigma(z)_{\alpha(z)-j,l}(x,\xi)\right) = \alpha'(z)\sigma(z)_{\alpha(z)-j,l}(x,\xi)\log[\xi] + p_{\alpha(z)-j,l}(z)(x,\xi)\,, \tag{1.22}$$

where $\sigma_{\alpha(z)-j,l}(z), p_{\alpha(z)-j,l}(z) \in \mathrm{CS}^{\alpha(z)-j}(U)$ are homogeneous in ξ of order $\alpha(z)-j$. Hence, $\partial_z(\sigma(z)_{\alpha(z)-j}) \in \mathrm{CS}^{\alpha(z)-j,m+1}(U)$. Iterating (1.22), $\partial_z^k(\sigma_{\alpha(z)-j}(z)(x,\xi))$ is thus seen to be a polynomial in $\log[\xi]$ of the form

$$\begin{aligned}&\left(\alpha'(z)\right)^k\sigma_{\alpha(z)-j,k}(z)(x,\xi)\log^{k+m}[\xi] + \cdots \\ &\qquad + |\xi|^{\alpha(z)-j}\partial_z^k\left(\sigma_{\alpha(z)-j,l}(z)\left(x, \tfrac{\xi}{|\xi|}\right)\right)\log^0[\xi]\end{aligned}$$

with each coefficient homogeneous of order $\alpha(z)-j$. This completes the proof. □

Thus, taking derivatives adds more logarithmic terms to each term $\sigma(z)_{\alpha(z)-j}(x,\xi)$, increasing the log-degree, but the order is unchanged. Specifically, $\sigma^{(k)}(z)_{\alpha(z)-j}$ takes the form

$$\sigma^{(k)}(z)_{\alpha(z)-j}(x,\xi) = \sum_{l=0}^{m+k}\sigma^{(k)}(z)_{\alpha(z)-j,l}(x,\xi)\log^l[\xi]\,, \tag{1.23}$$

where the terms $\sigma^{(k)}(z)_{\alpha(z)-j,l}(x,\xi)$ are positively homogeneous in ξ of degree $\alpha(z)-j$ for $|\xi| \geq 1$ and can be computed explicitly from the lower-order derivatives of $\sigma(z)_{\alpha(z)-j,m}(x,\xi)$. The following more precise inductive formulae will be needed in what follows.

Lemma 1.16. *Let $\sigma(z)(x,\xi) \in \mathrm{CS}(U,V)$ be a holomorphic family of classical symbols. Then for $|\xi| \geq 1$*

$$\begin{aligned}&\sigma^{(k+1)}_{\alpha(z)-j,k+1}(z)(x,\xi) = \alpha'(z)\sigma^{(k)}_{\alpha(z)-j,k}(z)(x,\xi)\,, \\ &\sigma^{(k+1)}_{\alpha(z)-j,l}(z)(x,\xi) = \alpha'(z)\sigma^{(k)}_{\alpha(z)-j,l-1}(z)(x,\xi) \\ &\qquad\qquad + |\xi|^{\alpha(z)-j}\partial_z\big(\sigma^{(k)}_{\alpha(z)-j,l}(z)(x,\xi/|\xi|)\big)\,, \quad 1 \leq l \leq k\,, \\ &\sigma^{(k+1)}_{\alpha(z)-j,0}(z)(x,\xi) = |\xi|^{\alpha(z)-j}\partial_z\big(\sigma^{(k)}_{\alpha(z)-j,0}(z)(x,\xi/|\xi|)\big)\,.\end{aligned}$$

Proof. From the above,

$$\sigma^{(k)}(z)_{\alpha(z)-j}(x,\xi) = \partial_z^k\big(\sigma(z)_{\alpha(z)-j}(x,\xi)\big) = \sum_{l=0}^{k} \sigma^{(k)}(z)_{\alpha(z)-j,l}(x,\xi)\log^l[\xi]\,,$$

so that

$$\sigma^{(k+1)}(z)_{\alpha(z)-j}(x,\xi) = \sum_{l=0}^{k} \partial_z\big(\sigma^{(k)}(z)_{\alpha(z)-j,l}(x,\xi)\big)\log^l[\xi]\,. \tag{1.24}$$

Hence, for $|\xi|\geq 1$,

$$\sum_{l=0}^{k+1}\sigma^{(k+1)}(z)_{\alpha(z)-j,l}(x,\xi)\log^l|\xi| = \sum_{r=0}^{k}\alpha'(z)\sigma^{(k)}(z)_{\alpha(z)-j,r}(x,\xi)\log^{r+1}|\xi| + |\xi|^{\alpha(z)-j}\partial_z\left(\sigma^{(k)}(z)_{\alpha(z)-j,r}\left(x,\tfrac{\xi}{|\xi|}\right)\right)\log^r|\xi|$$

where for the right side we apply (1.22) to each of coefficient on the right side of (1.24). Equating coefficients completes the proof. □

A corresponding result on the level of operators follows in a straightforward manner:

PROPOSITION 1.17. *Let $z\mapsto A(z)\in \mathrm{Cl}^{\alpha(z),m}(M,E)$ be a holomorphic family of log-polyhomogeneous ψdos. Then for any non-negative integer k, $A^{(k)}(z_0)$ lies in $\mathrm{Cl}^{\alpha(z_0),m+k}(M,E)$.*

EXAMPLE 1.18. For real numbers α, q with $q>0$, the function $\sigma(z)(x,\xi) = \psi(\xi)|\xi|^{\alpha-qz}$, where ψ is a smooth cut-off function which vanishes near the origin and is equal to 1 outside the unit ball, provides a holomorphic family of classical symbols; at any point $z=z_0\in\mathbb{C}$, we have

$$\sigma^{(k)}(z_0)(x,\xi) = (-q)^k\psi(\xi)\log^k|\xi|\,|\xi|^{\alpha-qz_0}$$

which lies in $\mathrm{CS}^{\alpha-qz_0,k}(U)$. More generally, if $Q\in\mathrm{Cl}^q(M,E)$ is a classical elliptic ψdo of order $q>0$ with principal angle θ, then one has for each $z\in\mathbb{C}$ the complex power $Q_\theta^{-z}\in\mathrm{Cl}^{-qz}(M,E)$ [Se1] represented in a local coordinate chart U by a classical symbol $\mathbf{q}(z)(x,\xi)\in\mathrm{CS}^{-qz}(U,V)$. Let $A\in\mathrm{Cl}^\alpha(M,E)$ be a coefficient classical ψdo represented in U by $\mathbf{a}(x,\xi)\in\mathrm{CS}^\alpha(U)$. Then $\sigma_{AQ_\theta^{-z}}(x,\xi)\in\mathrm{CS}^{\alpha-qz}(U,V)$ is a holomorphic family of symbols parametrized by $W=\mathbb{C}$ whose convergent Taylor expansion in $C^\infty(\mathbb{C}\times U,V)$ around each $z_0\in\mathbb{C}$ is from [O1, Lem. 2.1] given by

$$(\sigma_{AQ_\theta^{-z}})_{\alpha-qz-j}(x,\xi) = \sum_{k=0}^{\infty}\sum_{l=0}^{k}(-1)^k\big(\mathbf{a}\circ\log^k(\mathbf{q})\circ\mathbf{q}(z_0)\big)_{\alpha-qz_0-j,l}(x,\xi)\log^l|\xi|\,\frac{(z-z_0)^k}{k!}\,,$$

where $\circ$ denotes the usual mod($\mathrm{S}^{-\infty}$) symbol product, $\mathbf{q} := \mathbf{q}(-1)$ and $\log^k(\mathbf{q})(x,\xi) := (\log(\mathbf{q}) \circ \ldots \circ \log(\mathbf{q}))(x,\xi) \in \mathrm{CS}^{0,k}(U,V)$ with k factors.

1.4 A Laurent expansion for finite-part integrals of holomorphic symbols. The following theorem computes the Laurent expansion for finite-part integrals of holomorphic families of classical symbols of order $\alpha(z)$ in terms of local canonical and residue densities. This extends Proposition 3.4 in [KV], and results of [Gu, W], where the pole, the first coefficient in the expansion, was identified as the residue trace. The proof uses the property that each term of the Taylor series of a holomorphic family of classical symbols has an asymptotic symbol expansion, allowing the Laurent expansion of $-\!\!\!\!\!\!\int \sigma(z)(x,\xi)\, đ\xi$ to be computed through Lemma 1.8. Notice that although the Taylor expansion in the C^∞ topology gives no control over the symbol as $|\xi| \to \infty$, (1.15), (1.16) impose what is needed to ensure integrability requirements.

Definition 1.19. *A holomorphic function $\alpha : W \to \mathbb{C}$ defined on a domain $W \subset \mathbb{C}$ is said to be non-critical on*

$$P := \alpha^{-1}\big(\mathbb{Z} \cap [-n, +\infty[\,\big) \cap W$$

if $\alpha'(z_0) \neq 0$ at each $z_0 \in P$.

Theorem 1.20. (1) *Let U be an open subset of $\mathbb{R}^n$. Let $z \mapsto \sigma(z) \in \mathrm{CS}^{\alpha(z)}(U,V)$ be a holomorphic family of classical symbols parametrized by a domain $W \subset \mathbb{C}$ such that the order function α is non-critical on P. Then for each $x \in U$ the map $z \mapsto -\!\!\!\!\!\!\int_{T_x^*U} \sigma(z)(x,\xi) đ\xi$ is a meromorphic function on W with poles located in P. The poles are at most simple and for z near $z_0 \in P$ one has*

$$\begin{aligned}
-\!\!\!\!\!\!\int_{T_x^*U} \sigma(z)(x,\xi) đ\xi &= -\frac{1}{\alpha'(z_0)} \int_{S_x^*U} \sigma(z_0)_{-n}(x,\xi) đ_S\xi \,\frac{1}{(z-z_0)} \\
&\quad + \left(-\!\!\!\!\!\!\int_{T_x^*U} \sigma(z_0)(x,\xi) đ\xi - \frac{1}{\alpha'(z_0)} \int_{S_x^*U} \sigma'(z_0)_{-n,0}(x,\xi) đ_S\xi \right) \\
&\quad\quad + \frac{\alpha''(z_0)}{2\alpha'(z_0)^2} \int_{S_x^*U} \sigma(z_0)_{-n}(x,\xi) đ_S\xi + \sum_{k=1}^{K} \left(-\!\!\!\!\!\!\int_{T_x^*U} \sigma^{(k)}(z_0)(x,\xi) đ\xi \right. \\
&\quad\quad\quad \left. - \int_{S_x^*U} \mathcal{L}_k\big(\sigma(z_0), \cdots, \sigma^{(k+1)}(z_0)\big)_{-n,0}(x,\xi) đ_S\xi \right) \frac{(z-z_0)^k}{k!} \\
&\quad\quad\quad\quad + o\big((z-z_0)^K\big), \qquad (1.25)
\end{aligned}$$

where

$$\mathcal{L}_k\big(\sigma(z_0),\dots,\sigma^{(k+1)}(z_0)\big)=\sum_{j=0}^{k+1}\frac{p_{k+1-j}}{\alpha'(z_0)^{k+2-j}}\sigma^{(j)}(z_0)\in \mathrm{CS}^{\alpha(z_0),k+1}(U,V)\,, \tag{1.26}$$

and p_{k+1-j} *is an explicitly computable polynomial of degree* $k+1-j$ *in* $\alpha'(z_0),\dots,\alpha^{(k+1)}(z_0)$. *Furthermore, the coefficient of* $\frac{(z-z_0)^k}{k!}$ *in* (1.25) *is equal to* $\mathrm{fp}_{z=z_0}⨍_{T_x^*U}\sigma^{(k)}(z)$. *If* α *is a linear function* $\alpha(z)=qz+b$ *with* $q\neq 0$ *then* (1.25) *reduces to*

$$\begin{aligned}⨍_{T_x^*U}\sigma(z)(x,\xi)đ\xi &= -\frac{1}{q}\int_{S_x^*U}\sigma(z_0)_{-n}(x,\xi)đ_S\xi\,\frac{1}{(z-z_0)}\\ &\quad+\bigg(⨍_{T_x^*U}\sigma(z_0)(x,\xi)đ\xi-\frac{1}{q}\int_{S_x^*U}\sigma'(z_0)_{-n,0}(x,\xi)đ_S\xi\bigg)\\ &\quad+\sum_{k=1}^{K}\bigg(⨍_{T_x^*U}\sigma^{(k)}(z_0)(x,\xi)đ\xi\\ &\quad-\frac{1}{q(k+1)}\int_{S_x^*U}\sigma^{(k+1)}(z_0)_{-n,0}(x,\xi)đ_S\xi\bigg)\frac{(z-z_0)^k}{k!}+o\big((z-z_0)^K\big)\,. \end{aligned}\tag{1.27}$$

If $z_0\in W$ *but* $z_0\notin P$, *then* $⨍_{T_x^*U}\sigma(z)(x,\xi)đ\xi$ *is holomorphic at* $z=z_0$ *and* (1.25) *then simplifies to the Taylor expansion*

$$\begin{aligned}⨍_{T_x^*U}\sigma(z)(x,\xi)đ\xi &= ⨍_{T_x^*U}\sigma(z_0)(x,\xi)đ\xi\\ &\quad+\sum_{k=1}^{K}⨍_{T_x^*U}\sigma^{(k)}(z_0)(x,\xi)đ\xi\,\frac{(z-z_0)^k}{k!}+o\big((z-z_0)^K\big)\,.\end{aligned}\tag{1.28}$$

(2) *For any holomorphic family* $z\mapsto A(z)\in \mathrm{Cl}^{\alpha(z)}(M,E)$ *of classical* ψdos *parametrized by a domain* $W\subset\mathbb{C}$, *such that order function* α *is non-critical on* P, *the map* $z\mapsto \mathrm{TR}(A(z)):=\int_M dx\int_{T_x^*M}\mathrm{tr}_x(\sigma_{A(z)}(x,\xi))đ\xi$ *is a meromorphic function on* W *with poles located in* P. *The poles are at most simple and for* z *near* $z_0\in P$

$$\begin{aligned}\mathrm{TR}(A(z)) &= -\frac{1}{\alpha'(z_0)}\,\mathrm{res}\big(A(z_0)\big)\,\frac{1}{(z-z_0)}\\ &\quad+\int_M dx\left(\mathrm{TR}_x\big(A(z_0)\big)-\frac{1}{\alpha'(z_0)}\,\mathrm{res}_{x,0}(A'(z_0))\right)+\frac{\alpha''(z_0)}{2\alpha'(z_0)^2}\,\mathrm{res}\big(A(z_0)\big)\\ &\quad+\sum_{k=1}^{K}\int_M dx\big(\mathrm{TR}_x(A^{(k)}(z_0))-\mathrm{res}_{x,0}(\mathcal{L}_k(\sigma_{A(z_0)},\dots,\sigma_{A^{(k+1)}(z_0)}))\big)\frac{(z-z_0)^k}{k!}\\ &\quad+o\big((z-z_0)^K\big)\,.\end{aligned}\tag{1.29}$$

Furthermore, the coefficient of $(z-z_0)^k/k!$ *in* (1.29) *is equal to* $\mathrm{fp}_{z=z_0}\mathrm{TR}(A^{(k)}(z))$. *If* $A(z)$ *has order* $\alpha(z)=qz+b$ *with* $q\neq 0$ *then*

$$\begin{aligned}\mathrm{TR}(A(z)) = & -\frac{1}{q}\,\mathrm{res}\big(A(z_0)\big)\frac{1}{(z-z_0)} \\ & + \int_M dx\left(\mathrm{TR}_x\big(A(z_0)\big) - \frac{1}{q}\,\mathrm{res}_{x,0}\big(A'(z_0)\big)\right) \\ & + \sum_{k=1}^{K}\int_M dx\left(\mathrm{TR}_x(A^{(k)}(z_0)) - \frac{\mathrm{res}_{x,0}(\sigma_A^{(k+1)}(z_0))}{q\,(k+1)}\right)\frac{(z-z_0)^k}{k!} \\ & + o\big((z-z_0)^K\big)\,. \end{aligned} \tag{1.30}$$

If $z_0\in W$ *but* $z_0\notin P$, *then* $\mathrm{TR}(A(z))$ *is holomorphic at* $z=z_0$ *and* (1.29) *then simplifies to the Taylor expansion*

$$\mathrm{TR}(A(z)) = \mathrm{TR}\big(A(z_0)\big) + \sum_{k=1}^{K}\mathrm{TR}\big(A^{(k)}(z_0)\big)\,\frac{(z-z_0)^k}{k!} + o\big((z-z_0)^K\big)\,. \tag{1.31}$$

Remark 1.21. Since α is non-critical on P, we have from Proposition 1.17 and equation (1.22) that the operators $A^{(k)}(z_0)\in \mathrm{Cl}^{\alpha(z_0),k}(M,E)$ in equation (1.29) are not classical for $k\geq 1$.

Remark 1.22. At a point $z_0\in P$, $\alpha'(z_0)\neq 0$; writing $\alpha(z)=\alpha(z_0)+\alpha'(z_0)(z-z_0)+o(z-z_0)$ we find that α is injective in a neighborhood of z_0. As a consequence, $\mathbb{Z}$ being countable, so is the set of poles $P=\alpha^{-1}(\mathbb{Z}\cap[-n,+\infty)\,)\cap W$ countable.

Remark 1.23. Setting $\alpha(z)=z/(1+\lambda z)$ with $\lambda\in\mathbb{R}^*$ for $z\in\mathbb{C}\backslash\{-\lambda^{-1}\}$ gives rise to an additional finite part $\frac{\alpha''(0)}{2\alpha'(0)^2}\int_{S_x^*U}\sigma(0)_{-n}(x,\xi)đ_S\xi = \lambda\int_{S_x^*U}\sigma(0)_{-n}(x,\xi)đ_S\xi$ just as a rescaling $R\to e^{\lambda}R$ in the finite-part integrals gives rise to the extra term $\lambda\int_{S_x^*U}\sigma(0)_{-n}(x,\xi)đ_S\xi$ (see (1.10) with $k=0$ and $\mu=e^{\lambda}$).

Proof. Since the orders $\alpha(z)$ define a holomorphic map at each point of P, for any $z_0\in P$ there is a ball $B(z_0,r)\subset W\subset\mathbb{C}$ centered at $z_0\in W$ with radius $r>0$ such that $(B(z_0,r)\backslash\{z_0\})\bigcap P=\phi$. In particular, for all $z\in B(z_0,r)\backslash\{z_0\}$, the symbols $\sigma(z)$ have non-integer order. As a consequence, outside the set P, the finite-part integral $⨍_{T_x^*U}\sigma(z)(x,\xi)đ\xi$ is defined without ambiguity and $⨍_{T_x^*U}\sigma(z)(x,\xi)đ\xi\,dx$ defines a global density on M.

Since $z_0\in P$, there is some $j_0\in\mathbf{N}\cup\{0\}$ such that $\alpha(z_0)+n-j_0=0$. On the other hand, for $z\in B(z_0,r)\backslash\{z_0\}$ we have $\alpha(z)+n-j\neq 0$ and $N>\mathrm{Re}(\alpha(z))+n-1$ can be chosen uniformly to ensure that $\sigma_{(N)}(z)\in \mathrm{S}^{<-n}(U,V)$.

Hence, for $z \in B(z_0, r)\backslash\{z_0\}$, equation (1.9) yields (with $k=0$)

$$\begin{aligned}
&⨍_{T_x^*U} \sigma(z)(x,\xi) đ\xi \\
&\quad = \sum_{j=0}^{N} \int_{B_x^*(0,1)} \sigma(z)_{\alpha(z)-j}(x,\xi) đ\xi + \int_{T_x^*U} \sigma_{(N)}(z)(x,\xi) đ\xi \\
&\qquad - \sum_{j=0}^{N} \frac{1}{\alpha(z)+n-j} \int_{S_x^*U} \sigma(z)_{\alpha(z)-j}(x,\xi) đ_S\xi \\
&\quad = \sum_{j=0}^{N} \int_{B_x^*(0,1)} \sigma(z)_{\alpha(z)-j}(x,\xi) đ\xi + \int_{T_x^*U} \sigma_{(N)}(z)(x,\xi) đ\xi \\
&\qquad - \sum_{j=0, j\neq j_0}^{N} \frac{1}{\alpha(z)+n-j} \int_{S_x^*U} \sigma(z)_{\alpha(z)-j}(x,\xi) đ_S\xi \\
&\qquad\quad - \frac{1}{\alpha(z)-\alpha(z_0)} \int_{S_x^*U} \sigma(z)_{\alpha(z)-j_0}(x,\xi) đ_S\xi\,, \qquad (1.32)
\end{aligned}$$

where, in view of the growth conditions (1.15) and (1.16), it is not hard to see that each of the integrals on the right side of (1.32) is holomorphic in z. Since $\sigma_{\alpha(z)-j}(z)(x,\xi)$ is a holomorphic family of classical symbols, there is a Taylor expansion (1.13)

$$\sigma(z)_{\alpha(z)-j}(x,\xi) = \sum_{k=0}^{\infty} \sigma^{(k)}(z_0)_{\alpha(z_0)-j}(x,\xi)\, \frac{(z-z_0)^k}{k!} \qquad (1.33)$$

with coefficients in $\mathrm{CS}^{\alpha(z_0)-j,\,k}(U)$

$$\sigma^{(k)}(z_0)_{\alpha(z_0)-j}(x,\xi) := \partial_z^k\big(\sigma(z)_{\alpha(z)-j}\big)\big|_{z=z_0} = \big(\partial_z^k\sigma(z)\big)_{\alpha(z)-j}\big|_{z=z_0}\,, \qquad (1.34)$$

where the first equality is by definition, while the second equality is equation (1.20), and likewise there is a Taylor expansion of the remainder $\sigma_{(N)}(z)(x,\xi)$ with coefficients

$$\partial_z^k\left(\sigma_{(N)}(z)(x,\xi)\right)|_{z=z_0} = \big(\partial_z^k\sigma(z)\big)_{(N)}(x,\xi)|_{z=z_0}, \qquad (1.35)$$

where again the equality is consequent on equations (1.18) and (1.20). For any non-negative integer K we may therefore rewrite the first two lines of (1.32) as a polynomial $\sum_{k=0}^{K} a_k \frac{(z-z_0)^k}{k!}$ plus an error term of order $o((z-z_0)^K)$ with

$$a_k = \sum_{j=0}^{N} \int_{B_x^*(0,1)} \big(\partial_z^k \sigma(z)\big)_{\alpha(z)-j}(x,\xi)\big|_{z=z_0} đ\xi + \int_{T_x^*U} \big(\partial_z^k \sigma(z)\big)_{(N)}(x,\xi)|_{z=z_0} đ\xi$$
$$- \sum_{j=0, j\neq j_0}^{N} \partial_z^k|_{z=z_0} \bigg(\frac{1}{\alpha(z)+n-j} \int_{S_x^*U} \sigma(z)_{\alpha(z)-j}(x,\xi)\ đ_S\xi \bigg). \quad (1.36)$$

Here, since $j \neq j_0$, we use the fact that each factor in the terms of the final summation of (1.36) are holomorphic in a neighbourhood of z_0 (including at $z = z_0$). On the other hand, from (1.18),

$$\sigma^{(k)}(z_0)(x,\xi) := \partial_z^k \sigma(z)(x,\xi)|_{z=z_0} \sim \sum_{j\geq 0} \big(\partial_z^k \sigma(z)\big)_{\alpha(z)-j}(x,\xi)|_{z=z_0}\,,$$

while we know from (1.17) that $\sigma^{(k)}(z) \in \mathrm{CS}^{\alpha(z),k}(U)$. Hence (1.9) may be applied to see that

$$-\!\!\!\!\!\!\int_{T_x^*U} \sigma^{(k)}(z_0)(x,\xi) đ\xi$$
$$= \sum_{j=0}^{N} \int_{B_x^*(0,1)} \big(\partial_z^k \sigma(z)\big)_{\alpha(z)-j}(x,\xi)|_{z=z_0} đ\xi + \int_{T_x^*U} \big(\partial_z^k \sigma(z)\big)_{(N)}(x,\xi)|_{z=z_0} đ\xi$$
$$+ \sum_{j=0, j\neq j_0}^{N} \sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z_0)-j+n)^{l+1}} \int_{S_x^*U} \big(\partial_z^k \sigma(z)\big)_{\alpha(z)-j,l}(x,\xi)|_{z=z_0} đ_S\xi\,. \quad (1.37)$$

From the following lemma, we conclude that the expressions in (1.36) and (1.37) are equal.

Lemma 1.24. *For $j \neq j_0$, one has in a neighbourhood of z_0*

$$\partial_z^k \bigg(\frac{-1}{\alpha(z)+n-j} \int_{S_x^*U} \sigma(z)_{\alpha(z)-j}(x,\xi) đ_S\xi \bigg)$$
$$= \sum_{l=0}^{k} \frac{(-1)^{l+1} l!}{(\alpha(z)-j+n)^{l+1}} \int_{S_x^*U} \big(\partial_z^k \sigma(z)\big)_{\alpha(z)-j,l}(x,\xi) đ_S\xi\,. \quad (1.38)$$

Proof. We choose z in a neighbourhood of z_0 such that each of the factors on both sides of (1.38) are holomorphic. The equality holds trivially for $k = 0$. For clarity we check the case $k = 1$ before proceeding to the general inductive step. For $k = 1$, the left side of (1.38) is equal to

$$\frac{\alpha'(z)}{(\alpha(z)-j+n)^2} \int_{S_x^*U} \sigma(z)_{\alpha(z)-j}(x,\xi) đ_S\xi$$
$$- \frac{1}{\alpha(z)-j+n} \int_{S_x^*U} \partial_z \big(\sigma(z)_{\alpha(z)-j}\big)(x,\xi) đ_S\xi\,. \quad (1.39)$$

From (1.18) and (1.22), for $|\xi|\geq 1$,

$$\big(\partial_z\sigma(z)\big)_{\alpha(z)-j}(x,\xi)=\alpha'(z)\sigma(z)_{\alpha(z)-j}(x,\xi)\log|\xi|+p_{\alpha(z)-j}(z)(x,\xi)\,,$$

and hence $(\partial_z\sigma(z))_{\alpha(z)-j,1}(x,\xi)=\alpha'(z)\sigma(z)_{\alpha(z)-j}(x,\xi)$ for $|\xi|\geq 1$. The expression in (1.39) is therefore equal to

$$\frac{1}{(\alpha(z)-j+n)^2}\int_{S_x^*U}\big(\partial_z\sigma(z)\big)_{\alpha(z)-j,1}(x,\xi)đ_S\xi \\ -\frac{1}{\alpha(z)-j+n}\int_{S_x^*U}\partial_z\big(\sigma(z)_{\alpha(z)-j}\big)(x,\xi)đ_S\xi\,,$$

which is the right side of (1.38) for $k=1$.

Assume now that (1.38) holds for some arbitrary fixed $k\geq 0$. Then the left side of (1.38) for $k+1$ is equal to

$$\begin{aligned}\partial_z&\bigg(\partial_z^k\bigg(\frac{-1}{\alpha(z)+n-j}\int_{S_x^*U}\sigma(z)_{\alpha(z)-j}(x,\xi)đ_S\xi\bigg)\bigg)\\ &=\partial_z\bigg(\sum_{l=0}^k\frac{(-1)^{l+1}l!}{(\alpha(z)-j+n)^{l+1}}\int_{S_x^*U}\big(\partial_z^k\sigma(z)\big)_{\alpha(z)-j,l}(x,\xi)đ_S\xi\bigg)\\ &=\sum_{l=0}^k\frac{(-1)^l(l+1)!\alpha'(z)}{(\alpha(z)-j+n)^{l+2}}\int_{S_x^*U}\sigma^{(k)}(z)_{\alpha(z)-j,l}(x,\xi)đ_S\xi\\ &\qquad+\sum_{l=0}^k\frac{(-1)^{l+1}l!}{(\alpha(z)-j+n)^{l+1}}\int_{S_x^*U}\partial_z\big(\sigma^{(k)}(z)_{\alpha(z)-j,l}(x,\xi)\big)đ_S\xi\,, \end{aligned}\tag{1.40}$$

where, for the second equality we use the property that both of the factors in each summand on the right side of (1.38) are holomorphic near z_0, and in the notation of (1.23)

$$\big(\partial_z^k\sigma(z)\big)_{\alpha(z)-j}(x,\xi)=\sum_{r=0}^k\sigma^{(k)}(z)_{\alpha(z)-j,r}(x,\xi)\log^r[\xi]\,.$$

In that notation the right side of (1.38) for k replaced by $k+1$ reads

$$\sum_{l=0}^{k+1}\frac{(-1)^{l+1}l!}{(\alpha(z)-j+n)^{l+1}}\int_{S_x^*U}\sigma^{(k+1)}(z)_{\alpha(z)-j,l}(x,\xi)đ_S\xi\,,\tag{1.41}$$

while on the (co-)sphere S_x^*U where $|\xi|=1$ the identities of Lemma 1.16 become

$$\sigma^{(k+1)}_{\alpha(z)-j,k+1}(z)(x,\xi)=\alpha'(z)\sigma^{(k)}_{\alpha(z)-j,k}(z)(x,\xi)\,,$$
$$\sigma^{(k+1)}_{\alpha(z)-j,l}(z)(x,\xi)=\alpha'(z)\sigma^{(k)}_{\alpha(z)-j,l-1}(z)(x,\xi)+\partial_z\big(\sigma^{(k)}_{\alpha(z)-j,l}(z)(x,\xi)\big)\,,\ 1\leq l\leq k\,,$$
$$\sigma^{(k+1)}_{\alpha(z)-j,0}(z)(x,\xi)=\partial_z\big(\sigma^{(k)}_{\alpha(z)-j,0}(z)(x,\xi)\big)\,.$$

Substitution of these identities in (1.41) immediately shows (1.41) to be equal to (1.40). This completes the proof of Lemma 1.24. □

Returning to the proof of Theorem 1.20, from (1.36) and (1.37) and Lemma 1.24, we now have

$$a_k = -\!\!\!\!\!\!\int_{T_x^*U} \sigma^{(k)}(z_0)(x,\xi)đ\xi\,,$$

and so the first two lines of (1.32) may be replaced by

$$\sum_{k=0}^{K} -\!\!\!\!\!\!\int_{T_x^*U} \sigma^{(k)}(z_0)(x,\xi)đ\xi\, \frac{(z-z_0)^k}{k!} + o\big((z-z_0)^K\big)\,.$$

Hence (1.32) becomes

$$-\!\!\!\!\!\!\int_{T_x^*U} \sigma(z)(x,\xi)đ\xi = \sum_{k=0}^{K} -\!\!\!\!\!\!\int_{T_x^*U} \sigma^{(k)}(z_0)(x,\xi)đ\xi\, \frac{(z-z_0)^k}{k!} + o\big((z-z_0)^K\big) - \frac{1}{\alpha(z)-\alpha(z_0)} \int_{S_x^*U} \sigma(z)_{-n}(x,\xi)đ_S\xi\,. \quad (1.42)$$

To expand the sphere integral term in (1.42), since α is holomorphic we have in a neighbourhood of each $z_0 \in P$ a Taylor expansion

$$\alpha(z)-\alpha(z_0) = \sum_{l=1}^{L} \frac{\alpha^{(l)}(z_0)}{l!}\,(z-z_0)^l + o(z-z_0)^L$$

and hence since $\alpha'(z_0) \neq 0$ an expansion

$$\frac{1}{\alpha(z)-\alpha(z_0)} = \frac{1}{\alpha'(z_0)(z-z_0)} \cdot \frac{1}{1+\sum_{l=1}^{L} \frac{\alpha^{(l+1)}(z_0)}{\alpha'(z_0)}\frac{(z-z_0)^l}{(l+1)!} + o(z-z_0)^L}$$

$$= \frac{1}{\alpha'(z_0)} \cdot \frac{1}{(z-z_0)} - \frac{\alpha''(z_0)}{2\,\alpha'(z_0)^2} + \sum_{j=1}^{J} \beta_j(z_0)(z-z_0)^j + o(z-z_0)^J\,, \quad (1.43)$$

with $\beta_j(z_0)$ an explicitly computable rational function in $\alpha^{(k)}(z_0)$, $1 \le k \le j+1$ with denominator an integer power of $\alpha'(z_0)$. On the other hand, since $\alpha(z_0) - j_0 = -n$, the expansion (1.33) for $j = j_0$ becomes

$$\sigma(z)_{\alpha(z)-j_0}(x,\xi) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} \big(\sigma^{(k)}(z_0)\big)_{-n,l}(x,\xi)\log^l[\xi]\,\frac{(z-z_0)^k}{k!}\,. \quad (1.44)$$

Since $(\sigma^{(k)}(z_0))_{-n,l}(x,\xi)\log^l|\xi| = 0$ for $l \ge 1$ on S_x^*U, we find from the expansions (1.43) and (1.44)

$$\begin{aligned}
&\frac{1}{\alpha(z)-\alpha(z_0)}\int_{S_x^*U}\sigma(z)_{\alpha(z)-j_0}(x,\xi)\,đ_S\xi\\
&\quad=\frac{1}{\alpha'(z_0)}\cdot\int_{S_x^*U}\big(\sigma'(z_0)\big)_{-n,0}(x,\xi)\,đ_S\xi\,\frac{1}{(z-z_0)}\\
&\qquad-\sum_{k=0}^{K}\int_{S_x^*U}\mathcal{L}_k\big(\sigma(z_0),\sigma'(z_0),\dots,\sigma^{(k+1)}(z_0)\big)_{-n,0}(x,\xi)\,đ_S\xi\,\frac{(z-z_0)^k}{k!}\\
&\qquad\quad+o\left((z-z_0)^K\right),
\end{aligned}\tag{1.45}$$

where $\mathcal{L}_k\left(\sigma(z_0),\sigma'(z_0),\cdots,\sigma^{(k+1)}(z_0)\right)$ is readily seen to have the form in (1.26). In particular, the explicit formulae given for the first two terms in (1.43) lead to the formula

$$\begin{aligned}
&\mathcal{L}_0\big(\sigma(z_0),\sigma'(z_0)\big)(x,\xi)\\
&\quad=\frac{1}{\alpha'(z_0)}\int_{S_x^*U}\sigma'(z_0)_{-n,0}(x,\xi)\,đ_S\xi-\frac{\alpha''(z_0)}{2\alpha'(z_0)^2}\int_{S_x^*U}\sigma(z_0)_{-n}(x,\xi)\,đ_S\xi\,,
\end{aligned}$$

which with the contribution from the $k=0$ finite-part integral on the right-side of (1.42) gives the stated constant term in the expansion (1.25). The next term up, for example, is

$$\begin{aligned}
&\mathcal{L}_1\left(\sigma(z_0),\sigma'(z_0),\sigma''(z_0)\right)(x,\xi)\;=\\
&\frac{1}{\alpha'(z_0)}\int_{S_x^*U}\sigma''(z_0)_{-n,0}(x,\xi)\,đ_S\xi\;-\;\frac{\alpha''(z_0)}{2\,\alpha'(z_0)^2}\int_{S_x^*U}\sigma'(z_0)_{-n,0}(x,\xi)\,đ_S\xi\\
&\quad+\;\frac{3\alpha''(z_0)^2-2\alpha'''(z_0)\alpha'(z_0)}{12\,\alpha'(z_0)^3}\int_{S_x^*U}\sigma(z_0)_{-n}(x,\xi)\,đ_S\xi.
\end{aligned}$$

When $\alpha(z)=qz+b$ with $q\neq 0$ the right-side of (1.43) is $\frac{1}{q\,(z-z_0)}$ and so from (1.33) one then has $\mathcal{L}_k\left(\sigma(z_0),\sigma'(z_0),\cdots,\sigma^{(k+1)}(z_0)\right)=\frac{\sigma^{(k+1)}(z_0)}{q\,(k+1)!}$ and so (1.27) follows.

If $z_0\notin P$ then $\alpha(z)\in\mathbb{C}\backslash\mathbb{Z}$ and so the log-polyhomogeneous symbols $\mathcal{L}_k$ in (1.45) then have non-integer order and hence have no component of degree $-n$, and therefore vanish. Likewise the pole in (1.45) vanishes and so (1.25) simplifies, in this case, to (1.28). Alternatively, this can be seen in a simpler more direct way by using the linearity of the finite-part integral over log-polyhomogeneous symbols of non-integer order applied to the Taylor expansion of the symbol at z_0. (Indeed, in this case the term $j=j_0$ in (1.32) does not need to be treated separately from the sum in the previous line and (1.36) holds by linearity, from which Lemma 1.24 may then be inferred and now including the case $j=j_0$.)

This shows the first part of the theorem.

For the second part we use a partition of unity $\{(U_i, \phi_i) \mid i \in J\}$ such that for $i, j \in J$ there is an $l_{ij} \in J$ with $\mathrm{supp}(\phi_i) \cup \mathrm{supp}(\phi_j) \subset U_{ij} := U_{l_{ij}}$. We suppose trivialisations of $\pi : E \to M$ over each open set U_i. Then, with U_{ij} identified with an open subset of $\mathbb{R}^n$, one has $A(z) = \sum_{i,j} \phi_i A(z) \phi_j$ where $\phi_i A(z) \phi_j = \mathrm{Op}(\sigma_{(ij)}(z))$ is the localization of A over U_{ij} with amplitude

$$\sigma_{(ij)}(z)(x, y, \xi) \in \mathrm{CS}^{\alpha(z)}(U_{ij} \times U_{ij}, V)\,,$$

a local holomorphic family of symbols in (x, y) form. Each finite-part integral $⨍_{T_x U_{ij}} \sigma_{(ij)}(z)(x, x, \xi) đ\xi$ is well defined outside P, since $A(z)$ has non-integer order for those values of z. Using the linearity there of the canonical trace functional it follows that for $z \notin P$

$$\mathrm{TR}(A(z)) = \sum_{i,j} \int_{U_{ij}} ⨍_{T_x U_{ij}} \mathrm{tr}\left(\sigma_{(ij)}(z)(x, x, \xi)\right) đ\xi \, dx,$$

where tr is the trace on $\mathrm{End}(V)$, allowing (1.25) to be applied to each of the summands defined over the trivialising charts. Each locally defined coefficient in the Laurent expansion is seen by holomorphic continuation to define a global density on M in the way explained in Proposition 1.25. The first part of the theorem therefore yields that $\mathrm{TR}(A(z))$ is meromorphic with simple poles in P and since

$$\sigma^{(k)}_{A(z_0)} = \sigma_{A^{(k)}(z_0)} \tag{1.46}$$

the identity (1.29) now follows from the formula (1.25) applied to each localization.

The fact that the coefficients of $(z - z_0)^k / k!$ in the Laurent expansions of the meromorphic maps $z \mapsto ⨍_{T^*_x U} \sigma(z)$ and $z \mapsto \mathrm{TR}(A(z))$ correspond to the finite part at $z = z_0$ of their derivative at order k follows from the general property for a meromorphic function f on an open set $W \subset \mathbb{C}$ with Laurent expansion around z_0 given by

$$f(z) = \sum_{j=1}^{J} \frac{b_j}{(z - z_0)^j} + \sum_{k=0}^{K} a_k \frac{(z - z_0)^k}{k!} + o((z - z_0)^K)\,,$$

that

$$\mathrm{fp}_{z = z_0} f^{(k)}(z) = a_k\,. \tag{1.47}$$

Combined with the equality $\partial_z^k \mathrm{TR}(A(z)) = \mathrm{TR}(A^{(k)}(z))$ valid for $z \notin P$ we reach the conclusion.

Since the formulas (1.30), (1.31) now follow from (1.27) and (1.28), this ends the proof of the theorem. □

In passing from the local formula (1.25) to the global formula (1.29) in the proof of Theorem 1.20 we have implicitly used the following fact, yielding the Laurent coefficients to be global densities on M which can be integrated.

Proposition 1.25. *Let $c_k(x)$ denote the coefficient of $(z-z_0)^k/k!$ in the Laurent expansion (1.25). Then $c_k(x)dx$ is defined independently of the choice of local coordinates on M.*

Proof. By formula (1.47), the coefficient $c_k(x)$ of $(z-z_0)^k/k!$ in the Laurent expansion (1.25) with $\sigma(z)(x,\cdot)=\sigma_{A(z)}(x,\cdot)$ is identified with the finite part at z_0 of the k-th derivative of the map

$$z \mapsto I_{A(z)}(x) := \int\!\!\!\!\!\!-_{T_x^*U} \sigma_{A(z)}(x,\xi) đ\xi\,,$$

i.e. $c_k(x)=\mathrm{fp}_{z=z_0} I_{A^{(k)}(z)}(x)$. For $z\notin P$ the property (1.12) holds for the finite-part integral $I_{A(z)}(x)$ as well as for the finite-part integrals $I_{A^{(k)}(z)}(x)$ since the order of $A^{(k)}$ differs from that of $A(z)$ by an integer.

The map $z\mapsto I_{A^{(k)}(z)}(x)$ has a Laurent expansion $I_{A^{(k)}(z)}(x) = \sum_{j=1}^{k+1}\frac{b_j(x)}{(z-z_0)^j}+\sum_{k=0}^{K} c_k(x)\,\frac{(z-z_0)^k}{k!}+o((z-z_0)^k)$ and $(z-z_0)^{k+1}I_{A^{(k)}(z)}(x)$ can be extended to a holomorphic function in a small ball centered at z_0 with value $b_{k+1}(x)$ at z_0. Since property (1.12) holds for $I_{A^{(k)}(z)}(x)$ outside z_0 in this ball, it holds for the holomorphic extension on the whole ball and hence for $b_{k+1}(x)$. Using (1.12), we deduce that $b_{k+1}(x)\,dx$ is defined independently of the choice of local coordinates on M and so is the difference $\big(I_{A^{(k)}(z)}(x)-\frac{b_{k+1}(x)}{(z-z_0)^{k+1}}\big)dx$ for any z outside z_0 in a small ball centered at z_0. Iterating this argument, one shows recursively on the integer $1\le J\le k$ that $\big(I_{A^{(k)}(z)}(x)-\sum_{j=1}^{k+1-J}\frac{b_j(x)}{(z-z_0)^j}\big)dx$ is defined independently of the choice of local coordinates on M in a small ball centered at z_0. Consequently, the finite part $\big(\mathrm{fp}_{z=z_0}I_{A^{(k)}(z)}(x)\big)dx$ at z_0 is also defined independently of the choice of local coordinates. Since this finite part coincides with $k!\,c_k(x)$, we have that $c_k(x)dx$ is defined independently of the choice of local coordinates on M. □

Examining the singular and constant terms in the expansions of Theorem 1.20 we have the following corollaries.

First, the singular term yields the known identification of the residue trace with complex residue of the canonical trace, derived in [Gu], [W], [KV]. With the assumptions of Theorem 1.20:

COROLLARY 1.26. *The map $z \mapsto {-\!\!\!\!\!\!\int}_{T_x^*U}\sigma(z)(x,\xi)\bar{d}\xi$ is meromorphic with at most a simple pole at $z_0 \in P$ with complex residue*

$$\mathrm{Res}_{z=z_0}{-\!\!\!\!\!\!\int}_{T_x^*U}\sigma(z)(x,\xi)\bar{d}\xi = -\frac{1}{\alpha'(z_0)}\int_{S_x^*U}\sigma(z_0)_{-n}(x,\xi)\bar{d}_S\xi\,. \tag{1.48}$$

For the holomorphic family $z \mapsto A(z)$ of ψdos parametrized by W, the form $\frac{1}{\alpha'(z_0)}\int_{S_x^*U}(\sigma_{A(z_0)})_{-n}(x,\xi)\bar{d}_S\xi\,dx$ defines a global density on the manifold M and the map $z \mapsto \mathrm{TR}(A(z)) := \int_M dx\,\mathrm{TR}_x(A(z))$ is a meromorphic function with at most a simple pole at $z_0 \in P$ with complex residue

$$\mathrm{Res}_{z=z_0}\mathrm{TR}(A(z)) = -\frac{1}{\alpha'(z_0)}\,\mathrm{res}\,(A(z_0))\,. \tag{1.49}$$

Thus, consequent to Proposition 1.25, one infers here the global existence of the residue density for integer order operators from the existence of the canonical trace density for non-integer order operators and holomorphicity.

On the other hand, the constant term provides a *defect formula* for finite-part integrals.

With the assumptions of Theorem 1.20:

Theorem 1.27. *For a holomorphic family of symbols $z \mapsto \sigma(z) \in \mathrm{CS}(U,V)$ parametrized by a domain $W \subset \mathbb{C}$ and for any $x \in U$,*

$$\begin{aligned}\mathrm{fp}_{z=z_0}&{-\!\!\!\!\!\!\int}_{T_x^*U}\sigma(z)(x,\xi)\bar{d}\xi\\ &= {-\!\!\!\!\!\!\int}_{T_x^*U}\sigma(z_0)(x,\xi)\bar{d}\xi - \frac{1}{\alpha'(z_0)}\int_{S_x^*U}\sigma'(z_0)_{-n,0}(x,\xi)\bar{d}_S\xi\\ &\qquad + \frac{\alpha''(z_0)}{2\alpha'(z_0)^2}\int_{S_x^*U}\sigma(z_0)_{-n}(x,\xi)\bar{d}_S\xi\,.\end{aligned} \tag{1.50}$$

For the holomorphic family $z \mapsto A(z) \in \mathrm{Cl}(M,E)$ of ψdos parametrized by $W \subset \mathbb{C}$,

$$\begin{aligned}\mathrm{fp}_{z=z_0}\mathrm{TR}(A(z)) = \int_M dx\left(\mathrm{TR}_x\big(A(z_0)\big) - \frac{1}{\alpha'(z_0)}\,\mathrm{res}_{x,0}\big(A'(z_0)\big)\right)\\ + \frac{\alpha''(z_0)}{2\alpha'(z_0)^2}\,\mathrm{res}\big(A(z_0)\big)\,.\end{aligned} \tag{1.51}$$

REMARK 1.28. Since α is non-critical on P, from Proposition 1.17 if $z_0 \in P$ the operator $A'(z_0) \in \mathrm{Cl}^{\alpha(z_0),1}(M,E)$ in equation (1.51) is not classical.

REMARK 1.29. If $\mathrm{res}_{x,0}(A(z_0)) = 0$ then $\mathrm{fp}_{z=z_0}\mathrm{TR}_x(A(z)) = \lim_{z\to z_0}\mathrm{TR}_x(A(z))$. If this holds for all $x \in M$, then $\mathrm{TR}(A(z))$ is holomorphic at z_0 and $\mathrm{fp}_{z=z_0}\mathrm{TR}(A(z)) = \lim_{z\to z_0}\mathrm{TR}(A(z))$.

One therefore has the following statement on the existence of densities associated to the local canonical trace.

Theorem 1.30. *With the assumptions of Theorem 1.20, for a holomorphic family $z \mapsto A(z) \in \mathrm{Cl}(M,E)$ parametrized by a domain $W \subset \mathbb{C}$, and irrespective of the order $\alpha(z_0) \in \mathbb{R}$ of $A(z_0)$*

$$\left(\mathrm{TR}_x(A(z_0)) - \frac{1}{\alpha'(z_0)}\,\mathrm{res}_{x,0}(A'(z_0))\right) dx \tag{1.52}$$

defines a global density on M which integrates on M to $\mathrm{fp}_{z=z_0}\mathrm{TR}(A(z))$. If $\alpha(z_0) \notin \mathbb{Z}$ then (1.52) reduces to the canonical trace density on non-integer order classical ψdos of [KV].

Though this follows on the general grounds of Proposition 1.25, we have, for completeness, given a direct proof of Theorem 1.30 in Appendix A. This specializes to give the previously known existence of the canonical trace on non-integer order ψdos, recalled in section 1.2.

With the assumptions of Theorem 1.20:

Theorem 1.31. *Let $z \mapsto A(z) \in \mathrm{Cl}(M,E)$ be a holomorphic family of classical ψdos parametrised by $W \subset \mathbb{C}$ and let $z_0 \in W$. If either*

$$\mathrm{TR}_x(A(z_0))dx = \left(⨍_{T_x^*M} \mathrm{tr}_x(\sigma_{A(z_0)})(x,\xi)d\xi\right)dx$$

or

$$\mathrm{res}_x\big(A'(z_0)\big)dx := \int_{S_x^*M} \mathrm{tr}_x\big((\sigma_{A'(z_0)})_{-n}(x,\xi)\big)d_S\xi\, dx$$

defines a global density on M, then $\mathrm{TR}(A(z_0))$ and $\mathrm{res}(A'(z_0)) = \int_M \mathrm{res}_{x,0}(A'(z_0))dx$ are both well defined. The following defect formula then holds

$$\mathrm{fp}_{z=z_0}\mathrm{TR}(A(z)) = \mathrm{TR}(A(z_0)) - \frac{1}{\alpha'(z_0)}\,\mathrm{res}\big(A'(z_0)\big)\,. \tag{1.53}$$

This holds in the following cases:

(i) *If $A(z_0) \in \mathrm{Cl}^{\alpha(z_0),0}(M,E)$ satisfies one of the cases* (1), (2) *or* (3) *of Proposition 1.10 then $\mathrm{TR}(A(z_0))$ is defined and* (1.53) *holds. In case* (1) *this reduces to*

$$\mathrm{fp}_{z=z_0}\mathrm{TR}(A(z)) = \mathrm{TR}(A(z_0))\,. \tag{1.54}$$

(ii) *If $\mathrm{res}_{x,0}(A'(z_0)) = 0$ for all $x \in M$ then $\mathrm{TR}(A(z))$ is holomorphic at $z_0 \in W$, so that $\mathrm{fp}_{z=z_0}\mathrm{TR}(A(z)) = \lim_{z\to z_0}\mathrm{TR}(A(z))$, and* (1.54) *holds.*

(iii) *If $A(z_0)$ is a differential operator, and more generally whenever $\mathrm{TR}_x(A(z_0)) = 0$ for all $x \in M$,* (1.53) *reduces to*

$$\mathrm{fp}_{z=z_0}\mathrm{TR}(A(z)) = -\frac{1}{\alpha'(z_0)}\mathrm{res}\big(A'(z_0)\big)\,. \tag{1.55}$$

Remark 1.32. (1.53) can hold with both summands on the right-side of the equation non-zero. See Example 2.8.

Proof. The first statement is consequent to Theorem 1.30. Since $\mathrm{TR}_x(A(z_0))dx$ then defines a global density the transformation rule for finite-part integrals in Proposition 1.9 implies that

$$\int_{S_x^*M} \mathrm{tr}_x\big(\sigma_{A(z_0)}\big)_{-n,0}(x,\xi)\log|C^{-1}\xi|đ_S\xi = 0 \quad \forall\, C \in GL_n(\mathbb{C})\,,$$

and hence (taking $C = \lambda \cdot I$, $\lambda \in \mathbb{C}$) that

$$\mathrm{res}_{x,0}(A(z_0)) := \int_{S_x^*M} đ_S\xi\, \mathrm{tr}_x\left(\sigma_{A(z_0)}\right)_{-n,0}(x,\xi) = 0\,.$$

Equation (1.53) now follows from (1.51). Parts (i), (ii), (iii) are now obvious in view of Proposition 1.10 and Proposition 1.12 and the vanishing of the residue trace on non-integer order operators and on differential operators. □

2 Application to the Complex Powers

An operator $Q \in \mathrm{Ell}(M,E)$ of positive order is called *admissible* if there is a proper subsector of $\mathbb{C}$ with vertex 0 which contains the spectrum of the leading symbol $\sigma_L(Q)$ of Q. Then there is a half line $L_\theta = \{re^{i\theta}, r > 0\}$ (a spectral cut) with vertex 0 and determined by an Agmon angle θ which does not intersect the spectrum of Q. Let $\mathrm{Ell}^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ denote the subset of admissible operators in $\mathrm{Ell}(M,E)$ with positive order.

Let $Q \in \mathrm{Ell}^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ with spectral cut L_θ. For Re $z < 0$, the complex power Q_θ^z of Q is a bounded operator on any space $H^s(M,E)$ of sections of E of Sobolev class H^s defined by the contour integral:

$$Q_\theta^z = \frac{i}{2\pi}\int_{C_\theta} \lambda^z (Q - \lambda I)^{-1} d\lambda \tag{2.1}$$

where $C_\theta = C_{1,\theta,r} \cup C_{2,\theta,r} \cup C_{3,\theta,r}$. Here r is a sufficiently small positive number and $C_{1,\theta,r} = \{\lambda = |\lambda|e^{i\theta} \mid +\infty > |\lambda| \geq r\}$, $C_{2,\theta,r} = \{\lambda = re^{i\phi} \mid \theta \geq \phi \geq \theta - 2\pi\}$ and $C_{3,\theta,r} = \{\lambda = |\lambda|e^{i(\theta-2\pi)} \mid r \leq |\lambda| < +\infty\}$. Here $\lambda^z = \exp(z\log\lambda)$ where $\log\lambda = \log|\lambda| + i\theta$ on $C_{1,\theta,r}$ and $\log\lambda = \log|\lambda| + i(\theta - 2\pi)$ on $C_{3,\theta,r}$.

For $k \in \mathbf{N}$ the complex power Q^z is then extended to the half plane Re $z < k$ via the relation [Se1]

$$Q^k Q_\theta^{z-k} = Q_\theta^z\,.$$

The definition of a complex power depends in general on the choice of θ and yields for any $z \in \mathbb{C}$ an elliptic operator Q_θ^z of order $z \cdot \mathrm{ord}(Q)$. In spite of this θ-dependence, we may occasionally omit it in order to simplify notation.

Remark 2.1. For $z = 0$,

$$Q_\theta^0 = I - \Pi_Q\,,$$

where Π_Q is the smoothing operator projection

$$\Pi_Q = \frac{i}{2\pi}\int_{C_0} (Q-\lambda I)^{-1} d\lambda\,,$$

with C_0 a contour containing the origin but no other element of $\operatorname{spec}(Q)$, with range the generalized kernel $\{\psi \in C^\infty(M,E) \mid Q^N\psi = 0 \text{ for some } N \in \mathbb{N}\}$ of Q. (See [B], [W], presented recently in [Po]).

Let $Q \in \mathrm{Ell}^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ be of order q with spectral cut L_θ. For arbitrary $k \in \mathbb{Z}$, the map $z \to Q_\theta^z$ defines a holomorphic function from $\{z \in \mathbb{C}, \operatorname{Re} z < k\}$ to the space $\mathcal{L}(H^s(M,E) \to H^{s-k\cdot q}(M,E))$ of bounded linear maps and we can set

$$\log_\theta Q := \left[\frac{\partial}{\partial z} Q_\theta^z\right]_{z=0}.$$

From (1.22), in a local trivialisation $E_{|_U} \simeq U \times V$ of E over an open set U of M the symbol of $\log_\theta Q$ reads $\sigma_{\log_\theta Q}(x,\xi) = \operatorname{ord}(Q)\log|\xi|\mathrm{Id} + \rho(x,\xi)$ with $\rho \in \mathrm{Cl}^0(U,V)$, and so $\log_\theta Q \in \mathrm{Cl}^{0,1}(M,E)$ has order zero and log degree one. The logarithmic dependence is slight, for $P, Q \in \mathrm{Ell}^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$, of non-zero order p, q respectively and admitting spectral cuts L_θ and L_ϕ we have $\frac{\log_\theta P}{p} - \frac{\log_\phi Q}{q} \in \mathrm{Cl}^0(M,E)$. More generally, higher derivatives of the complex powers have symbols with polynomial powers of $\log|\xi|$ and it follows from Proposition 1.17 that

$$\log_\theta^k Q := \left[\frac{\partial^k}{\partial z^k} Q_\theta^z\right]_{z=0} \in \mathrm{Cl}^{0,k}(M,E)\,. \tag{2.2}$$

Theorem 1.20 leads to the following Laurent expansion.

Theorem 2.2. *Let $Q \in \mathrm{Ell}^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ with spectral cut θ and of order q and let $A \in \mathrm{Cl}^\alpha(M,E)$. On the half plane $\operatorname{Re}(z) > (\alpha+n)/q$ the local Schwartz kernel $K_{AQ_\theta^{-z}}(x,y)$ of AQ_θ^{-z} is well defined and holomorphic and the restriction to the diagonal $K_{AQ_\theta^{-z}}(x,x)dx = \int_{T_x^*M} \sigma_{AQ_\theta^{-z}}(x,\xi)đ\xi\, dx$ defines a global density, an element of $C^\infty(M,\mathrm{End}(E))$. There is a meromorphic extension of $K_{AQ_\theta^{-z}}(x,y)$ to all $z \in \mathbb{C}$,*

$$K_{AQ_\theta^{-z}}(x,x)|^{\mathrm{mer}} := ⨍_{T_x^*M} \sigma_{AQ_\theta^{-z}}(x,\xi)đ\xi\,, \tag{2.3}$$

with at most simple poles, each of which is located in $P := \{(\alpha - j)/q \mid j \in [-n, \infty[\cap \mathbb{Z}\}$. *For any* $x \in M$, *we have for* z *near* $(\alpha - j)/q \in P$

$$\begin{aligned}
K_{AQ_\theta^{-z}}(x,x)|^{\mathrm{mer}} &= \frac{1}{q}\int_{S_x^*M} \big(\sigma_{AQ_\theta^{(j-\alpha)/q}}\big)_{-n}(x,\xi)\,đ_S\xi \cdot \frac{1}{\big(z - \frac{\alpha-j}{q}\big)} \\
&+ \sum_{k=0}^{K} \frac{(-1)^k}{k!}\left(z - \frac{\alpha - j}{q}\right)^k \times \bigg(⨍_{T_x^*U} \sigma_{AQ_\theta^{(j-\alpha)/q}\log_\theta^k Q}(x,\xi)\,đ\xi \\
&\quad - \frac{1}{q(k+1)}\int_{S_x^*M} \big(\sigma_{AQ_\theta^{(j-\alpha)/q}\log_\theta^{k+1} Q}\big)_{-n,0}(x,\xi)\,đ_S\xi\bigg) \\
&\qquad + o\bigg(\left(z - \frac{\alpha-j}{q}\right)^K\bigg). \qquad (2.4)
\end{aligned}$$

It follows that the map $z \mapsto \mathrm{TR}(AQ_\theta^{-z}) := \int_M \mathrm{tr}_x(K_{AQ_\theta^{-z}}(x)|^{\mathrm{mer}})$ *is a meromorphic function with no more than simple poles located in* P, *and for* z *near* $(\alpha - j)/q \in P$

$$\begin{aligned}
\mathrm{TR}(AQ_\theta^{-z}) &= \frac{1}{q}\,\mathrm{res}(AQ_\theta^{\frac{j-\alpha}{q}}) \cdot \frac{1}{\big(z - \frac{\alpha-j}{q}\big)} + \sum_{k=0}^{K}\frac{(-1)^k}{k!}\left(z - \frac{\alpha-j}{q}\right)^k \\
&\times \int_M dx \left(\mathrm{TR}_x\big(AQ_\theta^{\frac{j-\alpha}{q}}\log_\theta^k Q\big) - \frac{1}{q(k+1)}\,\mathrm{res}_{x,0}(AQ_\theta^{\frac{j-\alpha}{q}}\log_\theta^{k+1} Q)\right) \\
&\qquad + o\bigg(\left(z - \frac{\alpha-j}{q}\right)^K\bigg). \qquad (2.5)
\end{aligned}$$

If $z_0 \notin P$ *then* $\mathrm{TR}(AQ_\theta^{-z})$ *is holomorphic at* z_0 *and for* z *in a small enough neighbourhood of* z_0

$$\mathrm{TR}(AQ_\theta^{-z}) = \sum_{k=0}^{K}\frac{(-1)^k}{k!}\,\mathrm{TR}(AQ_\theta^{z_0}\log_\theta^k Q)\,\frac{(z-z_0)^k}{k!} + o\big((z-z_0)^K\big)\,. \quad (2.6)$$

Proof. Since $\sigma(z) := \sigma_{AQ_\theta^{-z}}$ has order $\alpha(z) = \alpha - qz$, (1.27) of Theorem 1.20 can be applied to equation (2.3). Using (2.2) and Example 1.18, this yields (2.4). Applying the fibrewise trace tr_x and integrating over M yields equation (2.5). Equation (2.6), to which (2.5) reduces when $\alpha \notin \mathbb{Z}$, as the operators inside the local residue traces then have non-integer order, follows from (1.31); that $\mathrm{TR}_x(AQ_\theta^{z_0}\log_\theta^k Q)\,dx$ defines a global density on M in this case is known from [L]. □

Because of the identity with the generalized zeta-function

$$\zeta_\theta(A, Q, z) = \mathrm{TR}(AQ_\theta^{-z})$$

the expansion (2.5) is of particular interest near $z=0$, owing to the role of the Laurent coefficients there in geometric analysis.

Theorem 2.3. *If* $\mathrm{ord}(A)=\alpha\in[-n,\infty[\cap\mathbb{Z}$ *then* $0\in P$ *and one then has, near* $z=0$,

$$\zeta_\theta(A,Q,z)=\tfrac{1}{q}\,\mathrm{res}(A)\cdot\tfrac{1}{z}+\int_M dx\,\Big(\mathrm{TR}_x(A)-\tfrac{1}{q}\,\mathrm{res}_{x,0}(A\log_\theta Q)\Big)-\mathrm{tr}(A\,\Pi_Q)$$
$$+\sum_{k=1}^{K}(-1)^k\,\frac{z^k}{k!}\times\int_M dx\bigg(\mathrm{TR}_x(A\log_\theta^k Q)-\frac{1}{q(k+1)}\,\mathrm{res}_{x,0}(A\log_\theta^{k+1}Q)\bigg)$$
$$-\,\mathrm{tr}(A\log_\theta^k Q\Pi_Q)+o(z^K)\,. \tag{2.7}$$

If $\alpha\notin[-n,\infty[\cap\mathbb{Z}$ *then* $\zeta_\theta(A,Q,z)$ *is holomorphic at zero and one has for* z *near zero,*

$$\zeta_\theta(A,Q,z)=\sum_{k=0}^{K}(-1)^k\big(\mathrm{TR}(A\log_\theta^k Q)-\mathrm{tr}(A\log_\theta^k Q\Pi_Q)\big)\frac{z^k}{k!}+o(z^K)\,. \tag{2.8}$$

Remark 2.4. The formula (2.8) can also be deduced from exact formulas for the case $\alpha\notin\mathbb{Z}$ in [Gru1, §3]. All formulas presuppose the existence shown in [KV], [L] of the canonical trace for non-integer order ψdos with log-polyhomogeneous symbol.

Proof. The assumption $\alpha\in[-n,\infty[\cap\mathbb{Z}$ means that $(\alpha-j_0)/q=0$ for some $j_0\in[-n,\infty[\cap\mathbb{Z}$. Hence (2.7) is almost obvious from (2.5); the subtle point is to take care to replace $Q_\theta^{-(\alpha-j)/q}=Q_\theta^0$ by $I-\Pi_Q$, see Remark 2.1. Since the spectral projection Π_Q is a smoothing operator the term $\mathrm{TR}_x(A\,\log_\theta^k Q\,\Pi_Q)dx$ is an ordinary integral valued density and globally defined, yielding the term $\mathrm{tr}(A\,\log_\theta^k Q\,\Pi_Q)$. The formula (2.8) for A of non-integer order (to which (2.7) reduces in this case) is immediate from (2.6). □

We denote the coefficient of $\big(z-\frac{\alpha-j}{q}\big)^k/k!$ in the Laurent expansion of the generalized zeta function at $(\alpha-j)/q\in P$ by $\zeta_\theta^{(k)}(A,Q,(\alpha-j)/q)$. In the case $k=0$, we use the simpler convention of writing the constant term $\zeta_\theta^{(0)}(A,Q,(\alpha-j)/q):=\mathrm{fp}_{z=(\alpha-j)/q}\,\zeta_\theta(A,Q,z)$ as $\zeta_\theta(A,Q,(\alpha-j)/q)$. When $A=I$ write $\zeta_\theta(Q,(\alpha-j)/q):=\zeta_\theta(I,Q,(\alpha-j)/q)$.

Corollary 2.5. *For any operator* $A\in\mathrm{Cl}(M,E)$,

$$\zeta_\theta(A,Q,0)=\int_M dx\left(\mathrm{TR}_x(A)-\frac{1}{q}\,\mathrm{res}_{x,0}(A\log_\theta Q)\right)-\mathrm{tr}(A\,\Pi_Q)\,. \tag{2.9}$$

More generally, for any non-negative integer k

$$\zeta_\theta^{(k)}(A,Q,0) = (-1)^k \int_M dx \Big(\mathrm{TR}_x(A \log_\theta^k Q) - \frac{1}{q(k+1)} \,\mathrm{res}_{x,0}(A \log_\theta^{k+1} Q) \Big) + (-1)^{k+1} \,\mathrm{tr}(A \log_\theta^k Q \Pi_Q)\,. \quad (2.10)$$

If A has integer order $\alpha \in [-n,\infty) \cap \mathbb{Z}$ then

$$\zeta_\theta\left(A, Q, \frac{\alpha - j}{q}\right) = \int_M dx \left(\mathrm{TR}_x(A Q_\theta^{-\frac{\alpha-j}{q}}) - \frac{1}{q} \,\mathrm{res}_{x,0}\big(A Q_\theta^{-\frac{\alpha-j}{q}} \log_\theta Q\big) \right)$$

Applied to the complex powers, the general statement on the existence of densities associated to the canonical and residue traces of Theorem 1.30 now states that independently of the order of $A \in \mathrm{Cl}(M,E)$,

$$\Big(\mathrm{TR}_x(A) - \tfrac{1}{q} \,\mathrm{res}_{x,0}(A \log_\theta Q) \Big)\, dx$$

always defines a global density on M.

If A has non-integer order this reduces to the KV canonical trace density and (by (2.8)) the identity (2.10) loses its residue defect term and one then has the known formula (cf. [Gru1, Cor. 3.8])

$$\zeta_\theta^{(k)}(A,Q,0) = (-1)^k \mathrm{TR}(A \log_\theta^k Q) - (-1)^k \,\mathrm{tr}(A \log_\theta^k Q \Pi_Q)\,. \quad (2.11)$$

Applying Theorem 1.31 to the zeta function at $z = 0$ yields the following refinement of (2.9).

Theorem 2.6. *Let $Q \in \mathrm{Ell}^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ be a classical ψdo with spectral cut θ and of order q, and let $A \in \mathrm{Cl}^\alpha(M,E)$ be a classical ψdo of order α. If either $\mathrm{TR}_x(A)dx$ or $\mathrm{res}_x(A \log_\theta Q)dx$ defines a global density on M, then $\zeta_\theta(A,Q,z)$ is holomorphic at $z = 0$, $\mathrm{TR}(A)$ and $\mathrm{res}(A \log_\theta Q)$ both exist, and one has*

$$\zeta_\theta(A,Q,0) = \mathrm{TR}(A) - \tfrac{1}{q} \,\mathrm{res}(A \log_\theta Q) - \mathrm{tr}(A \Pi_Q)\,. \quad (2.12)$$

Proof. If $\mathrm{TR}_x(A)dx$ defines a global density then $\mathrm{res}(A)$ vanishes, as accounted for in the proof of Theorem 1.31, and so $\zeta_\theta(A,Q,z)$ is holomorphic at $z = 0$. The formula is obvious from (1.53). □

Notice that the assumptions of Theorem 2.6 also force $\mathrm{res}(A) = 0$.

The situation of Theorem 2.6 can be seen to hold for certain combinations of even-even and even-odd ψdos. First, it holds in the following circumstances.

Corollary 2.7. (i) *If A satisfies one of the cases* (1), (2) *or* (3) *of Proposition 1.10 then* $\mathrm{TR}(A)$ *is defined and* (2.12) *holds. In case* (1) *this reduces to*

$$\zeta_\theta(A,Q,0) = \mathrm{TR}(A) - \mathrm{tr}(A \Pi_Q)\,. \quad (2.13)$$

If Q is an even-even operator and has even order, then (2.13) *also holds when A satisfies case* (2) (*assume M is odd-dimensional*) *or* (3) (*assume*

M is even-dimensional) of Proposition 1.10. These facts are known from [Gru1].

(ii) *If A is a differential operator, and more generally whenever $\mathrm{TR}_x(A) = 0$ for all $x \in M$, (2.12) reduces to*

$$\zeta_\theta(A, Q, 0) = -\tfrac{1}{q}\,\mathrm{res}(A \log_\theta Q) - \mathrm{tr}(A\Pi_Q)\,. \tag{2.14}$$

Proof. Part (ii) follows from Proposition 1.12. For part (i), it is clear that (2.13) holds when $\mathrm{res}_x(A\log_\theta Q) = 0$ for each $x \in M$. This is evident for case (1) operators. If A satisfies case (2) (resp. case (3)) of Proposition 1.10 and if Q is even-even and of even order, then it is not hard to see that $\sigma_{A\log_\theta Q}(x, \xi)$ is also even-even (resp. even-odd) and hence $(\sigma_{A\log_\theta Q})_{-n,0}(x, \xi)$ vanishes when integrated over the $n-1$ sphere. □

Example 2.8. To see that (2.12) may hold with all three terms non-zero, take $A = D + S$ with D a differential operator and S a smoothing operator, and let $Q \in \mathrm{Ell}^{\mathrm{adm}}_{\mathrm{ord}>0}(M, E)$. Then $\mathrm{TR}(A) = \mathrm{tr}(S)$ and $\mathrm{res}(A\log_\theta Q) = \mathrm{res}(D \log_\theta Q)$ both exist (note Corollary 2.7 (ii)) and are non-zero in general. For example, if $Q = D \in \mathrm{Ell}^{\mathrm{adm}}_{\mathrm{ord}>0}(M, E)$ is invertible one has $\mathrm{res}(D\log_\theta D) = -\,\zeta_\theta(D, -1)$.

Remark 2.9. In Corollary 2.7 (i), if Q has odd order then (2.12) may hold with all three terms non-zero due to dependence on the choice of the spectral cut. The distinct behaviour for odd-order Q was kindly pointed out to the authors by Gerd Grubb.

Remark 2.10. Using Theorem 1.31 similar facts to those in Corollary 2.7 can be seen to hold for the $\zeta_\theta^{(k)}(A, Q, (\alpha - j)/q)$, see also [Gru1, §3]. The regularity of $\zeta_\theta(A, Q, z)$ at $z = 0$ in (ii) is proved in [GruS]. When $A = I$ the identity (2.14) was shown in [S]. On the other hand, when Q is a differential operator and taking $A = Q^m$ in (2.14) gives

$$\zeta_\theta(Q, -m) = -\tfrac{1}{q}\,\mathrm{res}(Q^m \log_\theta Q) - \mathrm{tr}(Q^m\Pi_Q)\,, \tag{2.15}$$

which was obtained in the case when Q is positive and invertible by other methods in [Lo]. Note that for sufficiently large m one has $\mathrm{tr}(Q^m\,\Pi_Q) = 0$.

Looking at the next term up in the Laurent expansion, around $z = 0$ the zeta function $\zeta_\theta(Q, z) = \mathrm{TR}(Q_\theta^{-z})$ is holomorphic and hence the ζ-determinant

$$\det_{\zeta,\theta} Q = \exp\big(-\zeta'_\theta(Q, 0)\big)\,,$$

is defined, where $\zeta'_\theta(Q, 0) = \partial_z \zeta_\theta(Q, z))_{|z=0}$.

Theorem 2.11. *One has*

$$\log \det_{\zeta,\theta}(Q) = \int_M dx \left(\mathrm{TR}_x(\log_\theta Q) - \tfrac{1}{2q}\,\mathrm{res}_{x,0}(\log^2_\theta Q) \right) - \mathrm{tr}(\log_\theta Q \Pi_Q)\,. \tag{2.16}$$

If M is odd-dimensional and Q is an even-even operator and has even order then one has (as known from [O2], [Gru1, §3], see also [KV, §4])

$$\log \det_{\zeta,\theta}(Q) = \mathrm{TR}(\log_\theta Q) - \mathrm{tr}(\log_\theta Q \Pi_Q)\,, \tag{2.17}$$

where $\mathrm{TR}(\log_\theta Q) = \int_M \mathrm{TR}_x(\log_\theta Q)dx$,

Proof. Examining the coefficient of z in the Laurent expansion (2.7) immediately yields (2.16). If Q is even-even and of even order then the classical component of the local symbol of $\log^2_\theta Q \in \mathrm{Cl}^{0,2}(M,E)$ also has even-even parity. Hence the local residue integral of the term of homogeneity $-n$ then vanishes, $\mathrm{TR}_x\,(\log_\theta Q)\,dx$ defines a global density on M, and (2.16) reduces to (2.17). □

2.1 The canonical trace on commutators and the residue trace on logarithms. The canonical trace TR is not defined on a commutator of classical ψdos which has integer order. Rather the following property holds.

Theorem 2.12. *Let $Q \in \mathrm{Ell}^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ be of order q and with spectral cut θ, and let $A \in \mathrm{Cl}^\alpha(M,E)$, $B \in \mathrm{Cl}^\beta(M,E)$ for any $\alpha, \beta \in \mathbb{R}$. Then*

$$\left(\mathrm{TR}_x\big([A,B]\big) - \tfrac{1}{q}\,\mathrm{res}_{x,0}\big([A, B\log_\theta Q]\big) \right) dx$$

defines a global density on M and one has

$$\int_M dx \left(\mathrm{TR}_x\big([A,B]\big) - \tfrac{1}{q}\,\mathrm{res}_{x,0}\big([A, B\log_\theta Q]\big) \right) = 0\,, \tag{2.18}$$

independently of the choice of Q.

Proof. Using the vanishing of TR in Proposition 1.13 (1), for $z \neq 0$ sufficiently close to 0 we have

$$\mathrm{TR}\big([A, BQ_\theta^{-z}]\big) = 0\,. \tag{2.19}$$

Hence the function $z\mathrm{TR}([A, BQ_\theta^{-z}])$ also vanishes identically for such nonzero z. But from (2.7), $z\mathrm{TR}([A, BQ_\theta^{-z}])$ extends holomorphically to include $z = 0$. By equation (2.19), this analytically continued function must also vanish at $z = 0$. It follows that $\mathrm{TR}([A, BQ_\theta^{-z}])$ is holomorphic near $z = 0$ and so (2.7) implies $\mathrm{fp}_{z=0}\mathrm{TR}([A, BQ_\theta^{-z}]) = \lim_{z\to 0}\mathrm{TR}([A, BQ_\theta^{-z}]) = 0$. Applying Proposition 1.31 to $A(z) = [A, BQ_\theta^{-z}]$ with $z_0 = 0$, we have by Theorem 1.27

$$0 = \mathrm{fp}_{z=0}\mathrm{TR}\big([A, BQ_\theta^{-z}]\big)$$

$$= \int_M dx \left(\mathrm{TR}_x\big([A, B(I-\Pi_Q)]\big) + \frac{1}{q}\,\mathrm{res}_{x,0}\big([A, B\log_\theta Q]\big)\right) ,$$

which is equation (2.18), since $\mathrm{TR}\left([A, B\Pi_Q)]\right) = \mathrm{tr}\left([A, B\Pi_Q)]\right) = 0$. □

Corollary 2.13. *Let $Q \in \mathrm{Ell}^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ be of order q and with spectral cut θ, and let $A \in \mathrm{Cl}^\alpha(M,E)$, $B \in \mathrm{Cl}^\beta(M,E)$. Then in cases* (1), (2) *and* (3) *of Proposition 1.13 the form $\mathrm{res}_x\left([A, B\log_\theta Q]\right) dx$ determines a global density on M and one has*

$$\mathrm{res}\big([A, B\log_\theta Q]\big) = 0\,,$$

independently of the choice of Q.

Remark 2.14. The independence from Q can also be seen for the residue trace term directly; given $Q_1, Q_2 \in \mathrm{Ell}^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ of order q_1 and q_2 respectively with common spectral cut θ, the difference

$$\Big(\tfrac{1}{q_1}\,\mathrm{res}_{x,0}\big([A, B\log_\theta Q_1]\big) - \tfrac{1}{q_2}\,\mathrm{res}_{x,0}\big([A, B\log_\theta Q_2]\big)\Big)\, dx$$

defines a global density which integrates to

$$\mathrm{res}\left(\left[A, B\left(\frac{\log_\theta Q_1}{q_1} - \frac{\log_\theta Q_2}{q_2}\right)\right]\right) = 0$$

since $\frac{\log_\theta Q_1}{q_1} - \frac{\log_\theta Q_2}{q_2}$ is a classical ψdo.

A useful consequence of Theorem 2.12 and Proposition 1.12 is

Corollary 2.15. *Let $Q \in \mathrm{Ell}^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ of order q and with spectral cut θ, and let $A, B \in \mathrm{Cl}(M,E)$. Whenever $\mathrm{TR}([A,B]) = \int_M dx \mathrm{TR}_x([A,B])$ is well defined then $\mathrm{res}_{x,0}\left([A, B\log_\theta Q]\right) dx$ is globally defined and one then has*

$$\mathrm{res}\big([A, B\log_\theta Q]\big) = q\,\mathrm{TR}\big([A,B]\big)\,. \tag{2.20}$$

In particular, if $[A,B]$ is a differential operator then $\mathrm{res}_{x,0}\left([A, B\log_\theta Q]\right) dx$ is globally defined and one has

$$\mathrm{res}\big([A, B\log_\theta Q]\big) = 0\,.$$

In that case, whenever $\mathrm{res}_{x,0}(AB\log_\theta Q)dx$ defines a global density, then so does $\mathrm{res}_{x,0}(B\log_\theta QA)dx$ and

$$\mathrm{res}(B\log_\theta QA) = \mathrm{res}(AB\log_\theta Q)\,.$$

In particular, since $\mathrm{res}\left(\log_\theta Q\right)$ exists [O1], for any invertible $A \in \mathrm{Cl}(M,E)$

$$\mathrm{res}\left(A^{-1}\log_\theta QA\right) = \mathrm{res}\left(\log_\theta Q\right)\,. \tag{2.21}$$

Remark 2.16. This proposition partially generalizes the fact [O1] that $\mathrm{res}_{x,0}\left([A, \log_\theta Q]\right) dx$ for A a classical ψdo defines a global density and $\mathrm{res}\left([A, \log_\theta Q]\right) = 0$, which when A is a differential operator follows from the corollary applied to $B = I$.

On the other hand, the well-known ([MN], [O1], [CDMP], [Gru2]) trace defect formula

$$\zeta_\theta\big([A,B],Q,0\big) = -\tfrac{1}{q}\operatorname{res}\big(A[B,\log_\theta Q]\big)\,. \tag{2.22}$$

follows easily by applying the same argument as in the proof of Theorem 2.12 to $C(z) = A\,[B,Q^{-z}]$. From (2.18) and (2.22) we infer

Corollary 2.17. *For classical* ψdos A *and* B

$$-\frac{1}{q}\operatorname{res}\big(A[B,\log_\theta Q]\big) = \int_M dx\left(\operatorname{TR}_x\big([A,B]\big) - \frac{1}{q}\operatorname{res}_x\big([A,B]\log_\theta Q\big)\right).$$

In cases (1), (2) *and* (3) *of Proposition 1.13 the form* $\operatorname{res}_x\left([A,B]\log_\theta Q\right)dx$ *determines a global density on* M *and one has*

$$\operatorname{res}\big(A[B,\log_\theta Q]\big) = \operatorname{res}\big([A,B]\log_\theta Q\big)\,.$$

While from Proposition 2.12 we conclude

Corollary 2.18. *The density* $\operatorname{res}_x\left([A,B\log_\theta Q] - [A,B]\log_\theta Q\right)dx$ *is globally defined on* M *for classical* ψdos A *and* B *and one has*

$$\operatorname{res}\big(A[B,\log_\theta Q]\big) = \operatorname{res}\big([A,B]\log_\theta Q - [A,B\log_\theta Q]\big)$$

Proof.

$$\begin{aligned}\frac{1}{q}\operatorname{res}\big(A[B,\log_\theta Q]\big) &= -\int_M dx\left(\operatorname{TR}_x\big([A,B]\big) - \frac{1}{q}\operatorname{res}_x\big([A,B]\log_\theta Q\big)\right)\\ &= -\int_M dx\Big(\operatorname{TR}_x\big([A,B]\big) - \frac{1}{q}\operatorname{res}_x\big([A,B\log_\theta Q]\big)\\ &\qquad + \frac{1}{q}\operatorname{res}_x\big([A,B]\log_\theta Q - [A,B\log_\theta Q]\big)\Big)\\ &= \frac{1}{q}\int_M dx\operatorname{res}_x\big([A,B]\log_\theta Q - [A,B\log_\theta Q]\big)\\ &= \frac{1}{q}\operatorname{res}\big([A,B]\log_\theta Q - [A,B\log_\theta Q]\big)\end{aligned}$$

□

We point out that Corollary 2.19 and (2.22) imply the following local index formulae.

Corollary 2.19. *Let* A *be an elliptic* ψdo *with parametrix* B. *Let* $Q \in \operatorname{Ell}^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ *be of order* q *and with spectral cut* θ. *Then, independently of the choice of* Q,

$$\operatorname{res}\big([A,B\log_\theta Q]\big) = \operatorname{res}\big(A[B,\log_\theta Q]\big)\,, \tag{2.23}$$

and are equal to $-q\,\operatorname{index}(A)$.

Proof. In this case $\operatorname{index}(A) = \operatorname{tr}([A,B])$ and since $[A,B]$ is smoothing equal to $\operatorname{TR}([A,B])$. The first equality thus follows from (2.20). Since

$AB = I + S$ where S is a smoothing operator, and since $\mathrm{res}_{x,0}(S \log_\theta Q)$ is therefore equal to zero, the second equality also follows. □

Appendix A: Proof of the Density Formula

The purpose here is to give a direct elementary proof of Theorem 1.30, which for the family $z \mapsto A(z) \in \mathrm{Cl}(M, E)$ parametrized by a domain $W \subset \mathbb{C}$ states that, irrespective of the order $\alpha(z_0) \in \mathbb{R}$ of $A(z_0)$,

$$\left(\mathrm{TR}_x(A) - \tfrac{1}{\alpha'} \mathrm{res}_{x,0}(A')\right) dx \tag{2.24}$$

defines a global density on M. Here, we have written $A = A(z_0)$, $A' = A'(z_0) := d/dz|_{z=z_0}(A(z))$, and $\alpha' = \alpha'(z_0)$.

From previous works [KV], it is known that $\mathrm{TR}_x(A(z_0))dx$ defines a global density on M when $\alpha(z_0)$ is not integer valued; this follows immediately from (2.24) and Proposition 1.17.

The method of proof uses a generalization of the method used in [O1] to show that the residue density is globally defined for any classical ψdo, and the method in [L] used to show that the canonical density is globally defined for classical ψdos of non-integer order. We will take A to be scalar valued for notational brevity, but the proof works in the same way for endomorphism valued operators; indeed it works equally for the pre-tracial density $(⨍_{T_xM} \sigma_A(x,\xi) đ\xi - \frac{1}{\alpha'} \int_{S_x^*M} (\sigma_{A'})_{-n,0}(x,\xi) đ_S\xi) dx$.

First, we have a lemma, generalizing Lemma C.1 in [O1].

Lemma 2.20. *Let $f(\xi)$ be a smooth function on $\mathbb{R}^n$ which is homogeneous of degree $-n$ for $|\xi| \geq 1$ and let T be an invertible linear map on $\mathbb{R}^n$. Then for $s \in \mathbb{C}$ and any non-negative integer k*

$$\int_{|\eta|=1} f(T\eta)|T\eta|^s \log^k |T\eta| đ_S\eta = \frac{(-1)^k}{|\det T|} \int_{|\xi|=1} f(\xi)|T^{-1}\xi|^{-s} \log^k |T^{-1}\xi| đ_S\xi\,.$$

Specifically, one has

$$\int_{|\eta|=1} f(T\eta) \log |T\eta| đ_S\eta = \frac{-1}{|\det T|} \int_{|\xi|=1} f(\xi) \log |T^{-1}\xi| đ_S\xi\,, \tag{2.25}$$

$$\int_{|\eta|=1} f(T\eta) đ_S\eta = \frac{1}{|\det T|} \int_{|\xi|=1} f(\xi) đ_S\xi\,. \tag{2.26}$$

Proof. It is enough to prove this for $k = 0$, differentiation with respect to s yields the general formula. We have, using the linearity of T,

$$\int_{1\leq|\eta|\leq 2} f(T\eta)|T\eta|^s d\eta = \int_{|\eta|=1} \int_{1\leq r\leq 2} f(rT\eta) r^s |T\eta|^s r^{n-1} dr\, đ_S\eta$$

$$= \left(\frac{2^s-1}{s}\right)\int_{|\eta|=1} f(T\eta)|T\eta|^s đ_S\eta\,.$$

On the other hand, changing variable,

$$\int_{1\le|\eta|\le 2} f(T\eta)|T\eta|^s d\eta = \frac{1}{|\det T|}\int_{1\le|T^{-1}\eta|\le 2} f(\eta)|\eta|^s d\eta$$
$$= \frac{1}{|\det T|}\int_{|\eta|=1}\int_{1/|T^{-1}\eta|\le r\le 2/|T^{-1}\eta|} f(r\eta)r^s|\eta|^s r^{n-1}dr\,đ_S\eta$$
$$= \frac{1}{|\det T|}\left(\frac{2^s-1}{s}\right)\int_{|\eta|=1} f(\eta)|T^{-1}\eta|^{-s}đ_S\eta\,. \qquad \square$$

Consider now a local chart on M defined by a diffeomorphism $x:\Omega\to U$ from an open subset Ω of M to an open subset U of $\mathbb{R}^n$. For $p\in\Omega$ we then have the local coordinate $x(p)\in\mathbb{R}^n$. Let $\kappa:U\to V$ be a diffeomorphism to a second open subset V of $\mathbb{R}^n$. Then $y(p)=\kappa(x(p))$ is also a local coordinate for Ω.

Let $a(x(p),\xi)=\tilde a(x(p),x(p),\xi)$ where $\tilde a(x(p),y(p),\xi)$ denotes the local amplitude of A in x-coordinates, and likewise let $b(y(p),\xi)$ denote the amplitude along the diagonal in y-coordinates. From [Hö] with $T(p):=(D\kappa_{x(p)})^t$ we have

$$\mathrm{TR}_{y(p)}(A)dy(p) := ⨍_{\mathbb{R}^n} b\big(y(p),\xi\big)đ\xi\, dy(p)$$
$$= ⨍_{\mathbb{R}^n} a\big(x(p),T(p)\xi\big)đ\xi\, dy(p)\,.$$

According to the transformation rule in Proposition 1.9, for $f\in \mathrm{CS}(V)$ and T an invertible linear map on $\mathbb{R}^n$,

$$⨍_{\mathbb{R}^n} f(T\xi)đ\xi = \frac{1}{|\det T|}\left(⨍_{\mathbb{R}^n} f(\xi)đ\xi - \int_{|\xi|=1} f(\xi)_{(-n)}\log|T^{-1}\xi|đ\xi\right),$$

with $f(\xi)_{(-n)}$ the homogeneous component of f of degree $-n$. Hence

$$\mathrm{TR}_{y(p)}(A)dy(p)$$
$$= \frac{1}{|\det T(p)|}\Bigg(⨍_{\mathbb{R}^n} a\big(x(p),\xi\big)đ\xi\, dy(p) - \int_{|\xi|=1} a\big(x(p),\xi\big)_{(-n)}\log\big|T(p)^{-1}\xi\big|đ\xi\, dy(p)\Bigg)$$
$$= ⨍_{\mathbb{R}^n} a\big(x(p),\xi\big)đ\xi\, dx(p) - \int_{|\xi|=1} a\big(x(p),\xi\big)_{(-n)}\log\big|T(p)^{-1}\xi\big|đ\xi\, dx(p)$$
$$= \mathrm{TR}_{x(p)}(A)dx(p) - \int_{|\xi|=1} a\big(x(p),\xi\big)_{(-n)}\log\big|T(p)^{-1}\xi\big|đ\xi\, dx(p)\,. \quad (2.27)$$

We turn now to the other component of (2.24) given in y-coordinates by

$$-\frac{1}{\alpha'}\int_{|\xi|=1} b'\big(y(p),\xi\big)_{(-n)} đ_S\xi\, dy(p)\,,$$

where $b'(y(p),\xi) = d/dz|_{z=z_0}(\sigma_A(z)(y(p),\xi))$ is the symbol derivative in y-coordinates and where $b'(y(p),\xi)_{(-n)}$ denotes its log-homogeneous (cf. (1.2)) component of degree $-n$. From [Hö] we have the asymptotic formula

$$b'\big(y(p),\xi\big) \sim \sum_{|\mu|\geq 0} \partial_\xi^\mu a'\big(x(p),T(p)\xi\big)\,\Psi_\mu(x,\xi)\,, \tag{2.28}$$

with $\Psi_\mu(x,\xi)$ polynomial in ξ of degree of at most $|\alpha|/2$. To begin with, suppose that $a(x(p),\xi)$ is homogeneous in ξ of degree $-n$. Then from (1.22) for $|\eta|\geq 1$,

$$a'\big(x(p),\eta\big) = \alpha' a\big(x(p),\eta\big)\log|\eta| + p_{-n}\big(x(p),\eta\big)\,, \tag{2.29}$$

with $p_{-n}(x(p),\eta)$ positively homogeneous in η of degree $-n$, and $a'(x(p),\eta) = a'(x(p),\eta)_{(-n)}$. Thus, if $a(x(p),\xi)$ is homogeneous in ξ of degree $-n$, by (2.28) and (2.29)

$$\begin{aligned}
-\frac{1}{\alpha'}\int_{|\xi|=1} b'\big(y(p),\xi\big)_{(-n)} đ_S\xi\, dy(p) &= -\frac{1}{\alpha'}\int_{|\xi|=1} a'\big(x(p),T(p)\xi\big) đ_S\xi\, dy(p)\\
&= -\frac{1}{\alpha'}\int_{|\xi|=1} \alpha' a\big(x(p),T(p)\xi\big)\log|T(p)\xi| đ_S\xi\, dy(p)\\
&\quad -\frac{1}{\alpha'}\int_{|\xi|=1} p_{-n}\big(x(p),T(p)\xi\big) đ_S\xi\, dy(p)\\
&= -\int_{|\xi|=1} a\big(x(p),T(p)\xi\big)\log|T(p)\xi| đ_S\xi\, dy(p)\\
&\quad -\frac{1}{\alpha'}\int_{|\xi|=1} p_{-n}\big(x(p),T(p)\xi\big) đ_S\xi\, dy(p)\,.
\end{aligned} \tag{2.30}$$

Using equations (2.25) and (2.26) of Lemma 2.20, (2.30) becomes

$$\begin{aligned}
-\frac{1}{\alpha'}\operatorname{res}_{y(p),0}(A')dy(p) &= \frac{1}{|\det T(p)|}\int_{|\xi|=1} a\big(x(p),\xi\big)\log\big|T(p)^{-1}\xi\big| đ_S\xi\, dy(p)\\
&\quad -\frac{1}{\alpha'}\frac{1}{|\det T(p)|}\int_{|\xi|=1} p_{-n}\big(x(p),\xi\big) đ_S\xi\, dy(p)\\
&= \int_{|\xi|=1} a\big(x(p),\xi\big)\log\big|T(p)^{-1}\xi\big| đ_S\xi\, dx(p)\\
&\quad -\frac{1}{\alpha'}\int_{|\xi|=1} p_{-n}\big(x(p),\xi\big) đ_S\xi\, dx(p)
\end{aligned}$$

$$= \int_{|\xi|=1} a\big(x(p),\xi\big) \log \big|T(p)^{-1}\xi\big| \,đ_S\xi \, dx(p) \\ - \frac{1}{\alpha'} \operatorname{res}_{x(p),0}(A') dx(p) \,, \qquad (2.31)$$

where the final equality follows from (2.29). Adding (2.27) and (2.31) we have when $a(x(p),\xi)$ is homogeneous in ξ of degree $-n$

$$\left(\mathrm{TR}_{y(p)}(A) - \frac{1}{\alpha'(z_0)} \operatorname{res}_{y(p),0}(A')\right) dy(p) \\ = \left(\mathrm{TR}_{x(p)}(A) - \frac{1}{\alpha'(z_0)} \operatorname{res}_{x(p),0}(A')\right) dx(p) \,, \quad (2.32)$$

proving the invariance of (2.24) in this case.

Next suppose that $a(x(p),\xi)$ is homogeneous in ξ of degree $\alpha > -n$. Then from (2.28) and since we can commute the z and μ derivatives

$$b'(y(p),\xi)_{(-n)} = \sum_{|\mu|\geq \alpha+n} \tfrac{d}{dz}\big|_{z=z_0} \partial_\xi^\mu \big(a(z)(x(p),T(p)\xi)\big)\, \Psi_{\mu,-n}(x,\xi)\,.$$

where $\Psi_{\mu,-n}(x,\xi)$ is a polynomial in ξ of degree $|\mu| - n - \alpha$. Hence

$$\int_{|\xi|=1} b'\big(y(p),\xi\big)_{(-n)} đ_S\xi \, dy(p) \\ = \sum_{|\mu|\geq\alpha+n} \frac{d}{dz}\Big|_{z=z_0} \int_{|\xi|=1} \partial_\xi^\mu \big(a(z)(x(p),T(p)\xi)\big)\, \Psi_{\mu,-n}(x,\xi) đ_S\xi \, dy(p) \\ = 0\,.$$

The final equality follows using the integration by parts property in Lemma C1 of [O1], which states that if $g(\xi)$ and $h(\xi)$ are homogeneous in ξ of degrees γ, δ where $\gamma + \delta = 1 - n$, then

$$\int_{|\xi|=1} \big(\partial_{\xi_j} g(\xi)\big) h(\xi) đ_S\xi = - \int_{|\xi|=1} g(\xi) \partial_{\xi_j} h(\xi) đ_S\xi \,,$$

along with the fact that $\Psi_{\mu,-n}(x,\xi)$ polynomial in ξ of degree $|\mu| - n - \alpha$.

This completes the proof that (2.24) is a density independent of coordinates.

Appendix B: Proof of Lemma 1.6 and Lemma 1.8

For a fixed $N \in \mathbb{N}$ chosen large enough such that $\mathrm{Re}(\alpha) - N - 1 < -n$, we write $\sigma(x,\xi) = \sum_{j=0}^{K_N} \sigma_{\alpha-j}(x,\xi) + \sigma_{(N)}(x,\xi)$ and split the integral accordingly as

$$\int_{B_x^*(0,R)} \sigma(x,\xi) đ\xi = \sum_{j=0}^{N} \int_{B_x^*(0,R)} \sigma_{\alpha-j}(x,\xi) đ\xi + \int_{B_x^*(0,R)} \sigma_{(N)}(x,\xi) đ\xi \,.$$

Since $\mathrm{Re}(\alpha) - N - 1 < -n$, $\sigma_{(N)}$ lies in $L^1(T_x^*U)$ and the integral $\int_{B_x^*(0,R)} \sigma_{(N)}(x,\xi)đ\xi$ converges when $R \to \infty$ to $\int_{T_x^*U} \sigma_{(N)}(x,\xi)đ\xi$. On the other hand, for any $j \leq N$,

$$\int_{B_x^*(0,R)} \sigma_{\alpha-j} = \int_{B_x^*(0,1)} \sigma_{\alpha-j} + \int_{D_x^*(1,R)} \sigma_{\alpha-j} \,. \tag{2.33}$$

Here $D_x^*(1,R) = B_x^*(0,R) \backslash B_x^*(0,1)$. The first integral on the r.h.s. converges and since $\sigma_{\alpha-j}(x,\xi) \sim \sum_{l=0}^{k} \sigma_{\alpha-j,l}(x,\xi) \log^l[\xi]$, the second integral reads:

$$\int_{D_x^*(1,R)} \sigma_{\alpha-j}(x,\xi)đ\xi = \sum_{l=0}^{k} \int_1^R r^{\alpha-j+n-1} \log^l r \, dr \cdot \int_{S_x^*U} \sigma_{\alpha-j,l}(x,\omega) d\omega \,.$$

Hence the following asymptotic behaviour:

$$\int_{D_x^*(1,R)} đ\xi \, \sigma_{\alpha-j}(x,\xi) \sim_{R\to\infty} \sum_{l=0}^{k} \frac{\log^{l+1} R}{l+1} \cdot \int_{S_x^*U} \sigma_{\alpha-j,l}(x,\omega) đ_S\xi$$

$$= \sum_{l=0}^{k} \frac{\log^{l+1} R}{l+1} \int_{S_x^*U} \sigma_{-n,l}(x,\xi) đ_S\xi \quad \text{if } \alpha - j = -n$$

$$\int_{D_x^*(1,R)} đ\xi \, \sigma_{\alpha-j}(x,\xi) \sim_{R\to\infty} \sum_{l=0}^{k} \Bigg(\sum_{i=0}^{l} \frac{(-1)^{i+1} \frac{l!}{(l-i)!} \log^i R}{(\alpha-j+n)^i} \cdot R^{\alpha-j+n} \int_{S_x^*U} \sigma_{\alpha-j,l}(x,\xi) đ_S\xi + (-1)^l l! \frac{R^{\alpha-j+n}}{(\alpha-j+n)^{l+1}} \cdot \int_{S_x^*U} \sigma_{\alpha-j,l}(x,\xi) đ_S\xi + \frac{(-1)^{l+1} l!}{(\alpha-j+n)^{l+1}} \cdot \int_{S_x^*U} \sigma_{\alpha-j,l}(x,\xi) đ_S\xi \Bigg) \text{if} \quad \alpha - j \neq -n \,.$$

Putting together these asymptotic expansions yields the statements with

$$C_x(\sigma) = \int_{T_x^*U} \sigma_{(N)} + \sum_{j=0}^{N} \int_{B_x^*(0,1)} \sigma_{a_j} + \sum_{j=0, a_j+n\neq 0}^{N} \sum_{l=0}^{L} \frac{(-1)^{l+1} l!}{(a_j+n)^{l+1}} \int_{S_x^*U} \sigma_{a_j,l} \,.$$

The μ-dependence follows from

$$\log^{l+1}(\mu R) = \log^{l+1} R \left(1 + \frac{\log \mu}{\log R}\right)^{l+1} \sim_{R\to\infty} \log^{l+1} R \sum_{k=0}^{l+1} C_{l+1}^k \left(\frac{\log \mu}{\log R}\right)^k .$$

The logarithmic terms $\sum_{l=0}^{k} \frac{1}{l+1} \int_{S_x^*U} \sigma_{-n,l}(x,\xi) đ_S\xi \log^{l+1}(\mu R)$ therefore contribute to the finite part by $\sum_{l=0}^{k} \frac{\log^{l+1}\mu}{l+1} \cdot \int_{S_x^*U} \sigma_{-n,l}(x,\xi) đ_S\xi$ as claimed in the lemma.

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