GAFA, Geom. funct. anal.
Vol. 17 (2007) 116 – 155
1016-443X/07/010116-40
DOI 10.1007/s00039-006-0590-7
ONLINE FIRST: February 2007

GAFA Geometric And Functional Analysis

ON THE EXPLICIT RECONSTRUCTION OF A RIEMANN SURFACE FROM ITS DIRICHLET–NEUMANN OPERATOR

Gennadi Henkin and Vincent Michel

Abstract. This article gives a complex analysis lighting on the problem which consists in restoring a bordered connected riemaniann surface from its boundary and its Dirichlet–Neumann operator. The three aspects of this problem, unicity, reconstruction and characterization are approached.

1 Statements of the Main Results

Let $\mathcal{X}$ be an open-bordered riemannian real surface (i.e. the interior of an oriented riemannian two-dimensional real manifold all of whose components have non-trivial one-dimensional smooth boundary) and g its metric. Using the boundary-control method, Belishev and Kurylev ([B1], [BK]) began the study of the inverse problem consisting in recovering $(\mathcal{X}, g)$ from the operators $N_\lambda : C^\infty(b\mathcal{X}) \ni u \mapsto (\partial\widetilde{u}_\lambda/\partial\nu)_{b\mathcal{X}}$ where $b\mathcal{X}$ is the boundary of $\mathcal{X}$, ν is the normal exterior unit to $b\mathcal{X}$ and $\widetilde{u}_\lambda$ is the unique solution of $\Delta_g U = \lambda U$ such that $U|_{b\mathcal{X}} = u$. The principal result of [BK] implies that the knowledge of $\lambda \mapsto N_\lambda$ on an non-empty open set of $\mathbb{R}_+$ determines $(\mathcal{X}, g)$ up to isometry. The important question whether $(\mathcal{X}, g)$ is uniquely determined by only one operator N_{λ_*} with $\lambda_* \neq 0$, remains open. This article mainly deals with the case of the Dirichlet–Neumann operator $N_{\mathcal{X}} := N_0$. Section 2 gives an intrinsic interpretation electrical impedance tomography on manifolds, EIT for short, in terms of the inverse Dirichlet–Neumann problem for twisted Laplacian. In dimension two, this clearly underlines how the complex structure of Riemannian surfaces is involved.

Two surfaces in the same conformal class which have the same oriented boundary and whose metrics coincide there, need to have the same Dirichlet–Neumann operator. Conversely, Lassas and Uhlmann [LU] have

---

Keywords and phrases: Riemann surface, complex Plateau problem, inverse Dirichlet to Neumann problem

AMS *Mathematics Subject Classification*: Primary 58j32, 32C25; Secondary 35R30

proved for a connected $\mathcal{X}$ that the conformal class and so the complex structure of $(\mathcal{X}, g)$ is determined by $N_{\mathcal{X}}$. Hence, it is relevant to consider $\mathcal{X}$ as a Riemann surface. In [B2], using also the full knowledge of $N_{\mathcal{X}}$, Belishev gives another proof of the above unicity by abstractly recovering $\mathcal{X}$ as the spectre of the algebra of boundary values of functions holomorphic on $\mathcal{X}$ and continuous on $\overline{\mathcal{X}} = \mathcal{X} \cup b\mathcal{X}$. It turns out that, in our Theorems 1 and 2, only three generic functions on the boundary and their images by $N_{\mathcal{X}}$ are sufficient for unicity to hold and to reconstruct $\mathcal{X}$ by integral Cauchy type formulas. Theorems 3a, 3b and 3c deal with characterizations of data of the type $(b\mathcal{X}, N_{\mathcal{X}})$ where $\mathcal{X}$ is a Riemann surface.

While the frame of bordered manifolds is sufficient for real analytic boundaries, characterization statements lead to consider a wider class of manifolds. In this article, $(\overline{\mathcal{X}}, \gamma)$ is a *Riemann surface with almost smooth boundary* if the following holds: $\overline{\mathcal{X}}$ is a compact metrizable topological manifold which is the closure of $\mathcal{X} = \overline{\mathcal{X}} \backslash \gamma$, $\mathcal{X}$ is a Riemann surface $h^2(\overline{\mathcal{X}}) < \infty$, where h^d is the d-dimensional Hausdorff measure, γ is a smooth real curve and the set $\overline{\mathcal{X}}_{\rm sing}$ of points of γ, where $\overline{\mathcal{X}}$ has no smooth boundary, satisfies $h^1(\overline{\mathcal{X}}_{\rm sing}) = 0$; $\overline{\mathcal{X}} \backslash \overline{\mathcal{X}}_{\rm sing}$ is denoted $\overline{\mathcal{X}}_{\rm reg}$.

Note that a Stokes formula holds automatically for such manifolds (see Lemma 11 in section 3). Note also that it could have been possible to allow singularities on γ itself, but we have avoided it for the sake of simplicity of statements. Likewise, we consider only smooth DN-data in the sequel.

If $(\overline{\mathcal{X}}, \gamma)$ is a Riemann surface with almost smooth boundary, classical results contained in [AS] imply Riemann's existence theorem: a real valued function u of class C^1 on γ has a unique continuous extension $\widetilde{u}$ to $\overline{\mathcal{X}}$ which is harmonic on $\mathcal{X}$, smooth on $\overline{\mathcal{X}}_{\rm reg}$ and satisfies $\int_{\mathcal{X}} i\, \partial \widetilde{u} \wedge \overline{\partial} \widetilde{u} < +\infty$. Moreover, $N_{\mathcal{X}} u$ still makes sense as the element of the dual space of $C^1(\gamma)$ which equals $\partial \widetilde{u} / \partial \nu$ on $\gamma \backslash \overline{\mathcal{X}}_{\rm sing}$ (see Proposition 12).

In the sequel, γ is a smooth compact oriented real curve without component reduced to a point, N is an operator from $C^1(\gamma)$ to the space of currents on γ of degree 0 and order 1 (i.e. functionals on C^1 1-forms on γ), τ is a smooth generating section of $T\gamma$ and ν is another vector field along γ such that the bundle $\mathcal{T}$ generated by (ν_x, τ_x), $x \in \gamma$, has rank 2; γ is assumed to be oriented by τ and $\mathcal{T}$ by (ν, τ).

The inverse Dirichlet–Neumann problem for $(\gamma, N, \mathcal{T})$ is to find, when it exists, an open riemaniann surface $(\mathcal{X}, g)$ with almost smooth boundary γ such that for all $x \in \gamma \cap \overline{\mathcal{X}}_{\rm reg}$, (ν_x, τ_x) is a positively oriented orthonormal basis of $T_x\overline{\mathcal{X}}$ and for all $u \in C^1(\gamma)$, $Nu = N_{\mathcal{X}} u$ in the sense of currents.

As these conditions do not distinguished between metrics g in the same conformal class, we look after $\mathcal{X}$ as a Riemann surface. The connection between real and complex analysis in the IDN-problem is realized through the operators L and θ defined for $u \in C^1(\gamma)$ by

$$Lu = \tfrac{1}{2}(Nu - i\,Tu) \quad \text{and} \quad \theta u = (Lu)(\nu^* + i\tau^*) \tag{1.1}$$

where T is the tangential derivation by τ and (ν_x^*, τ_x^*) is the dual basis of (ν_x, τ_x) for every $x \in \gamma$. Note that, in the sense of currents, the equality $Nu = N_{\mathcal{X}}u$ is equivalent to the identity $\partial\widetilde{u} = \theta u$, the tilde denoting, as all through this article, continuous harmonic extension to $\mathcal{X}$.

If $(\overline{\mathcal{X}}, \gamma)$ is a Riemann surface with almost smooth boundary, g is a hermitian metric on $\mathcal{X}$ for which (τ_x, ν_x) is a positively oriented orthonormal basis of $T_x\mathcal{X}$ for $x \in \gamma$ outside $\sigma = \overline{\mathcal{X}}_{\text{sing}}$ and if $\rho \in C^0(\overline{\mathcal{X}}) \cap C^\infty(\overline{\mathcal{X}}\backslash\sigma)$ is a defining function of γ in $\overline{\mathcal{X}}$, then $(\nu^*, \tau^*) = \frac{1}{|d\rho|_g}\,(d\rho, d^c\rho)$ on $\gamma\backslash\sigma$ where $d^c = i(\overline{\partial} - \partial\,)$ and $\partial\widetilde{u} = (Lu)\,|\partial\rho|_g^{-1}\,\partial\rho = \theta u$ on $\gamma\backslash\sigma$ for all $u \in C^1(\gamma)$.

Main hypothesis. In addition to the assumptions on γ, throughout this paper, we consider $u_0, u_1, u_2 \in C^\infty(\gamma)$ three real valued functions only ruled by the main hypothesis that

$$f = (f_1, f_2) = \big((Lu_\ell)/(Lu_0)\big)_{\ell=1,2} = \big((\theta u_\ell)/(\theta u_0)\big)_{\ell=1,2} \tag{1.2}$$

is an embedding of γ in $\mathbb{C}^2$ considered as the complement of $\{w_0 = 0\}$ in the complex projective plane $\mathbb{CP}_2$ with homogeneous coordinates $(w_0 : w_1 : w_2)$. Proposition 0 whose proof is omitted shows this is somehow generic:

PROPOSITION 0. *Assume γ, u_0, u_1 real analytic and that f_1 is non-constant on each connected component of γ. For any function $u_2 \in C^\omega(\gamma)$, one can construct $v_2 \in C^\omega(\gamma)$, arbitrarily close to u_2 in C^2 norm, such that $(f_1, (Lv_2)/(Lu_0))$ is an embedding of γ into $\mathbb{C}^2$.*

Assuming that $u = (u_\ell)_{0\leqslant\ell\leqslant2}$ satisfies the main hypothesis, we set $\theta u = (\theta u_\ell)_{0\leqslant\ell\leqslant2}$ and call $(\gamma, u, \theta u)$ a *restricted DN-datum* for an open Riemann surface $\mathcal{X}$ if $\mathcal{X}$ has almost smooth boundary γ, $(\partial\widetilde{u_\ell})|_{\gamma\backslash\sigma} = \theta u_\ell$ for $0 \leqslant \ell \leqslant 2$, and the well-defined meromorphic quotient F_ℓ of (1,0)-forms $(\partial\widetilde{u_\ell})/(\partial\widetilde{u_0})$ extends f_ℓ to $\mathcal{X}$ in the sense that, for every $x_0 \in \gamma$, $\lim_{x\to x_0,\, x\in\mathcal{X}} F(x)$ exists and equals $f(x_0)$. If γ and f are real analytic, this last property holds automatically.

We define an isomorphism between two Riemann surfaces with almost smooth boundary, $(\overline{\mathcal{X}}, \gamma)$ and $(\overline{\mathcal{X}'}, \gamma')$, as a map from $\overline{\mathcal{X}}$ to $\overline{\mathcal{X}'}$ which realizes a complex analytic isomorphism between $\mathcal{X}$ and $\mathcal{X}'$. As the definition of a Riemann surface with almost smooth boundary implies that its boundary is locally a Jordan curve in its double which is the compact Riemann surface obtained by gluing along its boundary its conjugate (see [AS]), a theorem of Caratheodory implies that if $\Phi : \mathcal{X} \to \mathcal{X}'$ is a complex analytic isomorphism, Φ and Φ^{-1} extend continuously to γ and γ' so that Φ becomes a homeomorphism from $\overline{\mathcal{X}}$ to $\overline{\mathcal{X}'}$. Hence, Φ is a diffeomorphism between manifolds with boundary from $\overline{\mathcal{X}}_{\mathrm{reg}} \cap \Phi^{-1}(\overline{\mathcal{X}'}_{\mathrm{reg}})$ to $\overline{\mathcal{X}'}_{\mathrm{reg}} \cap \Phi(\overline{\mathcal{X}}_{\mathrm{reg}})$.

The first theorem of this article is a significative improvement of results in [BK], [LU] on how unique $\mathcal{X}$ can be when a restricted DN-datum is specified.

Theorem 1. *Assume that $\mathcal{X}$ and $\mathcal{X}'$ are open Riemann surfaces with* restricted DN-datum $(\gamma, u, \theta u)$. *Then, there is an isomorphism of Riemann surfaces with almost smooth boundary between $\mathcal{X} \cup \gamma$ and $\mathcal{X}' \cup \gamma$ whose restriction on γ is the identity.*

Remarks. 1. If $E \subset \gamma$ and $h^1(E \cap c) > 0$ for each connected component c of γ, meromorphic functions are uniquely determined by their values on E and it follows that Theorem 1's conclusions hold when $N_{\mathcal{X}'} u_\ell = N_{\mathcal{X}} u_\ell$ is ensured only on E, and the meromorphic functions $(\partial \widetilde{u}_\ell)/(\partial \widetilde{u}_0)$ are continuous near γ. This includes [LU, Th. 1.1.i] which is stated for a connected $\mathcal{X}$.

2. The proof of Theorem 1 also contains the fact that two connected compact Riemann surfaces $\mathcal{Z}$ and $\mathcal{Z}'$ are isomorphic when they share the same real smooth curve γ which can be embedded into $\mathbb{C}^2$ by a map which extends meromorphically both to $\mathcal{Z}$ and $\mathcal{Z}'$ and continuously near γ.

The assumption on u_0, u_1 and u_2 is used only to ensure that the map f defined by (1.2) is an embedding of γ into $\mathbb{C}^2$ extending meromorphically to $\mathcal{X}$ into $F = (\partial \widetilde{u}_\ell / \partial \widetilde{u}_0)_{\ell=1,2}$. Moreover, Theorem 10 in section 3 implies that if $\mathcal{X}$ has an almost smooth boundary and solves the IDN-problem, the map F enables us to see $\overline{\mathcal{X}}$ as a normalization of the closure of a complex curve of $\mathbb{CP}_2 \backslash f(\gamma)$ uniquely determined by γ. This shows that in each characterization Theorem 3a, 3b, 3c, the constructed Riemann surface is, up to isomorphism, the only one which has a chance to solve the IDN-problem.

In this paper, complex curves are pure 1-dimensional complex analytic subsets of complex manfolds.

Our next result explains how to recover $F(\mathcal{X})$ and $\partial \widetilde{u}_\ell$ from θu_ℓ and the intersection of $F(\mathcal{X})$ with the lines $\Delta_\xi = \{z \in \mathbb{C}^2; z_2 = \xi\}$, $\xi \in \mathbb{C}$. Desingularization arguments then enable the reconstruction of $\mathcal{X}$ from $F(\mathcal{X})$.

Theorem 2. *If $\mathcal{X}$ is an open Riemann surface with* restricted DN-datum $(\gamma, u, \theta u)$, *the following hold:*

(1) *The map* f *defined by* (1.2) *has a meromorphic extension* F *to* $\mathcal{X}$ *and there are discrete sets* $\mathcal{A}$ *and* $\mathcal{B}$ *in* $\mathcal{X}$ *and* $\mathcal{Y} = F(\mathcal{X})\backslash f(\gamma)$ *respectively such that* $F : \mathcal{X}\backslash\mathcal{A} \to \mathcal{Y}\backslash\mathcal{B}$ *is one to one.*

(2) *Almost all* $\xi_* \in \mathbb{C}$ *has a neighborhood* W_{ξ_*} *such that, for all* ξ *in* W_{ξ_*}, $\mathcal{Y}_\xi = \mathcal{Y} \cap \Delta_\xi = \bigcup_{1\leqslant j\leqslant p}\{(h_j(\xi),\xi)\}$ *where* $h_1,\dots,h_p$ *are* p *mutually distinct holomorphic functions on* W_{ξ_*} *whose symmetric functions* $S_{h,m} = \sum_{1\leqslant j\leqslant p} h_j^m$ *are recovered by the Cauchy type integral formulas*

$$\frac{1}{2\pi i}\int_\gamma \frac{f_1^m}{f_2-\xi}df_2 = S_{h,m}(\xi) + P_m(\xi)\,, \quad m\in\mathbb{N}\,, \qquad (E_{m,\xi})$$

where P_m *is a polynomial of degree at most* m. *More precisely, the system* $E_\xi = (E_{m,\xi_\nu})_{\substack{0\leqslant m\leqslant B-1\\ o\leqslant\nu\leqslant A-1}}$ *enables explicit computation of* $h_j(\xi_\nu)$ *and* P_m *if* $A \geqslant B \geqslant 2p+1$ *and* $\xi_0,\dots,\xi_A$ *are mutually distinct points.*

(3) *For almost all* $\xi_* \in \mathbb{C}$, W_{ξ_*} *can be chosen so that* $\mathcal{B}\cap\bigcup_{\xi\in W_{\xi_*}}\mathcal{Y}_\xi = \varnothing$ *and* $\partial\widetilde{u}_\ell$, $0\leqslant\ell\leqslant 2$, *can be reconstructed in* $F^{-1}\big(\bigcup_{\xi\in W_{\xi_*}}\mathcal{Y}_\xi\big)$ *from the well-defined meromorphic quotient* $(\partial\widetilde{u}_\ell)/(\partial F_2)$ *thanks to the Cauchy-type formulas*

$$\frac{1}{2\pi i}\int_\gamma \frac{f_1^m}{f_2-\xi}\theta u_\ell = \sum_{1\leqslant j\leqslant p} h_j(\xi)^m \frac{\partial\widetilde{u}_\ell}{\partial F_2}\big(F^{-1}(h_j(\xi),\xi)\big) + Q_m(\xi) \quad (T_{m,\xi})$$

where m *is any integer and* Q *is a polynomial of degree at most* m.

REMARK. The number α of connected components of $\mathcal{X}$ can be computed by the following algorithm: let γ_1 be a component of γ and let λ_1 be a function which is zero on γ_j for $j \neq 0$ and non-constant on γ_1; then if $\mathcal{X}_1$ is the component of $\mathcal{X}$ whose boundary contains γ_1, $N\lambda_1 \neq 0$ on each component $\gamma_1,\dots,\gamma_k$ of γ which with γ_1 are the components of $b\mathcal{X}_1$. Iterating this with components γ different from $\gamma_1,\dots,\gamma_k$, yields a process with α steps.

The numerical resolution of (E_ξ) and the study of its stability requires an estimate of the number I_{Δ_ξ} of points of intersection, multiplicities taken into account, of $\mathcal{Y}$ with Δ_ξ. To achieve this, it is sufficient to estimate the number I_Δ of intersection points of $\mathcal{Y}$ with a $\mathbb{CP}_2$-line Δ, generic in the sense that Δ does not contain the germ of a component of $\mathcal{Y}$ near γ. Indeed, if L (resp. L_ξ) denotes a linear homogeneous form defining Δ (resp. Δ_ξ),

$$I_{\Delta_\xi} - I_\Delta = \frac{1}{2\pi i}\int_\gamma (L_\xi/L)^{-1}d(L_\xi/L)\,.$$

Thus, an a priori upper bound of I_Δ for any particular line Δ would be very useful. This open problem is related, because of the Ahlfors theorem on covering surface, to the computation of the genus $g_{\mathcal{X}}$ of $\mathcal{X}$ from some DN-datum when $\mathcal{X}$ is connected. Under the condition γ is connected, Belishev [B2] has shown that $2g_{\mathcal{X}}$ is the rank of $Id + (N_{\mathcal{X}}T^{-1})^2$ acting on the space of smooth functions on γ admitting a smooth primitive, T^{-1} being a primitive operator. A formula for $g_{\mathcal{X}}$ involving only the action of $N_{\mathcal{X}}$ on a finite generic set of functions has yet to be found.

The third aspect of the IDN-problem, characterization of which can and should be a DN-datum, has lead us to allow $\mathcal{X}$ to have only almost smooth boundary. Theorem 3a below explicitly characterizes the only right candidate for $\mathcal{X}$ while its part C gives a test which determines which $(\gamma, u, \theta u)$ are DN-data and which are not. To perform it, we need a *Green function for* $\overline{\mathcal{X}}$ relative to a domain $\mathcal{D}$ of $\mathcal{Z}$ containing $\overline{\mathcal{X}}$, that is a smooth symmetric function g defined on $\mathcal{D} \times \mathcal{D}$ without its diagonal such that each $g(\,.\,, z)$ is harmonic on $\mathcal{D}\backslash\{z\}$ and has singularity $\frac{1}{2\pi} \ln \operatorname{dist}(\,.\,, z)$ at z, the distance being computed in any hermitian metric on $\mathcal{Z}$.

Theorem 3a. *Assume that the main hypothesis is valid and consider*

$$G : \mathbb{C}^2 \ni (\xi_0, \xi_1) \mapsto \frac{1}{2\pi i} \int_\gamma f_1 \frac{d(\xi_0 + \xi_1 f_1 + f_2)}{\xi_0 + \xi_1 f_1 + f_2}\,. \tag{1.3}$$

A. *If an open Riemann surface $\mathcal{X}$ has restricted DN-datum $(\gamma, u, \theta u)$, then almost all point ξ_* of $\mathbb{C}^2$ has a neighborhood where one can find mutually distinct holomorphic functions $h_1, \dots, h_p$ such that*

$$0 = \frac{\partial^2}{\partial \xi_0^2}\Big(G - \sum_{1 \leqslant j \leqslant p} h_j\Big) \tag{1.4}$$

$$h_j \frac{\partial h_j}{\partial \xi_0} = \frac{\partial h_j}{\partial \xi_1}\,, \quad 1 \leqslant j \leqslant p\,. \tag{1.5}$$

B. *Conversely, assume γ is connected and the conclusion of* A *is satisfied in a connected neighborhood W_{ξ_*} of one point (ξ_{0*}, ξ_{1*}). Then, if $(\partial^2 G/\partial \xi_0^2)_{|W_{\xi_*}} \neq 0$, there is an open Riemann surface $\mathcal{X}$ with almost smooth boundary γ where f extends meromorphically. If $(\partial^2 G/\partial \xi_0^2)_{|W_{\xi_*}} = 0$, the same conclusion holds for a suitable orientation of γ.*

C. *Assume that $(\overline{\mathcal{X}}, \gamma)$ is a Riemann surface with almost smooth boundary. Let $\mathcal{Z}$ be the double of $\mathcal{X}$, $\mathcal{D}$ a smooth domain of $\mathcal{Z}$ containing $\overline{\mathcal{X}}$ and g a Green function for $\overline{\mathcal{X}}$ relatively to $\mathcal{D}$. Then, $(\gamma, u, \theta u)$ is actually a restricted DN-datum if and only if, for any $z \in \mathcal{D}\backslash\overline{\mathcal{X}}$,*

$$\int_\gamma u_\ell(\zeta)\, \partial_\zeta g(\zeta, z) + g(\zeta, z)\, \overline{\theta u_\ell(\zeta)} = 0\,, \quad 0 \leqslant \ell \leqslant 2\,. \tag{1.6}$$

Remarks. 1. The connectness of γ is essentially used to ensure that any possible solution to the IDN-problem has to be connected. Taking in account the remark following Theorem 2, one may weaken the connectness assumption on γ into the requirement that the given DN-datum ensures that possible solutions are connected. Then, the conclusions of Theorem 3a.B are still true (see the proof).

2. The proof includes that if γ and f are real analytic, $(\overline{\mathcal{X}}, \gamma)$ is a manifold with boundary in the classical sense.

3. Emphasizing f_2 instead of f_1, one can consider

$$G_2 : \xi \mapsto \frac{1}{2\pi i} \int_\gamma f_2 \frac{d(\xi_0 + \xi_1 f_1 + f_2)}{\xi_0 + \xi_1 f_1 + f_2} \, .$$

If h_j is linked to $h_{j,2}$ by $0 = \xi_0 + \xi_1 h_j + h_{j,2}$, $(h_1, \dots, h_p)$ satisfy (1.5) and (1.4) if and only $\frac{\partial^2}{\partial \xi_0^2}\big(G_2 - \sum_{1 \leqslant j \leqslant p} h_{j,2}\big) = 0$ and $h_j \frac{\partial h_{j,2}}{\partial \xi_0} = \frac{\partial h_{j,2}}{\partial \xi_1}$, $1 \leqslant j \leqslant p$.

4. Select $\mathcal{H} = \{h_1, \dots, h_p\}$ satisfying (1.5) and minimal for (1.4). Then, section 5.2 and Proposition 14 show that there is $\tau' \subset \delta = f(\gamma)$ such that $h^1(\tau') = 0$, and $\mathcal{X}$ is a normalization of the abstract curve $\mathcal{Y} \cup \tau'$ where, when $\mathcal{H} = \varnothing$, $\mathcal{Y}$ is the polynomial hull of δ in the affine complex plane

$$\mathbb{C}^2_{\xi_*} = \{ w \in \mathbb{CP}^2 \, ; \; \xi_* w = \xi_{0*} w_0 + \xi_{1*} w_1 + w_2 \neq 0 \} \, .$$

and, otherwise, $\mathcal{Y}$ is the analytic extension in $\mathbb{CP}_2 \backslash \delta$ of the union of the graphs of the functions $(1 : h_j : -\xi_0 - \xi_1 h_j)$, $1 \leqslant j \leqslant p$. Hence, when $\mathcal{H}$ is minimal, decomposition (1.4) of G is unique up to order, and $\operatorname{Card} \mathcal{H}$ is the minimal number p for which such a decomposition exists. Moreover, Theorem 10 implies that the only Riemann surfaces $\mathcal{X}$ which have a chance of solving the IDN-problem are normalizations of $\mathcal{Y}$.

5. With [HL1, Ex. 10.5], one can construct smooth restricted DN-data for which the solution of the IDN-problem is a manifold with only almost smooth boundary.

The vanishing of $\partial^2 G / \partial \xi_0^2$ in a connected neighborhood W_{ξ_*} of $\xi_* \in \mathbb{C}^2$ is known to be equivalent to the fact that $\delta = f(\gamma)$ satisfy the classical Wermer–Harvey–Lawson moment condition in $\mathbb{C}^2_{\xi_*}$: for all $k_1, k_2 \in \mathbb{N}$, $\int_\delta z_1^{k_1} z_2^{k_2} dz_2 = 0$ where $z = (w_j / \xi_* w)_{j=1,2}$ (see [DoH, Cor. 1.6.2]). When ξ_* belongs to the connected component of infinity of $\{ \xi \in \mathbb{C}^2 ; \, \forall w \in \delta, \, \xi w \neq 0 \}$, this moment condition is equivalent to the moment condition in $\mathbb{C}^2_{(1,0)}$ and also to the vanishing of G on this component. It is proved in [W] for the real analytic case and in [Bi], [HL1] for the smooth case, that for a suitable orientation of γ, this moment condition guarantees the existence in $\mathbb{C}^2_{\xi_*} \backslash \delta$ of a unique complex curve $\mathcal{Y}$ with finite mass and boundary $\pm\delta$ in the sense of currents. In [AlW], Alexander and Wermer have improved this Wermer–Bishop–Harvey–Lawson statement by showing that a closed oriented smooth connected real curve δ of $\mathbb{C}^2$ is, with its given orientation the boundary, in the sense of currents of a complex curve of finite mass in $\mathbb{C}^2_{\xi_*}\backslash\delta$, if and only if $\frac{1}{2\pi i}\int_\delta \frac{dA}{A} \geqslant 0$, for any polynomial A which does not vanish on γ. Hence, in case $(\partial^2 G/\partial\xi_0^2)_{|W_{\xi_*}} = 0$, it is sufficient to find one polynomial A such that $\int_{f(\gamma)} dA/A \neq 0$ to determine the correct orientation of γ.

Note that the case $(\partial^2 G/\partial\xi_0^2)_{|W_{\xi_*}} = 0$ occurs only for very special DN-data since it implies that for $\ell = 1, 2$, f_ℓ admits a $\mathbb{C}^2_{\xi_*}$-valued holomorphic extension to $\mathcal{X}$. Proposition 20 proposes another result of this kind for some other special DN-data when they are available.

To palliate the difficulty of computing Green functions, the theorem below proposes another way to achieve the same goals: select the right candidates for $\mathcal{X}$ and extension of θu_ℓ; check this yields a solution.

Theorem 3b. *Assume that the main hypothesis is valid. Let G be the function defined by* (1.3) *and let be $\widetilde{G}$ the form which in $\mathbb{CP}_2$ with homogenous coordinates $\eta = (\eta_0 : \eta_1 : \eta_2)$ is given by*

$$\widetilde{G} = \sum_{0\leqslant\ell\leqslant 2} \frac{1}{2\pi i}\left(\int_\gamma \frac{\theta u_\ell}{\eta_0 + \eta_1 f_1 + \eta_2 f_2}\right) d\eta_\ell = \sum_{0\leqslant\ell\leqslant 2} \widetilde{G}_\ell \, d\eta_\ell \,. \tag{1.7}$$

A. *If an open Riemann surface $\mathcal{X}$ has restricted DN-datum $(\gamma, u, \theta u)$, then,*

- (a1) *Almost all points $\eta_* = (\xi_{*0} : \xi_{*1} : 1)$ of $\mathbb{CP}_2$ have a neighborhood where $\widetilde{G}$ can be written as the sum of p holomorphic closed forms $g_j = \sum_{0\leqslant\ell\leqslant 2} g_{j,\ell}\, d\eta_\ell$ such that $(h_j) = (g_{j,1}/g_{j,0})_{1\leqslant j\leqslant p}$ satisfy* (1.5) *with the affine coordinates $\xi_0 = \eta_0/\eta_1$, $\xi_1 = \eta_1/\eta_2$.*
- (a2) *The form $\Theta_\ell = \partial\widetilde{u_\ell}$, $0 \leqslant \ell \leqslant 2$, satisfies*
$$\int_c \operatorname{Re}\Theta_\ell = 0 \tag{1.8}$$
for all c in the first homology group $H_1(\mathcal{X})$ of $\mathcal{X}$.

B.

- (b1) *Assume γ is connected and there is $\eta_* = (\xi_{*0} : \xi_{*1} : 1)$ and a connected neighborhood W_{ξ_*} of $\xi_* = (\xi_{*0}, \xi_{*1})$ such that* (a1) *is true for all $\eta \in W_{\eta_*} = \{(\xi_0 : \xi_1 : 1);\ (\xi_0, \xi_1) \in W_{\xi_*}\}$. Then, there exists an open Riemann surface $\mathcal{X}$, topologically bordered by γ, where f extends meromorphically, and each θu_ℓ extends weakly into a meromorphic $(1,0)$-form Θ_ℓ outside a set Σ of zero length.*

(b2) *In addition to* (a1)*, assume that* Θ_ℓ *satisfies* $\int i\,\Theta_\ell \wedge \overline{\Theta_\ell} < +\infty$ *and* (1.8)*. Then if* $(\partial^2 G/\partial \xi_0^2)_{|W_{\xi_*}} \neq 0$*,* $(\overline{\mathcal{X}}, \gamma)$ *is a manifold with almost smooth boundary; the same conclusion holds when* $\widetilde{G}|_{W_{\eta_*}} = 0$ *if* γ *has a suitable orientation. If* $(\partial^2 G/\partial \xi_0^2)_{|W_{\xi_*}} = 0$ *but* $\widetilde{G}|_{W_{\eta_*}} \neq 0$*, then either* $\mathcal{X}$ *is a domain with boundary* γ *in a normalization of an algebraic curve of* $\mathbb{CP}_2$*, or* $\overline{\mathcal{X}}$ *is a compact Riemann surface where* γ *is a slit, which means that* $\overline{\mathcal{X}} \backslash \gamma$ *is connected. In all cases,* u_ℓ *admits a continuous extension* $\widetilde{u_\ell}$ *to* $\overline{\mathcal{X}}$ *which is harmonic in* $\mathcal{X}$ *and such that* $\Theta_\ell = \partial \widetilde{u_\ell}$*, which means that* $N u_\ell$ *is actually the DN-datum of* $\mathcal{X}$ *for* u_ℓ*.*

That θu_ℓ extends weakly to Θ means that $\int_\gamma \varphi \theta u_\ell = \int_{\mathcal{X}} d(\varphi \Theta_\ell) = \int_{\mathcal{X}} (\overline{\partial} \varphi) \wedge \Theta_\ell$ holds for any Lipschitz function φ on $\overline{\mathcal{X}}$ which is a holomorphic function of f near points of Σ and singular points of $(\overline{\mathcal{X}}, \gamma)$; if $(\overline{\mathcal{X}}, \gamma)$ is a manifold with boundary, this definition means that $\Theta_\ell|_\gamma = \theta u_\ell$ in the usual sense.

REMARKS. 1. When (a1) holds, $h_{j,2} = \frac{g_{j,2}}{g_{j,0}}$ verify $h_j \frac{\partial h_{j,2}}{\partial \xi_0} = \frac{\partial h_{j,2}}{\partial \xi_1}$, $1 \leqslant j \leqslant p$.

2. Formulas $(E_{m,\xi})$ and $(T_{m,\xi})$ enable direct reconstruction of a projective presentation of $\mathcal{X}$ and forms Θ_ℓ.

3. Based on [D, Ex. 1], one can construct examples where (a1) is satisfied while the weak extension Θ_ℓ has essential singularities on some zero length set Σ.

Theorem 3b is obtained by a normalization of a singular version of the IDN-problem which is more explicit. When $\mathcal{X}$ is smooth, the harmonicity of a distribution U is equivalent to the fact that ∂U is holomorphic. For the case where $\mathcal{X}$ is a complex curve of an open set in $\mathbb{CP}_2$, we need two of the several non-equivalent definitions of holomorphic (1,0)-forms.

At first, we use the *weakly holomorphic* forms introduced by Rosenlicht [R] which can be defined as meromorphic (1,0)-forms ψ such that $\psi \wedge [\mathcal{X}]$ is a $\overline{\partial}$-closed current of $\mathbb{CP}_2$. Such forms ψ are also characterized by the fact that $p_* \psi$ is a usual holomorphic $(1,0)$-form for any holomorphic proper function $p : \mathcal{X} \to \mathbb{C}$. A distribution U is defined as *weakly harmonic* if ∂U is weakly holomorphic.

Now assume $\mathcal{X}$ lies in $\mathbb{CP}_2$ and that $\mathcal{X}$ is bounded in the sense of currents by γ. A distribution U on $\mathcal{X}$ is said almost smooth up to the boundary if it is the case near each $p \in \gamma$ where $(\overline{\mathcal{X}}, \gamma)$ is a manifold with boundary and if U has a restriction on γ in the sense of currents. When u is a smooth function on γ, a weakly harmonic extension of u to $\mathcal{X}$ is a weakly harmonic distribution U almost smooth up to boundary whose restriction on γ is u. Since two weakly harmonic extensions, U_1 and U_2, of u to $\mathcal{X}$ are equal when $\partial U_1 = \partial U_2$ on γ in the sense of currents, we consider a *weak Cauchy–Dirichlet problem*: a data is a smooth function u on γ and a smooth section λ of $T^*_\gamma \mathcal{X}$; a solution is a weakly harmonic function U almost smooth up to γ such that $u = U|_\gamma$ and $\lambda = (\partial U)_\gamma$ in the sense of currents; when it exists, such a U is unique and is denoted $\widetilde{u}$ as any harmonic extension in this article. In connection with this notion, we define a *weak restricted data* as a triplet $(\gamma, u, \theta u)$ where $u = (u_\ell)_{0\leqslant\ell\leqslant 2}$ (resp. $\theta u = (\theta u_\ell)_{0\leqslant\ell\leqslant 2}$) is a triplet of smooth functions (resp. (1,0)-forms) on γ such that $\theta u_\ell = (\partial \widetilde{u_\ell})_\gamma$ in the sense of currents.

The weak CD-problem has its own interest and arises naturally in the proof of Theorem 3b. However, the original IDN-problem requires a more restrictive notion of harmonicity. According to Griffiths [G], *holomorphic forms* (resp. *harmonic functions*) are, by definition, push forwards of holomorphic forms (resp. harmonic functions) on a normalization of $\mathcal{X}$. Equivalently, a real function U on $\mathcal{X}$ is harmonic if and only if U is harmonic in the regular part $\mathcal{X}_{\rm reg}$ of $\mathcal{X}$ and $\int_{\mathcal{X}_{\rm reg}} i\,\partial U \wedge \overline{\partial} U < +\infty$. This notion is close in spirit to a Riemann characterization of the harmonic function with given boundary value u as the smooth function extending u to $\overline{\mathcal{X}}$ and minimizing the preceding integral. We can now state a singular version of Theorem 3b.

Theorem 3c. *Consider in* $\mathbb{CP}_2\backslash\{w_0 = 0\}$ *a smooth oriented real curve* γ, *three functions* u_0, u_1, u_2 *in* $C^\infty(\gamma)$ *and* $\theta_0, \theta_1, \theta_2$ *three smooth sections of* $(T^{*1,0}\mathbb{CP}_2)|_\gamma$ *such that* $du_\ell = 2\,\mathrm{Re}\,\theta_\ell$, $0 \leqslant \ell \leqslant 2$, *and linked by the relations* $\theta_1 = z_1\theta_0$, $\theta_2 = z_2\theta_0$. *Let* G *and* $\widetilde{G}$ *be the form given by* (1.3) *and* (1.7) *but with* $(f_1, f_2) = (z_1, z_2)$.

A. *Assume* γ *bounds, in the sense of currents, a complex curve* $\mathcal{X}$ *of* $\mathbb{CP}_2\backslash\gamma$ *which has finite volume and weak restricted DN-datum* $(\gamma, u, \theta u)$. *Then,*

(a1) *The conclusions of Theorem 3b.A.a1 are valid.*

(a2) *The form* $\Theta_\ell = \partial\widetilde{u_\ell}$ *satisfies* (1.8) *for all* c *in* $H_1(\mathcal{X}_{\rm reg})$.

B.

(b1) *Conversely, assume that* γ *is connected and that* (a1) *is valid for one point* $\eta_* = (\xi_{0*} : \xi_{1*} : 1)$. *Then, there is in* $\mathbb{CP}_2\backslash\gamma$ *a complex curve* $\mathcal{X}$ *of finite mass where each* θ_ℓ *extends weakly on* $\mathcal{X}$ *into a weakly holomorphic* $(1,0)$*-form* Θ_ℓ. *Moreover, if* $(\partial^2 G/\partial\xi_0^2)_{|W_{\xi_*}} \neq 0$, *then* $\mathcal{X}$

has boundary γ in the sense of currents; the same conclusion holds if $\widetilde{G}|_{W_{\eta *}} = 0$ but for a suitable orientation of γ. If $(\partial^2 G/\partial\xi_0^2)_{|W_{\xi *}} = 0$ but $\widetilde{G}|_{W_{\eta *}} \neq 0$, either $\mathcal{X}$ is a domain in an algebraic curve of $\mathbb{CP}_2$ and has boundary γ in the sense of currents, either $\overline{\mathcal{X}}$ itself is an algebraic curve of $\mathbb{CP}_2$ where γ is a slit.

(b2) *If in addition* (1.8) *is satisfied by Θ_ℓ for all $c \in H_1(\mathcal{X}_{\mathrm{reg}})$, then u_ℓ has a (unique) weakly harmonic extension $\widetilde{u_\ell}$ and $\Theta_\ell = \partial\widetilde{u_\ell}$. If Θ_ℓ also satisfy $\int_{\mathcal{X}_{\mathrm{reg}}} i\,\Theta_\ell \wedge \overline{\Theta_\ell} < +\infty$, then $\widetilde{u_\ell}$ is harmonic.*

Remark. It is possible that $\mathcal{X}$ has zero boundary in the sense of currents. This occurs only in the exceptional case where $\overline{\mathcal{X}}$ is a compact complex curve of $\mathbb{CP}_2$ and (so is algebraic) where γ is a slit. In the other cases, $\mathcal{X}$ has boundary $\pm\gamma$ in the sense of currents, and a result of Chirka [Ch] gives that, outside a zero one Hausdorff-dimensional subset, $(\overline{\mathcal{X}}, \pm\gamma)$ is locally a manifold with boundary.

The proofs of the preceding theorems are given in sections 3 to 5. They use the results on the complex Plateau problem started in [W], [Bi], developed in [HL1], [H], [D] for $\mathbb{C}^n$ and in [He], [DoH], [HL] for $\mathbb{CP}_n$.

The non-constructive existence criteria of Theorems 3a, 3b and 3c may inspire one to seek a less general but more effective characterization. It has already been mentioned, after Theorem 3a, that in the special case $p = 0$, the condition $(\partial^2 G/\partial\xi_0^2)_{|W_{\xi *}} = 0$ together with the Alexander–Wermer moment criterion gives an effective tool but only when special DN-data are at hand.

For $p > 0$, the main result of [DoH] is that conditions of type (1.4) and (1.5) characterize the fact that a given closed, smooth and orientable real chain γ of $\mathbb{CP}_2$ is, with adequate orientation, the boundary of some holomorphic chain of $\mathbb{CP}_2\backslash\gamma$. These conditions have been qualified as mysterious in [HL] because the functions satisfying these relations are produced “deus ex machina”. The following criterion, which completes for a closed connected curve γ the one of [DoH], is obtained in [HL]: Suppose that the second coordinate f_2 of $\mathbb{C}^2$ does not vanish on γ, then there exists in $\mathbb{CP}_2\backslash\gamma$ a connected complex curve $\mathcal{X}$ with boundary $\pm\gamma$ in the sense of currents if and only if there exist $p \in \mathbb{N}$ and A_d in the space $\mathcal{O}(d)$ of holomorphic homogeneous polynomials of order d, $1 \leqslant d \leqslant p$, such that for ξ_0, in some neighborhood of 0, $C_m(\xi_0) = \frac{1}{2\pi i}\int_\gamma \frac{f_1^m}{f_2+\xi_0} df_2$ satisfies

$$C_k = Q_{k,p}(C_1, \dots, C_p t) \bmod \mathcal{O}(k)\,, \quad k > p\,,$$

$$C_d(\xi_0)=\sum_{k>d}\frac{(-\xi_0)^k}{2\pi i}\int_\gamma \frac{f_1^d}{f_2^{k+1}}df_2+A_d(\xi_0)\,,\quad 1\leqslant d\leqslant p\,,$$

where $Q_{k,p}$ are universal homogeneous polynomials.

In section 6.1, Theorem 3a is developed for $p>0$ into Theorem 4 below which gives a more effective criterion for the Plateau problem in $\mathbb{CP}_2$ and also for the IDN-problem. This new criterion follows from considerations on sums of shock-wave functions modulo affine functions in ξ_0. Even if decompositions in sum of shock-wave functions are studied for ξ_0-affine functions, Theorem 4 does not consider the case where G is of that type since it corresponds to a plain case of (1.4).

If H and u are holomorphic functions on a simply connected domain D, we set $\mathcal{D}_H=\frac{\partial}{\partial\xi_1}-\frac{\partial H}{\partial\xi_0}$ and denote by $\mathcal{L}_H u$ the unique function $v\in\mathcal{O}(D)$ such that $\partial v/\partial\xi_0=\mathcal{D}_H u$ and $v(0,\,.\,)=0$; π_1 is the projection $(\xi_0,\xi_1)\mapsto\xi_1$.

Theorem 4. *Let f be defined by* (1.2) *and consider the function G defined by* (1.3)*. We assume that γ is connected and that f_2 does not vanish on γ so that G, which is assumed to be not affine in ξ_0, is defined in a simply connected neighborhood D of* 0 *in $\mathbb{C}^2$.*

A. *If $(\overline{\mathcal{X}},\gamma)$ is a Riemann surface with almost smooth boundary where f extends meromorphically, then the following assertions hold for G:*

(1) *There is $p\in\mathbb{N}^*$ and holomorphic functions $a,b,\lambda_1,\dots,\lambda_{p-1}$ on $\Delta=\pi_1(D)$ such that the integro-differential equation*
$$-\mathcal{D}_{G+L}\mathcal{L}_{G+L}^{p-1}t(G+L)+\sum_{1\leqslant j\leqslant p-1}\mathcal{D}_{G+L}\mathcal{L}_{G+L}^{p-1-j}\widetilde{\lambda}_j=0$$
is valid with $L=\xi_0\otimes a+1\otimes b$ and $\widetilde{\lambda}_j=1\otimes\lambda_j$, $1\leqslant j\leqslant p$;

(2) *For $s_k=-\mathcal{L}_{G+L}^{k-1}G+\mathcal{L}_{G+L}^{k-2}\widetilde{\lambda}_1+\cdots+\mathcal{L}_{G+L}^{0}\widetilde{\lambda}_{k-1}$, $1\leqslant k\leqslant p$, the discriminant of $T_{\xi_0,\xi_1}=X^p+\sum_{1\leqslant k\leqslant p}s_k(\xi_0,\xi_1)X^{p-k}$ does not vanish identically in D;*

(3) *$G=-s_1-L$;*

(4) *Here is $q\in\mathbb{N}$, $\alpha,\beta\in\mathbb{C}_q[\xi_1]$ such that $\alpha(0)=0$, $\deg\beta<q$ and $a=\frac{\alpha'}{1-\alpha}$, $b=\frac{\beta}{1-\alpha}$.*

Moreover, if p is the least integer such that (1), (2) *and* (3) *assertions holds, (γ,f) uniquely determines $(a,b,\lambda_1,\dots,\lambda_{p-1})$.*

B. *Assume* (1), (2) *and* (3) *hold for some $p\in\mathbb{N}^*$. Then, there exists an open Riemann surface $\mathcal{X}$ such that $\overline{\mathcal{X}}=\mathcal{X}\cup\gamma$ is a manifold with almost smooth boundary where f extends meromorphically. Moreover,* (4) *holds.*

Remarks. 1. Non-unicity of $(a, b, \lambda_1, \dots, \lambda_{p-1})$ solving (1), (2) and (3) means that $\mathcal{X}$ exists but p is not minimal.

2. It is possible that regardless of its orientation, γ is the almost smooth boundary of an open Riemann surface $\mathcal{X}$ where f extends meromorphically. It is the case when γ cuts a compact Riemann surface $\mathcal{Z}$ into two smooth domains and f is the restriction to γ of an analytic map from $\mathcal{Z}$ to $\mathbb{CP}_2$.

2 Intrinsic EIT on Riemann Surfaces

The inverse Dirichlet–Neuman problem, which goes back to Calderon [C] and which is now called the Electrical-Impedance-Tomography problem, can be sketched like this: Suppose that a bounded domain $\mathcal{X}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is an ohmic conductor which means that the density of current j it may have is proportional (in isotropic cases) to the electrical field $e = \nabla U$ where U is an electrical potential. The scalar function σ such that $j = \sigma e$ is then called the conductivity of $\mathcal{X}$; $\rho = 1/\sigma$ is the resistivity. When there is no time dependence and no source or sink of current, the equation $\operatorname{div} j = 0$ holds and Calderon's problem is then to recover σ on the whole of $\mathcal{X}$ from the operator $C^\infty(\gamma) \ni u \mapsto (\sigma \nabla \widetilde{u})_\gamma$, $\widetilde{u}$ being the unique solution of $\operatorname{div}(\sigma \nabla \widetilde{u}) = 0$ with boundary value u.

In what follows, linking the Calderon problem to the Belishev problem mentioned in the introduction, we formulate the EIT-problem for a more general setting than the case of domains in $\mathbb{R}^n$. The second part of this section, despite the fact it is also quite elementary, seems to be new and underlines how complex structure is involved in the dimension-two case.

General dimension. Assume that $\mathcal{X}$, an open oriented bordered manifold of dimension n with boundary γ, is given with a volume form μ and a conductivity σ modelled as a tensor from $T^*\mathcal{X}$ to $\Lambda^{n-1}T^*\mathcal{X}$ (see [Sy]). The gradient associated to σ relative to μ is the differential operator which to any $f \in C^1(\mathcal{X})$ associates the tangent vector field $\nabla_{\mu,\sigma} f$ characterized by

$$(\nabla_{\mu,\sigma} f) \lrcorner \mu = \sigma(df)$$

where $\lrcorner$ is the interior product. When $U \in C^1(\mathcal{X})$ is some given potential, the density of physical current J is by definition

$$J = \nabla_{\mu,\sigma} U .$$

That $\nabla_{\mu,\sigma} U$ truly models the density of current is the assumption of Ohm's law. If $\mathcal{X}$ has no source or sink of currents and if U has no time dependence, the flux of current through the boundary of any domain is zero. Using

Stokes' formula, this can be modeled by the simplified Maxwell equation

$$0 = \operatorname{div}_\mu J = \operatorname{div}_\mu(\nabla_{\mu,\sigma} U) \tag{2.1}$$

where div_μ is the divergence with respect to the volume form μ; if t is a differentiable vector field, $\operatorname{div}_\mu t$ is defined by $d(t \lrcorner \mu) = (\operatorname{div}_\mu t)\mu$. Going back to the definition of gradient and divergence, we see that (2.1) is equivalent to the intrinsic equation formulated in [Sy] for domains in $\mathbb{R}^n$,

$$d\sigma(dU) = 0\,. \tag{2.2}$$

Since μ is no longer involved, the usual DN-operator has to be replaced by the operator Θ which to $u \in C^1(\gamma)$ associates $\sigma(du)_\gamma$ which is a section of $\Lambda^{N-1} t_\gamma^* \mathscr{X}$. The electrical impedance tomography problem, is then to reconstruct $(\mathscr{X}, \sigma)$ from its DN-map Θ. Of course, the two other aspects of this problem, unicity and characterization, also have to be studied.

The problem in such generality is still wide open; almost all publications are about domains in $\mathbb{R}^3$. In such a case, (2.1) is generally written in euclidean global coordinates. However, when $\mathscr{X}$ is a manifold, (2.2) yields the same equation in any chart (W, x); setting $\sigma dx_j = \sum_{1 \leqslant k \leqslant n} \sigma_{kj} (-1)^k dx_{\widehat{k}}$ with $dx_{\widehat{k}} = \bigwedge_{j \neq k} dx_j$, (2.2) becomes

$$\sum_{1 \leqslant k \leqslant n} \sum_{1 \leqslant j \leqslant n} \frac{\partial}{\partial x_k} \left(\sigma_{k,j} \frac{\partial U}{\partial x_j} \right) = 0\,. \tag{2.3}$$

When the conductivity σ is symmetric ($\sigma(a) \wedge b \equiv \sigma(b) \wedge a$) and invertible tensor, it is possible to design a natural metric $g_{\mu,\rho}$ associated to the resistivity map $\rho = \sigma^{-1}$ by the well-defined quotient of n-forms,

$$g_{\mu,\sigma^{-1}}(t) = \frac{\sigma^{-1}(t \lrcorner \mu) \wedge (t \lrcorner \mu)}{\mu}\,, \quad t \in T\mathscr{X}\,. \tag{2.4}$$

If (W, x) is any coordinates chart for $\mathscr{X}$, a direct calculus in x-coordinates shows that for $t = \Sigma t_k \partial/\partial x_k$ (2.4) becomes

$$g_{\mu,\rho}(t) = \sum_{k,\ell} t_\ell t_k \lambda \rho_{k,\ell}\,, \tag{2.5}$$

where $(\rho_{k,\ell})$ is the matrix of the resistivity $\rho = \sigma^{-1}$ when, at any given point z, the chosen basis for $\Lambda^{n-1} T_z^* \mathscr{X}$ and $T_z^* \mathscr{X}$ are $((-1)^k dx_{\widehat{k}})$ and (dx_k) respectively. When $(\sigma_{j,k})$ is positive definite, $g_{\mu,\rho}$ is a metric on $\mathscr{X}$.

When $n \geqslant 3$, there is a specially adequate choice of metric and volume.

Proposition 5. *Assume $n \geqslant 3$. Then one can correctly design a global volume form μ by letting it be defined by $\mu = [\det(\rho_{k,\ell})]^{\frac{-1}{n-2}} dx_1 \wedge \cdots \wedge dx_n$ in any coordinates chart (W, x) for $\mathscr{X}$. For this specific volume form, σ is the Hodge star operator of $g_{\mu,\rho}$ and μ is the riemannian volume form of $g_{\mu,\rho}$.*

This statement, already pointed out by Bossavit and Lee–Uhlmann (see [Bo] and [LU]) for domains in affine spaces, follows from calculus in coordinates.

The interest of Proposition 5 is to state the strict equivalence between the IDN-problem for riemannian manifolds and the EIT-problem when $n \geqslant 3$. When $\dim \mathcal{X} \geqslant 3$ and $\overline{\mathcal{X}}$ is a riemannian real analytic manifold with boundary, Lassas and Uhlmann have proved in [LU] that the DN-operator uniquely determines $\mathcal{X}$ and its metric.

The two-dimensional case. We now assume $n = 2$ and $\sigma = \rho^{-1}$ is symmetric and positive so that $(\mathcal{X}, g_{\mu,\rho})$ becomes a riemannian manifold whose volume form is thereafter denoted by $V_{\mu,\rho}$. Let us emphasize the complex structure associated to the conformal class of $(\mathcal{X}, g_{\mu,\rho})$ by choosing isothermal coordinates charts, that is holomorphic charts (see e.g. [V]). In such a chart (W, z),

$$g_{\mu,\rho} = \kappa_{\mu,\rho}(dx \otimes dx + dy \otimes dy) = \mathrm{Re}(\kappa_{\mu,\rho} dz \otimes d\overline{z}) \,,$$

where $x = \mathrm{Re}\, z$, $y = \mathrm{Im}\, z$ and $\kappa_{\mu,\rho} \in C^1(W, \mathbb{R}_+^*)$. Hence, in these coordinates, $(\sigma_{k,\ell}) = s\,\mathrm{diag}(1,1)$ with $\kappa_{\mu,\rho} = \lambda/s$ and $\lambda \in C^1(W, \mathbb{R}_+^*)$ is defined by $\mu = \lambda\, dx \wedge dy = \lambda \frac{i}{2} dz \wedge d\overline{z}$. Note that s is a global-positive function on $\mathcal{X}$ since it is the well-defined quotient of volume forms,

$$s = \mu / V_{\mu,\rho} \,.$$

Note also that s does not depend on μ and that (2.2), since $\sigma dU = s d^c U$, evolves into

$$d(s d^c U) = 0 \,, \tag{2.6}$$

where $d^c = i(\overline{\partial} - \partial)$, $\overline{\partial}$ and ∂ being the usual global differential operators associated to the complex structure of the conformal class of $(\mathcal{X}, g_{\mu,\rho})$. Hence, we have proved the following which generalizes a result written by Sylvester [Sy] for domains in $\mathbb{R}^2$.

PROPOSITION 6. *Let $\mathcal{X}$ be a real two dimensional manifold equipped with a symmetric and positive tensor $\sigma : T^*\mathcal{X} \to T^*\mathcal{X}$. Then, there is a complex structure on $\mathcal{X}$ and $s \in C^1(\mathcal{X}, \mathbb{R}_+^*)$, called scalar conductivity, such that* (2.2) *is equivalent to* (2.6).

The beginning of this paper has shown that the data $\partial U/\partial \nu$ is equivalent to the data $(\partial U)_\gamma$ which don't involve any metric. Since the knowledge of $(\partial U)_\gamma$ is equivalent to the knowledge of $(s d^c U)_\gamma$, we consider $(s d^c U)_\gamma$ as the DN-datum. We can now state an intrinsic IDN-problem for two-dimensional ohmic conductors; for the sake of simplicity, we limit ourselves to manifolds with boundary and smooth data.

A *two-dimensional ohmic conductor* is a couple $(\mathcal{X}, \rho)$ where $\mathcal{X}$ is an open-oriented bordered two-dimensional real surface (with boundary γ), the conductivity $\sigma = \rho^{-1}$ is a positive definite tensor from $T^*\mathcal{X}$ to $T^*\mathcal{X}$, and $\mathcal{X}$ is equipped with the complex structure associated to the riemannian metric $g_{\mu,\rho}$ defined by (2.4) where μ is any volume form of $\mathcal{X}$. In this setting, the scalar conductivity is the function $s = \mu/V_{\mu,\rho}$ where $V_{\mu,\rho}$ is the volume associated to $g_{\mu,\rho}$. The DN-operator is the operator $\theta_{\mathcal{X},\rho}$ defined by

$$\theta_{\mathcal{X},\rho} : C^1(\gamma) \ni u \mapsto (sd^c\widetilde{u})_\gamma \in T^*_\gamma\mathcal{X} \,,$$

where $\widetilde{u}$ is the unique solution of the following Dirichlet problem:

$$U|_\gamma = u \quad \text{and} \quad d(s\, d^cU) = 0 \,. \tag{2.7}$$

The IDN-problem associated to this setting is threefold:

Unicity. Assume that two dimensional ohmic conductors $(\mathcal{X},\rho)$ and $(\mathcal{X}',\rho')$ share the same boundary γ and the same DN-operator θ. Is it true that there is a diffeomorphism $\varphi : \overline{\mathcal{X}} \to \overline{\mathcal{X}'}$ between manifolds with boundaries such that $\varphi : \mathcal{X} \to \mathcal{X}'$ is analytic and $s = s' \circ \varphi$ where s and s' are scalar conductivities of $\mathcal{X}$ and $\mathcal{X}'$?

Reconstruction. Assume that $(\mathcal{X}, \rho)$ is a two dimensional ohmic conductor. How can one reconstruct, from its DN-operator, a two dimensional ohmic conductor $(\mathcal{X}', \rho')$ which is isomorphic (in the above sense) to $(\mathcal{X}, \rho)$?

Characterization. Let γ be a smooth abstract real curve, L a complex line bundle along γ and θ an operator from $C^1(\gamma)$ to the space of smooth sections of L. Find a non-trivial, necessary and sufficient condition on (γ, L, θ) which ensures that there exists a two dimensional ohmic conductor $(\mathcal{X}, \rho)$ such that $L = \Lambda^{1,0}T^*_\gamma\mathcal{X}$ and $\theta = \theta_{\mathcal{X},\rho}$.

All these problems are open. In the particular case of constant scalar conductivity σ, the Dirichlet problem (2.7) becomes

$$U|_\gamma = u \quad \text{and} \quad dd^cU = 0 \,,$$

where $dd^c = i\partial\overline{\partial}$ is the usual Laplacian. Hence, with Theorems 1 to 3c, our article gives a complete answer to the EIT-problem with constant scalar conductivity.

Concerning the main results given in the literature about unicity, reconstruction and stability for the important case where $\mathcal{X}$ is a domain in $\mathbb{R}^2$ but the scalar conductivity is not constant, see [BrU], [M] and references therein. Note that the exact method of reconstruction for this case goes back to [N].

3 Unicity Under Existence Assumption

The notation and hypothesis are taken from Theorem and section 1; we equip $\mathcal{X}$ with a hermitian metric g. Harmonicity does not depend of the chosen hermitian metric. Hence, there is a compact subset σ of γ such that $h^1(\sigma) = 0$ and $(\overline{\mathcal{X}}, \gamma)$ is a manifold with boundary near each point of $\gamma\backslash\sigma$.

When $u \in C^\infty(\gamma)$, Proposition 12 implies that u has a continuous harmonic extension $\widetilde{u}$ with finite Dirichlet integral on $\mathcal{X}$. An elementary calculus gives then that for a fixed continuous defining function ρ of γ, smooth on $\overline{\mathcal{X}}\backslash\sigma$, the operator L defined by (1.1) determines for all $u \in C^\infty(\gamma)$ the trace on γ of the holomorphic $(1,0)$-form $\partial\widetilde{u}$: $\partial\widetilde{u} = (Lu)|\partial\rho|_g^{-1}\partial\rho$ on $\gamma\backslash\sigma$. With (1.2), this implies that f is the restriction to γ of a function $F = (F_1, F_2)$ meromorphic on $\mathcal{X}$, smooth on $\gamma\backslash\sigma$. Since $(\gamma, u, \theta u)$ is assumed to be a restricted DN-datum for $\mathcal{X}$, F is continuous in a neighborhood of γ in $\overline{\mathcal{X}}$.

The proof of Theorem 1 relies on the following lemmas which enable us to see $\overline{\mathcal{X}}$ as a normalization of $\overline{F(\mathcal{X})}$.

Lemma 7. *Set $\delta = f(\gamma)$. Then $\mathcal{Y} = F(\mathcal{X})\backslash\delta$ is a complex curve of $\mathbb{CP}_2\backslash\delta$ without compact component, which has finite mass and satisfies $d[\mathcal{Y}] = [\delta]$. Moreover, each regular point of $\overline{\mathcal{X}}$ has in $\overline{\mathcal{X}}$ a neighborhood V such that $F : V \to F(V)$ is a diffeomorphism between manifolds with smooth boundary.*

Proof. Since F is continuous in a neighborhood of γ in $\overline{\mathcal{X}}$, $\mathcal{Y}$ is a closed set of $\mathbb{CP}_2\backslash\delta$. As $\mathcal{Y}$ is also locally the image of a Riemann surface by an analytic map, $\mathcal{Y}$ is a complex curve of $\mathbb{CP}_2\backslash\delta$. Since $F_*[\mathcal{X}]$ is a locally flat current, the Federer support theorem (see [H, p. 316], [F, 4.1.15 & 4.1.20]) produces a locally integrable function λ on $\mathcal{Y}$ such that $F_*[\mathcal{X}] = \lambda[\mathcal{Y}]$ on the regular part $\mathcal{Y}_{\mathrm{reg}}$ of $\mathcal{Y}$; since $d^2 = 0$, λ is locally constant. Since f embeds γ into $\mathbb{C}^2$, each point x in γ which is a regular boundary point of $\overline{\mathcal{X}}$ has a neighborhood V of x in $\overline{\mathcal{X}}$ such that $F : V_x \to F(V_x)$ is a diffeomorphism between classical manifolds with boundary. Let F_W be the restriction of F to the Riemann surfaces $W = \cup V_x$ and $W' = F(W)$. The degree of F_W is at most 1 otherwise almost all points of W' would have at least two different preimages which would imply that df is zero at almost all points of γ. So this degree is 1 and $F_*[\mathcal{X}] = [\mathcal{Y}]$ on W'. Hence, $\lambda = 1$ on each connected component of $\mathcal{Y}$ and $d[\mathcal{Y}] = [\delta]$.

If $\mathcal{Y}$ contains a compact complex curve $\mathcal{Z}$, $F^{-1}(\mathcal{Z})$ is a complex curve in $\mathcal{X}$ without boundary and so is empty. The fact that $\mathcal{Y}$ has a finite mass follows from a theorem of Wirtinger (see [H, Lem. 1.5 p. 315]). □

As δ is smooth, the conclusion of Lemma 7 implies, thanks to [HL1], that δ contains a compact set τ such that $h^1(\tau) = 0$ and $(\overline{\mathcal{Y}}, \delta)$ is a manifold with boundary near points of $\delta \backslash \tau$. The lemma below described how $\mathcal{Y}$ is near a point y of τ.

Lemma 8. *Assume $\mathcal{Y}$ is a complex curve of $\mathbb{CP}_2 \backslash \delta$ with finite mass satisfying $d[\mathcal{Y}] = \pm[\delta]$. Let y be a point of σ and U a domain containing y. Then, among the components of $\mathcal{Y} \cap U$, $\mathcal{C}^U_{y,1}, \dots, \mathcal{C}^U_{y,m_U}$, one, says $\mathcal{C}^U_{y,1}$, satisfies $d[\mathcal{C}^U_{y,1}] = \pm[\delta]|_{U_y}$ whereas for $j \geqslant 2$, $\overline{\mathcal{C}^U_{y,j}} \cap U$ is a complex curve of U.*

Proof. [HL1, Th. 4.7] implies that for each j there is $n_j \in \mathbb{Z}$ such that $d[\mathcal{C}^U_{y,j}] = n_j[\delta]$ on U. As $d[\mathcal{Y}] = \pm[\delta]$, $\Sigma d[\mathcal{C}^U_{y,j}] = \pm 1$ and at least one $\mathcal{C}^U_{y,j}$, says $\mathcal{C}^U_{y,1}$, is such that $n_j \neq 0$. Because $h^1(\sigma) = 0$, $\delta \cap U$ contains a point q not in σ. Then, if V is a sufficiently small ball centered at q, $\mathcal{Y} \cap V$ is submanifold of V with boundary $\delta \cap V$, and $\mathcal{Y} \cap V$ has only one connected component which can be nothing else than $\mathcal{C}^U_{y,1} \cap V$. Hence $n_1 = \pm 1$. Since two different bordered Riemann surfaces of some open set of $\mathbb{CP}_2$ meet at most in a set of zero one-dimensional Hausdorff measure, this implies that $n_j = 0$ for $j \neq 1$. Thus, if $j \geqslant 2$, $d[\mathcal{C}^U_{y,j}] = 0$ and with [H, Th. 2.1, p. 37] we conclude that $\overline{\mathcal{C}^U_{y,j}} \cap U$ is a complex curve of U_y. □

If $y \in \delta$, we denote by m_y the limit of m_U (see Lemma 8) when the diameter of U goes to 0, U neighborhood of y; if $m_y \geqslant 2$, then $y \in \tau$. A point y of δ is called a *strong singularity of* $\overline{\mathcal{Y}}$ if y is not a regular point of $\mathcal{C}^U_{y,1}$ and a *weak singularity* of $\overline{\mathcal{Y}}$ if y is regular point of $\mathcal{C}^U_{y,1}$ but $m_y \geqslant 2$.

We denote by τ_1 (resp. τ_2) the sets of points where $\overline{\mathcal{Y}}$ has weak (resp. strong) singularity. Then $\tau = \tau_1 \cup \tau_2$ and $\tau_1 \cap \tau_2 = \varnothing$. Note that both τ_1 and τ_2 may contain points y where $m_y \geqslant 2$.

We denote by $\overline{\mathcal{Y}}_{\rm sing} = \mathcal{Y}_{\rm sing} \cup \tau$ the singular locus of $\overline{\mathcal{Y}}$, that is the set of points of $\overline{\mathcal{Y}}$ where $\overline{\mathcal{Y}}$ is not a smooth manifold with boundary and we set $\mathcal{B} = f(\sigma) \cup \overline{\mathcal{Y}}_{\rm sing}$, $\mathcal{A} = F^{-1}(\mathcal{B})$ and $\mathcal{X}_\circ = \mathcal{X} \backslash F^{-1}(\delta) = \mathcal{X} \backslash F^{-1}(\tau_2)$.

Lemma 9. *The map $F : \overline{\mathcal{X}} \to \overline{\mathcal{Y}}$ is a normalization in the following sense: $F : \mathcal{X}_\circ \to \mathcal{Y}$ is a (usual) normalization and $F : \overline{\mathcal{X}} \backslash \mathcal{A} \to \overline{\mathcal{Y}} \backslash \mathcal{B}$ is a diffeomorphism between manifolds with boundary.*

Proof. Since $\mathcal{X}_\circ = \mathcal{X} \backslash F^{-1}(\delta)$, the properness of $F|_{\mathcal{X}_\circ}$ and the finiteness of its fibers are elementary. For each connected component $\mathcal{C}$ of $\mathcal{X}_\circ \backslash \mathcal{A}$, the degree $m_{\mathcal{C}}$ of $F : \mathcal{C} \to F(\mathcal{C})$ as a Riemann surfaces morphism is finite and $F_*[\mathcal{C}] = \delta_{\mathcal{C}}[F(\mathcal{C})]$. Reasoning as in Lemma 7's proof, we get $m_{\mathcal{C}} = 1$.

As $\mathcal{Y}_{\rm sing}$ contains all the points of $\mathcal{Y}$ which has more than one preimage by F, $F : \mathcal{C} \to F(\mathcal{C})$ is thus an isomorphism. Let $\mathcal{C}'$ be another connected component of $\mathcal{X}_\circ \backslash \mathcal{A}$ and assume that $F(\mathcal{C})$ and $F(\mathcal{C}')$ meet at q. Since $q \notin \mathcal{B}$, the germs of $F(\mathcal{C})$ and $F(\mathcal{C}')$ at q are equal. This leads to $F(\mathcal{C}) = F(\mathcal{C}')$ which yields the contradiction $d[\mathcal{Y}] = 2[\delta]$ near regular boundary points of δ in $bF(\mathcal{C})$. Hence, $F : \mathcal{X}_\circ \backslash \mathcal{A} \to \mathcal{Y} \backslash \mathcal{B} = \mathcal{Y}_{\rm reg}$ is an isomorphism of complex manifolds. As $\mathcal{X}_\circ \cap \mathcal{A} = F^{-1}(\mathcal{Y}_{\rm sing})$ has empty interior, $F : \mathcal{X}_\circ \to \mathcal{Y}$ is a usual normalization.

Set $\widetilde{\mathcal{X}} = \overline{\mathcal{X}} \backslash \mathcal{A}$, $\widetilde{\mathcal{Y}} = \overline{\mathcal{Y}} \backslash \mathcal{B}$, $\widetilde{\tau} = \tau \cup f(\sigma)$ and $\widetilde{\delta} = \delta \backslash \widetilde{\tau}$; by definition of $\mathcal{B}$, $\widetilde{\mathcal{Y}}$ is a manifold with smooth boundary $\widetilde{\delta} = \delta \backslash \widetilde{\tau}$ and $\widetilde{\mathcal{X}}$ has smooth boundary $\gamma \backslash \widetilde{\sigma}$ where $\widetilde{\sigma} = f^{-1}(\tau) = \sigma \cup f^{-1}(\tau)$. The map $F : \widetilde{\mathcal{X}} \to \widetilde{\mathcal{Y}}$ is onto by construction. It is injective because the maps $F : \mathcal{X}_\circ \to \mathcal{Y}$ and $f : \gamma \to \delta$ are so and because if $x_1 \in \mathcal{X}$ and $x_2 \in \gamma$ have the same image y by F, then $m_y \geqslant 2$, $y \in \tau$ and $x_1, x_2 \in \mathcal{A}$. Since $\widetilde{\mathcal{Y}} \backslash \widetilde{\tau} = \mathcal{Y}_{\rm reg}$ and $F : \mathcal{X}_\circ \backslash \mathcal{A} \to \mathcal{Y}_{\rm reg}$ is a diffeomorphism, the fact that $F : \widetilde{\mathcal{X}} \to \widetilde{\mathcal{Y}}$ is a diffeomorphism between manifolds with boundary has only to be check locally near boundary points. If $x \in \gamma \backslash \widetilde{\sigma}$, then $y = f(x) \notin \tau$ and the last conclusion of Lemma 7 implies that there are open neighborhoods V and W of x and y in $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$ such that $F : V \to W$ is a diffeomorphism between manifolds with boundary. □

3.1 Proof of Theorem 1. Let L' be the operator defined by (1.1) when N is changed for N', let us denote F' the meromorphic extension of f to $\mathcal{X}'$ and let $\mathcal{Y}' = F'(\mathcal{X}') \backslash \delta$ where $\delta = f(\gamma)$. By Lemma 7, the sets $\mathcal{Y}'$ and $\mathcal{Y}$ are two complex curves of $\mathbb{CP}_2 \backslash \delta$ which has no compact component and both are bordered by $[\delta]$ in the sense of currents. Hence they are identical by a consequence of a Harvey–Shiffman theorem (see [DoH, Prop. 1.4.1]).

Taking Lemma 9 into account and the fact that $\mathcal{B} \cap \mathcal{Y} = \mathcal{Y}_{\rm sing} = \mathcal{B}' \cap \mathcal{Y}$, this implies that $\Phi = F^{-1} \circ F'$ is an analytic isomorphism between $\mathcal{X}' \backslash F'^{-1}(\mathcal{Y}_{\rm sing})$ and $\mathcal{X}' \backslash F^{-1}(\mathcal{Y}_{\rm sing})$. Using the properness of $F : \mathcal{X}_\circ \to \mathcal{Y}$ and $F' : \mathcal{X}'_\circ \to \mathcal{Y}$, we conclude that Φ extends holomorphically to $\mathcal{X}'$. Likewise, $\Psi = F'^{-1} \circ F$ extends holomorphically to $\mathcal{X}$. As $\Phi(\Psi(x')) = x'$ and $\Psi(\Phi(x)) = x$ for almost all $x' \in \mathcal{X}'$ and $x \in \mathcal{X}$, the extension of Φ is an isomorphism from $\mathcal{X}$ to $\mathcal{X}'$.

As F and F' extend f to $\mathcal{X}$ and $\mathcal{X}'$, Φ extends continuously to γ by the identity map on γ. Set $\sigma = \overline{\mathcal{X}}_{\rm sing}$ and $\sigma' = \overline{\mathcal{X}'}_{\rm sing}$ and let x' be in $\gamma \backslash (\sigma \cup \sigma')$. Then if $y = f(x) \notin \tau = \delta \cap \overline{\mathcal{Y}}_{\rm sing}$, Φ is a diffeomorphism between neighborhoods of x in $\overline{\mathcal{X}}$ and $\overline{\mathcal{X}'}$ because F (resp. F') is a diffeomorphism between a manifold with boundary from a neighborhood of x in $\overline{\mathcal{X}}$ (resp. in $\overline{\mathcal{X}'}$) to a neighborhood of y in $\overline{\mathcal{Y}}$. If $y \in \tau$, then the last conclusion of

Lemma 7 implies that $y \in \tau_1$ so that there is a open neighborhood U of y, a component $\mathcal{C}_{y,1}^U$ of $\mathcal{Y} \cap U$ and open neighborhoods V and V' of x in $\overline{\mathcal{X}}$ and $\overline{\mathcal{X}'}$ such that $F : V \to \mathcal{C}_{y,1}^U$ and $F' : V' \to \mathcal{C}_{y,1}^U$ are diffeomorphisms between manifolds with smooth boundary. Hence, $\Phi : V \to V'$ is a diffeomorphism between manifolds with smooth boundary. Finally, Φ realizes a diffeomorphism between manifolds with smooth boundary from $\mathcal{X}' \backslash (\sigma \cup \sigma')$ to $\mathcal{X} \backslash (\sigma \cup \sigma')$ and the proof is complete. □

The proof contains the following variation of Theorem 1.

Theorem 10. *Assume that $\mathcal{X}$ and $\mathcal{X}'$ are open Riemann surfaces with almost smooth boundary γ such that the map f defined by* (1.2) *is an embedding of γ into $\mathbb{CP}_2$ and has a meromorphic extension F to $\mathcal{X}$ and F' to $\mathcal{X}'$ which are continuous near γ. Then $F(\mathcal{X}) \backslash f(\gamma) = F'(\mathcal{X}')/f(\gamma) \overset{def}{=} \mathcal{Y}$ is a complex curve of $\mathbb{CP}_2 \backslash \delta$ without compact component, which has finite mass and satisfies $d[\mathcal{Y}] = [\delta]$. Moreover, $\overline{\mathcal{X}}$ and $\overline{\mathcal{X}'}$ are normalizations of $\overline{\mathcal{Y}}$ in the sense of Lemma 9.*

Thus, Riemann surfaces constructed in the converse part of Theorems 3a, 3b and 3c are the only possible candidates for a solution to the IDN-problem.

4 Existence and Reconstruction, Proof of Theorem 2

We first prove that the Stokes formula holds in almost smoothly bordered manifolds.

Lemma 11. *Let $(\overline{\mathcal{X}}, \gamma)$ be a Riemann surface with almost smooth boundary. Then for any 1-form φ which is continuous on $\overline{\mathcal{X}}$ such that $d\varphi$ exists as an integrable differential on $\mathcal{X}$, we have*

$$\int_{\mathcal{X}} d\varphi = \int_{\gamma} \varphi \,. \tag{4.1}$$

Proof. Set $\sigma = \overline{\mathcal{X}}_{\text{sing}}$. Since $h^2(\overline{\mathcal{X}}) < \infty$ and $h^1(\sigma) = 0$, there is an increasing sequence $(\mathcal{X}_k)$ of smooth open sets of $\mathcal{X}$ such that $(h^1(b\mathcal{X}_k)\backslash\gamma)$ and $(h^2(\overline{\mathcal{X}}\backslash\mathcal{X}_k))$ both have limit zero and $\overline{\mathcal{X}}\backslash\overline{\mathcal{X}_k}$ is contained in a 2^{-k}-neighborhood of σ. Let φ be as above. Since $d\varphi$ is integrable and $\lim h^2(\overline{\mathcal{X}}\backslash\mathcal{X}_k) = 0$, $(\int_{\mathcal{X}_k} d\varphi)$ has limit $\int_{\mathcal{X}} d\varphi$. As $b\mathcal{X}_k = (\gamma \cap \overline{\mathcal{X}_k}) \cup [(b\mathcal{X}_k)\backslash\gamma]$ and $\lim h^1((b\mathcal{X}_k)\backslash\gamma) = 0$, $(\int_{(b\mathcal{X}_k)\backslash\gamma} \varphi)$ converges to 0. Hence, $\lim \int_{\gamma\cap\overline{\mathcal{X}_k}} \varphi = \int_{\gamma} \varphi$ and the classical Stokes formula for φ and $\mathcal{X}_k$ yields (4.1). □

We now prove a variation of Riemann's existence theorem.

Proposition 12. *Let $(\overline{\mathcal{X}}, \gamma)$ be a Riemann surface with almost smooth boundary and u a real valued lipschitzian function on γ. Then u has a unique continuous harmonic extension $\widetilde{u}$ of u to $\mathcal{X}$ and $\widetilde{u}$ has finite Dirichlet integral $\int i\,\partial\widetilde{u} \wedge \overline{\partial}\widetilde{u}$. Moreover, $N_{\mathcal{X}}u$ defined as $\partial\widetilde{u}/\partial\nu$ on $\gamma\backslash\overline{\mathcal{X}}_{\mathrm{sing}}$ admits an extension on γ as a current of order 1 on γ.*

Proof. Following the lines of Riemann's method for harmonic extension of smooth functions, we first construct an adequate space $W^1(\mathcal{X})$.

Since $(\overline{\mathcal{X}}, \gamma)$ is at least a topological bordered manifold, for every fixed point x in γ, we can choose in $\mathcal{X}$ an open set Δ_x whose closure in $\overline{\mathcal{X}}$ is a neighborhood of x and which is mapped by a complex coordinate φ_x into the closure of the unit disk $\mathbb{D}$ of $\mathbb{C}$, φ_x being a homeomorphism from $\overline{\Delta_x}$ to $\overline{\mathbb{D}}$. Note that, if $x' \in \gamma \cap \overline{\Delta_x}$ is a regular point of $\overline{\mathcal{X}}$, φ_x has to be a diffeomorphism between manifolds with boundary from a neighborhood of x' to a neighborhood of $\varphi_x(x')$ in $\overline{\mathbb{D}}$. If $x \in \mathcal{X}$, we choose a conformal open disk $\varphi_x : \Delta_x \to \mathbb{D}$ of $\mathcal{X}$ centered at x. With the help of a continuous partition of unity, we can now construct a continuous hermitian metric h on $\mathcal{X}$ by gluing together the local metrics $(\varphi_x)_* dz \wedge d\overline{z}$ where z is the standard coordinate of $\mathbb{C}$. We then denote by $W^1(\mathcal{X})$ the Sobolev space of functions in $L^2(\mathcal{X}, h)$ with finite Dirichlet integral.

By construction, any function A in $W^1(\mathcal{X})$ is such that for each $x \in \gamma$, $B_x = (\varphi_x)_* A|_{\Delta_x}$ is square integrable for the standard metric of $\mathbb{D}$. Since the values of Dirichlet integrals are conformal invariants, it follows that B_x is in the standard Sobolev space $W^1(\mathbb{D})$ and hence admits a boundary value b_x on $\mathbb{T} = b\mathbb{D}$ which is in $W^{1/2}(\mathbb{T})$. As b_x is punctually defined almost everywhere, $a_x = b_x \circ \varphi$ is defined almost everywhere in $\gamma \cap \overline{\Delta_x}$. The constructions made for each $x \in \gamma$ glue together to form a function defined almost everywhere in γ which we call the boundary value of A.

We consider now the subset F of $W^1(\mathcal{X})$ with boundary value u. It is closed and non-empty since, by a result of McShane [Mc], u admits a Lipschitz extension to $\mathcal{X}$. It follows now from classical arguments that the Dirichlet integral can be minimized in F at some function $\widetilde{u}$ which has to be harmonic in $\mathcal{X}$. It remains only to show that $\widetilde{u}$ is continuous on $\overline{\mathcal{X}}$. If $x \in \gamma$, what precedes implies that $v_x = (u_x \circ \varphi_x^{-1})|_{\mathbb{T}}$ is in $W^{1/2}(\mathbb{T})$, continuous near $\varphi_x(x)$ and is the boundary value of $\widetilde{v}_x = \widetilde{u} \circ \varphi_x^{-1}$. Hence, the classical Poisson formula for the disc implies that, near $\varphi_x(x)$ in $\overline{\mathbb{D}}$, $\widetilde{v}_x$ is continuous up to $\mathbb{T}$ with restriction v_x on $\mathbb{T}$. Since φ_x is an homeomorphism, we get that $\widetilde{u}$ is continuous at x with value $u(x)$.

Let θu be the form defined by (1.1). The Stokes formula (4.1) implies that, if $\varphi \in C^1(\gamma)$ and Φ is a Lipschitz extension of φ on $\mathcal{X}$,

$$\int_\gamma \varphi\theta u = -\int_{\mathcal{X}} \partial\widetilde{u}\wedge\overline{\partial}\Phi\,.$$

As the last integral is independent of the Lipschitz extension of φ, this means that θu and hence $N_{\mathcal{X}}u$, are well-defined currents of order 1. □

Now assume that the hypotheses of Theorem 2 are true. Lemma 7 points out that F projects γ on a smooth curve δ of $\mathbb{C}^2$ which bounds in the sense of currents a complex curve $\mathcal{Y}$ of $\mathbb{CP}_2\backslash\delta$ which has finite mass and no compact component, and Theorem 10 implies that for some subset $\mathcal{X}_\circ$ of $\mathcal{X}$, with discrete complement in $\mathcal{X}$, $F:\mathcal{X}_\circ\to\mathcal{Y}$ is a usual normalization. Hence, if $\mathcal{B}=\mathcal{Y}_{\mathrm{sing}}$ and $\mathcal{A}=F^{-1}(\mathcal{B})$, $F:\mathcal{X}\backslash\mathcal{A}\to\mathcal{Y}\backslash\mathcal{B}$ is one to one. This is part 1 of Theorem 2.

Before proving the second claim of Theorem 2, we recall that $\mathbb{CP}_2$ is equipped with homogenous coordinates w and $\mathbb{C}^2$ identified with $\{w_0\neq 0\}$ have affine coordinates $z_1=w_1/w_0$ and $z_2=w_2/w_0$. Set $\Delta_\infty=\{w_0=0\}$, $\mathcal{Y}_\infty=\mathcal{Y}\cap\Delta_\infty$ and, if $\xi\in\mathbb{C}$, we set $\Delta_\xi=\{w_2=\xi w_0\}$ and $\mathcal{Y}_\xi=\mathcal{Y}\cap\Delta_\xi$. Set

$$\Omega_\xi^m=\frac{z_1^m}{z_2-\xi}dz_2=\frac{w_1^m}{w_0^m}\frac{dw_2}{w_2-\xi w_0}-\frac{w_1^m}{w_0^{m+1}}\frac{w_2dw_0}{w_2-\xi w_0}\,.$$

Applying the Stokes formula either for $\mathcal{Y}$ or $\mathcal{X}$, it turns out that $\frac{1}{2\pi i}\int_\gamma\frac{f_1^m}{f_2-\xi}df_2$ equals $S_m\left(\xi\right)+P_m\left(\xi\right)$ where

$$S_m(\xi)=\sum_{z\in\mathcal{Y}_\xi}\mathrm{Res}(\eta^*\Omega_\xi^m,z)\,,\quad P_m(\xi)=\sum_{z\in\mathcal{Y}_\infty}\mathrm{Res}(\eta^*\Omega_\xi^m,z)\,,$$

and $\eta:\mathcal{Y}\to\mathbb{CP}_2$ is the canonical injection.

For almost all ξ_* in $\mathbb{C}$, $\mathcal{Y}$ meets Δ_{ξ_*} transversely only in $\mathbb{C}^2\cap\mathcal{Y}_{\mathrm{reg}}$; for such a fixed ξ_*, set $p=\mathrm{Card}\,\mathcal{Y}_{\xi_*}$ and $\mathcal{Y}_{\xi_*}=\{z_{1*},\dots,z_{p*}\}$. For ξ in a sufficiently small connected neighborhood W_{ξ_*} of ξ_*, $\mathcal{Y}_\xi$ lies then in $\mathbb{C}^2\cap\mathcal{Y}_{\mathrm{reg}}$ and can be written $\{z_1(\xi),\dots,z_p(\xi)\}$ with $z_j(\xi)=(h_j(\xi),\xi)$ where h_j is holomorphic in W_{ξ_*} and has value z_{j*} at ξ_*, $1\leqslant j\leqslant p$. Direct calculation shows (see [DoH]) that the poles of Ω_ξ^m in $\mathbb{C}^2$ are $z_1(\xi),\dots,z_p(\xi)$ with residue $h_1(\xi)^m,\dots,h_p(\xi)^m$. Hence, $S_m=S_{h,m}$ in V.

Reasoning as in Lemma 13 in the next section, we can assume, without loss of generality, that $\mathcal{Y}$ meets Δ_∞ transversely and that $\mathcal{Y}_\infty\subset\mathcal{Y}_{\mathrm{reg}}$. In this situation, a direct calculus (see [DoH]) gives that, at $y\in\mathcal{Y}_\infty$, Ω_ξ^m has a pole of order $m+1$ with a residue which is a polynomial in ξ of degree at most m. Hence, P_m is a polynomial in ξ of degree at most m; formula $(E_{m,\xi})$ is proved.

If $A\geqslant B$ and $\xi_0,\dots,\xi_{A-1}$ are mutually distinct, the Vandermonde matrix $(\xi_\nu^\mu)_{0\leqslant\nu,\mu\leqslant B-1}$ is invertible and, hence, the system $(E_{m,\xi_\nu})_{0\leqslant\nu\leqslant B-1}$

enables us to write the coefficients of P_m as a linear combination of the $S_{h,m}(\xi_\nu)$, $0 \leqslant \nu \leqslant B-1$. Introducing this result in $(E_\xi) = (E_{m,\xi_\nu})_{\substack{0\leqslant m\leqslant B-1\\ 0\leqslant \nu\leqslant B-1}}$, we get a linear system which, since $AB - \frac{1}{2}B(B+1) \geqslant pA$ when $B \geqslant 2p+1$, enables us to compute, for a generic ξ, the unknowns $S_{h,m}(\xi_\nu)$ and, thanks to the Newton–Girard formulas, the elementary symmetric functions of $h_1(\xi_\nu), \dots, h_p(\xi_\nu)$; finally we get the intersection points $(h_j(\xi_\nu), \xi_\nu)$ of $\mathcal{Y}$ with Δ_{ξ_ν}.

We prove the third assertion of Theorem 2. Almost all ξ_* in $\mathbb{C}^2$ have a connected neighborhood W_{ξ_*} such that there is a compact of $\mathbb{CP}_2\backslash(\delta \cup \mathcal{Y}_{\text{sing}})$ containing all $\mathcal{Y}_\xi$ when $\xi \in W_{\xi_*}$. When ξ_* is such and $\xi \in W_{\xi_*}$, the form $\Phi_\xi^{m,\ell} = \frac{F_1^m}{F_2-\xi}\partial\widetilde{u}_\ell$ may have poles of order at most m at infinity, i.e. in $\{w_0 = 0\} \cap \mathcal{X}$ and while its other poles lie in a compact of $\mathcal{X}$. Since $\widetilde{u}_\ell$ is the continuous harmonic extension of u_ℓ on $\mathcal{X}$, $\int_{\mathcal{X}} i\,\partial\widetilde{u}_\ell \wedge \overline{\partial}\widetilde{u}_\ell < +\infty$ by Proposition 12 and we can apply the Stokes formula (4.1) to it on $\mathcal{X}$. This gives $(T_{m,\xi})$ after a residue calculus.

Remark. The $\frac{1}{2}B(B+1)$ coefficients of the polynomials P_k come from the residues of the intersection points of $\mathcal{Y}$ with $\{w_0 = 0\}$. In the generic case, $\mathcal{Y}$ is given near theses points as the graph of holomorphic functions $\psi_1, \dots, \psi_q$ of the variable w_0/w_2, it appears that the coefficients of P_k are ruled by the derivatives of order at most k at 0 of the ψ_ℓ. The reconstruction of $\mathcal{X}$ is thus possible with a non-linear system with only $pA + q(B+1)$ unknowns.

5 Proofs of Characterizations Theorem 3a, 3b and 3c

The proofs of Theorems 3a, 3b and 3c follow a similar schema. The function f defined by (1.2) embeds γ into a smooth real curve $\delta = f(\gamma)$ of $\mathbb{C}^2$. The necessary conditions for the existence of a solution to the IDN-problem for γ are drawn from the fact that this existence implies that δ bounds a "concrete" Riemann surface in $\mathbb{CP}_2$ or $\mathbb{C}^2$. The sufficient part of Theorem 3c reconstructs the concrete but singular solution to the IDN-problem; a normalization gives then the sufficient part of 3b. The proof of Theorem 3a follows a similar scheme.

5.1 Proof of Theorem 3a.A. Assume that $\mathcal{X}$ is an open bordered riemannian surface of finite volume with restricted DN-datum $(\gamma, u, \theta u)$. Then the functions F_j $(j = 1, 2)$ which are the well-defined quotients of forms $(\partial\widetilde{u}_j)/(\partial\widetilde{u}_0)$ are meromorphic, and letting $F = (F_1, F_2)$, Lemma 7 implies that $\mathcal{Y} = F(\mathcal{X})\backslash\delta$, $\delta = f(\gamma)$, is a complex curve of finite volume, without a compact component and bordered by $[\delta]$ in the sense of currents. Moreover, the function G has the expression

$$G(\xi_0,\xi_1)=\frac{1}{2\pi i}\int_\delta \Omega_\xi\,,\quad \Omega_\xi=\frac{w_1}{w_0}\frac{d\Lambda_\xi(w)}{\Lambda_\xi(w)}-\frac{w_1}{w_0^2}dw_0\,,$$

where $\delta=f(\gamma)$, $(w_0:w_1:w_2)$ are homogenous coordinates for $\mathbb{CP}_2$ and $\Lambda_\xi(w)=\xi_0w_0+\xi_1w_1+w_2$.

For almost all $\xi_*=(\xi_{0*},\xi_{1*})$ and for all ξ in a sufficiently small connected neighborhood W_{ξ_*} of ξ_*, $\mathcal{Y}$ meets $\Delta_\xi=\{\Lambda_\xi=0\}$ transversely, $\mathcal{Y}_\xi=\mathcal{Y}\cap\Delta_\xi\subset\mathbb{C}^2\cap\mathcal{Y}_{\mathrm{reg}}$ so that there exists $p=\operatorname{Card}\mathcal{Y}_{\xi_*}$ holomorphic functions $H_j=(1:h_j:h_{j,2}):W_{\xi_*}\to\mathbb{CP}_2$ such that $\mathcal{Y}_\xi=\{H_j(\xi),\,1\leqslant j\leqslant p\}$ and $(h_j)_{1\leqslant j\leqslant p}$ are mutually distinct. Direct calculations shows that these functions satisfy the shock-wave equation (1.5); this lemma, which goes back to Darboux, is proved in [DoH, Lem. 2.4].

Let $\eta:\mathcal{Y}\to\mathbb{CP}_2$ the canonical injection. Since $\eta^*\Omega_\xi$ may only have poles in $\mathcal{Y}_\xi\cup\mathcal{Y}_\infty$, the Stokes formula gives that, near ξ_*, $G=H+L$ where

$$H(\xi)=\sum_{z\in\mathcal{Y}_\xi}\operatorname{Res}(\eta^*\Omega_\xi,z)\,,\quad L(\xi)=\sum_{z\in\mathcal{Y}_\infty}\operatorname{Res}(\eta^*\Omega_\xi,z)\,.$$

By construction, $\eta^*\Omega_\xi$ has residue $h_j(\xi)$ at $z=H_j(\xi)\in\mathcal{Y}_\xi$, and it remains only to know that L is affine in ξ_0 to prove Theorem 3a. The second part of the lemma below is needed in the proof of Theorem 4.

Lemma 13. *If W_{ξ_*} is small enough, $L=\Sigma h_j-G$ is affine in ξ_0. In addition, there is an integer q such that L is the limit in $\mathcal{O}(W_{\xi^*})$ of a continuous one parameter family of ξ_0-affine functions which are the sum of q mutually distinct shock-wave functions.*

Proof. With no loss of generality, we assume $\xi_*=0$ for the proof. For small complex parameters ε, we consider the homogeneous coordinates $w^\varepsilon=(w_0+\varepsilon w_1:w_1:w_2+\varepsilon w_1)$. For ε in a sufficiently small neighborhood of 0, the intersection of $\mathcal{Y}$ with the zero set of $\Lambda_\xi:w\mapsto\xi_0+\xi_1w_1^\varepsilon+w_2^\varepsilon$ is still generic in the sense that it is transverse and lies in $\{w_2^\varepsilon\neq0\}\cap\mathcal{Y}_{\mathrm{reg}}$. Hence, setting $\Omega_\xi^\varepsilon=\frac{w_1}{w_0^\varepsilon}\frac{d\Lambda_\xi^\varepsilon(w)}{\Lambda_\xi^\varepsilon(w)}-\frac{w_1}{(w_0^\varepsilon)^2}dw_0^\varepsilon$, the function $G^\varepsilon:\xi\mapsto\frac{1}{2\pi i}\int_\delta\Omega_\xi^\varepsilon$ is, on W_{ξ_*}, the sum of p mutually distinct shock-wave functions $h_1^\varepsilon,\dots,h_p^\varepsilon$. For generic ε, $\mathcal{Y}$ meets $\Delta_\infty^\varepsilon=\{w_0^\varepsilon=0\}$ transversely and $\mathcal{Y}_\infty^\varepsilon=\mathcal{Y}\cap\{w_0=0\}$ lies in $\mathcal{Y}_{\mathrm{reg}}\cap\{w_2^\varepsilon\neq0\}$. Hence, [DoH, Lem. 2.3.1] implies that $L^\varepsilon=\Sigma h_j^\varepsilon-G^\varepsilon$ is affine in ξ_0. The dependence of G^ε is clearly holomorphic in ε. The same holds for each h_j^ε since what precedes has shown that $h_j^\varepsilon(\xi)=\operatorname{Res}(\eta^*\Omega_\xi^\varepsilon,H_j(\xi))=\frac{1}{2\pi i}\int_{\mathcal{Y}\cap\partial U_j}\eta^*\Omega_\xi^\varepsilon$ where U_j is any sufficiently small neighborhood of $H_j(\xi)$ in $\mathbb{CP}_2$ whose boundary is smooth and transverse to $\mathcal{Y}$. Hence L^ε is holomorphic in ε and has to be affine in ξ_0 when $\varepsilon = 0$.

Let q be the number of points in $\mathcal{Y}_\infty$ counted with their multiplicities; when ξ is generic, q is either defined by

$$p - q = \frac{1}{2\pi i}\int_\gamma \frac{d(\xi_0 + \xi_1 f_1 + f_2)}{\xi_0 + \xi_1 f_1 + f_2}\,. \tag{5.1}$$

For sufficiently small generic ε, [DoH, Lem. 2.3.1] gives more precisely that $L^\varepsilon = \sum_{1\leqslant j\leqslant q} h_j^{\varepsilon,\infty}$ with

$$h_j^{\varepsilon,\infty} = -\operatorname{Res}(\eta^*\Omega_\xi, z_j^\varepsilon) = \frac{-\xi_0\psi_j^\varepsilon(0) + \psi_j^{\varepsilon\prime}(0)}{1 + \xi_1\psi_j^\varepsilon(0)}\,,$$

where $\mathcal{Y}_\infty^\varepsilon = \{z_1^\varepsilon, \dots, z_q^\varepsilon\}$ and $\psi_j^\varepsilon \in \mathcal{O}(U^\varepsilon)$, U^ε open neighborhood of 0 in $\mathbb{C}$, enable us to give in the affine coordinates $\zeta^\varepsilon = (w_j^\varepsilon/w_2^\varepsilon)_{j=0,1}$ the set $\mathcal{Y}$ as a graph above U^ε: $\mathcal{Y} \cap V_j^\varepsilon = \{(\zeta_0^\varepsilon : \psi_j^\varepsilon(\zeta_0^\varepsilon) : 1)\,;\ \zeta_0^\varepsilon \in U^\varepsilon$. Each $h_j^{\varepsilon,\infty}$ is clearly a shock-wave function, that is a solution to $h_{\xi_1} = h_{\xi_0}h$. □

Remark. When ε goes to a non-generic value, the fact that L is a sum of q shock-wave functions may not be preserved as section 6.1 shows.

5.2 Proof of Theorem 3a.B. Assume that γ satisfies (1.4) in a connected neighborhood W_{ξ_*} of one point $(\xi_{0*} : \xi_{1*} : 1)$ of $\mathbb{CP}_2$. If $(\partial^2 G/\partial\xi_0^2)_{|W_{\xi_*}} = 0$, then γ satisfies the classical Wermer–Harvey–Lawson moment condition in $\mathbb{C}^2_{\xi_*} = \mathbb{CP}_2\backslash\{\xi_{0*}w_0 + \xi_{1*}w_1 + w_2 = 0\}$ (see [DoH, Cor. 1.6.2]) and [W], [HL1] imply that if δ is suitably oriented, the polynomial hull of δ in $\mathbb{C}^2_{\xi_*}$ is the unique complex curve $\mathcal{Y}$ of finite mass of $\mathbb{C}^2_{\xi_*}\backslash\delta$ such that $d[\mathcal{Y}] = [\delta]$.

Now assume $(\partial^2 G/\partial\xi_0^2)_{|W_{\xi_*}} \neq 0$. Then we can choose a minimal $\mathcal{H} = \{h_1, \dots, h_p\}$ in the sense that no proper subset of $\mathcal{H}$ satisfies (1.4). Although it is not explicitly mentioned by their authors, the heart of the arguments of [DoH, Th. II, p. 390] is that $\pm[\delta] = d[\mathcal{Y}]$ where $\mathcal{Y}$ is the analytic extension $\mathcal{Y}$ in $\mathbb{CP}_2\backslash\delta$ of the union Γ of the graphs Γ_j of the functions

$$H_j : \xi \mapsto \big(1 : h_j(\xi) : -\xi_0 - \xi_1 h_j(\xi)\big)\,, \quad 1 \leqslant j \leqslant p\,.$$

This fact, not totally explicit in [He, p. 64], can be recovered *a posteriori* by a kind of trick which was used in [DoP] and is developed later in the proof of Theorem 3c: for the curve $\widetilde{\gamma}$ which is the union of γ with the boundaries of Γ_j negatively oriented, one goes back to the $\mathbb{C}^2_{\xi_*}$-case where $(\partial^2 G/\partial\xi_0^2)_{|W_{\xi_*}} = 0$. If $d[\mathcal{Y}]$ is $-[\delta]$ and not $[\delta]$, then the same arguments which have proved Theorem 3a.A would give that the functions h_j, geometrically defined as the first coordinates of points of intersection of $\mathcal{Y}$ with generic lines Λ_ξ, should satisfy not only the shock-wave equation $h_{\xi_1} = h_{\xi_0}h$

but also the "negative" shock-wave equation $h_{\xi_1} = -h_{\xi_0}h$. As this is impossible, $d[\mathcal{Y}] = [\delta]$.

In both cases, we have found (up to a change of orientation if $\partial^2 G/\partial\xi_0^2$ vanish on W_{ξ_*}) a complex curve $\mathcal{Y}$ of finite mass of $\mathbb{CP}_2\backslash\delta$ such that $d[\mathcal{Y}] = [\delta]$.

As δ is smooth, we know from [HL1] that there is in δ a compact set τ such that $h^1(\tau) = 0$ and for which each point of $y \in \delta\backslash\tau$ has a neighborhood U_y where $\overline{\mathcal{Y}} \cap U_y$ is a closed bordered submanifold of U_y with boundary $\delta \cap U_y$. Lemma 8 in section 3 describes how $\mathcal{Y}$ is near points of τ. Using the notation and definitions introduced after its proof, we let τ_2 (resp. τ') be the set of y in δ where $\overline{\mathcal{Y}}$ has a strong singularity (resp. $m_y \geqslant 2$) and define $\widetilde{\mathcal{Y}}$ as the abstract complex curve $\mathcal{Y} \cup \tau'$.

Consider a normalization $\pi : \mathcal{X} \to \widetilde{\mathcal{Y}}$; Lemma 8 implies that π is an open mapping. Let $\mathcal{Z}$ be the disjoint and abstract union $\mathcal{X} \cup \gamma$. If $x \in \gamma$, we define a neighborhood of x in $\mathcal{Z}$ as a subset of $\mathcal{Z}$ which contains a set of the kind $\pi^{-1}(C_{y,1}^U)$ where $y = f(x)$ and U is a neighborhood of y in $\mathbb{CP}_2$. Then $(\mathcal{Z}, \gamma)$ is a compact metrizable topological manifold with boundary which has finite 2-dimensionnal Hausdorff measure and smooth boundary outside $\sigma = f^{-1}(\tau_2)$. Since f is an embedding, $h^1(\sigma) = 0$ and $(\mathcal{Z}, \gamma)$ is a manifold with almost smooth boundary. Moreover, it follows by construction that the meromorphic extension $F : \mathcal{Z} \to \overline{\mathcal{Y}}$ of f to $\mathcal{X}$ defined by $F|_{\mathcal{X}} = \pi$ is a normalization of $\overline{\mathcal{Y}}$ in the sense of Lemma 9. □

Remark. When γ is real analytic, [HL1, Th. II] implies that $\mathcal{C}_{y,1}^U$ is a manifold with boundary in the classical sense. So, in that case, $(\mathcal{Z}, \gamma)$ is a classical manifold with boundary.

The following proposition which clarifies some results of [DoH] justifies the fourth remark after Theorem 3a.

Proposition 14. *Assume δ is connected. Then if $(\partial^2 G/\partial\xi_0^2)_{|W_{\xi_*}} = 0$, the polynomial hull of γ in $\mathbb{C}^2 = \mathbb{CP}_2\backslash\{\xi_{0*}w_0 + \xi_{1*}w_1 + w_2 = 0\}$ has boundary $\pm[\gamma]$. If $(\partial^2 G/\partial\xi_0^2)_{|W_{\xi_*}} \neq 0$ and no proper subset of $\mathcal{H} = \{h_1, \dots, h_p t\}$ satisfies* (1.4)*, then the analytic extension $\mathcal{Y}$ in $\mathbb{CP}_2\backslash\delta$ of the union Γ of the graphs Γ_j of the functions $H_j : \xi \mapsto (1 : h_j(\xi) : -\xi_0 - \xi_1 h_j(\xi))$, $1 \leqslant j \leqslant p$, is the complex curve which has minimal volume among complex curves $\mathcal{Z}$ such that $dt[\mathcal{Z}] = [\delta]$.*

Proof. The preceding proof contains the above conclusion except the minimality of volume of $\mathcal{Y}$ when $(\partial^2 G/\partial\xi_0^2)_{|W_{\xi_*}} \neq 0$ and $\mathcal{H}$ is minimal. In that case, let $\mathcal{Y}'$ be a complex curve of $\mathbb{CP}_2\backslash\delta$ with boundary $[\delta]$ and minimal volume. Then $\mathcal{Y}=\mathcal{Y}'\cup\mathcal{Z}$ where $\mathcal{Z}$ is a union of compact complex curves of $\mathbb{CP}_2$. But the intersection of any compact curve with a line Λ_ξ is described, for ξ in a neighborhood of a generic ξ_*, as a finite union of graphs of function $(1:g_j:g_{j,2})$ whose second homogeneous coordinate satisfy the shock-wave equation and such that Σg_j is affine in ξ_0 (see [He, §2]). Since $\mathcal{H}$ is minimal, no such g_j belongs to $\mathcal{H}$ and it appears that Γ has to be contained in $\mathcal{Y}'$ Hence, $\mathcal{Y}\subset\mathcal{Y}'$ and, finally, $\mathcal{Y}=\mathcal{Y}'$. □

5.3 Proof of Theorem 3a.C. Let $\mathcal{X}$, $\mathcal{Z}$, $\mathcal{D}$ and g be as in the statement. Let $u\in C^1(\gamma)$ and for $z\in\mathcal{D}\backslash\gamma$ set

$$\Omega_z=u\partial_\zeta g_z+g_z\overline{\theta u}\,.$$

If $\mathcal{Y}$ is a smooth domain in $\mathcal{Z}$, the Stokes formula (4.1) implies that value of $\mathbf{1}_{\mathcal{Y}}(z)\widehat{u}(z)+\int_{\mathcal{Y}}\overline{\partial}\widehat{u}\wedge\partial_\zeta g_z-\int_{(\partial\mathcal{Y})\backslash\gamma}u\partial_\zeta g_z$ does not depend of the Lipschitz extension $\widehat{u}$ of u to $\mathcal{Z}$ and equals $\int_{\gamma\cap\mathcal{Y}}u\partial_\zeta g_z$ when $\mathcal{X}_{\text{sing}}=\varnothing$. Hence we can take it as a definition for $\int_{\gamma\cap\mathcal{Y}}u\partial_\zeta g_z$ in the general case. Let then F be the function defined for $z\in\mathcal{D}\backslash\gamma$ by

$$F(z)=\frac{2}{i}\int_{\zeta\in\gamma}\Omega_z(\zeta)\,,\quad \Omega_z=u\partial_\zeta g_z+g_z\overline{\theta u}\,,$$

where $g_z=g(\,.\,,z)$.

Since $\widetilde{u},u$ and θu are continuous, the conclusion of part C follows from the lemma below which gives $\widetilde{u}=F|_{\mathcal{X}}$ and $\theta u=\partial\widetilde{u}$ on $\gamma\backslash\sigma$ if $F|_{\mathcal{D}\backslash\overline{\mathcal{X}}}=0$.

LEMMA 15. *$F_+=F|_{\mathcal{X}}$ and $F_-=F|_{\mathcal{D}\backslash\overline{\mathcal{X}}}$ are real valued harmonic functions such that*

$$u=F_+-F_-\quad\text{and}\quad\overline{\theta u}=\overline{\partial}F_+-\overline{\partial}F_-\quad\text{on}\quad\gamma\backslash\sigma\,. \tag{5.2}$$

Proof. The harmonicity of F is a simple consequence of the properties of g. Now fix p in $\gamma\backslash\sigma$ and in a neighborhood $\mathcal{U}$ of p in $\mathcal{D}$, a holomorphic chart $\mathcal{U}\to U$ centered at p; a hat "$\widehat{\ }$" denotes hereafter the coordinate expression of a function, a form or a set. For $z\in U\backslash\gamma$ let us write $F(z)=\varphi(z)+R_1(z)$ with $\varphi(z)=-2i\int_{\gamma\cap\mathcal{U}}\Omega_z$ and R_1 is smooth on $\mathcal{U}$. If y and x are the coordinates of $\zeta\in\gamma\cap\mathcal{U}$ and $z\in U\backslash\gamma$, $\widehat{g}(y,x)=g(\zeta,z)$ can be written in the form $\widehat{g}(y,x)=\frac{1}{2\pi}\ln|y-x|+h(y,x)$ where h is a smooth function on $U\times U$, harmonic in each variable. Hence

$$\widehat{\varphi}(x)=\frac{1}{2\pi i}\int_{\widehat{\gamma}\cap U}\frac{\widehat{u}(y)}{y-x}dy+\int_{\widehat{\gamma}\cap U}\frac{\ln|y-x|}{\pi i}\overline{\widehat{\theta u}(y)}+R_2(x)$$

where $R_2\in C^0(U)$. The second integral has no jump across $\widehat{\gamma}$ and from the classical Sohotsky–Plemelj formula, we know that the first integral has jump $\widehat{u}$ across $\widehat{\gamma}$ in the sense of distribution and pointwise near each regular boundary point. Since

$$\overline{\partial}\widehat{\varphi}(x) = \frac{1}{2\pi i} d\overline{x} \int_{\widehat{\gamma}\cap U} \frac{1}{\overline{x}-\overline{y}} \overline{\widehat{\theta u}(y)} + \overline{\partial} R_2(x)\,,$$

the jump of $\overline{\partial}\widehat{\varphi}$ through $\widehat{\gamma}$ is likewise $\overline{\widehat{\theta u}}$.

In order to check that $F(z) \in \mathbb{R}$ when $z \in \mathcal{D}\backslash\gamma$, we let $L_{\mathcal{X}}$ be the DN-operator of $\mathcal{X}$ and we note that since $\partial g_z = (L_{\mathcal{X}} g_z)(\nu^* + i\tau^*)$ and $\theta u = (Lu)(\nu^* + i\tau^*)$, $-\operatorname{Im} F(z) = \int_\gamma (u\,\tau g_z + g_z \tau u)\tau^* = \int_\gamma d(u g_z|_\gamma) = 0$. □

5.4 Proof of Theorem3c.A. We assume that γ is in the sense of currents the boundary of a complex curve $\mathcal{X}$ of $\mathbb{CP}_2\backslash\gamma$ for which $(\gamma, u, \theta u)$ is a restricted DN-datum. Then, as in the proof of Theorem 3a.A, for almost all $(\xi_{*0} : \xi_{*1} : 1)$ in $\mathbb{CP}_2$, there exists a neighborhood W_{ξ_*} of $\xi_* = (\xi_{*0}, \xi_{*1})$ such that, for every $\xi \in W_{\xi_*}$, $\mathcal{X} \cap \Delta_\xi$ lies in $\mathbb{C}^2$ and equals $\Gamma \cap \Delta_\xi$ where Γ is the union of the graphs Γ_j of $H_j = (1 : h_j : h_{j,2}) : W_{\xi_*} \to \mathbb{CP}_2$, $1 \leqslant j \leqslant p$ where H_j is holomorphic in W_{ξ_*}. Since we are concerned only by generic ξ_*, we can suppose that $\Gamma_j = \{(\varphi_j(z_2), z_2)\,;\; z_2 \in U_j\}$ where U_j is a neighborhood of $z^*_{j,2} = h_{j,2}(\xi_*)$ and $\varphi_j \in \mathcal{O}\{U_j\}$. The decomposition sought for $\widetilde{G}$ in (a1) can be then found in [He]. However, this residues calculus is needed in part B, and we include it here.

When $\xi \in W_{\xi_*}$, $\widetilde{G}_\ell(\xi_0 : \xi_1 : 1)$ is the sum of the residues of

$$\Lambda_\ell = \frac{z_\ell}{\xi_0 + \xi_1 z_1 + z_2}\Theta_0$$

(for convenience $z_\ell = 1$ if $\ell = 0$) in $\mathcal{X}$. Set $\Theta_0 = A_j(z_2)dz_2$ in each Γ_j and let us abbreviate $H_j(\xi)$ in z_j. Then $z_{j,2}$ is the only pole of Λ_ℓ in Γ_j. It is a simple one and the residue of Λ_ℓ at it is

$$g_{j,\ell} = \frac{z_{j,\ell} A_j(z_{j,2})}{\xi_1 \varphi'_j(z_{j,2}) + 1} = z_{j,\ell}\, g_{j,0} \tag{5.3}$$

where $z_{j,\ell} = 1$ if $\ell = 0$. As Γ_j is also parametrized by H_j, we can set

$$g_j = \sum_{0\leqslant\ell\leqslant 2} g_{j,\ell}\, d\eta_\ell \quad \text{on} \quad \Gamma_j\,,$$

and get that $g_{j,1}/g_{j,0} = z_{j,1} = h_j$ satisfy (1.5); $g_{j,2}/g_{j,0} = z_{j,2} = h_{j,2}$ satisfy then $h_j \frac{\partial h_{j,2}}{\partial \xi_0} = \frac{\partial h_{j,2}}{\partial \xi_1}$ because $h_{j,2} = -\xi_0 - \xi_1 h_j$. Note that the dependence in ξ of g can be made clearer if the identity $h_j - \varphi_j(-\xi_0 - \xi_1 h_j) = 0$ is used. Indeed, this relation implies

$$(1 + \xi_1\varphi'_j)\partial_{\xi_0} h_j + \varphi'_j = 0 \quad \text{and} \quad (1 + \xi_1\varphi'_j)\partial_{\xi_1} h_j + h_j\varphi'_j = 0\,.$$

Hence $\varphi'_j = -(\partial_{\xi_0} h_j)/(1 + \xi_1 \partial_{\xi_0} h_j)$ and

$$\frac{1}{1 + \xi_1\varphi'_j} = \frac{\partial_{\xi_1} h_j}{\partial_{\xi_0} h_j} \frac{1 + \xi_1\partial_{\xi_0} h_j}{h_j} = 1 + \xi_1 \partial_{\xi_0} h_j = \partial_{\xi_0} h_{j,2}\,,$$

since h_j satisfies $h_j\partial_{\xi_0}h_j = \partial_{\xi_1}h_j$. So, instead of (5.3), we now have

$$g_{j,\ell} = A_j(H_j)h_{j,\ell}\frac{\partial h_{j,2}}{\partial \xi_0}\,, \quad 1 \leqslant j \leqslant p\,,\ 0 \leqslant \ell \leqslant 2\,. \tag{5.4}$$

where $h_{j,0} = 1$ for convenience.

From the definition we get that when expressed in the affine coordinates ξ, g_j is given by the integral formula

$$2\pi i\, g_j = \left(\int_{\partial\Gamma_j} \frac{\Theta_0}{\xi_0 + \xi_1 z_1 + z_2}\right)d\xi_0 + \left(\int_{\partial\Gamma_j} \frac{z_1\Theta_0}{\xi_0 + \xi_1 z_1 + z_2}\right)d\xi_1$$

from which it is clear that g_j is closed.

To achieve the proof of part A, it is enough to remark that (a2) is a direct consequence of the fact $\operatorname{Re}\Theta_\ell = d\widetilde{u_\ell}$ is exact.

5.5 Proof of Theorem 3c.B. Assume that the hypothesis of (b1) is true and γ is connected.

Case $\widetilde{G}|_{W_{\eta*}} \neq 0$. This mean $\mathcal{H} \neq \varnothing$ when $\mathcal{H}$ is minimal in the sense that no proper subset of $\mathcal{H}$ gives a decomposition of $\widetilde{G}$ with the same properties. Let Γ be the union of the graphs Γ_j of the functions $H_j = (1 : h_j : h_{j,2})$, $1 \leqslant j \leqslant p$, where $h_{j,2} = -\xi_0 - \xi_1 h_j$. If needed, we can choose another ξ_* so that Γ does not meet Λ_{ξ_*} in $\{w_0 = 0\}$. Then for any ξ in a neighborhood Ω of ξ_*, the $H_j(\xi)$ are mutually distinct and are the points of $\Gamma \cap L_\xi$. Finally, we assume, which it is not a restriction, that Γ has a smooth oriented boundary $\partial\Gamma$.

Let $\widetilde{\gamma}$ be the union of $\partial\Gamma$ with opposite orientation and γ and let φ_η be the linear function $z \mapsto \eta_0 + \eta_1 z_1 + \eta_2 z_2$. From the hypothesis we get directly

$$\frac{1}{2\pi i}\int_\gamma \varphi_\eta^{-1}\theta_\ell = \sum_{1\leqslant j\leqslant p} g_{j,\ell} = \sum_{1\leqslant j\leqslant p} g_{j,\ell}h_{j,\ell}$$

where $h_{j,\ell} = 1$ if $\ell = 0$. On the other hand, if we set

$$\Theta_0 = (\partial_{\xi_0}h_{j,2})^{-1}g_{j,0}dh_{j,2}\,, \quad \text{on} \quad \overline{\Gamma_j}\,,\ 1 \leqslant j \leqslant p \tag{5.5}$$

and $z_0 = 1$, the residues calculus made in the proof of part A implies that

$$\int_{\partial\Gamma} z_\ell\varphi_\eta^{-1}\theta_0 = \sum_{1\leqslant j\leqslant p} g_{j,0}h_{j,\ell}\,, \quad 1 \leqslant j \leqslant p\,.$$

Hence, $\int_{\widetilde{\gamma}} \varphi_\eta\theta_\ell = 0$. As $\widetilde{\gamma}$ is contained in the affine space $E_{\xi_*} = \mathbb{CP}_2\backslash\Lambda_{\xi_*}$, we can apply [He, Cor. 4.2, p. 265] and [D, Prop. 1] and get in $E_{\xi_*}\backslash\widetilde{\gamma}$ a complex curve $\widetilde{\mathcal{X}}$ of finite volume where θ_0 extends weakly in a weakly holomorphic form Θ_0 satisfying

$$\int_{\mathcal{X}} (\overline{\partial}\varphi) \wedge \Theta_0 = \int_{\mathcal{X}} d(\varphi\Theta_0) = \int_\gamma \varphi\theta_0 \tag{5.6}$$

holds for any φ smooth in a neighborhood of $\overline{\mathcal{X}}$ and analytic near σ.

Note that in [He], [D], (5.6) is in fact obtained only for φ smooth on $\mathcal{X}$ and holomorphic in a neighborhood of γ, but (5.6) follows from this together with (5.4) and the residue relations (very close in spirit to the relations $T_{m,\xi}$)

$$\frac{1}{2\pi i}\int_\gamma z_1^m \frac{\theta_0}{\xi_0+\xi_1 z_1+z_2} = \sum_{1\leqslant j\leqslant p} h_j^m(\xi) g_{j,0}(\xi)\,,$$

where $h_j(\xi)$, $h_{j,2}(\xi)$ and $g_{j,0}(\xi)$ are as above. Indeed, for generic ξ, these relations enable the computation of $g_{j,0}(\xi)$ by a kramerian system and hence imply the smoothness of Θ_0 near points of $\gamma\backslash\sigma$. However, this precision is not essential in the sequel.

By construction, $\mathcal{X} = \widetilde{\mathcal{X}} \cup \overline{\Gamma}$ is a complex curve of $\mathbb{CP}_2\backslash\gamma$ where θ_0 extends as a weakly holomorphic form Θ_0, this extension coinciding in Γ with the form defined by (5.5). If $\ell = 1,2$, the form $\Theta_\ell = z_\ell\Theta_0$ is a weakly holomorphic extension of θ_ℓ to $\mathcal{X}$.

Let $\varepsilon > 0$ and let $\mathcal{W}_\varepsilon$ an ε-neighborhood of Λ_{ξ_*}. As $\mathcal{X}_\varepsilon = \mathcal{X}\backslash\overline{\mathcal{W}_\varepsilon}$ lies in the affine space $E_{\xi_*}\backslash\mathcal{W}_\varepsilon$, [HL1, Th. 4.7] implies that

$$d[\mathcal{X}_\varepsilon] = n_\varepsilon[\gamma] + \sum_{1\leqslant j\leqslant p} n_{\varepsilon,j}[\gamma_{\varepsilon,j}] \tag{5.7}$$

where for $1 \leqslant j \leqslant p$, $n_\varepsilon, n_{\varepsilon,j} \in \mathbb{Z}$ and $\gamma_{\varepsilon,j}$ is the (smooth) boundary of the smooth (manifold) $\mathcal{W}_\varepsilon \cap \Gamma_j$. If $0 < \varepsilon' < \varepsilon$,

$$\begin{aligned}\sum_{1\leqslant j\leqslant p} d\big[\Gamma_j \cap (\mathcal{W}_\varepsilon\backslash\mathcal{W}_{\varepsilon'})\big] &= -d[\mathcal{X}_\varepsilon] + d[\mathcal{X}_{\varepsilon'}]\\ &= (-n_\varepsilon + n_{\varepsilon'})[\gamma] + \sum_{1\leqslant j\leqslant p}(-n_{\varepsilon,j} + n_{\varepsilon',j})[\gamma_{\varepsilon,j}]\,.\end{aligned}$$

Hence, $n_\varepsilon = n_{\varepsilon'} \overset{def}{=} n$ and as each $\Gamma_j \cap \mathcal{W}_\varepsilon\backslash\mathcal{W}_{\varepsilon'}$ is a smooth manifold with boundary, $n_{\varepsilon,j} = n_{\varepsilon',j} = -1$. Now taking limits in (5.7) when ε goes to zero, we get $d[\mathcal{X}] = n[\gamma]$. We suppress from $\mathcal{X}$ any compact component it may have and still denote the result by $\mathcal{X}$; note that $\mathcal{X}$ now has to be connected. Since $\mathcal{X}$ is a complex curve of $\mathbb{CP}_2\backslash\gamma$, [Ch] implies that if $n \neq 0$, there is in γ a compact set σ such that $h^1(\sigma) = 0$ and $(\overline{\mathcal{X}}, \pm\gamma)$ is a manifold with boundary near points of $\gamma\backslash\sigma$; as $\mathcal{X}$ is connected, this implies $n = \pm 1$. When $n = 0$, the structure theorems of Harvey–Shiffman [H] imply that $\mathcal{Z} = \overline{\mathcal{X}}$ is then a complex compact curve of $\mathbb{CP}_2$; since γ is smooth, γ is locally a Jordan curve of $\mathcal{Z}_{\text{reg}}$ and the points where γ may meet the finite set $\mathcal{Z}_{\text{sing}}$ are only self-intersection points of $\mathcal{Z}$.

Since $\mathcal{X} \cap \Lambda_\xi = \{(1 : h_j(\xi) : h_{j,2}(\xi)),\ 1 \leqslant j \leqslant p\}$, the Stokes formula gives $G = \Sigma h_j$ (see Proof of Theorem 3a.A).

Assume that $(\partial^2 G/\partial\xi_0^2)_{|W_{\xi_*}} \neq 0$. Then $n \neq 0$ because otherwise, for ξ closed to ξ_*, the intersection of $\mathcal{X}$ with the line Λ_ξ would have to be the intersection with Λ_ξ of a compact Riemann surface, namely $\overline{\mathcal{X}}$, which by a theorem of Reiss would force Σh_j to be affine in ξ_0 (see [GH, Ch. 5.2] or [He, §2]). Reasoning as in the proof of 3a.B, we also eliminate the possibility $d[\mathcal{X}] = -[\gamma]$ because it would imply that if $\mathcal{H} = \{h_1, \dots, h_p\}$ is minimal in the sense that no proper subset of $\mathcal{H}$ satisfy (a1), each h_j also satisfies $h_{j,y} = -h_{j,x}h_j$. Hence $n = 1$.

When $(\partial^2 G/\partial\xi_0^2)_{|W_{\xi_*}} = 0$, that is when Σh_j is affine in ξ_0, and when $\overline{\mathcal{X}}$ is not an algebraic curve where γ is a slit, n has to be non-null and hence is ±1. Since $\widetilde{G}|_{W_{\eta_*}} \neq 0$, the minimal $\mathcal{H}$ in the above sense is not empty and reasoning likewise, we get $d[\mathcal{X}] = [\gamma]$.

Case $\widetilde{G}|_{W_{\eta_*}} = 0$. This means that the minimal $\mathcal{H}$ is empty. Then, we can apply [He, Th. 4.2, p. 264] in the affine case (see [D] for a generalization and a detailed proof in this case) to get in $\mathbb{C}^2\backslash\gamma$ (here $\mathbb{C}^2$ is the complement of $\Lambda_{\xi_*} = \{\xi_{*0}w_0 + \xi_{1*}w_1 + w_2 = 0\}$ in $\mathbb{CP}_2$) a complex curve $\mathcal{X}$ of finite volume where θ_0 extends weakly in a weakly holomorphic form Θ_0 satisfying (5.6).

Since θ_0 yields a non-zero measure on γ which, because $\widetilde{G}|_{W_{\eta_*}} = 0$, is orthogonal to all polynomials of $\mathbb{C}^2 \sim \mathbb{CP}_2\backslash\Lambda_{\xi_*}$, we can apply Bishop [Bi] and [S] (Wermer originates and solves this problem [W] for the real analytic case) to get that $\mathcal{X}$ is the polynomial hull $\widetilde{\gamma}$ of γ in $\mathbb{C}^2$ and $d(\pm[\mathcal{X}]) = [\gamma]$. [HL1] (see also [Ch]) implies then that γ contains a compact set σ such that $h^1(\sigma) = 0$ and $(\overline{\mathcal{X}}, \gamma)$ is a manifold with boundary near points of $\gamma\backslash\sigma$.

To prove (b2), we go back to the assumption that (a1) is true and we assume in addition that Θ_ℓ is holomorphic and satisfies (1.8) for all $c \in H_1(\mathcal{X}_{\mathrm{reg}})$. Then, there exists $U_\ell \in C^\infty(\mathcal{X}_{\mathrm{reg}})$ such that $\Theta_\ell = dV_\ell$ on $\mathcal{X}_{\mathrm{reg}}$; since Θ_ℓ is a (1,0)-form, $\partial V_\ell = dV_\ell = \Theta_\ell$. We know from the preceding point that $d[\mathcal{X}] = n[\gamma]$ where $n \in \{0, 1\}$, up to a change of orientation of γ when $\widetilde{G}|_{W_{\eta_*}} = 0$. Assume at first that $n = 1$. As d is elliptic up to the boundary, V_ℓ has to be smooth up to the boundary in the classical sense near points of γ outside σ and the preceding equality (5.6) yields $dv_\ell = du_\ell$ where $v_\ell = V_\ell|_\gamma$. Hence, z_0 is a point of γ where $(\overline{\mathcal{X}}, \gamma)$ is a manifold with boundary, there is a constant c such that $v_\ell = u_\ell + c$ near z_0 in γ. Since $h^1(\sigma) = 0$ and $dv_\ell = du_\ell$ is smooth we have $v_\ell(z) - v_\ell(z_0) = \int_{\gamma_{z_0,z}} du_\ell$ where $\gamma_{z_0,z}$ is the positively oriented path of γ starting at z_0 and ending at z. Hence, $v_\ell(z) - u_\ell(z_0) - c = u_\ell(z) - u_\ell(z_0)$ and $v_\ell(z) = u_\ell(z) + c$. This implies that U_ℓ is a weakly-harmonic extension of u_ℓ. When $n = 0$, $\overline{\mathcal{X}}$ is two-sided locally near points of γ and each local side has the same boundary regularity as in the case $n = 1$. Hence, we can reason as in this case and get that there is a weakly-harmonic extension of u_ℓ to $\mathcal{X}$.

When we assume also that $\int_{\mathcal{X}_{\mathrm{reg}}} \Theta_\ell \wedge \overline{\Theta_\ell} < +\infty$, Θ_ℓ is holomorphic in the sense that its pullback to any normalization $\pi : \mathcal{Z} \to \mathcal{X}$ of $\mathcal{X}$ is holomorphic and not only meromorphic. The isolated singularities that the pullback V_ℓ of U_ℓ may have in $\mathcal{Z}$ are removable because dV_ℓ is smooth. So U_ℓ is harmonic on $\mathcal{X}$ and U_ℓ is the harmonic extension of u_ℓ to $\mathcal{X}$. The proof is complete.

5.6 Proof of Theorem 3b. The map f enable us to embed the abstract IDN-problem of Theorem 3b in the projective but concrete frame of Theorem 3c. So Theorem 3b.A is a direct consequence of 3c.A.

For the converse part B.b1, we apply 3c.B to $\delta = f(\gamma)$ and $\theta_\ell = \theta u_\ell$, $\ell = 0, 1, 2$. We get in $\mathbb{CP}_2 \backslash \delta$ an irreducible complex curve $\mathcal{Y}$ such that $d[\mathcal{Y}] = n[\delta]$ where $n \in \{0, 1\}$, up to a change of orientation when $\widetilde{G}|_{W_{\eta*}} = 0$; in addition, each θ_ℓ extends weakly to $\mathcal{Y}$ into a weakly holomorphic (1,0)-form $\Theta_\ell^{\mathcal{Y}}$.

When $n = 1$, the boundary regularity of $\mathcal{Y}$ mentioned in the proof of Theorem 3c.B enables us to apply readily the construction made in the proof of Theorem 3a: adding to $\mathcal{Y}$ a subset σ' of γ of zero one-dimensional Hausdorff measure, we get an abstract complex curve $\widetilde{\mathcal{Y}}$ which can be normalized in the classical sense into an abstract Riemann surface $\mathcal{X}$; γ can then be topologically glued to $\mathcal{X}$ so that $(\overline{\mathcal{X}}, \gamma)$ becomes a manifold with almost smooth boundary where the pullback F to $\mathcal{X}$ of the meromorphic map $\mathbb{CP}_2 \ni z \mapsto (z_1, z_2)$ gives a meromorphic extension of f to $\mathcal{X}$. Since the forms $\Theta_\ell^{\mathcal{Y}}$ are meromorphic on $\mathcal{Y}$, $\Theta_\ell = F^* \Theta_\ell^{\mathcal{Y}}$ is well defined and meromorphic outside $\mathcal{X} \backslash F^{-1}(\sigma')$ which has zero length.

When $n = 0$, $\mathcal{Z} = \overline{\mathcal{X}}$ is an algebraic curve and one can use a standard normalization of $\mathcal{Z}$ to get the same kind of conclusions.

The supplementary hypothesis of part B.b2 forces each Θ_ℓ to have only removable singularities. Reasoning as in the proof of 3c.B.b2, (1.8) implies that $\Theta_\ell = dV_\ell = \partial V_\ell$ for some harmonic function V_ℓ smooth up to regular boundary points of $\overline{\mathcal{X}}$ (when $n = 0$, γ cuts locally $\mathcal{X}$ into two domains and this means that each restriction of V_ℓ to these domains is smooth up to γ) and that there is a constant c such that $U_\ell + c$ agrees with u_ℓ on γ. □

6 Characterizations, Effective or Affine

6.1 Explicit integro-differential characterization. In this section where Theorem 4 is proved, (ξ_0, ξ_1) is replaced by the simpler (x, y) and we reason in a neighborhood of $(0, 0)$. $\mathcal{O}_0$ is the set of one variable holomorphic functions near 0 is denoted by, $\mathcal{O}_0 \otimes \mathbb{C}_d[Z]$ stands for the set of polynomials of degree at most d with independent variable Z and coefficients in $\mathcal{O}_0$. An element of $\mathcal{O}_0 \otimes \mathbb{C}_d[X]$ (resp. $\mathcal{O}_0 \otimes \mathbb{C}_d[Y]$) should be thought of as a function of the type $(x, y) \mapsto \sum_{0 \leqslant j \leqslant d} \lambda_j(x) y^j$ (resp. $(x, y) \mapsto \sum_{0 \leqslant j \leqslant d} \lambda_j(y) x^j$) where each $\lambda_j \in \mathcal{O}_0$.

If h is differentiable, the derivative of h with respect to one of its variable u is denoted h_u. If U is an open set in $\mathbb{C}^n$, $\mathcal{O}(U)$ is the space of holomorphic functions in U; if $h \in \mathcal{O}(U)$ and $0 \in U$, we set $h_{x^{-0}, y^0} = h$ and in any simply connected neighborhood of 0 in U, we denote by $h_{x^{-\alpha-1}, y^\beta}$ $(\alpha, \beta \in \mathbb{N})$ the function which vanishes at 0 and satisfies $\partial(h_{x^{-\alpha-1}, y^\beta})/\partial x = h_{x^{-\alpha}, y^\beta}$; $h_{x, {}^\alpha y^{-\beta-1}}$ is defined similarly.

A two variable function h is called a *shock-wave function* on a domain D of $\mathbb{C}^2$ if it is holomorphic and satisfies $h_y = h_x h$ on D.

A *p-algebroïde* function on D is a p-uple $h = (h_1, \dots, h_m)$ of functions from D to $\mathbb{C}$ for which one can find p holomorphic functions $a_0, \dots, a_{p-1}$ in D such that for $z \in D$, $h_1(z), \dots, h_p(z)$ are the roots, with multiplicities, of the polynomial $T^h = X^p + a_{p-1}(z) X^{p-1} + \cdots + a_1(z) X + a_0(z)$.

A *p-multivaluate shock-wave function* on D is a p-algebroïde function h on D such that $\Sigma = \{\operatorname{Discr} T^h = 0\}$ is a hypersurface of D and for any $z_* \in D \backslash \Sigma$, the holomorphic functions $h_1^*, \dots, h_p^*$ which near z_* describe the roots of T_z^h are non-null shock-wave functions; the first symmetric function of T^h, that is the sum of the roots of T^h, is called the *trace* of T^h of h. Traces of p-multivaluate shock-wave functions are called *p-shock-wave functions*.

Lemma 16. *Consider p mutually distinct functions $h_1, \dots, h_p$ holomorphic in a domain D of $\mathbb{C}^2$. Then, each h_j is a shock-wave function if and only the functions $\sigma_k = (-1)^k \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant p} h_{j_1} \cdots h_{j_k}$ satisfy the following system:*

$$\sigma_p \sigma_{1,x} + \sigma_{p,y} = 0 \quad \textit{and} \quad \sigma_k \sigma_{1,x} + \sigma_{k,y} = \sigma_{k+1,x}\,, \quad 1 \leqslant k \leqslant p-1\,. \tag{6.1}$$

Proof. Set $T = X^p + \sum_{1 \leqslant k \leqslant p} \sigma_k X^{p-k}$ and $T' = \partial T / \partial X$. The relations $0 = (Th)_x = (Th)_y$ and $h^p = -\sum_{1 \leqslant k \leqslant p} \sigma_k h^{p-k}$ yields $(T'h)(h_y - h h_x) = Sh$ with

$$S = \sum_{1 \leqslant k \leqslant p-1} [\sigma_{k+1,x} - \sigma_k \sigma_{1,x} - \sigma_{k,y}] X^{p-k} - (\sigma_p \sigma_{1,x} + \sigma_{p,y})\,.$$

Since $\deg S \leqslant p-1$, the fact that each h_j is a shock-wave function implies that the coefficients of S vanish in a non-empty open set and thus in the domain D. If $S = 0$ at every point of D, then each h_j verifies $h_y - h_x h = 0$ in the domain D because $T' h_j \not\equiv 0$. □

Proposition 17. *Let D be a simply connected domain of $\mathbb{C}^2$ containing 0, Δ its image by the projection $(x,y) \mapsto y$ and $H \in \mathcal{O}(D)$. When u is differentiable, we set*

$$\mathcal{D}_H u = e^{H_{x,y^{-1}}} \partial(u e^{-H_{x,y^{-1}}})/\partial y \quad \text{and} \quad \mathcal{L}_H u = (\mathcal{D}_H u)_{x^{-1}} .$$

The following two assertions are equivalent:

1. *H is a p-shock-wave function in D.*
2. *There exists $\lambda_1, \dots, \lambda_{p-1} \in \mathcal{O}(\Delta)$ such that for $\widetilde{\lambda}_j(x,y) = \lambda_j(y)$, $1 \leqslant j \leqslant p-1$,*

$$\mathcal{D}_H \mathcal{L}_H^{p-1} H = \mathcal{D}_H \mathcal{L}_H^{p-2} \widetilde{\lambda}_1 + \cdots + \mathcal{D}_H \mathcal{L}_H^0 \widetilde{\lambda}_{p-1} \tag{6.2}$$

$$\operatorname{Discr} T_z \not\equiv 0 \tag{6.3}$$

where $T_z = X^p + \sum_{1 \leqslant k \leqslant p} s_k(z) X^{p-k}$ with

$$s_k = -\mathcal{L}_H^{k-1} H + \mathcal{L}_H^{k-2} \widetilde{\lambda}_1 + \cdots + \mathcal{L}_H^0 \widetilde{\lambda}_{k-1} , \quad 1 \leqslant k \leqslant p . \tag{6.4}$$

More precisely, in case (2) is true, T determines a p-multivaluate shock-wave function with trace H. Conversely, if H is the trace of a p-multivaluate shock-wave function T, the p holomorphic functions which near a point z_* in $\{\operatorname{Discr} T \neq 0\}$ describes the roots of T_z have symmetric functions $(-1)^k s_k$, $1 \leqslant k \leqslant p$ which satisfy (6.2) and (6.4).

Proof. (1) Assume H is a p-shock-wave function in D. Then, H is the first symmetric function of some $T \in \mathcal{O}(D) \otimes \mathbb{C}_p[Z]$, $\Sigma = \{\operatorname{Discr} T = 0\} \neq D$ and for any fixed $z_* \in D \backslash \Sigma$, $\deg T_{z_*} = p$ and the holomorphic functions $h_1^*, \dots, h_p^*$ which near z_* describe the roots of T_z are non-null shock-wave functions with no common value on a sufficient small convex neighborhood $W = U \times V$ of z_*. For $k \in \{1, \dots, p\}$ and on W, set $\rho_k = \sigma_k e^{-H_{x,y^{-1}}}$ where σ_k is defined in Lemma 16. Then (6.1) implies that $\rho_{p,y} = 0$ and that for $k \in \{1, \dots, p-1\}$,

$$(\rho_{k+1} e^{H_{x,y^{-1}}})_x = -H_x \sigma_k + \sigma_{k,y} = e^{H_{x,y^{-1}}} \frac{\partial}{\partial y} \sigma_k e^{-H_{x,y^{-1}}} = e^{H_{x,y^{-1}}} \rho_{k,y}$$

which yields $\lambda_k \in \mathcal{O}(V)$ such that

$$\rho_{k+1}(x,y) e^{H_{x,y^{-1}}} = [e^{H_{x,y^{-1}}} \rho_{k,y}]_{x^{-1}} + \lambda_k(y) .$$

Since $\sigma_1 = -H$, we get $e^{-H_{x,y^{-1}}} [e^{H_{x,y^{-1}}} \rho_{1,y}]_{x^{-1}} = -e^{-H_{x,y^{-1}}} \mathcal{L}_H H$. Setting $\widetilde{\lambda}_0 = -H$, we obtain $\rho_2 = e^{-H_{x,y^{-1}}} (\mathcal{L}_H \lambda_0 + \widetilde{\lambda}_1)$ and a straightforward finite recurrence gives (6.4) with $(s_k) = (\sigma_k)$. In particular, $k = p$ yields (6.2), because $\rho_{p,y} = 0$; the discriminant of T_z doesn't vanish in W because $h_1, \dots, h_p$ have no common value. Since (6.4) also reads $\sigma_{k+1} = \mathcal{L}_H \sigma_k + \widetilde{\lambda}_k$ we obtain that $\lambda_k = \sigma_{k+1}(0, .)$ on V. Hence, $\lambda_1, \dots, \lambda_p$ do not depend on z_* so that they are well-defined holomorphic functions on Δ.

(2) Assume now (2) is true. We only have to check that $T = X^p - \sum_{1\leqslant k\leqslant p} s_k X^{p-k}$ is actually a p-multivaluate shock-wave function. Formulas (6.4) also read $s_{k+1} = \mathcal{L}_H s_k + \widetilde{\lambda}_k$ for $1 \leqslant k \leqslant p-1$ and (6.2) means that $\mathcal{D}_H s_p = 0$. Hence $0 = s_{p,y} - H_x s_p = s_{p,y} - s_{1,x} s_p$ and $s_{k+1,x} = \mathcal{D}_H s_k = s_{k,y} - H_x s_k = s_{k,y} + s_{1,x} s_k$. So, if $z_* \in D$ is outside $\Sigma = \{\operatorname{Discr} T = 0\}$, Lemma 16 implies that the holomorphic functions $h_1, \dots, h_p$ which near z_* describe the roots of T_z are mutually distinct shock-wave functions. □

The following describes p-shock-wave functions which are affine in x.

Proposition 18. *Let D be a simply connected domain of $\mathbb{C}^2$ containing 0, $a, b \in \mathcal{O}(D)$ and $H = x \otimes a + 1 \otimes b$. Then, H is a $\mathcal{O}(D)$-limit of affine p-shock-wave functions if and only if there exists $Q_0, Q_1 \in \mathbb{C}_{p-1}[Y]$ such that*

$$a = \frac{Q_1}{1 - Q_{1,y^{-1}}} \quad \text{and} \quad b = \frac{Q_0}{1 - Q_{1,y^{-1}}} \tag{6.5}$$

When (6.5) *and*

$$\operatorname{Discr}(1 - Q_{1,y^{-1}}) \neq 0 \tag{6.6}$$

are satisfied, the decomposition of H in elementary fractions gives H as a sum of rational shock-wave functions.

Proof. First assume that $H = \sum_{1\leqslant j\leqslant p} h_j$ where $h_1, \dots, h_p$ are mutually distinct shock-wave functions; with the notation of Proposition 17, $(-1)^1 s_1, \dots, (-1)^p s_p$ are the symmetric functions of $h_1, \dots, h_p$ and satisfy the relations $\mathcal{D}_H s_p = 0$ and $s_{k+1} = \mathcal{L}_H s_k + \widetilde{\lambda}_k$, $1 \leqslant k \leqslant p-1$. There exist sequences of holomorphic functions, $(\lambda_{j,k})_{k\in\mathbb{N}}$, $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$, each satisfying the recurrence $u_{k+1} = u'_k - a u_k$, such that

$$\mathcal{L}_H^k \widetilde{\lambda}_j = \frac{x^k}{k!} \otimes \lambda_{j,k}\,, \qquad \mathcal{L}_H^k H = \frac{x^{k+1}}{(k+1)!} \otimes a_k + \frac{x^k}{k!} \otimes b_k\,, \qquad k \in \mathbb{N}\,.$$

Hence, (6.2) yields the vanishing of the x-polynomial

$$\frac{x^p}{p!} \otimes a_p + \frac{x^{p-1}}{(p-1)!} \otimes b_p - \sum_{1\leqslant j\leqslant p-1} \frac{x^{p-1-j}}{(p-1-j)!} \otimes \lambda_{j,p-j}$$

and ensures $a_p = b_p = \lambda_{j,p-j}$, $1 \leqslant j \leqslant p-j$. So, one can find $Q_1, Q_0 \in \mathbb{C}_{p-1}[Y]$ and $\Lambda_j \in \mathbb{C}_{p-j-1}[Y]$, $1 \leqslant j \leqslant p-1$, such that with $A = a_{y^{-1}}$

$$a_k = Q_1^{(k)} e^A\,, \quad b_k = Q_0^{(k)} e^A, \quad \lambda_{j,k} = \Lambda_j^{(k)} e^A,\ k \in \mathbb{N}\,.$$

Thus $a = a_0 = Q_1 e^A$, $1 - e^{-A} = Q_{1,y^{-1}}$ and hence, (6.5). To achieve the general case, it is now sufficient to prove the last statement of the proposition. Assume (6.5) and (6.6) are true for some $Q_0, Q_1 \in \mathbb{C}_{p-1}[Y]$. Then, the decomposition in elementary fraction of H is Σh_j where $h_j(x,y) = \frac{q_j x + c_j}{1-q_j y}$, $1 \leqslant j \leqslant p$ and $q_1^{-1}, \dots, q_p^{-1}$ are the roots of $1 - Q_{1,y^{-1}}$. It is quite evident that $h_1, \dots, h_p$ are mutually distinct shock-wave functions. □

Remark 1. If $x \otimes a + 1 \otimes b$ is the sum of p-mutually distinct shock-wave functions $h_1, \dots, h_p$, each h_j is algebraic since for $1 \leqslant k \leqslant p$, (6.4) yields $s_k(x,y) = [1 - Q_{1y^{-1}}(y)]^{-1} S_k(x,y)$ where S_k is the polynomial defined by

$$S_k = -\frac{x^k \otimes Q_1^{(k-1)}}{k!} - \frac{x^{k-1} \otimes Q_0^{(k-1)}}{(k-1)!} + \sum_{1 \leqslant j \leqslant k-1} \frac{x^{k-1-j} \otimes \Lambda_j^{(k-1-j)}}{(k-1-j)!}\,.$$

Remark 2. When Q_1 is allowed to be non-generic, $1 - Q_{1,y^{-1}}$ can only be written in the form $1 - Q_{1,y^{-1}} = \prod_{1 \leqslant j \leqslant m} (1 - q_j y)^{\alpha_j}$ with $\alpha_1, \dots, \alpha_m \in \mathbb{N}^*$. Hence, if $Q_0 \in \mathbb{C}_{p-1}[Y]$ and (a,b) is defined by (6.5), there is constants $c_{j,\ell}$ such that

$$H \overset{def}{=} x \otimes a + 1 \otimes b = \sum_{1 \leqslant j \leqslant m} \sum_{1 \leqslant \ell \leqslant \alpha_j} h_{j,\ell}$$

where $h_{j,\ell} = \frac{q_j x}{1-q_j y} + \frac{c_{j,\ell}}{(1-q_j y)^\ell}$, $1 \leqslant j \leqslant m$, $1 \leqslant \ell \leqslant \alpha_j$. Each $h_{j,\ell}$ is now a rational *generalized shock-wave function* in the sense it is a solution of the equation

$$h_y - h_x h = (\ell - 1) \kappa h_x^{\ell+1}$$

where $\kappa \in \mathbb{C}$ is equal to $c_{j,\ell}/q_j^\ell$.

Generalized shock-wave functions arise when an affine function $H = x \otimes a + 1 \otimes b$ has coefficient a and b given by (6.5) not constrained to (6.6). In that case, H is a limit of *rational p-shock-wave functions*, that is of sums of p mutually distinct rational shock-wave functions. The lemma below, which is an elementary consequence of Proposition 18, proves the converse and so, shows that generalized shock-wave functions occur naturally.

Lemma 19. *Let $(H^t)_{t \in T} = (x \otimes a^t + 1 \otimes b^t)_{t \in T}$ be a continuous family of holomorphic affine functions in a simply connected domain D such that the set T_{reg} of parameters t for which H_t is a rational p-shock-wave function is dense in T. Then there exists in $\mathbb{C}_{p-1}[Y]$ continuous family of holomorphic polynomials (Q_1^t) and (Q_0^t) such that for any $t \in T$, $a^t = Q_1^t (1 - Q_{1,y^{-1}}^t)^{-1}$ and $b^t = Q_{1,0}^t (1 - Q_{1,y^{-1}}^t)^{-1}$. Hence, H^t is a p-shock (resp. p-generalized) shock-wave function when* $\mathrm{Discr}(1 - Q_{1,y^{-1}}^t) \neq 0$ *(resp.* $\mathrm{Discr}(1 - Q_{1,y^{-1}}^t) = 0$*).*

6.1.1 Proof of Theorem 4.

(1) Assume that there is an open Riemann surface $\mathcal{X}$ such that $\overline{\mathcal{X}} = \mathcal{X} \cup \gamma$ is a manifold with almost smooth

boundary. Then, Theorem 3a gives that almost every point $(\xi_{0*}, \xi_{1*}, 1)$ of $\mathbb{CP}_2$ has a neighborhood W_{ξ_*} for which one can find an integer p and p mutually distinct shock-wave functions $h_1, \dots, h_p$ on W_{ξ_*} such that $L = \Sigma h_j - G$ is affine in ξ_0. Set $s_k = (-1)^k \sum_{1\leqslant j_1<\cdots<j_k\leqslant p} h_{j_1}\cdots h_{j_k}$ and $T = X^p + \sum_{1\leqslant k\leqslant p} s_k X^{p-k}$. Then $\operatorname{Discr} T \not\equiv 0$ and Proposition 17 implies that property (1) of Theorem 4 holds.

Now assume that $\widetilde{p}$ is the least integer q such that there is a q-multivaluate shock-wave function whose trace differs from $G|_{W_{\xi_*}}$ only by a ξ_0-affine function. Let $\widetilde{T}$ be a $\widetilde{p}$-multivaluate shock-wave function with trace $\widetilde{H}$ such that $\widetilde{L} = \widetilde{H} - G|_{W_{\xi_*}}$ is affine in ξ_0. Let $\widetilde{h}_1, \dots, \widetilde{h}_{\widetilde{p}}$ be the holomorphic function on W_{ξ_*} which describes the roots of $\widetilde{T}$. Then $\{\widetilde{h}_1, \dots, \widetilde{h}_{\widetilde{p}}\}$ is minimal in the sense of the fourth remark below Theorem 3a.

When $\widetilde{p} \geqslant 1$, this remark says that $\mathcal{X}$ is a normalization of the analytic extension $\mathcal{Y}$ in $\mathbb{CP}_2 \backslash f(\gamma)$ of the union of the graphs of the functions $(1 : \widetilde{h}_j : -\xi_0 - \xi_1 \widetilde{h}_j)$, $1 \leqslant j \leqslant \widetilde{p}$, and for any $\xi \in W_{\xi_*}$, the intersection of $\mathcal{Y}$ with the projective lines $\xi_0 w_0 + \xi_1 w_1 + w_2 = 0$ is

$$\left\{(1 : \widetilde{h}_j(\xi) : \xi_0 - \xi_1 \widetilde{h}_j(\xi))\,;\ 1 \leqslant j \leqslant \widetilde{p}\right\}.$$

Since, by Proposition 10, $\mathcal{Y}$ is uniquely determined by (γ, f), each $\widetilde{h}_j$ and so, each symmetric function $\widetilde{s}_k$ of $\widetilde{h}_1, \dots, \widetilde{h}_k$, is uniquely determined by (γ, f). If $\lambda_1, \dots, \lambda_{\widetilde{p}}$ are any one-variable holomorphic functions such that assertion (2) of Proposition 17 holds, with $\widetilde{H}$ instead of H, then $\lambda_{k-1}(\xi_1) = \widetilde{s}_k(0, \xi_1)$, $1 \leqslant k \leqslant p-1$. Hence, $\lambda_1, \dots, \lambda_{\widetilde{p}}$ are uniquely determined by (γ, f).

Lemma 13 implies now that L is affine in ξ_0 and is, for some integer q, the limit of a continuous one parameter family of ξ_0-affine q-shock-wave functions. Thanks to Lemma 19, this implies that $L(\xi_0, \xi_1) = \xi_0 \frac{\alpha'(\xi_1)}{1-\alpha(\xi_1)} + \frac{\beta(\xi_1)}{1-\alpha(\xi_1)}$ where $\alpha \in \mathbb{C}_q[\xi_1]$ vanish at zero and $b \in \mathbb{C}_{q-1}[\xi_1]$.

(2) Assume now that property (1) of Theorem 4 holds. Set $T = X^p + \sum 1 \leqslant k \leqslant p s_k X^{p-k}$ where now s_k are defined by (6.4). Proposition 17 implies then that T is determines a p-multivaluate shock-wave function whose trace is $G + L$. Since L is affine in ξ_0, G satisfies the hypothesis of Theorem 3a.B. As G is not affine in ξ_0 by hypothesis, γ, with its given orientation, is the boundary of an open Riemann surface $\mathcal{X}$ with the sought after properties. Hence, (4) has to be satisfied from the direct part of Theorem 4.

6.1.2 Particular case of Theorem 4. For minimal p equal to 2, it turns out that Theorem 4 says that γ bounds almost smoothly an open Riemann surface $\mathcal{X}$ where f extends meromorphically if and only if for some constants $c, \alpha_1, \alpha_2, \beta_0, \beta_1$

$$\mathcal{D}_G\mathcal{L}_G G = cC + \alpha_1 A_1 + \alpha_2 A_2 + \beta_0 B_0 + \beta_1 B_1 \tag{6.7}$$

where

$$C = (G_x - g_x)\mathrm{e}^{g_{x,y^{-1}}}\,,$$

$$A_1 = y\mathcal{D}_G\mathcal{L}_G G + (x\mathcal{L}_G G)_x\,, \quad A_2 = y^2\mathcal{D}_G\mathcal{L}_G G + (2xy\mathcal{L}_G G + x^2 G)_x\,,$$

$$B_0 = \mathcal{D}_G G - \mathcal{D}_G g\,, \quad B_1 = yB_0 + g - G\,.$$

6.2 Affine characterization. A characterization for an affine presentation is possible but very special data have to be selected. Assume $\mathcal{X}$ is a Riemann surface with almost smooth boundary γ. When $u \in C^\infty(\gamma)$, a straightforward computation gives that $d^c\widetilde{u} = (Nu)\tau^*$ as forms of $\gamma\backslash\overline{\mathcal{X}}_{\mathrm{sing}}$. So, Lemma 11 implies that

$$\int_\gamma (Nu)\tau^* = \int_{\mathcal{X}} dd^c\widetilde{u} = 0\,. \tag{6.8}$$

Hence $(Nu)\tau^*$ has a primitive v on γ and the holomorphic extension to $\mathcal{X}$ of $h = u + iv$ is equivalent to the moments condition

$$\forall \Psi \in H^{1,0}(\overline{\mathcal{X}})\,, \quad \int_\gamma h\Psi = 0\,. \tag{6.9}$$

If $H \in \mathcal{O}(\overline{\mathcal{X}})^{N+1}$ is such that $h = H|_\gamma$ embeds γ in $\mathbb{C}^{N+1}$, $\mathcal{Y} = H(\mathcal{X})\backslash h(\gamma)$ is a complex curve of $\mathbb{C}^{N+1}\backslash h(\gamma)$ which has finite volume and has boundary $h(\gamma)$ and γ has to satisfy the Harvey–Lawson-moments condition

$$\forall k_0, \dots, k_N \in \mathbb{N}\,, \quad \int_\gamma h_0^{k_0} \cdots h_N^{k_N}\, dh_0 = 0 \tag{6.10}$$

which by the way also contains (6.8).

Proposition 20. *Let $u_0, \dots, u_N \in C^\infty(\gamma)$ and let $v_\ell \in C^\infty(\gamma)$ be a primitive of $(Nu_\ell)\tau^*$, $0 \leqslant \ell \leqslant N$. Assume that $h = (u_\ell + iv_\ell)_{0\leqslant\ell\leqslant N}$ is an embedding of γ into $\mathbb{C}^{N+1}$. Then* (6.10) *is a necessary condition to the existence of a Riemann surface $\mathcal{X}$ with almost smooth boundary γ, such that each u_ℓ extends to $\mathcal{X}$ as a harmonic function with harmonic conjugate function. The converse is true when γ is connected and suitably oriented.*

Remark. Of course, the above conclusion means that $\mathcal{X}$ is a Riemann surface where each h_ℓ extends holomorphically.

Proof. If $\mathcal{X}$ exists with the required properties, $\delta = h(\gamma)$ bounds, in the sense of current, $h(\mathcal{X})\backslash\delta$ which is a complex curve of finite volume of $\mathbb{C}^n\backslash\delta$. Cauchy theorem implies then that (6.10) is verified. If (6.10) is satisfied, [HL1] produces a holomorphic 1-chain Y such that $dY = [\delta]$; since γ is connected, $Y = [\mathcal{Y}]$ for a suitable orientation of γ. A normalization of $\overline{\mathcal{Y}}$ constructed as in the proof of Theorem 3a gives a suitable $\mathcal{X}$. □

References

[AS] L.V. Ahlfors, L. Sario, Riemann Surfaces, Princeton Mathematical Series, No. 26, Princeton University Press, Princeton, NJ, 1960.

[AlW] H. Alexander, J. Wermer, Linking numbers and boundaries of varieties, Ann. of Math. 151 (2000), 125–150

[B1] M.I. Belishev, Boundary control in reconstruction of manifolds and metrics (the BC method), Inverse Probl. 13, (1997) R1-R45.

[B2] M.I. Belishev, The Calderon problem for two dimensional manifolds by the BC-method, SIAM J. Math. Anal. 35:1 (2003), 172–182.

[BK] M.I. Belishev, Y.V. Kurylev, To the reconstruction of a Riemaniann manifold via its spectral data (BC-method), Comm. Partial Differential Equations 17:5-6 (1992), 767–804.

[Bi] E. Bishop, Holomorphic completions, analytic continuation, and the interpolation of semi-norms, Ann. of Math. (2) 78 (1963), 468–500.

[Bo] A. Bossavit, On the notion of anisotropy of constitutive laws: Some implications of the "Hodge implies metric" result, COMPEL 20:1 (2001), 233–239.

[BrU] R. Brown, G. Uhlmann, Uniqueness in the inverse conductivity problem for non-smooth conductivities in two dimensions, Comm. Part. Diff. Equations 22 (1997), 1009–1027.

[C] A.-P. Calderon, On an inverse boundary problem, Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro), Soc. Brasil, Rio de Janeiro (1980), 65–76.

[Ch] E.M. Chirka, Regularity of the boundaries of analytic sets (in Russian), Mat. Sb. (N.S.) 117:3 (159) (1982), 291–336, 431.

[D] T.-C. Dinh, Orthogonal measures on the boundary of a Riemann surface and polynomial hull of compacts of finite length, J. Funct. Anal. 157 (1998), 624–649.

[DoH] P. Dolbeault, G. Henkin, Chaînes holomorphes de bord donné dans $\mathbb{CP}_n$, Bull. Soc. Math. France 125 (1997), 383–445.

[DoP] P. Dolbeault, J.-B. Poly, Variations sur le problème du bord dans $\mathbb{CP}_n$, preprint (1995).

[F] H. Federer, Geometric Measure Theory, Die Grundlehren der mathematischen Wissenschaften 153, Springer-Verlag, New York Inc., 1969.

[G] P. Griffiths, Variations on a theorem of Abel, Invent. Math. 35 (1976), 321–390.

[GH] P. Griffiths, J. Harris, Principles of Algebraic Geometry, Pure and Applied Mathematics, Wiley-Interscience, New York, 1978.

[H] R. Harvey, Holomorphic chains and their boundaries, Proc. Symp. Pure Math. 30 (1977), 309–382.

[HL1] F.R. Harvey, H.B. Lawson, Boundaries of complex analytic varieties, Ann. of Math. 102 (1975), 233–290.

[HL] F.R. Harvey, H.B. Lawson, Boundaries of varieties in projective manifolds, J. Geom. Anal. 14:4 (2004), 673–695.

[He] G. Henkin, The Abel–Radon transform and several complex variables, Ann. of Math. Stud., Princeton Univ. Press, Princeton, NJ (1995), 223–275.

[LU] M. Lassas, G. Uhlmann, On determining a Riemannian manifold from the Dirichlet-to-Neumann map, Ann. Sci. École Norm. Sup. 34 (2001), 771–787.

[M] N. Mandache, Exponential instability in an inverse problem for the Schrödinger equation, Inverse Problems 17 (2001), 1435–1444.

[Mc] E.J. McShane, Extension of range of functions, Bull. Am. Math. Soc. 40 (1934), 837–842.

[N] R. Novikov, Multidimensional inverse spectral problem for the equation $-\Delta\psi + (v - Eu)\psi = 0$, Funct. Anal. and Appl. 22 (1988), 263–272.

[R] M. Rosenlicht, Generalized Jacobian varieties, Ann. of Math. (2) 59 (1954), 505–530.

[S] G. Stolzenberg, Volumes, Limits and Extensions of Analytic Varieties, Springer-Verlag, NY, 1966.

[Sy] J. Sylvester, An anisotropic inverse boundary value problem, Comm. Pure Appl. Math. 43 (1990), 201–232.

[V] I.N. Vekua, Generalized Analytic Functions, Pergamon Press, 1962.

[W] J. Wermer, The hull of a curve in $\mathbb{C}^n$, Ann. of Math. 68:3 (1958), 550–561.

Gennadi Henkin, Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris Cedex 05, France `henkin@math.jussieu.fr`

Vincent Michel, Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris Cedex 05, France `michel@math.jussieu.fr`

Received: May 2005
Revision: October 2005
Accepted: October 2005