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TOWARDS RELATIVE INVARIANTS OF REAL SYMPLECTIC FOUR-MANIFOLDS

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Abstract. Let (X, ω, c_X) be a real symplectic four-manifold with real part $\mathbb{R}X$. Let $L \subset \mathbb{R}X$ be a smooth curve such that $[L]=0 \in H_1(\mathbb{R}X;\mathbb{Z}/2\mathbb{Z})$. We construct invariants under deformation of the quadruple (X, ω, c_X, L) by counting the number of real rational J-holomorphic curves which realize a given homology class d, pass through an appropriate number of points and are tangent to L. As an application, we prove a relation between the count of real rational J-holomorphic curves done in [W2] and the count of reducible real rational curves done in [W3]. Finally, we show how these techniques also allow us to extract an integer valued invariant from a classical problem of real enumerative geometry, namely about counting the number of real plane conics tangent to five given generic real conics.

1 Statement of the Results

Let (X, ω, c_X) be a *real symplectic four-manifold*, that is a triple made of a smooth compact four-manifold X, a symplectic form ω on X and an involution c_X on X such that $c_X^*\omega = -\omega$. The fixed point set of c_X is called *the real part of* X and is denoted by $\mathbb{R}X$. It is assumed to be nonempty here so that it is a smooth lagrangian surface of (X, ω). We label its connected components by $(\mathbb{R}X)_1, \dots, (\mathbb{R}X)_N$. Let $L \subset \mathbb{R}X$ be a smooth curve which represents 0 in $H_1(\mathbb{R}X;\mathbb{Z}/2\mathbb{Z})$, and $B \subset \mathbb{R}X$ be a surface having L as a boundary.

1.1 Definitions. Let $l \gg 1$ be an integer large enough and $\mathcal{J}_\omega$ be the space of almost complex structures of X which are tamed by ω and of class C^l. Let $\mathbb{R}\mathcal{J}_\omega$ be the subspace of $\mathcal{J}_\omega$ made of almost complex structures J for which the involution c_X is J-antiholomorphic. These two

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spaces are separable Banach manifolds which are non-empty and contractible (see §1.1 of [W2] for the real case). Assume that the first Chern class $c_1(X)$ of the symplectic four-manifold (X,ω) is not a torsion element in $H^2(X;\mathbb{Z})$ and let $d \in H_2(X;\mathbb{Z})$ be a homology class satisfying $c_1(X)d > 1$, $c_1(X)d \neq 4$ and $(c_X)_* d = -d$. Let $\underline{x} = (x_1, \dots, x_{c_1(X)d-2}) \in X^{c_1(X)d-2}$ be a *real configuration* of $c_1(X)d - 2$ distinct points of X, that is an ordered subset of distinct points of X which is globally invariant under c_X. For $j \in \{1, \dots, N\}$, we set $r_j = \#(\underline{x} \cap (\mathbb{R}X)_j)$ and $r = (r_1, \dots, r_N)$ so that the N-tuple r encodes the equivariant isotopy class of $\underline{x}$. We will assume throughout the paper that $r \neq (0, \dots, 0)$. Finally, denote by I the subset of those $i \in \{1, \dots, c_1(X)d - 2\}$ for which x_i is fixed by the involution c_X, so that $I \neq \emptyset$. For each $i \in I$, choose a line T_i in the tangent plane $T_{x_i}\mathbb{R}X$. Let $J \in \mathbb{R}\mathcal{J}_\omega$ be generic enough. Then, as in [W3], we denote by $\mathcal{C}usp^d(J,\underline{x})$ (resp. $\mathcal{R}ed^d(J,\underline{x})$, $\mathcal{T}an^d(J,\underline{x})$) the finite set of real rational cuspidal (resp. reducible, whose tangent line at some point x_i, $i \in I$, is T_i) J-holomorphic curves which realize the homology class d and pass through $\underline{x}$. Likewise, we denote by $\mathcal{T}an^d_L(J,\underline{x})$ the finite set of real rational J-holomorphic curves which realize the homology class d, pass through $\underline{x}$ and are tangent to L. Note that the genericity assumption on $J \in \mathbb{R}\mathcal{J}_\omega$ implies that the non-trivial point of contact of the curve with L is unique and of order two. Also, all these curves have only transversal double points as singularities lying outside of $\underline{x}$, with the exception of elements of $\mathcal{C}usp^d(J,\underline{x})$ which have in addition a unique real ordinary cusp. Let $C \in \mathcal{T}an^d_L(J,\underline{x}) \cup \mathcal{C}usp^d(J,\underline{x}) \cup \mathcal{R}ed^d(J,\underline{x}) \cup \mathcal{T}an^d(J,\underline{x})$, we define the *mass* of C and denote by $m(C)$ its number of real isolated double points. Here, a real double point is said to be *isolated* when it is the local intersection of two complex conjugated branches, whereas it is said to be *non-isolated* when it is the local intersection of two real branches. Let $C \in \mathcal{T}an^d_L(J,\underline{x})$ and y be its point of contact with L. Then, either $\mathbb{R}C$ is locally included in B near y, or its intersection with B is locally restricted to $\{y\}$. We define the *contact index* $\langle C, B\rangle$ to be -1 in the first case and $+1$ in the second. Likewise, if $C \in \mathcal{C}usp^d(J,\underline{x})$ (resp. $C \in \mathcal{T}an^d(J,\underline{x})$), then its cuspidal point (resp. its tangent line T_i, $i \in I$) is unique and we define $\langle C, B\rangle$ to be -1 if it is outside B or $+1$ if it is inside. Finally, if C belongs to $\mathcal{R}ed^d(J,\underline{x})$ and C_1, C_2 denote its irreducible components, then both these components are real and we set

$$\text{mult}_B(C) = \sum_{y \in \mathbb{R}C_1 \cap \mathbb{R}C_2} \langle y, B \rangle \,,$$

where $\langle y, B\rangle$ equals -1 if y is outside B or $+1$ if it is inside.

1.2 Statement of the results. We set

$$\Gamma_r^{d,B}(J,\underline{x}) = \sum_{C \in \cup \mathcal{T}an_L^d(J,\underline{x}) \cup \mathcal{T}an^d(J,\underline{x}) \cup \mathcal{C}usp^d(J,\underline{x})} (-1)^{m(C)} \langle C, B\rangle - \sum_{C \in \mathcal{R}ed^d(J,\underline{x})} (-1)^{m(C)} \,\text{mult}_B(C) \,.$$

Theorem 1.1. *Let (X, ω, c_X) be a real symplectic four-manifold and $B \subset \mathbb{R}X$ be a surface with boundary L. The connected components of $\mathbb{R}X$ are labelled by $(\mathbb{R}X)_1, \ldots, (\mathbb{R}X)_N$. Let $d \in H_2(X;\mathbb{Z})$ be such that $c_1(X)d > 1$ and $c_1(X)d \neq 4$, and $\underline{x} \subset X \setminus L$ be a real configuration of $c_1(X)d - 2$ distinct points. For $j \in \{1, \ldots, N\}$, denote by r_j the cardinality of $\underline{x} \cap (\mathbb{R}X)_j$ and by $r = (r_1, \ldots, r_N)$, which is assumed to be different from $(0, \ldots, 0)$. Finally, let $J \in \mathbb{R}\mathcal{J}_\omega$ be generic enough so that the integer $\Gamma_r^{d,B}(J,\underline{x})$ is well defined. Then, this integer $\Gamma_r^{d,B}(J,\underline{x})$ neither depends on the choice of J, nor on the choice of $\underline{x}$.*

From this theorem, the integer $\Gamma_r^{d,B}(J,\underline{x})$ can be denoted without ambiguity by $\Gamma_r^{d,B}$, and when it is not well defined, we set $\Gamma_r^{d,B} = 0$. Note that the condition $c_1(X)d \neq 4$ is to avoid appearance of multiple curves, see Remark 1.11 of [W3].

Remark 1.2. 1) In particular, the integer $\Gamma_r^{d,B}(J,\underline{x})$ does not depend on the relative position of $\underline{x}$ with respect to B, it only depends on r.

2) When $B = \emptyset$, $\Gamma_r^{d,B} = -\Gamma_r^d$, where Γ_r^d is the invariant defined in [W3]. Theorem 1.1 then follows from Theorem 0.1 of [W3]. In fact, Theorem 0.1 of [W3] is nothing but the particular case $B = \emptyset$ of Theorem 1.1. Note that this case $B = \emptyset$ is, however, of a slightly different nature since $\Gamma_r^{d,\emptyset} = -\Gamma_r^d$ is an absolute invariant whereas, as soon as $B \neq \emptyset$, the first term in the right-hand side defining $\Gamma_r^{d,B}$ makes it a relative invariant in the spirit of [LR] and [IP].

3) One has $\Gamma_r^{d,B} = -\Gamma_r^{d,\mathbb{R}X \setminus B}$.

We denote by $\Gamma^{d,B}(T)$ the generating function $\sum_{r \in \mathbb{N}^N} \Gamma_r^{d,B} T^r \in \mathbb{Z}[T_1, \ldots, T_N]$, where $T^r = T_1^{r_1} \ldots T_N^{r_N}$. This polynomial function is of the same parity as $c_1(X)d$ and each of its monomials actually only depends on one indeterminate. It follows from Theorem 1.1 that the function

$\Gamma^B : d \in H_2(X;\mathbb{Z}) \mapsto \Gamma^{d,B}(T) \in \mathbb{Z}[T]$ only depends on the quadruple (X,ω,c_X,B). Moreover, it is invariant under deformation of this quadruple, that is if ω_t is a continuous family of symplectic forms on X for which $c_X^*\omega_t = -\omega_t$ and B_t is an isotopy of compact surfaces in $\mathbb{R}X$, then this function is the same for all (X,ω_t,c_X,B_t).

Theorem 1.3. *Under the hypothesis of Theorem 1.1, assume that B is a disk in $\mathbb{R}X$. Then $2\chi^d_{r+1} = \Gamma^{d,B}_r + \Gamma^d_r$. Moreover,*

1) *If (X,ω,c_X) is the complex projective plane equipped with its standard symplectic form and real structure, then $\Gamma^{d,B}_r = \Gamma^d_r$.*
2) *If (X,ω,c_X) is the hyperboloid $(\mathbb{C}P^1\times\mathbb{C}P^1, \omega_{\mathbb{C}P^1}\oplus\omega_{\mathbb{C}P^1}, conj\times conj)$, then $\Gamma^{d,B}_r = 2\chi^d_{r+1} + \Gamma^d_r$.*

(Remember that the integer χ^d_{r+1} has been defined in [W1,2] and the integer Γ^d_r in [W3]. Note that when $\mathbb{R}X$ is connected, $r \in \mathbb{N}^*$.)

Corollary 1.4. *Under the hypothesis of Theorem 1.3, we have $\chi^d_{r+1} = \Gamma^d_r = \Gamma^{d,B}_r$ in the case of the complex projective plane and $\Gamma^{d,B}_r = 2\chi^d_{r+1}$, $\Gamma^d_r = 0$ in the case of the hyperboloid.* □

The first equality of this corollary has been announced in [W3, Prop. 0.3]. It provides a relation between the count of real rational J-holomorphic curves done in [W2] and the count of reducible and cuspidal curves done in [W3]. Does such a relation have a complex analog?

1.3 More tangency conditions, the case of conics. It is possible to extend the above results to curves having more than one tangency condition with L, at least in the case of plane conics (see also §4.3). We illustrate this phenomenon here on the following classical problem of real enumerative geometry, solved by De Joncquières in 1859: there are 3264 conics which are tangent to five given generic conics in the complex projective plane. If the five given conics are real, then the number of real conics tangent to them of course depends on the choice of the conics. We however show here how it is possible to extract an integer valued invariant from this problem.

Let $B_1,\dots,B_5$ be five embedded disks in $\mathbb{R}P^2$ which are transversal to each other and $L_i = \partial B_i$, $i \in \{1,\dots,5\}$. Let $J \in \mathbb{R}\mathcal{J}_\omega$ be generic enough. We denote by $\mathcal{C}on(J)$ the finite set of real conics tangent to $L_1,\dots,L_5$ and by $\mathcal{C}on_{red}(J)$ the finite set of real reducible conics, that is pairs of J-holomorphic lines, tangent to four out of these five curves $L_1,\dots,L_5$. Let $C \in \mathcal{C}on(J)$, we set $\langle C,B\rangle = \prod_{i=1}^5 \langle C,B_i\rangle$. In the same way, let $C \in \mathcal{C}on_{red}(J)$ and $i_1,\dots,i_4 \in \{1,\dots,5\}$ be such that C is tangent to $L_{i_1},\dots,L_{i_4}$. We set $\langle C,B\rangle = \prod_{j=1}^4 \langle C,B_{i_j}\rangle$ and $\mathrm{mult}_B(C) = +1$ if the singular point of C belongs to B_{i_5} and -1 otherwise. Set

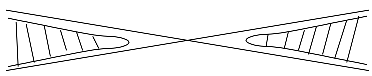
$$\Gamma^B(J) = \sum_{C \in \mathcal{C}on(J)} \langle C, B \rangle - \sum_{C \in \mathcal{C}on_{red}(J)} \langle C, B \rangle \operatorname{mult}_B(C) \in \mathbb{Z} .$$

Theorem 1.5. *The integer $\Gamma^B(J)$ does not depend on the generic choice of $J \in \mathbb{R}\mathcal{J}_\omega$. Moreover, it is invariant under isotopy of $B = B_1 \cup \cdots \cup B_5$.*

In particular, during such an isotopy, the five curves $L_1, \dots, L_5$ have to remain transversal to each other. Note that there are only finitely many isotopy classes of five real conics in the plane. How does Γ^B depend on the isotopy classes will be studied in §4.2, Proposition 4.1. The integer Γ^B is computed in the following cases.

PROPOSITION 1.6. *Let $B_1, \dots, B_5$ be five disjoint disks in $\mathbb{R}P^2$, then $\Gamma^B = 272$. The same holds when $B_1, \dots, B_5$ are close to a generic configuration of five real double lines of the plane.*

Here, a disk is said to be close to a double line with equation $y^2 = 0$ in the plane if it has an equation of the form $\{y^2 \le \epsilon^2 x^2 - \delta\}$ for small ϵ and δ's.



COROLLARY 1.7. *Let $L_1, \dots, L_5$ be five real generic plane conics whose isotopy class is given by Proposition 1.6. Then, the number of real conics tangent to $L_1, \dots, L_5$ is bounded from below by* 32.

Proof. The number of lines tangent to two different generic conics is four, they correspond to the intersection points between the two dual conics. The number of real reducible conics tangent to four out of the five conics $L_1, \dots, L_5$ thus does not exceed $240 = 5 * 3 * 4 * 4$. The result follows now from the definition of Γ^B and Proposition 1.6. □

Hence, this Corollary 1.7 provides lower bounds in real enumerative geometry. Note that this number of real conics does not admit any non-trivial upper bound. Indeed, F. Ronga, A. Tognoli and T. Vust have found a configuration of five real conics close to the double edges of some pentagon such that all the 3264 conics tangent to them are real, see [RTV].

The paper is organized as follows. The first paragraph is devoted to the construction of the moduli space $\mathbb{R}\mathcal{M}^d_L$ of real rational pseudo-holomorphic curves which realize the homology class d and are tangent to L. The second paragraph is devoted to the proof of the results of §1.2 and the third paragraph to the proof of the results of §1.3.

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2 Moduli Space of Real Rational Pseudo-Holomorphic Curves Tangent to L

Let $d \in H_2(X;\mathbb{Z})$ be such that $(c_X)_* d = -d$ and $c_1(X)d > 1$, $c_1(X)d \neq 4$. Let τ be an order two permutation of the set $\{1,\dots,c_1(X)d-2\}$ having one fixed point at least. Let $c_\tau : (x_1,\dots,x_{c_1(X)d-2}) \in X^{c_1(X)d-2} \mapsto (c_X(x_{\tau(1)}),\dots,c_X(x_{\tau(c_1(X)d-2)})) \in X^{c_1(X)d-2}$ be the associated real structure of $X^{c_1(X)d-2}$, its fixed point set is denoted by $\mathbb{R}_\tau X^{c_1(X)d-2}$. Let $L \subset \mathbb{R}X$ be a smooth curve such that $[L] = 0 \in H_1(\mathbb{R}X;\mathbb{Z}/2\mathbb{Z})$ and $B \subset \mathbb{R}X$ be a surface having L as a boundary. Finally, let g be a riemannian metric on X, invariant under c_X and for which L is a geodesic. We denote by ∇ the associated Levi–Civita connection on TX.

2.1 Moduli space $\mathbb{R}\mathcal{P}_L^*$ of real rational pseudo-holomorphic maps tangent to L. Let S be an oriented sphere of dimension two and *conj* be a smooth involution conjugated to the complex conjugation of $\mathbb{C}P^1$. Denote by $\mathbb{R}S$ the fixed point set of *conj* and by $\mathbb{R}\mathcal{J}_S$ the space of complex structures of class C^l of S which are compatible with its orientation and for which *conj* is J-antiholomorphic. Let $\xi \in \mathbb{R}S$, $\vec{\xi} \in T_\xi\mathbb{R}S \setminus \{0\}$ and $\underline{z} = (z_1,\dots,z_{c_1(X)d-2}) \in S^{c_1(X)d-2}$ be an ordered set of $c_1(X)d-2$ distinct points of $S \setminus \{\xi\}$. We assume that $\underline{z}$ is globally invariant under *conj* and that the permutation of $\{1,\dots,c_1(X)d-2\}$ induced by *conj* is τ. We set

$$\mathbb{R}\mathcal{P}_L = \big\{(u,J_S,J,\underline{x}) \in L^{k,p}(S,X) \times \mathbb{R}\mathcal{J}_S \times \mathbb{R}\mathcal{J}_\omega \times \mathbb{R}_\tau X^{c_1(X)d-2} \;\big|\; u_*[S] = d,$$
$$u(\underline{z}) = \underline{x},\ du + J\circ du \circ J_S = 0,\ c_X \circ u = u \circ conj,\ u(\xi) \in L \text{ and } d_\xi u(\vec{\xi}) \in T_{u(\xi)}L\big\},$$

where $1 \ll k \ll l$ is large enough and $p > 2$.

Let $\mathbb{R}\mathcal{P}_L^* \subset \mathbb{R}\mathcal{P}_L$ be the space of *non-multiple* pseudo-holomorphic maps, that is the space of quadruples $(u, J_S, J, \underline{x})$ for which u cannot be written $u' \circ \Phi$ where $\Phi : S \to S'$ is a non-trivial ramified covering and $u' : S' \to X$ a pseudo-holomorphic map.

Proposition 2.1. *The space $\mathbb{R}\mathcal{P}_L^*$ is a separable Banach manifold of class C^{l-k} with tangent bundle*

$$
\begin{aligned}
&T_{(u,J_S,J,\underline{x})}\mathbb{R}\mathcal{P}_L^* \\
&= \big\{(v,\dot J_S,\dot J,\dot{\underline{x}}) \in T_{(u,J_S,J,\underline{x})}(L^{k,p}(S,X)\times \mathbb{R}\mathcal{J}_S \times \mathbb{R}\mathcal{J}_\omega \times \mathbb{R}_\tau X^{c_1(X)d-2}) \ \big| \\
&\qquad v(\underline{z}) = \dot{\underline{x}}\,,\ dc_X\circ v = v\circ \mathrm{conj},\ v(\xi)\in T_{u(\xi)}L\,,\ \nabla_{\vec\xi}v \in T_{u(\xi)}L \\
&\qquad\qquad\qquad \textit{and}\ \ Dv + J\circ du\circ \dot J_S + \dot J\circ du\circ J_S = 0\big\}\,.
\end{aligned}
$$

Here, $T_uL^{k,p}(S,X) = \{v \in L^{k,p}(S,E_u)\}$ where $E_u = u^*TX$ and $D : v\in L^{k,p}(S,E_u)\mapsto \nabla v + J\circ\nabla v\circ J_S + \nabla_v J\circ du\circ J_S \in L^{k-1,p}(S,\Lambda^{0,1}S\otimes E_u)$ is the associated Gromov operator (see [MS, Prop. 3.1.1]).

Proof. If we remove the Cauchy–Riemann equation $du + J\circ du\circ J_S = 0$ from the definition of $\mathbb{R}\mathcal{P}_L^*$, then the corresponding space $\mathbb{R}\mathcal{A}_L^*$ is a separable Banach manifold of class C^{l-k}. After differentiation, the equation $d_\xi u(\vec\xi) = \lambda(u)\zeta(u)$, where ζ is a unitary vector field tangent to L, becomes $\nabla_{\vec\xi}v = d\lambda(u)\zeta(u) + \lambda(u)\nabla_v\zeta(u)$. Since L is a geodesic for g and v is collinear to ζ, the term $\nabla_v\zeta(u)$ vanishes and $\nabla_{\vec\xi}v\in T_{u(\xi)}L$. We have to prove that the space of non-multiple pseudo-holomorphic maps is a Banach submanifold of $\mathbb{R}\mathcal{A}_L^*$. This follows from the fact that the section $\sigma_{\overline\partial} : (u,J_S,J,\underline{x})\mapsto du + J\circ du\circ J_S$ of the bundle $L^{k-1,p}(S,\Lambda^{0,1}S\otimes E_u)$ vanishes transversely, the proof of the latter being the same as the one of Proposition 3.2.1 of [MS]. □

2.2 Normal sheaf. Remember that the $\mathbb{C}$-linear part of the Gromov operator D is some Cauchy-Riemann operator denoted by $\overline\partial$. The latter induces a holomorphic structure on the bundle $E_u = u^*TX$ which turns the morphism $du : TS \to E_u$ into an injective homomorphism of analytic sheaves (see [IvS], Lemma 1.3.1). Likewise, the $\mathbb{C}$-antilinear part of D is some order 0 operator denoted by R and defined by the formula $R_{(u,J_S,J,\underline{x})}(v) = N_J(v,du)$ where N_J is the Nijenhuis tensor of J. Denote by $\mathcal{O}_S(E_u)$ (resp. $\mathcal{O}_S(TS)$) the sheaf of analytic $\mathbb{Z}/2\mathbb{Z}$-equivariant sections of E_u (resp. TS). Also, denote by $\mathcal{N}_u$ the quotient sheaf $\mathcal{O}_S(E_u)/du(\mathcal{O}_S(TS))$ so that it fits in the following exact sequence of analytic sheaves $0\to\mathcal{O}_S(TS)\to\mathcal{O}_S(E_u)\to\mathcal{N}_u\to 0$. Denote by E_u^L the sheaf of $\mathbb{Z}/2\mathbb{Z}$-equivariant analytic sections of E_u which satisfy $v(\xi)\in T_{u(\xi)}L$ and $\nabla_{\vec\xi}v\in T_{u(\xi)}L$.

LEMMA 2.2. *Let w be a real vector field on S which vanishes at ξ. Then $du(w)\in E_u^L$.*

Proof. Denote by $v = du(w)$, then $v\in E_u$ and $v(\xi) = du(w(\xi)) = 0$. Moreover, $\nabla_{\vec\xi}v = (\nabla_{\vec\xi}du)(w) + du(\nabla_{\vec\xi}w)$. The first term vanishes since $w(\xi) = 0$ and the second belongs to $T_{u(\xi)}L$. □

We denote by $\mathcal{N}_u^L$ the quotient sheaf $\mathcal{O}_S(E_u^L)/du(\mathcal{O}_S(TS_{-\xi}))$, so that we have the exact sequence $0 \to \mathcal{O}_S(TS_{-\xi}) \to \mathcal{O}_S(E_u^L) \to \mathcal{N}_u^L \to 0$. Denote by $\widetilde{E}_u^L$ the sheaf of $\mathbb{Z}/2\mathbb{Z}$-equivariant sections of E_u for which $v(\xi) \in T_{u(\xi)}L$, so that $E_u^L \subset \widetilde{E}_u^L$.

LEMMA 2.3. *If $d_\xi u \neq 0$, then the quotient of $\widetilde{E}_u^L$ by $du(\mathcal{O}_S(TS))$ is the sheaf $\mathcal{N}_{u,-\xi}$. If $d_\xi u = 0$, but $\nabla_{\vec{\xi}} du(\vec{\xi}) \notin T_{u(\xi)}L$, then this quotient is the sheaf $N_u = \mathcal{O}_S(E_u)/du(\mathcal{O}_S(TS) \otimes \mathcal{O}_S(\xi))$.*

Proof. The first part follows from the fact that the condition $v(\xi) \in T_{u(\xi)}L$ for a section v of E_u reads in the quotient as a section of $\mathcal{N}_u$ which vanishes at ξ, since $T_{u(\xi)}L \subset Im(d_\xi u)$. In the second case, the normal sheaf $\mathcal{N}_u$ splits as $N_u \oplus \mathcal{N}_u^{sing}$, where $\mathcal{N}_u^{sing}$ is the skyscraper part $du(\mathcal{O}_S(TS) \otimes \mathcal{O}_S(\xi))/du(\mathcal{O}_S(TS))$. From the hypothesis, the cuspidal point at ξ is non-degenerated and has a tangent line distinct from $T_{u(\xi)}L$. Thus, $du(TS \otimes \mathcal{O}_S(\xi)) \not\subset \widetilde{E}_u^L$ and the skyscraper part $\mathcal{N}_u^{sing}$ does not belong to the quotient $\widetilde{E}_u^L/du(\mathcal{O}_S(TS))$. The projection $\widetilde{E}_u^L/du(\mathcal{O}_S(TS)) \subset \mathcal{N}_u$ onto N_u induced by $N_u \oplus \mathcal{N}_u^{sing} \to N_u$ provides the required isomorphism. □

As soon as $d_\xi u \neq 0$, we deduce the exact sequence $0 \to \mathcal{O}_S(TS_{-\xi}) \to \widetilde{E}_u^L \to \mathcal{N}_{u,-\xi} \oplus T_\xi L \to 0$, where $T_\xi L$ is the skyscraper sheaf $du(\mathcal{O}_S(TS))/du(\mathcal{O}_S(TS_{-\xi}))$. We deduce the inclusion $\mathcal{N}_u^L \subset \mathcal{N}_{u,-\xi} \oplus T_\xi L$.

PROPOSITION 2.4. 1) *Assume that $d_\xi u \neq 0$. Then, the skyscraper part $T_\xi L$ is included in $\mathcal{N}_u^L$ if and only if $\nabla_{\vec{\xi}} du(\vec{\xi}) \in T_{u(\xi)}L$, that is if $u(\xi)$ is a degenerated point of contact between $u(S)$ and L. In this case, the projection $\mathcal{N}_{u,-\xi} \oplus T_\xi L \to \mathcal{N}_{u,-\xi}$ restricted to $\mathcal{N}_u^L$ has image $\mathcal{N}_{u,-2\xi}$. Otherwise, this projection establishes an isomorphism between $\mathcal{N}_u^L$ and $\mathcal{N}_{u,-\xi}$.*

2) *Assume that $d_\xi u = 0$ but $\nabla_{\vec{\xi}} du(\vec{\xi}) \notin T_{u(\xi)}L$. Then, the sheaf $\mathcal{N}_u^L$ is isomorphic to $\mathcal{N}_u$.*

Proof. If $d_\xi u \neq 0$, the skyscraper part $T_\xi L$ is generated by $du(\mathcal{O}_S(TS))$. Let w be a real vector field on S, we have to see under which condition $du(w) \in E_u^L$. From the relation

$$\nabla_{\vec{\xi}}(du(w)) = (\nabla_{\vec{\xi}} du)(w) + du(\nabla_{\vec{\xi}} w)\,, \tag{1}$$

it is necessary and sufficient that $\nabla_{\vec{\xi}} du(\vec{\xi}) \in T_{u(\xi)}L$. In this case, the connection ∇ induces at ξ a derivation ∇^ξ of sections of the sheaf $\mathcal{N}_u$ such that the relations $v(\xi) \in T_{u(\xi)}L$ and $\nabla_{\vec{\xi}} v \in T_{u(\xi)}L$ reads in the quotient

$v(\xi) = 0$ and $\nabla^{\xi} v = 0$. Thus, the projection $\mathcal{N}_{u,-\xi} \oplus T_{\xi} L \to \mathcal{N}_{u,-\xi}$ restricted to $\mathcal{N}_u^L$ has image $\mathcal{N}_{u,-2\xi}$. Otherwise, it induces an isomorphism.

Assume now that $d_{\xi} u = 0$ but $\nabla_{\vec{\xi}} du(\vec{\xi}) \notin T_{u(\xi)} L$. Then, if w is a real vector field on S such that $w(\xi) \neq 0$, $du(w) \notin E_u^L$ from (1). The exact sequence $0 \to \mathcal{O}_S(TS) \to \widetilde{E}_u^L \to \mathcal{N}_u \to 0$ restricts thus as $0 \to \mathcal{O}_S(TS_{-\xi}) \to E_u^L \to \mathcal{N}_u \to 0$, hence the result. □

Denote by $\mathcal{O}_S(TS_{-\underline{z}})$ (resp. $\mathcal{O}_S(E_{u,-\underline{z}})$, $\mathcal{O}_S(E^L_{u,-\underline{z}})$, $\mathcal{N}_{u,-\underline{z}}$, $\mathcal{N}^L_{u,-\underline{z}}$, $T_{u(\xi),-\underline{z}} L$) the subsheaf of sections of $\mathcal{O}_S(TS)$ (resp. $\mathcal{O}_S(E_u)$, $\mathcal{O}_S(E_u^L)$, $\mathcal{N}_u$, $\mathcal{N}_u^L$, $T_{u(\xi)} L$) which vanish at $\underline{z}$. Remember that the operator $D : L^{k,p}(S, E^L_{u,-\underline{z}}) \to L^{k-1,p}(S, \Lambda^{0,1} S \otimes E_u)$ induces a quotient operator $\overline{D} : L^{k,p}(S, \mathcal{N}^L_{u,-\underline{z}}) := L^{k,p}(S, E^L_{u,-\underline{z}})/du(L^{k,p}(S, TS_{-\underline{z}}) \to L^{k-1,p}(S, \Lambda^{0,1} S \otimes N_u^L)$. From the short exact sequence of complexes

$$\begin{array}{ccccccc} 0 \to & L^{k,p}(S, TS_{-\xi-\underline{z}}) & \stackrel{du}{\to} & L^{k,p}(S, E^L_{u,-\underline{z}}) & \to & L^{k,p}(S, \mathcal{N}^L_{u,-\underline{z}}) & \to 0 \\ & \downarrow \overline{\partial}_S & & \downarrow D & & \downarrow \overline{D} & \\ 0 \to & L^{k-1,p}(S, \Lambda^{0,1} S \otimes TS) & \stackrel{du}{\to} & L^{k-1,p}(S, \Lambda^{0,1} S \otimes E_u^L) & \to & L^{k-1,p}(S, \Lambda^{0,1} S \otimes N_u^L) & \to 0, \end{array}$$

we deduce the long exact sequence $0 \to H^0(S, TS_{-\xi-\underline{z}}) \to H^0_D(S, E^L_{u,-\underline{z}}) \to H^0_{\overline{D}}(S, \mathcal{N}^L_{u,-\underline{z}}) \to H^1(S, TS_{-\xi-\underline{z}}) \to H^1_D(S, E^L_{u,-\underline{z}}) \to H^1_{\overline{D}}(S, \mathcal{N}^L_{u,-\underline{z}}) \to 0$, where H^0_D, $H^0_{\overline{D}}$ (resp. H^1_D, $H^1_{\overline{D}}$) denote the kernels (resp. cokernels) of the operators D, $\overline{D}$ on the associated sheaves. In particular,

$$\begin{aligned} \mathrm{ind}_{\mathbb{R}}(\overline{D}) &= \mathrm{ind}_{\mathbb{R}}(D) - \mathrm{ind}_{\mathbb{R}}(\overline{\partial}_S) \\ &= \big(c_1(X)d + 2 - 2 - 2\#\underline{z}\big) - (3 - 1 - \#\underline{z}) \\ &= 0\,. \end{aligned}$$

2.3 Moduli space of real rational pseudo-holomorphic curves tangent to *L*. Denote by $\mathcal{D}iff^+_{\mathbb{R}}(S, z, \xi)$ the group of diffeomorphisms of class C^{l+1} of S, which preserve the orientation, fix $\underline{z} \cup \{\xi\}$ and commute with *conj*. This group acts on $\mathbb{R}\mathcal{P}^*_L$ by

$$\phi.(u, J_S, J, \underline{x}) = \big(u \circ \phi^{-1}, (\phi^{-1})^* J_S, J, \underline{x}\big)\,,$$

where $(\phi^{-1})^* J_S = d\phi \circ J_S \circ d\phi^{-1}$. Denote by $\mathbb{R}\mathcal{M}^d_L$ the quotient of $\mathbb{R}\mathcal{P}^*_L$ by this action. The projection $\pi : (u, J_S, J, \underline{x}) \in \mathbb{R}\mathcal{P}^*_L \mapsto (J, \underline{x}) \in \mathcal{J}_\omega \times X^{c_1(X)d-2}$ induces on the quotient a projection $\mathbb{R}\mathcal{M}^d_L \to \mathbb{R}\mathcal{J}_\omega \times \mathbb{R}_\tau X^{c_1(X)d-2}$ still denoted by π.

PROPOSITION 2.5. *The space $\mathbb{R}\mathcal{M}^d_L$ is a separable Banach manifold of class C^{l-k}, and π is Fredholm of vanishing index. Moreover, if $[u, J_S, J, \underline{x}] \in \mathbb{R}\mathcal{M}^d_L$, then we have the isomorphisms* $\ker d\pi|_{(u,J_S,J,\underline{x})} \cong H^0_D(S, \mathcal{N}^L_{u,-\underline{z}})$ *and* $\mathrm{coker}\, d\pi|_{(u,J_S,J,\underline{x})} \cong H^1_D(S, \mathcal{N}^L_{u,-\underline{z}})$.

Proof. The proof is analogous to the one of Corollary 2.2.3 of [S] and Proposition 3.2.1 of [MS]. The action of $\mathcal{D}iff^+_{\mathbb{R}}(S,z)$ on $\mathbb{R}\mathcal{P}^*_L$ is smooth, fixed point free and admits a closed supplement. From Proposition 2.1 thus follows that $\mathbb{R}\mathcal{M}^d_L$ is a separable Banach manifold of class C^{l-k}. Moreover,

$$\begin{aligned}\ker d\pi|_{[u,J_S,J,\underline{x}]} &= \big\{(v,\dot{J}_S,0,0)\in T_{(u,J_S,J,\underline{x})}\mathbb{R}\mathcal{P}^*_L \;\big|\; v(\underline{z})=0\big\}/T_{Id}\mathcal{D}iff^+_{\mathbb{R}}(S,z,\xi)\\ &= \big\{v\in L^{k,p}(S,E^L_{u,-\underline{z}}) \;\big|\; \exists\phi\in L^{k-1,p}(S,\Lambda^{0,1}S\otimes TS)\,,\\ &\qquad\qquad Dv=du(\phi)\big\}/du\big(L^{k,p}(S,TS_{-\underline{z}})\big)\\ &= H^0_D(S,\mathcal{N}^L_{u,-\underline{z}})\,.\end{aligned}$$

Likewise,

$$\begin{aligned}Imd\pi|_{[u,J_S,J,\underline{x}]} = \big\{&(\dot{J},\underline{\dot{x}})\in T_J\mathbb{R}\mathcal{J}_\omega\times T_{\underline{x}}\mathbb{R}_\tau X^{c_1(X)d-2}\;\big|\\ &\exists(v,\dot{J}_S)\in L^{k,p}(S,E^L_{u,-\underline{z}})\times T_{J_S}\mathbb{R}\mathcal{J}_S\,,\\ &\qquad Dv+J\circ du\circ\dot{J}_S=-\dot{J}\circ du\circ J_S\,,\; v(\underline{z})=\underline{\dot{x}}\big\}\,,\end{aligned}$$

so that

$$\operatorname{coker} d\pi|_{[u,J_S,J,\underline{x}]}\cong L^{k-1,p}(S,\Lambda^{0,1}S\otimes E^L_u)\times T_{\underline{x}}\mathbb{R}_\tau X^{c_1(X)d-2}/Im(\widehat{D}\times ev)\,,$$

where $\widehat{D}:(v,\dot{J}_S)\in L^{k,p}(S,E^L_u)\times T_{J_S}\mathbb{R}\mathcal{J}_S\mapsto Dv+J\circ du\circ\dot{J}_S\in L^{k-1,p}(S,\Lambda^{0,1}S\otimes E^L_u)$ and $ev:v\in L^{k,p}(S,E^L_u)\mapsto v(\underline{z})\in T_{\underline{x}}\mathbb{R}_\tau X^{c_1(X)d-2}$. In particular, $Imd\pi|_{[u,J_S,J,\underline{x}]}$ is closed and π is Fredholm. By definition, $\operatorname{coker} D=H^1_D(S,E^L_u)$. From the short exact sequence $0\to E^L_{u,-\underline{z}}\to E^L_u\stackrel{ev}{\to}T_{\underline{x}}\mathbb{R}_\tau X^{c_1(X)d-2}\to 0$, we deduce the long exact sequence $\to H^0_D(S,E^L_u)\to H^0(S,T_{\underline{x}}\mathbb{R}_\tau X^{c_1(X)d-2})\to H^1_D(S,E^L_{u,-\underline{z}})\to H^1_D(S,E^L_u)\to 0$. Hence, the cokernel of $D\times ev$ in $T_J\mathbb{R}\mathcal{J}_\omega\times T_{\underline{x}}\mathbb{R}_\tau X^{c_1(X)d-2}$ is isomorphic to $H^1_D(S,E^L_{u,-\underline{z}})$. From the long exact sequence given at the end of §2.2, we deduce that the cokernel of $\widehat{D}\times ev$ and hence the one of $d\pi|_{[u,J_S,J,\underline{x}]}$ is isomorphic to $H^1_D(S,\mathcal{N}^L_{u,-\underline{z}})$. □

Corollary 2.6. *The critical points $[u,J_S,J,\underline{x}]$ of π are those for which $u(S)$ has a point of contact of order greater than two with L at $u(\xi)$ or u has a cuspidal point outside ξ.* □

2.4 Generic critical points of π are non-degenerated.

Theorem 2.7. *Let $[u,J_S,J,\underline{x}]\in\mathbb{R}\mathcal{M}^d_L$ be such that $u(S)$ has a point of contact of order two with L at $u(\xi)$ and a unique real ordinary cuspidal point outside ξ. Then, $[u,J_S,J,\underline{x}]$ is a non-degenerated critical points of π. The same holds if $u(S)$ is immersed but has a point of contact of order three with L at $u(\xi)$.*

The critical points of π which appear in this Theorem 2.7 are said to be *generic.*

Proof. The proof of the first part of this theorem is the same as the one of Lemma 2.13 of [W2], it is not reproduced here. Let $[u, J_S, J, \underline{x}] \in \mathbb{R}\mathcal{M}_L^d$ be such that $u(S)$ is immersed but has a point of contact of order three with L at $u(\xi)$. We have to prove that the quadratic form $\nabla d\pi|_{[u,J_S,J,\underline{x}]} : \ker d\pi|_{[u,J_S,J,\underline{x}]} \times \ker d\pi|_{[u,J_S,J,\underline{x}]} \to \operatorname{coker}\ker d\pi|_{[u,J_S,J,\underline{x}]}$ is non-degenerated. We saw in the proof of Proposition 2.5 that the kernel and cokernel of the map $d\pi$ are the same as the ones of the morphism $-\widehat{D}_{\mathbb{R}} : (v, \dot{J}_S, \dot{J}, \dot{x}) \in T_{[u,J_S,J,\underline{x}]}\mathbb{R}\mathcal{M}_L^d \mapsto \dot{J} \circ du \circ J_S \in L^{k-1,p}(S, \Lambda^{0,1}S \otimes N_u^L)$. From the relation $Dv + J \circ du \circ \dot{J}_S + \dot{J} \circ du \circ J_S = 0$, we deduce that $\widehat{D}_{\mathbb{R}}(v, \dot{J}_S, \dot{J}, \dot{x}) = Dv + J \circ du \circ \dot{J}_S$. We then have to prove that $\nabla \widehat{D}_{\mathbb{R}}|_{[u,J_S,J,\underline{x}]} : H_D^0(S, \mathcal{N}_{u,-\underline{z}}^L)^2 \to H_D^1(S, \mathcal{N}_{u,-\underline{z}}^L)$ is non-degenerated. Let $(v, \dot{J}_S, 0, 0)$ be a generator of $H_D^0(S, \mathcal{N}_{u,-\underline{z}}^L)$. From Propositions 2.4 and 2.5, $v = du(w)$ for some real vector field w on S which does not vanish at ξ. We can assume that $\dot{J}_S$ vanishes in a neighbourhood of $z \cup \xi$. After differentiation of the relation $D \circ du = du \circ \overline{\partial}_S$, we deduce

$$\nabla_v D \circ du + D \circ (\nabla_v du) + \nabla_{\dot{J}_S} D \circ du = (\nabla_v du) \circ \overline{\partial}_S \mod (Im(du)) .$$

Moreover, $\nabla_{(v,\dot{J}_S,0,0)}\widehat{D} = \nabla_v D + (\nabla_v du) \circ J_S \circ \dot{J}_S + \nabla_{\dot{J}_S} D \mod (Im(du))$. Since the relation $Dv + J \circ du \circ \dot{J}_S = 0$ forces $\overline{\partial}_S(w) + J_S\dot{J}_S = 0$, we get (compare Lemma 2.13 of [W2] and Theorem 1.8 of [W3])

$$(\nabla_{(v,\dot{J}_S,0,0)}\widehat{D})(v) + D(\nabla_v du)(w) = 0 \mod (Im(du)) .$$

From Proposition 2.4, $\mathcal{N}_{u,-\underline{z}}^L \cong \mathcal{N}_{u,-\underline{z}-2\xi}$. From Riemann–Roch duality, $H_D^1(S, \mathcal{N}_{u,-\underline{z}-2\xi})_{+1}^* \cong H_D^0(S, K_S \otimes \mathcal{N}_{u,-\underline{z}-2\xi})_{-1}$, see [W2, Lem. 1.7]. Let ψ be a generator of $H_D^0(S, \mathcal{N}_{u,-\underline{z}-2\xi})_{-1}$ so that $D^*\psi$ is a linear combination of Dirac sections of N_u^* at $z \cup \xi$ as well as of the derivative δ'_ξ of the Dirac section at ξ. Note that since $H_D^0(S, K_S \otimes \mathcal{N}_{u,-\underline{z}-\xi}) = 0$, the coefficient a_ξ of δ'_ξ in $D^*\psi$ does not vanish. We have

$$\begin{aligned}\langle \psi, \nabla d\pi((v, \dot{J}_S), (v, \dot{J}_S)) \rangle &= -\langle \psi, \nabla \widehat{D}((v, \dot{J}_S), (v, \dot{J}_S)) \rangle \\ &= \langle \psi, D(\nabla_v du)(w) \rangle \\ &= \langle D^*\psi, (\nabla_v du)(w) \rangle .\end{aligned}$$

Choose a local chart at $u(\xi)$ such that L is conjugated to the first coordinate axis of $\mathbb{R}^2 \subset \mathbb{C}^2$. Without loss of generality, we can assume that the first coordinate axis is J-holomorphic and that the metric g is constant in this chart, so that $\nabla = d$. The map u writes then $z \mapsto ((z-\xi)+o(|z-\xi|),(z-\xi)^3+o(|z-\xi|^3))$ in a neighbourhood of ξ. Thus, $\nabla_v du(w) = \nabla_w v$, considered as a section of the normal bundle of u, has a simple zero at ξ. Since w vanishes at z, we deduce that $\langle D^*\psi, (\nabla_v du)(w)\rangle = a_\xi\langle \delta'_\xi, (\nabla_v du)(w)\rangle$. Now since the vanishing order of $\nabla_v du(w)$ at ξ is one, $\langle \delta'_\xi, (\nabla_v du)(w)\rangle \neq 0$, hence the result. □

2.5 Gromov compactification $\mathbb{R}\overline{\mathcal{M}}^d_L$ of $\mathbb{R}\mathcal{M}^d_L$. The projection $\pi : \mathbb{R}\mathcal{M}^d_L \to \mathbb{R}\mathcal{J}_\omega \times \mathbb{R}_\tau X^{c_1(X)d-2}$ is not proper in general. Its lack of properness is described by the following lemma which follows from Gromov's compactness theorem (see [MS, Thm. 5.5.5]).

LEMMA 2.8. *Let $[u^n, J^n_S, J^n, \underline{x}^n]$ be a sequence of elements of $\mathbb{R}\mathcal{M}^d_L$ such that $(J^n, \underline{x}^n)$ converges to $(J^\infty, \underline{x}^\infty)$. Then, after possibly extracting a subsequence, we have one of the following:*

1) *This sequence $[u^n, J^n_S, J^n, \underline{x}^n]$ converges in $\mathbb{R}\mathcal{M}^d_L$.*
2) *The sequence $u^n(S)$ converges to some irreducible curve, tangent to L, but the point of contact belongs to $\underline{x}^\infty$.*
3) *The sequence $u^n(S)$ converges to some reducible curve. Moreover, in this case, the reducible curve is either tangent to L, or has two of its irreducible components which intersect on L.* □

3 Proofs of Theorems 1.1 and 1.3

Let $(J^0, \underline{x}^0)$ and $(J^1, \underline{x}^1)$ be two generic elements of $\mathbb{R}\mathcal{J}_\omega \times \mathbb{R}_\tau X^{c_1(X)d-2}$ so that the integers $\Gamma^{d,B}_r(J^0, \underline{x}^0)$ and $\Gamma^{d,B}_r(J^1, \underline{x}^1)$ are well defined. We have to prove that they coincide.

3.1 Choice of a path γ. Remember that by definition, a *stratum of codimension* $k \geq 0$ of a separable Banach manifold M is the image of a separable Banach manifold L under a Fredholm map Φ of index $-k$ such that the limits of sequences $\Phi(x_n)$ where $(x_n)_{n\in\mathbb{N}}$ diverges in N belong to a countable union of strata of higher codimensions. In particular, Φ is not assumed to be proper.

PROPOSITION 3.1. *The subset of elements $[u, J_S, J, \underline{x}]$ of $\mathbb{R}\mathcal{M}^d_L$ for which $u(S)$ has only transversal double points as singularities, outside $\underline{x} \cup L$, and a unique point of contact of order two with L, is a dense open subset of $\mathbb{R}\mathcal{M}^d_L$. The four followings are substrata of codimension one of $\mathbb{R}\mathcal{M}^d_L$.*

1) *Curves having only transversal double points as singularities, outside $\underline{x} \cup L$, and a unique point of contact with L which is of order three.*

2) *Curves having a unique real ordinary cusp and transversal double points as singularities, outside* $\underline{x} \cup L$*, and a unique point of contact with* L *which is of order two.*
3) *Curves having a unique real ordinary cusp on* L *and transversal double points outside* $\underline{x} \cup L$ *as singularities. These curves are not tangent to* L *and the tangent line of the curve at the cusp is distinct from the one of* L*.*
4) *Curves having a real ordinary triple point or real ordinary tacnode or a transversal double point on* $\underline{x} \cup L$ *or two points of contact with* L*.*

The set of curves not listed above belongs to a countable union of strata of codimension greater than one of $\mathbb{R}\mathcal{M}_L^d$*.*

Proof. The proof is the same as the one of Proposition 2.7 of [W2]. It is left to the reader. □

Let $\gamma : t \in [0,1] \mapsto (J^t, \underline{x}^t) \in \mathbb{R}\mathcal{J}_\omega \times \mathbb{R}_\tau X^{c_1(X)d-2}$ be a generic path transversal to $\pi_\mathbb{R}$. Denote by $\mathbb{R}\mathcal{M}_\gamma = \mathbb{R}\mathcal{M}_L^d \times_\gamma [0,1]$, $\mathbb{R}\overline{\mathcal{M}}_\gamma$ its Gromov compactification and $\pi_\gamma : \mathbb{R}\overline{\mathcal{M}}_\gamma \to [0,1]$ the associated projection.

Proposition 3.2. *As soon as* γ *is generic enough, the elements of* $\mathbb{R}\overline{\mathcal{M}}_\gamma \setminus \mathbb{R}\mathcal{M}_\gamma$ *are either irreducible curves* $[u^t, J_S^t, J^t, \underline{x}^t]$ *such that* $\underline{x}^t \cap L$ *is non-empty, or reducible curves* C^t *having two irreducible components* C_1^t, C_2^t*, both real, and only transversal double points as singularities, outside* $\underline{x}$*. Moreover, we have the following alternative:*

1) *Either* C^t *has a unique point of contact with* L *which is of order two and outside its singular points.*
2) *Or* C^t *has a unique double point on* L *which is an intersection point of* $\mathbb{R}C_1^t$ *and* $\mathbb{R}C_2^t$*. In this case, it is not tangent to* L*.*

Finally, if we denote by $m_i = \#(\underline{x}^t \cap C_i^t)$ *and* $d_i = [C_i^t] \in H_2(X;\mathbb{Z})$*,* $i \in \{1,2\}$*, so that* $m_1 + m_2 = c_1(X)d - 2$*, then either* $m_1 = c_1(X)d_1 - 1$ *or* $m_1 = c_1(X)d_1 - 2$*.*

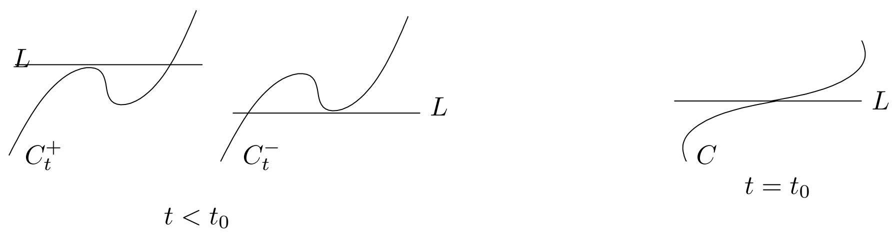
Proof. The proof is the same as the ones of Proposition 2.9, Corollary 2.10 and Proposition 2.11 of [W2], as well as Corollary 1.12 of [W3]. It is not reproduced here. □

Remark 3.3. Remember that to cover the case $r = (0, \dots, 0)$, one should take into account real reducible curves made of two complex conjugated components, see Remark 1.9 of [W3]. It would then be possible to extend Theorem 1.1 to this case provided an analog of Theorem 3.2 of [W3] is proved, see Remark 3.5 of [W3].

From now on, we fix a choice of γ generic enough so that $\mathbb{R}\overline{\mathcal{M}}_\gamma$ consists of curves listed in Propositions 3.1 and 3.2.

3.2 Neighbourhood of curves having an order three point of contact with *L*.

PROPOSITION 3.4. *Let $C = [u, J_S, J, \underline{x}] \in \mathbb{R}\mathcal{M}_\gamma$ be a curve having an order three point of contact with L and $t_0 = \pi_\gamma(C)$. Then, there exist $\eta > 0$ and a neighbourhood W of C in $\mathbb{R}\mathcal{M}_\gamma$ such that for every $t \in]t_0 - \eta, t_0[$, $\pi_\gamma^{-1}(t) \cap W$ consists of two curves C_t^+, C_t^- having same mass and for which $\langle C_t^+, B\rangle = -\langle C_t^-, B\rangle$ and for every $t \in]t_0, t_0 + \eta[$, $\pi_\gamma^{-1}(t) \cap W = \emptyset$, or vice versa.*



Proof. From Theorem 2.7, C is a non-degenerated critical point of π_γ. Since $\mathbb{R}\mathcal{M}_\gamma$ is of dimension one, this implies that there exist $\eta > 0$ and a neighbourhood W of C in $\mathbb{R}\mathcal{M}_\gamma$ such that for every $t \in]t_0 - \eta, t_0[$, $\pi_\gamma^{-1}(t) \cap W$ consists of two curves and for every $t \in]t_0, t_0 + \eta[$, $\pi_\gamma^{-1}(t) \cap W = \emptyset$, or vice versa. The only thing to prove is that in the first case, the two curves C_t^+, C_t^- have the same mass and satisfy $\langle C_t^+, B\rangle = -\langle C_t^-, B\rangle$. The former is obvious. Choose a parameterization $\lambda \in]-\sqrt{\eta}, \sqrt{\eta}[\mapsto C_\lambda = [u^\lambda, J_S^\lambda, J^\lambda, \underline{x}^\lambda] \in \mathbb{R}\mathcal{M}_\gamma$ such that $\pi_\gamma(C_\lambda) = t_0 - \lambda^2$. Fix a local chart $0 \in]-1,1[$ of $\xi \in \mathbb{R}S$ and $0 \in \mathbb{R}^2$ of $u^0(\xi) \in \mathbb{R}X$. We can assume that in this second chart, L is identified with the first coordinate axis and B with the upper half plane of $\mathbb{R}^2$. The one parameter family $(u^\lambda)_{\lambda \in]-\sqrt{\eta},\sqrt{\eta}[}$ reads as a map $f : (\lambda, z) \in]-\sqrt{\eta}, \sqrt{\eta}[\times]-1,1[\mapsto f(\lambda, z) \in \mathbb{R}^2$. Denote by $f_1(\lambda, z)$ and $f_2(\lambda, z)$ the two coordinates of $f(\lambda, z)$. These maps of class C^{l-k}, satisfy $f_1(0, z) = z + o(|z|)$, $f_2(0, z) = z^3 + o(|z|^3)$, $f_2(\lambda, 0) = 0$ and $\frac{\partial}{\partial z} f_2(\lambda, z)|_{z=0} = 0$. Moreover, $\frac{\partial}{\partial\lambda} C_\lambda|_{\lambda=0}$ generates the kernel of $d\pi_\gamma|_{C_\lambda}$. It thus follows from Proposition 2.5 that $\frac{\partial}{\partial\lambda} f(\lambda, z)|_{\lambda=0} = \frac{\partial}{\partial z} f(\lambda, z)|_{\lambda=0} = (1 + o(1), 3z^2 + o(|z|^2))$. We deduce that the order three jet of f_2 writes $f_2(\lambda, z) = z^2(z + a\lambda) + o(||(\lambda, z)||^3)$, for some $a \in \mathbb{R}^*$. Hence, when $\lambda > 0$ (resp. $\lambda < 0$), the sign of $f_2(\lambda, z)$ in a neighbourhood of $z = 0$ is the one of a (resp. its opposite). In particular, as soon as $\lambda \neq 0$, $\langle C_\lambda, B\rangle = -\langle C_{-\lambda}, B\rangle$. □

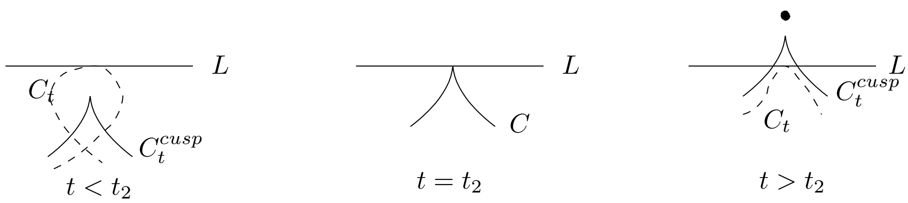
3.3 Neighbourhood of curves having a cuspidal point.

Proposition 3.5. *Let $C = [u, J_S, J, \underline{x}] \in \mathbb{R}\mathcal{M}_\gamma$ be a curve having a real ordinary cusp outside L and $t_1 = \pi_\gamma(C)$. Then, there exist $\eta > 0$ and a neighbourhood W of C in $\mathbb{R}\mathcal{M}_\gamma$ such that for every $t \in]t_1 - \eta, t_1[$, $\pi_\gamma^{-1}(t) \cap W$ consists of two curves C_t^+, C_t^- such that $m(C_t^+) = m(C_t^-) + 1$ and $\langle C_t^+, B\rangle = \langle C_t^-, B\rangle$ and for every $t \in]t_1, t_1 + \eta[$, $\pi_\gamma^{-1}(t) \cap W = \emptyset$, or vice versa.*

Proof. From Theorem 2.7, C is a non-degenerated critical point of π_γ. Since $\mathbb{R}\mathcal{M}_\gamma$ is of dimension one, this implies that there exist $\eta > 0$ and a neighbourhood W of C in $\mathbb{R}\mathcal{M}_\gamma$ such that for every $t \in]t_1 - \eta, t_1[$, $\pi_\gamma^{-1}(t) \cap W$ consists of two curves and for every $t \in]t_1, t_1 + \eta[$, $\pi_\gamma^{-1}(t) \cap W = \emptyset$, or vice versa. The only thing to prove is that $m(C_t^+) = m(C_t^-) + 1$. The proof of this is readily the same as the one of Proposition 2.16 of [W2]. It is not reproduced here. □

Proposition 3.6. *Let $C = [u, J_S, J, \underline{x}] \in \mathbb{R}\mathcal{M}_\gamma$ be a curve having a real ordinary cusp on L and $t_2 = \pi_\gamma(C)$. Then, there exist $\eta > 0$ and a neighbourhood W of C in $\mathbb{R}\mathcal{M}_\gamma$ such that for every $t \in]t_2 - \eta, t_2 + \eta[\setminus \{t_2\}$, $\pi_\gamma^{-1}(t) \cap W$ is reduced to one element $\{C_t\}$. Moreover, $\langle C_t, B\rangle$ does not depend on $t \in]t_2 - \eta, t_2 + \eta[\setminus \{t_2\}$. Likewise, C extends to a one parameter family C_t^{cusp} of cuspidal real rational J^t-holomorphic curves which pass through $\underline{x}^t$ and realize d. Assume that for $t \in]t_2 - \eta, t_2[$ (resp. $t \in]t_2, t_2 + \eta[$), $\mathbb{R}C_t^{cusp}$ does not intersect locally L (resp. intersects L locally in two points) near the cusp of C. Then for $t \in]t_2 - \eta, t_2[$, $m(C_t) = m(C)$ and for $t \in]t_2, t_2 + \eta[$, $m(C_t) = m(C) + 1$.*

Note that after changing the parameterization $t \mapsto 2t_2 - t$ if necessary, we can always assume that for $t \in]t_2 - \eta, t_2[$ (resp. $t \in]t_2, t_2 + \eta[$), $\mathbb{R}C_t^{cusp}$ does not intersect locally L (resp. intersects locally L in two points) near the cusp of C.

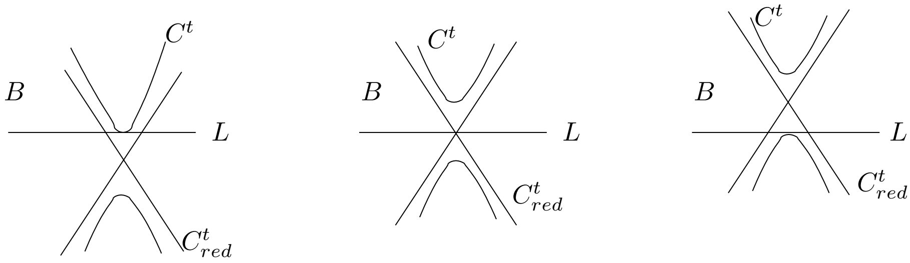


Proof. Remember that the choice of γ implies that the tangent line of C at the cusp is distinct from the one of L. Without loss of generality, we can assume that J, $\underline{x}$ are constant and that L (and the metric g) moves along a one parameter family L_t which crosses the cuspidal point of C. This indeed can be realized equivalently by fixing L and having J, $\underline{x}$ moving along one parameter families $\phi_t^* J$, $\phi_t(\underline{x})$ where ϕ_t is some $\mathbb{Z}/2\mathbb{Z}$-equivariant hamiltonian flow of X. The family of curves C_t^{cusp} is then nothing but the constant family C. Moreover, from Proposition 2.16 of [W2], the curve C extends to a one parameter family C^λ, $\lambda \in]-\epsilon, \epsilon[$, of real rational J-holomorphic curves which pass through $\underline{x}$ and realize d. These curves C^λ have an isolated real double point near the cusp of C when $\lambda < 0$ and a non-isolated one when $\lambda > 0$. Moreover, the latter form a one parameter family of loops which fill some disk of $\mathbb{R}X$ centered at the cusp of C (compare [W3, Lem. 3.3]). This follows from the fact that the intersection points between two curves in this family are located near their double points and at $\underline{x}$. Since for $t \in]t_2 - \eta, t_2[$, L_t is locally disjoint from C, there does exist some curve C^λ, $\lambda > 0$, in this family which is tangent to L_t, as soon as η is small enough. It has the same mass as C. From Corollary 2.6, C is a regular point of π_γ. The first part of the proposition is thus proved. Now for each $\lambda < 0$ close enough to 0, there should exist some $t \in]t_2 - \eta, t_2 + \eta[$ such that L_t is tangent to C^λ. From what precedes, t has to be greater than t_2 and the proposition is proved, since $m(C^\lambda) = m(C) + 1$ when $\lambda < 0$. □

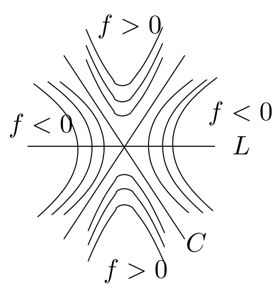
3.4 Neighbourhood of reducible curves. Let $C \in \mathbb{R}\overline{\mathcal{M}}_\gamma$ be a reducible curve and C_1, C_2 be its irreducible components. For $i \in \{1, 2\}$, denote by $d_i = [C_i] \in H_2(X; \mathbb{Z})$, $\underline{x}_i = \underline{x} \cap C_i$ and $m_i = \#\underline{x}_i$. From Proposition 3.2, $m_1 \in \{c_1(X)d_1 - 2, c_1(X)d_1 - 1\}$. Denote by $t_3 = \pi_\gamma(C)$ and assume that $\mathbb{R}C_1 \cap \mathbb{R}C_2 \cap L = \{y\}$ and that $m_1 = c_1(X)d_1 - 1$. Then, there exists $\eta > 0$ such that the curves C deforms to a one parameter family of real reducible J^t-holomorphic curves C^t_{red}, $t \in]t_3 - \eta, t_3 + \eta[$, which pass through $\underline{x}^t$, where $(J^t, \underline{x}^t) = \gamma(t)$. The nodal point y then deforms to a one parameter family of real non-isolated double point y^t of $\mathbb{R}C^t_{red}$. Without loss of generality, we can assume that $y^t \notin B$ if $t \in]t_3 - \eta, t_3[$ and $y^t \in B$ if $t \in]t_3, t_3 + \eta[$.

PROPOSITION 3.7. *Let $C^{t_3} = C_1^{t_3} \cup C_2^{t_3} \in \mathbb{R}\overline{\mathcal{M}}_\gamma$ be a real reducible curve and $t_3 = \pi_\gamma(C^{t_3})$. Assume that $\mathbb{R}C_1^{t_3} \cap \mathbb{R}C_2^{t_3} \cap L = \{y^{t_3}\}$ and that $m_1 = c_1(X)d_1 - 1$ with the above notation. Denote by C^t_{red} (resp. y^t), $t \in]t_3 - \eta, t_3 + \eta[$, the associated one parameter family of real reducible J^t-holomorphic curves (resp. of real double point of C^t_{red}). Assume that $y^t \notin B$ if $t \in]t_3 - \eta, t_3[$ and $y^t \in B$ if $t \in]t_3, t_3 + \eta[$. Then, as soon as η is small enough, there exists a neighbourhood W of C in $\mathbb{R}\overline{\mathcal{M}}_\gamma$ such that* *for every* $t \in]t_3 - \eta, t_3[$ *(resp.* $t \in]t_3, t_3 + \eta[$*),* $\sum_{C \in (\pi_\gamma^{-1}(t) \cap W)} \langle C, B \rangle = -1$ *(resp.* $\sum_{C \in (\pi_\gamma^{-1}(t) \cap W)} \langle C, B \rangle = +1$*).*

Note that all the curves C^t close to C^{t_3} are obtained topologically by smoothing the non-isolated real double point y^{t_3} of C^{t_3}. Thus, they have the same mass as C^{t_3}.



Proof. Without loss of generality, we can assume that $J^t, \underline{x}^t$ are constant and that L (and the metric g) moves along a one parameter family L_t which crosses the double point y^{t_3} of C^{t_3}. This indeed can be realized equivalently by fixing L and having J, $\underline{x}$ moving along one parameter families $\phi_t^* J$, $\phi_t(\underline{x})$ where ϕ_t is some $\mathbb{Z}/2\mathbb{Z}$-equivariant hamiltonian flow of X. The family of curves C^t_{red} is then nothing but the constant family C. Moreover, from Proposition 2.14 of [W2], the curve C^{t_3} extends to a one parameter family $C^{t_3}_\lambda$, $\lambda \in]-\epsilon, \epsilon[$, of real rational J-holomorphic curves which pass through $\underline{x}$ and realize d. These curves are obtained topologically by smoothing the real double point y^{t_3} of C^{t_3}. The intersection points between two different curves in this family $(C^{t_3}_\lambda)_{\lambda \in]-\epsilon,\epsilon[}$ are located near the double points of C^{t_3} and at $\underline{x}$. Thus, a neighbourhood U of y^{t_3} in $\mathbb{R}X$ is foliated by curves $C^{t_3}_\lambda \cap U$ and this foliation looks like the level sets of an index one critical point of some Morse function $f : U \to \mathbb{R}$.

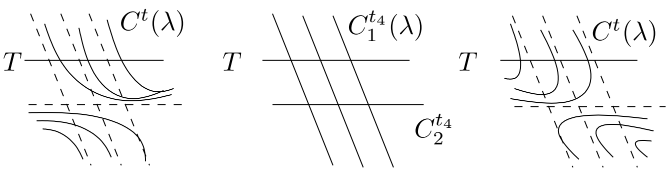


We can assume that $L \cap U$ belongs to the domain $f \leq 0$. Let $(t_-, t_+) \in]t_3 - \eta, t_3[\times]t_3, t_3 + \eta[$, restricting U and ϵ if necessary, we can assume that L_{t_-} and L_{t_+} are transversal to all the level sets $f \leq 0$. The number of maxima minus the number of minima of f restricted to $L_{t_\pm}$ is then equal to one, provided the latter have been chosen generic. Now each maximum (resp. minimum) of f restricted to L_{t_-} corresponds to a curve C^{t_-} having contact index $\langle C^{t_-}, B\rangle = +1$ (resp. $\langle C^{t_-}, B\rangle = -1$). Likewise, each maximum (resp. minimum) of f restricted to L_{t_+} corresponds to a curve C^{t_+} having contact index $\langle C^{t_+}, B\rangle = -1$ (resp. $\langle C^{t_+}, B\rangle = +1$), hence the result. □

PROPOSITION 3.8. *Let $C^{t_4} = C_1^{t_4} \cup C_2^{t_4} \in \mathbb{R}\overline{\mathcal{M}}_\gamma$ be a reducible curve and $t_4 = \pi_\gamma(C^{t_4})$. Assume that $\mathbb{R}C_1^{t_4} \cap \mathbb{R}C_2^{t_4} \cap L = \{y^{t_4}\}$ and that $m_1 = c_1(X)d_1 - 2$ with the notation of Proposition 3.7. Then, there exist $\eta > 0$ and a neighbourhood W of C^{t_4} in $\mathbb{R}\overline{\mathcal{M}}_\gamma$ such that for every $t \in]t_4 - \eta, t_4 + \eta[\setminus \{t_4\}$, $\sum_{C\in(\pi_\gamma^{-1}(t)\cap W)} \langle C, B\rangle = 0$.*

Note that once more, all the curves in W have the same mass. Note also that $C_1^{t_4}$ belongs to a one parameter family $C_1^{t_4}(\lambda)$ of J^{t_4}-holomorphic curves which pass through $\underline{x}_1^{t_4} = \underline{x}^{t_4} \cap C_1^{t_4}$ and realize d_1, whereas $C_2^{t_4}$ does not deform to any J^t-holomorphic curve for $t \neq t_4$.

Proof. Without loss of generality, we can assume that $\underline{x}^t$ is constant. Let U be a small neighbourhood of y^{t_4}, it is foliated by the curves $C_1^{t_4}(\lambda) \cap U$. Choose a transversal T to this foliation which is disjoint from $C_2^{t_4} \cap U$. From Proposition 2.14 of [W2], as soon as η is small enough, there is one and only one J^t-holomorphic real rational curve which pass through $\underline{x}^t$ and realize d through every point of T. This produces a one parameter family of disjoint J^t-holomorphic real rational curves $\mathbb{R}C^t(\lambda) \cap U$, $\lambda \in T$.



Each of these curves $\mathbb{R}C^t(\lambda) \cap U$ has two connected components, which produce two functions partially defined on L to T. To get the result, it is enough to observe that the number of maxima minus the number of minima of these functions are either $+1$ and -1, or 0 and 0. □

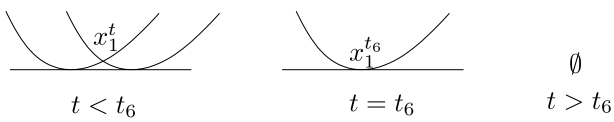
PROPOSITION 3.9. *Let $C^{t_5} = C_1^{t_5} \cup C_2^{t_5} \in \mathbb{R}\overline{\mathcal{M}}_\gamma$ be a real reducible curve tangent to L and $t_5 = \pi_\gamma(C^{t_5})$. Let R be the number of real intersection points between $\mathbb{R}C_1^{t_5}$ and $\mathbb{R}C_2^{t_5}$. Then, there exist $\eta > 0$ and a neighbourhood W of C^{t_5} in $\mathbb{R}\overline{\mathcal{M}}_\gamma$ such that for every $t \in]t_5 - \eta, t_5 + \eta[\setminus \{t_5\}$,*

$\pi_\gamma^{-1}(t)\cap W$ consists of exactly R curves each of them obtained by smoothing a different real intersection point between $\mathbb{R}C_1^{t_5}$ and $\mathbb{R}C_2^{t_5}$.

Proof. The proof is the same as the one of Proposition 2.14 of [W2], it is not reproduced here. The only argument which slightly differs from the one in [W2] is to show that for every real intersection point between $\mathbb{R}C_1^{t_5}$ and $\mathbb{R}C_2^{t_5}$, there is at most one J^t-holomorphic curve in $\pi_\gamma^{-1}(t)\cap W$ obtained by smoothing this point. Actually, if there were two of them, they would intersect at $\underline{x}^t$, at two points near each double point of C^{t_5} but the one smoothed and near the tangency point with L. This would produce more than d^2 intersection points, which is impossible. □

3.5 Neighbourhood of the case $\underline{x}\cap L\neq\emptyset$.

Proposition 3.10. *Let $C^{t_6}\in\mathbb{R}\overline{\mathcal{M}}_\gamma$ be an irreducible curve tangent to L and $t_6=\pi_\gamma(C^{t_6})$. Assume that $\underline{x}^{t_6}\cap L=\{x_1^{t_6}\}$. Assume that J^t is constant and that only the point x_1^t actually depends on t. Then, there exist $\eta>0$ and a neighbourhood W of C^{t_6} in $\mathbb{R}\overline{\mathcal{M}}_\gamma$ such that for every $t\in]t_6-\eta,t_6+\eta[\setminus\{t_6\}$, $\pi_\gamma^{-1}(t)\cap W$ consists of two curves having same mass and same contact index with L if x_1^t is locally on the same side of L as C^{t_6}, and $\pi_\gamma^{-1}(t)\cap W$ is empty otherwise.*



Proof. The moduli space of real rational J^{t_6}-holomorphic curves which pass through $\underline{x}^{t_6}\setminus\{x_1^{t_6}\}$ and realize d is one dimensional, and C^{t_6} is a regular point in this space. Thus, all the elements in this moduli space close to C^{t_6} are located on the same side of L as C^{t_6} itself. If x_1^t is not on this side, we deduce that $\pi_\gamma^{-1}(t)\cap W=\emptyset$ as soon as W is small enough. Denote by $C^{t_6}(\lambda)$ the curves in this moduli space and let U be a small neighbourhood of $x_1^{t_6}$ in $\mathbb{R}X$. Then, the curves $(\mathbb{R}C^{t_6}(\lambda)\cap U)\setminus L$ have two connected components, which produce two different foliations of one side of L in $U\setminus L$ if U is small enough. Thus, if x_1^t is on this side, then $\#(\pi_\gamma^{-1}(t)\cap W)=2$. In this case, the two curves in $\pi_\gamma^{-1}(t)\cap W$ have obviously same mass and same contact index with L. □

3.6 Proofs of Theorems 1.1 and 1.3.

3.6.1 Proof of Theorem 1.1. Let $(J^0,\underline{x}^0)$ and $(J^1,\underline{x}^1)$ be two generic elements of $\mathbb{R}\mathcal{J}_\omega\times\mathbb{R}_\tau X^{c_1(X)d-2}$ so that the integers $\Gamma_r^{d,B}(J^0,\underline{x}^0)$ and

$\Gamma_r^{d,B}(J^1,\underline{x}^1)$ are well defined. Let $\gamma : t\in[0,1]\mapsto(J^t,\underline{x}^t)\in\mathbb{R}\mathcal{J}_\omega\times\mathbb{R}_\tau X^{c_1(X)d-2}$ be a generic path chosen in §3.1 joining $(J^0,\underline{x}^0)$ to $(J^1,\underline{x}^1)$. Then, from genericity arguments of §3.1, we know that the integer $\Gamma_r^{d,B}(J^t,\underline{x}^t)$ is well defined for every $t\in[0,1]$ but a finite number of parameters $0<t_0<t_1<\cdots<t_k<1$ corresponding to the following phenomena.

Concerning the first term in the definition of $\Gamma_r^{d,B}(J^t,\underline{x}^t)$:

1) appearance of a unique real ordinary triple point or a unique real ordinary tacnode on an irreducible curve tangent to L;
2) appearance of a transversal double point of an irreducible curve tangent to L on $\underline{x}^t\cup L$;
3) appearance of a real ordinary cusp of an irreducible curve on L;
4) appearance of a an irreducible curve tangent to L which is a critical point of π_γ given by Theorem 2.7;
5) a sequence of curves of $\mathbb{R}\mathcal{M}_\gamma$ degenerates on a reducible curve given by Propositions 3.7, 3.8 or 3.9;
6) One has $\underline{x}^t\cap L\neq\emptyset$.

Concerning the last three terms in the definition of $\Gamma_r^{d,B}(J^t,\underline{x}^t)$:

a) One of those considered in [W3].
b) A cuspidal curve has its cusp on L but with a tangent line distinct from the one of L.
c) A reducible curve has one of the intersection points between its irreducible components on L but is not tangent to L.
d) One has $\underline{x}^t\cap L\neq\emptyset$.

We have to prove that the integer $\Gamma_r^{d,B}(J^t,\underline{x}^t)$ does not change while crossing one of these parameters $0<t_0<t_1<\cdots<t_k<1$. In the cases 1, 2, a, this is proven in the same way as in [W2,3]. In the cases 3, b, it follows from Proposition 3.6. Note that here the first term in the definition of $\Gamma_r^{d,B}(J^t,\underline{x}^t)$ is not invariant. The term on cuspidal curves allows us to compensate for this lack of invariance. In the case 4, it follows from Propositions 3.4, 3.5. In the cases 5, c, it follows from Propositions 3.7, 3.8 and 3.9. Note that here once more, in the case c, the first term in the definition of $\Gamma_r^{d,B}(J^t,\underline{x}^t)$ is not invariant. The term on reducible curves allows to compensate for this lack of invariance. In the cases 6, d, it follows from Proposition 3.10. Here the first term in the definition of $\Gamma_r^{d,B}(J^t,\underline{x}^t)$ is not invariant, this lack of invariance is compensated thanks to the term on $\mathcal{T}an^d(J,\underline{x})$. Indeed, using the notation of Proposition 3.10, we can assume that for $t=t_6$, the tangent line T_1 and $T_{x_1^{t_6}}L$ coincide. There is then a one to one correspondence between the curves tangent to L at $x_1^{t_6}$ and the elements of $\mathcal{T}an^d(J, \underline{x}^{t_6})$ having T_1 as a tangent line. During the deformation $t \in]t_6 - \eta, t_6 + \eta[$, the latter deform continuously. Now if x_1^t is locally on the same side of L as C^{t_6}, the two curves given by Proposition 3.10 are counted with respect to the sign $-(-1)^{m(C)}\langle x_1^t, B\rangle$ while the corresponding curve in $\mathcal{T}an^d(J, \underline{x}^t)$ is counted with respect to the sign $(-1)^{m(C)}\langle x_1^t, B\rangle$. The total contribution of these curves to $\Gamma_r^{d,B}(J^t, \underline{x}^t)$ is thus $-(-1)^{m(C)}\langle x_1^t, B\rangle$ in this case while it is just $(-1)^{m(C)}\langle x_1^t, B\rangle$ when x_1^t is locally on the opposite side of L as C^{t_6}. Since the sign $\langle x_1^t, B\rangle$ changes as x_1^t crosses L, these contributions are the same and $\Gamma_r^{d,B}(J^t, \underline{x}^t)$ is invariant. □

3.6.2 Proof of Theorem 1.3. Denote by $B(y, \epsilon)$ a disk of $\mathbb{R}X$ centered at $y \in X$ and having radius $\epsilon > 0$. Fix a generic $(J, \underline{x}) \in \mathbb{R}\mathcal{J}_\omega \times \mathbb{R}_\tau X^{c_1(X)d-2}$. When ϵ converges to zero, $B(y, \epsilon) \to y$ and the three last terms in the definition of $\Gamma_r^{d,B}(J, \underline{x})$ converge to $-\Gamma_r^d(J, \underline{x})$ since all the curves do not move and all the special points are outside $B(y, \epsilon)$. At the same time, the first term converges to a sum over real rational J-holomorphic curves which pass through $\underline{x} \cup \{y\}$ and realize d. Each of these curves are irreducible and immersed and deforms in exactly two curves tangent to $\partial B(y, \epsilon)$ for $\epsilon \ll 1$. Moreover, the latter are tangent from the outside of $B(y, \epsilon)$ and we deduce the relation $\Gamma_r^{d,B} = 2\chi_r^d - \Gamma_r^d$.

Likewise, when $X = \mathbb{C}P^2$ and ϵ converges to $+\infty$, the three last terms in the definition of $\Gamma_r^{d,B}(J, \underline{x})$ converge to $\Gamma_r^d(J, \underline{x})$ since all the curves do not move, and this time all the special points are inside $B(y, \epsilon)$. At the same time, the first term converges to a sum over real rational J-holomorphic curves which pass through $\underline{x}$, realize d and are tangent to the line at infinity. Each of these curves is irreducible and immersed and deforms in exactly two curves tangent to $\partial B(y, \epsilon)$ for $\epsilon \gg 1$. Moreover, one of these two curves is tangent from the outside of $B(y, \epsilon)$ and one from the inside, so that we get the relation $\Gamma_r^{d,B} = \Gamma_r^d$.

Finally, when $X = \mathbb{C}P^1 \times \mathbb{C}P^1$ and ϵ converges to $+\infty$, the boundary of $B(y, \epsilon)$ accumulates on the union of a section B_∞ and a fibre F_∞ of $\mathbb{R}P^1 \times \mathbb{R}P^1$. Then, the three last terms in the definition of $\Gamma_r^{d,B}(J, \underline{x})$ converge to $\Gamma_r^d(J, \underline{x})$ as before. At the same time, the first term converges to a sum over real rational J-holomorphic curves which pass through $\underline{x}$, realize d and are either tangent to $B_\infty \cup F_\infty$, or pass through $B_\infty \cap F_\infty$. Each of the curves tangent to $B_\infty \cup F_\infty$ are irreducible and immersed and deforms in exactly two curves tangent to $\partial B(y, \epsilon)$ for $\epsilon \gg 1$, one from the outside, the other one from the inside. Likewise, each of the curves passing through $B_\infty \cap F_\infty$ is irreducible and immersed and deforms in exactly two curves tangent to $\partial B(y,\epsilon)$ for $\epsilon \gg 1$, both from the outside. We hence get the relation $\Gamma_r^{d,B} = 2\chi_r^d + \Gamma_r^d$. □

4 On Real Conics Tangent to Five Generic Real Plane Conics

4.1 Proofs of Theorem 1.5 and Proposition 1.6.

Proof of Theorem 1.5. The proof is similar to the one of Theorem 1.1. We construct the universal moduli space $\mathbb{R}\mathcal{C}_L$ of real pseudo-holomorphic conics tangent to $L_1,\dots,L_5$. It is a separable Banach manifold of class C^{l-k} equipped with a Fredholm projection $\pi_\mathbb{R} : \mathbb{R}\mathcal{C}_L \to \mathbb{R}\mathcal{J}_\omega$ having vanishing index. Let J_0 and J_1 be two generic elements of $\mathbb{R}\mathcal{J}_\omega$ so that $\Gamma^B(J_0)$ and $\Gamma^B(J_1)$ are well defined, and $\gamma : t \in [0,1] \mapsto J^t \in \mathbb{R}\mathcal{J}_\omega$ be a generic path joining J_0 to J_1. Denote by $\mathbb{R}\mathcal{C}_\gamma = \mathbb{R}\mathcal{C}_L \times_\gamma [0,1]$, $\mathbb{R}\overline{\mathcal{C}}_\gamma$ its Gromov compactification and $\pi_\gamma : \mathbb{R}\overline{\mathcal{C}}_\gamma \to [0,1]$ the associated projection. Genericity arguments similar to the ones of §3.1 show that the elements of $\mathbb{R}\overline{\mathcal{C}}_\gamma$ are smooth real conics having a unique point of contact of order two with each L_i, $i \in \{1,\dots,5\}$, but a finite number of them which may be

1) smooth real conics which are bitangent to $L_1 \cup \dots \cup L_5$, every point of contact being of order at most two;
2) smooth real conics which have a point of contact of order three with one curve L_i, $i \in \{1,\dots,5\}$, the other ones being non-degenerated;
3) reducible conics made of two real lines, one of them being tangent to three curves L_i, $i \in \{1,\dots,5\}$, and the other one to the two remaining ones. These points of contact are non-degenerated and outside the singular point of the conic.
4) reducible conics of $\mathcal{C}on_{red}$ tangent to four curves $L_{i_1},\dots,L_{i_4}$ and whose singularity lie on the fifth curve L_{i_5}.

Likewise, the universal moduli space $\mathbb{R}\mathcal{C}_L^{red}$ of real reducible pseudo-holomorphic conics tangent to four curves $L_{i_1},\dots,L_{i_4}$ out of the five $L_1,\dots,L_5$ is a separable Banach manifold of class C^{l-k}. Denote by $\mathbb{R}\mathcal{C}_\gamma^{red} = \mathbb{R}\mathcal{C}_L^{red} \times_\gamma [0,1]$. It is a one-dimensional compact manifold whose elements are couples of real lines having four points of contacts with $L_{i_1},\dots,L_{i_4}$ which are of order two, but a finite number of them which may be

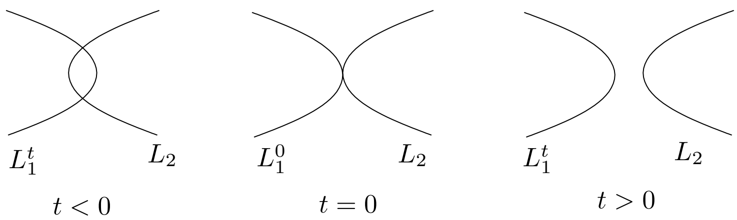
a) tangent to the five curves $L_1,\dots,L_5$, with non-degenerated points of contacts;

b) tangent to $L_{i_1}, \dots, L_{i_4}$ but with one point of contact of order three;
c) tangent to $L_{i_1}, \dots, L_{i_4}$ with their singular point on L_{i_5};
d) tangent to $L_{i_1}, \dots, L_{i_4}$ with their singular point on $L_{i_1} \cup \dots \cup L_{i_4}$.

The only thing to check is that the value of $\Gamma^B(J_t)$ does not change while t crosses one of the special values listed in 1-4 and a-d. In the cases 1, a and d, it is easy to check. In the cases 2, *b*, it follows from Proposition 3.4. In the case 3, the proof is the same as the one of Proposition 3.9. Finally, in the cases 4, c, the proof is the same as the one of Proposition 3.7. □

Proof of Proposition 1.6. From Theorem 1.5, we can assume that the five disjoint disks are of radius ϵ small, and have ϵ converging to zero so that they contracts onto five distinct points $y_1, \dots, y_5$. The conics tangent to $L_1, \dots, L_5$ degenerate onto conics passing through $y_1, \dots, y_5$. From [G], there is only one such J-holomorphic conic. Reversing this process as in the proof of Theorem 1.3, each conic passing through y_i deforms into two conics which are tangent to B_i from the outside, for ϵ small enough. As soon as ϵ is small enough, the first term in the definition of Γ^B then equals $2^5 = 32$. Likewise, elements of $\mathcal{C}on_{red}$ degenerate onto reducible conics passing through four out of the five points $y_1, \dots, y_5$. There are five ways to choose these four points, three couples of lines passing through these four points and each of these couples deforms into $2^4 = 16$ reducible conics tangent to the four associated disks B_i from the outside, as soon as ϵ is small enough. Since the singular point of these conics is outside $B = B_1 \cup \dots \cup B_5$, the second term in the definition of Γ^B equals $-5*3*16 = -240$. We deduce that $\Gamma^B = 32 + 240 = 272$. Likewise, if $B_1, \dots, B_5$ are close to five generic double lines of the plane we can have the curves $L_1, \dots, L_5$ degenerate onto five couples of real lines $L_i^1 \cup L_i^2$ close to the double lines and intersecting each other at $x_1, \dots, x_5$. Every conic tangent to L_i degenerates onto a conic tangent to $L_i^1 \cup L_i^2$ or a conic which passes through x_i. Now each conic tangent to L_i^1 deforms to a conic tangent to L_i^2 since L_i^1 and L_i^2 are as close to each other as we wish. Hence these conics come by pairs, one deforming to a conic tangent from the inside of B_i and the other one from the outside. Hence, the only conics which contributes to the first term of Γ^B correspond to the ones passing through $x_1, \dots, x_5$. Their contribution is 32 as before. In the same way, the second term of Γ^B equals -240 as before, as soon as $B_1, \dots, B_5$ are close enough to $(L_1^1 \cup L_1^2), \dots, (L_5^1 \cup L_5^2)$. Hence the result. □

4.2 How does Γ^B depend on the isotopy class of B? Let $B_2, \dots, B_5$ be four disks of $\mathbb{R}P^2$ transversal to each other and $(B_1^t)_{t \in]-\epsilon, \epsilon[}$ be a smooth one-parameter family of disks which are transversal to $B_2, \dots, B_5$ for $t \in]-\epsilon, \epsilon[\setminus\{0\}$ and which have an order two point of contact x with B_2 for $t = 0$. We can assume that for $t \in]-\epsilon, 0[$ (resp. for $t \in]0, \epsilon[$), the curves $L_1^t = \partial B_1^t$ and $L_2 = \partial B_2$ have two intersection points (resp. do not intersect) in a neighbourhood of x.



Denote by $B^t = B_1^t \cup B_2 \cup \dots \cup B_5$, the integer Γ^{B^t} is well defined for $t \in]-\epsilon, \epsilon[\setminus\{0\}$. We have to compare the values of Γ^{B^t} for $t < 0$ and $t > 0$. Denote by $\mathcal{C}on(J,x)$ the finite set of real conics which are tangent to L_3, L_4, L_5, pass through x and are tangent at x to L_1^0 and L_2. Likewise, denote by $\mathcal{C}on_{red}(J,x)$ the finite set of real reducible conics made of the J-holomorphic line T_x which is tangent to L_1^0 and L_2 at x and of a real J-holomorphic line tangent to two curves out of the three curves L_3, L_4, L_5.

PROPOSITION 4.1. *Let $B^t = B_1^t \cup B_2 \cup \dots \cup B_5$ be a one parameter family of five disks in $\mathbb{R}P^2$ as above and $(t_-, t_+) \in]-\epsilon, 0[\times]0, \epsilon[$. Then,*

$$\Gamma^{B^{t_+}} = \Gamma^{B^{t_-}} + 2\langle L_2, B_1^0\rangle \Big(\sum_{C \in \mathcal{C}on(J,x)} \prod_{j=2}^{5} \langle C, B_j\rangle \Big) - 2\langle L_2, B_1^0\rangle \Big(\sum_{C \in \mathcal{C}on_{red}(J,x)} \prod_{j=2}^{5} \langle C, B_j\rangle \Big).$$

Proof. We can have locally L_2 degenerate on a half line. The conics tangent to L_2 degenerate then on conics tangent to the half line and conics passing through the vertex s of this half line. As in the proof of Theorem 1.3, the contribution to Γ^{B^t} of conics tangent to the half line vanishes. As t goes to zero, s converges to x and the conics passing through s and tangent to B_1^t, B_3, B_4, B_5 converge to conics passing through the vertex s and tangent to B_1^0, B_3, B_4, B_5. If the order two point of contact of the latter with B_1^0 is outside x, they can be deformed for $t \in]-\epsilon, \epsilon[$. If on the contrary such a conic C belongs to $\mathcal{C}on(J,x) \cup \mathcal{C}on_{red}(J,x)$, it follows from Proposition 3.10 that it deforms for $t \in]-\epsilon, 0]$ (resp. $t \in [0, \epsilon[$) if and only if $\langle C, B_1^0\rangle = -\langle L_2, B_1^0\rangle$ (resp. $\langle C, B_1^0\rangle = \langle L_2, B_1^0\rangle$), that is, if and only if C and L_2 are locally on opposite sides of B_1^0 (resp. on the same side of B_1^0). Hence the result. □

4.3 Final remarks. 1) The results of §4 take advantage of the fact that a pseudo-holomorphic conic cannot be cuspidal and may have two irreducible components at most. To extend the results of §1.2 to pseudo-holomorphic curves having $s > 1$ tangency conditions with L would seem to require the introduction of 4^s terms in the definition of $\Gamma^{d,B}$. These terms consist of curves having s_1 tangency conditions with L, s_2 cusps, $s_3 + 1$ irreducible components and s_4 tangency conditions with the lines T_i, $i \in I$, where $s_1 + \cdots + s_4 = s$. One should then study the collisions between these tangency conditions, cusps, etc., which has not been done here.

2) In contrast with the works [W2,3], the moduli space $\mathbb{R}\mathcal{M}_L^d$ does not appear here as the fixed point set of some $\mathbb{Z}/2\mathbb{Z}$-action on some complexified moduli space $\mathcal{M}_L^d$. For such a purpose, we should have complexified L to some surface $L_{\mathbb{C}}$ in X and restricted ourselves to almost complex structures J for which $L_{\mathbb{C}}$ is J-antiholomorphic, as in [LR] and [IP]. The advantage not to do so here was to get immediately some invariant for any $J \in \mathbb{R}\mathcal{J}_\omega$ without any restrictions.

3) The condition that L is smooth and bounded by a smooth surface B is of course too restrictive. We reduced our study to this case for convenience. For example, one could replace the embedding $B \to \mathbb{R}X$ with some smooth map with finitely many ramification points and which maps the boundary L of B to some immersed curve with transversal double points as singularities. The index $\langle x, B\rangle$ for $x \in \mathbb{R}X$ should then be defined as twice the number of preimages of x in B less one. Since every step of the proof of Theorem 1.1 is local, it readily extends to this case. However, when L is any immersed curve, say for example the figure-eight curve in the projective plane, it is not clear to me how to extend the results presented here.

4) Likewise, how does $\Gamma_r^{d,B}$ depend on r can be understood exactly in the same way as in §3 of [W2] by introducing curves passing through $c_1(X)d - 3$ distinct points but having a double point at one special point of this configuration, see Theorem 3.2 of [W2]. The proof of Theorem 3.2 of [W2] adapts here without any change.

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