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EIGENVALUES AND HOMOLOGY OF FLAG COMPLEXES AND VECTOR REPRESENTATIONS OF GRAPHS

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Abstract. The flag complex of a graph $G=(V,E)$ is the simplicial complex $X(G)$ on the vertex set V whose simplices are subsets of V which span complete subgraphs of G. We study relations between the first eigenvalues of successive higher Laplacians of $X(G)$. One consequence is the following:

Theorem: *Let $\lambda_2(G)$ denote the second smallest eigenvalue of the Laplacian of G. If $\lambda_2(G) > \frac{k}{k+1}|V|$ then $\tilde{H}^k(X(G);\mathbb{R})=0$.*

Applications include a lower bound on the homological connectivity of the independent sets complex $\mathrm{I}(G)$, in terms of a new graph domination parameter $\Gamma(G)$ defined via certain vector representations of G. This in turns implies Hall type theorems for systems of disjoint representatives in hypergraphs.

1 Introduction

Let $G=(V,E)$ be a graph with $|V|=n$ vertices. The *Laplacian* of G is the $V\times V$ positive semidefinite matrix L_G given by

$$L_G(u,v)=\begin{cases}\deg(u) & u=v\,,\\ -1 & uv\in E\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Let $0=\lambda_1(G)\le\lambda_2(G)\le\cdots\le\lambda_n(G)$ denote the eigenvalues of L_G. The second smallest eigenvalue $\lambda_2(G)$, called the *spectral gap*, is a parameter of central importance in a variety of problems. In particular it controls the expansion properties of G and the convergence rate of a random walk on G (see e.g. [Bo]). The *flag complex* of G is the simplicial complex $X(G)$ on the vertex set V whose simplices are all subsets $\sigma\subset V$ which form a complete subgraph of G. Topological properties of $X(G)$ play key roles in recent results in matching theory (see below).

In this paper we study relations between $\lambda_2(G)$, the cohomology of $X(G)$, and a new graph domination parameter $\Gamma(G)$ which is defined via certain vector representations of G. As an application we obtain Hall type theorems for systems of disjoint representatives in families of hypergraphs.

For $k \geq -1$ let $C^k(X(G))$ denote the space of real valued simplicial k-cochains of $X(G)$ and let $d_k : C^k(X(G)) \to C^{k+1}(X(G))$ denote the coboundary operator. For $k \geq 0$ define the reduced k-dimensional Laplacian of $X(G)$ by $\Delta_k = d_{k-1}d_{k-1}^* + d_k^* d_k$ (see section 2 for details). Let $\mu_k(G)$ denote the minimal eigenvalue of Δ_k. Note that $\mu_0(G) = \lambda_2(G)$. Our main result is the following:

Theorem 1.1. *For $k \geq 1$,*

$$k\mu_k(G) \geq (k+1)\mu_{k-1}(G) - n\,. \tag{1}$$

As a direct consequence of Theorem 1.1 we obtain

Theorem 1.2. *If $\lambda_2(G) > \frac{kn}{k+1}$ then $\tilde{H}^k(X(G), \mathbb{R}) = 0$.*

Remarks. 1. Theorem 1.2 is related to a well-known result of Garland (Theorem 5.9 in [G]) and its extended version by Ballmann and Świątkowski (Theorem 2.5 in [BS]). Roughly speaking, these results (in their simplest untwisted form) guarantee the vanishing of $\tilde{H}^k(X;\mathbb{R})$ provided that for *each* $(k-1)$-simplex τ in X, the spectral gap of the 1-skeleton of the link of τ is sufficiently large. Theorem 1.2 is, in a sense, a global counterpart of this statement for flag complexes.

2. Let $n = r\ell$, where $r \geq 1$, $\ell \geq 2$, and let G be the Turán graph $T_r(n)$, i.e. the complete r-partite graph on n vertices with all sides equal to ℓ. The flag complex $X(T_r(n))$ is homotopy equivalent to the wedge of $(\ell-1)^r$ $(r-1)$-dimensional spheres. It can be checked that $\mu_k(T_r(n)) = \ell(r-k-1)$ for all $0 \leq k \leq r-1$, hence (1) is satisfied with equality. Furthermore, $\lambda_2(G) = \ell(r-1) = \frac{r-1}{r}n$ while $\tilde{H}^{r-1}(X(G)) \neq 0$. Therefore the assumption in Theorem 1.2 cannot be replaced by $\lambda_2(G) \geq \frac{kn}{k+1}$.

We next study some graph theoretical consequences of Theorem 1.2. The *independence complex* $\mathrm{I}(G)$ of G is the simplicial complex on the vertex set V whose simplices are all independent sets $\sigma \subset V$. Thus $\mathrm{I}(G) = X(\overline{G})$ where $\overline{G}$ denotes the complement of G. Recent work on hypergraph matching, starting in [AH] with later developments in [A], [ABZ], [ACK], [M1,2], has utilized topological properties of $\mathrm{I}(G)$ to derive new Hall type theorems for hypergraphs. The main ingredient in these developments are lower bounds on the homological connectivity of $\mathrm{I}(G)$. For a simplicial complex Z let $\eta(Z) = \min\{i : \tilde{H}^i(Z, \mathbb{R}) \neq 0\} + 1$. It turns out that various domination parameters of G may be used to provide lower bounds on $\eta(\mathrm{I}(G))$. For a subset of vertices $S \subset V$ let $N(S)$ denote all vertices that are adjacent to at least one vertex of S and let $N'(S) = S \cup N(S)$. S is a *dominating set* if $N'(S) = V$. S is a *totally dominating set* if $N(S) = V$. Here are a few domination parameters:

- The *domination number* $\gamma(G)$ is the minimal size of a dominating set.
- The *total domination number* $\tilde{\gamma}(G)$ is the minimal size of a totally dominating set.
- The *independent domination number* $i\gamma(G)$ is the maximum, over all independent sets I in G, of the minimal size of a set S such that $N(S) \supset I$.
- The *strong fractional domination number* $\gamma_s^*(G)$ is the minimum of $\sum_{v\in V} f(v)$ over all nonnegative functions $f : V \to \mathbb{R}$, such that $\sum_{uv\in E} f(u) + \deg(v)f(v) \geq 1$ for every vertex v.

Some known lower bounds on η are $\eta(\mathrm{I}(G)) \geq \tilde{\gamma}(G)/2$ [M1], $\eta(\mathrm{I}(G)) \geq i\gamma(G)$ [AH], $\eta(\mathrm{I}(G)) \geq \gamma_s^*(G)$ [M2].

Here we introduce a new domination parameter, defined by vector representations. It is similar in spirit to the Θ function defined by Lovász [L]. It uses vectors to mimic domination, in a way similar to that in which the Θ function mimics independence of sets of vertices. It is defined as follows. A *vector representation* of a graph $G = (V, E)$ is an assignment P of a vector $P(v) \in \mathbb{R}^\ell$ for some fixed ℓ to every vertex v of the graph, such that the inner product $P(u) \cdot P(v) \geq 1$ whenever u, v are adjacent in G and $P(u) \cdot P(v) \geq 0$ if they are not adjacent. We shall identify the representation with the matrix P whose v-th row is the vector $P(v)$.

Let $\mathbf{1}$ denote the all 1 vector in $\mathbb{R}^V$. A non-negative vector α on V is said to be *dominating for P* if $\sum_{v\in V} \alpha(v)P(v) \cdot P(u) \geq 1$ for every vertex u, namely $\alpha PP^T \geq \mathbf{1}$. (Note that taking α to be the characteristic function of some totally dominating set satisfies this condition regardless of the representation.) The *value* of P is

$$|P| = \min\left\{\alpha \cdot \mathbf{1} : \alpha \geq 0\,,\ \alpha PP^T \geq \mathbf{1}\right\}.$$

The supremum of $|P|$ over all vector representations P of G is denoted by $\Gamma(G)$. Our main application of Theorem 1.2 is the following

Theorem 1.3. $\eta(\mathrm{I}(G)) \geq \Gamma(G)$.

Remark. One natural vector representation of G is obtained by taking $P(v) \in \mathbb{R}^E$ to be the edge incidence vector of the vertex v. For this representation $|P| = \gamma_s^*(G)$ hence $\Gamma(G) \geq \gamma_s^*(G)$. The bound $\eta(\mathrm{I}(G)) \geq \gamma_s^*(G)$

was previously obtained in [M2]. Theorem 1.3 is however stronger and often gives much sharper estimates for $\eta(\mathrm{I}(G))$, see e.g. the case of cycles described in section 4.

We next use Theorem 1.3 to derive a new Hall type result for hypergraphs. Let $\mathcal{F} \subset 2^V$ be a hypergraph on a finite ground set V. The *width* $w(\mathcal{F})$ of $\mathcal{F}$ is the minimal t for which there exist $F_1, \dots, F_t \in \mathcal{F}$ such that for any $F \in \mathcal{F}$, $F_i \cap F \neq \emptyset$ for some $1 \leq i \leq t$.

The *fractional width* $w^*(\mathcal{F})$ of $\mathcal{F}$ is the minimum of $\sum_{E\in\mathcal{F}} f(E)$ over all non-negative functions $f : \mathcal{F} \to \mathbb{R}$ with the property that for every edge $E \in \mathcal{F}$ the sum $\sum_{F\in\mathcal{F}} f(F)|E \cap F|$ is at least 1. A *matching* in $\mathcal{F}$ is a subhypergraph $\mathcal{M} \subset \mathcal{F}$ such that $F \cap F' = \emptyset$ for all $F \neq F' \in \mathcal{M}$. Let $\{\mathcal{F}_i\}_{i=1}^m$ be a family of hypergraphs. A *system of disjoint representatives* (SDR) of $\{\mathcal{F}_i\}_{i=1}^m$ is a matching $F_1, \dots, F_m$ such that $F_i \in \mathcal{F}_i$ for $1 \leq i \leq m$. Haxell [H] proved the following:

Theorem 1.4 [H]. *If $\{\mathcal{F}_i\}_{i=1}^m$ satisfies $w(\cup_{i\in I}\mathcal{F}_i) \geq 2|I| - 1$ for all $\emptyset \neq I \subset [m]$, then $\{\mathcal{F}_i\}_{i=1}^m$ has an SDR.*

Here we use Theorem 1.3 to show

Theorem 1.5. *If $\{\mathcal{F}_i\}_{i=1}^m$ satisfies $w^*(\cup_{i\in I}\mathcal{F}_i) > |I|-1$ for all $\emptyset \neq I \subset [m]$, then $\{\mathcal{F}_i\}_{i=1}^m$ has an SDR.*

The *matching number* $\nu(\mathcal{F})$ of a hypergraph $\mathcal{F}$ on the vertex set V is the cardinality $|\mathcal{M}|$ of a largest matching $\mathcal{M}$ in $\mathcal{F}$. The *fractional matching number* $\nu^*(\mathcal{F})$ is the maximum of $\sum_{E\in\mathcal{F}} f(E)$ over all non-negative functions $f : \mathcal{F} \to \mathbb{R}$ such that $\sum_{F\ni v} f(F) \leq 1$ for all $v \in V$. A hypergraph $\mathcal{F}$ is *r-uniform* if $|F| = r$ for all $F \in \mathcal{F}$. The following extension of Hall's theorem to hypergraphs was conjectured in [AK] and proved by Aharoni and Haxell [AH].

Theorem 1.6 [AH]. *If $\{\mathcal{F}_i\}_{i=1}^m$ is a family of r-uniform hypergraphs which satisfies $\nu(\cup_{i\in I}\mathcal{F}_i) > r(|I| - 1)$ for all $\emptyset \neq I \subset [m]$, then $\{\mathcal{F}_i\}_{i=1}^m$ has an SDR.*

Observe that if $\mathcal{F}$ is r-uniform then $w^*(\mathcal{F}) \geq \nu^*(\mathcal{F})/r$. Theorem 1.5 thus implies the following improvement of Theorem 1.6:

Theorem 1.7. *If $\{\mathcal{F}_i\}_{i=1}^m$ is a family of r-uniform hypergraphs which satisfies $\nu^*(\cup_{i\in I}\mathcal{F}_i) > r(|I| - 1)$ for all $\emptyset \neq I \subset [m]$, then $\{\mathcal{F}_i\}_{i=1}^m$ has an SDR.*

The paper is organized as follows. In section 2 we recall some topological terminology and the simplicial Hodge theorem. Theorems 1.1 and 1.2 are proved in section 3. The proofs utilize the approach of Garland [G] and its exposition by Ballmann and Świątkowski [BS]. In section 4 we relate the Γ parameter to homological connectivity and prove Theorem 1.3. In section 5 we recall a homological Hall type condition (Proposition 5.1) for the existence of colorful simplices in a colored complex. Combining this condition with Theorem 1.3 then yields the proof of Theorems 1.5.

2 Topological Preliminaries

Let X be a finite simplicial complex on the vertex set V. Let $X(k)$ denote the set of k-dimensional simplices in X, each taken with an arbitrary but fixed orientation. A simplicial k-cochain is a real valued skew-symmetric function on all ordered k-simplices of X. For $k \geq 0$ let $C^k(X)$ denote the space of k-cochains on X. The i-face of an ordered $(k+1)$-simplex $\sigma = [v_0, \ldots, v_{k+1}]$ is the ordered k-simplex $\sigma_i = [v_0, \ldots, \widehat{v_i}, \ldots, v_{k+1}]$. The coboundary operator $d_k : C^k(X) \to C^{k+1}(X)$ is given by

$$d_k\phi(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i)\,.$$

It will be convenient to augment the cochain complex $\{C^i(X)\}_{i=0}^{\infty}$ with the (-1)-degree term $C^{-1}(X) = \mathbb{R}$ with the coboundary map $d_{-1} : C^{-1}(X) \to C^0(X)$ given by $d_{-1}(a)(v) = a$ for $a \in \mathbb{R}$, $v \in V$. Let $Z^k(X) = \ker(d_k)$ denote the space of k-cocycles and let $B^k(X) = \mathrm{Im}(d_{k-1})$ denote the space of k-coboundaries. For $k \geq 0$ let $\tilde{H}^k(X) = Z^k(X)/B^k(X)$ denote the k-th reduced cohomology group of X with real coefficients. For each $k \geq -1$ endow $C^k(X)$ with the standard inner product $(\phi, \psi) = \sum_{\sigma \in X(k)} \phi(\sigma)\psi(\sigma)$ and the corresponding L^2 norm $\|\phi\| = \big(\sum_{\sigma \in X(k)} \phi(\sigma)^2\big)^{1/2}$.

Let $d_k^* : C^{k+1}(X) \to C^k(X)$ denote the adjoint of d_k with respect to these standard inner products. The reduced k-Laplacian of X is the mapping

$$\Delta_k = d_{k-1}d_{k-1}^* + d_k^* d_k : C^k(X) \to C^k(X)\,.$$

Note that if G denotes the 1-skeleton of X and J is the $V \times V$ all-ones matrix, then the matrix $J + L_G$ represents Δ_0 with respect to the standard basis. In particular, the minimal eigenvalue of Δ_0 equals $\lambda_2(G)$.

The space of harmonic k-cochains $\tilde{\mathcal{H}}^k(X) = \ker \Delta_k$ consists of all $\phi \in C^k(X)$ such that both $d_k\phi$ and $d_{k-1}^*\phi$ are zero. The simplicial version of Hodge theorem is the following well-known

Proposition 2.1. *$\tilde{\mathcal{H}}^k(X) \cong \tilde{H}^k(X)$ for $k \geq 0$.*

In particular, $\tilde{H}^k(X) = 0$ iff the minimal eigenvalue of Δ_k is positive.

3 Eigenvalues of Higher Laplacians

Let $X = X(G)$ be the flag complex of a graph $G = (V, E)$ on $|V| = n$ vertices. For an i-simplex $\eta \in X$ let $\deg(\eta)$ denote the number of $(i+1)$-simplices in X which contain η. The *link* of a simplex $\sigma \in X$ is the complex

$$\mathrm{lk}(\sigma) = \{\tau \in X : \sigma \cup \tau \in X\,,\ \sigma \cap \tau = \emptyset\}\,.$$

For two ordered simplices $\sigma \in X$, $\tau \in \mathrm{lk}(\sigma)$ let $\sigma\tau$ denote their ordered union.

CLAIM 3.1. *For* $\phi \in C^k(X)$

$$\|d_k\phi\|^2 = \sum_{\sigma \in X(k)} \deg(\sigma)\phi(\sigma)^2 - 2 \sum_{\eta \in X(k-1)} \sum_{vw \in \mathrm{lk}(\eta)} \phi(v\eta)\phi(w\eta)\,.$$

Proof. Recall that for $\tau \in X(k+1)$ we denoted by τ_i the ordered k-simplex obtained by removing the i-th vertex of τ. Thus

$$\begin{aligned}
\|d_k\phi\|^2 &= \sum_{\tau \in X(k+1)} d_k\phi(\tau)^2 = \sum_{\tau \in X(k+1)} \sum_{i=0}^{k+1} (-1)^i \phi(\tau_i) \sum_{j=0}^{k+1} (-1)^j \phi(\tau_j) \\
&= \sum_{\tau \in X(k+1)} \sum_{i=0}^{k+1} \phi(\tau_i)^2 + \sum_{\tau \in X(k+1)} \sum_{i \neq j} (-1)^{i+j} \phi(\tau_i)\phi(\tau_j) \\
&= \sum_{\sigma \in X(k)} \deg(\sigma)\phi(\sigma)^2 - 2 \sum_{\eta \in X(k-1)} \sum_{vw \in \mathrm{lk}(\eta)} \phi(v\eta)\phi(w\eta)\,.
\end{aligned}$$

□

For $\phi \in C^k(X)$ and a vertex $u \in V$ define $\phi_u \in C^{k-1}(X)$ by

$$\phi_u(\tau) = \begin{cases} \phi(u\tau) & u \in \mathrm{lk}(\tau)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

CLAIM 3.2. *For* $\phi \in C^k(X)$

$$\begin{aligned}
&\sum_{u \in V} \|d_{k-1}\phi_u\|^2 \\
&\quad = \sum_{\sigma \in X(k)} \Big(\sum_{\tau \in \sigma(k-1)} \deg(\tau) \Big) \phi(\sigma)^2 - 2k \sum_{\tau \in X(k-1)} \sum_{vw \in \mathrm{lk}(\tau)} \phi(v\tau)\phi(w\tau)\,.
\end{aligned}$$

Proof. Applying Claim 3.1 with $\phi_u \in C^{k-1}(X)$ we obtain

$$||d_{k-1}\phi_u||^2 = \sum_{\tau \in X(k-1)} \deg(\tau)\phi_u(\tau)^2 - 2 \sum_{\eta \in X(k-2)} \sum_{vw \in \mathrm{lk}(\eta)} \phi_u(v\eta)\phi_u(w\eta)\,.$$

Hence

$$\begin{aligned}
&\sum_{u\in V}\|d_{k-1}\phi_u\|^2\\
&\qquad=\sum_{u\in V}\sum_{\tau\in X(k-1)}\deg(\tau)\phi_u(\tau)^2-2\sum_{u\in V}\sum_{\eta\in X(k-2)}\sum_{vw\in \mathrm{lk}(\eta)}\phi_u(v\eta)\phi_u(w\eta)\\
&\qquad=\sum_{\sigma\in X(k)}\Big(\sum_{\tau\in\sigma(k-1)}\deg(\tau)\Big)\phi(\sigma)^2\\
&\qquad\qquad\qquad\qquad-2\sum_{\eta\in X(k-2)}\sum_{vw\in \mathrm{lk}(\eta)}\sum_{u\in \mathrm{lk}(v\eta)\cap \mathrm{lk}(w\eta)}\phi(vu\eta)\phi(wu\eta)\\
&\qquad=\sum_{\sigma\in X(k)}\Big(\sum_{\tau\in\sigma(k-1)}\deg(\tau)\Big)\phi(\sigma)^2-2k\sum_{\tau\in X(k-1)}\sum_{vw\in \mathrm{lk}(\tau)}\phi(v\tau)\phi(w\tau)\,.
\end{aligned}$$

The last equality follows from the fact that since X is a flag complex, if $\eta\in X(k-2)$, $vw\in \mathrm{lk}(\eta)$ and $u\in \mathrm{lk}(v\eta)\cap \mathrm{lk}(w\eta)$, then $vw\in \mathrm{lk}(u\eta)$. □

Claims 3.1 and 3.2 imply

$$\begin{aligned}
&k\Big(\|d_k\phi\|^2-\sum_{\sigma\in X(k)}\deg(\sigma)\phi(\sigma)^2\Big)\\
&\qquad\qquad=\sum_{u\in V}\|d_{k-1}\phi_u\|^2-\sum_{\sigma\in X(k)}\Big(\sum_{\tau\in\sigma(k-1)}\deg(\tau)\Big)\phi(\sigma)^2. \qquad (2)
\end{aligned}$$

Claim 3.3. *For $\phi\in C^k(X)$*

$$\sum_{u\in V}\|d^*_{k-2}\phi_u\|^2=k\|d^*_{k-1}\phi\|^2. \qquad (3)$$

Proof. For $\tau\in X(k-1)$

$$d^*_{k-1}\phi(\tau)=\sum_{v\in \mathrm{lk}(\tau)}\phi(v\tau)\,.$$

Therefore

$$\|d^*_{k-1}\phi\|^2=\sum_{\tau\in X(k-1)}d^*_{k-1}\phi(\tau)^2=\sum_{\tau\in X(k-1)}\Big(\sum_{v\in \mathrm{lk}(\tau)}\phi(v\tau)\Big)^2. \qquad (4)$$

Substituting ϕ_u in (4) we obtain

$$\begin{aligned}
\sum_{u\in V}\|d^*_{k-2}\phi_u\|^2&=\sum_{u\in V}\sum_{\eta\in X(k-2)}\Big(\sum_{v\in \mathrm{lk}(\eta)}\phi_u(v\eta)\Big)^2\\
&=\sum_{\eta\in X(k-2)}\sum_{u\in \mathrm{lk}(\eta)}\Big(\sum_{v\in \mathrm{lk}(u\eta)}\phi(vu\eta)\Big)^2
\end{aligned}$$

$$= k \sum_{\tau \in X(k-1)} \Big(\sum_{v \in \mathrm{lk}(\tau)} \phi(v\tau) \Big)^2 = k \| d_{k-1}^* \phi \|^2 . \qquad \square$$

Let $\phi \in C^k(X)$. Summing (2) and (3) we obtain the following key identity:

$$k(\Delta_k \phi, \phi) = \sum_{u \in V} (\Delta_{k-1}\phi_u, \phi_u) - \sum_{\sigma \in X(k)} \Big(\sum_{\tau \in \sigma(k-1)} \deg(\tau) - k \deg(\sigma) \Big) \phi(\sigma)^2 . \quad (5)$$

To estimate the right-hand side of (5) we need the following:

Claim 3.4. *For $\sigma \in X(k)$*

$$\sum_{\tau \in \sigma(k-1)} \deg(\tau) - k \deg(\sigma) \le n \, . \quad (6)$$

Proof. Recall that $N(v)$ is the set of neighbors of v in G. Let $\sigma = [v_0, \ldots, v_k]$ then for any $I \subset \{0, \ldots, k\}$

$$\deg\big([v_i : i \in I]\big) = \Big| \bigcap_{i \in I} N(v_i) \Big| .$$

Therefore

$$\sum_{\tau \in \sigma(k-1)} \deg(\tau) - k \deg(\sigma) = \sum_{i=0}^{k} \Big| \bigcap_{j \neq i} N(v_j) \Big| - k \Big| \bigcap_{j=0}^{k} N(v_j) \Big| . \quad (7)$$

The claim now follows since each $v \in V$ is counted at most once on the right-hand side of (7). $\square$

Proof of Theorem 1.1. Let $0 \neq \phi \in C^k(X)$ be an eigenvector of Δ_k with eigenvalue $\mu_k(G)$. By double counting

$$\sum_{u \in V} \|\phi_u\|^2 = (k+1)\|\phi\|^2 . \quad (8)$$

Combining (5),(6) and (8) we obtain

$$k\mu_k(G)\|\phi\|^2 = k(\Delta_k\phi, \phi) \ge \sum_{u \in V} (\Delta_{k-1}\phi_u, \phi_u) - n \sum_{\sigma \in X(k)} \phi(\sigma)^2$$

$$\ge \mu_{k-1}(G) \sum_{u \in V} \|\phi_u\|^2 - n\|\phi\|^2 = \big((k+1)\mu_{k-1}(G) - n\big)\|\phi\|^2 . \quad \square$$

Proof of Theorem 1.2. Inequality (1) implies by induction on k that $\mu_k(G) \ge (k+1)\mu_0(G) - kn$. Therefore, if $\mu_0(G) = \lambda_2(G) > \frac{kn}{k+1}$ then $\mu_k(G) > 0$ and $\tilde{H}^k(X(G), \mathbb{R}) = 0$ follows from the simplicial Hodge theorem. $\square$

4 Vector Domination and Homology

Let $G = (V, E)$ be a graph with $|V| = n$. We first reformulate Theorem 1.2 in terms of the independence complex $\mathrm{I}(G)$.

Theorem 4.1. $\eta(\mathrm{I}(G)) \geq n/\lambda_n(G)$.

Proof. Let $\ell = \lceil n/\lambda_n(G) \rceil$. Since $\lambda_n(G) = n - \lambda_2(\overline{G})$ it follows that $\lambda_2(\overline{G}) > \frac{\ell-2}{\ell-1} n$. Therefore by Theorem 1.2, $\tilde{H}^i(\mathrm{I}(G)) = \tilde{H}^i(X(\overline{G})) = 0$ for $i \leq \ell - 2$. Hence $\eta(\mathrm{I}(G)) \geq \ell$. □

The proof of Theorem 1.3 depends on Theorem 4.1 and the following

CLAIM 4.2. *Let P be a vector representation of $G = (V, E)$. Then*

$$\lambda_n(G) \leq \max_{u \in V} P(u) \cdot \sum_{v \in V} P(v) \,.$$

Proof. Let $x = (x(v) : v \in V)$ be a vector in $\mathbb{R}^V$. Then

$$\begin{aligned} x^T L_G x &= \sum_{uv \in E} \big(x(u) - x(v)\big)^2 \\ &\leq \tfrac{1}{2} \sum_{(u,v) \in V \times V} \big(x(u) - x(v)\big)^2 P(u) \cdot P(v) \\ &= \sum_{u \in V} x(u)^2 P(u) \cdot \sum_{v \in V} P(v) - \Big\| \sum_{v \in V} x(v) P(v) \Big\|^2 \\ &\leq \|x\|^2 \max_{u \in V} P(u) \cdot \sum_{v \in V} P(v) \,. \end{aligned}$$

The claim follows since $\lambda_n(G) = \max\{x^T L_G x / ||x||^2 : 0 \neq x \in \mathbb{R}^V\}$. □

Let $\mathbb{Z}_+$ denote the positive integers and let $\mathbb{Q}_+$ denote the positive rationals. For a vector $\mathbf{a} = (a(v) : v \in V) \in \mathbb{Z}_+^V$ let $G_{\mathbf{a}}$ denote the graph obtained by replacing each $v \in V$ by an independent set of size $a(v)$. Formally $V(G_{\mathbf{a}}) = \{(v, i) : v \in V,\ 1 \leq i \leq a(v)\}$ and $\{(u, i), (v, j)\} \in E(G_{\mathbf{a}})$ if $\{u, v\} \in E$. The projection $(v, i) \to v$ induces a homotopy equivalence between $\mathrm{I}(G_{\mathbf{a}})$ and $\mathrm{I}(G)$. In particular $\eta(\mathrm{I}(G_{\mathbf{a}})) = \eta(\mathrm{I}(G))$.

Proof of Theorem 1.3. Let P be a representation of G. By linear programming duality

$$\begin{aligned} |P| &= \min\{\alpha \cdot \mathbf{1} : \alpha \geq 0 \,,\ \alpha P P^T \geq \mathbf{1}\} \\ &= \max\{\alpha \cdot \mathbf{1} : \alpha \geq 0 \,,\ \alpha P P^T \leq \mathbf{1}\} \\ &= \sup\{\alpha \cdot \mathbf{1} : \alpha \in \mathbb{Q}_+^V \,,\ \alpha P P^T \leq \mathbf{1}\} \,. \end{aligned}$$

Let $\alpha \in \mathbb{Q}_+^V$ such that $\alpha PP^T \leq \mathbf{1}$. Write $\alpha = \mathbf{a}/k$ where $k \in \mathbb{Z}_+$ and $\mathbf{a} = (a(v) : v \in V) \in \mathbb{Z}_+^V$. Let $N = |V(G_\mathbf{a})| = \sum_{u\in V} a(u)$. Consider the representation Q of $G_\mathbf{a}$ given by $Q((u,i)) = P(u)$ for $(u,i) \in V(G_\mathbf{a})$. By Claim 4.2

$$\begin{aligned}\lambda_N(G_\mathbf{a}) &\leq \max_{(u,i)\in V(G_\mathbf{a})} Q((u,i)) \cdot \sum_{(v,j)\in V(G_\mathbf{a})} Q((v,j)) \\ &= \max_{u\in V} P(u) \cdot \sum_{v\in V} a(v)P(v) \leq k\,.\end{aligned}$$

Hence by Theorem 4.1

$$\alpha \cdot \mathbf{1} = \frac{1}{k}\sum_{v\in V} a(v) = \frac{N}{k} \leq \frac{N}{\lambda_N(G_\mathbf{a})} \leq \eta(\mathrm{I}(G_\mathbf{a})) = \eta(\mathrm{I}(G))\,. \qquad \square$$

REMARKS. 1. Let C_n denote the n-cycle on the vertex set $V = \{0,...,n-1\}$. For $\epsilon > 0$ and $i \in V$ let

$$a_\epsilon(i) = \begin{cases} \epsilon & i \equiv 0 \bmod 3\,, \\ 1 & i \equiv 1 \bmod 3\,, \\ \epsilon^{-1} & i \equiv 2 \bmod 3\,. \end{cases}$$

Consider the representation P_ϵ of C_n given by

$$P_\epsilon(i) = a_\epsilon(i)e_i + a_\epsilon(i+1)^{-1}e_{i+1}$$

where $e_0, \dots, e_{n-1}$ are orthogonal unit vectors and the indices are cyclic modulo n. Let $\alpha \in \mathbb{R}^V$ be given by

$$\alpha(i) = \begin{cases} \frac{1}{2+\epsilon^2} & i \equiv 0,1 \bmod 3 \text{ and } i < n-1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then $\alpha P_\epsilon P_\epsilon^T \leq \mathbf{1}$. Hence by linear programming duality

$$|P_\epsilon| \geq \sum_{i=0}^{n-1} \alpha(i) = \begin{cases} \frac{2k}{2+\epsilon^2} & n = 3k, 3k+1\,, \\ \frac{2k-1}{2+\epsilon^2} & n = 3k-1\,. \end{cases}$$

Thus

$$\Gamma(C_n) \geq \sup_{\epsilon>0} |P_\epsilon| \geq \begin{cases} k & n = 3k, 3k+1\,, \\ k - \frac{1}{2} & n = 3k-1\,. \end{cases}$$

Theorem 1.3 now implies

$$\eta\big(\mathrm{I}(C_n)\big) \geq \big\lceil \Gamma(C_n) \big\rceil = \left\lfloor \frac{n+1}{3} \right\rfloor. \qquad (9)$$

This lower bound is in fact tight for all n (see Claim 3.3 in [M2]). Note that for C_n the bound $\eta(\mathrm{I}(G)) \geq \gamma_s^*(G)$ is weaker since $\gamma_s^*(C_n) = n/4$.

2. It can be shown that for any graph $\Gamma(G) \geq \sup\{\gamma_s^*(G_{\mathbf{a}}) : \mathbf{a} \in \mathbb{Z}_+^V\}$. We do not know of examples with strict inequality.

5 A Hall Type Theorem for Fractional Width

Let Z be a simplicial complex on the vertex set W and let $\bigcup_{i=1}^m W_i$ be a partition of W. A simplex $\tau \in Z$ is *colorful* if $|\tau \cap W_i| = 1$ for all $1 \leq i \leq m$. For $W' \subset W$ let $Z[W']$ denote the induced subcomplex on W'. The following Hall's type sufficient condition for the existence of colorful simplices appears in [AH] and in [M1].

Proposition 5.1. *If for all $\emptyset \neq I \subset [m]$*

$$\eta\Big(Z\Big[\bigcup_{i\in I} W_i\Big]\Big) \geq |I|$$

then Z contains a colorful simplex.

Let G be a graph on the vertex set W with a partition $W = \bigcup_{i=1}^m W_i$. A set $S \subset W$ is *colorful* if $S \cap W_i \neq \emptyset$ for all $1 \leq i \leq m$. The induced subgraph on $W' \subset W$ is denoted by $G[W']$. Combining Theorem 1.3 and Proposition 5.1 we obtain the following:

Theorem 5.2. *If $\Gamma\big(G\big[\bigcup_{i\in I} W_i\big]\big) > |I| - 1$ for all $\emptyset \neq I \subset [m]$ then G contains a colorful independent set.*

Let $\mathcal{F} \subset 2^V$ be a hypergraph, possibly with multiple edges. The *line graph* $G_{\mathcal{F}} = (W, E)$ associated with $\mathcal{F}$ has vertex set $W = \mathcal{F}$ and edge set E consisting of all $\{F, F'\} \subset \mathcal{F}$ such that $F \cap F' \neq \emptyset$. A matching in $\mathcal{F}$ corresponds to an independent set in $G_{\mathcal{F}}$. For each $F \in \mathcal{F}$ let $P(F) \in \mathbb{R}^V$ denote the incidence vector of F. P is clearly a vector representation of $G_{\mathcal{F}}$ and satisfies $|P| = w^*(\mathcal{F})$. Thus $\Gamma(G_{\mathcal{F}}) \geq w^*(\mathcal{F})$.

Proof of Theorem 1.5. Let $\mathcal{F}$ denote the disjoint union of the $\mathcal{F}_i$'s, and consider the graph $G_{\mathcal{F}} = (W, E)$ with the partition $W = \cup_{i=1}^m W_i$ where $W_i = \mathcal{F}_i$. Then for any $\emptyset \neq I \subset [m]$,

$$\Gamma\big(G_{\mathcal{F}}[\cup_{i\in I} W_i]\big) = \Gamma(G_{\cup_{i\in I}\mathcal{F}_i}) \geq w^*(\cup_{i\in I}\mathcal{F}_i) > |I| - 1\,.$$

Theorem 5.2 implies that $G_{\mathcal{F}}$ contains a colorful independent set, hence $\{\mathcal{F}_i\}_{i=1}^m$ contains an SDR. □

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