

ON K-STABILITY OF REDUCTIVE VARIETIES

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Abstract. G. Tian and S.K. Donaldson formulated a conjecture relating GIT stability of a polarized algebraic variety to the existence of a Kähler metric of constant scalar curvature. In [D3] Donaldson partially confirmed it in the case of projective toric varieties. In this paper we extend Donaldson’s results and computations to a new case, that of reductive varieties.

Introduction

Around 1997 G. Tian and S.K. Donaldson formulated a conjecture relating GIT stability of a polarized algebraic variety to the existence of a Kähler metric of constant scalar curvature, see [T1,2,3] and [D1,2,3]. The general idea of such a relationship has been known as a “folklore conjecture” for a long time. It comes naturally from earlier works of Yau and Tian.

Here, we will refer to the Donaldson’s version of the conjecture (1.1 below), since we will make a strong use of notation and results developed in [D3]. In this beautiful paper Donaldson partially confirmed Conjecture 1.1 in the case of projective toric varieties, where his arguments are based on the following foundation blocks:

- (1) A very well-known correspondence between projective toric varieties and lattice polytopes (see e.g. [F], [O]);
- (2) A somewhat lesser-known correspondence between convex piecewise linear functions on polytopes and equivariant degenerations of projective toric varieties (see e.g. [KSZ] or [Al]); and
- (3) The general framework developed by Guillemin and Abreu [G], [A], which relates invariant Kähler metrics on projective toric varieties to convex functions on the corresponding polytopes.

Now let G be a complex reductive group with Weyl group W . Then to every W -invariant maximal-dimensional lattice polytope P in the weight lattice of a maximal subtorus of G one can, in a rather elementary way, associate an equivariant projective normal compactification V_P of G , generalizing the correspondence (1) above.

On the other hand, in [AlB1,2] the first author and M. Brion built a theory of degenerations parallel to (2), and obtained an answer that is formally very similar: Any W -invariant rational convex PL function on P defines a W -invariant subdivision of P and a degenerating family $\mathcal{V} \rightarrow \mathbb{C}$ in which every fiber \mathcal{V}_t with $t \neq 0$ is isomorphic to V and the special fiber \mathcal{V}_0 is a *stable reductive variety* in the sense of [AlB1,2].

The purpose of this paper is to extend the Guillemin–Abreu–Donaldson theory and results and computations of [D3] to this new and much larger class of projective G -compactifications and reductive varieties. In particular, our Theorem 3.7 is a generalization of one of the main results of [D3]. We use Theorem 3.7 together with Donaldson’s examples [D3] of polarized toric surfaces that do not admit metrics of constant scalar curvature to produce new *non-toric* examples. Instead of 2-dimensional toric, our examples are 8-dimensional reductive varieties.

The paper is organized as follows: In section 2 we review the relevant definitions and results from [D3]. In section 3 we recall some results about reductive varieties from [AlB1,2]. In section 4 we extend several of Donaldson’s results from [D3] to the case of reductive varieties. Section 5 contains examples of reductive varieties which do not admit Kähler metrics of constant scalar curvature.

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Throughout the paper, we will use the following:

NOTATION 0.1. K will denote a compact real reductive group of dimension n with a maximal subtorus T of dimension r . We will denote by $G = K_{\mathbb{C}}$ and $H = T_{\mathbb{C}}$ their complexifications. We choose a Borel subgroup B of G and an opposite Borel subgroup B^{-} so that $B \cap B^{-} = H$.

Let $\Lambda \simeq \mathbb{Z}^r$ denote the group of characters of H . It comes with an action of the Weyl group W and with a decomposition of $\Lambda_{\mathbb{R}}$ into Weyl chambers. $\Lambda_{\mathbb{R}}^{+}$ will denote the positive chamber, and $r_i \in \Lambda$ the simple roots.

1 Basic Definitions and Results from [D3]

The precise formulation of the conjecture relating K-stability and existence of Kähler metric of constant scalar curvature was given by Donaldson in [D3]:

CONJECTURE 1.1. *A smooth polarized projective variety (V, L) admits a Kähler metric of constant scalar curvature in $c_1(L)$ if and only if it is K-stable.*

The definition of K-stability [D3, Def.2.1.2] involves another space V_0 which is allowed to be a general scheme. Let \mathcal{F} be an ample line bundle over a projective scheme W , and suppose one has a fixed \mathbb{C}^* -action on the pair (W, \mathcal{F}) . For each positive integer k one has a vector space

$$H_k = H^0(W, \mathcal{F}^k)$$

with a \mathbb{C}^* -action. From this, one obtains integers $d_k = \dim H_k$ and w_k , the weight of the induced action on the highest exterior power of H_k . The integers d_k, w_k are, for large k , given by polynomial functions of k , with rational coefficients $d_k = Q(k)$, $w_k = P(k)$, say. Define $F(k) = w_k/kd_k$. For large enough k one has an expansion

$$F(k) = F_0 + F_1k^{-1} + F_2k^{-2} + \dots,$$

with rational coefficients F_i . The *Futaki invariant* of the \mathbb{C}^* -action on (W, \mathcal{F}) is defined to be the coefficient F_1 .

DEFINITION 1.2. A *test configuration* or *test family* for (V, L) of exponent r consists of

1. a scheme \mathcal{V} with a \mathbb{C}^* -action;
2. a \mathbb{C}^* -equivariant line bundle $\mathcal{L} \rightarrow \mathcal{V}$;
3. a flat \mathbb{C}^* -equivariant map $\pi : \mathcal{V} \rightarrow \mathbb{C}$, where \mathbb{C}^* acts on \mathbb{C} by multiplication in the standard way;

such that any fiber $V_t = \pi^{-1}(t)$ for $t \neq 0$ is isomorphic to V and the pair (V, L^r) is isomorphic to $(V_t, \mathcal{L}|_{V_t})$.

DEFINITION 1.3. The pair (V, L) is *K-stable* if for each test configuration for (V, L) the Futaki invariant of the induced action on $(V_0, \mathcal{L}|_{V_0})$ is less than or equal to zero, with equality if and only if the configuration is a product configuration.

The pair (V, L) is *equivariantly K-stable* if one restricts oneself only to equivariant test configurations. For a toric variety this means H -invariant families, and in our case this will mean $G \times G$ -equivariant families.

See also the discussion in [D3] showing that K-stability is closely related to asymptotic GIT stability.

Following [D3] we briefly recall some facts about Mabuchi functional. It is a real-valued function \mathcal{M} on the set of Kähler metrics in the same Kähler class $[\omega_0]$, defined up to the addition of an overall constant. The metrics of constant scalar curvature are critical for the Mabuchi functional. The functional is defined through the formula for its variation at a metric $\omega = \omega_0 + i\bar{\partial}\partial\psi$ with respect to an infinitesimal change $\delta\psi$ in the Kähler potential:

$$\delta\mathcal{M} = \int_V (S - a)\delta\psi \frac{\omega^n}{n!}.$$

Here, S is the scalar curvature of ω and a is the average value of the scalar curvature. Therefore $\delta\mathcal{M}$ is not changed if one adds a constant to $\delta\psi$, and so depends only on the variation of the metric.

DEFINITION 1.4. A functional I_V on the set of Kähler potentials is defined by the formula

$$\delta I_V = 2 \int_V \delta\psi \frac{\omega^n}{n!}.$$

Assume that $D = \sum D_i$ is an effective divisor on V representing $c_1(V)$, with smooth components D_i (such D exists for any smooth toric variety V). Let χ be a meromorphic form with $(\chi) = -D$ and let $\nu = |\chi|^{-2}$.

PROPOSITION 1.5 [D3, 3.2.4]. For any metric $\omega = \omega_0 + 2i\bar{\partial}\partial\psi$ on V

$$\mathcal{M}(\omega) = \mathcal{L}_a(\omega) + \int_V \log \nu \frac{\omega^n}{n!} \text{ where } \mathcal{L}_a(\omega) = -I_D(\psi) + aI_V(\psi)$$

is the linear part of the Mabuchi functional.

Now, let (V, L) be a polarized projective toric variety with a moment polytope P . Via the Guillemin–Abreu theory, an equivariant Kähler potential on V corresponds to a convex function u on the interior of P with a logarithmic asymptotic behavior near the boundary. Namely, if faces are defined by linear inequalities $\delta_k(x) \geq 0$ then

$$u = \sum \delta_k(x) \log \delta_k(x) + \text{a smooth function}.$$

The lattice determines standard measures $d\mu$ on the polytope P and $d\sigma$ on faces of P . In [D3, 3.2.6] Donaldson establishes the following formula for the linear part of Mabuchi functional:

$$\mathcal{L}_a = (2\pi)^n \left(\int_{\partial P} u d\sigma - a \int_P u d\mu \right)$$

On the other hand, in [D3, 4.2.1] Donaldson shows that Futaki invariant of an equivariant test configuration defined by a rational PL function f is given by a very similar formula

$$-F_1 = \frac{1}{2 \text{Vol } P} \left(\int_{\partial P} f d\sigma - a \int_P f d\mu \right).$$

Thus, one obtains an ingenious connection between two seemingly very different invariants. Donaldson uses this connection to prove, among other things, the following:

PROPOSITION 1.6 [D3, Prop. 7.1.2]. *Suppose there is a function $f \in C^1$ with $\mathcal{L}_a(f) < 0$. Then the Mabuchi functional is not bounded below on the invariant metrics, and the manifold V does not admit any Kähler metric of constant scalar curvature in the given cohomology class.*

Our aim will be to generalize these results to the reductive case.

2 Overview of Reductive Varieties

2.1 Complexes of polytopes and (stable) reductive varieties. We recall some of the results and constructions of [AlB2].

Lattice points $\lambda \in \Lambda^+$ are in bijection with irreducible G -representation E_λ . The algebra of regular functions on G can be written canonically as

$$\mathbb{C}[G] = \bigoplus_{\lambda \in \Lambda} \text{End } E_\lambda \quad \text{and} \quad \text{End } E_\lambda \cdot \text{End } E_\mu \subset \bigoplus \text{End } E_{\lambda+\mu-\sum_{n_i \geq 0} n_i r_i}.$$

Let $P \subset \Lambda$ be a maximal-dimensional W -invariant polytope with vertices in Λ . Let $P^+ = P \cap \Lambda_{\mathbb{R}}^+$ be the part of P lying in the positive chamber, and $\text{Cone } \Delta^+ \subset \mathbb{R} \oplus \Lambda_{\mathbb{R}}$ be the cone over $(1, P^+)$. The vector space

$$R_P = \bigoplus_{\lambda \in \mathbb{Z} \oplus \Lambda \cap \text{Cone } \Delta^+} \text{End } E_\lambda$$

has a natural structure of a subalgebra in $\mathbb{C}[\mathbb{C}^* \times G]$. It is finitely generated e.g. by [AlB1, 4.8].

One defines $V_P = \text{Proj } R_P$ and $L_P = \mathcal{O}(1)$. Then V_P is a normal projective equivariant $G \times G$ -compactification of group G (with $G \times G$ acting on G by left and right multiplication: $(g_1, g_2) \cdot g = g_1^{-1} g g_2$), and L_P is a $G \times G$ -linearized ample sheaf on V_P .

The fixed point set $(V^{\text{diag } H}, L^{\text{diag } H})$ is a toric variety with a H -linearized ample line bundle. It corresponds to the same polytope P , and it comes with an action by WH , a semidirect product of W and H . The $G \times G$ -orbits of V are in bijection with WH -orbits of $V^{\text{diag } H}$ and with W -orbits of faces of the polytope P .

EXAMPLE 2.1. Take $G = \mathrm{PGL}_n$ embedded into $V = \mathbb{P}\mathrm{Mat}_{n \times n} = \mathbb{P}^{n^2-1}$. In this case, $\Lambda_{\mathbb{R}}$ is \mathbb{R}^{n-1} divided into $n!$ Weyl chambers. The polytope P is a simplex with a vertex on one of the rays of the positive chamber $\Lambda_{\mathbb{R}}^+$, and the other $n - 1$ vertices are its reflections under $W = S_n$.

EXAMPLE 2.2. In the previous example, take $n = 2$. The polytope P is an interval symmetric about the origin. The variety $V = \mathbb{P}^3$ is the wonderful compactification of De Concini–Procesi for PGL_2 .

EXAMPLE 2.3. For any semisimple group G , consider a point in the interior of $\Lambda_{\mathbb{R}}^+$ and let P be the convex hull of its W -reflections. The corresponding variety V_P is called a wonderful compactification of G .

For $G = \mathrm{PGL}_3$ and $W = S_3$, P is a hexagon and P^+ is a 4-gon. Note that the wonderful compactification of PGL_3 is not toric.

[AIB1, Sec.5] and [AIB2, Sec.2] generalize this picture to the case of a W -invariant complex of polytopes, i.e. a finite W -invariant collection $\Delta = \{P_i\}$ of lattice polytopes in $\Lambda_{\mathbb{R}}$ such that the intersection $P_i \cap P_j$ of any two polytopes is a union of faces of both. Note that the polytopes need not be maximal-dimensional and that reductive varieties in general do not contain G .

Theorem 2.4 [AIB2, Thm. 2.8]. (1) *A complex of polytopes Δ defines a family of polarized stable reductive varieties $\{(V_{\Delta,t}, L_{\Delta,t})\}$ parameterized by a certain cohomology group $H^1(\Delta/W, \mathrm{Aut})$. The choice $t = 1$ gives a distinguished “untwisted” member of this family.*

(2) $V_{\Delta,t}^{\mathrm{diag} H}$ is a stable toric variety, a union of ordinary toric varieties corresponding to polytopes P_i in Δ , together with a W -action.

(3) *The $G \times G$ -orbits of $V_{\Delta,t}$ are in a bijection with WH -orbits of $V_{\Delta,t}^{\mathrm{diag} H}$ and with the set Δ/W .*

(4) *Variety $V_{\Delta,t}$ is irreducible iff Δ/W contains a unique maximal polytope. In this case, $H^1(\Delta/W, \mathrm{Aut}) = \{1\}$. Variety V_{Δ} is a G -compactification iff Δ consists of one W -invariant polytope P of maximal dimension plus its faces.*

Note that here we use the notion of stable reductive varieties as defined in [AIB2] and not the notion of K-stability.

2.2 Nonsingular reductive varieties. In toric geometry, a lattice polytope P corresponds to a projective toric variety V_P together with a linearized ample sheaf L_P . It is well known that the variety V_P is nonsingular if and only if P is a *Delzant polytope*, i.e. at every vertex precisely $\dim P$ edges meet and the integral generators of these edges form a basis

of the lattice. We will need the following generalization of this result to the case of reductive varieties, which we will use in the case of a single W -invariant polytope.

PROPOSITION 2.5. *Let $\Delta = \{P_i\}$ be a W -invariant complex of polytopes, and $V = V_\Delta$ be the corresponding (stable) reductive variety.*

- (1) *If V is nonsingular, then Δ is a disjoint union of Delzant polytopes plus its faces.*
- (2) *If Δ is a disjoint union of Delzant polytopes none of whose vertices lie on the supporting hyperplanes of $\Lambda_{\mathbb{R}}^+$, plus its faces, then V is nonsingular.*

Proof. If H is a torus acting on a smooth variety Z , then the fixed point set Z^H is also nonsingular. By Theorem 2.4, $V^{\text{diag } H}$ is a stable toric variety, a union of ordinary toric varieties corresponding to $P_i \in \Delta$. It is nonsingular if and only if the maximal polytopes are disjoint and each of them is Delzant.

In the opposite direction, if none of the polytopes P_i contain vertices on the supporting hyperplanes of the positive chamber, then the variety V is toroidal, i.e. locally analytically it is isomorphic to \mathbb{A}^N times the toric variety corresponding to the polytopes P_i . Hence, it is nonsingular if P_i are Delzant. \square

2.3 Canonical class. According to [B2], the anticanonical divisor of any spherical variety for group G can be written as

$$-K_V = \partial_G V + \partial_B G.$$

Here $\partial_G V = \sum D_v$ is the reduced sum of G -invariant divisors, and $\partial_B G = \sum n_\rho D_\rho$ is the union of B -invariant, non- G -invariant divisors (“colors”), with uniquely defined positive coefficients n_ρ . For spherical varieties in general the coefficients n_ρ can be arbitrary, and [B2] provides the precise formula. However, for reductive varieties, which are spherical for the group $G \times G$, all coefficients $n_\rho = 2$, see [AlB2, Sec.5.2]. We will call $\partial_{G \times G} V$ the *vertical* and $\partial_{B \times B^-} V$ the *horizontal* parts of the anticanonical divisor (this notation comes from the behavior of these boundaries under the moment map). The above formula then becomes

$$-K_V = D_{\text{vert}} + D_{\text{hor}}.$$

If Δ is a complex of polytopes satisfying the conditions of Proposition 2.5 (2) then all codimension one faces also satisfy 2.5(2). Hence, every

D_v is a smooth reductive variety. In addition, since V is toroidal, the linear system $|D_{\text{hor}}|$ is basepoint-free. Hence, there exists a smooth divisor representing D_{hor} .

2.4 One-parameter degenerations. Let f be a convex rational W -invariant PL function on P . In the same fashion as in the toric case, f defines a $(\dim V + 1)$ -dimensional “test” family $\mathcal{V} \rightarrow \mathbb{C}$ such that every fiber \mathcal{V}_t for $t \neq 0$ is isomorphic to V . The construction, contained in [ALB2, Sec. 4.2], is as follows.

After replacing the polytope P by a large divisible multiple NP , i.e. replacing L by L^N , one can assume that the domains of linearity of f are lattice polytopes P_i .

Consider a bigger polytope \tilde{P} in $\mathbb{R} \oplus \Lambda_{\mathbb{R}}$ which is bounded from below by $(0, P)$ and from above by the graph of $R - f$ for some $R \gg 0$. Then, by the above construction, the variety $V_{\tilde{P}}$ is a projective compactification of the group $\mathbb{C}^* \times G$, and it comes with a natural map π to \mathbb{P}^1 . The test family \mathcal{V} is $V_{\tilde{P}} \setminus \pi^{-1}(\infty)$. The special fiber $\pi^{-1}(0)$ is a stable reductive variety for the complex of polytopes $\Delta = \{P_i\}$.

3 Linear Part of Mabuchi Functional and Futaki Invariant

Every symplectic variety V with a Hamiltonian action of a compact Lie group K admits a moment map $m : V \rightarrow k^*$ to the dual of the Lie algebra. The moment map commutes with the K -action on V and with the coadjoint action of K on k^* . Therefore, the image is a union of coadjoint orbits.

Let $t \subset k$ be the Cartan subalgebra and let $t^* \subset k^*$ be a splitting of the projection $k^* \rightarrow t^*$. Since every coadjoint orbit intersects the positive Weyl chamber $(t^*)^+$ at one point, one obtains a continuous map $\pi : V \rightarrow (t^*)^+ = \Lambda_{\mathbb{R}}^+$. By a theorem of Guillemin–Sternberg [GS] the image of V under the moment map m_T for a maximal torus $T \subset K$ is a polytope, and by a result of Kirwan [Ki] the image $\pi(V)$ is a polytope. Both polytopes live in the space $\Lambda_{\mathbb{R}}$ and both are traditionally called moment polytopes. For us, $\pi(V)$ will be important.

In the case when V is a projective spherical variety for the complexified Lie group $G = K_{\mathbb{C}}$, some general facts about the moment map were established in [B1] via representation theory. In particular, $\pi(V)$ identifies with Brion’s moment polytope of a spherical variety. For a reductive variety corresponding to a polytope P one thus obtains the moment polytope P^+ .

Let $\lambda \in \Lambda_{\mathbb{R}}^+$ be an integral vector. It corresponds to an irreducible

G -representation E_λ . According to the Weyl character formula, $\dim E_\lambda$ is given by a polynomial

$$h(x) = h_e(\lambda) + h_{e^{-1}}(\lambda) + \dots$$

written as a sum of its homogeneous parts.

Any reductive variety is spherical for the action of $G \times G$. For each $\lambda \in \Lambda^+$ the corresponding $G \times G$ -representation is $\text{End } E_\lambda$. Let us write its dimension as the sum of its homogeneous parts

$$\dim(\text{End } E_\lambda) = H(\lambda) = h^2(\lambda) = H_d(\lambda) + H_{d-1}(\lambda) + \dots$$

and extend each H_d to a polynomial function on $\Lambda_{\mathbb{R}}^+$.

In the previous section, we wrote $-K_V$ and the $B \times B^-$ -boundary of V in the form $D_{\text{vert}} + D_{\text{hor}}$. The vertical part D_{vert} is a union of $G \times G$ -invariant divisors, each of them is a reductive variety corresponding to a face of polytope P modulo the W -action. Under the map π , D_{vert} maps to a union of faces of P^+ . As in the toric case, for every face $F \subset P$ we will denote by $d\sigma$ the Euclidean measure on F induced by the sublattice $\Lambda_F = \Lambda \cap \mathbb{R}F$.

The horizontal part D_{hor} is not $G \times G$ -invariant, and it is easy to see that under the map π it maps surjectively onto P^+ .

LEMMA 3.1. (1) *The push-forward of the Liouville measure on V is*

$$\pi_* \frac{\omega^n}{n!} = (2\pi)^r H_d(x) d\mu.$$

(2) *Similarly, the push-forward of the Liouville measure on D_{vert} is*

$$\pi_* \frac{\omega^{n-1}}{(n-1)!} = (2\pi)^r H_d(x) d\sigma.$$

(3) *The push-forward of the Liouville measure on D_{hor} is*

$$\pi_* \frac{\omega^{n-1}}{(n-1)!} = (2\pi)^r \cdot 2H_{d-1}(x) d\mu.$$

Proof. This comes out straight from the Riemann–Roch theorem applied to $H^0(L^s)$ on the variety V and on a single coadjoint orbit \mathcal{O}_λ for $s \gg 0$. The push-forward of the Liouville measure is given by integrating $\omega^n/n!$. On the other hand, the Riemann–Roch theorem gives

$$H^0(L^s) = \int ch(L^s) \cdot \text{Td}(\mathcal{T}) = s^n \frac{c_1(L)^n}{n!} - s^{n-1} \frac{1}{2} K \frac{c_1(L)^{n-1}}{(n-1)!} + \dots$$

So, the volume of a single coadjoint orbit \mathcal{O}_λ is given by $H_d(\lambda)$ and the volume of the restriction of the horizontal part of $-K_V$ to $m^{-1}(\mathcal{O}_\lambda)$ by

$2H_{d-1}(\lambda)$. Reductive varieties are spherical for $G \times G$ -action, so in the decomposition of $H^0(V, L^s)$ every $\text{End } E_\lambda$, $\lambda \in sP^+$ appears with multiplicity one; and the formula follows. \square

PROPOSITION 3.2. *$K \times K$ -invariant Kähler metrics on V are in a bijection with W -invariant symplectic convex potentials u on P which have the same behavior near the boundary of P as in the toric case (i.e. $u = \sum l_i \log l_i +$ a smooth function).*

Proof. By [GS, Thm. 3.1], $\omega = i\partial\bar{\partial}\phi$ for a some potential ϕ on V . The restriction ϕ_T of this potential to the toric variety $V^{\text{diag } T}$ is W -invariant, and hence by the toric case, applying the Legendre transform, gives a symplectic potential u on P^+ . Clearly, it has to be W -invariant.

Vice versa, every W -invariant symplectic potential u on P gives a W -invariant potential ϕ_T on $V^{\text{diag } T}$. By $K \times K$ -action, it extends to a unique potential ϕ on V . Here, we are using the fact that for a spherical variety the preimages of coadjoint orbits under the moment map are precisely the $K \times K$ -orbits. \square

Theorem 3.3. *Let f be a convex rational W -invariant PL function on P . Then the Futaki invariant of the corresponding test family is given by the formula*

$$-F_1(f) = \frac{1}{2 \int_{P^+} H_d} \left(\int_{\partial P^+} f H_d d\sigma + 2 \int_{P^+} f H_{d-1} d\mu - a \int_{P^+} f H_d d\mu \right),$$

where

$$a = \frac{\int_{\partial P^+} H_d d\sigma + 2 \int_{P^+} H_{d-1} d\mu}{\int_{P^+} H_d d\mu}.$$

Proof. Same as the proof of Proposition 4.2.1 of [D3]. The difference is that this time an integral point $\lambda \in P^+$ represents not a one-dimensional vector space, as in the toric case, but the vector space $\text{End } E_\lambda$ whose dimension is $H(\lambda)$. We use the next lemma to estimate the sum, and get

$$\begin{aligned} A &= \int_{P^+} (R - f) H_d d\mu \\ B &= \frac{1}{2} \int_{\partial^+} (R - f) H_d d\sigma + \int_{P^+} (R - f) H_{d-1} d\mu \\ C &= \int_{P^+} H_d d\mu \\ D &= \frac{1}{2} \int_{\partial P^+} H_d d\sigma + \int_{P^+} H_{d-1} d\mu. \end{aligned}$$

Substituting $F_1 = (AD - BC)/2C^2$ gives the formula. \square

REMARK 3.4. In the toric case one has $H_d = 1$, $H_{d-1} = 0$, $P^+ = P$ and the formula reduces to that of [D3, 4.2.1].

LEMMA 3.5. *Let $P \subset \mathbb{R}^n$ be a lattice polytope of dimension n and $H_d : \mathbb{R}^n \rightarrow \mathbb{R}$ be a homogeneous polynomial function of degree d . Then for $k \gg 0$,*

$$\sum_{\lambda \in kP \cap \mathbb{Z}^n} H_d(\lambda) = k^{n+d} \int_P H_d d\mu + \frac{1}{2} k^{n+d-1} \int_{\partial P} H_d d\sigma + O(k^{n+d-2}).$$

Proof. Elementary. For a monomial $x_1^{a_1} \dots x_n^{a_n}$ the computation easily reduces to the case of a polytope of dimension $n + d$ and the function $H_0 = 1$, i.e. to counting integral points in a polytope, where the analogous formula is well known (cf. [D3, A1]). Alternatively, this is a corollary of a general formula of Pukhlikov and Khovanski [PK]. \square

Theorem 3.6. *Let u be a W -invariant symplectic potential on P corresponding to a Kähler form ω . Then the linear part of the Mabuchi functional is given by the formula*

$$\mathcal{L}_a(u) = (2\pi)^r \left(\int_{\partial P^+} u H_d d\sigma + 2 \int_{P^+} u H_{d-1} d\mu - a \int_{P^+} u H_d d\mu \right),$$

where

$$a = \frac{\int_{\partial P^+} H_d d\sigma + 2 \int_{P^+} H_{d-1} d\mu}{\int_{P^+} H_d d\mu}.$$

Proof. As in the proof of Proposition 3.2.4 and Lemma 3.2.6 of [D3], we use the formula from Proposition 1.5 and compute

$$\begin{aligned} I_D(\psi) - aI_V(\psi) &= I_{D_{\text{vert}}}(\psi) + I_{D_{\text{hor}}}(\psi) - aI_V(\psi) \\ &= \int_{D_{\text{vert}}} \psi \frac{\omega^{n-1}}{(n-1)!} + \int_{D_{\text{hor}}} \psi \frac{\omega^{n-1}}{(n-1)!} - a \int_{D_{\text{hor}}} \psi \frac{\omega^n}{n!}. \end{aligned}$$

Applying Lemma 3.1 completes the computation. \square

Therefore, for equivariant compactifications of G , similar to the toric case, the two seemingly unrelated functions, Futaki invariant of an equivariant test configuration and the linear part of Mabuchi functional, are given by the same formula. Although they are defined on different sets, convex PL functions in the first case and convex C^∞ -functions with a prescribed boundary behavior in the second, we can approximate one class of functions by another one freely. Hence, in the cases when the sign of F_1 , resp. \mathcal{L}_a , is definite, we get the equivalence between (equivariant) K-stability and boundedness of Mabuchi functional from below. As a corollary, we obtain

Theorem 3.7 (cf. [D3, Prop. 7.1.2]). *Let V be a smooth equivariant compactification of a complex reductive group G . Suppose there is a function $f \in C^1$ with $\mathcal{L}_a(f) < 0$. Then the Mabuchi functional is not bounded below on the invariant metrics and the manifold V does not admit any Kähler metric of constant scalar curvature in the given cohomology class.*

REMARK 3.8. This theorem also applies, with the same proof, to a larger class of smooth reductive varieties described by Proposition 2.5(2).

4 Examples of G -Compactifications without CSC Metrics

We modify the example in [D3, 7.2] appropriately to get a new series of varieties V_n without a Kähler metric of constant scalar curvature.

Donaldson starts with a triangle $(0, 0)$, $(0, 1)$, $(1, 0)$. Working with the corner $A = (0, 0)$ first, he considers the part of the first quadrant bounded by the x - and y -axes and points $B = (0, 1/4)$, $C = (1/4, 0)$ and $D_n = (r_n, r_n)$, where $(n-2)/4(3n-5)$. He then smooths out the corners B and C by adding very short segments of various slopes so that the resulting polyhedral body is Delzant, i.e. at every vertex the integral generators of edges form a basis of the standard lattice \mathbb{Z}^2 .

He then translates this picture symmetrically to the other three corners of the original triangle, and calls the resulting polytope P_n . It is clear that P_n is a Delzant polytope which is a “smoothing” of a 9-gon, and that the corresponding smooth toric variety V_n can be obtained by repeating blow-ups of the projective plane. The polytope represents an ample sheaf L_n on V_n . Donaldson shows that P_n does not contain a metric of constant curvature in the cohomology class $c_1(L_n)$.

We take $G = PGL_3$. Then Weyl group is S_3 acting on $\Lambda = \mathbb{Z}^2$ and $\Lambda_{\mathbb{R}}$ is divided into 6 cones, the chambers. Take a point A far in the interior of the positive chamber, at the same distance from both sides. Reflect it symmetrically to obtain vertices of a hexagon. At each of the 6 vertices, repeat Donaldson’s construction close to the corners.

The resulting polytope P_n is then a Delzant polytope approximating a polytope with $3 \cdot 6 = 18$ vertices. The corresponding polarized variety V_n is a compactification of G and has dimension 8. It is nonsingular by Proposition 2.5.

The Weyl character formula gives

$$H_d = x^2 y^2 (x + y)^2, \quad H_{d-1} = xy(x + y)^3 + 2x^2 y^2 (x + y).$$

Here, (x, y) are natural coordinates on $\Lambda_{\mathbb{R}}^+ \cap \Lambda \simeq \mathbb{Z}_{\geq 0}^2$.

We claim that P_n does not have a metric of constant curvature in the cohomology class $c_1(L_n)$ by exhibiting a convex function f on it with $\mathcal{L}_a(f) < 0$. We use the formula of Theorem 3.6 to compute it.

First, we need to observe that the function $2H_{d-1}/H_d$ reaches its minimum near the corner A , therefore near the corner D_n of P^+ the function $2H_d - aH_d$ is negative.

Define the function $f = f_{\epsilon, n} \geq 0$ to be a convex function which is equal to zero on the inside of P_n and has a simple crease at the distance ϵ to the point D_n . Define it at other corners by symmetry, to be W -invariant. We claim that for $n \gg 1/\epsilon \gg 0$ one has $\mathcal{L}_a(f) < 0$, as required. Indeed, the expressions

$$\int_{P^+} H_d d\mu, \quad \int_{\partial P^+} H_d d\sigma, \quad \int_{P^+} H_{d-1} d\mu,$$

all have finite positive limits near D_n . The contribution of ∂P^+ to $\mathcal{L}_a(f)$ in Theorem 3.6 is $O(\epsilon/n)$, as Donaldson's computation shows, and is positive. Finally, the contribution of the integral over P^+ is approximately $\epsilon^2(2H_{d-1} - aH_d) d\mu$. As we observed, this is negative and grows as $\text{const} \cdot \epsilon^2$. Since $\epsilon^2 \gg \epsilon/n$, we get $\mathcal{L}_a(f) < 0$. We note that the varieties V_n are not toric. They are, however, reductive varieties.

In conclusion, we note that Donaldson [D3] includes a number of conjectures and tentative results in the toric case. Using Theorems 3.3 and 3.6 we can generalize them to our situation. Once they are established in the toric case, we expect that they can be proved in the reductive case, as well.

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