

FUNCTIONAL EQUATIONS FOR LOCAL NORMAL ZETA FUNCTIONS OF NILPOTENT GROUPS

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with Appendix by
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Abstract. We give explicit formulae for the local normal zeta functions of torsion-free, class-2-nilpotent groups, subject to conditions on the associated Pfaffian hypersurface which are generically satisfied by groups with small centre and sufficiently large abelianization. We show how the functional equations of two types of zeta functions – the Weil zeta function associated to an algebraic variety and zeta functions of algebraic groups introduced by Igusa – match up to give a functional equation for local normal zeta functions of groups. We also give explicit formulae and derive functional equations for an infinite family of class-2-nilpotent groups known as Grenham groups, confirming conjectures of du Sautoy.

1 Introduction

In [GrSS] Grunewald, Segal and Smith introduced the concept of a zeta function of an infinite, finitely generated, torsion-free nilpotent group G (a \mathfrak{F} -group, in short). To any family \mathcal{X} of subgroups of G they associate the abstract Dirichlet series

$$\zeta_{\mathcal{X}}(s) = \sum_{H \in \mathcal{X}} |G : H|^{-s} = \sum_{n=1}^{\infty} a_n(\mathcal{X}) n^{-s},$$

where

$$a_n(\mathcal{X}) = |\{H \in \mathcal{X} \mid |G : H| = n\}|$$

(and $\infty^{-s} = 0$). Among the families whose study was initiated in [GrSS] are

$$\begin{aligned} \mathcal{S}(G) &= \{\text{all subgroups of finite index in } G\}, \\ \mathcal{N}(G) &= \{\text{all normal subgroups of finite index in } G\}, \\ \mathcal{H}(G) &= \{H \in \mathcal{S}(G) \mid \widehat{H} \cong \widehat{G}\}, \end{aligned}$$

where \widehat{G} denotes the profinite completion of the group G . These give rise to the zeta functions

$$\begin{aligned} \zeta_G^{\leq}(s) &= \zeta_{\mathcal{S}(G)}(s), \\ \zeta_G^{\triangleleft}(s) &= \zeta_{\mathcal{N}(G)}(s), \\ \widehat{\zeta}_G(s) &= \zeta_{\mathcal{H}(G)}(s). \end{aligned}$$

All these zeta functions satisfy an Euler product decomposition

$$\zeta_G^*(s) = \prod_{p \text{ prime}} \zeta_{G,p}^*(s), \quad * \in \{\leq, \triangleleft, \widehat{}\},$$

into a product of *local* zeta functions, where

$$\zeta_{G,p}^*(s) = \sum_{n=0}^{\infty} a_{p^n}^* p^{-ns}$$

and

$$a_{p^n}^{\leq} = a_{p^n}(\mathcal{S}), \quad a_{p^n}^{\triangleleft} = a_{p^n}(\mathcal{N}), \quad a_{p^n}^{\widehat{}} = a_{p^n}(\mathcal{H}).$$

It is the local normal zeta functions $\zeta_{G,p}^{\triangleleft}(s)$ we will concentrate on in this paper.

One of the main results of [GrSS] establishes the rationality of the local zeta functions in p^{-s} . It relies on the presentation of local zeta functions as p -adic integrals (recall that the *Hirsch length* of a \mathfrak{T} -group G is the number of infinite cyclic factors in a decomposition series for G):

Theorem 1 [GrSS, Prop. 3.1 and Thm. 4.1]. *Given a \mathfrak{T} -group G of Hirsch length $h(G) = n$. For $* \in \{\leq, \triangleleft, \widehat{}\}$ and almost all primes p*

$$\zeta_{G,p}^*(s) = (1 - p^{-s})^{-n} \int_{V_p^*} |m_{11}|^{s-1} \dots |m_{nn}|^{s-n} |dx|,$$

where $|m| = p^{-v_p(m)}$, v_p is the valuation on \mathbb{Z}_p , $|dx|$ is the normalized additive Haar measure on $\mathbb{Z}_p^{d(d+1)/2} \cong Tr_n(\mathbb{Z}_p)$, the triangular $n \times n$ -matrices over \mathbb{Z}_p with diagonal entries m_{ii} and suitable subsets $V_p^* \subseteq Tr_n(\mathbb{Z}_p)$.

Rationality is a consequence of the observation that the subsets V_p^* are *definable* in the language of fields. In this situation, a theorem of Denef’s [D1] is applicable which in turn relies on an application of Macintyre’s quantifier elimination for the theory of \mathbb{Q}_p [M] and on Hironaka’s theorem [H] on resolution of singularities in characteristic zero.

A major challenge in the field is to understand how the local zeta functions vary with the prime p . The zeta function $\zeta_G^*(s)$ is called *finitely uniform* if there are finitely many rational functions $V_i(X, Y) \in \mathbb{Q}(X, Y)$, $1 \leq i \leq r$, such that for each prime p there exists an i such that

$$\zeta_{G,p}^*(s) = V_i(p, p^{-s}), \tag{1}$$

and *uniform* if $r = 1$.

(Finite) uniformity is not typical for zeta functions of nilpotent groups, however: du Sautoy and Grunewald linked the question of the local factors' dependence on the prime p to the classical problem of counting points on varieties mod p . In [GrS1] they identify local zeta functions of groups as special cases of a more general class of p -adic integrals they call *cone integrals*:

Definition 1 [GrS1, Def.1.2]. *Let $f_i, g_i, i \in \{0, 1, \dots, l\}$, be polynomials over \mathbb{Q} in n variables. The condition*

$$\psi(\mathbf{x}) : v_p(f_i(\mathbf{x})) \leq v_p(g_i(\mathbf{x})) \text{ for } i \in \{1, \dots, l\} \tag{2}$$

is called a cone condition. A p -adic integral of the form

$$Z_{(\mathbf{f}, \mathbf{g})}(s, p) = \int_{\{\mathbf{x} \in \mathbb{Z}_p^n \mid \psi(\mathbf{x}) \text{ holds}\}} |f_0(\mathbf{x})|^s |g_0(\mathbf{x})| |dx|$$

is called a cone integral. The vector (\mathbf{f}, \mathbf{g}) is called cone integral data.

If the condition (2) is trivial (and $g_0 \equiv 1$) we recover Igusa's local zeta function (cf. [I1, Appendix], [D2]) as a special case of a cone integral. Writing local zeta functions of nilpotent groups as cone integrals not only allowed the authors of [GrS1] to dispense with the – in general mysterious – model-theoretic black box of quantifier elimination. It also enabled them to give an – in principal – very explicit expression for these functions.

Theorem 2 [GrS1, Thm.1.6]. *Let G be a \mathfrak{T} -group, and $* \in \{\leq, \triangleleft, \widehat{\cdot}\}$. There exists an algebraic variety Y^* defined over \mathbb{Q} , with irreducible components $E_i^*, i \in T^* := \{1, \dots, t^*\}$, all of which are smooth and intersect normally, and rational functions $P_I^*(X, Y) \in \mathbb{Q}(X, Y), I \subseteq T^*$ such that, for almost all primes p ,*

$$\zeta_{G,p}^*(s) = \sum_{I \subseteq T^*} c_{p,I}^* P_I^*(p, p^{-s}), \tag{3}$$

where

$$c_{p,I}^* = |\{a \in \overline{Y^*}(\mathbb{F}_p) : a \in \overline{E_i^*} \text{ if and only if } i \in I\}|,$$

and \overline{Y} means the reduction mod p of the variety Y .

The varieties Y^* arise as resolutions of singularities of the hypersurfaces $(\prod_{i=0}^l f_i^* g_i^* = 0)$, where $(\mathbf{f}^*, \mathbf{g}^*)$ is the cone integral data for the respective cone integrals. Theorem 2 suggests that finite uniformity should be the exception rather than the rule for zeta functions of nilpotent groups. But the question what varieties may appear in (3) in general remains wide open. Only recently du Sautoy presented the first example of a \mathfrak{T} -group G

for which neither $\zeta_G^{\leq}(s)$ nor $\zeta_G^{\triangleleft}(s)$ are finitely uniform, but depend on the number of \mathbb{F}_p -points of an elliptic curve [S2,3].

The first explicit formulae for non-uniform normal zeta functions, including du Sautoy’s examples, appeared in [V3]. (By abuse of language we will say ‘non-uniform’ for ‘not finitely uniform’.) Rather than evaluating cone integrals we introduced in this paper a much less coordinate-dependent calculus to compute local normal zeta functions of \mathfrak{T} -groups of nilpotency class 2. It is based on an enumeration of vertices in the affine Bruhat–Tits buildings associated to $Sl_n(\mathbb{Q}_p)$. In section 3 we will recall from [V3] what is necessary to make the current paper self-contained.

The examples computed in [V3] suggested that non-uniform zeta functions, too, might satisfy certain *local functional equations*. The (finitely uniform) examples computed before had all featured a functional equation of the form

$$V_i(X^{-1}, Y^{-1}) = (-1)^{l_i} X^{m_i} Y^{n_i} V_i(X, Y) \tag{4}$$

for integers l_i, m_i, n_i and $V_i(X, Y)$ as in (1) above, which defied explanation for $* \in \{\leq, \triangleleft\}$. In the non-uniform examples given in [V3], the uniform components showed symmetries like (4), too, matching with the functional equation of the Weil zeta function counting the number of \mathbb{F}_q -points of algebraic varieties to give a functional equation for the local zeta functions. But the nature of these uniform components and their functional equations remained poorly understood. In this paper we try to shed some light onto this phenomenon.

2 Statement of Results

Let now G be a \mathfrak{T} -group of nilpotency class 2 (a \mathfrak{T}_2 -group, in short) with derived group $G' := [G, G]$ and centre $Z(G)$. Only for simplicity we make the following

Assumption 1. G/G' and G' are torsion-free abelian of rank d and d' , respectively, and $G = Z(G)$.

Indeed, in general both $Z(G)/G'$ and G/G' are finitely generated abelian groups. But as we are only looking to prove results about all but finitely many of the local zeta functions $\zeta_{G,p}^{\triangleleft}(s)$ we loose nothing by restricting ourselves to primes p not dividing the orders of the respective torsion parts. And as one checks with no difficulty that

$$\zeta_{G \times \mathbb{Z}^r}^{\triangleleft}(s) = \zeta_G^{\triangleleft}(s) \cdot \zeta(s - n) \zeta(s - (n + 1)) \dots \zeta(s - (n + r - 1)) ,$$

where $h(G) = n$ and $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ is the Riemann zeta function, we may indeed assume that $Z(G) = G'$. Under Assumption 1 the group G has a presentation

$$G = \langle x_1, \dots, x_d, y_1, \dots, y_{d'} \mid [x_i, x_j] = M(\mathbf{y})_{ij} \rangle, \tag{5}$$

where $M(\mathbf{y})$ is an anti-symmetric $d \times d$ -matrix of \mathbb{Z} -linear forms in $\mathbf{y} = (y_1, \dots, y_{d'})$, and all other commutators are trivial. Note that we adopted additive notation for words in $Z(G)$. Conversely, of course, every such matrix $M(\mathbf{y})$ defines a \mathfrak{T}_2 -group via (5).

If the polynomial $\text{Pf}(M(\mathbf{y})) := \sqrt{\det(M(\mathbf{y}))}$ is not identical zero we call the hypersurface \mathfrak{P}_G in $\mathbb{P}^{d'-1}$ defined over the integers by $\text{Pf}(M(\mathbf{y})) = 0$ the *Pfaffian hypersurface* associated to G . Given a fixed prime number p we then denote by $\overline{\mathfrak{P}_G}$ its reduction modulo p . Our main result is

Theorem 3. *Assume that $\text{Pf}(M(\mathbf{y})) \in \mathbb{Z}[\mathbf{y}]$ is non-zero and irreducible. Assume that the Pfaffian hypersurface \mathfrak{P}_G is smooth and contains no lines. For a prime p let*

$$n_{\mathfrak{P}_G}(p) = |\mathfrak{P}_G(\mathbb{F}_p)|$$

denote the number of \mathbb{F}_p -rational points of \mathfrak{P}_G . Then there are (explicitly determined) rational functions $W_0(X, Y), W_1(X, Y) \in \mathbb{Q}(X, Y)$ such that if \mathfrak{P}_G has good reduction mod p

$$\zeta_{G,p}^{\triangleleft}(s) = W_0(p, p^{-s}) + n_{\mathfrak{P}_G}(p) \cdot W_1(p, p^{-s}). \tag{6}$$

Corollary 1. *The following functional equation holds:*

$$\zeta_{G,p}^{\triangleleft}(s)|_{p \rightarrow p^{-1}} = (-1)^{d+d'} p^{\binom{d+d'}{2} - (2d+d')s} \zeta_{G,p}^{\triangleleft}(s). \tag{7}$$

We would like to remark that the conditions of Theorem 3 on the \mathfrak{T}_2 -group G are *generically* satisfied for small d' and large d ($= 2r$, say): A presentation as in (5) specifies a \mathbb{Z} -linear embedding of a $\mathbb{P}^{d'-1}$ into the projective space $\mathbb{P}(\mathcal{S})$ over the vector space \mathcal{S} of anti-symmetric $(d \times d)$ -matrices. The Pfaffian hypersurface \mathfrak{P}_G is just the intersection of this $\mathbb{P}^{d'-1}$ with the universal Pfaffian hypersurface $\mathcal{X}_r \subset \mathbb{P}(\mathcal{S})$ of singular matrices. The singular locus of the latter consists of matrices of rank $\leq 2r - 4$ and has codimension 6. A *generic* $\mathbb{P}^{d'-1} \subset \mathbb{P}(\mathcal{S})$ therefore intersects \mathcal{X}_r along a *smooth* hypersurface in $\mathbb{P}^{d'-1}$ of degree r if $d' \leq 6$ (this is remark (8.3) in [B]).

More work is required to see that for all d' and for $d > 4d' - 10$ a generic Pfaffian hypersurface \mathfrak{P}_G will not contain lines. This follows immediately from the following proposition. It is due to Arnaud Beauville and we cordially thank him for his contribution of its proof as an Appendix to this paper.

PROPOSITION 1. *A generic Pfaffian hypersurface of degree $r > 2n - 3$ in \mathbb{P}^n contains no lines.*

In view of this remark it seems to us as if *smoothness* of the Pfaffian \mathfrak{P}_G was the most restrictive condition on the \mathfrak{T}_2 -group G in Theorem 3. In [V2] we examine examples showing that for singular Pfaffians \mathfrak{P}_G the expressions for the zeta functions of G will reflect the rank-stratification of the determinantal variety \mathfrak{P}_G . In section 3 we shall comment briefly on the technical reasons for the condition on the Pfaffian hypersurface to contain no lines. The first inroads in overcoming these obstacles were made by Pirita Paaajanen, a student of du Sautoy, who derived the following explicit formula for the local normal zeta functions of $F_{2,4}$, the free class-2-nilpotent group on four generators. In this example the Pfaffian hypersurface \mathfrak{P}_G is a smooth quadric fourfold.

PROPOSITION 2 (Paaajanen [P]). *Let $G = F_{2,4}$. Then for all primes p*

$$\zeta_{G,p}^{\leq}(s) = W_0(p, p^{-s}) + n_{\mathfrak{P}_G}^{(1)}(p)W_1(p, p^{-s}) + n_{\mathfrak{P}_G}^{(2)}(p)W_2(p, p^{-s}) + n_{\mathfrak{P}_G}^{(3)}(p)W_3(p, p^{-s}),$$

where

$$\begin{aligned} n_{\mathfrak{P}_G}^{(1)}(p) &= (p^2 + 1)(p^2 + p + 1), \\ n_{\mathfrak{P}_G}^{(2)}(p) &= (p + 1)(p^2 + 1)(p^2 + p + 1), \\ n_{\mathfrak{P}_G}^{(3)}(p) &= 2(p^2 + 1)(p + 1), \end{aligned}$$

denotes the number of \mathbb{F}_p -rational points of the Fano varieties of $(i - 1)$ -dimensional subspaces on \mathfrak{P}_G , and $W_i(X, Y)$ are rational functions. The functional equation (7) holds.

Note added in proof. For explicit formulae for the normal zeta functions of all free class-2-nilpotent groups $F_{2,d}$, $d \geq 2$, see [V5].

The proof of Theorem 3 will be given in two steps: First we prove it in the case $n_{\mathfrak{P}_G}(p) = 0$ to obtain the rational function $W_0(p, p^{-s})$. Note that there are \mathfrak{T}_2 -groups for which $n_{\mathfrak{P}_G}(p) = 0$ for infinitely many primes p . (Indeed, if $G = H(\mathfrak{o}_K)$ is the Heisenberg group over the ring of integers of an algebraic number field K , these are exactly the *inert* primes. The remaining primes are harder to deal with as the associated Pfaffian will not be smooth.) Loosely speaking, if p is a prime for which \mathfrak{P}_G defines a smooth non-empty hypersurface in projective $(d' - 1)$ -space over \mathbb{F}_p without lines we will need to ‘correct’ the rational function $W_0(p, p^{-s})$ along the

$n_{\mathfrak{P}_G}(p)$ \mathbb{F}_p -points of \mathfrak{P}_G by the function $W_1(p, p^{-s})$ to obtain the p -th local normal zeta function. This will constitute the second step in the proof of Theorem 3.

To prove Corollary 1 we shall demonstrate that the functional equation (7) is due to the interplay of two phenomena: We will firstly recall how the *functional equation* of the Hasse–Weil zeta function associated to the hypersurface \mathfrak{P}_G gives rise to a symmetry of the expression $n_{\mathfrak{P}_G}(p)$ as a function of p (and how this zeta function’s rationality gives sense to the symbol $n_{\mathfrak{P}_G}(p^{-1})$ in the left-hand side of (7) in the first place). Secondly we shall show that the uniform components $W_i(X, Y)$, $i \in \{0, 1\}$, in (6) satisfy a symmetry of the form (4). Our main tool to achieve this will be Theorem 4 below, which is essentially due to Igusa [I2]. It establishes such a symmetry for a single rational function (8) which is defined in terms of flag varieties.

Theorem 4. *Let $n \geq 2$ be an integer, X_1, \dots, X_{n-1} independent indeterminates and q a prime power. For $I \subseteq \{1, \dots, n - 1\}$ let $b_I(q) = |\mathcal{F}_I(\mathbb{F}_q)| \in \mathbb{Z}[q]$ denote the number of \mathbb{F}_q -points of the projective variety of flags in \mathbb{F}_q^n of type I . Set*

$$F_n(q, \mathbf{X}) := \sum_{I \subseteq \{1, \dots, n-1\}} b_I(q) \prod_{i \in I} \frac{X_i}{1 - X_i}. \tag{8}$$

Then

$$F_n(q^{-1}, \mathbf{X}^{-1}) = (-1)^{n-1} q^{-\binom{n}{2}} F_n(q, \mathbf{X}). \tag{9}$$

(See section 3 for the definition of ‘flag of type I ’.)

We will show how (the crucial factors of) the rational functions $W_i(p, p^{-s})$ in (6) may be derived from functions of type (8) by suitable substitutions of variables. The $W_i(p, p^{-s})$ ‘inherit’ the functional equation (9) as the quotient $F_n(q^{-1}, \mathbf{X}^{-1})/F_n(q, \mathbf{X})$ is independent of the ‘numerical data’ \mathbf{X} .

The zeta functions introduced by Igusa [I2] are defined in terms of root systems of algebraic groups. In fact we only need the most basic of these, the one associated to Gl_n . Our formulation (and our elementary proof) of Theorem 4 in the language of flag varieties seems natural from the point of view taken in [GrS1], whereas a connection to algebraic groups seems elusive in the context of normal zeta functions of groups.

In [SL], however, du Sautoy and Lubotzky interpret the zeta functions $\zeta^{\sim}(s)$ (where G is again a general \mathfrak{T} -group) as p -adic integrals over the algebraic automorphism group of the Lie algebra associated to G . A generalisation of Igusa’s work [I2] allows them to derive uniformity as well

as local functional equations of these zeta functions for certain classes of \mathfrak{T} -groups.

In [DM] Denef and Meuser prove a functional equation for the Igusa local zeta function associated to a homogeneous polynomial. These zeta functions also have an expression of the form (3). However, the ‘uniform components’ occurring in this context show rather less structure than the rational functions $W_i(X, Y)$ in Theorem 3 of the current paper.

As another application of Theorem 4 we derive both explicit formulae and local functional equations for the normal zeta functions of another infinite family of \mathfrak{T}_2 -groups known as ‘Grenham groups’ (cf. [S1, Chap. 5.8], or [GrS2, Chap. 6.3]). For $n \geq 2$ let

$$G_n := \langle x_1, \dots, x_n, y_1, \dots, y_{n-1} \mid [x_i, x_n] = y_i, 1 \leq i \leq n, \text{ all other } [,] \text{ trivial} \rangle.$$

The group G_n may be thought of as $n - 1$ copies of the discrete Heisenberg group – the group of 3×3 -upper uni-triangular matrices with integer entries – with one off-diagonal entry identified in each copy.

Theorem 5. For all primes p

$$\zeta_{G_n, p}^{\triangleleft}(s) = \zeta_{\mathbb{Z}_p^n}(s) \zeta_p((2n - 1)s - n(n - 1)) F_{n-1}(p^{-1}, \mathbf{X}),$$

where $\mathbf{X} = (X_1, \dots, X_{n-2})$ and

$$X_i = p^{-(2(n-i)-1)s + (n+i)(n-i-1)} \text{ for } i \in \{1, \dots, n - 2\}.$$

In particular, the following functional equation holds:

$$\zeta_{G_n, p}^{\triangleleft}(s)|_{p \rightarrow p^{-1}} = -p^{\binom{2n-1}{2} - (3n-1)s} \zeta_{G_n, p}^{\triangleleft}(s). \tag{10}$$

In the forthcoming paper [V1] we use our method to compute explicitly all the *subgroup* zeta functions $\zeta_{G_n}^{\leq}(s)$ and prove that

$$\zeta_{G_n, p}^{\leq}(s)|_{p \rightarrow p^{-1}} = -p^{\binom{2n-1}{2} - (2n-1)s} \zeta_{G_n, p}^{\leq}(s). \tag{11}$$

The functional equations (10) and (11) were conjectured by du Sautoy (Conjecture 5.41 in [S1]).

We will prove Theorem 4 in section 4.1 by an argument using the Schubert cell decomposition of flag varieties. The proofs of Theorem 3 and Theorem 5 will occupy sections 4.2 and 4.3, respectively. The point of view taken is the one developed in [V3] (where also the special cases of Theorem 3 for $d' \in \{2, 3\}$ were proved). There the *Cartan decomposition* for lattices in the centre of G was used to interpret the local zeta functions as generating functions associated to certain weight functions on the vertices of the Bruhat–Tits building $\Delta_{d'}$ for $Sl_{d'}(\mathbb{Q}_p)$, exhibiting its dependence on the geometry of the Pfaffian hypersurface \mathfrak{P}_G . We will recall briefly the

main results of [V3] in section 3 together with some basic definitions and observations about lattices and flags.

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3 Flags and Lattices

In this section we give a brief summary of the method developed in [V3] to compute local normal zeta functions.

Let G be a \mathfrak{A}_2 -group satisfying Assumption 1. For a fixed prime p , the computation of the p -th normal local zeta function of G comes down to an enumeration of those lattices in the \mathbb{Z}_p -Lie algebra (with Lie brackets induced by taking commutators)

$$G_p := (G/Z(G) \oplus Z(G)) \otimes_{\mathbb{Z}} \mathbb{Z}_p,$$

which are *ideals* in G_p . We call a lattice $\Lambda \subseteq \mathbb{Z}_p^n$ *maximal* (in its homothety class) if $p^{-1}\Lambda \not\subseteq \mathbb{Z}_p^n$. The key observation is the following

Lemma 1 [GrSS, Lemma 6.1]. *For each lattice $\Lambda' \subseteq G'_p$ put $X(\Lambda')/\Lambda' = Z(G_p/\Lambda')$. Then*

$$\begin{aligned} \zeta_{G,p}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^d}(s) \sum_{\Lambda' \subseteq G'_p} |G'_p : \Lambda'|^{d-s} |G_p : X(\Lambda')|^{-s} \\ &= \zeta_{\mathbb{Z}_p^d}(s) \zeta_p((d + d')s - dd') \underbrace{\sum_{\substack{\Lambda' \subseteq G'_p \\ \Lambda' \text{ maximal}}} |G'_p : \Lambda'|^{d-s} |G_p : X(\Lambda')|^{-s}}_{=: A(p, p^{-s}), \text{ say}}. \end{aligned}$$

Corollary 2.

$$\begin{aligned} A(p, p^{-s})|_{p \rightarrow p^{-1}} &= (-1)^{d'-1} p^{\binom{d'}{2}} A(p, p^{-s}) \\ \iff \zeta_{G,p}^{\triangleleft}(s)|_{p \rightarrow p^{-1}} &= (-1)^{d+d'} p^{\binom{d+d'}{2} - (2d+d')s} \zeta_{G,p}^{\triangleleft}(s). \quad \square \end{aligned}$$

Recall that (homothety classes defined by) maximal lattices are in one-to-one correspondence with vertices of the Bruhat–Tits building $\Delta_{d'}$ for $Sl_{d'}(\mathbb{Q}_p)$ (e.g. [Ga, §19]). (Two lattices $\Lambda, \Lambda' \subseteq \mathbb{Q}_p^{d'}$ are called *homothetic* if there is a non-zero constant $c \in \mathbb{Q}_p$ such that $c\Lambda = \Lambda'$.) To derive

an explicit formula for $A(p, p^{-s})$ requires a quantitative understanding of two integer-valued functions w and w' on the vertex set of the simplicial complex $\Delta_{d'}$. We write

$$A(p, p^{-s}) = \sum_{[\Lambda']} p^{dw([\Lambda']) - sw'([\Lambda'])}, \tag{12}$$

where, for a homothety class $[\Lambda']$ of a maximal lattice Λ' in $G'_p \cong \mathbb{Z}_p^{d'}$ we define

$$\begin{aligned} w([\Lambda']) &:= \log_p (|G'_p : \Lambda'|), \\ w'([\Lambda']) &:= w([\Lambda']) + \log_p (|G_p : X(\Lambda')|). \end{aligned}$$

In order to describe the dependence of these functions on the lattice Λ' we will introduce some notation. For an integer $m \in \mathbb{N}_{>0}$ we set $[m] := \{1, \dots, m\}$. A maximal lattice $\Lambda' \subseteq \mathbb{Z}_p^{d'}$ is said to be of type $\nu(\Lambda') = (I, \mathbf{r}_I)$ if

$$I = \{i_1, \dots, i_l\} \subseteq [d' - 1], \quad \mathbf{r}_I = (r_{i_1}, \dots, r_{i_l}), \quad i_1 < \dots < i_l, \tag{13}$$

and $r_{i_j} \in \mathbb{N}_{>0}$ for $j \in \{1, \dots, l\}$ and Λ' has elementary divisors

$$\left(\underbrace{1, \dots, 1}_{i_1}, \underbrace{p^{r_{i_1}}, \dots, p^{r_{i_1}}}_{i_2 - i_1}, \underbrace{p^{r_{i_1} + r_{i_2}}, \dots, p^{r_{i_1} + r_{i_2}}}_{i_3 - i_2}, \dots, \underbrace{p^{\sum_{j=1}^l r_{i_j}}, \dots, p^{\sum_{j=1}^l r_{i_j}}}_{d' - i_l} \right) =: (p^\nu). \tag{14}$$

By slight abuse of notation we may say that a maximal lattice is of type I if it is of type (I, \mathbf{r}_I) for some positive vector \mathbf{r}_I , and that the homothety class $[\Lambda']$ has type I if its maximal element has type I , in which case we write $\nu([\Lambda']) = I$. For computations it will be advantageous to write

$$\begin{aligned} A(p, p^{-s}) &= \sum_{I \subseteq [d' - 1]} A_I(p, p^{-s}), \quad \text{where} \tag{15} \\ A_I(p, p^{-s}) &:= \sum_{\nu([\Lambda']) = I} p^{dw([\Lambda']) - sw'([\Lambda'])}. \end{aligned}$$

Notice that the lattice's index – and thus $w([\Lambda'])$ – is given by

$$|\mathbb{Z}_p^{d'} : \Lambda'| = p^{\sum_{i \in I} r_i (d' - i)} = p^{w([\Lambda'])}. \tag{16}$$

We shall explain how the evaluation of w' may be reduced to solving linear congruences. The group $\Gamma = Sl_{d'}(\mathbb{Z}_p)$ acts transitively on the set of maximal lattices of fixed type. If we choose a basis for the \mathbb{Z}_p -module G'_p , represent lattices as the row span of $d' \times d'$ -matrices and denote by Γ_ν the stabilizer in Γ of the lattice generated by the diagonal matrix whose entries are given by the vector (14), the orbit-stabiliser theorem gives us a 1–1

correspondence

$$\{\text{maximal lattices of type } (I, \mathbf{r}_I)\} \xleftrightarrow{1-1} \Gamma/\Gamma_\nu. \tag{17}$$

This correspondence allows us to describe $|G_p : X(\Lambda')|$ – and thus $w'([\Lambda'])$ – for a maximal lattice Λ' in terms of $M(\mathbf{y})$, the matrix of commutators in a presentation for G as in (5).

Theorem 6 [V3, §2.2]. *Let Λ' correspond to the coset $\alpha\Gamma_\nu$ under (17), where $\alpha \in \Gamma$ with column vectors α^j , $j = 1, \dots, d'$. Then $|G_p : X(\Lambda')|$ equals the index of the kernel of the following system of linear congruences in G_p/G'_p :*

$$\forall i \in \{1, \dots, d'\} \quad \bar{\mathbf{g}}M(\alpha^i) \equiv 0 \pmod{(p^\nu)_i}, \tag{18}$$

where $\bar{\mathbf{g}} = (\bar{g}_1, \dots, \bar{g}_d) \in G_p/G'_p \cong \mathbb{Z}_p^d$ and $(p^\nu)_i$ denotes the i -th entry of the vector (p^ν) given in (14).

A flag of type I in $\mathbb{P}^{d'-1}(\mathbb{F}_p)$, $I \subseteq [d' - 1]$ as in (13), is a sequence $(V_i)_{i \in I}$ of incident vector spaces

$$\mathbb{P}^{d'-1}(\mathbb{F}_p) > V_{i_1} > \dots > V_{i_l} > \{0\}$$

with $\text{codim}_{\mathbb{F}_p}(V_i) = i$. A flag is called *incomplete* (or *partial*) if $I \neq [d' - 1]$, and *complete* otherwise. The flags of type I form a projective variety \mathcal{F}_I , whose number of \mathbb{F}_p -points is given by $b_I(p) \in \mathbb{Z}[p]$, a polynomial whose leading term equals $p^{\dim \mathcal{F}_I}$. These polynomials are easily expressed in terms of p -binomial coefficients, but we will not make use of this fact here. It is easy to see that for all $I \subseteq [n - 1]$

$$b_I(p^{-1}) = p^{-\dim \mathcal{F}_I} b_I(p). \tag{19}$$

Given $\nu = (I, \mathbf{r}_I)$ as above, let $f(I, \mathbf{r}_I, p) = |\Gamma/\Gamma_\nu|$ be the number of maximal lattices in $\mathbb{Z}_p^{d'}$ of type ν . Using (17) one easily proves

Lemma 2. $f(I, \mathbf{r}_I, p) = b_I(p) p^{\sum_{i \in I} r_i(d'-i) - \dim \mathcal{F}_I}$.

Given $\alpha \in \Gamma$ and I as in (13), let $\bar{\alpha}$ denote the reduction mod p and define vector spaces

$$V_i := \langle \overline{\alpha^{i+1}}, \dots, \overline{\alpha^{d'}} \rangle < \mathbb{P}^{d'-1}(\mathbb{F}_p), \quad i \in I.$$

Clearly $\text{codim}(V_i) = i$. We will call the flag $(V_i)_{i \in I}$ of type I *the flag associated to Λ'* if $\nu(\Lambda') = I$ and Λ' corresponds to $\alpha\Gamma_\nu$ under (17). (It is indeed straightforward to show that this is well defined, i.e. independent of the coset representative α). Given a fixed point $x \in \mathbb{P}^{d'-1}(\mathbb{F}_p)$, we call a maximal lattice Λ' *a lift of x* if its associated flag contains x as 0-dimensional member and we shall write $x = P([\Lambda'])$. Note that then necessarily $d' - 1 \in \nu([\Lambda'])$.

We can now explain why we made the assumption that \mathfrak{P}_G should be smooth and contain no lines. In the latter case, (18) is equivalent to

$$\begin{aligned} \bar{\mathfrak{g}} &\equiv 0 \pmod{p^{\sum_{i \in I \setminus \{d'-1\}} r_i}} \\ p^{-\sum_{i \in I \setminus \{d'-1\}} r_i} \bar{\mathfrak{g}} M(\alpha^{d'}) &\equiv 0 \pmod{p^{r_{d'-1}}} \end{aligned} \tag{20}$$

(where the congruence (3) is regarded trivial if $d' - 1 \notin I$). This follows easily from the observation that under this assumption the vectors $\alpha^1, \dots, \alpha^{d'-1}$ may always be chosen to lie outside the Pfaffian hypersurface, i.e. such that $v_p(\det(M(\alpha^i))) = 0$ for $i \in \{1, \dots, d' - 1\}$. In the case $d' - 1 \in I$ we are left to analyse the elementary divisors of the matrix $M(\alpha^{d'})$. But by an easy geometrical argument (cf. [V2, Lemma 1]) one sees that if the Pfaffian is smooth this matrix always has a $(d - 2) \times (d - 2)$ -unit minor. (This may or may not be the case if \mathfrak{P}_G is *singular* (cf. [V2]).) Thus the challenge to compute the weight function $w'([\Lambda'])$ is essentially reduced to the problem of determining the p -adic valuation $v_p(\det(M(\alpha^{d'})))$ (cf. equation (33)).

4 Proofs

4.1 Proof of Theorem 4. Choosing a common denominator for the sum $F_n(q, \mathbf{X})$ we write

$$F_n(q, \mathbf{X}) = \frac{f_n(q, \mathbf{X})}{\prod_{i=1}^{n-1} (1 - X_i)},$$

where

$$\begin{aligned} f_n(q, \mathbf{X}) &= \sum_{I \subseteq [n-1]} b_I(q) \prod_{i \in I} X_i \prod_{j \notin I} (1 - X_j) \\ &= \sum_{I \subseteq [n-1]} c_I(q) \prod_{i \in I} X_i, \text{ say.} \end{aligned} \tag{21}$$

Then

$$c_I(q) = \sum_{J \subseteq I \subseteq [n-1]} (-1)^{|I|-|J|} b_J(q) \in \mathbb{Z}[q].$$

For a subset $I \subseteq [n - 1]$ we define $I^c := [n - 1] \setminus I$. Theorem 4 will follow if we can prove

$$c_I(q^{-1}) = q^{-\binom{n}{2}} c_{I^c}(q) \quad \forall I \subseteq [n - 1]$$

or, equivalently, if $c_I(q) = \sum_k a_{k,I} q^k$, that

$$a_{k,I} = a_{\binom{n}{2}-k, I^c} \quad \forall 0 \leq k \leq \binom{n}{2}, \quad I \subseteq [n - 1]. \tag{22}$$

We shall prove (22) by showing that $a_{k,I}$ enumerates the k -dimensional Schubert cells of type I in the complete flag variety $\mathcal{F}_{[n-1]}$ and that post-multiplication by the longest word induces a 1-1-correspondence between k -dimensional cells of type I and $\binom{n}{2} - k$ -dimensional cells of type I^c .

Firstly recall that the variety $\mathcal{F}_{[n-1]}$ of complete flags in \mathbb{F}_q^n has a cell decomposition into Schubert cells Ω_w , labelled by $w \in S_n$, the symmetric group on n letters (e.g. [Ma, Chap. 3]). We write $w = (w_1 \dots w_n)$ if $w(i) = w_i$. Let us represent a complete flag by an $n \times n$ -matrix over \mathbb{F}_q : its r -th member is generated by the first r rows. For example ([Ma, p. 134]) an element of Ω_w , $w = (365142) \in S_6$ has a unique matrix representative of the form

$$\begin{pmatrix} * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & * & 1 \\ * & * & 0 & * & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Replacing the $*$'s by independent variables identifies the Schubert cell Ω_w with affine space over \mathbb{F}_q of dimension $l(w)$, where l denotes the usual length function, the number of inversions of w . The flag variety is now just the disjoint union of these affine spaces Ω_w .

The expression (21) involves the cardinalities of the 2^{n-1} varieties \mathcal{F}_I , $I \subset [n-1]$. The cell decomposition of $\mathcal{F}_{[n-1]}$ will allow us to accommodate all of the \mathcal{F}_I in one object by identifying them with certain unions of Schubert cells.

To that end we define the type $\nu(w)$ of a permutation w to be the smallest subset I of $[n-1]$ such that the natural surjection $G/B \rightarrow G/P_I$ of complete flags onto flags of type I is a bijection if restricted to Ω_w . Alternatively, given $w = (w_1 \dots w_n)$, set

$$\nu(w) := \{i \in [n-1] \mid w_{i+1} < w_i\}.$$

So for w as in our example above we have $\nu(w) = \{2, 3, 5\}$, the type of the longest word $w_0 = (n n - 1 \dots 2 1)$ equals $[n-1]$, and the type of the identity element is the empty set. We have a bijection of sets

$$\coprod_{\nu(w) \subseteq I} \Omega_w \xrightarrow{1-1} \mathcal{F}_I. \tag{23}$$

From (23) it follows immediately that

$$\sum_{\nu(w) \subseteq I} |\Omega_w| = b_I(q) \quad \text{and}$$

$$\sum_{\nu(w)=I} |\Omega_w| = c_I(q).$$

As the Schubert cells Ω_w are identified with some affine $l(w)$ -space, their cardinalities are just powers of q and $a_{k,I}(q)$, the k -th coefficient of the polynomial $c_I(q)$, counts the number of Schubert cells of type I of dimension k . Theorem 4 will follow from the following

PROPOSITION 3. *Let $w_0 = (n\ n-1\ \dots\ 2\ 1) \in S_n$ be the longest word. The bijection $w \mapsto w w_0$ induces bijections*

$$\{\Omega_w \mid \nu(w) = I\} \xrightarrow{1-1} \{\Omega_w \mid \nu(w) = I^e\}.$$

We have $l(w w_0) = \binom{n}{2} - l(w)$.

Proof. This follows easily from the definition of the type of a permutation and a comparison of the Schubert cells Ω_w and $\Omega_{w w_0}$ (e.g. as sets of matrices). □

4.2 Proof of Theorem 3 and Corollary 1.

4.2.1 The case $n_{\mathfrak{P}_G}(\mathbf{p}) = 0$. First we deal with the case that the Pfaffian hypersurface \mathfrak{P}_G has no \mathbb{F}_p -rational points. Thus $\det(M(\alpha))$ is a p -adic unit for all $\alpha \in \mathbb{Z}_p^{d'} \setminus p\mathbb{Z}_p^{d'}$ (i.e. for all column vectors of matrices in $Sl_{d'}(\mathbb{Z}_p)$) and (18) is equivalent to the single congruence

$$\bar{\mathbf{g}} \equiv 0 \pmod{p^{\sum_{j=1}^l r_{i_j}}}.$$

Hence

$$|G_p : X(\Lambda')| = p^{d \sum_{j=1}^l r_{i_j}} \tag{24}$$

and

$$w'([\Lambda']) = \sum_{i \in \nu([\Lambda'])} r_i(d + d' - i).$$

Thus

$$\begin{aligned} A(p, p^{-s}) &= \sum_{I \subseteq [d'-1]} A_I(p, p^{-s}) \\ &= \sum_{I \subseteq [d'-1]} \sum_{\mathbf{r}_I > 0} f(I, \mathbf{r}_I, p) \cdot p^{d \sum r_i(d'-i) - s \sum r_i(d+d'-i)} \\ &\stackrel{*}{=} \sum_{I \subseteq [d'-1]} \frac{b_I(p)}{p^{\dim \mathcal{F}_I}} \sum_{\mathbf{r}_I > 0} p^{\sum r_i(d+i)(d'-i) - s \sum r_i(d+d'-i)} \\ &\stackrel{**}{=} \sum_{I \subseteq [d'-1]} b_I(p^{-1}) \prod_{i \in I} \frac{X_i}{1 - X_i} r \end{aligned}$$

$$=: \sum_{I \subseteq [d'-1]} F_{d'}(I, p^{-1}, \mathbf{X}_I) = F_{d'}(p^{-1}, \mathbf{X}).$$

Here we used Lemma 2 for equality (*) and equation (19) to obtain (**). For $1 \leq i \leq d' - 1$, we made the substitutions

$$X_i := p^{(d+i)(d'-i)-s(d+d'-i)}. \tag{25}$$

Here \mathbf{X}_I stands for $(X_i)_{i \in I}$, $\mathbf{r}_I > 0$ for $\mathbf{r}_I \in \mathbb{N}_{>0}^I$ and \sum for $\sum_{i \in I}$. Theorem 3 and its Corollary 1 follow now from Theorem 4 together with Corollary 2.

4.2.2 The case $n_{\mathfrak{P}_G}(\mathbf{p}) > 0$. In this case $F_{d'}(p^{-1}, \mathbf{X})$ fails to represent the generating function $A(p, p^{-s})$, as (24) will not hold in general. We shall see, however, that the two rational functions agree ‘almost everywhere’ and we will show how to decompose them into summands in a geometrically meaningful way to see exactly where and how they differ. As we assume that the Pfaffian contains no lines we have (by (3))

$$A_I(p, p^{-s}) = F_{d'}(I, p^{-1}, \mathbf{X}_I) \text{ if } d' - 1 \notin I. \tag{26}$$

For a point $x \in \mathbb{P}^{d'-1}(\mathbb{F}_p)$ and $\mathbf{X}' := (X_i)_{i \in [d'-2]}$ we set

$$A(x, p, p^{-s}) := \sum_{d'-1 \in \nu([\Lambda']) \cap \mathfrak{P}([\Lambda']) = x} p^{w([\Lambda']) \cdot d - s w'([\Lambda'])} \tag{27}$$

$$F_{d',0}(p^{-1}, \mathbf{X}') := F_{d'-1}(p^{-1}, \mathbf{X}') p^{-(d'-1)} \frac{X_{d'-1}}{1 - X_{d'-1}}. \tag{28}$$

The rational function (27) might be thought of as the generating function (12) with summation restricted to the maximal lattices lifting a fixed point $x \in \mathbb{P}^{d'-1}(\mathbb{F}_p)$. It agrees with (28) if and only if $\det(M(x)) \neq 0 \in \mathbb{F}_p$, i.e. if $x \notin \overline{\mathfrak{P}_G}$,

$$A(x, p, p^{-s}) = F_{d',0}(p^{-1}, \mathbf{X}') \text{ if } x \notin \overline{\mathfrak{P}_G}. \tag{29}$$

If we write

$$A(p, p^{-s}) = \sum_{d'-1 \notin I} A_I(p, p^{-s}) + \sum_{x \in \mathbb{P}^{d'-1}(\mathbb{F}_p)} A(x, p, p^{-s}),$$

$$F_{d'}(p^{-1}, \mathbf{X}) = \sum_{d'-1 \notin I} F_{d'}(I, p^{-1}, \mathbf{X}_I) + \binom{d'}{1}_p F_{d',0}(p^{-1}, \mathbf{X}'),$$

we see that in order to prove Theorem 3 we are left with the challenge to prove

PROPOSITION 4. *Let $x \in \overline{\mathfrak{P}_G}$. Then*

$$A_{d'}(x, p, p^{-s}) - F_{d',0}(p^{-1}, \mathbf{X}') = F_{d'-1}(p^{-1}, \mathbf{X}') p^{-(d'-1)} \frac{pY - X_{d'-1}}{(1 - X_{d'-1})(1 - Y)}, \tag{30}$$

where

$$\begin{aligned} X_{d'-1} &:= p^{d+d'-1}T^{d+1} \quad (\text{in accordance with (25)}) \text{ and} \\ Y &:= p^{d+d'-2}T^{d-1}. \end{aligned}$$

In particular, the function $A_{d'}(x, p, p^{-s}) - F_{d',0}(p^{-1}, \mathbf{X}')$ is independent of x .

Indeed, together with equations (26) and (29) Proposition 4 clearly implies

$$\begin{aligned} A(p, p^{-s}) &= F_{d'}(p^{-1}, \mathbf{X}) + (A(p, p^{-s}) - F_{d'}(p^{-1}, \mathbf{X})) \\ &= F_{d'}(p^{-1}, \mathbf{X}) + n_{\mathfrak{P}_G}(p)(A_{d'}(x, p, p^{-s}) - F_{d',0}(p^{-1}, \mathbf{X}')) \\ &= F_{d'}(p^{-1}, \mathbf{X}) + n_{\mathfrak{P}_G}(p)F_{d'-1}(p^{-1}, \mathbf{X}')p^{-(d'-1)} \frac{pY - X_{d'-1}}{(1 - X_{d'-1})(1 - Y)}. \end{aligned} \tag{31}$$

Proof of Proposition 4. Let J_2 denote the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Locally around any of the $n_{\mathfrak{P}_G}(p)$ points of the Pfaffian mod p , the admissibility conditions (18) look like

$$\begin{aligned} \bar{\mathbf{g}} \cdot \text{diag} \left(\left(\begin{matrix} 0 & x_1 \\ -x_1 & 0 \end{matrix} \right), J_2, \dots, J_2 \right) &\equiv 0 \pmod{p^{\sum_{j=1}^l r_{i_j}}} \tag{32} \\ \bar{\mathbf{g}} &\equiv 0 \pmod{p^{\sum_{j=1}^{l-1} r_{i_j}}}, \end{aligned}$$

where $\mathbf{x} = (x_1 : \dots : x_{d'}) \in \mathbb{P}^{d'-1}(\mathbb{Z}_p/(p^{r_{d'-1}}))$, $\mathbf{x} \equiv (0 : 1 : \dots : 1) \pmod{p}$ and we have

$$w'([\Lambda']) = \sum_{i \in \nu([\Lambda'])} r_i(d + d' - i) - 2 \min \{ r_{d'-1}, v_p(x_1) \}. \tag{33}$$

Therefore

$$A_{d'}(x, p, p^{-s}) = B_0(p, p^{-s})F_{d'-1}(p^{-1}, \mathbf{X}'), \tag{34}$$

say, where

$$B_0(p, p^{-s}) := \sum_{\substack{r_{d'-1} > 0 \\ (\mathbf{x}) \in \mathbb{P}^{d'-1}(\mathbb{Z}_p/(p^{r_{d'-1}})) \\ \mathbf{x} \equiv (0:1:\dots:1) \pmod{p}}} p^{dr_{d'-1} - s(1+d-2 \min\{r_{d'-1}, v_p(x_1)\})}. \tag{35}$$

We see that in the present case the weight function $w'([\Lambda'])$ given in (33) depends on more than just the lattice's type. To get an explicit expression for the function $B_0(p, p^{-s})$, we must find a way to eliminate the term ‘min’ in the sum (35). In other words, we must answer the following question: Given $(a, b) \in \mathbb{N}_{>0}^2$, how many of the lifts $(x_1 : \dots : x_{d'})$ of the point $(0 : 1 : \dots : 1) \in \mathbb{P}^{d'-1}(\mathbb{F}_p)$ to points mod p^a are there with $v_p(x_1) = b$?

Lemma 3. For $(a, b) \in N := \{(x, y) \in \mathbb{N}_{>0}^2 \mid x \geq y \geq 1\}$ let

$$\lambda(a, b) := \{\mathbf{x} \in \mathbb{P}^{d'-1}(\mathbb{Z}_p/(p^a)) \mid \mathbf{x} \equiv (0 : 1 : \dots : 1) \pmod p, v_p(x_1) = b\}.$$

Then

$$\lambda(a, b) = \begin{cases} p^{(d'-2)(a-1)}, & \text{if } (a, b) \in \Delta, \\ p^{(d'-2)(a-1)+a-b}(1-p^{-1}), & \text{if } (a, b) \in N \setminus \Delta, \end{cases}$$

where $\Delta := \{(x, y) \in N \mid x = y\}$.

Proof. This is easy to check in an affine chart. □

Thus

$$\begin{aligned} B_0(p, p^{-s}) &= \sum_{(a,b) \in \Delta} p^{(d'-2)(a-1)+ad-s(d-1)a} \\ &\quad + (1-p^{-1}) \sum_{(a,b) \in N \setminus \Delta} p^{(d'-2)(a-1)+a-b+ad-s((d+1)a-2b)} \\ &= p^{-(d'-2)} \left(\frac{Y}{1-Y} \left(1 + (1-p^{-1}) \frac{X_{d'-1}}{1-X_{d'-1}} \right) \right) \\ &= p^{-(d'-1)} \frac{Y(p-X_{d'-1})}{(1-Y)(1-X_{d'-1})}, \end{aligned} \tag{36}$$

where $X_{d'-1}$ and Y are defined as in the statement of Proposition 4, which now follows from routine computations combining the identities (28), (34) and (36). □

With equation (31) we have given an explicit formula for the generating function $A(p, p^{-s})$, which, by Lemma 1, is tantamount to the local normal zeta function, completing the proof of Theorem 3. The functional equation also follows swiftly from (31), Theorem 4 and Corollary 2. Indeed, let V be any non-singular, projective variety over \mathbb{F}_p of dimension n . If $b_{V,e}$, $e \geq 1$, denotes the number of \mathbb{F}_{p^e} -rational points of V it is a well-known consequence of the *rationality* of the Weil zeta function

$$Z_V(u) = \exp \left(\sum_{e=1}^{\infty} \frac{b_{V,e} u^e}{e} \right),$$

that there are complex numbers $\beta_{r,j}$, $r = 0, \dots, 2n$, $j = 1, \dots, B_r$, $B_r \in \mathbb{N}$, such that

$$b_{V,e} = \sum_{r=0}^{2n} (-1)^r \sum_{j=1}^{t_r} \beta_{r,j}^e,$$

and that the function

$$\mathbb{N}_{\geq 0} \rightarrow \mathbb{N}$$

$$e \mapsto b_{V,e}$$

has a *unique* extension to \mathbb{Z} (cf. [DM, Lemma 2]). The *functional equation* of the Weil zeta function

$$Z_V(1/p^n u) = \pm (p^{n/2} u)^\chi Z_V(u),$$

where $\chi = \sum_{i=1}^{2n} (-1)^i B_i$, implies the 1-1-correspondences

$$\left\{ \frac{p^n}{\beta_{r,j}} \mid 1 \leq j \leq B_r \right\} \xleftrightarrow{1-1} \{ \beta_{2n-r,i} \mid 1 \leq i \leq B_{2n-r} \},$$

for $0 \leq j \leq 2n$ (cf. [I3, p. 213]). This gives

$$b_{V,-e} = p^{-en} b_{V,e}$$

formally. (An instance of which we have also seen in (19) where $b_{V,e}$ was given as a polynomial in p^e .) Corollary 1 now follows immediately if we set $V = \mathfrak{P}_G$, $e = 1$, $n = d' - 2$, $n_{\mathfrak{P}_G}(p^e) = b_{V,e}$.

4.3 Proof of Theorem 5. The proof of Theorem 5 would have been presented in section 4.2.1 had this not interrupted the proof of Theorem 3. Here $d = n$ and $d' = n - 1$. The essential observation is that again the weight function $w'([\Lambda'])$ only depends on the lattice's *type*. It was indeed explicitly calculated in [V4, Chap. 5.2], as

$$w'([\Lambda']) = \sum_{i \in \nu([\Lambda'])} r_i (2(d' - i) + 1).$$

This allows us to write

$$\begin{aligned} A(p, p^{-s}) &= \sum_{I \subseteq [d'-1]} A_I(p, p^{-s}) \\ &= \sum_{I \subseteq [d'-1]} \sum_{\mathbf{r}_I > 0} f(I, \mathbf{r}_I, p) \cdot p^{d \sum r_i (d'-i)} T^{\sum r_i (2(d'-i)+1)} \\ &= \sum_{I \subseteq [d'-1]} \frac{b_I(p)}{p^{\dim \mathcal{F}_I}} \sum_{\mathbf{r}_I > 0} p^{\sum r_i (d+i)(d'-i)} T^{\sum r_i (2(d'-i)+1)} \\ &= F_{d'}(p^{-1}, \tilde{\mathbf{X}}). \end{aligned}$$

Here, for $1 \leq i \leq d' - 1$, we made the substitutions

$$\tilde{X}_i := p^{(d+i)(d'-i)} T^{2(d'-i)+1}.$$

Again $\mathbf{r}_I > 0$ stands for $\mathbf{r}_I \in \mathbb{N}_{>0}^I$ and \sum for $\sum_{i \in I}$. The result follows from Theorem 4 and Corollary 2.

Appendix: Lines on Pfaffian Hypersurfaces

BY A. BEAUVILLE

The aim of this appendix is to prove that a general pfaffian hypersurface of degree $r > 2n - 3$ in \mathbb{P}^n contains no lines (Proposition 1). By a simple dimension count (see Corollary 4 below), it suffices to show that the variety of lines contained in the universal pfaffian hypersurface (that is, the hypersurface of degenerate forms in the space of all skew-symmetric forms on a given vector space) has the expected dimension. We will deduce this from an explicit description of the pencils of degenerate skew-symmetric forms, which is the content of the proposition below.

We work over an algebraically closed field k . We will need an elementary lemma:

Lemma 4. *Given a pencil of skew-symmetric forms on a n -dimensional vector space, there exists a subspace of dimension $\lfloor \frac{n+1}{2} \rfloor$ which is isotropic for all forms of the pencil.*

Proof. By induction on n , the cases $n = 0$ and $n = 1$ being trivial. Let $\varphi + t\psi$ be our pencil; we can assume that φ is degenerate. Let D be a line contained in the kernel of φ , and let D^\perp be its orthogonal with respect to ψ . Then φ and ψ induce skew-symmetric forms $\bar{\varphi}$ and $\bar{\psi}$ on D^\perp/D ; by the induction hypothesis there exists a subspace of dimension $\lfloor \frac{n-1}{2} \rfloor$ in D^\perp/D which is isotropic for $\bar{\varphi}$ and $\bar{\psi}$. The pull-back of this subspace in D^\perp has dimension $\lfloor \frac{n+1}{2} \rfloor$ and is isotropic for φ and ψ . \square

The following result must be well known, but I have not been able to find a reference:

PROPOSITION 5. *Let V be a vector space of dimension $2r$, and $(\varphi_t)_{t \in \mathbb{P}^1}$ a pencil of degenerate skew-symmetric forms on V . There exists a subspace $L \subset V$ of dimension $r + 1$ which is isotropic for φ_t for all $t \in \mathbb{P}^1$.*

Proof. Again we prove the proposition by induction on r , the case $r = 1$ being trivial. The associated maps $\Phi_t : V \rightarrow V^*$ form a pencil of singular linear maps. By a classical result in linear algebra (see [G, Chap. XII, Thm. 4]), there exist subspaces $K \in V$ and $L' \in V^*$, with $\dim K = \dim L' + 1$, such that $\Phi_t(K) \subset L'$ for all t ; equivalently, there exist subspaces K and L of V , with $\dim K + \dim L = 2r + 1$, which are orthogonal for each φ_t . Replacing (K, L) by $(K \cap L, K + L)$ we may assume $K \subset L$; the pencil (φ_t) restricted to L is singular on K , hence induces a pencil $(\bar{\varphi}_t)$ on L/K . Put $\dim K = p$, so that $\dim(L/K) = 2r + 1 - 2p$. By the above lemma there is a subspace

of L/K , of dimension $r + 1 - p$, which is isotropic for each $\bar{\varphi}_t$. Its pull-back in L has dimension $r + 1$ and is isotropic for each φ_t . \square

Let us give a few consequences of Proposition 5. We keep our vector space V of dimension $2r$; we denote by \mathcal{S}_r the space of skew-symmetric forms on V , and by \mathcal{X}_r the hypersurface of degenerate forms in $\mathbb{P}(\mathcal{S}_r)$.

Corollary 3. *The variety of lines contained in \mathcal{X}_r is irreducible, of codimension $r + 1$ in the Grassmannian of lines of $\mathbb{P}(\mathcal{S}_r)$.*

Proof. The $(r + 1)$ -planes of V are parametrized by a Grassmannian \mathcal{G} of dimension $r^2 - 1$. For such a plane L the space $\mathcal{S}_{r,L}$ of forms $\varphi \in \mathcal{S}_r$ vanishing on L has dimension

$$\dim \mathcal{S}_{r,L} = \dim \Lambda^2 V^* - \dim \Lambda^2 L^* = r(2r - 1) - \frac{r(r + 1)}{2} = \frac{3r(r - 1)}{2}.$$

Let \mathcal{P} be the Grassmannian of lines in $\mathbb{P}(\mathcal{S}_r)$ (that is, the variety of pencils of skew-symmetric forms). Consider the locus $Z \in \mathcal{P} \times \mathcal{G}$ of pairs (ℓ, L) with $\ell \in \mathcal{S}_{r,L}$. The projection $Z \rightarrow \mathcal{G}$ is a smooth fibration; its fibre above a point $L \in \mathcal{G}$ is the Grassmannian of lines in $\mathbb{P}(\mathcal{S}_{r,L})$, which has dimension $2 \dim \mathcal{S}_{r,L} - 4$. Thus Z is smooth, irreducible, of dimension $r^2 - 1 + 2 \dim \mathcal{S}_L - 4 = 4r^2 - 3r - 5$.

Let $\mathcal{P}_{\text{sing}}$ be the subvariety of \mathcal{P} consisting of lines contained in \mathcal{X}_r (that is, the subvariety of singular pencils). The content of Proposition 5 is that $\mathcal{P}_{\text{sing}}$ is the image of Z under the projection to \mathcal{P} . Thus $\mathcal{P}_{\text{sing}}$ is irreducible, of dimension $\leq 4r^2 - 3r - 5$, or equivalently, since $\dim \mathcal{P} = 2 \dim \mathcal{S}_r - 4 = 4r^2 - 2r - 4$, of codimension $\geq r + 1$. On the other hand, $\mathcal{P}_{\text{sing}}$ is defined locally by $(r + 1)$ equations in \mathcal{P} , given by the coefficients of the polynomial $\text{Pf}(\varphi_t)$ of degree r . The corollary follows. \square

Observe that $r + 1$ is the number of conditions that the requirement to contain a given line imposes on a hypersurface of degree r in projective space. In other words, Corollary 3 says that the hypersurface \mathcal{X}_r behaves like a general hypersurface of degree r as far as the dimension of its variety of lines is concerned.

Let L be a vector space, of dimension $n + 1$, and $\ell = (\ell_{ij})$ a $(2r \times 2r)$ -skew-symmetric matrix of linear forms on L . The hypersurface X_ℓ in $\mathbb{P}(L)$ ($= \mathbb{P}^n$) defined by $\text{Pf}(\ell_{ij}) = 0$ is called a *pfaffian hypersurface*. It is defined by the equation $\text{Pf}(\ell_{ij}) = 0$, of degree r .

Corollary 4. *If $r > 2n - 3$ and the forms ℓ_{ij} are general enough, X_ℓ contains no lines.*

Proof. The matrix (ℓ_{ij}) defines a linear map $u : L \rightarrow \mathcal{S}_r$, which is injective when the forms ℓ_{ij} are general enough (observe that $\dim L < \dim \mathcal{S}_r$). Thus we can identify L to its image in \mathcal{S}_r , and X_ℓ to the hypersurface $\mathcal{X}_r \cap \mathbb{P}(L)$ in $\mathbb{P}(L)$.

Let G be the Grassmann variety of $(n+1)$ -dimensional vector subspaces of \mathcal{S} , and F the variety of lines contained in \mathcal{X}_r . Consider the incidence variety $Z \in F \times G$ of pairs (ℓ, L) with $\ell \in \mathbb{P}(L)$. The fibre of the projection $Z \rightarrow G$ at a point $L \in G$ is the variety of lines contained in $\mathcal{X}_r \cap \mathbb{P}(L) = X_\ell$.

Put $N := \dim \mathcal{S}_r$. We have $\dim F = 2N - 4 - (r+1)$ by Corollary 3; the projection $Z \rightarrow F$ is a fibration of relative dimension $(n-1)(N-n-1)$. This gives $\dim Z = 2N - 4 - (r+1) + (n-1)(N-n-1)$, while $\dim G = (n+1)(N-n-1)$. Thus

$$\dim Z - \dim G = 2n - 3 - r < 0,$$

hence the general fibre of the projection $Z \rightarrow G$ is empty. \square

Note that ‘ (ℓ_{ij}) general enough’ means ‘for (ℓ_{ij}) in a certain Zariski open subset of $(L^*)^N$ ’. In particular, suppose that our vector space L comes from a vector space L_0 over an infinite subfield k_0 of k ; then *the matrices $(\ell_{ij}) \in (L_0^*)^N$ such that X_ℓ contains no lines are Zariski dense in the parameter space $(L^*)^N$ for $r > 2n - 3$.*

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