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CONFORMAL ASSOUAD DIMENSION AND MODULUS

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Abstract. Let $\alpha \geq 1$ and let (X, d, μ) be an α-homogeneous metric measure space with conformal Assouad dimension equal to α. Then there exists a weak tangent of (X, d, μ) with uniformly big 1-modulus.

1 Introduction

The existence of families of curves with non-vanishing modulus in a metric measure space is a strong and useful property, perhaps most appreciated in the study of abstract quasiconformal and quasisymmetric maps, Sobolev spaces, Poincaré inequalities, and geometric rigidity questions. See for example [BoK3], [BouP2], [Ch], [HK], [HKST], [K2,3], [KZ], [P1], [Sh], [T2] and the many references contained therein. In this paper we give an existence theorem for families of curves with non-vanishing modulus in weak tangents of certain metric measure spaces, and show that this existence is fundamentally related to the conformal Assouad dimension of the underlying metric space.

Theorem 1.0.1. *Let $\alpha \geq 1$ and let (X, d, μ) be an α-homogeneous metric measure space with conformal Assouad dimension equal to α. Then there exists a weak tangent of (X, d, μ) with uniformly big 1-modulus.*

The terminology of Theorem 1.0.1 will be explained in section 2, and several applications will be given momentarily. For the moment we note that the notion of uniformly big 1-modulus is new, and describes those metric measure spaces that are rich in curves at every location and scale, as measured in a scale-invariant way by modulus. We recall that a *weak tangent* of a metric measure space is any metric measure space obtained as a pointed measured Gromov–Hausdorff limit of a sequence of re-scalings of the original metric measure space taken at various locations; and the *conformal Assouad dimension* of a metric space (X, d) is defined as the infimal Assouad dimension amongst all metric spaces quasisymmetrically

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homeomorphic to (X, d). Heinonen [H, Theorem 14.16] has shown that the conformal Assouad dimension of a uniformly perfect complete metric space (X, d), and in particular an Ahlfors regular complete metric space, can be more familiarly expressed as the infimal Hausdorff dimension amongst all Ahlfors regular subsets of Euclidean space quasisymmetrically homeomorphic to (X, d). Theorem 1.0.1 can be strengthened in this setting of Ahlfors regular metric spaces by an argument involving a result of Tyson [T2] to obtain the following.

COROLLARY 1.0.2. *Let $\alpha \geq 1$ and let (X, d) be an Ahlfors α-regular metric space. Then the conformal Assouad dimension of (X, d) is equal to α if and only if there exists a weak tangent of (X, d) that has non-vanishing p-modulus for some $p \geq 1$. In this case there further exists a weak tangent of (X, d) with uniformly big 1-modulus.*

Theorem 1.0.1 can be applied to deduce stronger results for metric spaces with sufficient symmetry. This is demonstrated in the following corollaries.

COROLLARY 1.0.3. *Let $\alpha \geq 1$, let (X, d, μ) be a compact α-homogeneous metric measure space with conformal Assouad dimension equal to α, and suppose that (X, d) admits a uniformly quasi-Möbius action $G \curvearrowright X$ for which the induced action on the space of distinct triples of X is cocompact. Then there exists $x \in X$ and a quasi-Möbius homeomorphism from $X \setminus \{x\}$ to an α-homogeneous metric measure space with uniformly big 1-modulus. If we further assume that (X, d) is Ahlfors α-regular, then (X, d) has uniformly big α-modulus.*

Bonk and Kleiner [BoK2] have shown that an Ahlfors α-regular metric space with non-vanishing α-modulus, $\alpha > 1$, which admits a quasi-Möbius action that is both fixed point free and has an induced cocompact action on the space of distinct triples of X, is then Loewner; or equivalently by Heinonen and Koskela [HK] admits a $(1, \alpha)$-Poincaré inequality. Bonk and Kleiner [BoK2] then applied Corollary 1.0.3 in conjunction with several deep results to show that a Gromov hyperbolic group acts discretely, cocompactly, and isometrically on hyperbolic 3-space if the following holds: The boundary of the Gromov hyperbolic group is homeomorphic to the 2-sphere, and quasisymmetric homeomorphic to Ahlfors α-regular space with conformal Assouad dimension α. A well-known conjecture of Cannon [C] postulates that the same conclusion holds without the assumption that the boundary is quasisymmetrically homeomorphic to an Ahlfors α-regular space with conformal Assouad dimension α. We recall that solving

Cannon's conjecture would solve a large part of Thurston's hyperbolization conjecture; see [C].

Another notion of self-symmetry, expressed by *big pieces of itself* (BPI) geometry, and introduced by David and Semmes [DS], describes those Ahlfors regular metric spaces that possess certain quantitative bi-Lipschitz self-similarity on sets of positive measure. There is a natural notion of BPI equivalence amongst BPI metric spaces. An example of one such equivalence class is given by uniformly α-rectifiable metric spaces, $\alpha \in \mathbf{N}$, which are those BPI metric spaces that are BPI equivalent to $\mathbf{R}^\alpha$. The collection of BPI metric spaces exhibit a wide range of relations with modulus. For example, there are BPI metric spaces (X,d) that have uniformly big p-modulus for every $p > \alpha$, but not for $p = \alpha$, where α is the Hausdorff dimension of (X,d). Nonetheless, the next corollary demonstrates that the realm of BPI metric spaces is a natural setting for uniformly big 1-modulus.

COROLLARY 1.0.4. *Let $p \geq 1$. Every BPI metric space with non-vanishing p-modulus is BPI equivalent to a BPI metric space with uniformly big* 1-*modulus.*

The quintessential example of a space with uniformly big 1-modulus is the product of any Ahlfors regular metric space with the unit interval. By this example we see that uniformly big 1-modulus is a weaker condition than the $(1,1)$-Poincaré inequality of Heinonen and Koskela [HK]. Nonetheless, many properties that hold on spaces that exhibit a $(1,1)$-Poincaré inequality also hold on spaces with uniformly big 1-modulus. In particular, quasisymmetric maps between Ahlfors regular metric spaces with uniformly big 1-modulus exhibit certain quantitative bi-Lipschitz behavior, even when the map is not defined on the whole space. This topic is studied in [KL] where we answer and generalize related question of Semmes [S, Questions 4.8 and 4.9]. Corollary 1.0.4 plays a crucial role in this analysis.

The above applications of Theorem 1.0.1 involve exploiting the symmetry of a metric space in order to establish the existence of curves. Theorem 1.0.1 can also be applied in its contrapositive form to give upper bounds for the conformal Assouad dimension of a metric space. The problem of calculating the conformal Assouad dimension, or the conformal (Hausdorff) dimension, of an arbitrary metric space has been considered in a variety of circumstances. See for example [Bou], [BouP1], [BT], [G1], [P1], [T3] and [H, Section 15]. Theorem 1.0.1 implies that the conformal Assouad dimension of certain metric measure spaces is strictly less than their Assouad dimension whenever all the weak tangents of the given metric measure space do not have uniformly big 1-modulus. In practice this condition is often simple to check, especially when the given metric space is sufficiently self-similar. As an example consider the *Sierpinski n-carpet*, $n \in \mathbf{N}$, which is the classical fractal obtained by subdividing the cube $[0,1]^n \subset \mathbf{R}^n$ into 3^n congruent cubes, removing the middle cube, and continuing in a self-similar manner.

COROLLARY 1.0.5. *The conformal Assouad dimension, and therefore also the conformal (Hausdorff) dimension, of the Sierpinski n-carpet is strictly less than the Hausdorff dimension of the Sierpinski n-carpet, for every $n \in \mathbf{N}$.*

As far as we know the above result is new for $n \geq 3$ – Bruce Kleiner recently informed us that he previously proved that the conformal Hausdorff dimension of the Sierpinski square and Menger sponge is each strictly less than their respective Hausdorff dimension. Our proof is general and can be applied to most fractals in Euclidean space that are constructed by processes similar to that of the Sierpinski carpet. For example, Theorem 1.0.1 can also be used to show that the conformal Assouad dimension of both the Menger sponge and Sierpinski gasket (triangle) are strictly less than their respective Hausdorff dimensions. Recently the second author has used other means to show that the conformal Assouad dimension of the Sierpinski triangle is equal to 1. The method of Theorem 1.0.1 can also be employed to improve the upper bounds on the conformal Assouad dimension given in Corollary 1.0.5. This application will be pursued elsewhere.

In this current paper, Theorem 1.0.1 is deduced from stronger results that may be of independent interest. We now briefly describe these.

- Theorem 4.0.5 states that an α-homogeneous metric measure space, $\alpha \geq 1$, with non-vanishing p-modulus, $p \geq 1$, contains a Borel set A of positive measure, such that every tangent of A with non-vanishing measure, has uniformly big 1-modulus. Thus there is no need here to use weak tangents. Theorem 4.0.5 is further quantified in Remark 4.0.6.
- Theorem 5.0.10 outlines quantitative conditions which guarantee that the Assouad dimension of a metric measure space can be reduced through a quasisymmetric homeomorphism. Roughly speaking, the condition specifies that at every location and scale there should be an annulus, so that the discrete modulus of discrete curves connecting the annulus's inner and outer part is sufficiently small. Originally in Proposition 3.3.3 this condition is expressed in terms of the (classical)

modulus of certain annuli on weak tangents of the given metric measure space.

- Proposition 5.1.1 outlines general conditions that guarantee a metric measure space is quasisymmetrically homeomorphic to a metric space with Assouad dimension equal to α', for any given fixed $\alpha' > 0$. See also Remark 5.1.2.
- Both Theorem 4.0.5 and 5.0.10 rely on the notion of discrete modulus that is developed in section 3. The idea of discretizing modulus is not new. It was developed by Pansu [P1,2] and Tyson [T1,2] in its most abstract form to-date on Ahlfors regular metric spaces. As far as we understand, the particular technique of discretization developed in this paper is different and new. This is partly evidenced by the fact that it applies to a more general class of metric measure spaces than previously.

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1.2 Outline. Theorem 1.0.1 and all the corollaries are easily deduced in section 6 from Theorem 4.0.5 and Theorem 5.0.10. The proof of these latter two theorems comprises most of the paper, and relies on the theory of discrete modulus developed in section 3.

Concerning the proof of Theorem 5.0.10, we remark that a qualitative constraint on all weak tangents of a metric spaces usually gives some sort of uniform control. For example, a Lipschitz function whose weak tangents are all injective, is necessarily bi-Lipschitz (see [DS, Section 14.3]). In our case, the hypotheses of Theorem 5.0.10 gives a uniform control which we exploit through the use of discrete modulus (see Proposition 3.3.3). This enables us to conformally and semi-discretely deform the metric at every location and scale in a way that decreases the measure while still preserving relatively large distances (see Proposition 5.1.1). The desired quasisymmetric homeomorphism is then constructed by simultaneously implementing each of these conformal deformations (see Proposition 5.2.1). The above described uniform control is encapsulated in the parameter Ψ. A similar theme occurs in the proof of Theorem 4.0.5 where again the relevant parameter is Ψ.

2 Preliminary Definitions and Remarks

In this section we recall standard terminology and make standard remarks. With regards to language, we say $\alpha, \beta > 0$ are *comparable* with comparability constant $C > 0$, whenever $C^{-1}\alpha \leq \beta \leq C\alpha$. When we say $C > 0$ is a *varying constant* that depends only on some data $\mathcal{D}$, this means that C denotes a positive variable, whose value may vary between each usage, but is then fixed and depends only on $\mathcal{D}$. For example when we say that property $\mathcal{P}$ holds whenever $\alpha < C\beta$ for some $\alpha, \beta \in \mathbf{R}$, we mean that there exists $L > 0$ that depends only on $\mathcal{D}$, such that property $\mathcal{P}$ holds whenever $\alpha < L\beta$. We let $[r]$ denote the greatest integer less than r, for any $r \in \mathbf{R}$.

2.1 Metric measure spaces and quasisymmetric maps. We refer the reader to [H] for detailed discussion on all topics covered in this subsection. A ball in a metric space (X, d) centered at $x \in X$ and with radius $r > 0$, is a set of the form

$$B(x, r) = \big\{ y \in X : d(y, x) < r \big\} .$$

An embedding $f : X \to Y$ between metric spaces (X, d) and (Y, ρ) is said to be *η-quasisymmetric* (or just *quasisymmetric*), for some homeomorphism $\eta : [0, \infty) \to [0, \infty)$, if

$$\frac{\rho(f(x), f(y))}{\rho((f(x), f(z))} \leq \eta \left(\frac{d(x, y)}{d(x, z)} \right),$$

for every $x, y, z \in X$ with $x \neq z$. A metric space (X, d) is called *uniformly perfect* if there is a constant $C \geq 1$ such that for each $x \in X$ and $r > 0$, we have

$$B(x, r) \setminus B(x, r/C) \neq \varnothing \quad \text{whenever} \quad X \setminus B(x, r) \neq \varnothing .$$

A *metric measure space* (X, d, μ) consists of a set X, a metric d on X, and a Borel measure μ supported on X. A metric measure space (X, d, μ) is said to be *Ahlfors α-regular*, for some $\alpha > 0$, with constant $C > 0$ (which we call the *Ahlfors regularity constant of* (X, d, μ)), if

$$C^{-1} r^{\alpha} \leq \mu\big(B(x, r)\big) \leq C r^{\alpha}, \tag{1}$$

for every $x \in X$ and $0 < r < \operatorname{diam} X$. Any measure that satisfies (1) will be comparable to the α-Hausdorff measure. Here we say that two Borel measures μ and ν, defined on a space X, are comparable with constant C to mean that $C^{-1}\mu(A) \leq \nu(A) \leq C\mu(A)$ for every Borel set $A \subset X$. Thus Ahlfors regularity is in fact a metric condition, and so we describe a metric space as being Ahlfors α-regular if the α-Hausdorff measure μ on (X, d) satisfies (1) for every $x \in X$ and $0 < r < \operatorname{diam} X$. In this case, and unless otherwise specified, we always denote the α-Hausdorff measure on (X, d) by μ.

The measure μ of a metric measure space (X, d, μ) is said to be α-*homogeneous* (or *homogeneous*), for some $\alpha > 0$, if there exists a constant $C > 0$ (which we call an *homogeneity constant*) such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq C\left(\frac{r}{R}\right)^{\alpha},$$

whenever $0 < r < R < \operatorname{diam} X$, $x \in X$, and $y \in B(x, R)$. In this case we also say that (X, d, μ), and also μ, is α-homogeneous. A measure is homogeneous if and only if it is doubling; a measure is *doubling* if there exists $L > 0$ such that

$$\mu\big(B(x, 2r)\big) \leq L\mu\big(B(x, r)\big),$$

for every $x \in X$ and $0 < r < \operatorname{diam} X$. In this paper we prefer to use the first definition because of the immediate relation to the Assouad dimension.

Recall that the *Assouad dimension* of a metric space is the infimum of all numbers $\beta > 0$ such that there exists $C > 0$ so that every set of diameter r can be covered by at most $C\epsilon^{-\beta}$ sets of diameter at most ϵr. If (X, d, μ) is α-homogeneous, then the Assouad dimension of (X, d) is at most α. Conversely, Vol'berg and Konyagin [VK], and Luukkainen and Saksman [LS], have shown that any complete metric space with Assouad dimension equal to $\alpha_0 \geq 0$ carries an α-homogeneous measure for every $\alpha > \alpha_0$. As mentioned in the introduction, the *conformal Assouad dimension* of a metric space is the infimal Assouad dimension amongst all metric spaces quasisymmetrically homeomorphic to (X, d). The following theorem demonstrates that the conformal Assouad dimension of a complete uniformly perfect metric space (X, d), and in particular any Ahlfors regular metric space, can be more familiarly expressed as the infimal Hausdorff dimension amongst all Ahlfors regular subsets of Euclidean space quasisymmetrically homeomorphic to (X, d).

Theorem 2.1.1 (Heinonen [H, Theorem 14.16]). *Let (X, d) be a complete, uniformly perfect metric space of finite Assouad dimension α_0. Then, for each $\alpha > \alpha_0$, there exists a quasisymmetric homeomorphism of X onto a closed Ahlfors α-regular metric subset of some Euclidean space.*

The notion of conformal Assouad dimension is related to Pansu's concept [P1] of *conformal dimension* defined as the infimal Hausdorff dimension amongst all metric spaces quasisymmetrically homeomorphic to (X, d). To avoid confusion we will always refer to conformal dimension in Pansu's sense as conformal (Hausdorff) dimension.

2.2 Dyadic decompositions. David [D] and Christ [Chr] have constructed generalized dyadic decompositions for any homogeneous metric measure space. The existence of the dyadic decomposition described here can easily be inferred from their constructions. Let (X,d,μ) be α-homogeneous, for some $\alpha>0$, and assume for the moment that (X,d) is unbounded. Then a *dyadic decomposition* is a collection $\triangle=\cup_{j\in\mathbf{Z}}\triangle_j$ satisfying the following conditions. Each $\triangle_j$ is a collection of *cubes*, each of which is an ordered pair (A,j) where A is a Borel subset of X. We often identify (A,j) with the set A when this causes no ambiguity. We define a function $\mathrm{rad}:\triangle\to Z$ by $\mathrm{rad}(A,j)=2^{-j}$, and call $\mathrm{rad}(A,j)$ the *radius* of (A,j). In general the diameter of a cube can vanish, and so will not be comparable to the radius. We now state the properties of $\triangle$. There exists a constant $C>0$ that depends only on α and the homogeneity constant of μ such that:

1. We have $X=\bigcup_{Q\in\triangle_j}Q$, for every $j\in\mathbf{Z}$.
2. We have $Q\cap R=\varnothing$ whenever $j\in\mathbf{Z}$, and $Q,R\in\triangle_j$ with $Q\neq R$.
3. Let $Q\in\triangle_j$ and $R\in\triangle_k$ for some $k\le j$. Then either $Q\subset R$ or $Q\cap R=\varnothing$.
4. For each $Q\in\triangle$, there exists $w\in Q$ such that
$$B(w,C^{-1}\,\mathrm{rad}\,Q)\subset Q\subset B(w,C\,\mathrm{rad}\,Q)\,.$$
5. For each $Q\in\triangle$ and $0<\tau<1$, we have $\mu(\partial_\tau Q)\le C\tau^{1/C}\ \mu(Q)$, where here and after we let
$$\partial_\tau Q=\left\{x\in Q:\mathrm{dist}(x,X\setminus Q)\le\tau\,\mathrm{rad}\,Q\right\}\\ \cup\left\{x\in X\setminus Q:\mathrm{dist}(x,Q)\le\tau\,\mathrm{rad}\,Q\right\},$$
and for future reference define $\partial_\tau\triangle_j=\cup_{Q\in\triangle_j}\partial_\tau Q$ for every $j\in\mathbf{Z}$.
6. For each $Q\in\triangle$, the metric measure subspace $(\overline{Q},d,\mu)$ is α-homogeneous with homogeneity constant C.

Let
$$\ell_\triangle=\sup_{R\in\triangle}\left\{\mathrm{diam}\,R(\mathrm{rad}\,R)^{-1}\right\},$$
and observe that from the properties of the dyadic decomposition we have $\ell_\triangle\le C$.

If (X,d) is bounded, there is a similar decomposition, except that we restrict ourselves to $j\in\mathbf{Z}$ that satisfy $j\le j_0$, where $j_0\in Z$ is determined by $2^{j_0}\le\mathrm{diam}\,X\le 2^{j_0+1}$. In this case we take X to be the top cube of the decomposition. We remark that in the case when (X,d) is unbounded, we can exhaust (X,d,μ) by closed and bounded α-homogeneous metric measure subspaces each of which has homogeneity constant bounded above by C. This follows from property 6 of the dyadic decomposition, and can be seen by taking the closure of a union of a sufficiently large, but also uniformly bounded in number, collection of cubes from each $\triangle_j$.

2.3 Modulus. We refer the reader to [H, Chapter 7] for a complete discussion of the following topics. A *curve* in a metric space (X, d) is a continuous map γ of an interval $I \subset \mathbf{R}$ into X, and is said to be *rectifiable* if it has finite *length*, which we denote by $\ell(\gamma)$. For a given family Γ of curves in a metric measure space (X, d, μ), and $p \geq 1$, the *p-modulus of* Γ is given by

$$\mathrm{mod}_p(\Gamma) = \inf \int_X \rho^p \, d\mu \,,$$

where the infimum is taken over all Borel functions $\rho : X \to [0, \infty]$ that satisfy

$$\int_\gamma \rho \, ds \geq 1 \,,$$

for all locally rectifiable curves $\gamma \in \Gamma$. Such functions are said to be *admissible* for Γ. A metric measure space (X, d, μ) is said to have *non-vanishing* p-modulus if there exists a collection of non-constant curves in X with positive p-modulus. Otherwise we say (X, d, μ) has *vanishing* p-modulus. Unless otherwise specified, when speaking of modulus in relation to an Ahlfors α-regular metric space, where $\alpha > 0$, we assume that the measure used in the calculation of modulus is the α-Hausdorff measure.

The following theorem of Tyson emphasizes the relation between quasisymmetric homeomorphism and modulus, and will be utilized in our proof.

Theorem 2.3.1 (Tyson [H, Theorem 15.10]). *Let $\alpha \geq 1$, and let (X, d) be an Ahlfors α-regular metric space with non-vanishing α-modulus. Then the conformal (Hausdorff) dimension of (X, d) is equal to α.*

2.4 Convergence of metric (measure) spaces, tangents, and weak tangents. We now briefly recall several notions of convergence for metric spaces and metric measure spaces that are, up to a subsequence, equivalent to the usual definition for Gromov–Hausdorff convergence and measured Gromov–Hausdorff convergence. We review this topic in order to establish a language for describing converging metric spaces that is convenient for our application. See [G2], [Pe], [ChiY] for more information.

A sequence (F_n) of nonempty closed subsets of a metric space (Z, l) is said to *converge to* another nonempty closed subset $F \subset Z$ if

$$\lim_{n\to\infty} \sup_{z \in F_n \cap B(q,R)} \mathrm{dist}(z, F) = 0 \,,$$

and

$$\lim_{n\to\infty} \sup_{z\in F\cap B(q,R)} \operatorname{dist}(z, F_n) = 0\,,$$

for all $q \in Z$ and $R > 0$. These suprema are interpreted to vanish when the relevant sets of competitors $F_n \cap B(z,R)$ and $F \cap B(z,R)$ are empty. Recall that a metric space is said to be *proper* if the closure of every ball in that space is compact. Such spaces are complete. A *pointed metric space* consists of a metric space and a point in that space, and likewise a *pointed metric measure space* consists of a metric measure space and a point in that space. We denote the push-forward of a measure μ under a map ι by $\iota_*\mu$. We say a sequence of measures (μ_n) defined on a topological space X *converges weakly* to some measure μ if for every continuous compactly supported function $f : X \to \mathbf{R}$, we have

$$\int_X f\,d\mu_n \to \int_X f\,d\mu\,,$$

as $n \to \infty$.

Definition 2.4.1 (Convergence of compact metric measure spaces). A sequence of compact metric measure spaces $\{(X_n, d_n, \mu_n)\}$ is said to *converge to* another compact metric measure space (X, d, μ) if the following holds: There exists a compact metric space (Z, l) and isometric embeddings $\iota : X \to Z$ and $\iota_n : X_n \to Z$, for each $n \in \mathbf{N}$, such that $(\iota_n(X_n))$ converges to $\iota(X)$ as subspaces of Z, and such that $((\iota_n)_*\mu_n)$ converges weakly to $\iota_*\mu$.

Definition 2.4.2 (Convergence of proper pointed metric measure spaces). A sequence of proper pointed metric measure spaces $\{(X_n, d_n, \mu_n, p_n)\}$ is said to *converge to* another proper pointed metric measure space (X, d, μ, p) if the following holds: There exists a proper pointed metric space (Z, l, q) and isometric embeddings $\iota : X \to Z$ and $\iota_n : X_n \to Z$ for each $n \in \mathbf{N}$ such that $\iota(p) = \iota_n(p_n) = q$, such that $(\iota_n(X_n))$ converges to $\iota(X)$ as subspaces of Z, and such that $((\iota_n)_*\mu_n)$ converges weakly to $\iota_*\mu$.

We define the convergence for sequences of compact metric spaces and proper pointed metric spaces in the same manner as above, except that we omit mention of any measure. The following lemmas can be deduced from variants of Gromov's compactness theorem [G2, Proposition 5.2]; see also [DS, Lemma 8.28], [ChiY, Theorem 2.1.7], [Fuk1, Proposition 2.10], and [Fuk2, Lemma 2.4].

Lemma 2.4.3. *Let $\{(X_n, d_n, \mu_n, p_n)\}$ be a sequence of α-homogeneous, complete pointed metric measure spaces each with homogeneity constant C, for some $\alpha, C > 0$, and suppose that for every $n \in \mathbf{N}$ we have $L^{-1} \le \mu_n(B(p_n, R)) \le L$ for some fixed $R, L > 0$. Then a subsequence of*

$\{(X_n, d_n, \mu_n, p_n)\}$ *converges to a complete pointed metric measure space that is* α*-homogeneous with homogeneity constant* C.

Lemma 2.4.4. *The collection of complete pointed metric measure spaces that are Ahlfors* α*-regular with Ahlfors regularity constant* C*, for some* $\alpha, C > 0$*, is compact under the convergence of pointed metric measure spaces.*

We now recall the notion of tangents and weak tangents of metric measure spaces.

Definition 2.4.5 (Tangents of metric measure spaces). A complete pointed metric measure space (Y, l, ν, y) is said to be a *tangent* of a metric measure space (X, d, μ) if there exists sequences (r_n) and (s_n) of positive reals, where (r_n) converges to 0, such that $\{(\overline{X}, d/r_n, s_n\mu, x)\}$ converges to (Y, l, ν, y). Here we denote the completion of X by $\overline{X}$.

Definition 2.4.6 (Weak tangents of metric measure spaces). A pointed metric measure space (Y, l, ν, y) is said to be a *weak tangent* of a metric measure space (X, d, μ) if there exists a sequence $(x_n) \subset X$ and sequences (r_n) and (s_n) of positive reals, where each $0 < r_n < \operatorname{diam}(X, d)$, such that $\{(\overline{X}, d/r_n, s_n\mu, x_n)\}$ converges to (Y, l, ν, y). (Notice that (r_n) or (s_n) need not tend to zero.)

We define the tangents and weak tangents of a metric space as above, except that we suppress any mention of a measure.

We now describe the notion of convergence for maps between converging metric spaces. Let (f_n) be a sequence of maps given by $f_n : X_n \to Y_n$ for each $n \in \mathbf{N}$, where $\{(X_n, d_n, p_n)\}$ and $\{(Y_n, l_n, q_n)\}$ are sequences of pointed metric spaces that converge to (X, d, p) and (Y, l, q), respectively. By saying (f_n) *converges to* f, we mean that there exists isometric embeddings, like those described in Definition 2.4.2, that realize the convergence of the relevant sequences of metric spaces such that the following holds: After identifying each metric space with its image under each embedding, we have $\lim f_n(x_n) = f(x)$ whenever (x_n) is a sequence, with $x_n \in X_n$ for each $n \in \mathbf{N}$, that converges to some $x \in X$. The following lemma can be deduced via a variant of the proof of the Arzelá–Ascoli theorem; compare with [H, Corollary 10.30].

Lemma 2.4.7. *Let* $\{(X_n, d_n, p_n)\}$ *and* $\{(Y_n, l_n, q_n)\}$ *be sequences of proper pointed metric spaces that converge to* (X, d, p) *and* (Y, l, q)*, respectively. Let* $f_n : X_n \to Y_n$ *be an* η*-quasisymmetric homeomorphism for each* $n \in \mathbf{N}$*, where* η *is fixed. Further suppose that* $f_n(p_n) = q_n$*, that there exists* $C > 0$*,* *and that there exists a sequence* (x_n), *where each* $x_n \in X_n$, *such that*

$$C^{-1} \leq d_n(p_n, x_n) \leq C \quad \textit{and} \quad C^{-1} \leq l_n\big(q_n, f(x_n)\big) \leq C\,,$$

for every $n \in \mathbf{N}$. *Then, after passing to a subsequence, we have* (f_n) *converges to some* η*-quasisymmetric homeomorphism between* X *and* Y.

3 Discrete Modulus

We now present a discretization of modulus, and establish upper semi-continuity type relationships with the usual modulus.

3.1 Discrete curves and the definition of discrete modulus. A finite sequence $(x_i)_{i=1}^N$ of points in a metric space (X, d) is a *discrete* ϵ*-curve* (or just a discrete curve), where $N \in \mathbf{N}$ and $\epsilon > 0$, if $d(x_i, x_{i+1}) \leq \epsilon$ whenever $1 \leq i \leq N-1$. We often write the elements of a discrete curve γ as $\gamma(1), \ldots, \gamma(\#\gamma)$ where $\#S$ denotes the cardinality of a set S; for example $\#(x_i)_{i=1}^N = N$. We define the length of a discrete curve γ by

$$\ell(\gamma) = \sum_{i=1}^{\#\gamma - 1} d(x_i, x_{i+1})\,.$$

A discrete curve is *degenerate* if it consists of only one member. A function $\gamma : [0, L) \to X$, where $L > 0$, is said to be a *parameterization* of a discrete curve $(x_i)_{i=1}^N$ if there exists a sequence $0 = t_1 \leq t_2 \leq \cdots \leq t_N = L$ such that

$$d(x_i, x_{i+1}) = t_{i+1} - t_i\,, \tag{2}$$

for every $1 \leq i \leq N-1$, and such that $\gamma(r) = x_i$ whenever $t_i \leq r < t_{i+1}$ and $1 \leq i \leq N-1$.

A sequence (γ_n) of discrete curves in (X, d) is said to *converge to* a curve γ in (X, d) if the following holds: There exists $L > 0$, and there exists parameterizations $\gamma : [0, L) \to X$ and $\gamma_n : [0, L) \to X$ for each $n \in \mathbf{N}$, such that when viewed as functions, we have (γ_n) converges to γ uniformly. Given a sequence of families Γ_n of discrete curves, we define the *limit supremum* of (Γ_n) (written as $\limsup_{n\to\infty} \Gamma_n$) to be the collection of all curves obtained as a limit of a subsequence of all such sequences (γ_n), where $\gamma_n \in \Gamma_n$ for each $n \in \mathbf{N}$.

We call a discrete ϵ-curve $(x_i)_{i=1}^N$ *properly discrete* if $d(x_i, x_{i+1}) \geq \epsilon/10$ for every $1 \leq i \leq N-1$. We say that a sequence (γ_n) of properly discrete ϵ-curves in (X, d), where $\epsilon > 0$, *converges to* a properly discrete curve γ in (X, d) if $\#\gamma = \#\gamma_n$ for all but finitely many $n \in \mathbf{N}$, and if $(\gamma_n(i))$ converges to $\gamma(i)$ for every $1 \leq i \leq \#\gamma$. Given a sequence (Γ_n) of families of properly discrete ϵ-curves, we define the *discrete limit supremum* of (Γ_n) (written as dis-$\limsup_{n\to\infty}\Gamma_n$) to be the collection of properly discrete ϵ-curves obtained as a limit of a subsequence of all sequences (γ_n), where $\gamma_n \in \Gamma_n$ for each $n \in \mathbf{N}$.

To describe the convergence for sequences of curves, and sequences of families of curves, between converging metric spaces, we embed the given sequence of converging metric spaces into a proper metric space (Z, l), as described in Definition 2.4.2, and then transfer the definitions across. In this setting the definitions of convergence depend on the particular choice of the space (Z, l) and the corresponding embeddings. In practice we always assume one such embedding is chosen and fixed.

DEFINITION 3.1.1 (Discrete modulus). Define the *discrete p-modulus at scale ϵ* of a family Γ of discrete curves in a metric measure space (X, d, μ), where $\epsilon > 0$ and $p \geq 1$, by

$$\operatorname{mod}_p(\Gamma, \epsilon) = \inf \int_X \rho^p \, d\mu \,,$$

where the infimum is taken over all continuous functions $\rho : X \to [0, \infty)$ that satisfy

$$1 \leq \sum_{i=1}^{N-1} d(x_i, x_{i+1}) \inf_{y \in B(x_i, \epsilon)} \rho(y) \,,$$

for every discrete ϵ-curve $\{x_i\}_{i=1}^N$ in Γ. Each such function ρ will be said to be *discretely admissible at scale ϵ* for Γ. To simplify notation we define the discrete p-modulus at scale ϵ of any set Γ to be the discrete p-modulus at scale ϵ of the family of discrete curves contained in Γ.

3.2 Upper semi-continuity properties of discrete modulus. The following proposition is applied in the proof of Theorem 1.0.1, specifically in Lemma 4.0.7 and Lemma 3.3.3.

PROPOSITION 3.2.1. *Let $p \geq 1$, and let $\{(X_n, d_n, \mu_n)\}$ be a sequence of compact metric measure spaces that converges to a compact metric measure space (X, d, μ) with $\mu(X) < \infty$. Then we have*

$$\operatorname{mod}_p\Big(\limsup_{n\to\infty} \Gamma_n\Big) \geq \limsup_{n\to\infty} \operatorname{mod}_p(\Gamma_n, \epsilon_n) \,, \tag{3}$$

whenever (ϵ_n) is a sequence of positive reals that converges to 0 and where (Γ_n) is a sequence of families of curves with each curve in Γ_n in (X_n, d_n). We also have

$$\operatorname{mod}_p\Big(\text{dis-}\limsup_{n\to\infty} \Lambda_n, \epsilon\Big) \geq \limsup_{n\to\infty} \operatorname{mod}_p(\Lambda_n, \epsilon) \,, \tag{4}$$

whenever $\epsilon > 0$ *and where* (Λ_n) *is a sequence of families of properly discrete* ϵ*-curves with each discrete curve in* Λ_n *in* (X_n, d_n).

REMARK 3.2.2. The definitions for the convergence of sequences of discrete curves may appear cumbersome and overly restrictive. However, relaxing these definitions would give weaker results in the above proposition. That is, relaxing the definition of convergence would possibly increase the size of $\text{dis-lim sup}_{n\to\infty}\Lambda_n$, and thereby possibly increase the left-hand side of (3).

REMARK 3.2.3. We now describe a useful consequence of the above proposition for convergence involving metric spaces that are not necessarily compact. Let $\{(X_n, d_n, \mu_n, x_n)\}$ be a sequence of proper pointed metric measure spaces that converges to a proper pointed metric measure space (X, d, μ, x) with μ finite on bounded sets. Further let $y_n \in X_n$ for each $n \in \mathbf{N}$, and suppose that (y_n) converges to $y \in X$ (here we have implicitly adopted a collection of embeddings into some metric space (Z, l) as described in Definition 2.4.2, and view this convergence as taking place in (Z, l)). We can then apply Proposition 3.2.1 to conclude that (3) and (4) hold whenever each Γ_n is now a family of discrete curves in $B_n(y_n, r)$, and each Λ_n is now a family of properly discrete ϵ-curves in $B_n(y_n, r)$, where $\epsilon, r > 0$ are fixed. Here $B_n(w, s)$ denotes the ball in (X_n, d_n) with center $w \in X_n$ and radius $s > 0$.

To see this, pass to a subsequence of $\{(X_n, d_n, \mu_n, y_n)\}$ so that the lim sup in the right-hand side of (3) is achieved, and let $\hat{\mu}_n = \mu \llcorner B_n(y_n, 3r)$ be the restriction of μ to $B_n(y_n, 3r)$ for each $n \in \mathbf{N}$. Pass to another subsequence so that the sequence (B_n) comprised of the completion of each $B_n(y_n, 4r)$ converges to some $A \subset X$ and so that $(\hat{\mu}_n)$ converges weakly to some Borel measure $\hat{\mu}$ on X. Observe that $\{(B_n, d_n, \hat{\mu}_n)\}$ converges to $(A, d, \hat{\mu})$, and also that $\mu(A) < \infty$. It follows that (3) holds with the modulus of Γ and each Γ_n calculated using the measures $\hat{\mu}$ and $\hat{\mu}_n$, respectively.

Now each curve in Γ, where $\Gamma = \limsup_{n\to\infty} \Gamma_n$, is contained in $B(x, 2r)$ for n sufficiently large, and moreover $\mu = \hat{\mu}$ on $B(x, 2r)$. Therefore $\text{mod}_p(\Gamma)$ is equivalently defined by both measures μ and $\hat{\mu}$. We conclude that (3) holds with modulus for Γ and each Γ_n calculated using the measures μ and μ_n, respectively. A similar argument can be used to verify (4) in this setting. This completes the proof of the claim of this remark.

We now turn to the proof of Proposition 3.2.1. Assume the hypotheses of Proposition 3.2.1, and let $\Gamma = \limsup_{n\to\infty} \Gamma_n$ and $\Lambda = \text{dis-lim sup}_{n\to\infty}\Lambda_n$. Fix a compact metric space (Z, l) and isometric embeddings $\iota : X \to Z$ and $\iota_n : X_n \to Z$ for each $n \in \mathbf{N}$, that satisfy the conditions of Definition 2.4.1, and that realize $\iota\Gamma = \limsup_{n\to\infty}(\iota\Gamma_n)$ and $\iota\Lambda = \limsup_{n\to\infty}(\iota\Lambda_n)$. Identify all relevant sets, measures, and curves, with their images under ι and each ι_n.

LEMMA 3.2.4. *Let $\tau > 0$, let $g : Z \to [0,\infty)$ be continuous and admissible for Γ, and suppose that Γ does not contain any constant curves. Then there exists $N \in \mathbf{N}$ such that $g+\tau$ is admissible at scale ϵ_n for Γ_n whenever $n \geq N$.*

Proof. In order to get a contradiction suppose that the lemma does not hold. Then after passing to a subsequence we have $g+\tau$ is not admissible at scale ϵ_n for Γ_n for every $n \in \mathbf{N}$. Consequently there exists ϵ_n-curves $\gamma_n \in \Gamma_n$ for each $n \in \mathbf{N}$ such that

$$\sum_{i=1}^{\#\gamma_n - 1} d(x_i, x_{i+1})\big(\tau + g_n(x_i)\big) < 1\,, \tag{5}$$

where $g_n : Z \to [0,\infty)$ is given by

$$g_n(x) = \inf_{d(y,x)\leq\epsilon_n} g(y)\,,$$

for every $x \in Z$. It follows that $\ell(\gamma_n) \leq \tau^{-1}$.

Pass to a subsequence so that $(\ell(\gamma_n))$ converges to some $l \geq 0$, and let $L = \sup_n \ell(\gamma_n)$. Fix parameterizations $\gamma_n : [0,\ell(\gamma_n)) \to X$, and extend γ_n to a function defined on $[0,L)$, by defining it to be equal to the endpoint of $\gamma_n(\#\gamma_n)$ on $[\ell(\gamma_n), L)$. By a variant of the Arzelá–Ascoli theorem, and using (2) from the definition of the parameterization of a discrete curve, and the fact that Z is compact, we can again pass to a subsequence so that (γ_n) converges uniformly to some 1-Lipschitz function $\gamma : [0,L) \to X$. By definition we have $\gamma \in \Gamma$, and thus from our assumption that Γ contains no constant curve, we have $l > 0$.

Observe that for every $n \in \mathbf{N}$ we have

$$\int_0^{\ell(\gamma_n)} g_n \circ \gamma_n(t)dt = \sum_{i=1}^{\#\gamma_n - 1} d(x_i, x_{i+1}) g_n(x_i)\,.$$

Recall that g is continuous and that (g_n) converge uniformly to g. We therefore have

$$\int_0^l g\circ\gamma(t)dt \leq \liminf_{n\to\infty} \int_0^l g_n \circ \gamma_n(t)dt = \liminf_{n\to\infty} \int_0^{\ell(\gamma_n)} g_n\circ\gamma_n(t)dt\,.$$

Since γ is 1-Lipschitz, we have

$$\int_\gamma g\,ds \leq \int_0^l g\circ\gamma(t)dt\,.$$

The last three inequalities together with (5) imply that

$$\int_\gamma g\, ds \le 1 - l\tau < 1\,.$$

Since $\gamma \in \Gamma$, this contradicts the fact that g is admissible for Γ. This completes the proof. □

The proof of the following lemma is similar to the above proof, and so omitted.

LEMMA 3.2.5. *Let $\tau > 0$, let $h : Z \to [0,\infty)$ be continuous and admissible for Λ, and suppose that Λ does not contain any constant curves. Then there exists $N \in \mathbf{N}$ such that $h+\tau$ is admissible at scale ϵ for Λ_n whenever $n \ge N$.*

In order to continue we need a result of the first author. To state the result we recall that a sequence of rectifiable curves (γ_n) on a metric space (Y,e) is said to *converge to* another curve γ contained in Y, if there exist uniformly Lipschitz parameterizations of the given curves, all with the same domain, so that when viewed as functions the sequence (γ_n) converges uniformly to γ. Also, a family of curves is said to be *closed* if the limit of every sequence of converging curves in the family, is again a member of the family.

PROPOSITION 3.2.6 [K3]. *Let $q \ge 1$, let Υ be a closed family of curves contained in a proper metric measure space (Y,e,ν) with $\nu(Y) < \infty$, such that there exists a bounded subset of Y that meets every curve in Υ, and let $\tau > 0$. Then there exists a continuous function $h : Y \to (0,\infty)$ admissible for Υ, such that*

$$\int_Y h^q\, d\mu \le \operatorname{mod}_q(\Upsilon) + \tau\,.$$

The first claim of Proposition 3.2.1 is trivially true if Γ contains degenerate curves. (We use the convention here that $\inf \emptyset = \infty$). We therefore exclude this case. Let $\tau > 0$. Notice that Γ is necessarily closed, and that by hypothesis we have $\mu(X) < \infty$. Proposition 3.2.6 then asserts that there exists a continuous function $g : X \to [0,\infty)$ that is admissible for Γ, such that

$$\int_Z g^p\, d\mu \le \operatorname{mod}_p(\Gamma) + \tau\,. \tag{6}$$

Since $\mu(X) < \infty$, we can further assume that g is bounded away from 0. Extend g to a continuous and non-negative function defined on Z, and so that $\inf_Z g > 0$. Lemma 3.2.4 asserts that there exists $N \in \mathbf{N}$ such that $(1+\tau)g$ is discretely admissible for Γ_n at scale ϵ_n whenever $n > N$. We therefore have

$$\operatorname{mod}_p(\Gamma_n, \epsilon_n) \leq \int_Z (1+\tau)^p g^p \, d\mu_n \, .$$

It then follows from (6) and the weak convergence of (μ_n) to μ, that

$$\limsup_{n\to\infty} \operatorname{mod}_p(\Gamma_n, \epsilon_n) \leq (1+\tau)^p \int_X g^p \, d\mu \leq (1+\tau)^p \big(\operatorname{mod}_p(\Gamma) + \tau \big) \, .$$

This completes the proof of (3). We can similarly deduce (4) from Lemma 3.2.5. This completes the proof of Proposition 3.2.1.

3.3 Annulus systems with infinitesimally small modulus. In this section we establish restraints in terms of discrete modulus for metric measure spaces whose weak tangents have quantitatively small modulus. Ultimately this will be used to build quasisymmetric homeomorphisms that lower the Assouad dimension of the given metric measure space. We urge the dispirited readers to skip the following definitions (on their first reading), and replace the hypotheses of Proposition 3.3.3 by the simplified assumptions that the given metric measure space (X, d, μ) is α-homogeneous, for some $\alpha \geq 1$, and that all the weak tangents of (X, d, μ) have vanishing p-modulus for some $p \geq 1$.

Given $0 < t < T$, and a point x in a metric space (X, d), define the *(x,t,T)-annulus* by

$$A(x,t,T) = B(x,T) \setminus B(x,t) \, .$$

Also let $\Theta(x,t,T)$ denote the family of both curves and discrete curves in X that meet both

$$X \setminus B(x,T) \quad \text{and} \quad \big\{ y \in X : d(x,y) \leq t \big\} \, .$$

We say that an annulus $A(x,t,T)$ in any weak tangent of a metric space (X,d) is the *re-scaled limit* of $\{A(x_n,t_n,T_n)\}$, where $(x_n) \subset X$ and $(t_n), (T_n) \subset (0,\infty)$, if the following holds. We have $(X_\infty, d_\infty, x_\infty)$ is the limit of the sequence comprised of the completion of $(X, d/r_n, x_n)$ for each $n \in \mathbf{N}$, for some sequence (r_n) of positive reals, and for some $(x_n) \subset X$, and we have $T = \lim_{n\to\infty} T_n/r_n < \infty$ and $t = \lim_{n\to\infty} t_n/r_n > 0$. (Here we implicitly assume that the completion of each $(X, d/r_n)$ is proper.) If (X, d, μ) is a metric measure space, then we say that a measure μ_∞ on $(X_\infty, d_\infty, x_\infty)$ is *associated* to $A(x_\infty, t, T)$ if there exists a sequence of positive reals (s_n) such that some subsequence of $\{(X_n, d/r_n, \mu/s_n, x_n)\}$ converges to $(X_\infty, d_\infty, \mu_\infty, x_\infty)$. We remark that the re-scaled limit of an annulus is generally different from the generalized Hausdorff limit of a sequence of sets each given by the annulus.

Definition 3.3.1. A collection of annuli $\mathcal{A}$ in a metric space (X,d) is called a λ-*annulus system* of (X,d) (or just an *annulus system*), for some

$0 < \lambda < 1$, if for every $x \in X$ and $r > 0$, there exists $A(y, \lambda s, s) \in \mathcal{A}$ for some $y \in B(x, r)$ and $\lambda r \leq s \leq r$. Furthermore, every annulus in $\mathcal{A}$ is of this form.

Definition 3.3.2. An annulus system $\mathcal{A}$ of a metric measure space is said to have *infinitesimally small (η, p)-modulus*, for some $\eta > 0$ and $p \geq 1$, if we have

$$\operatorname{mod}_p\big(\Theta(x,t,T)\big) \leq \eta\nu\big(B(x,T)\big)T^{-p},$$

whenever $A(x, t, T)$ is a re-scaled limit of a sequence of annuli in $\mathcal{A}$, and where ν is any non-vanishing measure associated to $A(x, t, T)$.

Proposition 3.3.3. *Let $\eta, \sigma > 0$ and $\alpha, p \geq 1$, and let (X, d, μ) be an α-homogeneous metric measure space that admits an annulus system $\mathcal{A}$ with infinitesimally (η, p)-small modulus. Then there exists $\Psi \geq 1$ such that whenever $A(x, t, T) \in \mathcal{A}$ for some $x \in X$ and $0 < t < T$, there exists a continuous function $g : X \to [(\Psi T)^{-1}, \Psi T^{-1}]$ that is discretely admissible at scale $\Psi^{-1}T$ for $\Theta(x, t, T)$, such that*

$$\int_X g^p \, d\mu \leq (1+\sigma)\eta\mu\big(B(x,T)\big)T^{-p}. \tag{7}$$

Proof. Without loss of generality we can assume that (X, d) is proper. (Observe that the hypotheses of Proposition 3.3.3 are preserved under completion. Whereas, a complete homogeneous metric measure space is necessarily proper). In order to get a contradiction suppose that the conclusion of the proposition is false. Then there exists a sequence of annuli given by $A(x_n, t_n, T_n) \in \mathcal{A}$ for every $n \in \mathbf{N}$, where $(x_n) \subset X$ and $(t_n), (T_n) \subset (0, \infty)$, such that the following holds. First we have $\lambda = \lim_{n\to\infty} t_n/T_n$ for some $\lambda > 0$. More importantly we have that there is no continuous function $g : X \to [(nT_n)^{-1}, nT_n^{-1}]$ that is discretely admissible at scale $n^{-1}T_n$ for $\Theta(x_n, t_n, T_n)$, such that

$$\int_X g^p \, d\mu \leq (1+\sigma)\eta\mu\big(B(x_n,T_n)\big)T_n^{-p}.$$

For each $n \in \mathbf{N}$, write $X_n = X$ and define the metric d_n on X_n by $d_n = d/T_n$. Let $\mu_n = \mu/\mu(B(x_n, T_n))$. We now rewrite the above statement for the metric measure space (X, d, μ) in terms of the metric measure spaces $\{(X_n, d_n, \mu_n)\}$. For every $n \in \mathbf{N}$, there is no continuous function $g : X_n \to [n^{-1}, n]$ that is discretely admissible at scale n^{-1} for $\Theta(x_n, \lambda, 1)$, such that

$$\int_{X_n} g^p \, d\mu_n \leq (1+\sigma)\eta\,.$$

The desired contradiction will be achieved by showing that this last statement is false for some $n \in \mathbf{N}$.

Observe that $\{(X_n, d_n, \mu_n, x_n)\}$ is a sequence of α-homogeneous metric measure spaces with uniform homogeneity constant. Thus by Lemma 2.4.3 we can pass to a subsequence so that $\{(X_n, d_n, \mu_n, x_n)\}$ converges to a weak tangent $(X_\infty, d_\infty, \mu_\infty, x_\infty)$ of (X, d, μ, x). Fix a proper metric space (Z, l) and isometric embeddings $\iota : X \to Z$ and $\iota_n : X_n \to Z$ for each $n \in \mathbf{N}$, that realize this convergence (as described in Definition 2.4.2). Identify all relevant sets, measures, and curves, with their images under ι and each ι_n.

Let $\tau > 0$. By assumption $\operatorname{mod}_p(\Theta(x_\infty, \lambda, 1)) \le \eta$. Therefore Proposition 3.2.6 provides a bounded and continuous function $g : B_\infty(x_\infty, 2) \to \mathbf{R}$ that is admissible for $\Theta(x_\infty, \lambda, 1)$, such that

$$\int_{B_\infty(x_\infty,2)} g^p \, d\mu_\infty < \eta + \tau \, .$$

Here $B_\infty(y, r)$ denotes the ball in (X_∞, d_∞) with center $y \in X_\infty$ and radius $r > 0$, and for each $n \in \mathbf{N}$, we let $B_n(w, r)$ denote the ball in (X_n, d_n) with center $w \in X_n$ and radius $r > 0$. Since $\mu_\infty(B_\infty(x_\infty, 2)) < \infty$ we can further arrange for $1/M \le g \le M$ for some $M \in \mathbf{N}$. Now extend g to a continuous function on Z so that $1/M \le g \le M$.

Since g is admissible for $\Theta(x_\infty, \lambda, 1)$, there exists $N \in \mathbf{N}$ such that $(1+\tau)g$ is discretely admissible at scale n^{-1} for $\Theta(x_n, \lambda, 1)$ whenever $n \ge N$. This follows from Lemma 3.2.4; see also Remark 3.2.3. It follows from the weak convergence of (μ_n) to μ that

$$\limsup_{n\to\infty} \int_{B_n(x_n,1)} (1+\tau)^p g^p d\mu_n \le \int_{B_\infty(x_\infty,2)} (1+\tau)^p g^p d\mu \le (1+\tau)^p(\eta+\tau) \, .$$

For an appropriate choice of $\tau > 0$ and then a sufficiently large choice of $n \in \mathbf{N}$, the above properties of g contradict the assumption on (X_n, d_n, μ_n). This completes the proof. □

4 Uniformly Big 1-modulus on the Tangent Space

We now define the notion of uniformly big modulus and then show that this property occurs in a certain quantitative sense on tangents of homogeneous metric measure spaces with non-vanishing modulus.

Definition 4.0.4. A metric measure space (X, d, μ) is said to have *uniformly big p-modulus with constant δ*, for some $p \ge 1$ and $\delta > 0$, if μ is non-vanishing, and we have

$$\operatorname{mod}_p(\Gamma) \ge \delta \mu\big(B(x, r)\big) r^{-p},$$

whenever $x \in X$ and $0 < r \le \operatorname{diam} X$, and whenever Γ is the collection of all curves in $B(x, r)$ with diameter at least δr.

We now list some immediate observations about uniformly big modulus. Every homogeneous metric measure space that admits a $(1,p)$-Poincaré inequality, for some $p \geq 1$, as defined by Heinonen and Koskela [HK], has uniformly big p-modulus (see [K3] for the proof of related claims). The converse is not true, as demonstrated by the product of an Ahlfors regular metric space with the unit interval. Like the Poincaré inequality, every metric space with uniformly big p-modulus has uniformly big q-modulus for $q \geq p$. This follows from an application of Jensen's inequality. In particular the strongest of these conditions is to have uniformly big 1-modulus. Again similar to the Poincaré inequality, and by an application of a theorem of Tyson [T2, Theorem 6.4], the property of having uniformly big α-modulus, for some $\alpha \geq 1$, is quantitatively preserved by quasisymmetric homeomorphisms between Ahlfors α-regular metric spaces.

Theorem 4.0.5. *Let $\alpha, p \geq 1$ and let (X,d,μ) be an α-homogeneous metric measure space with non-vanishing p-modulus. Then there exists a Borel set $A \subset X$ of positive measure such that every tangent of (X,d,μ) with non-vanishing measure, at any member of A, has uniformly big 1-modulus.*

REMARK 4.0.6. Theorem 4.0.5 is quantitative in the following sense. First assume that (X,d,μ) is a bounded and α-homogeneous metric measure space, $\alpha \geq 1$, and that $\operatorname{mod}_p(\Gamma) > 0$ for some $\tau > 0$ and $p \geq 1$, where Γ is the collection of rectifiable curves in X with endpoints at least a distance τ apart. Then there exists a Borel set $A \subset X$ such that every tangent of (X,d,μ) with non-vanishing measure, at any member of A, has uniformly big 1-modulus. Moreover, both the uniformly big 1-modulus constant and $\mu(A)$ depend only on p, τ, $\operatorname{mod}_p(\Gamma)$, $\operatorname{diam} X$, $\mu(X)$, α, and the homogeneity constant of μ.

Recall that every α-homogeneous metric measure space (X,d,μ) can be exhausted by closed and bounded homogeneous metric measure subspaces with uniform control on the homogeneity constants that depend only on α and the homogeneity constant of (X,d,μ). Now observe that if (X,d,μ) has non-vanishing p-modulus, then so too does one of these exhausting subspaces. This is true because modulus is countably subadditive. Also observe that when $p = 1$, we can obtain

$$\operatorname{mod}_1(\Gamma)^q \mu(X)^{1-q} \leq \operatorname{mod}_q(\Gamma)\,,$$

by an application of Jensen's inequality, for every $q \geq 1$, and for any family Γ of curves in X. Finally notice that claims of Remark 4.0.6 are scale invariant. We conclude that in order to prove Theorem 4.0.5, it suffices to verify

Remark 4.0.6 for every α-homogeneous metric measure space (X, d, μ) with $\operatorname{diam} X = 1$ and $\mu(X) = 1$, where $\alpha, \tau > 0$ and $p > 1$. We now do this.

Let $C > 0$ denote a varying constant whose value depends only on p, τ, α, and the homogeneity constant of (X, d, μ). Let $0 < \eta \leq 1$, and make the further assumption on X that there exists a Borel set $E \subset X$ such that $\mu(E) < \eta$, and such that if $F \subset X \setminus E$ is a Borel set each of whose tangent (with non-vanishing measure) has uniformly big 1-modulus with constant at least $C\eta$, then $\mu(F) = 0$. We shall deduce from this assumption that

$$\operatorname{mod}_p(\Gamma) < C\eta^C. \tag{8}$$

This then verifies Remark 4.0.6 by the contrapositive. Recall here that Γ is the collection of rectifiable curves in X with endpoints at least a distance τ apart.

Given $\theta, \vartheta > 0$ and a subset E of a metric space, we let $\Lambda(E, \theta, \vartheta)$ denote the family of properly ϑ-discrete curves in E with diameter at least θ. Let $\triangle = \cup_{n\in\mathbf{N}}\triangle_n$ be a dyadic decomposition of (X, d, μ). Next define a function $\mathcal{B} : \mathbf{R}^+ \times \mathbf{N} \to \mathcal{P}(\triangle)$ by setting $\mathcal{B}(\Psi, m)$, for $\Psi \geq 0$ and $m \in \mathbf{N}$, to be the collection of maximal cubes in $\cup_{k\geq m}\triangle_m$ such that

$$\operatorname{mod}_1\big(\Lambda(Q, \eta \operatorname{rad} Q, \Psi \operatorname{rad} Q), \Psi \operatorname{rad} Q\big) < \eta\mu(Q)(\operatorname{rad} Q)^{-1}. \tag{9}$$

(Here $\mathcal{P}(\triangle)$ denotes the power set of $\triangle$). The following two lemmas independently establish results for $\mathcal{B}$ that are then applied in Lemma 4.0.9.

LEMMA 4.0.7. *We have*

$$\limsup_{\Psi\to 0}\limsup_{m\to\infty}\mu\big(X \setminus (\cup\mathcal{B}(\Psi, m) \cup E)\big) = 0\,. \tag{10}$$

Proof. In order to get a contradiction suppose that (10) does not hold. It follows that there exists $\sigma > 0$ and there exist sequences $(\Psi_n), (m_n) \subset (0, \infty)$ that converge to 0 and ∞, respectively, such that for every $n \in \mathbf{N}$, we have $\mu(A_n) \geq \sigma$, where A_n is defined to consist of the points of density of $X \setminus (\cup\mathcal{B}(\Psi_n, m_n) \cup E)$. (Here we use the fact that almost every point of a measurable set is a point of density – this holds because μ is doubling; see [H, Theorem 1.8].) By assumption $\mu(X) < \infty$. Therefore $A = \limsup_{n\to\infty} A_n$ is a Borel subset of X with $\mu(A) \geq \sigma$ such that every member of A is in turn a member of an infinite sub-collection of (A_n).

To complete the proof it suffices to show that every tangent of (X, d, μ) with non-vanishing measure, at every member of A, has uniformly big 1-modulus with constant $C\eta$. This then contradicts our assumption on E. Let $a \in A$ and let $(X_\infty, d_\infty, \mu_\infty, a_\infty)$ be a tangent of (X, d, μ) at a with non-vanishing measure. Then there exists sequences (r_j) and (s_j) of positive reals that converge to 0, such that $\{(X_j, d_j, \mu_j, a_j)\}$ converges to

$(X_\infty, d_\infty, \mu_\infty, a_\infty)$, where $X_j = X$, $d_j = d/r_j$, $\mu_j = \mu/s_j$ and $a_j = a$, for each $j \in \mathbf{N}$, and where μ_∞ is non-vanishing.

Fix $x \in X_\infty$ and $R > 0$. In order to demonstrate $(X_\infty, d_\infty, \mu_\infty)$ has uniformly big 1-modulus with constant η, we need to show that the collection of curves in $B(x, CR)$ with diameter at least $C\eta R$ has 1-modulus at least $C\eta\mu(B(x, CR))R^{-1}$. Due to Proposition 3.2.1 (see also Remark 3.2.3), this will follow if we can demonstrate that

$$\operatorname{mod}_1\big(\Lambda(B(x, CR), C\eta R, C\Psi_n R), C\Psi_n R\big) \geq C\eta\mu\big(B(x, R)\big)R^{-1}, \tag{11}$$

for some subsequence of (Ψ_n).

Pass to a subsequence of (A_n) so that A_n has a point of density at a for every $n \in \mathbf{N}$, and then fix $n \in \mathbf{N}$. Since a is a point of density of A_n, and because μ is homogeneous, it is a standard fact that there exists a sequence of points (x_j) in A_n that converge to x (see [K1, Lemma 6.2.3]), where we view $x_j \in X_j$ for each $j \in \mathbf{N}$. Here we have implicitly adopted a collection of embeddings as described in Definition 2.4.2. Now fix an integer $j \geq m_n$ and fix $Q \in \triangle_{[\log_2 Rr_j]}$ that contains x_j. It then follows from $x_j \in X \setminus \cup\mathcal{B}(\Psi_n, m_n)$ that

$$\operatorname{mod}_1\big(\Lambda(Q, \eta 2^{[\log_2 Rr_j]}, \Psi_n 2^{[\log_2 Rr_j]}), \Psi_n 2^{[\log_2 Rr_j]}\big) \geq C\eta\mu(Q)(Rr_j)^{-1}.$$

Observe also that $Q \subset B(x_j, CRr_j)$. Consequently we can rewrite the above inequality in terms of the metric measure space (X_j, d_j, μ_j) to get

$$\operatorname{mod}_1\big(\Lambda(B_j(x_j, CR), C\eta R, C\Psi_n R), C\Psi_n R\big) \geq C\eta\mu_j\big(B(x_j, R)\big)R^{-1}.$$

Here $B_j(x_j, CR)$ is the ball in (X_j, d_j) centered at x_j and with radius CR. Since

$$\mu_\infty\big(B(x, R)\big) \leq C \limsup_{j\to\infty} \mu_j\big(B(x_j, R)\big)\,,$$

we can now apply Proposition 3.2.1 (see also Remark 3.2.3) to conclude that (11) holds. This completes the proof. □

LEMMA 4.0.8. *Let $\Psi \leq \eta$ and $Q \in \triangle$, and suppose that* (9) *holds. Then there exists a continuous function $g : Q \to [0, 10\Psi^{-1}]$ such that*

$$\fint_Q g\, d\mu \leq 2\eta\,, \tag{12}$$

and such that for every curve γ in X that intersects $Q \setminus \partial_{3\eta} Q$ and $X \setminus Q$, there exists a subcurve γ' of γ in Q such that

$$\int_{\gamma'} g\, ds \geq \operatorname{rad} Q\,. \tag{13}$$

Proof. It follows from (9) that there exists a continuous function $h : Q \to (0, \infty)$ that is $(\Psi \operatorname{rad} Q)$-admissible for $\Lambda(Q, \eta \operatorname{rad} Q, \Psi \operatorname{rad} Q)$, such that

$$\int_Q h\, d\mu \leq 2\eta\mu(Q)(\operatorname{rad} Q)^{-1}\,.$$

Since each curve in $\Lambda(Q, \eta \operatorname{rad} Q, \Psi \operatorname{rad} Q)$ is properly $(\Psi \operatorname{rad} Q)$-discrete, we can further assume that $\sup h \le 10(\Psi \operatorname{rad} Q)^{-1}$. We emphasize that this last estimate is crucial, and is the motivation behind our current use and development of discrete modulus.

Let γ be a rectifiable curve in X that intersects $Q \setminus \partial_{3\eta} Q$ and $X \setminus Q$, and then let γ' be a closed subcurve of γ in Q with $\operatorname{diam} \gamma' \ge 2\eta \operatorname{rad} Q$. We would like to show that

$$\int_{\gamma'} h \, ds \ge 1 \,. \tag{14}$$

To do this we inductively define a discrete curve $(x_i)^N \subset \gamma'$ for some $N \in \mathbf{N}$. Let $x_1 = \gamma(0)$. Assume x_{i-1} has been chosen for some $i \ge 1$, and let $x_i = \gamma(t)$ where

$$t = \sup \bigl\{ r \ge 0 : \gamma(r) \in B(x_{i-1}, \Psi \operatorname{rad} Q) \bigr\} \,.$$

We halt the inductive process when $d(\gamma(t), \gamma(0)) \ge \eta \operatorname{rad} Q$ and then let $N = i$. The inductive sequence will always halt because of the diameter constraint for γ'. It follows by construction that $d(x_i, x_{i+1}) = \Psi \operatorname{rad} Q$ for $1 \le i \le N-1$. Consequently $\{x_i\}_{i=1}^N \in \Lambda(Q, \eta \operatorname{rad} Q, \Psi \operatorname{rad} Q)$. Due to the discrete admissibility of h for this collection of discrete curves, we have

$$1 \le \sum_{i=1}^{N-1} d(x_i, x_{i+1}) \inf_{y \in B(x_i, \Psi \operatorname{rad} Q)} h(y) \,.$$

From this we conclude that (14) holds. The claim of the lemma can now easily be verified for $g : Q \to \mathbf{R}$ defined by $g(x) = h(x) \operatorname{rad} Q$ for every $x \in Q$. This completes the proof. $\square$

In the following we denote the characteristic function of any set U by χ_U. Recall that Γ is the collection of rectifiable curves in X with endpoints at least a distance τ apart.

LEMMA 4.0.9. *There exists $\Psi > 0$ such that for every $M \in \mathbf{N}$, there exists $m > M$, there exists an open set $U \subset X$, and there exists $\rho : X \to [0, 10\Psi^{-1}]$, such that $\mu(U) < C\eta^C$, such that $C(\rho + \chi_U)$ is admissible for Γ, and such that*

$$⨍_Q \rho \, d\mu \le C\eta \,, \tag{15}$$

for every $Q \in \cup_{k \le m} \triangle_k$.

Proof. It follows Lemma 4.0.7 that there exists $\Psi \le \eta$ such that for every $M \in \mathbf{N}$, there are integers $l > m > M$ such that

$$\mu\bigl(X \setminus (\cup \mathcal{B} \cup E)\bigr) < \eta \,,$$

where

$$\mathcal{B} = \mathcal{B}(\Psi, m) \cap \bigcup_{k \le l} \triangle_k \,.$$

Since $\mathcal{B}$ is an essentially disjoint collection of cubes, it follows from the properties of the dyadic decomposition that

$$\mu\Big(\bigcup_{Q \in \mathcal{B}} \partial_{3\eta} Q \Big) \le \sum_{Q \in \mathcal{B}} C\eta^C \mu(Q) \le C\eta^C.$$

Also, by assumption we have $\mu(E) < \eta$. Consequently there exists an open set $U \subset X$ such that $\mu(U) \le C\eta^C$, and such that

$$\big(X \setminus \cup\mathcal{B}\big) \cup U \cup \bigcup_{Q \in \mathcal{B}} \partial_{3\eta} Q \subset U \,.$$

Recall that $\Psi \le \eta$. Therefore for every $Q \in \mathcal{B}$ there is a function $g_Q : Q \to [0, 10\Psi^{-1}]$ that satisfies the assertions of Lemma 4.0.8. Define $\rho : X \to [0, 10\Psi^{-1}]$ by

$$\rho(x) = \begin{cases} g_Q(x) \,, & \text{if } x \in Q \text{ for some } Q \in \mathcal{B} \,, \\ 0 \,, & \text{otherwise} \,. \end{cases}$$

Because of (12) we have (15) holds for every $Q \in \mathcal{B}$. Also notice that ρ vanishes on $X \setminus \cup\mathcal{B}$. Finally recall that $\mathcal{B} \subset \cup_{k \ge m} \triangle_k$. We therefore conclude that (15) holds for every $Q \in \cup_{k \le m} \triangle_k$.

To complete the proof it remains to establish that $C(\rho + \chi_U)$ is admissible for Γ. Fix $\gamma \in \Gamma$. We can assume that γ is closed, and write $\gamma : [0, L] \to X$ for some $L > 0$. We now inductively define sequences $\{b_i\}_{i=0}^{N}, \{a_i\}_{i=1}^{N+1} \subset [0, L]$ for some $N \in \mathbf{N}$, to be used to decompose γ into a consecutive union of paths, that are either deeply contained in some $Q \in \mathcal{B}$, or that live the entirety of their life on U. First let $b_0 = 0$. Assume that b_{i-1} is defined for some integer $i \ge 1$ and then let

$$a_i = \inf \big\{ a > b_{i-1} : \gamma(a) \in \cup\mathcal{B} \setminus U \big\} \,.$$

Now let

$$b_i = \sup \big\{ b \le L : \gamma(b) \in Q \big\} \,,$$

where Q is the unique cube in $\mathcal{B}$ that contains $\gamma(a_i)$. We halt the inductive process when the infimum in the definition of a_i is taken over an empty set, and then let $N = i$ and $a_N = L$. The inductive process will always halt. Otherwise each $\gamma(a_i)$ is contained in a distinct member of $\mathcal{B}$ for each $i \in \mathbf{N}$, whereas by construction the cardinality of $\mathcal{B}$ is finite.

From our construction of ρ and because of (13), we have that for each $1 \le i \le N - 1$ there exists a subcurve γ' of γ in Q such that

$$d\big(\gamma(a_i), \gamma(b_i)\big) \le \operatorname{rad} Q \le \int_{\gamma'} \rho \, ds \,.$$

Whereas for each $0 \le i \le N$ we have $\gamma((b_i, a_{i+1})) \subset U$ and consequently that
$$d\big(\gamma(b_i), \gamma(a_{i+1})\big) \le \ell\big(\gamma((b_i, a_{i+1}))\big) \le \int_{\gamma(b_i)}^{\gamma(a_{i+1})} \chi_U \, ds \,,$$
where the last integral is along the curve γ from $\gamma(b_i)$ to $\gamma(a_{i+1})$. These last two estimates together with the triangle inequality imply that
$$\begin{aligned}\tau \le d\big(\gamma(0), \gamma(a_N)\big) &\le \sum_{i=1}^{N-1} d\big(\gamma(a_i), \gamma(b_i)\big) + \sum_{i=0}^{N-1} d\big(\gamma(b_i), \gamma(a_{i+1})\big) \\ &\le \int_\gamma \rho + \chi_U \; ds \,.\end{aligned}$$
This completes the proof. $\square$

We are now sufficiently equipped to prove (8) and thereby complete the proof of Theorem 4.0.5. A consequence of Lemma 4.0.9 is that there exists a sequence $(m_n) \subset \mathbf{N}$ that converges to ∞, there exists a sequence (U_n) in X, and there exists a sequence of nonnegative Borel functions (ρ_n) defined on X, such that the following holds for every $n \in \mathbf{N}$: We have $\mu(U_n) < C\eta^C$, we have $C(\rho_n + \chi_{U_n})$ is admissible for Γ, we have $\sup_X \rho_n < 10\Psi^{-1}$, and we have
$$⨍_Q \rho_n \, d\mu \le C\eta \tag{16}$$
for every $Q \in \cup_{k \le m_n} \triangle_k$. In particular and since $\mu(X) < \infty$, the sequences $(\|\rho_n\|_p)$ and $(\|\chi_{U_n}\|_p)$ are uniformly bounded. By reflexivity there exists $\rho, h \in L^p(X)$ such that after passing to a subsequence, the sequences (ρ_n) and (χ_{U_n}) converge weakly in $L^p(X)$ to ρ and h, respectively. A consequence of a theorem of Fuglede [Fu, Theorem 3(f)] is that there exists $\Gamma' \subset \Gamma$ with $\mathrm{mod}_p(\Gamma') = 0$, such that $C(\rho + h)$ is admissible for every curve in $\Gamma \setminus \Gamma'$ (see the comments following the proof of [Fu, Theorem 3(f)]; see also the proof of [HKST, Theorem 8.8]). The above argument is where $p > 1$ is used.

Therefore in order to establish (8) it needs to be shown that
$$\int_X (\rho + h)^p \, d\mu \le C\eta^C.$$
To see this holds observe that
$$\int_X h^p \, d\mu \le \sup_n \int_X (\chi_{U_n})^p \, d\mu = \sup_n \mu(U_n) \le C\eta^C.$$
Whereas since (16) holds for every $Q \in \triangle$ whenever $n \in \mathbf{N}$ is sufficiently large, we have
$$⨍_Q \rho \, d\mu = \lim_{n\to\infty} \frac{1}{\mu(Q)} \int_X \rho_n \chi_Q \, d\mu \le C\eta^C,$$

and therefore $\rho \leq C\eta^C$ almost everywhere in X. This establishes (8) and thereby completes the proof of Remark 4.0.6 and Theorem 4.0.5.

5 Lowering the Assouad Dimension Through Quasisymmetric Homeomorphisms

The proof of the following theorem is formally given in section 5.3, and relies on the accumulating results of Proposition 3.3.3, Proposition 5.1.1, and Proposition 5.2.1. The non-quantitative version of Theorem 5.0.10 is applied in section 6 to prove Theorem 1.0.1. Like before we encourage the dispirited readers to, on their first reading, replace the technical hypotheses of the following theorem with the simplified assumptions that the given metric measure space (X, d) is α-homogeneous for some $\alpha \geq 1$, and that all the weak tangents of (X, d) have vanishing α-modulus.

Theorem 5.0.10. *Let $\alpha, \eta > 0$ and $0 < \lambda < 1$, and let (X, d, μ) be an α-homogeneous metric measure space that admits a λ-annulus system with infinitesimally (η, α)-small modulus. Further suppose that $\eta \leq L\lambda^{2\alpha}$ where $L > 0$ is a constant that depends only on α and the homogeneity constant of μ. Then there exists $0 < \alpha' < \alpha$ such that (X, d) is quasisymmetrically homeomorphic to a metric space with Assouad dimension equal to α'.*

Remark 5.0.11. Observe that the hypotheses of the above theorem is trivially satisfied if every weak tangent of the given metric space has vanishing α-modulus. It is ultimately a consequence of Theorem 1.0.1 that every weak tangent of a metric space (X, d) has vanishing modulus, whenever (X, d) satisfies the hypotheses of Theorem 5.0.10 and is Ahlfors regular, and η is small enough. Thus we see that for Ahlfors regular metric spaces there is a clean dichotomy between admitting a weak tangent with uniformly big 1-modulus and admitting an annulus system with (sufficiently) infinitesimally small modulus.

Remark 5.0.12. The hypotheses of the above theorem can be expanded to include collections of α-homogeneous metric measure spaces $\mathcal{X}$, as long as we have the same homogeneity bounds for each member of $\mathcal{X}$, and where we have modulus estimates for those annuli obtained as *generalized* rescaled limits of sequences of annuli. By this we mean that the sequence of annuli can be comprised of annuli from different members of $\mathcal{X}$. The conclusion is then that there exists $0 < \alpha' < \alpha$ such that every member of $\mathcal{X}$ is quasisymmetrically homeomorphic to some Ahlfors α'-regular metric space. This claim can be easily inferred from the proof of Theorem 5.0.10, and we leave the details to the reader.

5.1 Discrete insulation. Given a dyadic decomposition $\triangle$ of a homogeneous metric measure space, we define a rough hyperbolic distance function $H : \triangle \times \triangle \to [0,\infty)$ by

$$H(Q,R) = \log\left(\left(\frac{\operatorname{dist}(Q,R)}{\operatorname{rad}(Q)+\operatorname{rad}(R)}\right)+2\right) + \left|\log\left(\frac{\operatorname{rad}(Q)}{\operatorname{rad}(R)}\right)\right|,$$

for each $Q,R \in \triangle$. Also define the ϕ-*refinement* of $\triangle$ for some $\phi \in \mathbf{N}$, to be $\diamondsuit = \cup_n \diamondsuit_n$ where each $\diamondsuit_n = \triangle_{n\phi}$. We further denote $\partial_\delta \diamondsuit_n = \partial_\delta \triangle_{n\phi}$ and $\partial_\delta \diamondsuit = \cup_n \partial_\delta \diamondsuit_n$ for every $\delta > 0$. Recall that $\partial_\delta \triangle$ is defined in section 2.2.

PROPOSITION 5.1.1. *Assume the hypotheses of Theorem 5.0.10, further suppose that* $\operatorname{diam} X = 1$, *let* $\Psi > 0$ *be as determined by Proposition 3.3.3, and fix a dyadic decomposition* $\triangle$ *of* (X,d,μ). *Then there exists* $\delta,\varphi,\phi_0 > 0$, *that depend only on the parameters of the hypotheses, such that for every* $\phi \in \mathbf{N}$ *that satisfies* $\phi \geq \phi_0$, *there exists* $0 < \alpha' < \alpha$ *that depends only on* ϕ *and the parameters of the hypotheses, and there exists a function* $\rho : \diamondsuit \to [0,\infty)$ *where* $\diamondsuit = \cup_{n\in\mathbf{N}} \diamondsuit_n$ *is the* ϕ*-refinement of* $\triangle$, *such that following conditions hold.*

1. *We have*
$$\sum_{\substack{T\subset Q\\ T\in\diamondsuit_{n+1}}} \rho(T)^{\alpha'} \mu(T)^{\alpha'/\alpha} = \mu(Q)^{\alpha'/\alpha}, \tag{17}$$
whenever $n \in \mathbf{N}$ *and* $Q \in \diamondsuit_n$.
2. *Let* γ *be a discrete* $\ell_\triangle 2^{-(n+1)\phi+1}$*-curve such that* $\gamma \cap \partial_\delta \diamondsuit_n \neq \varnothing$, *for some* $n \in \mathbf{N}$. *Then we have*
$$d\big(\gamma(1),\gamma(\#\gamma)\big) \leq \sum_{i=1}^{\#\gamma} d\big(\gamma(i),\gamma(i+1)\big) \min_T \rho(T)\,, \tag{18}$$
where each minimum is taken over
$$\left\{T \in \diamondsuit_{n+1} : T \cap B(\gamma(i), \ell_\triangle 2^{-(n+1)\phi+2}) \neq \varnothing\right\}. \tag{19}$$
3. *We have*
$$\prod_{\substack{T\in\diamondsuit\\ Q\subset T}} \rho(T) \leq \varphi^{H(Q,R)\phi^{-1}} \prod_{\substack{T\in\diamondsuit\\ R\subset T}} \rho(T)\,,$$
for every $Q,R \in \diamondsuit$. (*We emphasize that* φ *depends on* Ψ *but not* ϕ.)

REMARK 5.1.2. The conditions set forth in Proposition 5.1.1 are sufficient to guarantee the existence of a quasisymmetric homeomorphism that lowers the Assouad dimension of the given metric space. Specifically, let $\delta,\varphi > 0$

and $\alpha \geq \alpha' > 0$, let (X,d,μ) be a bounded α-homogeneous metric measure space, and let $\rho : \triangle \to [0,\infty)$ be a function where $\triangle$ is some dyadic decomposition of (X,d). Then there exists $\phi_0 > 0$ that depends only on δ, φ, α and the homogeneity constant of (X,d), such that if for some integer $\phi \geq \phi_0$ we have the above conditions 1, 2 and 3 hold, then (X,d) is quasisymmetrically homeomorphic to an α'-homogeneous metric measure space. If we further suppose (X,d) is Ahlfors α-regular we can then get that (X,d) is quasisymmetric to an Ahlfors α'-regular metric space. This claim is essentially established in the proof of Proposition 5.2.1.

Proof. Let $\Psi > 0$ be as determined by Proposition 3.3.3, and let $C > 0$ denote a varying constant whose value depends only on λ, Ψ, α, and the homogeneity constant of μ. Next fix $0 \leq \delta \leq 1$ that depends only on C, so that for every $Q \in \triangle$, there exists $x \in Q$ so that $B(x, \delta 2^{\mathrm{rad}(Q)+4}) \subset Q$. We will shortly construct the desired function $\rho : \diamondsuit \to \mathbf{R}$ where $\diamondsuit$ is a ϕ-refinement of $\triangle$ for some $\phi \in \mathbf{N}$. During the construction we specify a finite number of lower bounds for ϕ, that depend only on η and C, and that guarantee ρ satisfies the claims of the proposition. We then implicitly take $\phi_0 > 0$ to be the maximum of these bounds.

Let $n \in \mathbf{N}$ and $Q \in \diamondsuit_n$. It follows from Proposition 3.3.3, the properties of $\triangle$ and δ, and the definition of an annulus system, that there exists $w \in Q$, and there exists $g : X \to [\Psi^{-1}\delta^{-1}2^{n\phi}, \Psi(\lambda\delta)^{-1}2^{n\phi}]$ such that we have $B(w, \delta 2^{-n\phi+3}) \subset Q$, we have g is discretely admissible at scale $\Psi^{-1}\lambda\delta 2^{-n\phi}$ for $\Theta(w, \lambda^2\delta 2^{-n\phi}, \delta 2^{-n\phi})$, and we have

$$\int_{B(w,\delta 2^{-n\phi+1})} g^\alpha \, d\mu \leq C\eta\mu\big(B(w,\delta 2^{-n\phi})\big)(\delta 2^{-n\phi})^{-\alpha}. \tag{20}$$

Recall that $\Theta(x,t,T)$ is a collection of curves, and $A(x,t,T)$ is an annulus, both of which are defined in section 3.3. In our case above, we have that the pertinent annulus provided by the annulus system with infinitesimally small modulus is contained in $A(w, \lambda^2\delta 2^{-n\phi}, \delta 2^{-n\phi})$.

We now let

$$E_1 = A(w, \lambda^2\delta 2^{-n\phi-1}, \delta 2^{-n\phi+1}),$$
$$E_2 = X \setminus B(w, \lambda^2\delta 2^{-n\phi-1}),$$
$$E_3 = B(w, \lambda^2\delta 2^{-n\phi-1}),$$

define $h : X \to \mathbf{R}$ by

$$h(x) = \max\left\{\delta 2^{-n\phi+2} g(x)\chi|_{E_1}(x), \chi|_{E_2}(x), \omega\chi|_{E_3}(x)\right\},$$

for every $x \in X$, and let $\rho(T) = \min_{y\in T} h(y)$ for each $T \in \diamondsuit_{n+1}$ contained in Q. The value of $\omega \geq \eta$ will momentarily be determined. In the meantime we assume that $\omega = \eta$.

Notice that

$$\sum_{T\subset Q}\rho(T)^\alpha\mu(T)\le\sum_{T\cap E_1\ne\varnothing}\rho(T)^\alpha\mu(T)+\sum_{T\cap E_2\ne\varnothing}\mu(T)+\sum_{T\cap E_3\ne\varnothing}\eta\mu(T)\,,\quad(21)$$

where here and in the relevant equations below we only sum over $T\in\Diamond_{n+1}$. Due to (20), the first term on the right-hand side of the above equation is dominated by

$$\int_{B(w,\delta^{-1}2^{-n\phi+1})}(\delta 2^{-n\phi+2}g)^\alpha d\mu\le C\eta\mu(Q)\,.$$

Using the homogeneity of μ, we now specify that ϕ is chosen sufficiently large as determined by C, so that the second term on the right hand side of (21) is dominated by $(1-C(\lambda^2\delta)^\alpha)\mu(Q)$. By hypotheses we have that $\eta<L\lambda^{2\alpha}$ holds with $L>0$ sufficiently small as determined by α and the homogeneity constant of μ, so that

$$\sum_{T\subset Q}\rho(T)^\alpha\mu(T)\le\big(1-C(\lambda^2\delta)^\alpha+C\eta\big)\mu(Q)\le(1-\sigma)\mu(Q)\,,$$

for some $\sigma>0$ that depends only on C. From this we conclude that

$$\begin{aligned}&\sum_{T\subset Q}\rho(T)^{\alpha'}\mu(T)^{\alpha'/\alpha}\mu(Q)^{-\alpha'/\alpha}\\&\quad\le\max_{\substack{R\in\Diamond_{n+1}\\R\subset Q}}\big\{\rho(R)^{\alpha'-\alpha}\mu(R)^{\frac{\alpha'-\alpha}{\alpha}}\mu(Q)^{\frac{\alpha-\alpha'}{\alpha}}\big\}\sum_{T\subset Q}\rho(T)^\alpha\mu(T)\mu(Q)^{-1}\\&\quad\le\eta^{\alpha'-\alpha}(C2^\phi)^{\frac{\alpha-\alpha'}{\alpha}}(1-\sigma)\,,\end{aligned}$$

for every positive $\alpha'>\alpha$.

Choose $0<\alpha'<\alpha$ sufficiently close to α so that the right-hand side of the above equation is at most 1. This choice of α' depends on all of the above parameters except Q. Now increase the value of ω so that $\eta\le\omega\le 1$ and so that (17) holds. To see that we can do this observe that if $\rho=1$ everywhere on Q then (17) would hold with "=" replaced by "$\ge$". Since the choice of α' did not depend on Q, the above argument defines a function $\rho:\Diamond\to\mathbf{R}$ that satisfies (17) for every $Q\in\Diamond$. This completes the proof of condition 1 of Proposition 5.1.1.

We now verify the remaining conditions of Proposition 5.1.1. Continue to let $n\in N$, and let γ be a discrete $\ell_\triangle 2^{-(n+1)\phi+1}$-curve in X. We seek to verify condition 2 of Proposition 5.1.1. To do so we require that ϕ satisfies

$$\ell_\triangle 2^{-\phi+4}\le\Psi^{-1}\lambda^2\delta\,.\quad(22)$$

This first guarantees that γ can be decomposed into a consecutive union of discrete curves γ_i each of which has an endpoint in $\partial_\delta Q$, and satisfies $\gamma_i\subset\partial_\delta Q\cup Q$ for some $Q\in\Diamond_n$. It therefore suffices to assume that γ

satisfies these same conditions. We continue to let w denote the center of the annulus in Q, and let g denote the discretely admissible function on Q, both defined in the first part of the proof. If $\gamma \cap B(w, \lambda^2\delta 2^{-n\phi}) = \varnothing$, it then follows from (22) that $\rho(T) \geq 1$ whenever T is contained in the set (19) and $i = 1, \dots, \#\gamma$. Thus in this case γ trivially satisfies (18).

Otherwise we define

$$a = \min\left\{i \in \mathbf{N} : \gamma(i) \subset B(w, \delta 2^{-n\phi+1})\right\},$$
$$b = \max\left\{i \in \mathbf{N} : \gamma(i) \subset B(w, \delta 2^{-n\phi+1})\right\}.$$

Like before we have that the discrete curves $\gamma(1),\dots,\gamma(a)$ and $\gamma(b),\dots,\gamma(\#\gamma)$ satisfy (18). Therefore, to complete the proof of condition 2 of Proposition 5.1.1, it remains to establish (18) for the discrete curve given by $\beta = \{\gamma(i)\}_{i=a}^{b}$. Observe that $\beta \in \Theta(w, \lambda^2\delta 2^{-n\phi}, \delta 2^{-n\phi})$. Inequality (22) guarantees that g is discretely admissible at scale $\Psi^{-1}\lambda\delta 2^{-n\phi}$ for $\{\beta\}$. Therefore

$$1 \leq \sum_{i=a}^{b-1} d\big(\gamma(i), \gamma(i+1)\big) \inf_{y \in B(\gamma(i), \Psi^{-1}\lambda\delta 2^{-n\phi})} g(y)\,. \tag{23}$$

Observe that

$$S \subset B\big(x, \ell_{\triangle} 2^{-(n+1)\phi+3}\big) \quad \text{whenever} \quad S \cap B\big(x, \ell_{\triangle} 2^{-(n+1)\phi+2}\big) \neq \varnothing\,,$$

for every $x \in X$ and $S \in \diamondsuit_{n+1}$. For such x and S, it then follows from (22) that $S \subset B(x, \Psi^{-1}\lambda\delta 2^{-n\phi})$ and therefore that

$$\delta 2^{-n\phi+2} \inf_{y \in B(x, \Psi^{-1}\lambda\delta 2^{-n\phi})} g(y) \leq \min_T \rho(T)\,,$$

where the minimum on the right-hand side of the inequality is taken over

$$\left\{T \in \diamondsuit_{n+1} : T \cap B(x, 2^{-(n+1)\phi+2}) \neq \varnothing\right\}.$$

This with (23) implies (18) and so completes the verification of condition 2 of Proposition 5.1.1.

In order to complete the proof it remains to establish condition 3 of Proposition 5.1.1. To do this we use the following consequence from the definition of ρ. If ϕ is sufficiently large depending on C (and in particular δ), then $\rho(S) = 1$ whenever $S \cap \partial_\delta T \neq \varnothing$, for every $n \in \mathbf{N}$, $S \in \diamondsuit_{n+1}$, and $T \in \diamondsuit_n$. Now fix $1 \leq n \leq m$ and let $Q \in \diamondsuit_n$ and $R \in \diamondsuit_m$. To prove the condition it suffices to show that

$$C^{-H(Q,R)\phi^{-1}} u(R) \leq u(Q) \leq u(R) C^{H(Q,R)\phi^{-1}}, \tag{24}$$

where $u : \diamondsuit \to \mathbf{R}$ is given by

$$u(T) = \prod_{\substack{T \in \diamondsuit \\ T \subset W}} \rho(W)$$

for every $T \in \triangle$, and where here and below we further allow $C > 0$ to depend on η.

Let S be the cube in $\diamondsuit_n$ that contains R. Notice that whenever $l \in \mathbf{N}$ satisfies

$$l \leq n - C\phi^{-1} \log \left(\frac{\operatorname{dist}(S,Q)}{\operatorname{rad} S} + 2 \right), \tag{25}$$

we have that either both S and Q are contained in the same member of $\diamondsuit_l$, or that both S and Q have a nontrivial intersection with $\partial_\delta \diamondsuit_l$. Consequently for every $l \in \mathbf{N}$ that satisfies (25), and for every $W, E \in \diamondsuit_l$ that contain Q and S, respectively, we have $\rho(W) = \rho(E)$. This with the estimate $\max\{\rho, \rho^{-1}\} \leq C$ then implies that

$$C^{-\phi^{-1} \log\left(\frac{\operatorname{dist}(S,Q)}{\operatorname{rad} S} + 2 \right)} u(S) \leq u(Q) \leq C^{\phi^{-1} \log\left(\frac{\operatorname{dist}(S,Q)}{\operatorname{rad} S} + 2 \right)} u(S) .$$

Inequality (24) then follows because

$$\frac{\operatorname{dist}(S,Q)}{\operatorname{rad} S} + 1 \quad \text{and} \quad \left(\frac{\operatorname{dist}(Q,R)}{\operatorname{rad} Q + \operatorname{rad} R} \right) + 1$$

are comparable with comparability constant that depends only on C, and because

$$C^{-\log\left(\frac{\operatorname{rad} Q}{\operatorname{rad} R}\right)\phi^{-1}} u(R) \leq u(S) \leq C^{\log\left(\frac{\operatorname{rad} Q}{\operatorname{rad} R}\right)\phi^{-1}} u(R) .$$

This last inequality is a consequence of the fact that $R \subset S$ and $\rho \leq \max\{C, C^{-1}\}$. This completes the proof of condition 3 of Proposition 5.1.1 and therefore also Proposition 5.1.1. □

5.2 A weight-loss program. We now use the results of Proposition 5.1.1 to construct the desired quasisymmetric homeomorphism.

PROPOSITION 5.2.1. *Assume the hypotheses of Proposition 5.1.1. Then there exists $0 < \alpha' < \alpha$, and there exists a metric θ on X such that (X, θ) supports an α'-homogeneous measure ν, and such that the identity map from (X, d) to (X, θ) is quasisymmetric. Moreover, the quasisymmetric regularity function for this identity map, the homogeneity constant of ν, and α', depend only on the parameters of the hypotheses (which include Ψ).*

We suppose without loss of generality that (X, d) is proper. (To see this there is no loss of generality, observe that the hypotheses of Theorem 5.0.10 are preserved under completion. Whereas, a complete homogeneous metric measure space is necessarily proper.) We continue with the notation established in Proposition 5.1.1 and in particular suppose $\phi \in \mathbf{N}$ is sufficiently large as is required by Proposition 5.1.1. We also continue to specify lower bounds for ϕ that depend only on the parameters of the hypotheses. To realize the conclusion of the proof we then take ϕ to be fixed and greater than this finite collection of lower bounds. Thus the value of ϕ and therefore also α', and the value of the other variables in the proof that initially depend on both ϕ and the parameters of the hypotheses of Proposition 5.2.1, ultimately depend only on the parameters of the hypotheses. We now let $C > 0$ denote a varying constant whose value depends only on the parameters of the hypotheses (and therefore not ϕ or α').

Let $H = X \times \mathbf{N}$ and $H_n = X \times \{1, \dots, n\}$ for every $n \in \mathbf{N}$. We refer to each $H \times \{n\}$ as a *level* of H. Define $b : H \to \mathbf{R}$ by

$$b(x,k) = \min_{T} \prod_{\substack{T \subset R \\ R \in \diamondsuit}} \rho(R)\,,$$

for every $x \in X$ and $k \in \mathbf{N}$, where the minimum is taken over

$$\bigl\{T \in \diamondsuit_k : T \cap B(x, \ell_\triangle 2^{-k\phi+1}) \neq \varnothing\bigr\}\,. \tag{26}$$

We remark that we here define b as a minimum over nearby cubes, so that later we can prove condition 3 of Lemma 5.2.3. Despite its convoluted definition we still retain good estimates on b. The next estimate is a direct consequence of the definition of b and condition 3 of Proposition 5.1.1.

Lemma 5.2.2. *There exists $\sigma \geq 1$ that depends only on C such that*

$$b(x,k) \leq \max\Bigl\{\sigma, \sigma^{|k-l| + \log\left(\frac{d(x,y)^{1/\phi}}{2^{-k}+2^{-l}}+1\right)}\Bigr\} b(y,l)\,,$$

whenever $x, y \in X$ and $k, l \in \mathbf{N}$.

A finite sequence $\{(x_j, k_j)\}_{j=1}^N \subset H$, where $N \in \mathbf{N}$, is said to be *conductible* if for every $1 \leq j \leq N-1$, either

- we have $d(x_j, x_{j+1}) \leq \ell_\triangle 2^{-k_j\phi+1}$ and $k_j = k_{j+1}$, or
- we have $x_j = x_{j+1}$ and $|k_j - k_{j+1}| = 1$.

Define the *ν-length* of such sequences by

$$\ell_\nu\bigl(\{(x_j,k_j)\}_{j=1}^N\bigr) = \sum_{j=1}^{N-1} \nu\bigl((x_j,k_j),(x_{j+1},k_{j+1})\bigr)\,,$$

where $\nu : H \times H \to \mathbf{R}$ is defined by

$$\nu\bigl((x,k),(y,l)\bigr) = \begin{cases} d(x,y)\min\{b(x,k), b(y,k)\}\,, & \text{if } k = l\,,\\ \sigma\ell_\triangle 2^{-\min\{k,l\}\phi} b(x, \max\{k,l\})\,, & \text{if } x{=}y \text{ and } |k{-}l|{=}1\,,\\ 0\,, & \text{otherwise}\,, \end{cases}$$

whenever $x, y \in X$ and $k, l \in \mathbf{N}$. For $x \in X$ and $j \in \mathbf{N}$ let $(x,j)^* = x$ and $(x,j)_* = j$, and given any sequence $\gamma \subset H$, let γ^* denote the discrete curve given by $\{\gamma(i)^*\}_{i=1}^{\#\gamma}$.

Roughly speaking, condition 1 of the following lemma requires that conductible sequences which move from one cube to another in the same level of H, pass through the boundary of one of the cubes. Condition 2 is an estimate that penalizes a conductible sequence for jumping between levels of H. Condition 3 asserts that length ℓ_ν is preserved for those conductible sequences that are contained in one level of H (for example $H_{n+1} \setminus H_n$), that pass through boundary of a cube Q in the level above (that is, in $\diamondsuit_n$), and that remain inside the union of Q and its boundary.

LEMMA 5.2.3. *1. Let $n \in \mathbf{N}$ and $Q \in \diamondsuit_n$, and let $\gamma \subset H_{n+1} \setminus H_n$ be a conductible sequence such that $\gamma^* \cap Q \neq \varnothing$, and such that $\gamma^* \cap \partial_\delta \diamondsuit_n = \varnothing$. Then $\gamma^* \subset Q$.*

2. We have

$$\nu\big((x,n),(y,n)\big) \leq \nu\big((x,n),(x,n+1)\big)\,,$$

whenever $n \in \mathbf{N}$, $Q \in \diamondsuit_n$ and $x, y \in Q$.

3. Let $Q \in \diamondsuit_n$, and let γ be a conductible sequence in $H_{n+1} \setminus H_n$ such that $\gamma^*(i) \in Q \cup \partial_\delta Q$ for all $1 \leq i \leq \#\gamma$, such that $\gamma^*(i) \in \partial_\delta Q$ for some $1 \leq i \leq \#\gamma$. Then

$$\nu\big((\gamma^*(1),n),(\gamma^*(\#\gamma),n)\big) \leq \ell_\nu(\gamma)\,.$$

Proof. Condition 1 of Lemma 5.2.3 can be seen to hold by specifying that ϕ is sufficiently large as determined by δ and the dimension and homogeneity constant of (X,d,μ) (and therefore really just C). We now prove condition 2 of Lemma 5.2.3. Observe that

$$b(x,n) \leq \sigma b(x,n+1) \quad \text{and} \quad d(x,y) \leq \ell_\triangle 2^{-n\phi}.$$

Consequently

$$\begin{aligned}\nu\big((x,n),(y,n)\big) &\leq d(x,y)b(x,n)\\ &\leq \ell_\triangle 2^{-n\phi}\sigma b(x,n+1)\\ &= \nu\big((x,n),(x,n+1)\big)\,.\end{aligned}$$

This proves condition 2 of Lemma 5.2.3.

We now prove condition 3 of Lemma 5.2.3. Fix $x, y \in Q \cup \partial_\delta Q$. By definition we have

$$b(x,n+1) = \prod_{\substack{S \subset R\\ R \in \diamondsuit}} \rho(R)\,,$$

for some $S \in \diamondsuit_{n+1}$ that intersects $B(x, \ell_\triangle 2^{-(n+1)\phi+1})$. Observe then that

$$W \cap B(y, \ell_\triangle 2^{-n\phi+1}) \neq \varnothing\,,$$

where $W \in \diamondsuit_n$ is the cube that contains S. Consequently

$$\min_T \rho(T) b(y,n) \leq \rho(S) \prod_{\substack{W \subset R\\ R \in \diamondsuit}} \rho(R) = b(x,n+1)\,, \tag{27}$$

where the minimum is taken over

$$\left\{T \in \diamondsuit_{n+1} : T \cap B(x, \ell_\triangle 2^{-(n+1)\phi+1}) \neq \varnothing\right\}.$$

Now, by hypothesis we have γ^* is a discrete $\ell_\triangle 2^{-(n+1)\phi+1}$-curve such that $\gamma \cap \partial_\delta \diamondsuit_n \neq \varnothing$. Consequently from condition 2 of Proposition 5.1.1 we have

$$d\big(\gamma^*(1), \gamma^*(\#\gamma)\big) \leq \sum_{i=1}^{\#\gamma} d\big(\gamma^*(i), \gamma^*(i+1)\big) \min_T \rho(T),$$

where the minimum is taken over (19). Condition 3 of Lemma 5.2.3 now follows from (27) applied with $y = \gamma^*(1)$ or $\gamma^*(\#\gamma)$, and $x = \gamma^*(i)$ or $\gamma^*(i+1)$, for each of the terms in the summand. In order to apply (27) we have here used the hypothesis that each $\gamma^*(i) \in Q \cup \partial_\delta Q$. This completes the proof. $\square$

The next lemma shows that the length of a conductible sequence does not get shorter by travelling into deeper levels of H.

Lemma 5.2.4. *Let $n \in \mathbf{N}$ and $x_1, x_2 \in X$, and let γ be a conductible sequence in H_{n+1} that begins at $(x_1, n+1)$ and ends at $(x_2, n+1)$, such that $\gamma^* \cap \partial_\delta \diamondsuit_n \neq \varnothing$. Then there exists a conductible sequence β in H_n that begins at (x_1, n) and ends at (x_2, n), such that $\ell_\nu(\beta) \leq \ell_\nu(\gamma)$.*

Proof. We begin by inductively defining increasing sequences $(a_i)_{i=1}^{N+1}$, $(b_i)_{i=0}^{N} \subset \mathbf{N}$ for some $N \in \mathbf{N}$, that will be used to decompose γ into more manageable subsequences. Let $a_1 = 1$ and $i \geq 1$ be an integer, and suppose that a_i has been defined. Then let

$$b_i = \min\left\{i > a_i : \gamma(i) \in H_n \text{ or } i = \#\gamma\right\},$$
$$a_{i+1} = \min\left\{i \geq b_i : \gamma(i+1) \in H_{n+1} \setminus H_n\right\}.$$

We terminate the process when $b_i = \#\gamma$. Next let $\gamma_i = (\gamma(a_i), \ldots, \gamma(b_i))$ for $1 \leq i \leq N$, and $\delta_i = (\gamma(b_i), \ldots, \gamma(a_{i+1}))$ for $1 \leq i \leq N-1$.

We claim and will soon prove that for every $1 \leq i \leq N$, there exists a conductible sequence $\beta_i \subset H_n$ such that $\beta_i(1) = (\gamma_i^*(1), n)$ and $\beta_i(\#\beta) = (\gamma_i^*(\#\gamma_i), n)$, and such that $\ell_\nu(\beta_i) \leq \ell_\nu(\gamma_i)$. A consequence of the claim is that

$$\beta = \beta_1 \cup \delta_1 \cup \beta_2 \cup \delta_2 \cup \cdots \cup \delta_{N-1} \cup \beta_N \subset H_n$$

is conductible, and begins at (x_1, n) and ends at (x_2, n). The assertion of the lemma will then follow from the estimate

$$\ell_\nu(\beta) = \sum_{i=1}^{N} \ell_\nu(\beta_i) + \sum_{i=1}^{N} \ell_\nu(\delta_i) \leq \sum_{i=1}^{N} \ell_\nu(\gamma_i) + \sum_{i=1}^{N} \ell_\nu(\delta_i) = \ell_\nu(\gamma).$$

We now prove the claim. Fix $1 \leq i \leq N$. Consider the case when $\gamma_i \cap \partial_\delta \diamondsuit_n = \varnothing$. It follows from condition 1 of Lemma 5.2.3 that $\gamma^*(a_i)$ and $\gamma^*(b_i)$ are both contained in the same member of $\diamondsuit_n$. Consequently

$$\beta_i = \big((\gamma^*(a_i), n), (\gamma^*(b_i), n)\big)$$

is conductible. From our hypotheses we also see that $\gamma_i \neq \gamma$, and therefore either $\gamma(a_i)$ or $\gamma(b_i)$ is contained in H_n. We can now safely apply condition 2 of Lemma 5.2.3 to see that $\ell_\nu(\beta_i) \leq \ell_\nu(\gamma_i)$. We conclude that β_i satisfies the properties described in the claim.

Now consider the case when $\gamma_i \cap \partial_\delta \diamondsuit_n \neq \varnothing$. Condition 1 of Lemma 5.2.3 permits us to obtain an increasing sequence $(k_j)_{j=1}^M \subset \mathbf{N}$ for some $M \in \mathbf{N}$

- such that $k_1 = 1$ and $k_M = \#\gamma_i$,
- such that $\gamma_i^*(k_j) \in \partial_\delta \diamondsuit_n$ for every $2 \leq j \leq M-1$, and
- such that $d(\gamma_i^*(k_j), \gamma_i^*(k_{j+1})) \leq \ell_\triangle 2^{-n\phi+1}$ for every $1 \leq j \leq M-1$.

The point here is that condition 1 of Lemma 5.2.3 prevents γ_i from passing through a cube of $\diamondsuit_n$ without intersecting the boundary of that cube. Through successive application of condition 3 of Lemma 5.2.3 to each of the conductible sequences $(\gamma_i(l))_{l=k_j}^{k_{j+1}}$, we see that

$$\ell_\nu(\beta_i) = \sum_{j=1}^{M-1} \nu\big(\beta_i(j), \beta_i(j+1)\big) \leq \sum_{j=1}^{M-1} \ell_\nu\big((\gamma_i(l))_{l=k_j}^{k_{j+1}}\big) = \ell_\nu(\gamma_i)\,,$$

where $\beta_i = \{(\gamma_i(k_j), n)\}_{j=1}^M$. Again β_i satisfies the properties described in the claim. This completes the proof of the claim and therefore the lemma. □

For every $n \in \mathbf{N}$, define a function $\theta_n : H_n \times H_n \to [0, \infty)$ by

$$\theta_n\big((x,k),(y,l)\big) = \inf_\gamma \ell_\nu(\gamma)\,,$$

for every $x, y \in X$ and $1 \leq k, l \leq n$, where the infimum is taken amongst all conductible sequences γ in H_n that contain (x,k) and (y,l). Notice that every two members of H_n are mutually contained in some conductible sequence of H_n. Therefore θ_n is well defined as a function with bounded range. Moreover, from the definition of ν-length, and because ν is symmetric, we see that θ_n is symmetric and observes the triangle inequality. Since θ_n only vanishes on the diagonal, we conclude that θ_n is a metric on H_n.

The next two lemmas give estimates needed to obtain the desired metric θ and the quasisymmetric bounds. Roughly speaking, the next lemma makes use of the fact that the penalty was not too severe for those conductible sequences that jump between levels of H. In order to proceed we require that ϕ be chosen sufficiently large as determined by C so that $2^{-\phi}\sigma < 1/2$.

LEMMA 5.2.5. *Let $1 \leq m \leq n$ and $x_1, x_2 \in X$, and suppose that*
$$d(x_1, x_2) \leq 2^{-m\phi}.$$
Then we have
$$\theta_n\big((x_1, n), (x_2, n)\big) \leq Cb(x_1, m)2^{-m\phi}.$$

Proof. To prove the result we obtain an appropriate upper bound for the ν-length of the conductible sequence given by
$$\gamma = \big((x_1, n), (x_1, n-1), \ldots, (x_1, m), (x_2, m), (x_2, m-1), \ldots, (x_2, n)\big).$$
First observe that from the definition of ν we have
$$\nu\big((x_1, m), (x_2, m)\big) \leq b(x_1, m)2^{-m\phi}.$$
Whereas from the definition of ν and Lemma 5.2.2, we have
$$\begin{aligned}\nu\big((x_j, k), (x_j, k+1)\big) &= \sigma\ell_\triangle 2^{-k\phi} b(x_j, k+1) \\ &\leq C2^{-k\phi}\sigma^{k-m} b(x_1, m),\end{aligned} \tag{28}$$
for every $m \leq k \leq n$ and $j = 1, 2$. Since $2^{-\phi}\sigma < 1/2$, we conclude that
$$\begin{aligned}\ell_\nu(\gamma) &= \nu\big((x_1, m), (x_2, m)\big) + \sum_{j=1}^{2}\sum_{k=m}^{n} \nu\big((x_j, k), (x_j, k+1)\big) \\ &\leq Cb(x_1, m)2^{-m\phi}\left(1 + \sum_{j=1}^{2}\sum_{k=m}^{\infty}(2^{-\phi}\sigma)^{k-m}\right) \\ &\leq Cb(x_1, m)2^{-m\phi}.\end{aligned}$$
This completes the proof. □

LEMMA 5.2.6. *Let $1 \leq m \leq n$ and $x_1, x_2 \in X$, and suppose that*
$$2^{-m\phi} \leq d(x_1, x_2) \leq 2^{-(m-1)\phi}.$$
Then we have
$$\theta_n\big((x_1, n), (x_2, n)\big) \geq Cb(x_1, m)d(x_1, x_2).$$

Proof. Let γ be a conductible sequence in H_n that begins at (x_1, n) and ends at (x_2, n). Since $d(x_1, x_2) \geq 2^{-m\phi}$, there exists a positive integer $l \leq m + C$ such that x_1 and x_2 are contained in different cubes in $\diamondsuit_l$. For the case when $l > n$, we set $l = n$. Through successive application of Lemma 5.2.4 to γ, we obtain a conductible sequence $\beta \subset H_l$ such that β begins at (x_1, l) and ends at (x_2, l), and such that $\ell_\nu(\beta) \leq \ell_\nu(\gamma)$ (or if $l = n$ we let $\beta = \gamma$). To complete the proof it suffices to show that
$$b(x_1, m)d(x_1, x_2) \leq C\ell_\nu(\beta). \tag{29}$$

Observe that either

- there exists $1 \leq i \leq \#\beta - 1$ such that $\beta_*(i) = l$ and $\beta_*(i+1) = l - 1$, and such that $d(\beta^*(i), x_1) \leq \ell_\triangle 2^{-m\phi+2}$, or

- there exists $1 \le a \le \#\beta$ such that $\beta'^* \subset B(x_1, \ell_\triangle 2^{-m\phi+2})$, such that $\operatorname{diam} \beta'^* \ge 2^{-m\phi}$, where $\beta' = (\beta(1), \beta(2), \dots, \beta(a))$, and such that $\beta'_*(i) = l$ for $1 \le i \le a$.

In the first case, it follows from the definition of ν and ν-length that

$$b(\beta(i))2^{-l\phi} \le C\nu\big(\beta(i), \beta(i+1)\big) \le C\ell_\nu(\beta)\,.$$

Moreover, we can apply Lemma 5.2.2 to see that $b(x_1, m) \le Cb(\beta(i))$. Inequality (29) then follows from fact that $l \le m + C$.

In the second case we infer from Lemma 5.2.2 that

$$b(x_1, m) \le Cb\big(\beta'(j)\big)\,,$$

for every $1 \le j \le a$. It then follows from the triangle inequality, and the definition of ν-length, that

$$\begin{aligned} d(x_1, x_2)b(x_1, m) &\le C \sum_{j=1}^{\#\beta'-1} d\big(\beta'^*(j), \beta'^*(j{+}1)\big) \min\big\{b(\beta'(j)), b(\beta'(j{+}1))\big\} \\ &= C\ell_\nu(\beta') \le C\ell_\nu(\beta)\,. \end{aligned}$$

This completes the proof. □

From here on we keep ϕ fixed. Thus ϕ depends only on C, that is, the value of ϕ depends only on the parameters of the hypotheses of Proposition 5.2.1. We now allow future uses of C to depend not only on the parameters of the hypotheses of Proposition 5.2.1 as before, but also on ϕ and α'.

Let $x, y \in X$ and $n \in \mathbf{N}$ satisfy $n \ge -\log_{2^\phi} d(x, y)$. It follows from Lemma 5.2.5 with $m = [-\log_{2^\phi} d(x, y)]$, Lemma 5.2.2 with $l = 0$ and $k = m$, and the fact that $\sigma 2^{-\phi} < 1/2$, that

$$\theta_n\big((x, n), (y, n)\big) \le C\eta\big(d(x, y)\big)\,,$$

where the function $\eta : (0, 1] \to (0, \infty)$ is defined by the formula

$$\eta(t) = 2^{-\log_{2^\phi}(t^{-1}+2)}.$$

Observe that $\eta(t) \to 0$ as $t \to 0$. We can therefore apply the Arzelá–Ascoli theorem to obtain $\theta : X \times X \to (0, \infty)$ as a limit of a subsequence of $(\theta_n|_{(X,n)\times(X,n)})$. By Lemma 5.2.6 we see that θ vanishes only on the diagonal. We can now safely conclude that θ is a metric on X. It further follows from Lemma 5.2.5 together with Lemma 5.2.6 that

$$\theta(x, y) \approx b\big(x, [-\log_{2^\phi} d(x, y)]\big)d(x, y)\,, \tag{30}$$

for every $x, y \in X$. Here and below we write $s \approx r$, for some $s, r > 0$, to mean that s and r are comparable with comparability constant C.

LEMMA 5.2.7. *The identity map from (X, d) to (X, θ) is quasisymmetric with quasisymmetric regularity depending only on C.*

Proof. Let $x, y, z \in X$ with $x \neq z$. It then follows from (30) that

$$\frac{\theta(x,y)}{\theta(x,z)} \approx \frac{b(x,[-\log_{2^\phi} d(x,y)])d(x,y)}{b(x,[-\log_{2^\phi} d(x,z)])d(x,z)} \,. \tag{31}$$

In case $d(x,y) \leq d(x,z)$, it follows from Lemma 5.2.2 that

$$b\big(x,[-\log_{2^\phi} d(x,y)]\big) \leq C\sigma^{\log_{2^\phi}\left(\frac{d(x,z)}{d(x,y)}+2\right)} b\big(x,[-\log_{2^\phi} d(x,z)]\big) \,.$$

Consequently the right-hand side of (31) is bounded by

$$C\sigma^{\log_{2^\phi}\left(\frac{d(x,z)}{d(x,y)}+2\right)} 2^{-\phi\log_{2^\phi}\left(\frac{d(x,z)}{d(x,y)}\right)} \leq C\eta\left(\frac{d(x,y)}{d(x,z)}\right) .$$

If $d(x,y) \geq d(x,z)$, it follows from Lemma 5.2.2 that

$$\begin{aligned}\frac{b(x,[-\log_{2^\phi} d(x,y)])}{b(x,[-\log_{2^\phi} d(x,z)])} &\leq C\sigma^{\log_{2^\phi}\left(\frac{d(x,y)}{d(x,z)}+2\right)} \\ &\leq C\left(\frac{d(x,y)}{d(x,z)}\right)^C .\end{aligned}$$

Consequently the right-hand side of (31) is bounded by

$$\left(\frac{d(x,y)}{d(x,z)}\right)^C .$$

This completes the proof. □

To complete the proof of Proposition 5.2.1 it remains to show that (X,θ) admits an α'-homogeneous measure with homogeneity constant C. This is achieved in the following two lemmas. Define a function $\kappa : \diamondsuit \to \mathbf{R}$ by

$$\kappa(Q) = \mu(Q)^{\alpha'/\alpha} \prod_{\substack{R\in\diamondsuit \\ Q\subset R}} \rho(R)^{\alpha'},$$

for every $Q \in \diamondsuit$.

LEMMA 5.2.8. *The function κ extends to a Borel measure on (X,θ).*

Proof. For every $A \subset X$ and $\delta > 0$ let

$$\kappa_\delta(A) = \inf_{\mathcal{G}} \sum_{Q\in\mathcal{G}} \kappa(Q) \,,$$

where the infimum is taken over all covers $\mathcal{G} \subset \diamondsuit$ of A such that each member of $\mathcal{G}$ has diameter at most δ (say with respect to the metric d, although in what follows we could also use θ). By the Carathéodory construction [F, 2.10.1] there exists a Borel measure κ' on (X,θ) given by $\kappa'(A) = \lim_{\delta\to 0^+} \kappa_\delta(A)$ for every $A \subset X$. We need to show that $\kappa(Q) = \kappa'(Q)$ for every $Q \in \diamondsuit$. This is a consequence of the fact that for every $n \in \mathbf{N}$ and $Q \in \diamondsuit_n$ we have

$$\begin{aligned}\kappa(Q) &= \mu(Q)^{\alpha'/\alpha} \prod_{\substack{R\in\diamondsuit \\ Q\subset R}} \rho(R)^{\alpha'} \\ &= \sum_{\substack{T\subset Q \\ T\in\diamondsuit_{n+1}}} \mu(T)^{\alpha'/\alpha}\rho(T)^{\alpha'} \prod_{\substack{R\in\diamondsuit \\ Q\subset R}} \rho(R)^{\alpha'} \\ &= \sum_{\substack{T\subset Q \\ T\in\diamondsuit_{n+1}}} \mu(T)^{\alpha'/\alpha} \prod_{\substack{R\in\diamondsuit \\ T\subset R}} \rho(R)^{\alpha'} \\ &= \sum_{\substack{T\subset Q \\ T\in\diamondsuit_{n+1}}} \kappa(T)\,.\end{aligned}$$

Here we used condition 1 of Proposition 5.1.1 to get the second equality. This completes the proof. □

Here and after we let $B_\theta(y,s)$ denote the ball in (X,θ) with center $y \in X$ and radius $s > 0$.

LEMMA 5.2.9. *We have*

$$\frac{\kappa(B_\theta(y,r))}{\kappa(B_\theta(x,R))} \geq C\left(\frac{r}{R}\right)^{\alpha'}, \tag{32}$$

whenever $0 < r < R < \operatorname{diam} X$, $x \in X$, *and* $y \in B_\theta(x,R)$.

Proof. Fix x, y, r and R as above. Using the fact that (X,d) is proper, we fix $a, b \in X$ such that

$$d(a,y) = \min\bigl\{d(w,y) : w \in X \text{ and } \theta(w,y) \geq r\bigr\} \quad \text{and} \quad \theta(a,y) \geq r\,,$$
$$d(b,y) = \max\bigl\{d(w,y) : w \in X \text{ and } \theta(w,y) \leq 2R\bigr\} \quad \text{and} \quad \theta(b,y) \leq 2R\,.$$

Then we have

$$B\bigl(y, d(a,y)\bigr) \subset B_\theta(y,r) \quad \text{and} \quad B_\theta(x,R) \subset B\bigl(y, 2d(b,y)\bigr)\,.$$

Now fix $Q \in \diamondsuit$ such that $\operatorname{rad} Q \geq Cd(a,y)$ and $Q \subset B(y,d(a,y))$, and let

$$W = \bigcup\Bigl\{T \in \diamondsuit_{[-\log_{2\phi} d(b,y)]} : T \cap B(y, 2d(b,y)) \neq \varnothing\Bigr\}\,.$$

Then we have

$$Q \subset B_\theta(y,r) \quad \text{and} \quad B_\theta(x,R) \subset W\,. \tag{33}$$

From the properties of the dyadic decomposition we have that the union defining W is a union of at most C cubes. It then follows from the definition of κ and b and by Proposition 5.1.1.3 that

$$\frac{\kappa(Q)}{\kappa(W)} \approx \frac{\mu(Q)^{\alpha'/\alpha} b(y,[-\log_{2\phi} d(a,y)])^{\alpha'}}{\mu(W)^{\alpha'/\alpha} b(y,[-\log_{2\phi} d(b,y)])^{\alpha'}}\,.$$

Whereas, from the homogeneity of μ we get

$$\frac{\mu(Q)}{\mu(W)} \geq C \left(\frac{d(a,y)}{d(b,y)} \right)^{\alpha}.$$

Finally, it follows from (30) and the choice of a and b that

$$\frac{d(a,y)b(y,[-\log_{2\phi}]d(a,y))}{d(b,y)b(y,[-\log_{2\phi}]d(b,y))} \approx \frac{\theta(a,y)}{\theta(b,y)} \geq C\frac{r}{R}.$$

The last three inequalities together with (33) imply (32). This completes the proof. □

This completes the proof of Proposition 5.2.1.

5.3 Proof of Theorem 5.0.10. First apply Proposition 3.3.3. This determines $\Psi > 0$ that depends only on (X, d, μ), and which will be used in future applications of Proposition 5.2.1. Fix $x \in X$ and $r > 0$ so that $\operatorname{diam} B(x,r) > 0$. Recall that since (X,d) is a complete α-homogeneous metric measure space, we can exhaust X by a sequence of compact α-homogeneous metric measure subspaces $X_1 \subset X_2 \subset \dots$, with homogeneity constant uniformly bounded above by a bound that depends only on α and the homogeneity constant of μ. We organize this exhaustion so that $B(x,r) \subset X_n$ for every $n \in \mathbf{N}$. Fix $n \in \mathbf{N}$. Proposition 5.2.1 together with a re-scaling argument provides an α'-homogeneous metric measure space (Y_n, l_n, κ_n), and a quasisymmetric homeomorphism $f_n : X_n \to Y_n$, where $0 < \alpha' < \alpha$ depends only on Ψ and the parameters of our hypotheses.

From Proposition 5.2.1 we have that the homogeneity constant of κ_n, and the quasisymmetric regularity of f_n, are uniformly controlled. Since $B(x,r) \neq \{x\}$ and f_n is a homeomorphism, we can re-scale ρ_n so that $\operatorname{diam} f_n(B(x,r)) = 1$. Normalize each of the measures κ_n, and then pass to a subsequence so that $\{(Y_n, l_n, \kappa_n, y_n)\}$ converges to an α'-homogeneous, complete pointed metric measure space (Y, l, κ, y). Since κ is α'-homogeneous, the Assouad dimension of (Y, l) is at most α', and therefore strictly less than α. Now apply Lemma 2.4.7 to conclude that a subsequence of (f_n) converges to some quasisymmetric homeomorphism $f : X \to Y$. This completes the proof.

6 The Proof of Theorem 1.0.1 and the Corollaries

Proof of Theorem 1.0.1. It follows from the contrapositive of Theorem 5.0.10 that there exists a weak tangent of (X, d, μ) with non-vanishing α-modulus. Theorem 4.0.5 then implies that a tangent of this weak tangent, and therefore a weak tangent of (X, d, μ) has uniformly big 1-modulus. This completes the proof. $\square$

Proof of Corollary 1.0.2. Theorem 5.0.10 proves the first "only if" part of Corollary 1.0.2. To see the converse implication suppose that a weak tangent of (X, d) has non-vanishing p-modulus for some $p \geq 1$. Theorem 4.0.5 then asserts that there is a tangent of this weak tangent, and hence a weak tangent of (X, d), that has uniformly big 1-modulus. This proves the last claim of Corollary 1.0.2.

Suppose that (X, d) has conformal Assouad dimension strictly less than α. To complete the proof we need to show that every weak tangent of (X, d) has vanishing p-modulus for every $p \geq 1$. By Theorem 2.1.1 there exists a quasisymmetric homeomorphism from (X, d) to an Ahlfors α'-regular metric space for some $\alpha' < \alpha$. It then follows from Lemma 2.4.4 and Lemma 2.4.7 that every weak tangent of (X, d) is quasisymmetrically homeomorphic to an Ahlfors α'-regular metric space. In particular we have that each weak tangent of (X, d) has conformal Assouad dimension at most α', and therefore conformal (Hausdorff) dimension at most α'. Since $\alpha' < \alpha$, Theorem 2.3.1 of Tyson implies that each weak tangent of (X, d) has vanishing α-modulus. Thus by Theorem 4.0.5, and using the fact that a weak tangent of a weak tangent is a weak tangent of the original metric space, we see that every weak tangent of (X, d) has vanishing p-modulus for every $p \geq 1$. This completes the proof. $\square$

Proof of Corollary 1.0.3. We refer the reader to the work of Bonk and Kleiner [BoK1,2] for the definition and relevant properties of a uniformly quasi-Möbius action $G \curvearrowright X$ for which the induced action on the space of distinct triples of X is cocompact. Of significance here are the following two properties which hold for every non-compact weak tangent (Y, l) of (X, d) that contains at least three points.

- There exists $x \in X$ and a quasi-Möbius homeomorphism $f : X \setminus \{x\} \to Y$ that restricts to a quasisymmetric map, with uniform quasisymmetric regularity, on every ball $B(y, r)$ where $y \in X$ and $r = d(x, y)/2$.
- If (X, d) and (Y, l) are Ahlfors α-regular, then for every family of curves Γ in Y, we have
$$\operatorname{mod}_\alpha(\Gamma) \leq C \operatorname{mod}_\alpha\left(\{f^{-1} \circ \gamma : \gamma \in \Gamma\}\right),$$
where C is a constant that depends only on X, Y, and the quasi-Möbius regularity of f.

This last result is a special case of Theorem [BoK2, Theorem 2.7], which in turn follows from more general results of Tyson [T2, Theorem 6.4 and Lemma 9.2].

Assume the hypothesis of Corollary 1.0.3. Theorem 1.0.1 implies there exists a weak tangent of (X, d, μ) with uniformly big 1-modulus. We can apply Theorem 4.0.5 in order to get a non-compact weak tangent (Y, l, ν) with the same property. The first claim of Corollary 1.0.3 therefore follows from the first property listed above. For the case when (X, d) is Ahlfors α-regular, we claim that there exists $\delta > 0$ so that the collection of curves of diameter at least δr, contained in any ball $B(y, r)$ in X, where $y \in X$ and $r = d(x, y)/4$, has α-modulus at least δ. This then implies (X, d) has uniformly big α-modulus. Fix such a ball. As mentioned above, we have f restricts to a quasisymmetric map on $B(y, 2r)$ with uniform quasisymmetric regularity. This and the fact that (Y, l) is Ahlfors regular with uniformly big 1-modulus implies there exists $\delta > 0$, independent of y and r, such that the collection of curves in $f(B)$ with diameter at least $\delta \operatorname{diam} f(B)$ has α-modulus at least δ. The claim then follows from the second property of quasi-Möbius maps given above. This completes the proof. $\square$

Proof of Corollary 1.0.4. A key property of BPI metric spaces is that they are BPI equivalent to all of their weak tangents (see [DS, Corollary 9.9]). Since BPI spaces are by definition Ahlfors regular, Corollary 1.0.4 then follows from Corollary 1.0.2. This completes the proof. $\square$

Proof of Corollary 1.0.5. Let (X, d) be the Sierpinski n-carpet for some $n \in \mathbf{N}$, and observe that (X, d) is Ahlfors regular. It is a simple procedure to show that (X, d) has vanishing 1-modulus (see [S, pp. 29–34] for discussion on a similar topic), and moreover that every weak tangent of (X, d) contains a re-scaled isometric copy of (X, d). Therefore no weak tangent of (X, d) has uniformly big 1-modulus. Now apply Corollary 1.0.2 to conclude that the conformal Assouad dimension of (X, d) is strictly less than the Hausdorff dimension of (X, d). This completes the proof. $\square$

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