

LAGRANGIAN SPHERES IN $S^2 \times S^2$

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1 Introduction

The purpose of this paper is to consider the symplectic manifold $S^2 \times S^2$ with the direct sum symplectic form $\omega = \omega_0 \oplus \omega_0$, where ω_0 is the standard area form on S^2 . Then the antidiagonal $\overline{\Delta} = \{(x, -x) | x \in S^2\} \subset S^2 \times S^2$ is a Lagrangian submanifold, that is, $\omega|_L = 0$. It is the aim of this paper to demonstrate that any Lagrangian sphere L in $S^2 \times S^2$ must actually be isotopic to $\overline{\Delta}$ through Lagrangian spheres (or, in other words, Lagrangian isotopic to $\overline{\Delta}$). That is, we will show the following.

Theorem 1. *Let L be a Lagrangian sphere in $S^2 \times S^2$. Then there exists a Hamiltonian diffeomorphism of $S^2 \times S^2$ mapping L onto $\overline{\Delta}$.*

We note that any such Lagrangian sphere in $S^2 \times S^2$ is certainly homologous to $\overline{\Delta}$ since, by A. Weinstein's Lagrangian neighborhood theorem, all Lagrangian spheres have self-intersection number -2 .

The problem of studying Lagrangian knots in symplectic manifolds was first proposed by V.I. Arnold in [A] and some results are already known in this direction, see for example the survey [EP2]. We observe that $\Delta = \{(x, x) | x \in S^2\} \subset S^2 \times S^2$ is a symplectic submanifold and $S^2 \times S^2 \setminus \Delta$ is symplectomorphic to a neighbourhood of the zero-section in T^*S^2 , mapping $\overline{\Delta}$ to the zero-section. Y. Eliashberg and L. Polterovich have shown in [EP1] that any Lagrangian submanifold in T^*S^2 must be smoothly isotopic to the zero-section. On the other hand, P. Seidel in [S] has given examples of Lagrangian spheres in symplectic manifolds which are smoothly isotopic but not Lagrangian isotopic.

The main point of our proof will be to construct two transverse foliations of $S^2 \times S^2$ by spheres in the classes [point $\times S^2$] and [$S^2 \times$ point]. These spheres should be holomorphic with respect to some almost-complex structure compatible with ω and each sphere should intersect L transversally at

a single point. Of course, without the condition of intersecting L transversally we could take the standard foliation by spheres which are holomorphic with respect to a split complex structure $\mathbb{C}P^1 \times \mathbb{C}P^1$. To obtain our required foliations we will start with this complex structure but then deform it in a neighborhood of a contact hypersurface. As the almost-complex structure is deformed, Gromov showed in [G] that transverse foliations of holomorphic curves will continue to exist, but they will also be deformed and we will show that they eventually become transverse to L . The deformation of the almost-complex structure that we use is known as stretching-the-neck.

Taking a limit as the neck is stretched to infinite length is possible by some recent results due to F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki and E. Zehnder, see [BEHWZ], one obtains finite energy holomorphic curves in symplectic manifolds with cylindrical ends. One of these manifolds is $T^*L = T^*S^2$ and we describe in section 3 the resulting foliation of this manifold by holomorphic curves. In section 4 we discuss the behaviour of holomorphic curves as we deform the almost-complex structure (stretching-the-neck) and in section 5 use the accumulated information to reach our conclusion on the Lagrangian isotopy class. Various facts about finite energy holomorphic curves in symplectic manifolds with cylindrical ends and the relevant compactness theorem are gathered together first in section 2.

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2 Holomorphic Curves in Symplectic Manifolds with Cylindrical Ends

In this section we will state some theorems about holomorphic curves in open symplectic manifolds with cylindrical ends. The definitions and most of the proofs can be found in the series of papers by Hofer, Wysocki and Zehnder, [HoWZ2-5]. The generalizations to the slightly degenerate situation which we will study are taken from the paper of F. Bourgeois, [B1], see also [B2] for the proofs. Such a theory of holomorphic curves forms the basis of symplectic field theory, see [EGH].

We are interested in symplectic manifolds (W, ω) with noncompact ends symplectomorphic to either $((0, \infty) \times M, d(e^t \alpha))$ or $((-\infty, 0) \times M, d(e^t \alpha))$. Here M is a contact 3-manifold with α a contact form. Let X be the

corresponding Reeb vectorfield (which is uniquely defined by $X \lrcorner d\alpha = 0$ and $\alpha(X) = 1$) and $\xi = \{\alpha = 0\}$. The two types of ends are called convex or concave respectively.

Given such a (W, ω) , we equip it with a compatible almost-complex structure J . The compatibility condition means that $\omega(V, JV) > 0$ for all non-zero tangent vectors V , and at the point $(a, m) \in (0, \infty) \times M$ or $(-\infty, 0) \times M$, the almost-complex structure is defined by

$$J(a, m)(h, k) = (-\alpha(m)(k), J'(m)\pi k + hX(m)) \tag{1}$$

for $(h, k) \in T_{(a,m)}((-\infty, \infty) \times M)$, where $\pi : TM \rightarrow \xi$ denotes the projection along X , and J' is a fixed complex structure on ξ .

Finite energy holomorphic spheres are defined as follows.

Suppose $u : S^2 \setminus \Gamma \rightarrow W$ is a proper map, where $S^2 \setminus \Gamma$ denotes the Riemann sphere minus a finite set Γ of punctures.

The energy of u can be defined by

$$E(u) = \sup_{\phi} \int_{\mathbb{C}} u^* \omega_{\phi}$$

where the supremum is taken over all smooth, increasing functions $\phi : (-\infty, \infty) \rightarrow (0, 2)$ such that $\phi = e^t$ for t close to 0 and ω_{ϕ} is defined to be $d(\phi\alpha)$ on $(0, \infty) \times M$ and $(-\infty, 0) \times M$ and equal to the original ω elsewhere.

A finite energy holomorphic curve is then defined to be a J -holomorphic map u with $E(u) < \infty$.

Let η_t be the flow of the Reeb vectorfield X on M associated to α . Suppose that x is a periodic orbit of X of period T . Since $\eta_t^* \alpha = \alpha$, the differential $D\eta_T$ induces a linear map $L : \xi_{x(0)} \rightarrow \xi_{x(0)}$, and x is called nondegenerate if L does not contain 1 in its spectrum. The contact forms we will use do not have nondegenerate periodic orbits. In fact our contact manifold M will actually be foliated by periodic orbits, all with the same period, of the corresponding Reeb vectorfield. This is a special case of a Reeb flow of Morse–Bott type.

The following theorem is proven by Hofer, Wysocki and Zehnder in [HoWZ2, Theorems 1.2 and 1.4] in the nondegenerate case, and in [HoWZ5] in the Morse–Bott situation.

Theorem 2. *Let u be a finite energy holomorphic sphere from a Riemann surface with a noncompact end identified with $\mathbb{C} \setminus D$. We may assume that $u(\mathbb{C} \setminus D) \subset (-\infty, 0) \times M$ or $(0, \infty) \times M$. Let \bar{u} denote the projection of u to M . Then there exists a periodic orbit x of the Reeb vectorfield X , say*

of period T , and a sequence $R_k \rightarrow \infty$ such that

$$\bar{u}(R_k e^{\pm 2\pi i t/T}) \rightarrow x(t)$$

in $C^\infty(\mathbb{R})$. If x is nondegenerate or of Morse–Bott type then this limit exists for $R \rightarrow \infty$ and the asymptotic approach is exponential.

Punctures of a finite energy holomorphic sphere can be either positive or negative depending upon whether the image of the curve lies in $(-\infty, 0) \times M$ or $(0, \infty) \times M$ near the puncture. In Theorem 2 the sign in the exponent is $+1$ in the case of a convex end and -1 in the case of a concave end.

There is also a Fredholm theory for such curves. This was done in [HoWZ4] in the nondegenerate case and in our degenerate situation the result is again discussed in [B1], with the proofs given in [B2].

Let u be an embedded finite energy holomorphic sphere with positive ends asymptotic to Reeb orbits $\gamma_1^+, \dots, \gamma_{s^+}^+$ and negative ends asymptotic to Reeb orbits $\gamma_1^-, \dots, \gamma_{s^-}^-$. We are interested in the virtual dimension of the moduli space of finite energy spheres containing u , modulo reparameterizations. This is the index, $\text{index}(u)$, of a certain Fredholm operator.

For generic choices of almost-complex structure J satisfying equation (1) near the open ends, this index does indeed give the dimension of the moduli space of finite energy holomorphic spheres in a neighborhood of an embedded curve. As usual, virtual cycle techniques must be employed to deal with multiply covered curves, see the discussion in [B2]. In this paper, all of the finite energy curves we encounter will turn out to be embedded. The theorem below can be found in [B1] or [B2].

Theorem 3. *With u as above, the deformation index of u is given by*

$$\begin{aligned} \text{index}(u) = & -(2 - s^+ - s^-) + 2c_1(TW)[u] \\ & + \sum_{i=1}^{s^+} (\mu(\gamma_i^+) + \frac{1}{2}\text{dim}(\gamma_i^+)) - \sum_{i=1}^{s^-} (\mu(\gamma_i^-) - \frac{1}{2}\text{dim}(\gamma_i^-)) \end{aligned}$$

where $\mu(\gamma_i^\pm)$ is a generalized Conley–Zehnder index defined in [RS] and $\text{dim}(\gamma_i^\pm)$ is the dimension of the manifold of Reeb orbits containing γ_i^\pm .

The moduli space containing a sphere u will in general contain spheres asymptotic to a different set of Reeb orbits. The definitions of the Conley–Zehnder index and Chern class here are given with respect to a fixed trivialization along the Reeb orbits. More precisely, for each i we choose a symplectic trivialization of ξ along γ_i . With respect to this trivialization the Reeb flow gives a family of symplectic matrices $dh_t \in \text{Sp}(2, \mathbb{R})$ for $0 \leq t \leq T$

where T is the period of γ_i . We associate the index $\mu(\gamma_i)$ to this family following [RS]. Now, our trivialization along the γ_i naturally induces one of $TW|_{\gamma_i}$ (thinking of γ_i here as lying in $(0, \infty) \times M$ or $(-\infty, 0) \times M$) since $T_{\gamma_i(t)}W = \xi_{\gamma_i(t)} \oplus \mathbb{R} \frac{\partial}{\partial t} \oplus \mathbb{R}X$ and so of the complex line bundle $\bigwedge^2 TW$ along the γ_i . This can be used to define $c_1(TW)[u] = c_1(\bigwedge^2 TW)[u]$ as follows. We choose a section of $u^* \bigwedge^2 TW$ which coincides with our trivialization near the punctures and then count the numbers of zeros with multiplicity.

Finally we state a compactness theorem which will be needed in the course of our proof. Now let (W, ω, J) be a closed symplectic manifold with a compatible almost-complex structure J . The particular situation in which we will be interested is taking a limit when we deform the almost-complex structure in the neighbourhood of a contact-type hypersurface $\Sigma \subset W$, called stretching-the-neck.

A contact type hypersurface $\Sigma \subset W$ is an embedded contact manifold with a contact form α such that $\omega|_{\Sigma} = d\alpha$. This condition allows us to find a symplectic embedding of $((-\epsilon, \epsilon) \times \Sigma, d(e^t\alpha))$ to a neighbourhood V of Σ taking $\{0\} \times \Sigma$ onto Σ . By perturbing J near V we may assume that it coincides with the push-forward of an almost-complex structure given by formula (1) on $(-\epsilon, \epsilon) \times \Sigma$.

We suppose that Σ divides W into two symplectic manifolds W_1 and W_2 such that Σ is a convex boundary for W_1 and a concave boundary for W_2 . This means that the above embedding maps $(-\epsilon, 0) \times \Sigma$ into W_1 and $(0, \epsilon) \times \Sigma$ into W_2 .

Following [EGH] and [HoWZ6], we remove the tubular neighbourhood V of Σ from W and for each N replace it by gluing in a copy of $(-N, N) \times \Sigma$. We call the resulting manifold A_N and define an almost-complex structure J_N on A_N by again using formula (1) on $(-N, N) \times \Sigma$ and letting $J_N = J$ elsewhere. This almost-complex structure is compatible with a symplectic form ω_N on A_N and (A_N, ω_N) is symplectomorphic to (W, ω) via a symplectomorphism equal to the identity away from V .

We will need to study finite energy holomorphic spheres in three associated noncompact symplectic manifolds with cylindrical ends. Let \tilde{W}_1 be a completion of W_1 formed by gluing an end symplectomorphic to $((0, \infty) \times \Sigma, d(e^t\alpha))$ and equipped with a compatible almost-complex structure agreeing with J on the contact planes $\xi \subset T\Sigma$. Similarly define \tilde{W}_2 to be a completion of W_2 with end symplectomorphic to $((-\infty, 0) \times \Sigma, d(e^t\alpha))$ and equipped with a corresponding compatible almost-complex structure. Third we have the symplectization $S\Sigma = (\mathbb{R} \times \Sigma, d(e^t\alpha))$ which also has a

compatible almost-complex structure agreeing with J on the contact planes.

The following definitions and results are extracted from a more detailed discussion in [EGH], see also [B1]. For a proof see [BEHWZ] or [B2].

DEFINITION 4. *Let (S, j) be a genus 0 Riemann surface with nodes. Then a level k holomorphic map will consist of the following data:*

- (i) *A labelling of the components of $S \setminus \{\text{nodes}\}$ by integers $\{1, \dots, k\}$ called levels such that two components sharing a node have levels differing at most by 1. Let S_r be the union of components of level r .*
- (ii) *Finite energy holomorphic spheres $v_1 : S_1 \rightarrow \tilde{W}_1$, $v_r : S_r \rightarrow S\Sigma$, $2 \leq r \leq k - 1$, and $v_k : S_k \rightarrow \tilde{W}_2$. We require that each node shared by S_r and S_{r+1} is a positive puncture for v_r asymptotic to a Reeb orbit γ and a negative puncture for v_{r+1} asymptotic to the same Reeb orbit γ . Further v_r should extend continuously across each node within S_r .*

Suppose that $u_N : S^2 \rightarrow (A_N, J_N)$ are a sequence of J_N holomorphic curves where S^2 is the Riemann sphere with its complex structure i . We suppose that the curves in the sequence have bounded symplectic area. This is guaranteed if for instance they lie in a fixed homology class, the situation we encounter in this paper.

DEFINITION 5. *The sequence u_N converges to a level k holomorphic map from a Riemann surface with nodes (S, j) if there exist maps $\phi_N : S^2 \rightarrow S$ and sequences $t_N^r \in \mathbb{R}$, $r = 2, \dots, k - 1$, such that*

- (i) *the ϕ_N are diffeomorphisms except that they may collapse circles in S^2 to nodes in S , and $\phi_{N*}i \rightarrow j$ away from the nodes of S ;*
- (ii) *the sequences of maps $u_N \circ \phi_N^{-1} : S_1 \rightarrow \tilde{W}_1$, $u_N \circ \phi_N^{-1} + t_N^r : S_r \rightarrow S\Sigma$, $2 \leq r \leq k - 1$, and $u_N \circ \phi_N^{-1} : S_k \rightarrow \tilde{W}_2$ converge in the C^∞ topology to the corresponding maps v_r on compact subsets of S_r .*

In the above definition, as is necessary we are identifying $(-N, N) \times \Sigma \subset A_N$ with an increasing sequence of domains in $S\Sigma$, $W_1 \cup (-N, N) \times \Sigma \subset A_N$ with an increasing sequence of domains in \tilde{W}_1 and $W_2 \cup (-N, N) \times \Sigma \subset A_N$ with an increasing sequence of domains in \tilde{W}_2 .

Theorem 6. *There exists a subsequence $N(i)$ of N and a k such that the sequence $u_{N(i)}$ converges to a level k holomorphic map.*

3 Holomorphic Curves in T^*S^2

Using the round metric on S^2 , we can identify T^*S^2 with TS^2 . The aim of this section is to construct on TS^2 a convenient symplectic form and almost-complex structure and describe possible foliations by finite energy curves.

We will write $T^r S^2$ for the collection of tangent vectors of length r . The pullback λ of the Liouville form pdq from T^*S^2 restricts to a contact form α on $T^1 S^2$. The contact plane at $v \in T^1 S^2$ is spanned by horizontal and vertical vectors orthogonal to v and the corresponding Reeb flow coincides with the geodesic flow. In particular, it is periodic with period 2π . We will call a periodic orbit simple if its period is 2π .

Denote by $\pi : TS^2 \setminus S^2 \rightarrow T^1 S^2$ the projection along the fibers from the complement of the zero-section to the unit tangent bundle. Let $\phi : TS^2 \rightarrow [0, \infty)$ be a smooth increasing function such that $\phi|_{T^r S^2} = r$ for $r \leq 1$ and $\phi|_{T^r S^2} = e^r$ for r large. Then the 2-form defined by $\omega = d(\phi\pi^*\alpha)$ extends to a symplectic form on TS^2 equal to $d\lambda$ near the zero-section. Globally, (TS^2, ω) is symplectomorphic to $(T^*S^2, d(pdq))$ via a symplectomorphism fixing the zero-section.

We can think of (TS^2, ω) as one of our open symplectic manifolds with a cylindrical end. Observe that $SO(3) \equiv \text{Isom}(S^2)$ acts by differentials on TS^2 . This action is by symplectomorphisms preserving each $T^r S^2$.

We equip (TS^2, ω) with a compatible almost-complex structure J_0 satisfying the following conditions. The almost-complex structure J_0 should be invariant under the action of $SO(3)$ on TS^2 ; the contact planes $\xi = \ker(\pi^*\alpha)$ on $T^r S^2$ should be invariant under J_0 ; for r sufficiently large J_0 should be invariant under translation in the r direction and $J_0(\partial/\partial r) = X_r$. Here X_r is the Reeb vector field on $T^r S^2$ corresponding to the contact form $\pi^*\alpha$. For example, on the unit tangent bundle J_0 could be taken to be the standard almost-complex structure mapping vertical tangent vectors to their corresponding horizontal tangent vectors. For r large, J_0 coincides with one of the standard almost-complex structures on cylindrical ends given by formula (1).

We can now study finite energy holomorphic spheres in TS^2 . For the moment we will assume that the almost-complex structure J_0 is suitably perturbed to an almost-complex structure J such that the linearization of the Fredholm operator from Theorem 3 is surjective and so its index gives the dimension of the corresponding moduli space of finite energy spheres.

In the case of TS^2 there are no negative ends, so a finite energy sphere

u has only positive asymptotic limits, $\gamma_1, \dots, \gamma_s$. Suppose that γ_i covers a simple closed orbit $\text{cov}(\gamma_i)$ times. We suppose that u is embedded.

LEMMA 7. *The dimension of the moduli space of finite energy planes containing u is given by*

$$\text{index}(u) = 2(s - 1) + \sum_{i=1}^s 2 \text{cov}(\gamma_i).$$

Proof. We choose a global trivialization of ξ over T^1S^2 of horizontal and vertical tangent vectors in TS^2 . The induced trivialization of $\bigwedge^2 T(TS^2)$ over T^1S^2 extends over all of TS^2 because of the existence of a global splitting of $T(TS^2)$ into Lagrangian vertical and horizontal subspaces. Thus in the formula of Theorem 3 the Chern class term will always be zero. Further, $\dim(\gamma_i) = 2$ for all i since we have only one family of Reeb orbits. Thus to apply Theorem 3 it remains to compute the Conley–Zehnder index of an orbit γ with respect to this trivialization, say $\{H, V\}$ where H and V are unit horizontal and vertical vectors in ξ respectively. Given a vector $v \in \xi_{\gamma(0)}$, the image under the Reeb flow $d\eta_t(v) = \frac{d}{ds}|_{s=0}\gamma_s(t)$ where γ_s is a family of closed orbits with $\frac{d}{ds}|_{s=0}\gamma_s(0) = v$. Write $d\eta_t(v) = u_t H + w_t V \in \xi_{\gamma(t)}$. Since all Reeb orbits c in T^1S^2 correspond to geodesics \bar{c} in S^2 we observe that $(u_t, w_t) = (J(t), J'(t))$ where $J(t)$ is the component of the Jacobi field along $\bar{\gamma}$ corresponding to the variation $\bar{\gamma}_s$. This Jacobi field is perpendicular to $\bar{\gamma}$ since $d\eta_t(v) \in \xi_{\gamma(t)}$. The Jacobi equation for the sphere is $J'' + J = 0$ and hence $(u_t, w_t) = (\cos(t)u_0 + \sin(t)w_0, -\sin(t)u_0 + \cos(t)w_0)$ or equivalently

$$d\eta_t = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \in \text{Sp}(2, \mathbb{R}), \quad 0 \leq t \leq 2\pi \text{cov}(\gamma).$$

But this path has Conley–Zehnder index $2 \text{cov}(\gamma)$ and the result follows. \square

Actually we can note that there are general formulas relating Conley–Zehnder indices of closed Reeb orbits in unit tangent bundles with Morse indices of the corresponding geodesics, see the discussion in [Mo].

From the lemma we see that $\text{index}(u) \geq 2$ with equality if and only if u has a single puncture and simply covers a Reeb orbit at that puncture.

Now, there are various possible foliations of TS^2 by finite energy holomorphic spheres. For example, there exists a foliation by finite energy planes in which all curves are asymptotic to a cover of the same Reeb orbit (this was done in [H] in a nondegenerate situation but the same foliation exists here). We will obtain a foliation by a stretching-the-neck procedure.

This is described in section 4 and we prove there that the foliation is by finite energy planes asymptotic to simple Reeb orbits. Lemma 10 proves this and that there is a single plane asymptotic to each such orbit. Meanwhile in this section we derive some properties of such a foliation.

LEMMA 8. *Suppose that TS^2 is foliated by finite energy planes with a single plane asymptotic to each simple Reeb orbit. Then the intersection number of each plane with the zero-section is ± 1 .*

Proof. Think of TS^2 as a neighborhood U of $\overline{\Delta}$ in $S^2 \times S^2$ with smooth boundary $\Sigma = T^1S^2$. Then $S^2 \times S^2 \setminus U$ is a disk bundle over Δ and the boundaries of the disks are the Reeb orbits in Σ . The asymptotic behaviour of our finite energy planes allows us to compactify them to maps $(D, \partial D) \rightarrow (TS^2 = U, \Sigma)$, and we can glue the boundaries to the disks in the complement of TS^2 to obtain a foliation of $S^2 \times S^2$, smooth at least away from Σ . As the spheres in the foliation intersect Δ in a single point each and necessarily have self-intersection number 0, they lie in one of the classes $[S^2 \times \text{pt}]$ or $[\text{pt} \times S^2]$. In particular, the finite energy planes in TS^2 intersect $\overline{\Delta}$ with intersection number ± 1 . \square

Such a foliation of finite energy planes will be shown to exist with respect to any regular almost-complex structure J on TS^2 . We now consider a sequence of regular almost-complex structures converging smoothly to the $SO(3)$ invariant structure J_0 . The author does not know whether or not this structure can be assumed regular, that is, whether or not our index formula is still valid.

In any case, following [HoWZ6], by a similar procedure by which we will later stretch-the-neck, using the compactness theorem we can take a limit of a subsequence and find a J_0 -holomorphic finite energy plane through any point in TS^2 . Taking a diagonal sequence of almost-complex structures we can find disjoint and embedded planes through a dense set of points. There is no bubbling here since simple orbits already have minimal period and TS^2 contains no closed holomorphic curves. The planes stay disjoint and embedded in the limit by an application of positivity of intersections, see [M]. Since the limiting planes are still asymptotic to simple Reeb orbits they are not multiply covered. By a further limiting process we can include these planes in a foliation of TS^2 , and, again by positivity of intersections, planes in this foliation exhaust the possible limits of our finite energy planes in this subsequence up to reparameterization.

LEMMA 9. *The J_0 -holomorphic foliation of TS^2 is by finite energy planes asymptotic to distinct simple orbits of the Reeb flow. Each plane intersects the zero-section transversally in a single point.*

Proof. Let C be a plane in our foliation asymptotic to a Reeb orbit γ corresponding to a geodesic $\bar{\gamma}$, and let K be the S^1 subgroup of $SO(3) = \text{Isom}(S^2)$ which preserves $\bar{\gamma}$, that is, the group of rotations about a perpendicular axis. For the first part, we choose $L \subset SO(3)$ to be a small disk through the identity which is transverse to K . Then by the $SO(3)$ invariance of J_0 , for any $l \in L$ the plane $l.C$ is also a finite energy plane asymptotic to a Reeb orbit close to γ . Similarly to the argument in Lemma 8, we deduce that all such finite energy planes are disjoint. To see this, we observe that as the asymptotic limits are disjoint, so are the disks we can glue in $S^2 \times S^2 \setminus U$ to obtain spheres in $S^2 \times S^2$ as in Lemma 8. These spheres have intersection number $+1$ with Δ and nonnegative self-intersection number by the positivity of intersections (the spheres corresponding to C and $l.C$ intersect only in U). Together this implies a self-intersection number of 0 and that the planes are disjoint. Therefore the planes form a foliation of a neighborhood of C . Now, any other finite energy plane C' in our foliation asymptotic to γ but disjoint from C must intersect some of the curves $l.C$. But this contradicts positivity of intersections since C' must be homotopic to C (fixing γ in a suitable compactification).

For the second part, we also observe that $k.C$ must coincide with C for all $k \in K$ (for otherwise these planes would intersect some $l.C$ giving a contradiction as above). The orbits of K on S^2 consist of a point p , the antipodal point q to p , and circles around p . Our plane cannot intersect S^2 in a circle for this would imply the existence of a holomorphic disk in TS^2 with boundary on S^2 , a contradiction to Stokes' theorem since ω is positive on holomorphic curves whereas its primitive vanishes on the zero-section S^2 . Therefore the plane can intersect S^2 only at the points p and q . Now, K acts transitively on $T_p S^2 \subset T_p(TS^2)$ and $T_q S^2 \subset T_q(TS^2)$, so these intersections must be transversal (otherwise the plane would be tangent to S^2 at p or q , a contradiction since S^2 is Lagrangian while embedded holomorphic curves are symplectic). Thus each intersection is transversal and will contribute ± 1 to our intersection number. Since the total intersection number is also ± 1 we deduce that our finite energy planes (which are embedded) intersect S^2 transversally in a single point as claimed. \square

We close this section by remarking that the transversal intersection property will remain true for the regular almost-complex structures in our subsequence which are sufficiently close to J_0 by the smooth convergence ensured by the compactness theorem.

4 Stretching the Neck

Suppose that $L \subset S^2 \times S^2$ is a Lagrangian submanifold homologous to $\overline{\Delta}$. By Weinstein's theorem, a sufficiently small neighbourhood U of L can be symplectically embedded into T^*S^2 , taking L to the zero-section. Let $T^{\leq R}S^2$ denote the metric tube of radius R inside TS^2 , R large, and suppose that as in the previous section we have constructed a symplectic form $\omega = d(\phi\pi^*\alpha)$ on $T^{\leq R}S^2$ such that $\phi\pi^*\alpha$ is equal to the standard Liouville form near the zero-section.

Now, for ϵ small enough, $(T^{\leq R}S^2, \epsilon\omega)$ can be symplectically embedded into U , again sending the zero-section to L .

Let $\Sigma \subset U \subset S^2 \times S^2$ be the boundary of a tubular neighbourhood of $L \subset S^2 \times S^2$ symplectomorphic to $(T^{\leq R}S^2, \epsilon\omega)$. We can push forward our $SO(3)$ -invariant almost-complex structure on $(T^{\leq R}S^2, \epsilon\omega)$ and extend it to a compatible almost-complex structure on $S^2 \times S^2$. If necessary we will perturb this almost-complex structure slightly such that the index of the Cauchy–Riemann operator does indeed give the dimension of our moduli spaces of holomorphic curves in all cases after we stretch the neck.

As described in [G], for any such compatible almost-complex structure J on $S^2 \times S^2$ there exist two corresponding foliations \mathcal{F}_1 and \mathcal{F}_2 by J -holomorphic spheres in the classes $[\text{point} \times S^2]$ and $[S^2 \times \text{point}]$. Each sphere in \mathcal{F}_1 intersects each sphere in \mathcal{F}_2 transversally in a single point. We will now deform the almost-complex structure in a neighbourhood of Σ , stretching the neck as described in section 2 to get a sequence J_N of almost-complex structures on $S^2 \times S^2$. We note that our almost-complex structure is already in the standard form (1) near Σ . For each N we have corresponding J_N -holomorphic foliations \mathcal{F}_1 and \mathcal{F}_2 .

We apply the compactness theorem from section 2 as we take the limit $N \rightarrow \infty$. Different reparameterizations of a suitable subsequence converge to finite energy holomorphic curves from punctured spheres into one of three symplectic manifolds, namely TS^2 with our almost-complex structure, the symplectization $S\Sigma$ of Σ with its translation invariant almost-complex structure and the complement of U in $S^2 \times S^2$ completed with a cylindrical end symplectomorphic to $((-\infty, 0) \times \Sigma, d(e^t\alpha))$. The limit of a sequence of spheres can be thought of as a tree in which the vertices are finite energy spheres and the edges connect finite energy spheres with the same Reeb orbits as asymptotic limits. Choosing reparameterizations of spheres in \mathcal{F}_1 which pass through a chosen point of U , after taking a subsequence of $N \rightarrow \infty$, in the limit we can find a finite energy sphere

passing through any point of TS^2 . Now, taking a diagonal subsequence, we can find a collection of finite energy spheres in the limit which pass through a dense set of points in TS^2 . A second limiting process as in [HoWZ6] can be used to find a finite energy sphere through every point of TS^2 . These punctured spheres actually form a foliation and reparameterizations of any converging sequence of J_N -holomorphic spheres must converge to one of these finite energy spheres. This follows from the nature of the convergence. Any intersections or singular points would also be seen as intersections amongst J_N -holomorphic curves in the corresponding foliation \mathcal{F}_1 , see for instance [M]. This does not however immediately exclude the possibility of the image of a limiting curve arising as a branched cover. As is required for section 3, we want to show that this foliation is by planes asymptotic to simple Reeb orbits.

As a remark, we observe that whether or not such a foliation arises as a result of this limiting process, such a foliation is necessarily present in TS^2 with our symmetric compatible almost-complex structure. One way of seeing this would be to replace L with $\bar{\Delta}$ so that the whole arrangement in $S^2 \times S^2$ is invariant under the action of $SO(3)$. Then every Reeb orbit must be an asymptotic limit of a sphere in our foliation and thus the foliation is by planes invariant under 1-parameter subgroups and asymptotic to simple Reeb orbits.

Taking further diagonal subsequences, the limiting process also gives a foliation by finite energy spheres in the (completed) complement of U , say W . By studying this foliation and the spheres in the symplectization $S\Sigma$ of Σ , we are able to derive some information about our foliation of TS^2 . Now, fixing the trivialization along the Reeb orbits as before, the Chern class term $c_1(T(S^2 \times S^2))[u]$ in Theorem 3 is now equal to 2 for the components of finite energy spheres u mapping to W in our homology class. This is because $c_1(T(S^2 \times S^2))$ gives 2 when evaluated on spheres in the foliation \mathcal{F}_1 but our trivialization on TS^2 (and $S\Sigma$) is chosen such that it gives 0 on the punctured spheres in TS^2 and $S\Sigma$.

We can now observe that the component of the limiting holomorphic map which has image in W must be connected. For otherwise we could find such a finite energy sphere v in W with Chern class $c_1(T(S^2 \times S^2))[v] < 2$. This is a contradiction since such a sphere could be glued to some planes in TS^2 to produce a symplectic sphere in $S^2 \times S^2$ of Chern class less than 2. This also implies immediately that such spheres in W are not multiple covers and therefore that our index formula is valid (as the almost-complex structure was chosen generically).

The spheres u in W have only negative asymptotic ends, say $\gamma_1, \dots, \gamma_s$, and so we find by Theorem 3 and the computation of the Conley–Zehnder indices in Lemma 7 that the dimension of the corresponding moduli space is given by

$$\text{index}(u) = 2(s + 1) - \sum_{i=1}^s 2 \text{cov}(\gamma_i).$$

In particular, it is at most two, and equals two only if all of the asymptotic limits simply cover the Reeb orbits. Therefore an open dense region of W is foliated by curves having simple asymptotic limits. Further, generic simple Reeb orbits appear as negative asymptotic limits for curves in the foliation of W . (Fixing a set of these limits would necessarily give a moduli space of dimension less than two.)

In the symplectization $S\Sigma$ of Σ , the finite energy spheres which appear in our limits must have a single positive puncture. (The maximum principle implies that there must be at least one positive puncture.) This follows because the limit curve has a single component in W , but as we are dealing only with curves of genus 0, different positive asymptotic limits of curves in $S\Sigma$ could not be connected to the same component. If this positive puncture is asymptotic to a simple Reeb orbit then the curve must actually be a cylinder $\mathbb{R} \times \gamma$ over this orbit. This is because $\int u^* \pi^* d\alpha \geq 0$ for all curves in $S\Sigma$, where π denotes the projection onto Σ (in fact the formula $u^* \pi^* d\alpha \geq 0$ holds pointwise), and equality occurs if and only if the curve is a cylinder. But by the asymptotic convergence to Reeb orbits, this integral is just the difference between the periods of the positive and negative asymptotic limits. There is a negative asymptotic limit since γ is noncontractible and hence, since γ has minimal period, the integral is nonpositive.

Similarly, the components of our limiting curve in TS^2 must have a single positive asymptotic limit. Since for a generic curve (that is, its component in W passes through an open dense subset in W) its component in W has only simple negative asymptotic limits and any components in $S\Sigma$ are cylinders, the components of the curve in TS^2 are planes with a simple asymptotic limit (and in particular are not multiple covers). We find such finite energy planes asymptotic to an open dense set of Reeb orbits and use this information to deduce our final lemma as claimed in section 3.

LEMMA 10. *The asymptotic limits of the curves in our foliation of TS^2 are simple Reeb orbits, and there is a single curve asymptotic to each Reeb orbit.*

Proof. The moduli space of finite energy planes in TS^2 asymptotic to simple Reeb orbits does indeed have dimension 2 by the formula of Lemma 7,

and the part of the moduli space close to a given plane consists of planes asymptotic to the nearby Reeb orbits (since our almost-complex structure is assumed regular). Again, such nearby planes are automatically disjoint and as in Lemma 9 we see that there is a single plane asymptotic to each simple orbit. The moduli space of such planes is compact since their asymptotic limits have minimal period and no further bubbling is possible. Thus, including planes asymptotic to the remaining Reeb orbits gives a foliation of TS^2 which is our foliation as required. \square

Hence we can now apply the results of section 3 to deduce that if our almost-complex structure is chosen sufficiently close to J_0 each plane will intersect the zero-section S^2 transversally in a single point. Thus, after taking the subsequence, for N sufficiently large the J_N -holomorphic spheres in the foliation \mathcal{F}_1 must also intersect L transversally in a single point. Taking a further subsequence, we can assume that the same is true for spheres in the foliation \mathcal{F}_2 .

In conclusion, we have shown the existence of an almost-complex structure J_N on $S^2 \times S^2$, tamed by the standard symplectic form, such that the transverse foliations \mathcal{F}_1 and \mathcal{F}_2 by holomorphic spheres have the property that each leaf in each foliation intersects L transversally in a single point.

5 Conclusion of Proof

We want to construct a Lagrangian isotopy between L and $\overline{\Delta}$. In fact, since the group of symplectomorphisms of $S^2 \times S^2$ which act trivially on homology is connected (by a result of Gromov [G] it is homotopic to $SO(3) \times SO(3)$), it will suffice to construct a symplectomorphism taking L to $\overline{\Delta}$.

Corresponding to L we have an almost-complex structure J such that the holomorphic curves in the corresponding foliations \mathcal{F}_1 and \mathcal{F}_2 intersect L transversally in single points. Similarly, the standard foliations $\text{point} \times S^2$ and $S^2 \times \text{point}$ are holomorphic for the standard split complex structure J_0 and each curve intersects $\overline{\Delta}$ in a single point.

Now, there is a unique extension of any diffeomorphism $L \rightarrow \overline{\Delta}$ to a diffeomorphism ϕ of $S^2 \times S^2$ sending the transverse foliations corresponding to J onto those corresponding to J_0 . Provided that our initial diffeomorphism is chosen to be orientation preserving, ϕ will preserve the complex orientation on the leaves and act trivially on homology. Both $\phi^{-1*}\omega$ and ω itself are compatible with the almost complex structure ϕ_*J (in the second case because ϕ_*J preserves the foliations $\text{point} \times S^2$ and $S^2 \times \text{point}$ which

are orthogonal with respect to ω) and so $\omega_t = (1-t)\phi^{-1*}\omega + t\omega$ is a family of cohomologous symplectic forms on $S^2 \times S^2$ with respect to which $\bar{\Delta}$ is Lagrangian (the forms are clearly closed and they are nondegenerate since each is compatible with ϕ_*J).

Using Moser's method, we write $\omega_t = \phi^{-1*}\omega + d\beta_t$. Then $d\beta_t|_{\bar{\Delta}} = 0$ which implies that $\beta_t|_{\bar{\Delta}} = dh_t$ for some function $h_t : \bar{\Delta} \rightarrow \mathbb{R}$. We can extend the h_t smoothly to functions on $S^2 \times S^2$, replace our β_t by $\beta_t - dh_t$ and therefore assume that $\beta_t|_{\bar{\Delta}} = 0$ for all t .

Now let X_t be the unique solution of $X_t|_{\omega_t} = \frac{d\beta_t}{dt}$. Then $\mathcal{L}_{X_t}\omega_t = d(\frac{d\beta_t}{dt}) = \frac{d\omega_t}{dt}$ and since $\frac{d\beta_t}{dt}|_{\bar{\Delta}} = 0$ we deduce that X_t must be tangent to $\bar{\Delta}$ along $\bar{\Delta}$. It follows that the time-1 flow ψ of the time-dependent vector field X_t is a diffeomorphism preserving $\bar{\Delta}$ and such that $\psi^*\omega = \phi^{-1*}\omega$. Hence $\psi \circ \phi$ is our required symplectomorphism.

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